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Dynamic equations of motion for inextensible beams and plates

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Abstract The large deflections of cantilevered beams and rectangular plates are modeled and discussed. Traditional nonlinear elastic models (e.g., von Karman's) employ elastic restoring forces based on the effect of stretching on bending, and these are less applicable to cantilevers. Recent experimental work indicates that elastic cantilevers are subject to nonlinear inertial and stiffness effects. We review a recently established (quasilinear and nonlocal) cantilevered beam model, and consider some extensions to two spatial dimensions, namely inextensible plates. Our principal configuration is that of a thin, isotropic, homogeneous rectangular plate, clamped on the one edge and free on the remaining three. We proceed through the geometric and elastic modeling to obtain equations of motion via Hamilton's principle for the appropriately specified energies. We then enforce *effective* inextensibility constraints through Lagrange multipliers. Multiple plate analogs of the established 1D model are obtained, based on assumptions. In total, we present three distinct nonlinear partial differential equation models and, additionally, describe a class of "higher-order" models. Each model has particular advantages and drawbacks for both mathematical and engineering analyses. We conclude with a discussion of the various models, as well as some analytical problems.

Keywords Cantilever · Nonlinear elasticity · Inextensibility · quasilinear PDE · Energy harvesting

Mathematics subject classification 74B20 · 74K20 · 35L77 · 93A30

1 Introduction

The purpose of this manuscript is to obtain equations of motion—including boundary conditions—for the large deflections of elastic cantilevers. Cantilevers, as a class of elastic objects, have received less attention

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in the voluminous literature on beams and plates. In particular, there is a dearth of work analyzing the *large deflections* of cantilevers. Recent applications, as described below, have brought about the need for viable partial differential equation (PDE) models (i.e., distributed parameter systems) that capture the relevant dynamics. For instance, we are interested in the dynamics of a long, slender flapping beam. In terms of spatial distribution, we consider both beams (1D) and plates (2D) here. As we are interested in large deflections, models will be *nonlinear* in the displacement variables. The linear theory of elasticity is well established in both engineering and mathematics for beams and plates (e.g., [2,7,9,20,28,35]). Often, standard models—such as that of Euler–Bernoulli, Rayleigh, or Timoshenko for beams, and Kirchhoff–Love or von Karman for plates—are utilized because they are sufficiently accurate, while being analytically and computationally tractable. For cantilevered beams, we aim to produce a theory that generalizes the traditional linear theory in the realm of (nonlinear) large deflections. We will consider *planar* transverse and in-axis displacements for beams, and for plates, we will consider rectangular plates, with 3D displacements from an equilibrium given in a standard Euclidean frame.

Unlike traditional nonlinear theories of beams and plates, where the boundaries are fully restricted, we will not assume that nonlinear forces result from local stretching (extensibility), as would be the case for fully clamped or hinged structures (panels). Rather, we will assume certain *inextensibility* conditions—that the beam or plate does not stretch in specified ways throughout deflection. For a beam with clamped–free edges or free–free edges, the assumption of inextensibility (zero mid-plane strain) is motivated by the zero-strain boundary condition on the free edge(s). The enforcement of an *inextensibility constraint* via Lagrange multipliers in the dynamics will yield nonlinear inertial terms in the equations of motion. For consistency in *order considerations*, we will invoke higher-order expressions for the potential energy, yielding nonlinear stiffness terms in the equations of motion when Hamilton's principle is invoked.

We will specifically consider a *plate* with three free edges and one edge clamped. Clearly along the clamped edge, the plate is not free to extend and must have strain in the direction parallel to the clamped edge. The purpose of this investigation is to explore several alternative models based upon *a variety of assumptions* about the 2D mid-plane strain. As seen in the three principal nonlinear cantilever models derived below, the *issue of shear strain is quite subtle*. The most attractive *mathematical model* that generalizes the well-accepted inextensible beam theory will require an assumption of zero mid-plane shear strain, as well as zero mid-plane longitudinal strains. Clearly, this assumption must be violated locally along the clamped edge. We explore the mathematical consequences of this assumption and also consider several alternative assumptions, which lead to more complex mathematical models.

1.1 Motivating application

Under harmonic excitation [37], follower forces [34], or other boundary forces, cantilevers may experience large deflections (say, on the order of their length). We focus here on the application of *piezoelectric energy harvesters*. Slender cantilevers in axial[1] airflows experience a self-destabilization known as *flutter*, even at low flow velocities [17,19,42,44]. Beyond a critical flow velocity, the flow-structure system enters a *limit cycle oscillation* (LCO), and such dynamics can generate extractable power [15].

While interest in airfoil and panel flutter has been *immense* for some 75 years (see [13]), interest in the motion of cantilevers driven by axial flow has been modest, at least until about 20 years ago. In [1,24], the cantilever flutter of the human soft palate was considered, and, as in most engineering scenarios, the flutter thereof is undesirable, with a design goal of prevention. Lack of interest for this configuration seems to derive from the few practical motivations. However, recent interest in alternative energies, in particular energy harvesting [15,18], has provided reasons for the investigation of cantilever LCOs. In this application, dynamic instability is *encouraged* in order to extract energy from post-onset cantilever LCOs. This has been shown possible for flow-induced cantilever oscillations with affixed piezoelectric material [18,40,41,46].

This motivates the problem of characterizing *post-flutter* behaviors of cantilever (i.e., LCOs), which necessitates a viable set of nonlinear equations of motion. In reality, we must practically consider distributed and variable stiffness and damping effects, owing to the use and placement of piezoelectric patches and layers. Recent experiments [25,30,38] and others [18,41] have studied piezoelectric structures in elastic energy harvesters.

[1] Along the principal length of the beam or plate.

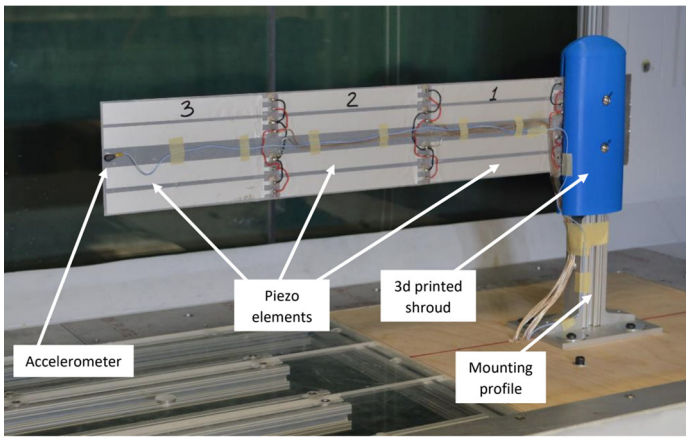

Fig. 1 Experimental setup from [30]

Thus, there is a need for a theory of large deflections for cantilevers. In physically applicable scenarios, the relevant theory should be truly 2D. To date, very little work has been done experimentally and analytically in this configuration, relative to the immense body of work on extensible plates.

1.2 Modeling and analysis of cantilever large deflections

The primary focus of this work is to produce nonlinear equations of motion. These equations can subsequently be analyzed numerically and analytically. As we have established, the models of interest are based on inextensibility, rather than extensibility. In the case of extensible structures, models and theory are well-established, especially in scenarios where the boundary is fully restricted (e.g., paneling [7,8,13]); such nonlinear restoring forces are based on the effect of local stretching on bending. In the case of beams, a prominent example is the Woinowsky–Krieger beam [47], which appends a semilinear extensible force to any of the standard linear beams such as those of Euler–Bernoulli, Rayleigh, shear, or Timoshenko [20]. We note some classical mathematical references for the analysis of such extensible beams, e.g., [3,16], as well as some more recent [22,23] that specifically address extensible cantilever systems. Alternatively, [28] provides a modeling account (with subsequent analysis of solutions and their stability) for an extensible beam that is intimately related to the so-called *full von Karman* model for plates.

Indeed, for plates, the linear theory of Kirchhoff–Love is well established, and the prominent nonlinear generalization for large deflections is the theory of von Karman [9] (and references therein). One can consult [5,7,35] for classical discussions, and the rigorous mathematical justifications of these models in [9,29]. The more modern references of [8,27] provide detailed discussion of solutions, stability, and dynamical systems aspects of von Karman dynamics (among others). We finally mention [26] for the analysis of the *full von Karman* plate, which is an extensible plate system that accounts for both in-plane and transverse inertial terms (in contrast to the scalar von Karman equations [8]); the full von Karman plate equations, as we shall see, have some mathematical similarities to the plate models derived herein.

We forgo an extensive discussion of the literature on the analysis of the aforementioned nonlinear beams and plate models and focus specifically here on recent work that *attempts to capture the effects of inextensibility for cantilevers*. It should be noted that the theory of von Karman is not viable (the operative hypotheses invoke the effect of stretching on bending) in the case of a cantilever. As such, a theory based on inextensibility must be developed from first principles. The earlier work of Païdoussis derives and discusses 1D inextensible pipes conveying fluid, along with associated numerical stability analyses [36,39]. Subsequently, the thesis [32] focuses on inextensible beams and plates, but primarily focuses on finite-dimensional descriptions through Rayleigh–Ritz considerations, not providing the explicit equations of motion in the standard Euclidean frame. The work of Dowell et al. further elaborated on inextensible beams and plates, developing several related approaches and investigating an inextensible theory experimentally and numerically in the sequence of papers [42,44,45]. Finally, the work of McHugh et al. [14,33,34] provided the first explicit calculus of variations derivation of the PDE equations of motion of an inextensible beam (in both the free–free and cantilever configurations). Apart from these papers, other aforementioned analyses primarily employ the Rayleigh–Ritz

procedure to obtain finite-dimensional equations of motion directly from specified energies, circumventing the PDE equations of motion. Numerical comparisons and computational studies are then performed. The discussion of boundary conditions for the 2D system (from first principles) is typically omitted, and the rigorous analysis of solutions, as well as the well-posedness thereof, is not made.

Recently, for the inextensible cantilever as derived in [14], a mathematical theory of solutions was developed [11,12]. In the present manuscript, we aim to take an analogous first step in this direction, by providing the PDE equations of motion for the "natural" beam extension to a 2D inextensible plate. We specifically follow the approaches for the beam, coming from references [12,14,33]. To begin our considerations, we will first revisit the setup for an inextensible cantilevered beam; this will provide a template for plate considerations. In doing so, we will include a novel "higher-order" beam model, which is motivated from a possible need for inclusion of such effects for the plate.

As we shall see, there is no single, clear plate extension of the our beam model. Indeed, there are certain modeling and order choices, each of which yields a different system for the equations of motion. The presence of 2D shear effects is a critical aspect of the analysis here. Providing a careful derivation of the equations of motion is indeed quite necessary, as we will see, since "natural" modeling choices produce nontrivial nonlinear boundary conditions for each plate model. This is to say: it is notable that for the 2D plate, the modeling hypotheses produce (nonlinear) boundary conditions, which are not those of a standard, linear, cantilevered plate. Moreover, some aesthetic simplifications that present themselves in 1D will be conspicuously absent in 2D.

