

Weak Solutions in Nonlinear Poroelasticity with Incompressible Constituents

Lorena Bociu*

Boris Muha†

Justin T. Webster‡

February 22, 2022

Abstract

We consider quasi-static nonlinear poroelastic systems with applications in biomechanics and, in particular, tissue perfusion. The nonlinear permeability is taken to be dependent on solid dilation, and physical types of boundary conditions (Dirichlet, Neumann, and mixed) for the fluid pressure are considered. The system under consideration represents a nonlinear, implicit, degenerate evolution problem, which falls outside of the well-known implicit semigroup monotone theory. Previous literature related to proving existence of weak solutions for these systems is based on constructing solutions as limits of approximations, and energy estimates are obtained only for the constructed solutions. In comparison, in this treatment we provide for the first time a direct, fixed point strategy for proving the existence of weak solutions, which is made possible by a novel result on the uniqueness of weak solutions of the associated linear system (where the permeability is given as a function of space and time). The uniqueness proof for the associated linear problem is based on novel energy estimates for arbitrary weak solutions, rather than just for constructed solutions. The results of this work provide a foundation for addressing strong solutions, as well as uniqueness of weak solutions for nonlinear poroelastic systems.

Keywords: nonlinear poroelasticity, implicit evolution equations, quasilinear parabolic, weak solutions, energy methods, incompressible constituents

2020 AMS: 35M13, 35A01, 35B65, 35Q86, 35Q92, 74F10, 76S05

Acknowledgments: L. Bociu was partially supported by NSF-DMS 1555062 (CAREER). J.T. Webster was partially supported by NSF-DMS 1907620. B. Muha was partially supported by the Croatian Science Foundation project IP-2019-04-1140. B. Muha and J.T. Webster thank UMBC's Office of Research Development for the generous support through a SURFF grant, as well as the Department of Mathematics and Statistics for facilitating their collaboration in October and November, 2020. The authors wish to sincerely thank the anonymous referees for their incisive and close reading of the manuscript, and several suggestions which dramatically improved the clarity and quality of the paper.

*2311 Stinson Dr., North Carolina State University, Raleigh, NC, 27695; lbociu@ncsu.edu

†University of Zagreb, Faculty of Mathematics, Croatia; borism@math.hr

‡1000 Hilltop Dr., University of Maryland, Baltimore County, Baltimore, MD, 21250; websterj@umbc.edu

1 Introduction

The fully dynamic Biot model in poroelasticity is a coupled, mixed hyperbolic-parabolic system that describes the behavior of a deformable saturated porous medium. The momentum balance equations for the elastic displacement $\mathbf{u}$ of the medium and the mass balance equation for the fluid pressure p , under the assumptions of full saturation and small deformations for the homogeneous porous medium, are given by

$$\begin{cases} \rho \mathbf{u}_{tt} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \alpha \nabla p = \mathbf{F}(x, t), \\ (c_0 p + \alpha \nabla \cdot \mathbf{u})_t - \nabla \cdot k \nabla p = S(x, t). \end{cases} \quad (1.1)$$

The key parameters in the system are: the density of the porous and permeable medium $\rho > 0$, the Lamé parameters λ and μ , the Biot-Willis constant $\alpha > 0$ which accounts for the pressure-deformation coupling, and the constrained storage coefficient $c_0 \geq 0$ which combines the porosity of the medium and the compressibility of both the fluid and solid constituents [41]. The given function $\mathbf{F}$ represents an elastic body force, while S is a given fluid source. This coupled system can describe the settlement of soils under load, wave propagation in fluid-saturated porous media, as well as perfusion in tissues and organs. Consequently, it has received a lot of attention in geophysics and civil engineering, and industrial and biomedical applications [2–4, 9, 11, 12, 14, 17–19, 30, 35–37, 46, 48, 50].

In most biological and biomechanical applications, the inertial effects (the accelerations of both fluid and solid) are negligible, so that one can focus on an elastic *quasi-static* deformation of the fluid-saturated porous medium [7, 11, 15, 19, 25, 29, 38]. In this scenario, the coupling is of elliptic-parabolic type, where the small deformations of the solid matrix are described by the Navier equations of linear elasticity, and the diffusive fluid flow is described by Duhamel’s equation:

$$\begin{cases} \mathcal{E} \mathbf{u} + \alpha \nabla p = \mathbf{F}(x, t) \\ [c_0 p + \alpha \nabla \cdot \mathbf{u}]_t + A p = S(x, t), \end{cases} \quad (1.2)$$

where $\mathcal{E}$ is an appropriate “elasticity” operator (described precisely in Section 4.1), while $A = -\nabla \cdot [k \nabla]$ is the diffusion operator. Moreover, due to the fact that biological tissues have a mass density close to that of water, one can work under the assumption of incompressible solid and fluid constituents.¹ Mathematically, this assumption translates into the following parameter simplifications: $c_0 = 0$ and $\alpha = 1$ [19]. In this case, the pressure equation in (1.2) can degenerate where $\nabla \cdot \mathbf{u} \equiv 0$. The coupled system (1.2) can be reduced to an implicit evolution equation by solving the elliptic equation for displacement $\mathbf{u}$ in terms of pressure p . There has been great interest in *implicit evolution equations* [39–42] (and references therein). In fact, general theories have been developed for implicit systems of monotone type

$$[Bp]_t + Ap \ni S, \quad (1.3)$$

¹The solid and the fluid phases cannot undergo volume changes at the microscale.

where the operator A and the pressure-to-dilation operator B may in fact be nonlinear [20,43]. As noted above, when B has a non-trivial kernel, this abstract system (1.3) is referred to as *degenerate* [41]. In the case of compressible constituents $c_0 > 0$, the operator $c_0 I + B$ becomes coercive [1] and invertible on L^2 , which permits simplification upon inversion. This case is referred to in [26,40,42] as a “regular” implicit equation. Thus the case of fluid-solid mixtures with compressible constituents ($c_0 > 0$) is fundamentally different from the scenario with incompressible fluid and solid constituents (see [6] for more detailed discussion).

A new challenge present in systems like (1.2), motivated by biological structures like tissues, organs, cartilages and bones, is the fact that the permeability k is not a constant parameter; rather, it is a function that depends on the pore architecture inside the body as well as the properties of the fluid [49]. For example, if a Newtonian fluid flows in the interstitial spaces of a pack of spherical particles, then the Carman-Kozeny formula is used, which states that k is a nonlinear function of the solid dilation $\nabla \cdot \mathbf{u}$, given by $k(y) \sim y^3(1 - y)^{-2}$ [27]. On the other hand, if a Newtonian fluid flows inside cylindrical pores, then the formula for capillary beds states that permeability is proportional to a quadratic function of $\nabla \cdot \mathbf{u}$ [11]. This nonlinear dependence of permeability on solid dilation introduces a quasilinearity into the dynamics that is not monotone in nature [6,7,13]. The latter fact disqualifies the nonlinear theory which has been developed in the above mentioned mathematical works [20,40,43], where the nonlinearity is monotone and depends directly on the pressure p .

Main Contributions. In this treatment we focus on quasi-static systems like (1.2) with incompressible constituents ($c_0 = 0$), nonlinear permeability k dependent on solid dilation, and physically-motivated pressure boundary conditions (Dirichlet, Neumann, and mixed). For a complete description of the PDE system under consideration, see Section 2.1. Existence and uniqueness criteria for weak solutions to these systems have been addressed recently in [6,7]. The proof presented in [7] is constructive, and based on Rothe’s method. The reference [6] shows existence of weak solutions through a multi-valued map fixed point argument in the simplified scenario of homogeneous boundary conditions for both solid displacement and fluid pressure. *In comparison, the present treatment provides a straightforward approach based on a fixed point map strategy, made possible by a novel result on the uniqueness of weak solution to the associated linear coupled system with given permeability $K(\mathbf{x}, t)$.* More specifically, as a preliminary step, we consider the analysis of a linear, time-dependent poroelasticity system, where the nonlinearity can be replaced with a given function of space and time:

$$-\operatorname{div}[k(\nabla \cdot \mathbf{u})\nabla] \mapsto -\operatorname{div}[K(\mathbf{x}, t)\nabla].$$

Then one deals with an implicit, time-dependent linear parabolic problem

$$[Bp]_t + A(t)p = S,$$

where the abstract work in [40, Chapter III.3] can be invoked to obtain existence of weak solutions. Regarding the issue of uniqueness of weak solution, from the point of view of abstract variational theory [40] or discretization approaches [7,13,51], one inherits the critical

problem that only *constructed* weak solutions satisfy energy estimates. Existing theory requires additional smoothness (time differentiability) of k in order to circumvent this issue [40, pp.115–117], which unfortunately is not available for the nonlinear problem of interest.

The crux of the matter here in proving uniqueness of weak solutions for the linear problem (without extra regularity assumptions) is obtaining an appropriate energy estimate for arbitrary weak solutions, rather than for just constructed solutions (as in [6, 40]). Formally, one can see from the dynamic Biot system (1.2) that the “natural” elasticity multiplier is $\mathbf{u}_t$, as it elicits cancellation of “cross” coupled terms; $\mathbf{u}_t$ remains the desired multiplier even in the quasi-static scenario. However, there is no clear temporal regularity associated to $\mathbf{u}_t$ in the latter case. Additionally, the implicit presentation (1.3) demonstrates a peculiarity in passing between temporal and spatial regularity in the equation, which must take place through the pressure-to-dilation B operator. To address these issues, inspired by [1, 41, 42], we effectively “mod out” $\text{Ker}(B)$ in the variational structure of the problem, in conjunction with a time mollification in the appropriate operator-theoretic framework. We also take advantage of the connection between the reduced, implicit formulation and the full quasi-static Biot formulation, to exploit the divergence structure (embedded Stokes problem) of the equations.

To summarize, we obtain these novel results: (i) uniqueness of weak solution for the time-dependent linear poroelasticity problem with $A(t) = -\nabla \cdot [K(\mathbf{x}, t)\nabla]$, without requiring additional time regularity on the permeability by providing (ii) a priori estimates for arbitrary weak solutions, rather than for just the constructed solutions. The aforementioned linear uniqueness problem is resolved in a way that can be utilized in order to obtain (iii) a direct fixed-point argument for the Biot system with permeability depending nonlinearly on the solid dilation, as was not possible in previous literature [6, 7, 13]. Additionally, we provide the first, clear functional framework for weak solutions, including a justification of the regularity and type of initial data taken, while addressing the degeneracy induced by the incompressible constituents ($c_0 = 0$) through appropriate modifications of the pressure state space.

2 Main Results and Discussion

2.1 PDE Model of Nonlinear Poroelasticity

We relegate our attention to the physical assumptions of full saturation of the porous media, negligible inertia, small deformations, and incompressible mixture components [7] (and references therein). Let $\Omega \subset \mathbb{R}^3$ be the fluid-solid mixture domain, of class $\mathcal{C}^2$, with boundary $\Gamma = \partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ and unit outward normal $\mathbf{n}$. Here Γ_D and Γ_N are Dirichlet and Neumann parts of the boundary (*with respect to the pressure variable*), respectively, and $\Gamma_N \cap \Gamma_D = \emptyset$ (although we permit their closures to intersect). The balance of momentum for the fluid-solid mixture and the balance of mass for the fluid are given by

$$-\nabla \cdot \mathbf{T}(\mathbf{u}, p) = \mathbf{F} \quad \text{in } \Omega \times (0, T) \quad (2.1)$$

$$\zeta_t + \nabla \cdot \mathbf{v} = S \quad \text{in } \Omega \times (0, T). \quad (2.2)$$

The notation used for the system variables along with the constitutive relations are described below. The variable $\mathbf{u}$ represents the solid displacement, while p is the Darcy fluid pressure and $\mathbf{v}$ is the associated Darcy velocity of the fluid.

We work here in the mathematically simplified framework of homogeneous Dirichlet conditions for the displacement, and we permit Dirichlet, Neumann, and mixed type conditions for the pressure. The total stress of the fluid-solid mixture is given by $\mathbf{T} = \sigma(u) - p\mathbf{I}$. The linearized stress tensor field $\sigma(u)$ is given by $\sigma(u) = 2\mu\varepsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}$, where the symmetrized gradient $\varepsilon(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$ represents the linearized strain tensor field, and λ and μ are the standard Lamé parameters. We use $\nabla\mathbf{u}$ to denote the Jacobian of $\mathbf{u}$, i.e., $\nabla\mathbf{u} = (\partial_j u^i)$, with $\nabla\mathbf{u}^T = (\partial_i u^j)$. The balance of linear momentum for the mixture (2.1) can be written equivalently as $-\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \nabla p = \mathbf{F}$.

