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Frieze varieties are invariant under Coxeter mutation

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Abstract. We define a generalized version of the frieze variety introduced by Lee, Li, Mills, Seceleanu and the second author. The generalized frieze variety is an algebraic variety determined by an acyclic quiver and a generic specialization of cluster variables in the cluster algebra for this quiver. The original frieze variety is obtained when this specialization is $(1, \ldots, 1)$.

The main result is that a generalized frieze variety is determined by any generic element of any component of that variety. We also show that the "Coxeter mutation" cyclically permutes these components. In particular, this shows that the frieze variety is invariant under the Coxeter mutation at a generic point.

The paper contains many examples which are generated using a new technique which we call an invariant Laurent polynomial. We show that a symmetry of a mutation of a quiver gives such an invariant rational function.

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Introduction

At the ARTA VI Conference in Mexico celebrating Jose Antonio de la Peña's 60th birthday, the second author presented his paper [2] defining "frieze varieties" of an acyclic quiver and, using a result of Jose Antonio de la Peña [3,6] on eigenvalues of the Coxeter matrix of the quiver, to prove the main result: The dimension of this frieze variety is equal to 0,1, or ≥ 2 if and only if the representation type of

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the quiver is finite, tame, or wild, respectively. In particular, the dimension of the frieze variety is 1 if and only if Q is an extended Dynkin quiver.

After the talk, the two authors discussed properties and examples of frieze varieties throughout the conference. This paper is a report on these discussions.

The basic idea was to generalize the notion of frieze varieties by allowing for arbitrary generic vectors $(b_1, b_2, \ldots, b_n)$ instead of the vector $(1, 1, \ldots, 1)$ as the initial condition for the defining recurrence. We show that, for any generic point $a_* = (a_1, \ldots, a_n)$ in any component of the (generalized) frieze variety, the Coxeter mutations of a_* (2.1) will be contained as a dense subset of the variety. (Theorems 2.3, 3.5)

We also show that the Coxeter mutation cyclically permutes the components of the (generalized) frieze variety (Theorem 3.2). In nice examples, we can use this cyclic permutation to generate all polynomials which define all components of the generalized frieze variety out of a single rational function. (Proposition 4.2)

Finally, we also observe that a symmetry of a mutation of a quiver decreases the dimension of the frieze variety. (Proposition 6.2)

1. Preliminaries

We recall the main result of [2]. Let Q be a connected finite quiver without oriented cycles and label the vertices $1, 2, \ldots, n$ such that if there is an arrow $i \to j$ then $i > j$.

DEFINITION 1.1. [2]

(1) For every vertex $i \in Q_0$ define positive rational numbers $f_i(t)$ $(t \in \mathbb{Z}_{\geq 0})$ recursively by $f_i(0) = 1$ and

$$f_i(t+1) = \frac{1 + \prod_{j \to i} f_j(t) \prod_{j \leftarrow i} f_j(t+1)}{f_i(t)}.$$

(2) For every $t \geq 0$, define the point $P_t = (f_1(t), \ldots, f_n(t)) \in \mathbb{C}^n$.

(3) The *frieze variety* $X(Q)$ of the quiver Q is the Zariski closure of the set of all points P_t $(t \in \mathbb{Z}_{\geq 0})$.

The main result in [2] is the following characterization of the finite–tame–wild trichotomy for acyclic quivers Q in terms of its frieze variety $X(Q)$.

THEOREM 1.2. [2] *Let Q be an acyclic quiver.*

(a) *If Q is representation finite then the frieze variety $X(Q)$ is of dimension 0.*

(b) *If Q is tame then the frieze variety $X(Q)$ is of dimension 1.*

(c) *If Q is wild then the frieze variety $X(Q)$ is of dimension at least 2.*

2. Definitions and main result

Let Q be a quiver as in section 1, or an acyclic valued quiver. Let $\mathcal{A}(Q)$ be the skew-symmetric (if Q is a quiver) or the skew-symmetrizable (if Q is a valued quiver) cluster algebra of Q and let $\mathbf{x} = (x_1, \ldots, x_n)$ be the initial cluster in $\mathcal{A}(Q)$. Let μ_k denote the mutation in direction k and let x'_k be the new cluster variable obtained by this mutation, thus $\mu_k(\mathbf{x}) = (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n)$. We define the *Coxeter mutation* to be the mutation sequence

$$\mu_* = \mu_n \circ \cdots \circ \mu_2 \circ \mu_1, \tag{2.1}$$

where the order $1, 2, \ldots, n$ of the vertices is as in section 1. It is shown in [2] that the point P_t in Definition 1.1 is equal to the specialization of the cluster $\mu_*^t(\mathbf{x})$ at $x_i = 1$.

For an arbitrary point $a_* = (a_1, \cdots, a_n) \in \mathbb{C}^n$ with $a_i \neq 0$ and any $1 \leq k \leq n$, let

$$\mu_k(a_1, \cdots, a_n) = (a_1, \cdots, a_{k-1}, a_k', a_{k+1}, \cdots, a_n) \in \mathbb{C}^n$$

where a_k' is obtained from the cluster variable x_k' by specializing the initial cluster variables $x_i = a_i$, $i = 1, 2, \ldots, n$. For generic a_* we will have $a_k' \neq 0$, so the Coxeter mutation can be repeated. Let $\mu_*(a_*) = \mu_n \circ \cdots \circ \mu_1(a_*)$.