In the end, we will present various sets of hypotheses, yielding a variety of different systems of equations of motion. *To the best knowledge of the authors, each of the plate systems here is novel*. In future work, we will numerically analyze these models to compare against [42,43]. Moreover, the theory developed in [11] will be adapted, at least in the cases where the 2D equations of motion most closely resemble those of the inextensible cantilevered beam in [11,14]. Indeed, one principal goal in this work is to address inextensibility in the standard coordinate framework—which is to say, to extend standard linear beam theory, rather than using more sophisticated geometric models (such as those of [2]). In this way, existing mathematical and numerical tools, as well as theory, can be utilized, viewing these models as "extensions" of the classical linear theory of elasticity. Finally, we provide a brief discussion of the numerical work from [33] in Sect. 4.

1.3 Outline of the remainder of the paper

To provide context for the approach we take to *inextensible plates*, we first give a summary of the work in [11,14] for the recent *inextensible cantilevered beams* in Sect. 2.

Section 3 is the principal modeling section. There, we walk through our choices for the potential energies, as well as the interpretation and enforcement of "inextensibility" in the plate. Ultimately, we present three distinct models in Sect. 3, with a clear discussion of the relevant modeling and order hypotheses. Each model is presented as a system of partial differential equations in the elastic deflection variables, as well as constraint variables. We conclude Sect. 3 with a description of how to obtain a class of higher-order models, though without explicitly producing the equations of motion.

Finally, in Sect. 4 we discuss and compare the derived models from both the engineering and analysis-of-PDE points of view.

1.4 Conventions and notation

We utilize the convention that spatial points $\mathbf{x}$ are identified with their position vector $\langle x, y, z\rangle$. We will sometimes write $x_1 = x,\ x_2 = y,\ x_3 = z$ when it is notationally expedient. The equilibrium (undeformed) domain under consideration will be cylindrical, namely $\{\mathbf{x} \in \Omega \times (-h/2, h/2)\}$. The domain Ω will be identified as the centerline, $z = x_3 = 0$. We make use of the partial derivative notation $\dfrac{\partial}{\partial x_i} = \partial_{x_i}$, as well as the standard 2D spatial gradient $\nabla = \langle \partial_x, \partial_y\rangle$, divergence $[\nabla\cdot]$, and Laplacian (in rectangular coordinates) $\Delta = \nabla\cdot\nabla = \sum_{i=1}^{2} \partial_{x_i}^2$.

2 Inextensible cantilevered beams

We begin our discussion with cantilevered beams. These are markedly simpler, owing to the lack of interaction in the chord-wise and span-wise dynamics. The exposition here mirrors [12,14]. By presenting a streamlined version of the 1D model and its procurement from precise geometric and order hypotheses, we elucidate the decisions which lead to the plate models in subsequent sections.

Consider a slender, isotropic, homogeneous beam, as is standard in Euler–Bernoulli beam theory [20]. We assume that the filaments of the beam remain perpendicular to the centerline throughout deflection (the Kirchhoff–Love hypothesis—see [20,28]). The variable $x \in [0, L]$ will represent the beam's centerline at equilibrium. We utilized the notation $\langle x + u(x,t), w(x,t)\rangle \in \mathbb{R}^2$ for the planar deflection that corresponds to the equilibrium point x at instant t. This is to say that u is the in-axis displacement of the beam, whereas w is the transverse displacement. In this way, the beam's displaced curve is parametrized by x with position vector $\langle x + u(x,t), w(x,t)\rangle$ at time t. If we let ε denote the strain associated with the beam centerline, we have the identity [10,28,35]:

$$\big[1+\varepsilon\big]^2 = (1+u_x)^2 + w_x^2. \tag{2.1}$$

2.1 Inextensibility

When the beam is *inextensible*, we take there to be no center-line extensional stress, i.e., $\varepsilon(x,t) = 0$. This yields the condition:

$$1 = (1+u_x)^2 + w_x^2. \tag{2.2}$$

By expanding (2.2), we see that if $w_x \sim \eta$, we will have $u_x \sim \eta^2$:

$$2u_x + u_x^2 + w_x^2 = 0.$$

Therefore, we may elect to drop terms of order $u_x^2 \sim \eta^4$, owing to their relative order. Approximating, then

$$0 = 2u_x + w_x^2 \;\;\Rightarrow\;\; u_x = -\frac{1}{2}w_x^2.$$

The above can be taken an *effective inextensibility constraint*, providing a direct relationship between u and w:

$$u(x,t) = -\frac{1}{2}\int_0^x [w_x(\xi,t)]^2 \mathrm{d}\xi. \tag{2.3}$$

We note that the above analysis can easily be extended to "higher order," as follows:

$$1 = (1+u_x)^2 + w_x^2 \;\;\Longrightarrow\;\; u_x = -1 \pm \sqrt{1-w_x^2}. \tag{2.4}$$

The binomial expansion can then be used to obtain expressions to higher order

$$u_x = -\frac{1}{2}w_x^2 - \frac{1}{8}w_x^4 - \frac{1}{16}w_x^6 \ldots, \quad |w_x| < 1.$$

Let us now fix the nomenclature for effective inextensibility constraints, with reference to the order of approximation.

Assumption 2.1 In referring to η^2-order and η^4-order approximations for the inextensible beam, we, respectively, assume one of the following relations:

$$\begin{cases} \eta^2\text{order}: & u_x = -\dfrac{1}{2}w_x^2 \\ \eta^4\text{order}: & u_x = -\dfrac{1}{2}w_x^2 - \dfrac{1}{8}w_x^4. \end{cases} \tag{2.5}$$

2.2 Energies

Define the elastic potential energy (E_{P}) via beam curvature κ and constant stiffness D [7,24,35,39][2] in the standard way:

$$E_{\mathrm{P}} \equiv \frac{D}{2}\int_0^L \kappa^2 \mathrm{d}x.$$

As discussed above, the beam's displaced configuration is parametrized by $x \in [0, L]$. As such, the curvature is given by [10]

$$\kappa = \frac{(1+u_x)w_{xx} - u_{xx}w_x}{[(1+u_x)^2 + w_x^2]^{3/2}}.$$

Substituting in the (full) inextensibility constraint (2.2) in to the curvature κ, we obtain:

$$\kappa = (1+u_x)w_{xx} - w_x u_{xx} = (1-w_x^2)^{1/2}w_{xx} - w_x\left(w_x w_{xx}(1-w_x^2)^{-1/2}\right) = \frac{w_{xx}}{(1-w_x^2)^{1/2}}.$$

In E_{P}, we may invoke the appropriate geometric series for $\kappa^2 = \dfrac{w_{xx}^2}{(1-w_x^2)}$ and truncate to a given order *in the nonlinear coefficient* of w_{xx}; we obtain $\kappa^2 \approx w_{xx}^2(1+w_x^2)$ in the case of the η^2-order approximation, and $\kappa^2 \approx w_{xx}^2(1+w_x^2+w_x^4)$ in the case of the η^4-order approximation.

Remark 2.1 Note that, at the stage of the potential energy simplification, we attempt to *match* the order of approximation for the inextensibility condition and the approximated potential energy.

The kinetic energy (E_{K}) is defined in the standard way, where we have mass-normalized the beam

$$E_{\mathrm{K}} \equiv \frac{1}{2}\int_0^L \left[w_t^2 + u_t^2\right]\mathrm{d}x.$$

2.3 Beam equations of motion for η^2-order

We consider displacements u and w (and virtual displacements δu and δw), which respect the essential boundary conditions at $x = 0$:

$$w,\ w_x,\ \delta w,\ \delta w_x\ :\ 0\ \text{ at }\ x=0; \quad u,\ \delta u\ :\ 0\ \text{ at }\ x=0.$$

The effective inextensibility constraint, defined as $f \equiv u_x + (1/2)w_x^2 = 0$, is enforced through the *Lagrange multiplier* λ. We will minimize the Lagrangian:

$$\mathcal{L} = E_K - E_P + \int_0^L \lambda f \mathrm{d}x. \tag{2.6}$$

Invoking Hamilton's principle for the time-integrated variation of (2.6) yields the Euler–Lagrange equations of motion and the associated boundary conditions (after the requisite integration by parts) for the displacements u and w.

$$\text{from arbitrariness of } \delta u: \quad u_{tt} + \lambda_x = 0; \tag{2.7}$$

$$\text{from arbitrariness of } \delta w: \quad w_{tt} - D\partial_x\left(w_{xx}^2 w_x\right) + D\partial_{xx}\left(w_{xx}\left[1+w_x^2\right]\right) + \partial_x(\lambda w_x) = 0. \tag{2.8}$$

For the (natural) boundary conditions at $x = L$ in w (with u and λ then inferred), we have:

$$\lambda(L) = 0; \quad (1+w_x^2(L))w_{xx}(L) = 0; \quad (1+w_x^2(L))w_{xxx}(L) + w_x(L)w_{xx}^2(L) = 0. \tag{2.9}$$

The standard free beam boundary conditions—$w_{xx}(L) = w_{xxx}(L) = 0$—are then obtained algebraically from (2.9), and from Eq. (2.7), we write $\lambda(L) - \lambda(x) = -\int_x^L u_{tt}(\xi)\mathrm{d}\xi$. We invoke the previously obtained $\lambda(L) = 0$ to obtain $\lambda(x) = \displaystyle\int_x^L u_{tt}(\xi)\mathrm{d}\xi$.

[2] Beam flexural rigidity, which can be given in terms of inertial and Young's coefficients, EI.

Remark 2.2 Here, and in the 2D sections which follow, the Lagrange multiplier(s) represent spatially distributed forces that serve to enforce the effective inextensibility constraint(s) through the beam/plate.

Presenting the system in (u, w, λ) yields some degree of redundancy, as one can fully eliminate both λ and u in describing the dynamics above. Indeed, we can solve for λ in terms of u from (2.7); then, we can recover u in terms of w from the constraint (2.3) to present the system solely in w. However, we present the system in its full form in anticipation of more complicated 2D systems to follow; in the models presented below, we will not be able to eliminate all non-transverse variables from the "complete" description. We can summarize, then, in the following description of the η^2 *inextensible beam* system:

$$u_{tt} + \lambda_x = 0 \tag{2.10}$$

$$w_{tt} - D\partial_x \left(w_{xx}^2 w_x\right) + D\partial_{xx} \left(w_{xx}\left[1 + w_x^2\right]\right) + \partial_x \left(\lambda w_x\right) = 0 \tag{2.11}$$

$$u_x = -\frac{1}{2} w_x^2 \tag{2.12}$$

$$w(0) = w_x(0) = 0; \; w_{xx}(L) = w_{xxx}(L) = 0; \tag{2.13}$$

$$u(0) = 0; \;\; u_x(L) = -\frac{1}{2} w_x^2(L) \tag{2.14}$$

$$\lambda(0) = \int_0^L u_{tt}(\xi) \mathrm{d}\xi; \;\; \lambda(L) = 0. \tag{2.15}$$

Of course, the above equations would be supplemented by appropriate initial conditions for w, namely

$$w(x, 0) = w_0(x), \;\; w_t(x, 0) = w_1(x).$$

Initial conditions for u are determined from those for w through the inextensibility constraint.