The so called *fluid content* is given here by the constitutive relation $\zeta = \nabla \cdot \mathbf{u}$. This is a simplification of the general Biot formula $\zeta = c_0 p + \alpha \nabla \cdot \mathbf{u}$ where c_0 is the constrained specific storage coefficient and α is the Biot-Willis coefficient [1–4, 41]; due to the fact that we have incompressible mixture components (as discussed above), we have that $c_0 = 0$ and $\alpha = 1$ [7, 19]. The *discharge velocity* has the following dependence on pressure and permeability: $\mathbf{v} = -k(\nabla \cdot \mathbf{u})\nabla p$, where the permeability $k(\cdot)$ is a nonlinear scalar function. In this consideration, we take a continuous function k , with positive lower and upper bounds (see Assumption 1). The body force $\mathbf{F}$ and source S are given functions of space and time.

Taking the above into account, the formulation of our problem becomes: *Given data d_0 , $\mathbf{F}$, and S , find solution $(\mathbf{u}, p)$ that satisfies:*

$$\begin{cases} -\Delta\mathbf{u} - 2\nabla(\nabla \cdot \mathbf{u}) + \nabla p = \mathbf{F} & \text{in } \Omega \times (0, T) \\ [\nabla \cdot \mathbf{u}]_t - \nabla \cdot [k(\nabla \cdot \mathbf{u})\nabla p] = S & \text{in } \Omega \times (0, T) \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma \times (0, T) \\ k\nabla p \cdot \mathbf{n} = 0 & \text{on } \Gamma_N \times (0, T) \\ p = 0 & \text{on } \Gamma_D \times (0, T) \\ [\nabla \cdot \mathbf{u}](0) = d_0 & \text{in } \Omega. \end{cases} \quad (2.3)$$

The Lamé parameters λ and μ have been set equal to 1, without loss of generality.

In using a fixed point argument (Section 5), we will consider linearizing the above system, taking $k = k(z)$, for a given $z \in L^2(0, T; L^2(\Omega))$. We refer to this linear system as $(2.3)_{\text{lin}}$.

$$\begin{cases} -\Delta\mathbf{u} - 2\nabla(\nabla \cdot \mathbf{u}) = -\nabla p + \mathbf{F} & \text{in } \Omega \times (0, T) \\ [\nabla \cdot \mathbf{u}]_t - \nabla \cdot [k(z)\nabla p] = S & \text{in } \Omega \times (0, T) \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma \times (0, T) \\ k\nabla p \cdot \mathbf{n} = 0 & \text{on } \Gamma_N \times (0, T) \\ p = 0 & \text{on } \Gamma_D \times (0, T) \\ [\nabla \cdot \mathbf{u}](0) = d_0 & \text{in } \Omega. \end{cases} \quad (2.3)_{\text{lin}}$$

Finally, for ease of discussion, let us denote an arbitrary linear system corresponding to a given permeability $K(\mathbf{x}, t)$. We will take $(2.3)_{\text{gen}}$ to be identical to the $(2.3)_{\text{lin}}$, but with pressure equation replaced by

$$[\nabla \cdot \mathbf{u}]_t - \nabla \cdot [K(\mathbf{x}, t) \nabla p] = S \text{ in } \Omega \times (0, T).$$

2.2 Notation and Function Spaces

The Sobolev space of order s defined on a domain D will be denoted by $H^s(D)$, with $H_0^s(D)$ denoting the closure of test functions $C_0^\infty(D) := \mathcal{D}(D)$ in the $H^s(D)$ norm (which we denote by $\|\cdot\|_{H^s(D)}$ or $\|\cdot\|_{s,D}$). When $s = 0$ we may further abbreviate the notation to $\|\cdot\|$. Vector valued spaces will be denoted as $\mathbf{L}^2(\Omega) \equiv [L^2(\Omega)]^n$ and $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$. We make use of the standard notation for the trace of functions $\gamma[w]$ as the map from $H^1(D)$ to $H^{1/2}(\partial D)$. We will make use of the spaces $L^2(0, T; U)$ and $H^s(0, T; U)$, when U is a Hilbert space. Associated norms (and inner products) will be denoted with the appropriate subscript, e.g., $\|\cdot\|_{L^2(0,T;U)}$, though we will simply denote L^2 inner products by $(\cdot, \cdot)$ when the context is clear.

We introduce the following notation for a variable state space for the fluid pressure, as a function of the pressure boundary conditions:

$$V = \begin{cases} V_D = \{p \in H^1(\Omega) : p|_{\Gamma_D} = 0\}, & \text{when } \Gamma_D \neq \emptyset, \\ V_N = H^1(\Omega) \cap [L^2(\Omega)/\mathbb{R}], & \text{when } \Gamma_D = \emptyset. \end{cases} \quad (2.4)$$

Note that $\Gamma_D = \Gamma \implies V = H_0^1(\Omega)$. The space $L^2(\Omega)/\mathbb{R}$ is isomorphic to the subspace of $L^2(\Omega)$ functions with zero average

$$L_0^2(\Omega) = \{u \in L^2(\Omega) : \int_{\Omega} u \, d\mathbf{x} = 0\}.$$

The gradient seminorm is a norm on V in all cases, first, by the Poincaré inequality when $\Gamma_D \neq \emptyset$, and then by the Poincaré-Wirtinger inequality when $\Gamma_D = \emptyset$ [10, 28]. Thus we topologize V in all cases by $\|p\|_V^2 := \int_{\Omega} |\nabla p|^2$.

Then the primary spaces in our analysis are thus denoted by

$$V \quad \mathbf{V} \equiv \mathbf{H}_0^1(\Omega), \quad \mathbb{V} \equiv V \times \mathbf{V}, \quad (2.5)$$

for the pressure p , displacement $\mathbf{u}$, and state $(p, \mathbf{u})$, respectively.

We define the (standard) linear operator $\mathcal{E} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ and bilinear form associated to elasticity as

$$\begin{aligned} \mathcal{E}\mathbf{u}(\mathbf{v}) &= e(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \sigma(\mathbf{u}) \cdot \epsilon(\mathbf{v}) \, d\Omega = \int_{\Omega} [Tr(\epsilon(\mathbf{u}))Tr(\epsilon(\mathbf{v})) + 2\epsilon(\mathbf{u}) \cdot \epsilon(\mathbf{v})] \, d\Omega \\ &= (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + (\nabla \mathbf{u}, \nabla \mathbf{v}) + (\nabla \mathbf{u}, \nabla \mathbf{v}^T). \end{aligned} \quad (2.6)$$

Above $A..B$ stands for the Frobenius scalar product for tensors, i.e., $A..B = A_{ij}B_{ij}$ taken with the Einstein convention.

2.3 Formal Statement of Results and Relationship to the Literature

In the literature there are different definitions of *weak solution* for Biot type systems [1, 7, 13, 34, 41, 51]. We provide a straightforward definition with clear utility in the analysis to follow.

Definition 1. [*Weak Solution*] A solution to (2.3) is a pair of functions

$$(p, \mathbf{u}) \in L^2(0, T; \mathbb{V})$$

for which $\zeta_t \in L^2(0, T; V')$, such that:

(a) the following variational form is satisfied in $L^2(0, T)$ for any $(q, \mathbf{v}) \in \mathbb{V}$:

$$e(\mathbf{u}, \mathbf{v}) + (\nabla p, \mathbf{v}) + (k(\zeta) \nabla p, \nabla q) + \frac{d}{dt}(\zeta, q) = \langle \mathbf{F}, \mathbf{v} \rangle_{\mathbf{V}' \times \mathbf{V}} + \langle S, q \rangle_{V' \times V}, \quad (2.7)$$

(b) the initial condition $\zeta(0) = d_0$ is satisfied in the sense of $C([0, T]; V')$, i.e.,

$$\lim_{t \searrow 0} \zeta(t) = d_0 \in V'.$$

Remark 2.1. The definition of a weak solution to $(2.3)_{\text{lin}}$ and $(2.3)_{\text{gen}}$ are obtained mutatis mutandis by replacing $k(\zeta)$ with $k(z(\mathbf{x}, t))$ and $K(\mathbf{x}, t)$.

To be consistent with other works that consider nonlinear (or time-dependent) permeability [6, 7, 13, 21], we assume continuity and L^∞ type bounds on the permeability, as well as continuity to permit $k(\cdot)$ to be considered as a Nemytskii operator.

Assumption 1. [*Assumptions on Permeability*] The permeability function $k : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist constants $k_1 > 0$ and $k_2 > 0$ such that

$$0 < k_1 \leq k(x) \leq k_2, \quad \forall x \in \mathbb{R}.$$

In the discussion that follows, we recall the distinction made in the Introduction between the case of compressible Biot constituents ($c_0 > 0$) and the incompressible constituents case ($c_0 = 0$). From a formal point of view, taking $c_0 = 0$ destroys the formal parabolic appearance of the equation, removing a conserved quantity that provides temporal regularity.

At this point, we note that several existence results are available for (2.3) and $(2.3)_{\text{gen}}$. Let us point out that, in the linear, time-dependent case for $(2.3)_{\text{gen}}$ with $A(t) = -\nabla \cdot [K(\mathbf{x}, t) \nabla]$, existence of weak solutions was obtained in [39] (later expounded in [40, p.116]). The conditions for existence in these references are quite general and permit $c_0 \geq 0$. Moreover, uniqueness results are available with the additional hypothesis that $K_t \in L^1(0, T; L^\infty(\Omega))$. (See also the more recent [26, 31] for a poroelastic plate model and construction of weak solutions.) The works [1, 41] provide an abstract framework for the case of constant permeability $k = \text{const.}$,

but, in spirit, are close to the linear analysis we present here. The reference [1] considers only the compressible case $c_0 > 0$ with homogeneous boundary data and no forcing terms; the later [41] utilizes implicit semigroup theory and accommodates $c_0 \geq 0$ as well as more general boundary conditions. Again, for constant permeability, [34] makes additional regularity hypotheses on the data and constructs solutions (partially smoother than in Definition 1) in a Galerkin framework.

The more recent works [6–8, 13] provide existence results for weak solutions to (a version of) the nonlinear problem (2.3). First, [13] works explicitly with $c_0 > 0$ and fully homogeneous Dirichlet boundary conditions; [7] considers mixed boundary conditions in all variables (a Lipschitz domain) and boundary sources, obtaining weak solutions for $c_0 = 0$, as well as accommodating the case of viscoelasticity in the porous matrix. Further work incorporating and analyzing viscoelasticity in Biot can be found in [5, 21, 44, 47]. Both nonlinear works [7, 13] utilize Rothe’s method for the construction of weak solutions. The only available uniqueness results (before the treatment at hand) for the *linear poroelastic problem* (2.3)_{gen} necessitate additional regularity for the permeability, precluding their ability to be used in constructing weak solutions for the nonlinear problem. Thus, without resolving the issue of uniqueness of weak solutions for the linear problem, one is forced to work in the context of multiple solutions. More recently, [6] considers the fully homogeneous Dirichlet boundary conditions in all variables and provides existence of weak solutions for $c_0 > 0$ using a multi-valued fixed point approach, and for $c_0 = 0$ via a limiting procedure. In [6], regularity criteria is given for uniqueness of smooth solutions, though such (strong) solutions are not constructed there, nor is a regularity theory developed. *We note that in all cases for poroelastic dynamics, uniqueness of weak solutions was left open for (2.3)_{gen} without making the strong assumption of time differentiability of the permeability K . Moreover, there is no unified treatment of the nonlinear poroelastic problem (2.3) in the literature, based on clear a priori energy estimates.*²

This brings us to the principal results for systems (2.3) and (2.3)_{lin} in the treatment at hand. The first results are for (2.3)_{lin}, where a given $z \in L^2(0, T; L^2(\Omega))$ yields a given permeability $k(z(\mathbf{x}, t))$. Several of the aforementioned existence results (e.g. [6, 40]) construct weak solutions with the properties below, including satisfying an energy inequality. Our first result states that any weak solution, with $\mathbf{u}$ continuous in time into $\mathbf{V}$, satisfies an energy inequality. This will permit us to obtain, in the standard way, the first uniqueness result for (2.3)_{lin} that does not place additional smoothness assumptions on the permeability. Namely, the energy estimate holds in an entire class of weak solutions, rather than for a particular solution constructed as a subsequential limit point. Additionally, this uniqueness will permit a well-defined fixed point mapping for the construction of weak solutions to the nonlinear system (2.3).