We propose the following generalization of the frieze variety.

Definition 2.1. (1) We say that $a_* \in \mathbb{C}^n$ is a *generic specialization* of $\mathbf{x}$ if all coordinates of $\mu_*^t(a_*)$ are nonzero for all $t \geq 0$.

(2) We refer to the set of all $\mu_*^t(a_*) \in \mathbb{C}^n$ for $t \geq 0$ as the *μ_*-orbit* of a_*.
(3) For any generic specialization a_* of $\mathbf{x}$, the *generalized frieze variety* $X(Q, a_*)$ is defined to be the Zariski closure in $\mathbb{C}^n$ of the μ_*-orbit of a_*.
(4) Let $\widetilde{X}(Q, a_*)$ be $X(Q, a_*)$ with zero dimensional components removed. So, $\widetilde{X}(Q, a_*)$ is empty when $X(Q, a_*)$ is finite, e.g. when Q has finite type. (Theorem 1.2, Remark 3.9.)

Remark 2.2. (1) By the well-known Laurent Phenomenon proved by Fomin and Zelevinsky in [1], the coordinates of $\mu_*^t(a_*)$ for any integer t are given by Laurent polynomials in $a_1, \cdots, a_n$. Therefore, $\mu_*^t(a_*)$ is defined for all t as long as $a_* \in (\mathbb{C}^\times)^n$, i.e. $a_i \neq 0$ for all i.

Moreover, by the positivity theorem proved in [4] (for acyclic quivers already in [5]), if a_* is a positive real vector, then all $\mu_*^t(a_*)$ are positive real vectors. In particular, every positive real vector is a generic specialization.

(2) There are negative integer vectors which are not generic. For example, the vector $a_* = (-1, -1)$ is not generic for the quiver $Q : 1 \leftarrow 2$ since $\mu_*(-1, -1) = (0, -1)$.
(3) The frieze variety of Q is $X(Q) = X(Q, (1, 1, \cdots, 1))$.
(4) We will see that all components of $X(Q, a_*)$ have the same dimension and, therefore, $\widetilde{X}(Q, a_*) = X(Q, a_*)$ when the set is infinite. (Theorem 3.3)
(5) The Zariski closure of the set of all $\mu_*^t(a_*)$ for all $t \in \mathbb{Z}$ is the union of $X(Q, a_*)$ and $X(Q^{op}, a_*)$. One could speculate that these two varieties should always be equal. In any case, our theorems apply to these varieties separately.

We will show that the frieze variety is invariant under mutation in the following sense.

Theorem 2.3. *If $a_* \in \mathbb{C}^n$ is a generic point on the frieze variety, then a_* is a generic specialization of $\mathbf{x}$ and $X(Q, a_*) = \widetilde{X}(Q)$. More precisely, for each component X_i of $X(Q)$ of dimension ≥ 1 there is a subset $U_i \subset X_i$ given as a countable intersection of open subsets U_i^d so that, for any a_* in any U_i we have $X(Q, a_*) = \widetilde{X}(Q)$.*

Corollary 2.4. *If $a_* \in X(Q)$ is a generic point then $\mu_*(a_*) \in X(Q)$.*

We note that frieze varieties often have nongeneric points. See, e.g., Remark 5.1 and the end of Example 3.8.

3. Proof of Theorem 2.3

We prove a more general result (Theorem 3.5 below) using the following lemmas.

LEMMA 3.1. *If b_* is a generic specialization of* $\mathbf{x}$ *and* $a_* \in \widetilde{X}(Q, b_*) \cap (\mathbb{C}^\times)^n$ *then*

$$\mu_*(a_*) \in \widetilde{X}(Q, b_*).$$

PROOF. Since $\widetilde{X}(Q, b_*)$ contains all but finitely many elements of $X(Q, b_*)$, $\widetilde{X}(Q, b_*) = X(Q, \mu_*^t(b_*))$ for sufficiently large $t > 0$. Replacing b_* with $\mu_*^t(b_*)$, we may therefore assume that $\widetilde{X}(Q, b_*) = X(Q, b_*)$ contains $\mu_*^t(b_*)$ for all $t \geq 0$.