2.4 Beam equations of motion to η^4-order

Repeating the steps in the previous section, we retain terms up to η^4 in the effective inextensibility condition and the curvature expression κ^2. We obtain—paying particular attention to the natural boundary conditions—the η^4 *inextensible beam* system (omitting initial conditions):

$$u_{tt} + \lambda_x = 0 \tag{2.16}$$

$$w_{tt} - D\partial_x \left([w_x + 2w_x^3] w_{xx}^2\right) + D\partial_x^2 \left(w_{xx}\left[1 + w_x^2 + w_x^4\right]\right) + \partial_x \left(\lambda w_x\right) = 0 \tag{2.17}$$

$$u_x = -\frac{1}{2} w_x^2 - \frac{1}{8} w_x^4 \tag{2.18}$$

$$w(0) = w_x(0) = 0; \; w_{xx}(L) = w_{xxx}(L) = 0; \tag{2.19}$$

$$u(0) = 0; \; u_x(L) = -\frac{1}{2} w_x^2(L) - \frac{1}{8} w_x^4(L) \tag{2.20}$$

$$\lambda(0) = \int_0^L u_{tt}(\xi) \mathrm{d}\xi; \;\; \lambda(L) = 0. \tag{2.21}$$

3 Inextensible cantilevered plates

Let $\mathbf{x} = \langle x, y \rangle \in \mathbb{R}^2$. We will let $\langle u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t) \rangle \in \mathbb{R}^3$ denote the mid-plate displacement (from equilibrium) of a rectangular, cantilevered plate that occupies (at equilibrium) the region $\Omega \times (-h/2, h/2)$. The coordinate u is span-wise displacement, with v being chord-wise; w is the transverse deflection. (Thus, $\langle x + u(\mathbf{x}, t), y + v(\mathbf{x}, t), w(\mathbf{x}, t) \rangle$ describes the position of the equilibrium point $\mathbf{x}$ at the instant t.) The physical quantity $\nu \in (0, 1/2)$ represents the so-called *Poisson Ratio*.

Consider the open rectangle $\Omega \equiv \{(x, y) \in (0, L_x) \times (0, L_y)\}$, representing the undeformed mid-plane ($z = 0$) of the thin, homogeneous, isotropic plate. Let us take the four components of the boundary $\partial\Omega = \Gamma$ (in standard orientation) to be given by

$$\Gamma_E = \{x = L_x,\ y \in (0, L_y)\}, \qquad \Gamma_N = \{x \in (0, L_x),\ y = L_y\},$$
$$\Gamma_W = \{x = 0,\ y \in (0, L_y)\}, \qquad \Gamma_S = \{x \in (0, L_x),\ y = 0\}.$$

We have the associated outward unit normals: $\mathbf{n}_E = \mathbf{e}_1$, $\mathbf{n}_N = \mathbf{e}_2$, $\mathbf{n}_W = -\mathbf{e}_1$, $\mathbf{n}_S = -\mathbf{e}_2$ and corresponding tangentials (respectively): $\mathbf{e}_2, -\mathbf{e}_1, -\mathbf{e}_2, \mathbf{e}_1$.

Remark 3.1 For reference, we provide the classical clamped-free conditions associated with a *linear* cantilever in this configuration, that is, the plate is clamped on Γ_W and free elsewhere. On Γ_N, Γ_S, the boundary conditions are:

$$\nu w_{xx} + w_{yy} = 0; \quad w_{yyy} + (2 - \nu) w_{xxy} = 0.$$

On Γ_E, the boundary conditions are:

$$w_{xx} + \nu w_{yy} = 0; \quad w_{xxx} + (2 - \nu) w_{yyx} = 0.$$

And, of course, on Γ_W we have the clamped conditions:

$$w = 0; \quad w_x = 0.$$

The boundary conditions here are markedly simpler than the general case [8,27] involving the well-known B_1 and B_2 operators, owing to the simplified rectangular geometry.

3.1 Full elastic potential energy and strain–displacement relations

In this section, we discuss the plate potential energy. We have in mind to simplify it in accordance with the relevant plate inextensibility constraints, as done analogously for the beam. We begin with the bulk description of the potential energy, namely that the potential energy E_{P} has the form:

$$E_{\mathrm{P}} = \frac{1}{2} \int_{-h/2}^{h/2} \int_{\Omega} \hat{\varepsilon}_{ij} \hat{\sigma}_{ij} \mathrm{d}\Omega \mathrm{d}z,$$

where $\hat{\varepsilon}$ and $\hat{\sigma}$ are the 3D strain and stress tensor (resp.) for 3D displacements. With the plate taken as homogeneous and isotropic, we invoke the standard Hookean stress–strain relation

$$\hat{\sigma}_{ij} = \frac{E}{1+\nu} \left(\hat{\varepsilon}_{ij} + \frac{\nu}{1-2\nu} \hat{\varepsilon}_{kk} \delta_{ij} \right), \quad i, j = 1, 2, 3, \tag{3.1}$$

where $E > 0$ is the Young's modulus, $\nu \in (0, 1/2)$ is the Poisson ratio, and the Einstein summation convention is in force (above δ_{ij} is that of Kronecker).

As is customary in thin plate theory, we assume the Kirchhoff–Love hypotheses.

Assumption 3.1 Assume $\hat{\sigma}_{33} = 0$. We also assume that plate filaments that are originally orthogonal to the mid-surface remain so throughout deflection and that said filaments have constant length; this induces $\hat{\varepsilon}_{13} = \hat{\varepsilon}_{23} = 0$.

The above assumption reduces E_{P} to the following expression [9,27], which we will use throughout this treatment:

$$E_{\mathrm{P}} = \frac{1}{2} \frac{E}{1-\nu^2} \int_{\Omega} \int_{-h/2}^{h/2} \left[\hat{\varepsilon}_{11}^2 + \hat{\varepsilon}_{22}^2 + 2\nu \hat{\varepsilon}_{11} \hat{\varepsilon}_{22} + \frac{1-\nu}{2} \hat{\varepsilon}_{12}^2 \right] \mathrm{d}z \mathrm{d}\Omega. \tag{3.2}$$

We now attempt to simplify, utilizing only mid-plane strains ε, curvature κ, and the higher-order tensor μ [35], yielding:

$$\hat{\varepsilon}_{11} = \varepsilon_{11} + z\kappa_{11} + z^2 \mu_{11} \tag{3.3}$$
$$\hat{\varepsilon}_{22} = \varepsilon_{22} + z\kappa_{22} + z^2 \mu_{22} \tag{3.4}$$
$$\hat{\varepsilon}_{12} = \varepsilon_{12} + z\kappa_{12} + z^2 \mu_{12}. \tag{3.5}$$

In all of what follows, we drop the terms scaled by z^2 above, consistent with [33,35].

Assumption 3.2 In the expression for strains $\hat{\varepsilon}_{ij}$ for $i, j = 1, 2$, we assume that each $z^2\mu_{ij}$ for $i, j = 1, 2$ are negligible.

It is necessary, at this point, to invoke strain–displacement relations [35]. Recalling that u, v and w denote the Lagrangian mid-plane displacements, we have

$$\varepsilon_{11} = u_x + \frac{1}{2}\left[u_x^2 + v_x^2 + w_x^2\right] \tag{3.6}$$

$$\varepsilon_{22} = v_y + \frac{1}{2}\left[u_y^2 + v_y^2 + w_y^2\right] \tag{3.7}$$

$$\varepsilon_{12} = u_y + v_x + u_x u_y + v_x v_y + w_x w_y, \tag{3.8}$$

which, as mentioned above, are the mid-plane extensible axial strains and the shear strain (resp.).

3.2 Inextensibility

The "true" inextensibility conditions would correspond to taking $\varepsilon_{ij} = 0$ [35,45]; this translates to no extensional stresses along the mid-plane of the plate. In that case, we have three conditions, taken from (3.6)–(3.8) by equating each to zero:

$$(1 + u_x)^2 + v_x^2 + w_x^2 = 1 \tag{3.9}$$

$$u_y^2 + (1 + v_y)^2 + w_y^2 = 1 \tag{3.10}$$

$$u_y + v_x + u_x u_y + v_x v_y + w_x w_y = 0. \tag{3.11}$$

We refer to these as the *full* plate inextensibility conditions. The first condition is a span-wise inextensibility constraint, the second is chord-wise, with the lattermost corresponding to shear.

The first two inextensibility conditions can be re-written (discarding non-physical square roots) as:

$$u_x = -1 + \sqrt{1 - (v_x^2 + w_x^2)} \tag{3.12}$$

$$v_y = -1 + \sqrt{1 - (u_y^2 + w_y^2)}. \tag{3.13}$$

These can be truncated through Taylor expansions about equilibrium. Namely (up to quartic terms):

$$u_x = -1 + \sqrt{1 - (v_x^2 + w_x^2)} = -\frac{1}{2}v_x^2 - \frac{1}{2}w_x^2 - \frac{1}{4}v_x^2 w_x^2 - \frac{1}{8}v_x^4 - \frac{1}{8}w_x^4 - \cdots \tag{3.14}$$

$$v_y = -1 + \sqrt{1 - (u_y^2 + w_y^2)} = -\frac{1}{2}u_y^2 - \frac{1}{2}w_y^2 - \frac{1}{4}u_y^2 w_y^2 - \frac{1}{8}u_y^4 - \frac{1}{8}w_y^4 - \cdots \tag{3.15}$$

Remark 3.2 It is clear that any order analysis associated with truncating these identities—such as what is done for the beam in Sect. 2.1—will involve the quantities v_x, u_y, w_x, and w_y. Specifically, the above invites comparisons of u_x to both v_x, w_x and v_y to both u_y, w_y.

3.2.1 Effective inextensibility conditions

In working with the potential energy, we will invoke approximated versions of the inextensibility constraints (3.9)–(3.11), analogously with the 1D beam. However, in the case of the plate, there are more complex order comparisons that can be made, and a variety of assumptions can be taken.

In the simplest case, we can assert the following η^2-type scaling assumption:

Assumption 3.3 Assume that $\partial_{x_i} w \sim \eta$ and assume that $[\partial_{x_i} u], [\partial_{x_i} v] \sim \eta^2$, for all $i = 1, 2$.

This is to say that we assume the in-plane gradient is of higher order than the slopes associated with transverse deflection. With this assumption, we can reduce the full inextensibility constraints in (3.9)–(3.11) up to the

order η^2 (thereby dropping terms of the form $[\partial_{x_i} u][\partial_{x_j} v]$, $[\partial_{x_i} u]^2$, $[\partial_{x_i} v]^2$). This produces a particular set of η^2-effective inextensibilty constraints:

$$u_x = -\frac{1}{2} w_x^2 \tag{3.16}$$

$$v_y = -\frac{1}{2} w_y^2 \tag{3.17}$$

$$u_y + v_x = -w_x w_y. \tag{3.18}$$

Remark 3.3 One can directly obtain, from the above equations, the identities:

$$4u_x v_y - 2u_y v_x = [u_y^2 + v_x^2] \quad \text{and} \quad w_{xx} w_{yy} = w_{xy}^2.$$

These are elaborated upon in [33,45]. We note that neither of these "composite" constraints over-constrain the model, since they are derived from (combinations of) prior equations that each eventually correspond (one-to-one) with a Lagrange multiplier variable.