We note that the proofs of the linear results for (2.3)_{lin} below are directly adapted to the situation of (2.3)_{gen} when $K(\mathbf{x}, t)$ in $L^\infty((0, T) \times \Omega)$. We choose the $z(\mathbf{x}, t) \mapsto k(z(\mathbf{x}, t))$ framework for our proofs because it is a direct step in obtaining a fixed point for the physically-

²In the case of nonlinear poro-visco-elasticity, viable energy estimates on constructed weak solutions are obtained in [7], from which uniqueness can be deduced. See also [5, 47].

motivated nonlinear problem. See Corollary 2.3 and Corollary 2.5.

Theorem 2.1. *Suppose that the permeability $k(\cdot)$ satisfies Assumption 1. Let $\mathbf{u}_0 \in \mathbf{V}$ with $d_0 = \nabla \cdot \mathbf{u}_0 \in L^2(\Omega)$, $z \in L^2(0, T; L^2(\Omega))$, $\mathbf{F} \in H^1(0, T; \mathbf{V}')$, and $S \in L^2(0, T; V')$. Then any weak solution to (2.3)_{lin} with additional regularity such that $\mathbf{u} \in C([0, T]; \mathbf{V})$ satisfies the estimate:*

$$\|\mathbf{u}(T)\|_{\mathbf{V}}^2 + 2 \int_0^T \int_{\Omega} k |\nabla p|^2 \leq 2 \left(\|\mathbf{F}(0)\|_{\mathbf{V}'}^2 + 2 \|\mathbf{F}(T)\|_{\mathbf{V}'}^2 + 2 \|\mathbf{u}_0\|_{\mathbf{V}}^2 + \frac{1}{k_1} \int_0^T \|S\|_{V'}^2 + \int_0^T \|\partial_t \mathbf{F}\|_{\mathbf{V}'}^2 \right) e^{2T}. \quad (2.8)$$

In particular, (2.3)_{lin} has a unique weak solution satisfying the assumptions above.

Remark 2.2. We note that, owing to the built in hypothesis that $\mathbf{u} \in C([0, T]; \mathbf{V})$, we will immediately have that, given a weak solution as above, $\lim_{t \searrow 0} \mathbf{u} = \mathbf{u}_0$.

We first point to the assumption on the data that $d_0 \in L^2(\Omega)$ specifically emanates from a $\mathbf{u}_0 \in \mathbf{V}$ such that $\nabla \cdot \mathbf{u}_0 = d_0$. This assumption is the same as the one taken in [6, 7, 39, 40, 51], and is typically a byproduct of the construction of the solution. We note that this condition seems somewhat peculiar, as the only term appearing under the time derivative in the dynamics (2.3)_{lin} is $\nabla \cdot \mathbf{u}$, and thus the natural data would be $[\nabla \cdot \mathbf{u}](0) = d_0$.

Remark 2.3. In the above estimate, taking $d_0 = 0$ (as well as $S = 0$ and $\mathbf{F} \equiv \mathbf{0}$) does not necessarily ensure that $\mathbf{u}$ or p are identically zero.

We address these issues, and resolve them, through the next result. Working abstractly on the *reduced form* of (2.3)_{lin} (given later in (4.9)), we can improve Theorem 2.2 and remove the excessive requirement that $\mathbf{u}_0 \in \mathbf{V}$.

Theorem 2.2. *Suppose that the permeability $k(\cdot)$ satisfies Assumption 1. Let $d_0 \in L_0^2(\Omega)$, $z \in L^2(0, T; L^2(\Omega))$, $\mathbf{F} \in H^1(0, T; \mathbf{V}')$, and $S \in L^2(0, T; V')$. Then:*

(i) *There exists a weak solution to (2.3)_{lin} satisfying the following estimate:*

$$\|\mathbf{u}\|_{L^\infty(0, T; \mathbf{V})}^2 + \|p\|_{L^2(0, T; V)}^2 + \|[\nabla \cdot \mathbf{u}]_t\|_{L^2(0, T; V')}^2 \lesssim \|d_0\|_{L^2(\Omega)}^2 + \|S\|_{L^2(0, T; V')}^2 + \|\mathbf{F}\|_{H^1(0, T; \mathbf{V}')}^2. \quad (2.9)$$

(ii) *Moreover, any weak solution to (2.3)_{lin} in the sense of Definition 1 has the property that $\mathbf{u} \in C([0, T]; \mathbf{V})$.*

The above theorem can be used to resolve the issue of uniqueness of arbitrary weak solutions in either case of $V = V_D$ or $V = V_N$. Indeed, we show that any weak solution, for $d_0 \in L_0^2(\Omega)$, will (a posteriori) have the property that $\mathbf{u} \in C([0, T]; \mathbf{V})$. Thus, extracting $\mathbf{u}(0)$, we can apply Theorem 2.1 to obtain uniqueness of the particular solution that satisfies (2.9).

Corollary 2.3. *Assume that the permeability $k(\cdot)$ satisfies Assumption 1. Let $d_0 \in L_0^2(\Omega)$, $z \in L^2(0, T; L^2(\Omega))$, $\mathbf{F} \in H^1(0, T; \mathbf{V}')$, and $S \in L^2(0, T; V')$. Then there exists a unique weak solution to (2.3)_{lin} that satisfies (2.9).*

With the results for the general linear problem established, we can simplify our proofs in [6, 7] and obtain the first direct fixed point construction for the existence of solutions to the quasilinear problem (2.3).

Theorem 2.4. *Let all assumptions of Theorem 2.2 hold. Assume additionally that $\mathbf{F} \in L^2(0, T; L^2(\Omega))$. Then there exists a weak solution to the nonlinear problem (2.3) that satisfies estimate (2.9). In addition, we have that $\|\mathbf{u}\|_{L^2(0, T; \mathbf{H}^2(\Omega))} \leq C(\text{data})$.*

Remark 2.4. The above theorem depends upon elliptic regularity for elasticity in the fixed point construction (to obtain compactness of the fixed point mapping). This is why also we require more regularity on the source of linear momentum $\mathbf{F}$ than the two previous results obtained for the linear problem. (See Section 2.4 for more discussion.)

We mention that the regularity criterion (in fact, a weak-strong uniqueness result) presented in [6, 8] remains valid here. A future work will explicitly use these results to construct strong solutions to the nonlinear problem (2.3) satisfying the requisite regularity to be unique.

Lastly, we present the linear result available in the general setting for a given permeability $K(\mathbf{x}, t)$, corresponding to (2.3)_{gen}.

Corollary 2.5. *Assume that the permeability K has the property that*

$$0 < \|K\|_{L^\infty(\Omega \times (0, T))} < +\infty.$$

Let $d_0 \in L_0^2(\Omega)$ and $\mathbf{F} \in H^1(0, T; \mathbf{V}')$, and $S \in L^2(0, T; V')$. Then there exists a unique weak solution to (2.3)_{gen} that satisfies (2.9).

2.4 Remarks on Regularity of Ω

For all results presented above we take the standing hypothesis that Ω is of class $\mathcal{C}^2$. However, this assumption is made for simplicity of exposition and can be relaxed without significantly changing the proofs.

More precisely, we use smoothness of the domain only to apply elliptic regularity for the elasticity equations. Since we do not use elliptic regularity in the proof of Theorem 2.1, this theorem is valid for arbitrary Lipschitz domains. Moreover, in the proofs of Theorems 2.2 and 2.4 elliptic regularity is only used for interpolation to prove $Bp \in C([0, T]; L_0^2(\Omega))$ and for spatial compactness in Aubin-Lions lemma, respectively. Note that, in both instances, full elliptic regularity is not needed, as it is enough to prove just ϵ gain of regularity over $\mathbf{H}^1(\Omega)$ of the elastic displacement, i.e., $\nabla \cdot \mathbf{u} \in L^2(0, T; H^\epsilon(\Omega))$ for some $\epsilon > 0$. Such regularity results are available in a variety of situation, e.g., polyhedral domains and mixed boundary condition for the elastic displacement (see e.g. [24, 32, 33]). Furthermore, some regularity of $\mathbf{F}$ can be sacrificed. Therefore, our analysis covers cases previously considered in the literature (e.g. [7, 41]), including those motivated by applications.

3 Energy Estimates for Weak Solutions: Proof of Theorem 2.1

We forgo the explicit construction of a weak solution for (2.3)_{lin}. Several viable and direct approaches are available, perhaps the most useful are [40, Chapter III.3] and [6]. The former utilizes a generalization of Lax-Millgram on an equivalent formulation of the problem, and

the latter is explicitly based on a spatial Galerkin's method. In either case, weak solutions are constructed and *the constructed* weak solution satisfies an energy inequality. Here, we are focusing on a general energy inequality itself. Moreover, with any a priori energy estimate holding (for approximants), a construction of weak solutions (as in Definition 1) follows.

Formally, the desired energy inequality in Theorem 2.1 is proved by formally taking the pair $(\partial_t \mathbf{u}, p)$ as a test function in a weak form (2.7). While p has sufficient regularity to be used as such, the quasi-static nature of the Biot dynamics does not permit $\partial_t \mathbf{u}$ as a multiplier in the elasticity equation for an arbitrary weak solution. Hence, we seek a mollification mechanism by which to allow such multiplication in the framework of any given weak solution.

In this argument, we are working with the full system as opposed to the reduced system, which we will use in the next section. We are attempting to gain $L^\infty(0, T; \mathbf{V})$ bounds on the displacement $\mathbf{u}$, and thus we assume that $\mathbf{u}(0) = \mathbf{u}_0 \in \mathbf{V}$, from which we will require that the initial condition $\nabla \cdot \mathbf{u}(0) = d_0 \in L_0^2(\Omega)$ is compatible, as discussed in the previous section. We will eliminate this requirement in the sequel.

We first prove a small mollification argument, followed by the desired energy estimate through mollification; finally, we conclude the uniqueness result directly.

3.1 Temporal $V' \times V$ Mollification

Let $h > 0$ and $j_h \in \mathcal{D}(\mathbb{R})$ such that $\text{supp}\{j_h\} \subset (-h, h)$, and $\int_{\mathbb{R}} j_h = 1$.³ For a locally integrable function f we denote by f_h its temporal regularization (mollification):

$$f_h(t) := \int_{\mathbb{R}} f(s) j_h(t - s) ds.$$

In order to apply the regularization procedure to the linear Biot system, we need to extend all variables from $(0, T)$ to $\mathbb{R}$. With a slight abuse of notation, we denote the extension in the same way as the original functions. The extensions are given in the following way:

$$\mathbf{u}(t) = \begin{cases} \mathbf{u}_0 & t \leq 0 \\ \mathbf{u}(t) & 0 < t < T \\ \mathbf{u}(T) & t \geq T \end{cases}, \quad \mathbf{F}(t) = \begin{cases} \mathbf{F}(0) & t \leq 0 \\ \mathbf{F}(t) & 0 < t < T \\ \mathbf{F}(T) & t \geq T \end{cases}, \quad p(t) = \begin{cases} p(0) & t \leq 0 \\ p(t) & 0 < t < T \\ p(T) & t \geq T \end{cases}. \quad (3.1)$$

Note that by our assumption on the data $\mathbf{F}$, and that weak solutions from Theorem 2.1 have that $\mathbf{u} \in C([0, T]; \mathbf{V})$, we conclude that the elasticity equation $(2.3)_1$ is satisfied in $\mathbf{V}'$ for every t , and thus $\nabla p \in C([0, T]; \mathbf{V}')$ for weak solutions corresponding to Theorem 2.1. From this, we infer that $p \in C([0, T]; L^2(\Omega))$ through the characterization of $\mathbf{V} = \mathbf{H}^{-1}(\Omega)$. Therefore, all extensions in (3.1) are well-defined. For such extensions we have:

Lemma 3.1. *For extended functions as the ones defined in (3.1), we have the following identity:*

$$\int_0^T \langle \nabla \cdot \partial_t \mathbf{u}_h, p \rangle_{V' \times V} = \int_0^T \langle [\nabla \cdot \mathbf{u}]_t, p_h \rangle_{V' \times V} + O(h).$$

³This is the traditional mollifier, sometimes denoted by η_h [22, 28].