The variety $X(Q, b_*)$ is given by a finite number of polynomials f_j. For any $a_* \in (\mathbb{C}^\times)^n$, $\mu_*(a_*) \in \mathbb{C}^n$ is well-defined and lies in $X(Q, b_*)$ if and only if $f_j(\mu_*(a_*)) = 0$ for all j. Since the coordinates of $\mu_*(x_*)$ are Laurent polynomials in $x_1, \cdots, x_n$, each $f_j(\mu_*(x_*))$ is also a Laurent polynomial in the x_i. So, there are monomials $g_j(x_*)$ with the property that

$$F_j(x_*) := f_j(\mu_*(x_*))g_j(x_*) \in \mathbb{C}[x_1, \cdots, x_n]. \tag{3.1}$$

The polynomials F_j have the property that, for any $a_* \in (\mathbb{C}^\times)^n$, $\mu_*(a_*) \in X(Q, b_*)$ if and only if $F_j(a_*) = 0$ for all j. Since $\mu_*^t(b_*) \in X(Q, b_*)$ for all $t \geq 0$, this implies that $F_j(\mu_*^t(b_*)) = 0$ for all $t \geq 0$. This implies that $F_j = 0$ on the Zariski closure of this set of points: $X(Q, b_*)$. Now let $a_* \in X(Q, b_*)$, then $F_j(a_*) = 0$ for all j, and by the above argument, $\mu_*(a_*) \in X(Q, b_*) = \widetilde{X}(Q, b_*)$ as claimed. □

In order to state the main result (Theorem 3.5 which will generalize Theorem 2.3), we need to consider the irreducible components of the variety $\widetilde{X}(Q, b_*)$. We will show that the the Coxeter mutation μ_* cyclically permutes these components.

THEOREM 3.2. *Let b_* be any generic specialization of the cluster* $\mathbf{x}$*, for example, $b_* = (1, 1, \cdots, 1)$. Choose $t_0 \geq 0$ so that $\mu_*^t(b_*) \in \widetilde{X}(Q, b_*)$ for all $t \geq t_0$. Then the components of $\widetilde{X}(Q, b_*) = X(Q, \mu_*^{t_0}(b_*))$ can be numbered $X_1, \cdots, X_m$ with the following properties.*

1. *For each $t \geq t_0$, $\mu_*^t(b_*) \in X_i$ if and only if $t \equiv i$ modulo m. In particular, for each $t \geq t_0$, $\mu_*^t(b_*)$ lies in exactly one X_i.*
2. *X_i is the closure of the set of all $\mu_*^{i+km}(b_*)$ for all integers $k \geq t_0/m$.*
3. *For any $a_* \in X_i \cap (\mathbb{C}^\times)^n$ we have $\mu_*(a_*) \in X_{i+1}$ (or X_1 if $i = m$).*

PROOF. By replacing b_* with $\mu_*^{t_0}(b_*)$ we may assume that $t_0 = 0$ and $\widetilde{X}(Q, b_*) = X(Q, b_*)$.

Let $X_1, \cdots, X_m$ be the components of $X(Q, b_*)$.

Consider the polynomials f_{jk} which define the component X_j. As in (3.1), there are polynomials F_{jk} with the property that, for any $a_* \in (\mathbb{C}^\times)^n$, $\mu_*(a_*) \in X_j$ if and only if $F_{jk}(a_*) = 0$ for all k. Then the polynomials F_{jk} and f_{ip} define a subvariety X_{ij} of X_i which contains all $a_* \in (\mathbb{C}^\times)^n \cap X_i$ so that $\mu_*(a_*) \in X_j$. By Lemma 3.1, $\mu_*(a_*) \in \widetilde{X}(Q, b_*) = \cup X_j$, for all $a_* \in (\mathbb{C}^\times)^n \cap X_i$. Therefore, X_i is the union of the subvarieties X_{ij}. Since X_i is irreducible, $X_i = X_{ij}$ for some j. In fact j is uniquely determined by i, but we don't need to verify this.

The equation $X_i = X_{ij}$ implies that, for any $a_* \in X_i \cap (\mathbb{C}^\times)^n$, $\mu_*(a_*) \in X_j$. For each i, choose one such j. Then $\pi(i) = j$ defines a mapping of the set $\{1, 2, \cdots, m\}$

to itself. Statement (3) will follow from this after we show that π is a cyclic permutation.

CLAIM 1. π is a permutation which is transitive, i.e. π is an m-cycle.

PROOF. Since $b_* \in X(Q, b_*)$, it must lie in one of the components; suppose that $b_* \in X_i$. Then $\mu_*(b_*) \in X_{\pi(i)}$ and $\mu_*^t(b_*) \in X_{\pi^t(i)}$. Thus, the μ_*-orbit of b_* lies in the union of all X_j where j is in the π-orbit of i. But the closure of the μ_*-orbit of b_* is the union of all the X_j. So π must be transitive, which also implies π is a permutation and, in particular, an m-cycle.

Since π is an m-cycle, we may reindex the sets X_i so that $\pi(i) = i+1$ for $i < m$ and $\pi(m) = 1$ and so that $b_* \in X_m$. □

CLAIM 2. For each $t \geq 0$, $\mu_*^t(b_*)$ lies in only one X_i: the one where $i \equiv t$ mod m.

PROOF. Suppose not. Then $\mu_*^t(b_*) \in X_i \cap X_j \subset X_j$ where $j \neq i$. By the division algorithm, any $s \geq t$ can be written as $s = p + km$ where $1 \leq p < m$. If $p \neq i$, then $\mu_*^s(b_*) \in X_p$. If $p = i$ then $\mu_*^s(b_*) \in X_j$. So, the set of all $\mu_*^s(b_*)$ for all $s \geq t$ lies in the union of all X_p for $p \neq i$ which is a contradiction. □

These claims prove Statement (1). Statement (2) follows from the definition of X_i. □

Theorem 3.2 can be strengthened as follows.