Remark 3.4 The latter equality in the previous remark does provide another interesting identity via the boundary conditions and integration by parts:

$$\int_\Omega w_{xy}^2 \mathrm{d}\mathbf{x} = \int_\Omega w_{xx} w_{yy} \mathrm{d}\mathbf{x} = \int_\Omega w_{xy}^2 \mathrm{d}\mathbf{x} + w_x w_{yy}\Big|_{x=0}^{x=L_x} - w_x w_{xy}\Big|_{y=0}^{y=L_y}.$$

As we have yet to derive the higher-order boundary conditions via a variational procedure, the only condition we can invoke is the essential condition that $w_x = 0$ on $\{x = 0\}$, which finally yields:

$$w_x(L_x) w_{yy}(L_x) = w_x w_{xy}\Big|_{y=0}^{y=L_y}.$$

Remark 3.5 Alternatively, in the choice of truncation above in Assumption 3.3, we could more closely follow the logic of our beam approximation in simplifying (3.9)–(3.11). Namely, we might assume only that $w_x \sim \eta$, from (3.9), and $w_y \sim \eta$ from (3.10). We could expand to see that $u_x \sim \eta^2$ with $v_x \sim \eta$, as well as $v_y \sim \eta^2$ with $u_y \sim \eta$. In this case, we can choose to retain terms in the Taylor expansions (3.14)–(3.15) up to order η^4:

$$u_x = -\frac{1}{2} v_x^2 - \frac{1}{2} w_x^2 - \frac{1}{4} v_x^2 w_x^2 - \frac{1}{8} v_x^4 - \frac{1}{8} w_x^4 \tag{3.19}$$

$$v_y = -\frac{1}{2} u_y^2 - \frac{1}{2} w_y^2 - \frac{1}{4} u_y^2 w_y^2 - \frac{1}{8} u_y^4 - \frac{1}{8} w_y^4 \tag{3.20}$$

$$u_y + v_x = -u_x u_y - v_x v_y - w_x w_y. \tag{3.21}$$

Note that, if we go up to order η^4, the shear inextensibility condition will be retained in full—no truncation is called for. In the final class of models in Sect. 3.6, we will discuss the use of higher-order approximations of the inextensibility constraints, analogous to the model in Sect. 2.4

3.2.2 Curvature expressions

To obtain expressions for κ_{ij} of the $(x, y) \in \Omega$ middle surface of the plate, we follow the Kirchhoff–Love hypotheses ($\hat{\varepsilon}_{13} = \hat{\varepsilon}_{23} = \hat{\varepsilon}_{33} = 0$), and tracking the work in [35], we obtain:

$$\kappa_{11} = (1 + u_x)\theta_x + v_x \psi_x + w_x \chi_x \tag{3.22}$$

$$\kappa_{22} = u_y \theta_y + (1 + v_y)\psi_y + w_y \chi_y \tag{3.23}$$

$$\kappa_{12} = (1 + u_x)\theta_y + (1 + v_y)\psi_x + u_y \theta_x + v_x \psi_y + w_x \chi_y + w_y \chi_x, \tag{3.24}$$

with θ, ψ, χ given by:

$$\theta = -(1 + v_y) w_x + v_x w_y; \quad \psi = -(1 + u_x) w_y + u_y w_x; \quad \chi = u_x + v_y + u_x v_y - u_y v_x. \tag{3.25}$$

Remark 3.6 As noted in [35], the above expressions are themselves geometric approximations, assuming that strains are small compared to 1. In [35], the relations $\hat{u} = u + z\theta$, $\hat{v} = v + z\psi$, and $\hat{w} = w + z\chi$ are plugged in to the strain relations for $\hat{\varepsilon}_{i3}$, then themselves simplified by geometric order considerations.

We can thus obtain:

$$\begin{aligned}
\kappa_{11} &= (1+u_x)[v_{xx}w_y + v_x w_{xy} - v_{xy}w_x - (1+v_y)w_{xx}] \\
&\quad + v_x[u_{xy}w_x + u_y w_{xx} - u_{xx}w_y - (1+u_x)w_{xy}] \\
&\quad + w_x[u_{xx} + v_{xy} + u_{xx}v_y + u_x v_{xy} - u_{xy}v_x - u_y v_{xx}] \qquad (3.26)\\
\kappa_{22} &= u_y[v_{xy}w_y + v_x w_{yy} - v_{yy}w_x - (1+v_y)w_{xy}] \\
&\quad + (1+v_y)[u_{yy}w_x + u_y w_{xy} - u_{xy}w_y - (1+u_x)w_{yy}] \\
&\quad + w_y[(1+v_y)u_{xy} + (1+u_x)v_{yy} - u_{yy}v_x - u_y v_{xy}] \qquad (3.27)\\
\kappa_{12} &= (1+u_x)[v_{xy}w_y + v_x w_{yy} - v_{yy}w_x - (1+v_y)w_{xy}] \\
&\quad + (1+v_y)[u_{xy}w_x + u_y w_{xx} - u_{xx}w_y - (1+u_x)w_{xy}] \\
&\quad + u_y[v_{xx}w_y + v_x w_{xy} - v_{xy}w_x - (1+v_y)w_{xx}] \\
&\quad + v_x[u_{yy}w_x + u_y w_{xy} - u_{xy}w_y - (1+u_x)w_{yy}] \\
&\quad + w_x[(1+v_y)u_{xy} + (1+u_x)v_{yy} - u_{yy}v_x - u_y v_{xy}] \\
&\quad + w_y[u_{xx} + v_{xy} + u_{xx}v_y + u_x v_{xy} - u_{xy}v_x - u_y v_{xx}]. \qquad (3.28)
\end{aligned}$$

In what follows, we will use these expressions along with the inextensibility conditions in order to produce the potential energy from which we will obtain the equations of motion. Let us first truncate these relations by dropping principal terms (second spatial derivatives) with coefficients of order higher than η^2 (according to the scaling in Assumption 3.3). After various cancellations, this yields :

$$\kappa_{11} = w_y v_{xx} + w_x u_{xx} - (1 + u_x + v_y)w_{xx} \qquad (3.29)$$

$$\kappa_{22} = u_{yy}w_x + w_y u_{yy} - (1 + u_x + v_y)w_{yy} \qquad (3.30)$$

$$\kappa_{12} = 2[v_{xy}w_y + u_{xy}w_x - (1 + u_x + v_y)w_{xy}]. \qquad (3.31)$$

Remark 3.7 Note that, in higher-order models, we may wish to retain more terms in the above.

3.3 Plate model I: three η^2-inextensibility conditions

We begin with the potential energy expression from Sect. 3.1. We invoke the three full inextensibility conditions in (3.9)–(3.11) to eliminate $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ from the energetic expression. Also, as previously mentioned, we drop all $z^2\mu_{ij}$ terms in the expressions for ε_{ij}. This yields:

$$E_{\mathrm{P}} = \frac{1}{2}\left[\frac{1}{12}\frac{Eh^2}{(1-\nu^2)}\right]\int_\Omega \left[\kappa_{11}^2 + \kappa_{22}^2 + 2\nu\kappa_{11}\kappa_{22} + \frac{1-\nu}{2}\kappa_{12}^2\right]\mathrm{d}\Omega. \qquad (3.32)$$

We then invoke the curvature expressions in (3.29)–(3.31) and input them in (3.32).

After truncating terms, we will obtain a potential energy (shown in the next section) for which we can enforce three *effective inextensibility relations* in (3.16)–(3.18). Then, utilizing Hamilton's principle as in Sect. 2, we will obtain a straightforward PDE model with clear equations of motion. On the other hand, enforcing the three effective inextensibilty constraints results in three Lagrange multiplier variables that cannot be simultaneously eliminated in the full description of the system.

3.3.1 Simplified energies

We employ the effective inextensibility constraints (3.16)–(3.18) in the curvature expressions given in (3.29). In particular, we can differentiate (3.16)–(3.18) variously, simplify, and rewrite κ_{ij} solely in w

$$\kappa_{11} = -w_{xx}\left[1 + \frac{1}{2}w_x^2 + \frac{1}{2}w_y^2\right]$$

$$\kappa_{22} = -w_{yy}\left[1 + \frac{1}{2}w_x^2 + \frac{1}{2}w_y^2\right]$$
$$\kappa_{12} = -2w_{xy}\left[1 + \frac{1}{2}w_x^2 + \frac{1}{2}w_y^2\right].$$

We then form the appropriate products of κ_{ij} as they appear in (3.32). In line with the η^2 analysis of the beam, we retain in E_{P} only terms with coefficients up to and including η^2. (And thus we drop expressions of the form $[\partial_{x_i} w]^2[\partial_{x_j} w]^2\partial^2_{x_k} w$.) The result is the potential energy we will employ for the two principal models in this treatment:

$$E_{\mathrm{P}} = \frac{D}{2}\int_0^{L_y}\int_0^{L_x}\left[1 + w_x^2 + w_y^2\right]\left[w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx}w_{yy} + 2(1-\nu)w_{xy}^2\right]\mathrm{d}x\mathrm{d}y, \tag{3.33}$$

where we have now denoted $D = \dfrac{1}{12}\dfrac{Eh^2}{(1-\nu^2)}$. We utilize the standard expression for the plate's kinetic energy (again with normalized mass density):

$$E_{\mathrm{K}} = \frac{1}{2}\int_0^{L_y}\int_0^{L_x}\left[u_t^2 + v_t^2 + w_t^2\right]\mathrm{d}x\mathrm{d}y.$$

3.3.2 Equations of motion

We introduce Lagrange multipliers $\lambda_i(\mathbf{x}, t)$, $i = 1, 2, 3$ acting to enforce the effective inextensibility constraints. Let λ_1 be associated with the axial (effective) inextensibility condition (3.16), λ_2 to the chord-wise condition (3.17), and λ_3 to the shear constraint (3.18). We invoke Hamilton's principle for the Lagrangian, written in the λ_i, E_{P}, and E_{K}, with virtual displacements δu, δv, and δw. The calculations are involved but analogous to those in Sect. 2 for the 1D dynamics. Essential boundary conditions are enforced for u, v, and w (and their virtual changes) on Γ_W.

In this presentation, since the variables $\lambda_i(\mathbf{x}, t)$ serve to enforce a constraint in the equations of motion, *we will retain the equations* (3.16)–(3.18) as part of the system:

$$u_x + \frac{1}{2}w_x^2 = 0; \quad v_y + \frac{1}{2}w_y^2 = 0; \quad u_y + v_x + w_x w_y = 0. \tag{3.34}$$

For the dynamics, we obtain:

$$u_{tt} + \partial_x(\lambda_1) + \partial_y(\lambda_3) = 0 \tag{3.35}$$
$$v_{tt} + \partial_y(\lambda_2) + \partial_x(\lambda_3) = 0 \tag{3.36}$$
$$w_{tt} + D\Delta[(1 + |\nabla w|^2)\Delta w] - D\nabla\cdot[|\Delta w|^2\nabla w] + \partial_x(\lambda_1 w_x) + \partial_y\left(\lambda_2 w_y\right) + \partial_x\left(\lambda_3 w_y\right) + \partial_y(\lambda_3 w_x) = 0. \tag{3.37}$$

The above system would be supplemented with appropriate initial displacement $w_0 = w(x, y; 0)$ and velocity $w_1(x, y) = w_t(x, y; 0)$, from which the initial conditions for u and v can be inferred through the relationships (3.16)–(3.17).