Proof. Let $f = \nabla \cdot \partial_t \mathbf{u} \in L^2(0, T; \mathbf{V}')$. Then

$$\begin{aligned} \int_0^T \langle f_h(t), p(t) \rangle_{V' \times V} dt &= \int_0^T \int_{t-h}^{t+h} \langle j_h(t-s)f(s), p(t) \rangle_{V' \times V} ds dt \\ &= \left(\int_{-h}^{T+h} ds \int_{s-h}^{s+h} dt - \int_{-h}^h ds \int_{s-h}^0 dt - \int_{T-h}^{T+h} ds \int_T^{s+h} dt \right) \langle f(s), j_h(s-t)p(t) \rangle_{V' \times V} \\ &= \int_0^T \langle f(s), p_h(s) \rangle_{V' \times V} ds - \left(\int_0^h ds \int_{s-h}^0 dt + \int_{T-h}^T ds \int_T^{s+h} dt \right) \langle f(s), j_h(s-t)p(t) \rangle_{V' \times V} \quad (3.2) \end{aligned}$$

since $f(s) = 0$ outside of $[0, T]$.

Now we have the following claims:

$$I_1 = \int_0^h ds \int_{s-h}^0 dt \langle f(s), j_h(s-t)p(t) \rangle_{V' \times V} \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad \text{and} \quad (3.3)$$

$$I_2 = \int_{T-h}^T ds \int_T^{s+h} dt \langle f(s), j_h(s-t)p(t) \rangle_{V' \times V} \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad (3.4)$$

We prove here only (3.3), as (3.4) follows similarly. First, assume that $p(0) \in V$ and recall that $f(s) = \partial_s \nabla \cdot \mathbf{u}$. Therefore we use IBP and rewrite I_1 as follows:

$$\begin{aligned} I_1 &= \int_0^h \langle \nabla \cdot \mathbf{u}(s), p(0)j_h(-h) \rangle_{V' \times V} ds + \int_0^h \langle \nabla \cdot \mathbf{u}(s), p(0) \int_{s-h}^0 j_h'(t-s)dt \rangle_{V' \times V} ds \\ &\quad + \langle \nabla \cdot \mathbf{u}(h) - \nabla \cdot \mathbf{u}(0), p(0) \int_{-h}^0 j_h(t)dt \rangle_{V' \times V} \\ &= \int_0^h \langle \nabla \cdot \mathbf{u}(s), p(0) \int_{s-h}^0 j_h'(t-s)dt \rangle_{V' \times V} ds + \langle \nabla \cdot \mathbf{u}(h) - \nabla \cdot \mathbf{u}(0), p(0) \int_{-h}^0 j_h(t)dt \rangle_{V' \times V}. \end{aligned}$$

Note that each term in the last equality has L^2 spatial regularity, and thus all of the $V' \times V$ duality pairings may be replaced by $L^2(\Omega)$ inner products and then estimated as follows:

$$\begin{aligned} \left| \int_0^h \left(\nabla \cdot \mathbf{u}(s), p(0) \int_{s-h}^0 j_h'(t-s)dt \right) ds \right| &\leq C \|p(0)\|_{L^2(\Omega)} h \sup_{[0, T]} \|\mathbf{u}\|_{\mathbf{V}} \xrightarrow{h \rightarrow 0} 0 \\ \left| \left(\nabla \cdot \mathbf{u}(h) - \nabla \cdot \mathbf{u}(0), p(0) \int_{-h}^0 j_h(t)dt \right) \right| &\leq C \|p(0)\|_{L^2(\Omega)} \|\mathbf{u}(h) - \mathbf{u}(0)\|_{\mathbf{V}} \xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

where in the last line we used the fact that $\mathbf{u} \in C([0, T]; \mathbf{V})$.

In the case where $p(0) \in L^2(\Omega)$ only, by density, take $p_n(0) \in V$ to be such that $p_n(0) \xrightarrow{n \rightarrow \infty} p(0) \in L^2(\Omega)$, and denote $p_n(t)$ as the extension analogous to (3.1). Perform the computations listed above with $p_n(0) \in V$, and then pass with the limit in n in the final

step. This finishes the proof of the claims.

Lastly, combining (3.2) with (3.3) and (3.4), we obtain that

$$\int_0^T \langle (\nabla \cdot \partial_t \mathbf{u})_h, p \rangle_{V' \times V} = \int_0^T \langle [\nabla \cdot \mathbf{u}]_t, p_h \rangle_{V' \times V}.$$

Moreover, we have that $(\nabla \cdot \partial_t \mathbf{u})_h = \nabla \cdot \partial_t \mathbf{u}_h$. This concludes the proof of the lemma. $\square$

We now apply the temporal mollification directly to the elasticity equation to obtain:

$$-\Delta \mathbf{u}_h - 2\nabla(\nabla \cdot \mathbf{u}_h) = -\nabla p_h + \mathbf{F}_h. \quad (3.5)$$

By the above discussion, this equation holds for every t in the sense of $\mathbf{V}'$.

We recall the bilinear form $e(\cdot, \cdot)$ associated with elasticity given in (2.6), and the corresponding norm on $\mathbf{V}$

$$\|\mathbf{u}\|_{\mathbf{V}}^2 \equiv \langle \mathcal{E} \mathbf{u}, \mathbf{u} \rangle_{V' \times V} = e(\mathbf{u}, \mathbf{u}).$$

We may test the regularized elasticity equation by $\partial_t \mathbf{u}_h \in C^\infty([0, T]; H_0^1(\Omega))$. The pressure equation (2.3)₂ (which holds in the sense of $L^2(0, T; V')$) may be tested against p_h which is similarly smooth in time into V . Summing the results of these integrations, we obtain the following equality which is valid in $L^2(0, T)$ (and hence *a.e.* t):

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_h\|_{\mathbf{V}}^2 + (\nabla p_h, \partial_t \mathbf{u}_h) + \langle [\nabla \cdot \mathbf{u}]_t, p_h \rangle_{V' \times V} + (k \nabla p, \nabla p_h) = \langle \mathbf{F}_h, \partial_t \mathbf{u}_h \rangle_{V' \times V} + \langle S, p_h \rangle_{V' \times V}. \quad (3.6)$$

Upon integration in time $\int_0^T dt$ and a temporal integration by parts we obtain:

$$\begin{aligned} & \frac{1}{2} \|\mathbf{u}_h(T)\|_{\mathbf{V}}^2 + \int_0^T (\nabla p_h, \partial_t \mathbf{u}_h) + \int_0^T \langle \nabla \cdot \partial_t \mathbf{u}, p_h \rangle_{V' \times V} + \int_0^T (k \nabla p, \nabla p_h) \\ &= - \int_0^T \langle \partial_t \mathbf{F}_h, \mathbf{u}_h \rangle_{V' \times V} - \langle \mathbf{F}_h(T), \mathbf{u}_h(T) \rangle_{V' \times V} + \langle \mathbf{F}_h(0), \mathbf{u}_h(0) \rangle_{V' \times V} \\ & \quad + \int_0^T \langle S, p_h \rangle_{V' \times V} + \frac{1}{2} \|\mathbf{u}_h(0)\|_{\mathbf{V}}^2. \end{aligned} \quad (3.7)$$

We observe that all terms above are well-defined for the regularity classes associated to a weak solution in the sense of Definition 1.

3.2 Limit Passage

We now note convergences that will allow us to pass with the limit in the equality (3.6).

Proposition 3.2. *Suppose $(\mathbf{u}, p)$ is a weak solution as in Definition 1 and k is as in Assumption 1. The following limits hold as $h \searrow 0$:*

$$1. \int_0^T (k \nabla p, \nabla p_h) \rightarrow \int_0^T \|k^{1/2} \nabla p\|^2.$$

$$2. \int_0^T \left((\nabla p_h, \partial_t \mathbf{u}_h) + \langle \nabla \cdot \partial_t \mathbf{u}, p_h \rangle_{V' \times V} \right) \rightarrow 0.$$

$$3. \frac{1}{2} \|\mathbf{u}_h(t)\|_{\mathbf{V}}^2 \rightarrow \frac{1}{2} \|\mathbf{u}(t)\|_{\mathbf{V}}^2 \quad \text{in } C([0, T]).$$

Proof. The first claim is a direct consequence of the elementary properties of convolution with the standard mollifiers [10, 22, 45]. Indeed, we note that $p \in L^2(0, T; V)$ as well as the fact that the permeability function $k(\cdot)$ is strictly bounded from below and above by Assumption 1.

Secondly, since p_h and $\partial_t \mathbf{u}_h$ are sufficiently smooth in space (owing to the fact that $\mathbf{u} \in L^2(0, T; \mathbf{V}) \implies \partial_t \mathbf{u}_h \in L^2(0, T; \mathbf{V})$) we can directly apply integration by parts with $\partial_t \mathbf{u}_h|_{\Gamma} = 0$ to obtain:

$$\int_0^T \int_{\Omega} \nabla p_h \cdot \partial_t \mathbf{u}_h = - \int_0^T \int_{\Omega} p_h (\nabla \cdot \partial_t \mathbf{u}_h).$$

With this observation, the claim reduces to:

$$\int_0^T \langle \nabla \cdot (\partial_t \mathbf{u} - \partial_t \mathbf{u}_h), p_h \rangle_{V' \times V} \rightarrow 0.$$

This is equivalent to

$$\underbrace{\int_0^T \langle \nabla \cdot (\partial_t \mathbf{u} - \partial_t \mathbf{u}_h), p_h - p \rangle_{V' \times V}}_I - \underbrace{\int_0^T \langle \nabla \cdot (\partial_t \mathbf{u} - \partial_t \mathbf{u}_h), p \rangle_{V' \times V}}_{II} \rightarrow 0.$$

We estimate the first term in the following way:

$$|I| \leq \underbrace{\|\nabla \cdot (\partial_t \mathbf{u} - \partial_t \mathbf{u}_h)\|_{L^2(0, T; V')}}_{\leq C} \underbrace{\|p - p_h\|_{L^2(0, T; V)}}_{\rightarrow 0} \rightarrow 0,$$

where the latter convergence follows again via the standard L^p mollifier property [10, 22]. Here we have also used $\nabla \cdot \mathbf{u}_t \in L^2(0, T; V')$ in Definition 1. For the second term, II , we first use the previous Lemma 3.1 to arrive at

$$\int_0^T \int_{\Omega} (\nabla \cdot \partial_t \mathbf{u}_h) p = \int_0^T \langle \nabla \cdot \partial_t \mathbf{u}, p_h \rangle_{V' \times V} + O(h).$$

Therefore the integral II can be treated in an analogous way as the first one:

$$II = \int_0^T \langle \nabla \cdot \partial_t \mathbf{u}, p - p_h \rangle_{V' \times V} \rightarrow 0.$$

Finally, let us prove the third property. By the assumption on the solution of Theorem 2.1, we have $\mathbf{u} \in C([0, T]; \mathbf{V})$ and again by the standard properties of mollification [10, Theorem. 4.21] we have that $\mathbf{u}_h \rightarrow \mathbf{u}$ strongly in $C([0, T]; \mathbf{V})$. Therefore by continuity of norm we

have the uniform convergence

$$\frac{1}{2}\|\mathbf{u}_h(t)\|_{\mathbf{V}}^2 \rightarrow \frac{1}{2}\|\mathbf{u}(t)\|_{\mathbf{V}}^2 \quad \text{in } C([0, T]).$$

□

3.3 Concluding the Proof of Theorem 2.1

Now we can proceed with the proof of Theorem 2.1. Using Proposition 3.2 and equation (3.6), and by taking $h \rightarrow 0$, we obtain that weak solution $(\mathbf{u}, p)$ from Theorem 2.1 satisfies the energy equality:

$$\frac{1}{2}\|\mathbf{u}(T)\|_{\mathbf{V}}^2 + \int_0^T \int_{\Omega} k |\nabla p|^2 = - \int_0^T \langle \partial_t \mathbf{F}, \mathbf{u} \rangle_{\mathbf{V}' \times \mathbf{V}} - \langle \mathbf{F}(s), \mathbf{u}(s) \rangle_{\mathbf{V}' \times \mathbf{V}} \Big|_{s=0}^{s=T} + \int_0^T \langle S, p \rangle_{V' \times V} + \frac{1}{2}\|\mathbf{u}_0\|_{\mathbf{V}}^2.$$