THEOREM 3.3. *For all generic specializations b_* of $\mathbf{x}$, all components of $X(Q, b_*)$ have the same dimension. In particular, if $X(Q, b_*)$ is infinite, then $\widetilde{X}(Q, b_*) = X(Q, b_*)$ and $t_0 = 0$ in Theorem 3.2 above.*

PROOF. The Coxeter mutation μ_* and its inverse are given by Laurent polynomials which are rational functions whose denominators are monomials. By Theorem 3.2 μ_* gives a bijection between dense subsets of X_i, X_{i+1} which are disjoint from the coordinate hyperplanes. Therefore, the components X_i of $\widetilde{X}(Q, b_*)$ are birationally equivalent and therefore have the same dimension since the dimension of an irreducible variety is the transcendence degree of its field of rational functions. When $X(Q, b_*)$ is finite, the Coxeter mutation clearly acts as a cyclic permutation of that set. So, assume $X(Q, b_*)$ is infinite.

To see that $\widetilde{X}(Q, b_*) = X(Q, b_*)$ in the infinite case, suppose not. Then there must one point $\mu_*^t(b_*)$ not in $\widetilde{X}(Q, b_*)$ so that $\mu_*^{t+1}(b_*) \in X_i$ for some i. Choose a regular function $f : \mathbb{C}^n \to \mathbb{C}$, i.e. a polynomial in n variables, which is zero on X_{i-1} and nonzero on the point $\mu_*^t(b_*)$. Composing with the rational morphism μ_*^{-1} gives a rational function on X_i whose denominator is a monomial and whose numerator is a polynomial function g. Moreover, g is zero on a dense subset of X_i by Theorem 3.2(2) and thus zero on all of X_i, but g is nonzero on $\mu_*^{t+1}(b_*) \in X_i$, since $g(\mu_*^{t+1}(b_*)) = f(\mu_*^t(b_*)) \neq 0$. This contradiction shows that $\mu_*^t(b_*) \in X_{i-1}$ as claimed. □

LEMMA 3.4. *For every component X_i of $X(Q, b_*)$ and every integer $d > 0$, there is a $p_d > d$ and a dense open subset U_i^d of X_i so that, for every $a_* \in U_i^d$, we have the following.*

(1) *For every $0 \leq t \leq p_d$, the coordinates of $\mu_*^t(a_*)$ are nonzero.*

(2) *Any polynomial of degree* $\leq d$ *which is zero on* $\mu_*^t(a_*)$ *for all* $0 \leq t \leq p_d$ *will also be zero on* $X(Q, b_*)$.

Before we prove this lemma, we will show that it implies the following generalization of Theorem 2.3. We use the notation U_i^∞ for the intersection of U_i^d over all $d > 0$.

THEOREM 3.5. *Let* $b_* \in \mathbb{C}^n$ *be any generic specialization of the cluster* $\mathbf{x}$ *of* Q. *Then, for generic* $a_* \in X(Q, b_*)$, *i.e. for* $a_* \in \bigcup U_i^\infty$, *we have* $X(Q, a_*) = X(Q, b_*)$.

PROOF. By (1) in Lemma 3.4, every element of U_i^∞ is a generic specialization of $\mathbf{x}$. By Lemma 3.1, the entire μ_*-orbit of a_* lies in $\widetilde{X}(Q, b_*) = X(Q, b_*)$. So, $X(Q, a_*) \subset X(Q, b_*)$.

If $X(Q, a_*) \neq X(Q, b_*)$ there must be a polynomial f which is zero on the μ_*-orbit of a_* but nonzero on $X(Q, b_*)$. Let $d = \deg f$. Given that f is zero on the μ_*-orbit of $a_* \in U_i^\infty \subset U_i^d$, we conclude by (2) in Lemma 3.4 that f is zero on $X(Q, b_*)$. This contradiction proves the theorem. □

REMARK 3.6. Theorem 2.3 follows from Theorem 3.5 by choosing $b_* = (1, 1, \ldots, 1)$.

It remains to prove the lemma:

PROOF OF LEMMA 3.4. We consider only the nontrivial case when $X(Q, b_*)$ is infinite. By Theorems 3.2 and 3.3, $X(Q, b_*) = X_0 \cup \cdots \cup X_{m-1}$ and $t_0 = 0$ in Theorem 3.2. By (1) in Theorem 3.2 we have that $\mu_*^i(b_*) \in X_i$. Since $\widetilde{X}(Q, b_*) = X(Q, b_*)$, a polynomial f will be zero on $X(Q, b_*)$ if and only if $f(\mu_*^t(b_*)) = 0$ for all $t \geq i$. The key point of the proof is to show that, for f of degree $\leq d$, we only need to check this condition for $t \leq p_d$ for some fixed $p_d > 0$. This is a linear condition on the coefficients of f. Since the rank of a linear system is a lower semi-continuous function, there will be an open subset U_i^d of X_i on which this system is defined (Condition (1)) and has maximum rank. This will be the desired set.