Taken as a system, the constraint equations in (3.34) and (3.35)–(3.37) constitute six equations in six principle unknowns. As such, boundary conditions should be available and complement these equations for each unknown, u, v, w and λ_i for $i = 1, 2, 3$.

Remark 3.8 Letting $\mathbf{u} = \langle u, v\rangle$ and $\Lambda = \begin{bmatrix}\lambda_1 & \lambda_3\\ \lambda_3 & \lambda_2\end{bmatrix}$, we have a nice vectorial description of the system that makes for an apt comparison against other nonlinear plate equations (e.g., the full von Karman equations of motion [9,26,27]):

$$\mathbf{u}_{tt} + \mathrm{div}\Lambda = 0 \tag{3.38}$$
$$w_{tt} + D\Big[\Delta\big[(1 + |\nabla w|^2)\Delta w\big] - \nabla\cdot\big(|\Delta w|^2\nabla w\big)\Big] + \mathrm{div}\Big(\Lambda\nabla w\Big) = 0, \tag{3.39}$$

where the action of Λ on the gradient is that of matrix multiplication in this presentation. Although the Λ terms here serve to capture inertial effects due to the enforcement of three constraints, the full von Karman equations have a comparable formal structure with respect to these terms. They differ, however, in the presence of the other nonlinear terms.

Formally comparing these equations against other standard plate dynamics, we, of course, note that the linear part of the plate corresponds to that of a Kirchhoff–Love or Euler–Bernoulli (linear) plate. The quasilinear terms in w in (3.38) seem novel, but are clear generalizations of the nonlinear stiffness terms appearing in the beam equation (2.11).

3.3.3 Boundary conditions

Here, we provide the boundary conditions for w and for the λ_i. From these, the boundary conditions for u and v can be inferred. On the clamped edge Γ_W, we again have the essential boundary condition

$$w = 0; \quad w_x = 0 \quad \text{on } \Gamma_W.$$

The minimization of the Lagrangian and the arbitrariness of virtual displacements yield the *natural* boundary conditions from the potential energy. For the second-order conditions in w, we obtain those of the standard linear free plate:

$$\begin{aligned} w_{xx} + \nu w_{yy} &= 0 \quad \text{on } \Gamma_E \\ w_{yy} + \nu w_{xx} &= 0 \quad \text{on } \Gamma_S \\ w_{yy} + \nu w_{xx} &= 0 \quad \text{on } \Gamma_N. \end{aligned}$$

The potential energy produces nonlinear forces (cubic-type)—and thus boundary conditions—along the free edges:

$$\begin{aligned} &(1-\nu)\left[(1+\nu)w_x w_{yy}^2 - 2w_x w_{xy}^2 - 4w_y w_{yy} w_{xy}\right] - \left[1 + w_x^2 + w_y^2\right]\left[w_{xxx} + (2-\nu)w_{yyx}\right] = 0 \quad \text{on } \Gamma_E \\ &(1-\nu)\left[(1+\nu)w_y w_{xx}^2 - 2w_y w_{xy}^2 - 4w_x w_{xx} w_{xy}\right] - \left[1 + w_x^2 + w_y^2\right]\left[w_{yyy} + (2-\nu)w_{xxy}\right] = 0 \quad \text{on } \Gamma_S \\ &(1-\nu)\left[(1+\nu)w_y w_{xx}^2 - 2w_y w_{xy}^2 - 4w_x w_{xx} w_{xy}\right] - \left[1 + w_x^2 + w_y^2\right]\left[w_{yyy} + (2-\nu)w_{xxy}\right] = 0 \quad \text{on } \Gamma_N. \end{aligned}$$

Remark 3.9 The above conditions are **clearly not** the linear boundary conditions associated with the free plate, as presented in Remark 3.1.

The Lagrange multipliers λ_i, as variables, also have boundary conditions. These are readily obtained from Hamilton's principle along the free edges:

$$\begin{aligned} \lambda_1(x, y) = 0 \text{ and } \lambda_3(x, y) = 0 \quad &\text{on } \Gamma_E \\ \lambda_2(x, y) = 0 \text{ and } \lambda_3(x, y) = 0 \quad &\text{on } \Gamma_S \\ \lambda_2(x, y) = 0 \text{ and } \lambda_3(x, y) = 0 \quad &\text{on } \Gamma_N. \end{aligned}$$

Finally, we can express the boundary conditions for the Lagrange multipliers on the clamped edge Γ_W using (3.35)–(3.36) and the equations above:

$$\begin{aligned} \lambda_1(0, y) &= \int_0^{L_x} \left[u_{tt} + \partial_y(\lambda_3)\right] \mathrm{d}x; \quad \lambda_2(0, y) = \int_y^{L_y} \left[v_{tt} + \partial_x(\lambda_3)\right] \mathrm{d}y \Big|_{x=0}; \\ \lambda_3(0, y) &= \int_0^{L_x} \left[v_{tt} + \partial_y(\lambda_2)\right] \mathrm{d}x, \end{aligned}$$

as well as

$$\begin{aligned} &\lambda_1(x, y) = \int_x^{L_x} \left[u_{tt} + \partial_y(\lambda_3)\right] \mathrm{d}x \Big|_{y=0} \quad \text{on } \Gamma_S, \qquad \lambda_1(x, y) = \int_x^{L_x} \left[u_{tt} + \partial_y(\lambda_3)\right] \mathrm{d}x \Big|_{y=L_y} \quad \text{on } \Gamma_N, \\ &\lambda_2(x, y) = \int_y^{L_y} \left[v_{tt} + \partial_x(\lambda_3)\right] \mathrm{d}y \Big|_{x=L_x} \quad \text{on } \Gamma_E. \end{aligned}$$

Remark 3.10 Note that, above, we have suppressed the dependence of the Lagrange variables on time.

3.3.4 Reduction of system (3.35)–(3.37)

To be consistent with the beam system analysis, one may attempt to eliminate the λ_i, as well as the in-plane variables u, v in the system. This would yield a dynamic equation in the principal elastic displacement w only (as may be done for beam dynamics, and also as is possible in the case of von Karman's equations via the *Airy Stress Function* [9,27]). Yet, for the plate dynamics above, there seems to be no clear way to accomplish this. We opt, here, to eliminate λ_1 and λ_2 and to use (3.16) and (3.17) to write u, v in terms of w. *This critically exploits the simplified nature of the span- and chord-wise effective inextensibility constraints at the* η^2-order. In addition, we can see that since the third (shear) constraint (3.18) does not permit us to explicitly solve for a displacement quantity, we will retain both w and λ_3 in our reduced system.

To that end, we integrate (3.35) from x to L_x and utilize the boundary condition $\lambda_1(L_x, y) = 0$. This yields:

$$\lambda_1(x, y) = \int_x^{L_x} \left[u_{tt} + \partial_y(\lambda_3)\right] \mathrm{d}x. \tag{3.40}$$

Similarly, we integrate (3.36) from y to L_y and use the condition $\lambda_2(x, L_y) = 0$. This gives:

$$\lambda_2(x, y) = \int_y^{L_y} [v_{tt} + \partial_x(\lambda_3)] \mathrm{d}y. \tag{3.41}$$

Substituting (3.40) and (3.41) into (3.37), we obtain:

$$\begin{aligned} w_{tt} - D\Big[\Delta[(1+|\nabla w|^2)\Delta w] - \nabla\cdot(|\Delta w|^2\nabla w)\Big] + \partial_x\left(w_x\int_x^{L_x} u_{tt}\right) + \partial_y\left(w_y\int_y^{L_y} v_{tt}\right) \\ + \partial_x\left(w_x\int_x^{L_x}\partial_y(\lambda_3)\right) + \partial_y\left(w_y\int_y^{L_y}\partial_x(\lambda_3)\right) + \partial_x(\lambda_3 w_y) + \partial_y(\lambda_3 w_x) = 0. \end{aligned} \tag{3.42}$$

In-plane inertial expressions, u_{tt} and v_{tt}, can be obtained, as in the case of the beam, by solving in (3.16), (3.17), and (3.18) and formally differentiating in time. Note that, using the essential boundary conditions at $x = 0$ for each of u, v, w, we have:

$$u(x, y) = -\frac{1}{2}\int_0^x w_x(\xi_1, y)^2 \mathrm{d}\xi_1 \tag{3.43}$$

$$v(x, y) - v(x, 0) = -\frac{1}{2}\int_0^y w_y(x, \xi_2)^2 \mathrm{d}\xi_2 \tag{3.44}$$

$$v(x, y) = -\int_0^x w_x(\xi_1, y) w_y(\xi_1, y)\mathrm{d}\xi_1 - \int_0^x u_y(\xi_1, y)\mathrm{d}\xi_1, \tag{3.45}$$

where we have suppressed the dependence on t above. From which we have an expression for $v(x, y)$, and thence an expression for $u(x, y)$, where both depend only on the transverse variable w:

$$v(x, y) = \int_0^x \int_0^{\xi_1} w_x(\zeta, y) w_{xy}(\zeta, y)\mathrm{d}\zeta\,\mathrm{d}\xi_1 - \int_0^x w_x(\xi_1, y) w_y(\xi_1, y)\mathrm{d}\xi_1. \tag{3.46}$$

From these, we obtain the inertial expressions

$$u_{tt} = -\int_0^x [w_{xt}^2 + w_x w_{xtt}]\mathrm{d}\xi_1 \tag{3.47}$$

$$v_{tt} = -\int_0^y [w_{yt}^2 + w_y w_{ytt}]\mathrm{d}\xi_2 + v_{tt}(y = 0). \tag{3.48}$$

The above formulae in (3.42)–(3.48) showcase of the principal strengths of the η^2-order effective inextensibility constraints: namely, we obtain straightforward equations of motion in two unknowns, with a clear connection to the η^2-order *beam dynamics* in Sect. 2.3. On the other hand, owing to the structure of the third effective inextensibility constraint, (3.18), it does not seem that systematic integrations and/or differentiations—coupled with the given boundary conditions—will allow a clean elimination of the λ_3 variable. Thus, any analytical

treatment of this system would seem to require both the w and λ_3 variables directly. This is to say, we would require retaining the constraint equation associated with λ_3:

$$u_y + v_x = \ -w_x w_y, \tag{3.49}$$

with u and v as above, written explicitly in terms of w. To this, initial and boundary conditions are affixed to w, as described above, and to λ_3, the boundary conditions described above.