We estimate:

$$\begin{aligned} \|\mathbf{u}(T)\|_{\mathbf{V}}^2 + 4 \int_0^T \int_{\Omega} k |\nabla p|^2 &\leq 2 \int_0^T \|\partial_t \mathbf{F}(s)\|_{\mathbf{V}'}^2 ds + 2 \int_0^T \|\mathbf{u}(s)\|_{\mathbf{V}}^2 ds + 2\|\mathbf{F}(0)\|_{\mathbf{V}'}^2 + 4\|\mathbf{F}(T)\|_{\mathbf{V}'}^2 \\ &\quad + \|\mathbf{u}_0\|_{\mathbf{V}}^2 + \frac{2}{k_1} \int_0^T \|S\|_{V'}^2 + 2 \int_0^T k_1 \|p\|_{V'}^2. \end{aligned}$$

The last term on the right-hand side can be absorbed into the left-hand side. Finally, by using the Grönwall inequality we obtain:

$$\|\mathbf{u}(T)\|_{\mathbf{V}}^2 + 2 \int_0^T \int_{\Omega} k |\nabla p|^2 \leq 2 \left(\|\mathbf{F}(0)\|_{\mathbf{V}'}^2 + 2\|\mathbf{F}(T)\|_{\mathbf{V}'}^2 + 2\|\mathbf{u}_0\|_{\mathbf{V}}^2 + \frac{1}{k_1} \int_0^T \|S\|_{V'}^2 + \int_0^T \|\partial_t \mathbf{F}\|_{\mathbf{V}'}^2 \right) e^{2T}. \quad (3.8)$$

Since the above can be applied to *any* weak solution in the sense of Defintion 1 having also the additional property that $\mathbf{u} \in C([0, T]; \mathbf{V})$, we can apply it to the difference of two such solutions. This provides a continuous dependence estimate. The standard argument then yields uniqueness of these solutions through the above estimate, if all data and sources are identified for two weak solutions.

This concludes the proof of Theorem 2.1.

Remark 3.1. At this juncture, uniqueness requires that all of the data for $\mathbf{u}_0$ vanish in order to deduce that the solution is identically zero; it is not sufficient (yet) that only the divergence $\nabla \cdot \mathbf{u}_0$ vanish to deduce that the solution is zero.

4 Reduced Problem and Proof of Theorem 2.2

As mentioned above, existence of weak solutions for the linear time-dependent problem in $(2.3)_{\text{lin}}$ can be obtained, for instance, from [39] in the context of implicit equations (see also [40]). Here we summarize the principal operators and the reduction of the linear system

to an implicit evolution equation (1.3), as they are essential in the exposition and proof of Theorem 2.2, which we give later in this section.

4.1 Operators and Functional Setup

Elasticity Operator. We will define an elasticity operator in the balance-of-momentum equation to invert, and thus write the solid displacement $\mathbf{u}$ as a direct function of p . Recall that, for $\mathbf{u} \in \mathbf{V}$ and a smooth function $\mathbf{v}$,

$$-(\operatorname{div} \sigma(\mathbf{u}), \mathbf{v}) = -(\operatorname{div}[2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}], \mathbf{v}) = e(\mathbf{u}, \mathbf{v}).$$

Thus, if we let $\mathbf{v} \in \mathbf{V}$ be an arbitrary test function in (2.1), we obtain the variational form of the elasticity equation (2.3)_{lin}:

$$e(\mathbf{u}, \mathbf{v}) = \int_{\Omega} p \mathbf{I} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega + \langle F, \mathbf{v} \rangle_{\mathbf{V}' \times \mathbf{V}}. \quad (4.1)$$

We note that $e(\cdot, \cdot)$ is symmetric, continuous and coercive on $\mathbf{V}$. If we let $f(\mathbf{v}) = \int_{\Omega} p \mathbf{I} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega + \langle F, \mathbf{v} \rangle_{\mathbf{V}' \times \mathbf{V}}$, then $f \in \mathbf{V}'$ directly, as we have the following estimate:

$$|f(\mathbf{v})| \leq C \|p\|_{L^2(\Omega)} \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega)} + C \|\mathbf{F}\|_{\mathbf{V}'} \|\mathbf{v}\|_{\mathbf{V}} \leq C \left(\|p\|_{L^2(\Omega)} + \|\mathbf{F}\|_{\mathbf{V}'} \right) \|\mathbf{v}\|_{\mathbf{V}}. \quad (4.2)$$

By direct application of Lax-Milgram, there exists unique solution $\mathbf{u} = \mathbf{u}(p, \mathbf{F}) \in \mathbf{V}$ to (4.1). Note that even though $p \in V \subset H^1(\Omega)$ (for all boundary conditions considered), (4.1) allows us to define $\mathbf{u}$ as a function of p for all $p \in L^2(\Omega)$, since $H^1(\Omega)$ is dense in $L^2(\Omega)$ and the above estimate (4.2) *depends only on the $L^2(\Omega)$ -norm of p* .

Hereafter we denote the resulting elasticity operator above by $\mathcal{E}(\mathbf{u}) = f$, i.e., $\mathcal{E} : \mathbf{V} \rightarrow \mathbf{V}'$ is the linear operator determined by the bilinear form $e(\cdot, \cdot)$ on $\mathbf{V}$. We have that $\mathcal{E}$ is an isomorphism in this setting. We summarize the above discussion in the following lemma.

Lemma 4.1. *Consider the elasticity problem:*

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{G} & \text{on } \Omega \\ \mathbf{u} = 0 & \text{on } \Gamma. \end{cases} \quad (4.3)$$

with distributed source $\mathbf{G} \in \mathbf{V}'$. Then there exists a unique weak solution $\mathbf{u} \in \mathbf{V}$ [16, 28] that satisfies the stability estimate

$$\|\mathbf{u}\|_{\mathbf{V}} \leq C \|\mathbf{G}\|_{\mathbf{V}'}, \quad \forall \mathbf{u} \in \mathbf{V}.$$

Moreover, since we have assumed Ω is of class $\mathcal{C}^2$, classical elliptic regularity applies [16, 45]. Hence, if $\mathbf{G} \in \mathbf{L}^2(\Omega)$, then the solution $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, and $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leq C \|\mathbf{G}\|_{\mathbf{L}^2(\Omega)}$.

Pressure-to-Dilation Map. The pressure-to-dilation map was introduced in the setting of Biot poroelasticity in [1, 40, 41]. Motivated by the elasticity problem in Lemma 4.1, we define the

operator $B : L^2(\Omega) \rightarrow L^2(\Omega)$ by

$$Bp = -\nabla \cdot \mathcal{E}^{-1}(\nabla p) = \nabla \cdot \mathbf{u}. \quad (4.4)$$

When $p \in H^s(\Omega)$ we have that $\nabla p \in \mathbf{H}^{s-1}(\Omega)$ [10, 28, 45], with $p \mapsto \nabla p$ continuous in this setting. In the specific case when $p \in L^2(\Omega)$, then $\nabla p \in \mathbf{H}^{-1}(\Omega) = \mathbf{V}'$. Invoking the properties of the elliptic operator $\mathcal{E}$, we see that $B \in \mathcal{L}(L^2(\Omega))$.

If $p \in V$ (either V_D or V_N), with Ω is smooth here, we have that

$$\nabla p \in L^2(\Omega) \implies \mathcal{E}(\mathbf{u}) = -\nabla p \in L^2(\Omega) \implies \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{V} \implies \nabla \cdot \mathbf{u} \in \nabla \cdot \mathbf{V} = V_N,$$

where in the last equality we used that the divergence operator is surjective onto $L_0^2(\Omega)$, e.g. [23, Theorem III.3.3].

Remark 4.1. Therefore, only in the case of purely Neumann boundary conditions for the fluid pressure, is the pressure solution space invariant under the pressure-to-dilation map. This is a key difference between the two cases considered for V , and has ramifications in the analysis.

We summarize the discussion of B in this setting where elliptic regularity holds for the pair $(\mathcal{E}, \Omega)$ in the following lemma:

Lemma 4.2. *Given $p \in V$ and $\mathbf{F} \in \mathbf{L}^2(\Omega)$, the corresponding solver $\mathcal{E}^{-1}(-\nabla p + \mathbf{F}) \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$ with associated bound. When $\mathbf{F} \equiv 0$ and $p \in V$, we have $Bp = \nabla \cdot \mathbf{u} \in V_N$ for $\mathcal{E}(\mathbf{u}) = -\nabla p$. From this we obtain that*

$$B : V \rightarrow V_N, \text{ continuously.}$$

We note some important kernel and range properties of the B operator [7, 41, 45]:

Lemma 4.3. *Considered as a mapping on $L^2(\Omega)$, $\text{Ker}(B) = \{\text{constants}\}$, and hence B is injective on $L_0^2(\Omega)$ as well as on V_N . With respect to ranges, we have $B(L^2(\Omega)) \subseteq L_0^2(\Omega)$. Thus $B \in \mathcal{L}(L_0^2(\Omega))$ and $B \in \mathcal{L}(V_N)$. Finally, we have that B is a self-adjoint, monotone operator when considered on $L^2(\Omega)$ or $L_0^2(\Omega)$.*

Remark 4.2. $B \in \mathcal{L}(L^2(\Omega))$, but it need not be coercive in that setting. B can be extended to a linear operator (still denoted by B) which lies in $\mathcal{L}(V_N')$. Such an extension fails for $V = V_D$, owing to the fact that for $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, the function $\nabla \cdot \mathbf{u}$ lands in $H^1(\Omega) \cap L_0^2(\Omega)$ and not $H_0^1(\Omega)$.

Proposition 4.4. *The operator B is an isomorphism on $L_0^2(\Omega)$.*

Proof. Let $q \in L_0^2(\Omega)$. Then, by definition of B , we have that $q = Bp$ if and only if there exists $\mathbf{u}$ such that $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ is a solution to the following Stokes problem:

$$\begin{aligned} -\mu \Delta \mathbf{u} + \alpha \nabla p &= (\lambda + \mu) \nabla q & \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= q & \text{in } \Omega \\ \mathbf{u} &= 0 & \text{on } \Gamma. \end{aligned}$$

We use classical existence theorem for the Stokes equation (see e.g. [45, Prop I.2.2. and Remark I.2.6]) to conclude that for every $q \in L_0^2(\Omega)$ there is a unique $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ satisfying the above equation and the following estimate:

$$\|\mathbf{u}\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C (\|\nabla q\|_{\mathbf{H}^{-1}(\Omega)} + \|q\|_{L^2(\Omega)}) \leq C \|q\|_{L^2(\Omega)}.$$

Therefore, we proved $\|Bp\|_{L^2(\Omega)} \geq \frac{1}{C} \|p\|_{L^2(\Omega)}$ which concludes the proof. $\square$

Remark 4.3. A more direct proof follows from the Bogovskii Theorem (e.g. [23, Theorem III.3.3]) which states that the divergence is surjective operator from $H_0^1(\Omega) \rightarrow L_0^2(\Omega)$. Therefore ∇ is an injection from $L_0^2(\Omega)$ into $\mathbf{H}^{-1}(\Omega)$. From these facts, we may deduce that the range of B is closed in $L_0^2(\Omega)$, and since B is self-adjoint with null kernel, the Closed Range Theorem guarantees that B is an isomorphism on $L_0^2(\Omega)$. (These observations are essentially used in the proof of the existence theorem for Stokes equation, yielding Proposition 4.4.)

Diffusion Operator $A(t)$. For $k \in L^\infty(\mathbb{R})$, we can define for each $z \in L^2(0, T; L^2(\Omega))$ the linear operator $A(t) : V \rightarrow V'$ through the bilinear form

$$A[p, q; k(z)] = (k(z)\nabla p, \nabla q), \quad \forall p, q \in V. \quad (4.5)$$

If k and z are given and smooth, then we have an unbounded operator $A(t) : L^2(\Omega) \rightarrow L^2(\Omega)$ with domain $\mathcal{D}(A(t)) = H^2(\Omega) \cap V$ and action given by

$$A(t)p = -\nabla \cdot [k(z)\nabla p], \quad \forall p \in \mathcal{D}(\Omega). \quad (4.6)$$

When $k \equiv \text{const}$, $A(t) = A$ is a multiple of the standard Laplacian (Dirichlet, Neumann, or mixed, depending on V) defined on $H^2(\Omega) \cap V$.