We now construct the linear system. With n, d fixed, consider the polynomial mapping

$$P_d : \mathbb{C}^n \to \mathbb{C}^{\binom{n+d}{n}}$$

which sends $x_* = (x_1, \cdots, x_n) \in \mathbb{C}^n$ to the sequence of all monomials in x_j of degree $\leq d$. For example, when $n = 2, d = 3$, we have:

$$P_3(x, y) = (1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3).$$

Then any polynomial function f on $\mathbb{C}^n$ of degree $\leq d$ is given as the composition of P_d with a linear mapping $f_* : \mathbb{C}^{\binom{n+d}{n}} \to \mathbb{C}$.

Let $B_d \subset \mathbb{C}^{\binom{n+d}{n}}$ be the vector space span of all $P_d(\mu_*^t(b_*))$ for all $t \geq m$. Then a polynomial f of degree $\leq d$ is zero on $X(Q, \mu_*^m(b_*)) = X(Q, b_*)$ if and only if $f_*(B_d) = 0$. So, $P_d(a_*) \in B_d$ for all $a_* \in X(Q, b_*)$. Let k be the dimension of B_d. Then B_d has a basis consisting of $P_d(\mu_*^{t_j}(b_*))$ for some $m \leq t_1 < t_2 < \cdots < t_k$. These vectors form a $\binom{n+d}{n} \times k$ matrix of rank k. So, there is some $k \times k$ minor M of this matrix which is nonzero. Since $t_j \geq m \geq i$, the entries of the matrix, being monomials in the coordinates of $\mu_*^{t_j - i}(\mu_*^i(b_*))$ for some j, are given as Laurent polynomials in the coordinates of $\mu_*^i(b_*) \in X_i$. Therefore, for each i, the minor M is a Laurent polynomial in the coordinates of $\mu_*^i(b_*)$. Let F_i be the numerator of this polynomial. Then $F_i(\mu_*^i(b_*)) \neq 0$.

Let $p_d = t_k - i$ and let V_i^d be the subset of X_i of all points a_* so that $\mu_*^t(a_*)$ is defined with nonzero coordinates for $0 \le t \le p_d$. Since this is an open condition and $\mu_*^i(b_*) \in V_i^d$, V_i^d is a dense open subset of X_i. By Lemma 3.1, $\mu_*^t(a_*) \in X(Q, b_*)$ for all $0 \le t \le p_d$. Therefore, $P_d(\mu_*^t(a_*)) \in B_d$ for all $0 \le t \le p_d$. The condition that the vectors $P_d(\mu_*^t(a_*))$, for $0 \le t \le p_d$ span B_d is an open condition which holds for $a_* = \mu_*^i(b_*) \in X_i \cap V_i^d$. Therefore, it holds on some Zariski open neighborhood of $\mu_*^i(b_*)$ in $X_i \cap V_i^d$. In fact, this condition will hold on the open subset U_i^d of $X_i \cap V_i^d$ on which $F_i \neq 0$.

Since $U_i^d \subset V_i^d$, then $a_* \in U_i^d$ will satisfy Condition (1). For any polynomial f of degree $\le d$ which is zero on $\mu_*^t(a_*)$ for all $0 \le t \le p_d$, the corresponding linear map f_* will vanish on the vector $P_d(\mu_*^t(a_*))$ for all $0 \le t \le p_d$. Since these vectors span B_d, $f_*(B_d) = 0$. This implies that f is zero on the set $X(Q, b_*)$, proving Condition (2) and concluding the proof of Lemma 3.4, Theorems 3.5 and 2.3. □

We illustrate some of the concepts in the proof of Lemma 3.4 with two examples.

Example 3.7. Consider the A_2 quiver

$$Q: \quad 1 \longleftarrow 2.$$

Then $X(Q)$ has only five points $\mu_*^i(b_*) = (1,1), (2,3), (2,1), (1,2), (3,2)$ for $i = 0, 1, 2, 3, 4$. Thus $m = 5$ and each X_i consists of one point. For $d = 2$, $\binom{n+d}{n} = 6$. So, the five vectors $P_2(\mu_*^i(b_*))$ do not span $\mathbb{C}^6$. These five vectors are the rows of the following 5×6 matrix. (The proof of Lemma 3.4 uses the transpose of this matrix.)

	1	x	y	x^2	xy	y^2
b_*	1	1	1	1	1	1
$\mu_*(b_*)$	1	2	3	4	6	9
$\mu_*^2(b_*)$	1	2	1	4	2	1
$\mu_*^3(b_*)$	1	1	2	1	2	4
$\mu_*^4(b_*)$	1	3	2	9	6	4

The span of these five vectors is $B_2 \subset \mathbb{C}^6$. This is a hyperplane perpendicular to the vector $(3, -2, -2, 1, -1, 1)$. Dot product with this vector gives a linear map $f_* : \mathbb{C}^6 \to \mathbb{C}$, composing with P_2 gives

$$f(x, y) = f_*(P_2(x, y)) = x^2 - xy + y^2 - 2x - 2y + 3.$$

This is the only quadratic polynomial which vanishes on the frieze variety $X(Q)$. The real points form an ellipse centered at $(2, 2)$ with major axis going from $(1, 1)$ to $(3, 3)$.

Here is another example which explains the minor M and numerator F.