3.4 Model II: partial use of effective shear constraint

In this model, we retain the treatment of the potential energy in Sect. 3.3.1. To wit: we retain all three full inextensibility constraints (3.9)–(3.11) in eliminating the in-plane strains ε_{ij} from the potential energy in (3.2). After this, we truncate all three inextensibility constraints to quadratic order (as in (3.16)–(3.18)). Finally, we truncate the expanded potential energy as before to arrive at the potential energy expression in (3.33). On the other hand, in developing the equations of motion, we elect (with foresight coming from those issues associated with λ_3 above) to *enforce only the first two η^2-effective inextensibility conditions*, (3.16) and (3.17) through Lagrange multipliers λ_1, λ_2. Thus, in this model, shear inextensibility is only implicitly enforced from the point of view of the choice of the potential energy. We utilize only two Lagrange multipliers in the derivation of the equations of motion.

In this presentation, the constraint variables λ_1, λ_2 serve to enforce the span- and chord-wise quadratic effective inextensibility constraints. Accordingly, *we retain these as part of the system*:

$$u_x + \frac{1}{2}w_x^2 = 0; \quad v_y + \frac{1}{2}w_y^2 = 0. \tag{3.50}$$

For the dynamics, Hamilton's principle yields identical nonlinear terms in w, owing to the use of E_{P} as in Sect. 3.3.2. Moreover, we do not retain any reference to the shear constraint in the equations, as there is no λ_3 present. The unforced equations of motion are then:

$$u_{tt} + \partial_x(\lambda_1) = 0 \tag{3.51}$$

$$v_{tt} + \partial_y(\lambda_2) = 0 \tag{3.52}$$

$$w_{tt} + D\left[\Delta[(1+|\nabla w|^2)\Delta w] - \nabla\cdot(|\Delta w|^2\nabla w)\right] + \partial_x(\lambda_1 w_x) + \partial_y\left(\lambda_2 w_y\right) = 0. \tag{3.53}$$

As before, the equations would be supplemented with appropriate initial displacement $w_0 = w(x, y; 0)$ and velocity $w_1(x, y) = w_t(x, y; 0)$, from which the initial conditions for u and v can be inferred.

3.4.1 Boundary conditions

The boundary conditions for w (which then yield conditions for u and v through (3.50)) are identical to Sect. 3.3.3. The boundary conditions of the Lagrange multipliers λ_1 in this case are:

$$\lambda_1(x, y) = \int_0^{L_y} u_{tt}\mathrm{d}y \quad \text{on } \Gamma_W, \qquad \lambda_1(x, y) = 0 \quad \text{on } \Gamma_E,$$

$$\lambda_1(x, y) = \int_x^{L_x} u_{tt}\mathrm{d}x\Big|_{y=0} \quad \text{on } \Gamma_S, \qquad \lambda_1(x, y) = \int_x^{L_x} u_{tt}\mathrm{d}x\Big|_{y=L_y} \quad \text{on } \Gamma_N.$$

And for λ_2:

$$\lambda_2(x, y) = \int_y^{L_y} v_{tt}\mathrm{d}y\Big|_{x=0} \quad \text{on } \Gamma_W, \qquad \lambda_2(x, y) = \int_y^{L_y} v_{tt}\mathrm{d}y\Big|_{x=L_x} \quad \text{on } \Gamma_E,$$

$$\lambda_2(x, y) = 0 \quad \text{on } \Gamma_S, \qquad \lambda_2(x, y) = 0 \quad \text{on } \Gamma_N.$$

3.4.2 Reduction of system (3.51)–(3.53)

One of the primary benefits for this system, discussed further in Sect. 4, is that the Lagrange multiplier variables can be fully eliminated (as with the inextensible beam). Indeed, one obtains from (3.51) and (3.52) that

$$\lambda_1(L_x, y) - \lambda_1(x, y) = -\int_x^{L_x} u_{tt} \mathrm{d}\xi; \quad \lambda_2(x, y) - \lambda_2(x, 0) = -\int_0^y v_{tt} \mathrm{d}\zeta.$$

At this point, we can invoke the boundary conditions for λ_i, as seen above on the appropriate edge of the plate, to conclude

$$\lambda_1(x, y) = \int_x^{L_x} u_{tt} \mathrm{d}\xi; \quad \lambda_2(x, y) = -\int_0^y v_{tt} \mathrm{d}\zeta. \tag{3.54}$$

These quantities may be substituted directly into the equations of motion for w, which results in the following *closed* system—with no reference to the Lagrange variables:

$$w_{tt} + D\Delta[(1 + |\nabla w|^2)\Delta w] - D\nabla \cdot [|\Delta w|^2 \nabla w] + \partial_x\Big(w_x \int_x^{L_x} u_{tt} \mathrm{d}\xi\Big) - \partial_y\Big(w_y \int_0^y v_{tt} \mathrm{d}\zeta\Big) = 0 \tag{3.55}$$

$$u_x + \frac{1}{2} w_x^2 = 0; \quad v_y + \frac{1}{2} w_y^2 = 0. \tag{3.56}$$

A complete description, then, would again provide the relevant initial and boundary conditions from the principal variable here, w.

3.5 Model III: complete omission of the shear constraint

In natural succession from the previous models, one may inquire:

What happens if we omit the third (shear) constraint in the derivation?

This is a reasonable subsequent step, as we have in the previous model only partially made use of the shear constraint. For the model in this section, we refrain entirely from making mention to a shear constraint. With only span- and chord-wise inextensibility enforced, we obtain the equations of motion. In particular, this showcases how convoluted the equations of motion become and demonstrates the usefulness of the shear constraint.

3.5.1 Addressing the potential energy

The two constraints approach forbids elongation in the x and y axes but *permits* shear strain. This translates into $\varepsilon_{11} = \varepsilon_{22} = 0$, but we will *not* take $\varepsilon_{12} = 0$. Immediately it is clear we will have more terms in the equations of motion. As before, after invoking the full inextensibility conditions $\varepsilon_{11} = 0$ and $\varepsilon_{22} = 0$, we will compute the associated potential energy and then truncate to a particular order.

Applying only two inextensibility constraints to eliminate $\varepsilon_{11}, \varepsilon_{22}$ into the bulk strain expressions (3.3)–(3.5) yields:

$$\begin{aligned}
\hat{\varepsilon}_{11} &= -z\left[w_{xx}\left(1 + u_x + v_y\right) - w_x u_{xx} - w_y v_{xx}\right] \\
\hat{\varepsilon}_{22} &= -z\left[w_{yy}\left(1 + u_x + v_y\right) - w_x u_{yy} - w_y v_{yy}\right] \\
\hat{\varepsilon}_{12} &= u_y + v_x + u_x u_y + v_x v_y + w_x w_y - 2z\left[w_{xy}\left(1 + u_x + v_y\right) - w_x u_{xy} - w_y v_{xy}\right].
\end{aligned}$$

We then invoke the η^2-order hypothesis to obtain the curvature expressions (3.29)–(3.31). We must also truncate

$$\varepsilon_{12} = u_y + v_x + u_x u_y + v_x v_y + w_x w_y \approx u_y + v_x + w_x w_y,$$

under the same η^2-hypothesis. And, as before, we invoke the (two) η^2-effective inextensibility constraints

$$u_x = -\frac{1}{2} w_x^2, \quad v_y = -\frac{1}{2} w_y^2,$$

from which we can differentiate to solve for u_{xx} and v_{yy}.

Remark 3.11 Without creating nonlocal conditions—and invoking boundary conditions, as yet undetermined–we can no longer obtain expressions for v_x, u_y as before. We note that this is a key point of distinction when we have imposed no notion of shear inextensibility.

Upon simplifying the terms as described above, we obtain:

$$\begin{aligned}
\hat{\varepsilon}_{11} &= z\left[-w_{xx}\left(1+\frac{1}{2}w_x^2-\frac{1}{2}w_y^2\right)+w_y v_{xx}\right]\\
\hat{\varepsilon}_{22} &= z\left[-w_{yy}\left(1-\frac{1}{2}w_x^2+\frac{1}{2}w_y^2\right)+w_x u_{yy}\right]\\
\hat{\varepsilon}_{12} &= u_y+v_x+w_x w_y-2zw_{xy}\left[1+\frac{1}{2}w_x^2+\frac{1}{2}w_y^2\right].
\end{aligned}$$

Then, integrating in z and simplifying, with coefficients:

$$\begin{aligned}
\int_{-h/2}^{h/2}[\hat{\varepsilon}_{11}^2+\hat{\varepsilon}_{22}^2]\mathrm{d}z &= \frac{h^3}{12}\Big[w_y^2v_{xx}^2+w_x^2u_{yy}^2-2w_yv_{xx}w_{xx}-2w_xu_{yy}w_{yy}\\
&\quad -w_y(w_x^2-w_y^2)v_{xx}w_{xx}+w_x(w_x^2-w_y^2)u_{yy}w_{yy}\\
&\quad +w_{xx}^2\Big(1+(w_x^2-w_y^2)+\frac{1}{4}(w_x^2-w_y^2)^2\Big)+w_{yy}^2\Big(1-(w_x^2-w_y^2)-\frac{1}{4}(w_x^2-w_y^2)^2\Big)\Big]\\
2\nu\int_{-h/2}^{h/2}[\hat{\varepsilon}_{11}\hat{\varepsilon}_{22}]\mathrm{d}z &= \frac{\nu h^3}{6}\Big[v_{xx}u_{yy}w_xw_y+w_{xx}w_{yy}w_xw_{xx}u_{yy}-w_yv_{xx}w_{yy}\\
&\quad -\frac{1}{4}w_{xx}w_{yy}(w_x^2-w_y^2)^2+\frac{1}{2}(w_x^2-w_y^2)(w_yv_{xx}w_{yy}-w_xw_{xx}u_{yy})\Big]\\
\frac{1-\nu}{2}\int_{-h/2}^{h/2}[\hat{\varepsilon}_{12}^2]\mathrm{d}z &= \frac{h(1-\nu)}{2}(u_y+v_x+w_xw_y)^2+\frac{h^3(1-\nu)}{6}\Big(1+\frac{1}{2}w_x^2+\frac{1}{2}w_y^2\Big)^2w_{xy}^2.
\end{aligned}$$

Recalling the full potential energy expression (3.2), we input the above expressions for the strains. Since we are operating at the η^2-level here for coefficients, we discard *any terms* with coefficients scaled by η^3 or higher. Note that, in the (new) situation where first derivative terms appear independent of second derivative terms, we choose to retain. This yields

$$\begin{aligned}
E_{\mathrm{P}} &= \frac{1}{2}\frac{E}{1-\nu^2}\int_\Omega \frac{h^3}{12}\Big[w_y^2v_{xx}^2+w_x^2u_{yy}^2-2w_yv_{xx}w_{xx}-2w_xu_{yy}w_{yy}\\
&\quad +w_{xx}^2\Big(1+w_x^2-w_y^2\Big)+w_{yy}^2\Big(1-w_x^2+w_y^2\Big)\Big]\\
&\quad +\frac{\nu h^3}{6}\Big[v_{xx}u_{yy}w_xw_y+w_{xx}w_{yy}w_xw_{xx}u_{yy}-w_yv_{xx}w_{yy}\Big]+\frac{h^3(1-\nu)}{6}\Big(1+w_x^2+w_y^2\Big)w_{xy}^2\\
&\quad +\frac{h(1-\nu)}{2}(u_y+v_x+w_xw_y)^2\,d\Omega.
\end{aligned}$$

Remark 3.12 The terms in final line above represent an interesting contribution, both in terms of their "h" scaling, i.e., thickness, and being detached from principal second derivative terms.