In the above setting, for a given $z \in L^2(0, T; L^2(\Omega))$, the bilinear form $A[\cdot, \cdot; k(z)]$ continuous, coercive, and symmetric on V .

Translation to Eliminate Source $\mathbf{F}$. Note that it is sufficient to solve the linear problem (2.3)_{lin} with $\mathbf{F} \equiv 0$ by a translation argument. Indeed, as the elasticity equation is elliptic and $\mathbf{F} \in H^1(0, T; \mathbf{V}')$, for a.e. $t \in [0, T]$ we can define

$$\mathbf{u}_{\mathbf{F}}(t) = \mathcal{E}^{-1}(\mathbf{F}(t)) \in \mathbf{V}. \quad (4.7)$$

Thus we have that $\mathbf{u}_{\mathbf{F}} \in H^1(0, T; \mathbf{V})$. Then, considering the variable $\mathbf{w} = \mathbf{u} - \mathbf{u}_{\mathbf{F}}$, we note that $\mathbf{u}$ solves (2.3)_{lin} if and only if $\mathbf{w}$ solves

$$\begin{cases} \mathcal{E}(\mathbf{w}) = -\nabla p & \in L^2(0, T; \mathbf{V}') \\ \nabla \cdot \mathbf{w}_t + A(t)p = S + \nabla \cdot \mathbf{u}_{\mathbf{F},t} & \in L^2(0, T; V') \\ \nabla \cdot \mathbf{w}(0) = d_0 - \nabla \cdot \mathbf{u}_{\mathbf{F}}(0) & \in L^2(\Omega). \end{cases} \quad (4.8)$$

Hence, by re-scaling $S \in L^2(0, T; V')$ and $d_0 = \zeta(0) \in L^2(\Omega)$, we obtain an equivalent linear

problem for a given z with $\mathbf{F} \equiv 0$.

4.2 Reduced Problem

Finally, using the pressure to dilation operator introduced above, we equivalently reformulate (2.3)_{lin} with $\mathbf{F} \equiv 0$ (as in [6]) as the initial boundary value problem

$$\begin{cases} [Bp]_t - \nabla \cdot [k(z)\nabla p] = S, & \in L^2(0, T; V') \\ Bp(0) = d_0, & \in V'. \end{cases} \quad (4.9)$$

We define a weak solution to (4.9)—which is valid for both $V = V_D$ or V_N —as follows:

Definition 2. *Given $z \in L^2(0, T; L^2(\Omega))$, we say that $p \in L^2(0, T; V)$ with $[Bp]' \in L^2(0, T; V')$ is a weak solution for (4.9) provided that*

1. *For every $q \in V$,*

$$\frac{d}{dt}(Bp, q) + A[p, q; k(z)] = \langle S, q \rangle_{V' \times V}. \quad (4.10)$$

2. *$[Bp](0) = d_0 \in V'$ in the sense of $C([0, T]; V')$.*

Note that since $Bp \in L^2(0, T; V)$ and $[Bp]' \in L^2(0, T; V')$, we have that $Bp \in C([0, T]; L_0^2(\Omega))$ and thus the initial condition above is well-defined.

As mentioned in the beginning of the section, the existence of a weak solution is obtained, e.g., in [39]. We thusly have the following theorem:

Theorem 4.5. *Let Assumption 1 be in force, $S \in L^2(0, T; V')$ and $d_0 \in L_0^2(\Omega)$. Then (4.9) has a weak solution, according to Definition 2.*

4.3 Estimates for Reduced Problem (4.9)

In this section we derive two a priori estimates for the reduced problem (as above) *with initial data only given in terms of $[Bp](0)$* . The first, a formal estimate, will hold on approximants, and any constructed solution therefrom will inherit this bound. We will then show: for any weak solution to (4.10) $p \in L^2(0, T; V)$ and $Bp \in H^1(0, T; V')$ taking only $[\nabla \cdot \mathbf{u}](0) = d_0 \in L_0^2(\Omega)$, we can infer the additional property that $\mathbf{u} \in C([0, T]; \mathbf{V})$ for $Bp = \nabla \cdot \mathbf{u}$. Putting these two facts together will allow us to markedly improve Theorem 2.1 by eliminating an unnecessary requirement on the data, as well as showing that the solution is unique, with the additional property that $\mathbf{u} \in C([0, T]; \mathbf{V})$.

The principle issue with this task is that B is not isomorphism on $L^2(\Omega)$ because $\text{Ker}(B) = \mathbb{R}$. In what follows we extensively use the fact that $L^2(\Omega) \equiv \mathbb{R} \oplus L_0^2(\Omega)$. We denote by $\mathcal{P} : L^2(\Omega) \rightarrow L_0^2(\Omega)$ the orthogonal projection on $L_0^2(\Omega)$ which is given by the standard formula:

$$\mathcal{P}f = f - \frac{1}{|\Omega|} \int_{\Omega} f. \quad (4.11)$$

Let us also define a symmetric bilinear form on $L^2(\Omega)$ (using self-adjointness of B)

$$\beta(p, q) := (Bp, q)_{L^2(\Omega)} = (p, Bq)_{L^2(\Omega)}, \quad p, q \in L^2(\Omega).$$

By Lemma 4.3, $|p|_B := \sqrt{\beta(p, p)}$ is a semi-norm on $L^2(\Omega)$. With this notation we can re-write the weak form (4.10) equivalently as

$$\frac{d}{dt} [\beta(p(t), q)] + A[p(t), q; k(z)] = \langle S(t), q \rangle_{V' \times V} \quad \text{in } \mathcal{D}'(0, T), \quad q \in V. \quad (4.12)$$

We now consider the two cases, $V = V_D$ or V_N separately (recall the definition in (2.4), and that V_D includes the mixed case). In each case below there are two main steps: (i) to show an improved, formal energy estimate (valid for approximants), and (ii) to show that, a posteriori, any weak solution as in Definition 1 has the additional property that $\mathbf{u} \in C([0, T]; \mathbf{V})$.

4.3.1 Neumann Case: $V = V_N$

In the (purely) Neumann case, we have $H \equiv L_0^2(\Omega)$ and $V = H^1(\Omega) \cap L_0^2(\Omega)$. Therefore, by Proposition 4.4, we have in this case that $\beta(\cdot, \cdot)$ is in fact a scalar product on H , and by the standard polarization identity, it is equivalent to the $L^2(\Omega)$ scalar product.

Remark 4.4. It is worth noting that this approach is essentially used in [1]. There, $\beta(\cdot, \cdot)$ is an equivalent inner product on $L^2(\Omega)$ since Dirichlet boundary conditions are taken with $c_0 > 0$. In that case, when $A(t) = A$ (constant), one obtains a unique weak solution $p \in L^2(V_D)$ if $p(0)$ or $Bp(0)$ is specified. Alternatively, using a modified, implicit semigroup approach, the same result can be obtained (as well as generalization to stronger solutions) [39, 41] for $c_0 \geq 0$. However, when $A(t)$ is truly time-dependent and $c_0 = 0$, uniqueness requires additional assumptions [40]. Moreover, as we shall see in the next section, we must work harder to permit specification of data as $Bp(0)$, since B is not, in general, invertible on $L^2(\Omega)$ nor does $\beta(\cdot, \cdot)$ induce a true inner product there.

Now, by taking p as a formal test function in (4.12) and integrating in time, we immediately obtain the estimate:

$$\|\beta(p, p)\|_{L^\infty(0, T)} + \|p\|_{L^2(0, T; V)}^2 \leq C \left(\|S\|_{L^2(0, T; V')}^2 + \|p(0)\|_{L^2(\Omega)}^2 \right). \quad (4.13)$$

Finally, by norm/inner-product equivalence,

$$c\|p(0)\|_{L^2(\Omega)}^2 \leq |p(0)|_B^2 = \beta(p(0), p(0)) = (Bp(0), p(0)) \leq C\|Bp(0)\|_{L^2(\Omega)}^2.$$

We have, in addition, that $\|p\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq C\|\beta(p, p)\|_{L^\infty(0, T)}$. Thus for any weak solution constructed from approximants (obeying (4.13)) we obtain the energy estimate:

$$\|\beta(p, p)\|_{L^\infty(0, T)} + \|p\|_{L^2(0, T; V)}^2 \leq C \left(\|S\|_{L^2(0, T; V')}^2 + \|Bp(0)\|_{L^2(\Omega)}^2 \right). \quad (4.14)$$

Now let us suppose that p is any weak solution (that is, not necessarily satisfying (4.14)). We obtain that $[Bp]_t \in L^2(0, T; V')$ directly from the definition of weak solution in Definition 2, with

$$\|Bp_t\|_{L^2(0, T; V')}^2 \lesssim \|S\|_{L^2(0, T; V')} + \|\nabla p\|_{L^2(0, T; L^2(\Omega))}^2.$$

Moreover, by boundedness of B on V we know that $Bp \in L^2(0, T; V)$, since, as a weak solution, $p \in L^2(0, T; V)$. Thus by the standard interpolation result for Bochner spaces [16, 22] for the triple $H^1(\Omega) \cap L_0^2(\Omega) = V \subset L_0^2(\Omega) \subset V'$, we infer that $Bp \in C([0, T]; L_0^2(\Omega))$ and then by the invertibility of B on $L_0^2(\Omega)$ as shown above in Lemma 4.3 we obtain that $p \in C([0, T]; L_0^2(\Omega))$. Now, since $\nabla p \in \mathbf{H}^{-1} = \mathbf{V}'$ (by the characterization of $\mathbf{H}^{-1}(\Omega)$), the corresponding elasticity equation $\mathcal{E}(\mathbf{u}) = -\nabla p$ is satisfied in $\mathbf{V}'$ for every $t \in [0, T]$. Therefore, interpreting the equation variationally through $e(\cdot, \cdot)$, we have $\mathbf{u}(t) \in \mathbf{V}$, $t \in [0, T]$ with:

$$\|\mathbf{u}(t)\|_{\mathbf{V}} \leq C\|\nabla p(t)\|_{\mathbf{V}'} \leq C\|p(t)\|_{L_0^2(\Omega)}. \quad (4.15)$$

Therefore we have proven $\mathbf{u} \in C([0, T]; \mathbf{V})$ and hence every weak solution satisfies assumptions of Theorem 2.1. Moreover, since any weak solution satisfies the hypotheses of Theorem 2.1—namely that $\mathbf{u} \in C([0, T]; \mathbf{V})$ —all weak solutions are in fact unique. Finally, since we have constructed a weak solution that satisfies the estimate (4.14), using Section 4.1, we may translate back to the full problem; we deduce, then that the unique weak solution as in Definition (1) satisfies the final estimate (2.9), only assuming that $Bp(0) = \nabla \cdot \mathbf{u}(0) \in L_0^2(\Omega)$ is given as data.

Remark 4.5. In the Neumann case we can formally integrate the second equation of $(2.3)_{lin}$ (or equivalently $(4.9)_1$), and use the divergence theorem to obtain the following necessary condition for the existence of solution: $\int_{\Omega} S = 0$. In Theorem 4.5 this condition is contained in assumption $S \in L^2(0, T; V_N')$. Informally, the functionals from $L^2(0, T; V_N')$ only “see” mean free part of the function since

$$\int_{\Omega} Sq = \int_{\Omega} \mathcal{P}Sq, \quad S \in L^2(\Omega), \quad q \in V_N.$$

Formally, since V_N is not dense in L^2 , functionals from V_N' cannot be extended to L^2 in a unique way and therefore L^2 cannot be embedded in V_N' .

4.3.2 Mixed Case

The same results as above hold for the mixed case $V = V_D$, but the proof is more subtle, as B is not an isomorphism on $H = L^2(\Omega)$ in this case. We use the fact that kernel of B over $L^2(\Omega)$ is one-dimensional, as well as the fact that the elasticity equation for $\mathbf{u}$ does not “see” additive constants.