Example 3.8. Consider the Kronecker quiver

$$Q: \quad 1 \Longleftarrow 2$$

Consider the frieze variety $X(Q)$. The first three points are $b_* = (1, 1)$, $\mu_*(b_*) = (2, 5)$, $\mu_*^2(b_*) = (13, 34)$.

Take $d = 1$. Then $\binom{n+d}{n} = 3$. In order to span $\mathbb{C}^3$ we need three vectors: $P_1(\mu_*^t(b_*))$ for $t = 0, 1, 2$. These are the rows of the following matrix.

	1	x	y
b_*	1	1	1
$\mu_*(b_*)$	1	2	5
$\mu_*^2(b_*)$	1	13	34

Since this has full rank, the determinant of this matrix (which is -15) is the maximal minor. However we need the minor as a Laurent polynomial in the coordinates of $\mu_*^i(b_*)$. Take $i = 1$ and write $\mu_*^1(b_*) = (y_1, y_2)$. Thus $y_1 = x_1', y_2 = x_2'$. In terms of the cluster y_1, y_2, the 3×3 matrix under consideration is:

$$\begin{bmatrix} 1 & \frac{y_1^4+y_2^2+2y_1^2+1}{y_1y_2^2} & \frac{y_1^2+1}{y_2} \\ 1 & y_1 & y_2 \\ 1 & \frac{y_2^2+1}{y_1} & \frac{y_1^4+2y_2^2+y_1^2+1}{y_1^1y_2} \end{bmatrix}$$

The determinant of this matrix is the rational function M. The numerator of M is the polynomial $F = y_1^2y_2^2M$. This is a polynomial of degree 8 in y_1, y_2. The reason we use these variables is because we are looking for points $a_* = (a_1, a_2)$ close to $\mu_*^1(b_*) = (2, 5)$. What we have already calculated is: $M(2, 5) = -15$.

Using invariant rational functions, the generalized frieze variety $X(Q, b_*)$ can be given as follows. The rational function

$$h(\mathbf{x}) = \frac{x_1^2 + x_2^2 + 1}{x_1x_2}$$

is equal to itself, as an element of $\mathbb{C}(\mathbf{x})$, on all iterated Coxeter mutations of the cluster $\mathbf{x}$. To see this, write it as:

$$h(x_1, x_2) = \frac{x_1 + x_1'}{x_2} = h(x_1', x_2)$$

which is invariant under Coxeter mutation since μ_1 switches the terms x_1, x_1' and similarly for μ_2. So, it is invariant under $\mu_2 \circ \mu_1$. At $b_* = (1, 1)$ it takes the value $h(1, 1) = 3$. This makes

$$x_1^2 + x_2^2 + 1 = 3x_1x_2 \tag{3.2}$$

at all points in $X(Q)$. Note that equation (3.2) is a specialization of the Markov equation $x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$.

Using this equation we can see that the value of M at any point (y_1, y_2) in $X(Q)$ is equal to -15. So, $F = -15y_1^2y_2^2$. The set U_1^d for $d = 1$ is given by $F(\mu_*^t(a_*)) \neq 0$ for three values of t, namely $t = -1, 0, 1$ since we are thinking of a_* as a specialization of $(y_1, y_2) = (x_1', x_2')$. This makes $U_1^d = V_1^d$ the complement in $X(Q)$ of the 12 points consisting of μ_*^{-t}, for $t = -1, 0, 1$, applied to the points $(0, \pm\sqrt{-1}), (\pm\sqrt{-1}, 0)$.

Remark 3.9. We observe that Theorem 1.2 does not always hold for the generalized frieze variety. For example, when Q is the Kronecker quiver considered above and $b_* = \left(\frac{\sqrt{2}i}{2}, \frac{\sqrt{2}i}{2}\right)$, $X(Q, b_*)$ consists of only two points, b_* and $-b_*$. However, we believe that, for almost all b_*, the analogue of Theorem 1.2 hold.

4. Construction from invariant rational functions

For any $k \geq 0$, the coordinates of $\mu_*^k(\mathbf{x})$ are Laurent polynomials in $\mathbf{x}$. Furthermore, each coordinate of $\mathbf{x}$ is given as a Laurent polynomial in $\mu_*^k(\mathbf{x})$. So, the set of values of $\mu_*^k(\mathbf{x})$ is not contained in any hypersurface in $\mathbb{C}^n$. So, for any rational function $h(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$ and any $t \geq 0$, we have another rational function $h(\mu_*^t(\mathbf{x})) \in \mathbb{C}(\mathbf{x})$ since the denominator of $h(\mu_*^t(\mathbf{x}))$ cannot be identically zero. Suppose, furthermore, that $h(\mathbf{x})$ is a Laurent polynomial in $\mathbf{x}$ and $a_* \in \mathbb{C}^n$ is a generic specialization of $\mathbf{x}$. Then $h(\mu_*^t(a_*))$ is a well-defined complex number for any $t \geq 0$. This is particularly useful when $h(\mathbf{x})$ is periodic in the following sense.