Remark 3.13 For reference, the terms we discarded above include:

$$\begin{aligned}
&\frac{h^3}{12}\Big[w_x(w_x^2-w_y^2)u_{yy}w_{yy}-w_y(w_x^2-w_y^2)v_{xx}w_{xx}+\frac{w_{xx}^2}{4}(w_x^2-w_y^2)^2-\frac{w_{yy}^2}{4}(w_x^2-w_y^2)^2\Big]\\
&\quad -\frac{\nu h^3}{24}w_{xx}w_{yy}(w_x^2-w_y^2)^2+\frac{\nu h^3}{12}(w_x^2-w_y^2)(w_yv_{xx}w_{yy}-w_xw_{xx}u_{yy})\\
&\quad +\frac{h^3(1-\nu)}{6}\Big(\frac{1}{2}w_x^2w_y^2+\frac{1}{4}w_x^4+\frac{1}{4}w_y^4\Big)w_{xy}^2.
\end{aligned}$$

Now, invoking the definition of the constant D, the potential energy can then be written:

$$E_{\mathrm{P}} = \frac{6D}{h}\int_0^{L_y}\int_0^{L_x}\Big\{\frac{h^2}{12}\big[w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx}w_{yy} + 2(1-\nu)w_{xy}^2\left(1 + w_x^2 + w_y^2\right) + (w_x^2 - w_y^2)(w_{xx}^2 - w_{yy}^2)$$
$$- 2w_y w_{xx}v_{xx} - 2w_x w_{yy}u_{yy} - 2\nu\left(w_y w_{yy}v_{xx} + w_x w_{xx}u_{yy}\right)\big]$$
$$+ \frac{1-\nu}{2}\big[u_y + v_x + w_x w_y\big]^2\Big\}\mathrm{d}x\mathrm{d}y. \tag{3.57}$$

3.5.2 Equations of motion

As before, λ_1 is used to enforce the span-wise effective constraint (3.16) and λ_2 the chord-wise (3.17). Utilizing the arbitrariness of the relevant virtual changes, we can gather the equations of motion. We recover both effective constraints (3.16) and (3.17) via the associated Lagrange multipliers λ_1 and λ_2 in Hamilton's principle. Following the procedure of the previous sections, the equations of motion for the in-plane displacements are:

$$u_{tt} + \partial_x(\lambda_1) - D\Big[w_{xyy}w_{yy} + 2w_{xy}w_{yyy} + w_x\partial_y^4 w + \nu w_{xyy}w_{xx} + 2\nu w_{xy}w_{xxy} + \nu w_x w_{xxyy}\Big]$$
$$- \frac{12D}{h^2}(1-\nu)\big[u_{yy} + w_x w_{yy}\big] = 0$$
$$v_{tt} + \partial_y(\lambda_2) - 2D\big[w_{yxx}w_{xx} + 2w_{yx}w_{xxx} + w_y\partial_x^4 w + \nu w_{yxx}w_{yy} + 2\nu w_{yx}w_{yyx} + \nu w_y w_{yyxx}\big]$$
$$- \frac{12D}{h^2}(1-\nu)\big[v_{xx} + w_{xx}w_y\big] = 0.$$

We rewrite these and include the equation for w:

$$u_{tt} + \partial_x(\lambda_1) - \frac{12D}{h^2}(1-\nu)\big[u_{yy} + w_x w_{yy}\big] - 2D\partial_y^2\big[w_x(w_{yy} + \nu w_{xx})\big] = 0 \tag{3.58}$$

$$v_{tt} + \partial_y(\lambda_2) - \frac{12D}{h^2}(1-\nu)\big[v_{xx} + w_{xx}w_y\big] - 2D\partial_x^2\big[w_y(w_{xx} + \nu w_{yy})\big] = 0 \tag{3.59}$$

$$w_{tt} + \partial_x(\lambda_1 w_x) + \partial_y(\lambda_2 w_y) - D\Big[\partial_x^4 w(1 + w_x^2 - w_y^2) + \partial_y^4 w(1 - w_x^2 + w_y^2) + 2w_{xxyy}(1 + w_x^2 + w_y^2)$$
$$- \nu w_{xxyy}(w_x^2 + w_y^2) - w_x\partial_y^4 u - w_y\partial_x^4 v + 4w_x w_{xx}w_{xxx} + 4w_y w_{yy}w_{yyy} - w_x w_{yy}w_{xyy}$$
$$- w_y w_{xx}w_{yxx} + (4-2\nu)w_x w_{xy}w_{xxy} + (4-2\nu)w_y w_{xy}w_{yyx} - 4w_x w_{xy}w_{yyy} - 4w_y w_{yx}w_{xxx}$$
$$- 2w_{xy}v_{xxx} - 2w_{xy}u_{yyy} + 4(1-\nu)w_x w_{xx}w_{xyy} + 4(1-\nu)w_y w_{yy}w_{xxy} + w_{xx}^3 + w_{yy}^3 + w_{xx}w_{yy}^2$$
$$+ w_{yy}w_{xx}^2 - (1+3\nu)w_{xx}w_{xy}^2 - (1+3\nu)w_{yy}w_{xy}^2\Big]$$
$$+ \frac{6D}{h^2}(1-\nu)\Big[w_{xx}w_y^2 + 2w_x w_y w_{xy} + 2u_y w_{xy} + v_{xx}w_y + 2v_x w_{xy} + w_x^2 w_{yy} + u_{yy}w_x\Big] = 0.$$

3.5.3 Boundary conditions

On the clamped edge Γ_W, we have:

$$w = 0; \quad w_x = 0; \quad u = 0; \quad v = 0.$$

For the second-order conditions, we have:

On Γ_E:

$$w_{xx} + \nu w_{yy} = 0, \quad w_{xx}(1 + w_x^2 - w_y^2) - w_y v_{xx} + \nu w_{yy} - \nu w_x u_{yy} = 0.$$

On Γ_S and Γ_N:

$$w_{yy} + \nu w_{xx} = 0, \quad w_{yy}(1 - w_x^2 + w_y^2) - w_x u_{yy} + \nu w_{xx} - \nu w_y v_{xx} = 0.$$

For the third-order conditions, we have:

On Γ_E:

$$\frac{h^2}{6} w_y \left[w_{xxx} + \nu w_{yyx} \right] + (1-\nu) \left[v_x + w_x w_y + u_y \right] = 0$$

$$\frac{h^2}{6} \Big\{ - w_x w_{xx}^2 - w_x w_{yy}^2 - w_{yy} u_{yy} - (2-\nu) w_x w_{xy}^2 - w_{xxx}(1 + w_x^2 - w_y^2) + 2 w_y w_{xx} w_{yx}$$
$$+ w_{yx} v_{xx} + w_y v_{xxx} - \nu w_{yyx} - \nu w_x^2 w_{xyy} - 2(1-\nu) \Big[w_{xyy}(1 + w_x^2 + w_y^2) + 2 w_y w_{xy} w_{yy} \Big] \Big\}$$
$$+ (1-\nu) \Big[w_x w_y^2 + u_y w_y + v_x w_y \Big] = 0.$$

On Γ_S and Γ_N:

$$\frac{h^2}{6} w_x \left[w_{yyy} + \nu w_{xxy} \right] + (1-\nu) \left[u_y + w_x w_y + v_x \right] = 0$$

$$\frac{h^2}{6} \Big\{ - w_y w_{yy}^2 - w_y w_{xx}^2 - w_{xx} v_{xx} - (2-\nu) w_y w_{xy}^2 - w_{yyy}(1 - w_x^2 + w_y^2) + 2 w_x w_{yy} w_{yx}$$
$$+ w_{yx} u_{yy} + w_x u_{yyy} - \nu w_{xxy} - \nu w_y^2 w_{yyx} - 2(1-\nu) \Big[w_{xxy}(1 + w_x^2 + w_y^2) + 2 w_x w_{xy} w_{xx} \Big] \Big\}$$
$$+ (1-\nu) \left[w_x^2 w_y + u_y w_x + v_x w_x \right] = 0.$$

The boundary conditions of the Lagrange multipliers λ_i are as follows:

$$\lambda_1(x, y) = 0 \ \text{ on } \ \Gamma_E; \quad \lambda_2(x, y) = 0 \ \text{ on } \ \Gamma_S; \quad \lambda_2(x, y) = 0 \ \text{ on } \ \Gamma_N;$$

$$\lambda_1(x, y) = \int_0^{L_y} \left\{ u_{tt} - \frac{12D}{h^2}(1-\nu) \left[u_{yy} + w_x w_{yy} \right] - 2D \partial_y^2 \left[w_x (w_{yy} + \nu w_{xx}) \right] \right\} \mathrm{d}y \Big|_{x=0} \quad \text{on } \ \Gamma_W$$

$$\lambda_1(x, y) = \int_x^{L_x} \left\{ u_{tt} - \frac{12D}{h^2}(1-\nu) \left[u_{yy} + w_x w_{yy} \right] - 2D \partial_y^2 \left[w_x (w_{yy} + \nu w_{xx}) \right] \right\} \mathrm{d}x \Big|_{y=0} \quad \text{on } \ \Gamma_S$$

$$\lambda_1(x, y) = \int_x^{L_x} \left\{ u_{tt} - \frac{12D}{h^2}(1-\nu) \left[u_{yy} + w_x w_{yy} \right] - 2D \partial_y^2 \left[w_x (w_{yy} + \nu w_{xx}) \right] \right\} \mathrm{d}x \Big|_{y=L_y} \quad \text{on } \ \Gamma_N$$

$$\lambda_2(x, y) = \int_y^{L_y} \left\{ v_{tt} - \frac{12D}{h^2}(1-\nu) \left[v_{xx} + w_{xx} w_y \right] - 2D \partial_x^2 \left[w_y (w_{xx} + \nu w_{yy}) \right] \right\} \mathrm{d}y \Big|_{x=0} \quad \text{on } \ \Gamma_W$$

$$\lambda_2(x, y) = \int_y^{L_y} \left\{ v_{tt} - \frac{12D}{h^2}(1-\nu) \left[v_{xx} + w_{xx} w_y \right] - 2D \partial_x^2 \left[w_y (w_{xx} + \nu w_{yy}) \right] \right\} \mathrm{d}y \Big|_{x=L_x} \quad \text{on } \ \Gamma_E.$$

3.6 Higher-order models

We conclude the central part of the treatment of inextensible plates by addressing the natural question of including higher-order terms in various modelling steps. We see that in the approaches above that, unlike for the beam, there are critical junctures where the order (of truncation) affects the inextensibility considerations, as well as (though not independent of) the analysis of the potential energy. On the one hand, it is possible to address some of the shortcomings of the previous three models via the inclusion of higher-order effects; on the other hand, as we have already seen, the simplest possible (and lowest order) truncations already yield models which are notably complex.