The first step is again to formally take the solution p as a test function in (4.12) and

integrate $\int_0^t$ to obtain the following formal equality (valid on approximants):

$$\|\beta(p, p)\|_{L^\infty(0, T)} + \int_0^t A[p(s), p(s); k(z(s))] ds = \int_0^t \langle S(s), p(s) \rangle_{V' \times V} ds + |p(0)|_B^2. \quad (4.16)$$

The last term will be critical to estimate, since $Bp(0)$ is the given initial condition rather than $p(0)$ here, and B is not invertible as before. We calculate

$$|p(0)|_B^2 = (Bp(0), p(0))_{L^2(\Omega)} = (Bp(0), \mathcal{P}p(0))_{L^2(\Omega)} \quad (4.17)$$

where we have used the assumption that $Bp(0) \in L_0^2(\Omega)$ and used orthogonality to obtain the above equality. We now note that $\|\mathcal{P}p(0)\| \leq C\|B\mathcal{P}p(0)\|$, since $\mathcal{P}p(0) \in L_0^2(\Omega)$ and, as before, B is an isomorphism on this space (see proof of Proposition 4.4). Moreover, we have $Bp(0) = B\mathcal{P}p(0)$ for all $p \in L^2(\Omega)$. Thus:

$$(Bp(0), \mathcal{P}p(0))_{L^2(\Omega)} \leq C\|Bp(0)\| \|B\mathcal{P}p(0)\| \leq C\|Bp(0)\|_{L^2(\Omega)}^2. \quad (4.18)$$

Since $Bp(0)$ is given as data in $L_0^2(\Omega)$, we deduce that the LHS of (4.16) is bounded by data, as in (4.14).

Now, again suppose that $p \in L^2(0, T; V_D)$ is any weak solution with $d_0 \in L_0^2(\Omega)$. Since B is not an isomorphism here, we cannot proceed in the same way as we did in the previous case to obtain that $\partial_t Bp$ lies in a suitable dual space. As a weak solution, we have immediately that $Bp_t \in L^2(0, T; V_D')$ and $Bp \in L^2(0, T; H^1(\Omega) \cap L_0^2(\Omega))$ (considering the range of B in Lemma 4.2). But, by restricting test functions to $V_D \cap L_0^2(\Omega) \subseteq V_D$ in the weak form (4.10) and estimating directly, we obtain that $Bp_t \in L^2(0, T; [V_D \cap L_0^2(\Omega)]')$. Again, by interpolation of $V_D \cap L_0^2(\Omega) \subseteq L_0^2(\Omega) \subseteq [V_D \cap L_0^2(\Omega)]'$, we obtain that $Bp \in C([0, T]; L_0^2(\Omega))$. However, at this stage, we know only that $p \in L^2(0, T; V_D)$, and thus direct “inversion” of B to obtain the result is not possible as before.

On the other hand, we note that $\mathcal{P}p \in L^2(0, T; V_D \cap L_0^2(\Omega))$ and that $Bp = B\mathcal{P}p$ (as before). Therefore, we obtain $\mathcal{P}p \in C([0, T]; L_0^2(\Omega))$ (with associated estimate). Finally, by the definition of $\mathcal{P}$, we observe that $\nabla p = \nabla \mathcal{P}p$, and therefore again conclude that the elasticity equation is satisfied for every $t \in [0, T]$. Analogous to the Neumann case, we then obtain $\mathbf{u}(t) \in \mathbf{V}$, and estimate (4.15) again holds. The final conclusion and estimate follows as does the conclusion of the Neumann case as at the end of Section 4.3.1. This concludes the proof of Theorem 2.2.

5 Nonlinear Problem

In this section we utilize the preceding constructions and estimates to obtain the existence of a weak solution in the sense of Definition 1 to the *nonlinear problem* (2.3). This constitutes the proof of Theorem 2.4, providing the first direct fixed point construction of solutions to the quasilinear Biot problem.

5.1 Fixed Point Map

We consider the abstract problem in $(2.3)_{\text{lin}}$, for a given $z \in L^2(0, T; L^2(\Omega))$ which yields $A(t) = -\nabla \cdot [k(z(t))\nabla(\cdot)]$, which is defined *a.e.* $t \in [0, T]$. For emphasis, we re-write the problem here, including an auxiliary variable ζ which will allow us to more clearly perform the fixed point argument. Recall that the space V is interpreted in a case-dependent way (2.4), but the argument below does not distinguish between these cases. For data

$$\mathbf{F} \in H^1(0, T; \mathbf{V}') \cap L^2(0, T; \mathbf{L}^2(\Omega)), \quad S \in L^2(0, T; V'), \quad d_0 \in L_0^2(\Omega)$$

consider the problem

$$\begin{cases} \mathcal{E}(\mathbf{u}) = -\nabla p + \mathbf{F} & \in L^2(0, T; L^2(\Omega)) \\ \zeta_t - \nabla \cdot [k(z(t))\nabla p] = S & \in L^2(0, T; V') \\ \zeta = \nabla \cdot \mathbf{u} & \in L^2(0, T; V_N) \\ [\nabla \cdot \mathbf{u}](0) = d_0 \in L_0^2(\Omega). \end{cases} \quad (5.1)$$

By Theorem 2.1, the above linear problem (with the associated regularity of data) has a *unique weak solution* written here as $(\mathbf{u}(z), \zeta(z), p(z))$. Let us define the following mapping:

$$\mathcal{F} : L^2(0, T; L^2(\Omega)) \rightarrow L^2(0, T; L^2(\Omega)), \quad \text{given by } \mathcal{F}(z) = \zeta(z),$$

where $\zeta(z) = \nabla \cdot \mathbf{u}(z)$ comes from the unique solution to (5.1) for the given z .

Lemma 5.1. *The map $\mathcal{F}$ introduced above is well-defined on $L^2(0, T; L^2(\Omega))$. This follows from existence and uniqueness of solution to this linear problem $(2.3)_{\text{lin}}$.*

Note that a fixed point of $\mathcal{F}$ would yield the existence of a weak solution to the nonlinear problem (2.3).

Lemma 5.2. *Suppose $\bar{z} \in L^2(0, T; L^2(\Omega))$ is a fixed point of $\mathcal{F}$. Then $(\mathbf{u}(\bar{z}), \bar{z}, p(\bar{z}))$ is a weak solution to (5.1), and thus we have a weak solution to (2.3) (as in Definition 1).*

We will apply Schauder's fixed point theorem.

5.2 Applying Schauder's Theorem

We proceed to establish a fixed point by employing the subspace version of Schauder directly.

Theorem 5.3. *The mapping $\mathcal{F} : L^2(0, T; L^2(\Omega)) \rightarrow L^2(0, T; L^2(\Omega))$ has a fixed point.*

Proof of Theorem 5.3. We must characterize the image of $\mathcal{F}$, and demonstrate compactness and continuity of the map.

Let $d_0 \in L_0^2(\Omega)$, $\mathbf{F} \in H^1(0, T; \mathbf{V}') \cap L^2(0, T; \mathbf{L}^2(\Omega))$, and $S \in L^2(0, T; V')$ be given. We consider the mapping $\mathcal{F} : L^2(0, T; L^2(\Omega)) \rightarrow L^2(0, T; L^2(\Omega))$ defined above. By the estimates

for *linear solutions* as established in Theorem 2.2, and a posteriori, by satisfying (1), we have and that for each $z \in L^2(0, T; L^2(\Omega))$ and $\zeta = \mathcal{F}(z)$

$$\zeta \in L^2(0, T; V), \quad \text{and} \quad \zeta_t \in L^2(0, T; V'),$$

with associated estimates.

Continuity. Let $z_n \rightarrow z \in L^2(0, T; L^2(\Omega))$, $\zeta_n = \mathcal{F}(z_n)$. We want to prove that ζ_n has a (strong) limit point $\zeta = \mathcal{F}(z)$.

First, by Assumption 1, the function $k(\cdot)$ considered as Nemytskii operator, has the property that $k(z_n) \rightarrow k(z) \in L^2(0, T; L^2(\Omega))$ —see [7, 13] for more discussion. Now, since $\zeta_n = \mathcal{F}(z_n)$, for the unique $Bp_n = \zeta_n$ we have by definition of $\mathcal{F}$, the estimates that provide a uniform-in- n bound on the quantities

$$\|p_n\|_{L^2(0, T; V)}, \quad \|p_n\|_{L^\infty(0, T; L^2(\Omega))}, \quad \|\beta(p_n, p_n)\|_{L^\infty(0, T)}.$$

From the bound on p_n in $L^2(0, T; V)$ we extract a weak subsequential limit point, i.e., $p_{n_k} \rightharpoonup p \in L^2(0, T; V)$. From this and the continuity of $B \in \mathcal{L}(L^2(0, T; L^2(\Omega)))$, we obtain that $\zeta_{n_k} = Bp_{n_k} \rightharpoonup Bp$. We define this latter quantity as $\zeta := Bp$, and hence $\zeta_{n_k} \rightharpoonup \zeta$. In addition, we obtain from the weak form, and the uniqueness of limits ensure that (perhaps passing to a further subsequence with the same label), $\zeta_{n_k} \rightharpoonup \zeta \in H^1(0, T; V')$.

We want to show that $\zeta = \mathcal{F}(z)$, and this is accomplished by passing with the limit on the subsequence n_k in the weak formulation (4.10). To that end, let us again consider the weak form evaluated on n_k , and restrict our spatial test functions to $q \in L^2(0, T; V) \cap L^\infty(0, T; W^{1, \infty}(\Omega))$:

$$\int_0^T \langle \zeta'_{n_k}(t), q(t) \rangle dt + \int_0^T A[p_{n_k}(t), q(t); z_{n_k}(t)] dt = \int_0^T \langle S(t), q(t) \rangle dt. \quad (5.2)$$

Limit passage on the first term on the LHS is immediate, identifying weak limits in the weak form. For the second term, more care must be taken. Consider:

$$\int_0^T (k(z_{n_k}) \nabla p_{n_k}, \nabla q(t)) dt = \int_0^T ([k(z_{n_k}) - k(z)] \nabla p_{n_k}, \nabla q(t)) dt + \int_0^T (k(z) \nabla p_{n_k}, \nabla q(t)) dt. \quad (5.3)$$

The first term on the RHS is handled through the Nemytskii property of $k(\cdot)$:

$$\begin{aligned} \int_0^T ([k(z_{n_k}) - k(z)] \nabla p_{n_k}, q(t)) dt &\leq C(\|q\|_{L^\infty(0, T; W^{1, \infty}(\Omega))}) \|k(z_{n_k}) - k(z)\|_{L^2(0, T; L^2(\Omega))} \|p_{n_k}\|_{L^2(0, T; V)} \\ &\leq C(q, \|p\|_{L^2(0, T; V)}) \|k(z_{n_k}) - k(z)\|_{L^2(0, T; L^2(\Omega))} \rightarrow 0, \end{aligned}$$

by the uniform bound on p_{n_k} in $L^2(0, T; V)$. Convergence of the second term in (5.3) is immediate, since by the boundedness of k we have $k(z) \nabla q \in L^2(0, T; L^2(\Omega))$; then, $\nabla p_{n_k} \rightharpoonup \nabla p \in L^2(0, T; L^2(\Omega))$.

Thus, we have shown that for $q \in L^2(0, T; V) \cap L^\infty(0, T; W^{1,\infty}(\Omega))$

$$\int_0^T (k(z_{n_k}) \nabla p_{n_k}, \nabla q(t)) dt \rightarrow \int_0^T (k(z) \nabla p, \nabla q(t)) dt,$$

and hence, passing to the limit as $k \rightarrow \infty$ in (5.2) we obtain for $\zeta = \zeta(z)$ the identity

$$\int_0^T \langle \zeta_t, q \rangle dt + \int_0^T (k(z) \nabla p, \nabla q(t)) dt = \int_0^T \langle S, q(t) \rangle dt \quad (5.4)$$

for all $q \in L^2(0, T; V) \cap L^\infty(0, T; W^{1,\infty}(\Omega))$, the latter being dense in $L^2(0, T; V)$. Thus we have shown that $(\zeta(z), p(z))$ satisfies the weak form of the pressure equation and hence we have constructed a weak solution $(\zeta(z), p(z))$ for $z \in L^2(0, T; L^2(\Omega))$. Obtaining the initial condition is also immediate from the definition of $\mathcal{F}$. Hence ζ_n has a *weak subsequential limit point* $\zeta = \mathcal{F}(z)$.