Definition 4.1. We say that a rational function $h(\mathbf{x})$ is *invariant under* μ_*^k if:

$$h(\mu_*^k(\mathbf{x})) = h(\mathbf{x}) \tag{4.1}$$

as an element of $\mathbb{C}(\mathbf{x})$. If $k > 0$ is minimal and $h(\mathbf{x})$ is Laurent, we say that $h(\mathbf{x})$ is an *invariant Laurent polynomial for Q of period k*. For each $t \geq 0$ we will use the notation:

$$h(\mu_*^t(\mathbf{x})) = \frac{f_t(\mathbf{x})}{g_t(\mathbf{x})} \tag{4.2}$$

Note that $f_t, g_t \in \mathbb{C}[\mathbf{x}]$ depend only on the residue class of t modulo the period k.

Proposition 4.2. *Let $h(\mathbf{x})$ be an invariant Laurent polynomial of period k. Let a_* be a generic specialization of $\mathbf{x}$. For each $t \geq 0$, let $c_t = h(\mu_*^t(a_*))$ and let $f_t(\mathbf{x}), g_t(\mathbf{x})$ be as in (4.2). For $0 \leq t < k$, let*

$$F_{j,t}(\mathbf{x}) := f_t(\mathbf{x}) - c_{t+j}g_t(\mathbf{x})$$

be the numerator of the rational function $h(\mu_*^t(\mathbf{x})) - h(\mu_*^{j+t}(a_*))$, and let X_j be the intersection of the k hypersurfaces given by $F_{j,t}(\mathbf{x}) = 0$, for $0 \leq j < k$.

Then the generalized quiver variety $X(Q, a_*)$ is contained in the union $X_0 \cup X_1 \cup \cdots \cup X_{k-1}$.

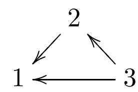
Proof. For any $0 \leq j < k$ and $s \geq 0$, let $b_* = \mu_*^{j+ks}(a_*)$. Then, for any $t \geq 0$, we have $h(\mu_*^t(b_*)) = h(\mu_*^{j+t}(a_*)) = c_{j+t}$. Since $F_{j,t}(\mathbf{x})$ is the numerator of $h(\mu_*^t(\mathbf{x})) - h(\mu_*^{j+t}(a_*))$, we get $F_{j,t}(b_*) = 0$. Therefore, $b_* = \mu_*^{j+ks}(a_*)$ lies in X_j for all $s \geq 0$ and the union of the X_j contains the entire μ_*-orbit of a_*. So, $X(Q, a_*) \subset \bigcup X_j$. □

Remark 4.3. Thus, the single Laurent polynomial $h(\mathbf{x})$ generates k^2 polynomials $F_{j,t}(\mathbf{x})$ giving k varieties X_j whose union contains $X(Q, a_*)$ and, in many cases, is equal to $X(Q, a_*)$ as shown in several examples below. In these examples, all of the rational functions $h(\mu_*^t(\mathbf{x})) = f_t(\mathbf{x})/g_t(\mathbf{x})$ are Laurent polynomials with positive integer coefficients. This is reflected in the fact that the monomials in the polynomials $F_{j,t}(\mathbf{x}) = f_t(\mathbf{x}) - c_{j+t}g_t(\mathbf{x})$ have the same sign except for one: $-c_{j+t}g_t(\mathbf{x})$. We note that this is not a general phenomenon since, e.g., when Q has finite type, $\mu_*^k(\mathbf{x}) = \mathbf{x}$ for some k and, therefore, every Laurent polynomial will be invariant with period dividing k.

5. Examples

To illustrate Proposition 4.2, we give two examples, both tame, where a single invariant Laurent polynomial $h(\mathbf{x})$ whose period k is one less than the number of vertices of Q gives the complete decomposition of $X(Q)$ as a union of k curves.

5.1. The affine quiver $\tilde{A}_2$. Let Q be the quiver:



This quiver has the property that $\mu_1 Q \cong Q$ after renumbering the vertices. In terms of the cluster variables (x, y, z), after one mutation, we get back the same quiver with new variables (y, z, x') where $x' = \frac{1}{x}(yz+1)$. For any rational function $h(x, y, z)$ let

$$h'(x, y, z) := h(y, z, x'),$$

where x' is the cluster variable obtained from the cluster (x, y, z) by mutation in x. We also use the notation $h' = h \circ \tilde{\mu}$ where

$$\tilde{\mu}(x, y, z) := (y, z, x'). \tag{5.1}$$

Note that $h^{(3)}(x, y, z) = h'''(x, y, z) = h(\mu_*(x, y, z))$.

For example, let $h(\mathbf{x})$ be the Laurent polynomial

$$h(x, y, z) = \frac{x+z}{y}$$

Then h', h'' are given by

$$h'(x, y, z) = h(y, z, x') = \frac{y + x'}{z} = \frac{xy + yz + 1}{xz}$$

$$h''(x, y, z) = h'(y, z, x') = h(z, x', y') = \frac{z + y'}{x'}$$

Observe that

$$\frac{h}{h''} = \frac{(x+z)x'}{y(z+y')} = \frac{yz + 1 + zx'}{yz + 1 + zx'} = 1$$

and thus $h'' = h$ and, consequently, $h^{(a)} = h$ if a is even, and $h^{(a)} = h'$ if a is odd. Therefore h and h' are invariant under μ_*^2:

$$h(\mu_*^2(x, y, z)) = h^{(6)}(x, y, z) = h(x, y, z),$$

and similarly, $h' \circ \mu_*^2 = h^{(7)} = h'$.