3.6.1 Higher-order inextensibility

We recall the "full" inextensibility constraints:

$$(1+u_x)^2+v_x^2+w_x^2=1 \tag{3.60}$$

$$u_y^2+(1+v_y)^2+w_y^2=1 \tag{3.61}$$

$$u_y+v_x+u_xu_y+v_xv_y+w_xw_y=0. \tag{3.62}$$

We can, of course, expand the first two to read:

$$2u_x+u_x^2+v_x^2+w_x^2=\ 0;\quad 2v_y+v_y^2+u_y^2+w_y^2=\ 0. \tag{3.63}$$

Remark 3.14 If we were to combine the span and chord constraints, we would have:

$$\nabla\cdot\langle u,v\rangle=\ -\frac{1}{2}\big[|\nabla u|^2+|\nabla v|^2+|\nabla w|^2\big] \tag{3.64}$$

$$u_y+v_x=\ -[u_xu_y+v_xv_y+w_xw_y]. \tag{3.65}$$

Now, we may proceed as we did before in Assumption 3.3 (and the discussion thereafter) and retain higher-order terms (up to η^4) in (3.14)–(3.15). In this case, we would obtain:

$$u_x=\ -\frac{1}{2}[w_x^2+v_x^2]-\frac{1}{8}w_x^4 \tag{3.66}$$

$$v_y=\ -\frac{1}{2}[w_y^2+u_y^2]-\frac{1}{8}w_y^4 \tag{3.67}$$

$$u_y+v_x=-u_xu_y-v_xv_y-w_xw_y. \tag{3.68}$$

On the other hand, more closely following the logic in 1D, we would have from (3.60) that $w_x^2\sim\eta^2$, and also then that $u_x\sim\eta^2$, so we would discard **only** u_x^2 , but not u_y^2. Similarly, from (3.61), we would discard v_y^2 only but not v_x^2. We can formalize this into a separate hypothesis.

Assumption 3.4 Assume that $\partial_{x_i}w,\ \partial_yu,\ \partial_xv\sim\eta$, and assume that $[\partial_xu],\ [\partial_yv]\sim\eta^2$, for all $i=1,2$.

Formulating the reduced η^4 constraints from the Taylor expansions in (3.14)–(3.15), we obtain:

$$u_x=\ -\frac{1}{2}v_x^2-\frac{1}{2}w_x^2-\frac{1}{4}v_x^2w_x^2-\frac{1}{8}v_x^4-\frac{1}{8}w_x^4 \tag{3.69}$$

$$v_y=\ -\frac{1}{2}u_y^2-\frac{1}{2}w_y^2-\frac{1}{4}u_y^2w_y^2-\frac{1}{8}u_y^4-\frac{1}{8}w_y^4 \tag{3.70}$$

$$u_y+v_x=-u_xu_y-v_xv_y-w_xw_y. \tag{3.71}$$

3.6.2 Higher-order potential energy

With either of the choices in the previous section, we might develop an η^4 potential energy. (This is principally a distinction between taking Assumptions 3.3 and 3.4.) In doing so, we would revisit the analysis as before, beginning with the potential energy expression

$$E_{\mathrm{P}}=\frac{1}{2}\frac{E}{1-\nu^2}\int_\Omega\int_{-h/2}^{h/2}\left[\hat{\varepsilon}_{11}^2+\hat{\varepsilon}_{22}^2+2\nu\hat{\varepsilon}_{11}\hat{\varepsilon}_{22}+\frac{1-\nu}{2}\hat{\varepsilon}_{12}^2\right]\mathrm{d}z\mathrm{d}\Omega. \tag{3.72}$$

We would reconsider the full strains, discarding the terms which are quadratic in z (as in Assumption 3.2):

$$\hat{\varepsilon}_{11}=\varepsilon_{11}+z\kappa_{11};\quad \hat{\varepsilon}_{22}=\varepsilon_{22}+z\kappa_{22};\quad \hat{\varepsilon}_{12}=\varepsilon_{12}+z\kappa_{12}. \tag{3.73}$$

Recalling the strain–displacement relations

$$\varepsilon_{11} = u_x + \frac{1}{2}\left[u_x^2 + v_x^2 + w_x^2\right] \tag{3.74}$$

$$\varepsilon_{22} = v_y + \frac{1}{2}\left[u_y^2 + v_y^2 + w_y^2\right] \tag{3.75}$$

$$\varepsilon_{12} = u_y + v_x + u_x u_y + v_x v_y + w_x w_y, \tag{3.76}$$

we may then choose to enforce inextensibility by taking each $\varepsilon_{ij} = 0, i, j = 1, 2$, or by only taking $\varepsilon_{11} = \varepsilon_{22} = 0$. In either case, we must then implement curvature expressions which are themselves "accurate" up to the order of η^4. For the sake of space, we do not reproduce the curvature expressions from (3.26)–(3.28) here, but suffice to say that one may implement either Assumption 3.3 or Assumption 3.4 *up to order* η^4. We demonstrate with (3.26):

$$\begin{aligned}\kappa_{11} = {} & (1+u_x)[v_{xx}w_y + v_x w_{xy} - v_{xy}w_x - (1+v_y)w_{xx}] \\ & + v_x[u_{xy}w_x + u_y w_{xx} - u_{xx}w_y - (1+u_x)w_{xy}] \\ & + w_x[u_{xx} + v_{xy} + u_{xx}v_y + u_x v_{xy} - u_{xy}v_x - u_y v_{xx}].\end{aligned} \tag{3.77}$$

Recall that under Assumption 3.3, up to order η^2, we obtained:

$$\kappa_{11} = \; w_y v_{xx} + w_x u_{xx} - (1 + u_x + v_y)w_{xx}.$$

Now, including all terms up to order η^4 (from either Assumption 3.3 or Assumption 3.4), we should retain all terms in (3.77). Indeed, coefficients on second-order terms are at or below η^4 order in their coefficients. Additionally, unlike the resulting analysis from the η^2 inextensibility assumption, we cannot explicitly solve for $u_{x_i x_j}$ and $v_{x_i x_j}$ to cleanly simplify the expressions in κ_{11}. Thus, pursuing an η^4-order model from either Assumption 3.3 or Assumption 3.4 will require retaining expressions for κ_{ij} in their entirety, resulting in a rather perilous expression for the potential energy E_{P}.

We do not pursue this line further here. We have included the discussion above to demonstrate a general method of building a model that consistently implements an order hypothesis across the inextensibility constraints. Moreover, we observe that it is possible to do so to ever-increasing order. On the other hand, as we can already see through Models 1–3 above, the degree of complexity in the PDE models, boundary conditions, and enforcement of the inextensibility constraints through Lagrange multipliers are notable at the level of even η^2.

4 Discussion and future work

We now provide some discussion of the plate models presented above.

- The principal benefit of building η^2-plate models is they maintain u and v as decoupled variables, in that one can solve for them in terms of w. (This was the case of the beam analysis.) This, of course, is tremendously useful in developing a clear and tractable set of equations of motion for analysis.
- The first point of departure from the beam theory presented in Sect. 2 is the emergence of nonlinear boundary conditions (Sect. 3.3.3); this is a necessary by-product of invoking Hamilton's principle with the chosen potential energy (very similar to what occurs in [23] in an extensible situation). For any future theory of existence and uniqueness of solutions for the inextensible plate mirroring that of the beam [11], nonlinear boundary conditions will be a central issue. This is especially true, as the nonlinear boundary conditions emerge in the notoriously troublesome higher-order plate conditions, and differ markedly from the linear theory. In any approach making using mode functions, one has to be cognizant of this point, in particular, from the point of view of convergence analysis and associated error near the free edges.
- Furthermore, even the *linear mode functions* associated with 2D cantilevered plates are notoriously challenging [4], far beyond the simple cantilever mode functions in 1D [22,23]. Moreover, with the aforementioned appearance of nonlinear boundary conditions, linear mode functions will not satisfy relevant inextensible plate boundary conditions, even if one could explicitly express such linear mode functions through separation [6,31]. It is common in the engineering community to assume mode shapes are multiples of the 1D beam mode shapes, although for plates with free edges this is not exact [4].

- The second major point of departure is the inability to neatly and cleanly resolve the equations through the elimination of the Lagrange multipliers in Plate Models I and III. The benefit of Model II is that it yields equations of motion that are written explicitly in the transverse deflection w; on the other hand, it has an inherent discrepancy between the enforcement of the shear inextensibility constraint in the potential energy (taken) and the Lagrange analysis (ignored as a constraint).
- The dissertation [33] demonstrated a numerical implementation of Model I described in this paper using a Rayleigh–Ritz global modal method. It was found that the model was less stiff than a nonlinear model analyzed by ANSYS, a commercial finite element method solver. Model III was proposed in the dissertation, and the same method was used to implement that model numerically. Using the potential energy associated with Model III resulted in a plate, which was far stiffer than the one modeled in ANSYS.
- In the discussion of cantilever large deflections, it is not physically appropriate to discard u_{tt} or v_{tt} in the modeling as one would do in the case of the scalar von Karman dynamics [8,27]. Such a decision would, of course, simplify the model; however, as nonlinear inertial effects are paramount here, doing so would be a poor modeling choice. For a cantilevered beam, it is easily observed that these "acceleration terms" are on the order of nonlinear stiffness terms for the first mode [42,45]. In general, this will be true for the cantilevered plate, though this is not the case for beams or plates, which are restrained on opposing boundaries (again, see the von Karman plate theory [9,27]).
- As a first mathematical step, one should solve the stationary problem(s) associated with the potential energy E_P in (3.33). This should be done from the point of view of internal and boundary loading. Some preliminary numerics—discussed above—have considered edge loading, but no thorough numerical or analytical studies have been undertaken. Mathematically, a first step will be developing a theory of strong solutions for the stationary version of Model I (dropping time derivatives and Lagrange multiplier terms). The resulting model is spatially quasilinear with nonlinear boundary conditions.
- It is apparent from the analysis in Model III and the discussion in Sect. 3.6 that higher-order methods can and should be developed. However, owing to the apparent complexity of such models, their development should be largely tailored to the context of what is to be modeled—for instance, paying close attention to the plate's aspect ratio.
- There is a substantial literature on cantilevered beams, which bend in two mutually orthogonal directions and also twist about the beam axis. This corresponds to assuming that there is inextensibility only for deformations along the beam axis and that v is only a function of x and t and $w = h(x,t) + y\alpha(x,t)$, where x is the spatial coordinate along the beam axis (span), y is the spatial coordinate along the chord axis and alpha is the twist about the x axis. For example, see [21]. Thus, this is a special case of a nonlinear plate theory (not yet developed here or elsewhere), if only rigid body translation and rotation are considered in the y direction.
- In all of our order assumptions herein, we began by declaring $\partial_{x_i} w \sim \eta$. From the physical point of view, one could make separate assumptions about w_x and w_y. An interesting consideration would permit different orders for the slopes of w in the x and y directions, and it would be relevant to further allow different orders in u and v.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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