To conclude the continuity of $\mathcal{F}$, we must improve the convergence of $\zeta_{n_k} \rightarrow \zeta$ to that of strong in $L^2(0, T; L^2(\Omega))$. This is done via the Lions-Aubin compactness theorem (see, for instance, [40]). In addition to the estimates in Theorem 2.2 for the sequence p_{n_k} , we obtain two additional uniform-in- k estimates from continuity of $B : V \rightarrow H^1(\Omega)$ and from satisfying the weak form of the pressure equation, namely:

$$\|\zeta_{n_k}\|_{L^2(0,T;H^1(\Omega))}^2 = \|Bp_{n_k}\|_{L^2(0,T;H^1(\Omega))}^2 \lesssim \|p\|_{L^2(0,T;V)}^2 \quad (5.5)$$

$$\|[\zeta_{n_k}]'\|_{L^2(0,T;V')}^2 = \|[Bp_{n_k}]'\|_{L^2(0,T;V')}^2 \lesssim \|p\|_{L^2(0,T;V)}^2 + \|S\|_{L^2(0,T;V')}^2. \quad (5.6)$$

By possibly passing to a further subsequence n_{k_m} (not affecting the previous steps in establishing the weak solution or associated estimates), we improve the convergence of $\zeta_{n_{k_m}} \rightarrow \zeta \in L^2(0, T; L^2(\Omega))$.

Compactness. We must show that the range of $\mathcal{F}$ is relatively compact in $L^2(0, T; L^2(\Omega))$. But, as in the previous step, this will follow from the Lions-Aubin compactness criterion. Indeed, for $\zeta = \mathcal{F}(z)$, ζ corresponds to a weak solution satisfying the above estimates. In particular, we obtain for any such $\zeta(z)$ there is an associated $(p(z), \mathbf{u}(z))$ such that:

$$\|\zeta\|_{L^2(0,T;H^1(\Omega))}^2 \leq C\|p\|_{L^2(0,T;V)}^2 \leq C[\|d_0\|_{L^2(\Omega)}^2 + \|S\|_{L^2(0,T;V')}^2] \quad (5.7)$$

$$\|\zeta'\|_{L^2(0,T;V')}^2 \leq C[\|p\|_{L^2(0,T;V)}^2 + \|S\|_{L^2(0,T;V')}^2] \leq C[\|d_0\|_{L^2(\Omega)}^2 + \|S\|_{L^2(0,T;V')}^2]. \quad (5.8)$$

A subset of $L^2(0, T; L^2(\Omega))$ which is bounded as in the previous two estimates is relatively compact by the Lions-Aubin criterion, and hence $\zeta = \mathcal{F}(z)$ lies in a compact set. This is the final hypothesis to be satisfied for applying the Schauder fixed point theorem.

Doing so, and applying Schauder's point theorem, yields the existence of a function $z \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V')$ and an associated weak solution $(\zeta(z), p(z))$ for which $z = \mathcal{F}(z)$. $\square$

Remark 5.1. We again note that, owing to the presence of the nonlinearity, regularity of the solution ζ —in particular of $\nabla \cdot \mathbf{u}$ —needs to be better than $L^2(0, T; L^2(\Omega))$. This is because we must obtain compactness in ζ to utilize the Nemytskii property of $k(\cdot)$. Moreover, if $d_0 \in V'$ only, this would preclude our ability to obtain such regularity, as this would seem to lower the evolution of $Bp = \nabla \cdot \mathbf{u}$ to the regularity of V' .

References

- [1] Auriault, J.L. and Sanchez-Palencia, E., 1977. Etude du comportement macroscopique d'un milieu poreux saturé déformable, *Journal de Mécanique*, 16(4), pp.575–603.
- [2] Biot, M. A., 1941. General theory of three-dimensional consolidation. *Journal of Applied Physics* 12(2), pp.155–164.
- [3] Biot, M.A. and Willis, D.G., 1957. The elastic coefficients of the theory of consolidation. J. ASME.
- [4] Biot, M.A., 1955. Theory of elasticity and consolidation for a porous anisotropic solid. *Journal of Applied Physics*, 26(2), pp.182–185.
- [5] Bociu, L., Guidoboni, G., Sacco, R. and Verri, M., 2019. On the role of compressibility in poroviscoelastic models. *Mathematical Biosciences and Engineering*, 16(5).
- [6] Bociu, B. and Webster, J.T., 2021. Nonlinear Quasi-static Poroelasticity. *Journal of Differential Equations*, to appear.
- [7] Bociu, L., Guidoboni, G., Sacco, R., and Webster, J.T., 2016. Analysis of Nonlinear Poro-Elastic and Poro-Visco-Elastic Models, *Archives of Rational Mechanics and Analysis*, 222, pp.1445–1519.
- [8] Bociu, L., Čanić, S., Muha, B. and Webster, J.T., 2021. Multilayered Poroelasticity Interacting with Stokes Flow. *SIAM Journal of Mathematical Analysis*, to appear.
- [9] Bosco E, Peerlings R H, Geers M G., 2015. Predicting hygro-elastic properties of paper sheets based on an idealized model of the underlying fibrous network. *Int. J. Solids Struct.*; 56: 43-52.
- [10] Brezis, H., 2010. *Functional analysis, Sobolev spaces and partial differential equations*, Springer Science & Business Media.
- [11] P. Causin, G. Guidoboni, A. Harris, D. Prada, R. Sacco, S. Terragni, *A poroelastic model for the perfusion of the lamina cribrosa in the optic nerve head*, *Mathematical Biosciences* 257 (2014) 33-41, *Multiscale models and methods in biomedicine*.
- [12] Canic S, Hartley CJ, Rosenstrauch D, Tambaca J, Guidoboni G, A. Mikelic., 2006. Blood Flow in Compliant Arteries: An Effective Viscoelastic Reduced Model, *Numerics and Experimental Validation. Ann. Biomed. Eng.*; 34: 575 - 592
- [13] Cao, Y., Chen, S., and Meir, A.J., 2013. Analysis and numerical approximations of equations of nonlinear poroelasticity, *DCDS-B*, 18, pp. 1253–1273.
- [14] Cardoso L., Fritton SP, Gailani G, Benalla M, Cowin SC., 2013. Advances in assessment of bone porosity, permeability, and interstitial fluid flow. *J. Biomech.*; 46: 253265.

- [15] D. Chapelle, J. Sainte-Marie, J.-F. Gerbeau, I. Vignon-Clementel, A poroelastic model valid in large strains with applications to perfusion in cardiac modeling, *Computational Mechanics* 46(1) (2010) 91-101.
- [16] Ciarlet, P.G., 1988. *Three-dimensional elasticity, Vol. 1, Elsevier*.
- [17] Coussy O., 2004 Poromechanics. John Wiley and Sons.
- [18] Cowin SC., 1999. Bone Poroelasticity. *J.Biomech.*;32:218-238.
- [19] Detournay, E. and Cheng, A.H.D., 1993. Fundamentals of poroelasticity. *Analysis and design methods (pp. 113-171)*. Pergamon.
- [20] Di Benedetto, E. and Showalter, R.E., 1981. Implicit degenerate evolution equations and applications. *SIAM Journal on Mathematical Analysis*, 12(5), pp.731–751.
- [21] van Duijn, C. and Mikelić, A., 2019. Mathematical Theory of Nonlinear Single-Phase Poroelasticity. Preprint hal-02144933, Lyon June.
- [22] Evans, E., 2010. *Partial Differential Equations, 2nd Ed., AMS*, Graduate Studies in Mathematics, 19.
- [23] Giovanni Paolo Galdi. 2011. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, volume 501. Springer.
- [24] P. Grisvard, 2011. *Elliptic problems in nonsmooth domains, SIAM*, 410pp
- [25] G. Guidoboni, A. Harris, L. Carichino, Y. Arieli, B. A. Siesky, Effect of intraocular pressure on the hemodynamics of the central retinal artery: a mathematical model, *Mathematical Biosciences and Engineering* 11(3) (2014) 523-546.
- [26] Gurvich, E. and Webster, J.T., 2021. Weak Solutions for a Poro-elastic Plate System. *Applicable Analysis*, published online (July 2021).
- [27] C. Hsu, P. Cheng, *Thermal dispersion in a porous medium*, International Journal of Heat and Mass Transfer 33 (8) (1990) 1587-1597.
- [28] Kesavan, S., 2019. *Topics in functional analysis and applications*. New Age International Publishers, Vol. 3.
- [29] A-RA Khaled, K. Vafai, The role of porous media in modeling flow and heat transfer in biological tissues, *International Journal of Heat and Mass Transfer* 46(26) (2003) 4989-5003
- [30] Lewis R W, Schrefler B A. 1987 The Finite Element Method in the Deformation and Consolidation of Porous Media. Wiley.
- [31] Marciniak-Czochra, A. and Mikelić, A., 2015. A rigorous derivation of the equations for the clamped Biot-Kirchhoff-Love poroelastic plate. *Archive for Rational Mechanics and Analysis*, 215(3), pp.1035-1062.
- [32] Mazzucato, A.L. and Nistor, V., 2010. Well-posedness and regularity for the elasticity equation with mixed boundary conditions on polyhedral domains and domains with cracks. *Archive for rational mechanics and analysis*, 195(1), pp.25–73.
- [33] Nicaise, S., 1992. About the Lamé system in a polygonal or a polyhedral domain and a coupled problem between the Lamé system and the plate equation. I: Regularity of the solutions. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 19(3), pp.327–361.

- [34] Owczarek, S., 2010. A Galerkin method for Biot consolidation model. *Mathematics and mechanics of solids*, 15(1), pp.42–56.
- [35] Prosi M, Zunino P, Perktold K, Quarteroni A., 2005. Mathematical and numerical models for transfer of low-density lipoproteins through the arterial walls: a new methodology for the model set up with applications to the study of disturbed luminal flow. *J. Biomech.*; 38: 903-917.
- [36] Rice, J.R. and Cleary, M.P., 1976. Some basic stress-diffusion solutions for fluid saturated elastic porous media with compressible constituents. *Rev. Geophys. Space Phys.*, 14, 227-241.
- [37] Rudnicki, J.W. and Hsu, T.-C., 1988. Pore pressure changes induced by slip on permeable and impermeable faults. *J. Geophys. Res.*, 93, 3275-3285.
- [38] R. Sacco, G. Guidoboni, A.G. Mauri, A Comprehensive Physically Based Approach to Modeling in Bioengineering and Life Sciences, Academic Press (2019), London (UK)
- [39] Showalter, R.E., 1974. Degenerate evolution equations and applications, *Indiana University Mathematics Journal*, 23(8), pp.655—677.
- [40] Showalter, R.E., 1996. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, AMS, Mathematical Surveys and Monographs, 49.
- [41] Showalter, R.E., 2000. Diffusion in poro-elastic media, *JMAA*, 251, pp. 310–340.
- [42] Showalter, R.E., 2010. *Hilbert space methods in partial differential equations*. Courier Corporation.
- [43] Su, N. and Showalter, R.E., 2001. Partially saturated flow in a poroelastic medium, *DCDS-B*, 1, pp. 403–420.
- [44] R. E. Showalter and Ulisse Stefanelli, 2004 Diffusion in Poro-Plastic Media, *Mathematical Methods in the Applied Sciences* 27, pp. 2131-2151
- [45] Temam, R., 2001. *Navier-Stokes equations: theory and numerical analysis*. American Mathematical Soc..
- [46] Terzaghi K. 1951. Theoretical soil mechanics. Chapman And Hall, London.
- [47] Verri, M., Guidoboni, G., Bociu, L. and Sacco, R., 2018. The role of structural viscoelasticity in deformable porous media with incompressible constituents: Applications in biomechanics. *Mathematical Biosciences and Engineering: MBE*, 15(4), pp.933–959.
- [48] Verruijt A, Van Baars S. 2007. Soil mechanics. Delft, the Netherlands: VSSD.
- [49] S. Whitaker, The Method of Volume Averaging, Theory and Applications of Transport in Porous Media, vol. 13, Springer, Netherlands, 1999.
- [50] Yamamoto, T., Koning, H.L., Sellmeijer, H. and Van Hijum, E., 1978. On the response of a poro-elastic bed to water waves. *J. Fluid Mech.*, 87, 193-206.
- [51] Zenisek, A., 1984. The existence and uniqueness theorem in Biot’s consolidation theory, *Appl. Math.*, 29, pp. 194–211.