Thus, $h(\mathbf{x})$ is an invariant Laurent polynomial for Q of period 2. So, Proposition 4.2 applies with

$$c_0 = h(1, 1, 1) = 2$$
$$c_1 = h'(1, 1, 1) = 3.$$

So, $X(Q)$ is contained in the union of two curves $X_0 \cup X_1$ where X_0 is given by the polynomial equations $F_{00} = F_{01} = 0$ where

$$F_{00}(x, y, z) = Num(h(x, y, z) - c_0) = x + z - 2y$$

$$F_{01}(x, y, z) = Num(h'(x, y, z) - c_1) = xy + yz + 1 - 3xz$$

and X_1 is given by $F_{11} = F_{12} = 0$ where

$$F_{10}(x, y, z) = Num(h(x, y, z) - c_1) = x + z - 3y$$

$$F_{11}(x, y, z) = Num(h'(x, y, z) - c_0) = xy + yz + 1 - 2xz.$$

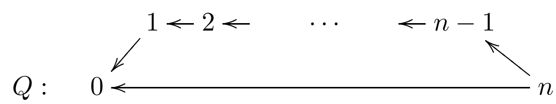
From these equations it is easy to verify the observation from [2] that X_0 is a nonsingular degree 2 curve. Indeed the equation $F_{00} = 0$ is equivalent to the linear equation $z = 2y - x$ which reduced the second equation to $F_{01}(x, y, z) = F_{01}(x, y, 2y - x) = 0$ which is a nondegenerate quadratic in two variables. Thus

X_0 is a nonsingular curve in $\mathbb{C}^3$ containing the infinite set of points $\mu_*^{2k}(1,1,1)$ for $k \geq 0$. So, it must be the closure of this set. Similarly, the curve X_1 must be the closure of the set of all $\mu_*^{2k+1}(1,1,1)$. We therefore see that the frieze variety $X(Q)$ has two components given by the above four polynomials. These polynomials come from an example worked out in [2], but here all four polynomials come from the same Laurent polynomial h.

Remark 5.1. We note that frieze varieties often have nongeneric points. For example, the first component X_0 of the frieze variety discussed above contains the point $(0, \sqrt{2}i/2, \sqrt{2}i)$. Mutation at the first vertex gives $x' = (yz+1)/x = 0/0$ which is undefined. However, $\tilde{\mu}$ (defined in (5.1) above) sends the 0-component of $X(Q)$ to the 1-component X_1. So, the value of x' can be computed from the linear equation $F_{10}(y,z,x') = 0$:

$$x' = 3z - y = \frac{5}{2}\sqrt{2}i.$$

5.2. The affine quiver $\tilde{A}_n$. More generally, consider the quiver:



for $n \geq 3$. Let h be the Laurent polynomial:

$$h(x_0, x_1, \cdots, x_n) = \frac{x_{n-2} + x_n}{x_{n-1}}.$$

Mutation gives:

$$h'(x_0, \cdots, x_n) := h(x_1, x_2, \cdots, x_n, x_0) = \frac{x_{n-1} + x_0'}{x_n} = \frac{x_{n-1}x_0 + x_1x_n + 1}{x_0x_n}.$$

Mutating k times for $3 \leq k \leq n$ ($k = 2$ is given in (5.2) below) gives

$$h^{(k)}(x_0, x_1, \cdots, x_n) = h(x_k, x_{k+1}, \cdots, x_n, x_0', \cdots, x_{k-1}') = \frac{x_{k-3}' + x_{k-1}'}{x_{k-2}'}$$

For $k = n$, $h^{(n)} = h$ since the quotient is:

$$\frac{h}{h^{(n)}} = \frac{(x_{n-2} + x_n)x_{n-2}'}{x_{n-1}(x_{n-3}' + x_{n-1}')} = \frac{x_{n-1}x_{n-3}' + 1 + x_nx_{n-2}'}{x_{n-1}x_{n-3}' + 1 + x_nx_{n-2}'} = 1$$

This gives us a cleaner formula for $h^{(k)}$ for $2 \leq k \leq n$ using $2 - n \leq k - n \leq 0$:

$$h^{(k)}(\mathbf{x}) = h^{(k-n)}(\mathbf{x}) = h(x_{k+1}', \cdots, x_n', x_0, x_1, \cdots, x_k) = \frac{x_{k-2} + x_k}{x_{k-1}}. \tag{5.2}$$

The equation $h = h^{(n)}$ also implies that $h^{(nk)} = h$, for all $k \geq 0$. So,

$$h(\mu_*^k(\mathbf{x})) = h^{k(n+1)}(\mathbf{x}) = h^{(k)}(\mathbf{x}).$$

In particular $h(\mu_*^n(\mathbf{x})) = h^{(n)}(\mathbf{x}) = h(\mathbf{x})$. So, $h(\mathbf{x})$ is an invariant Laurent polynomial for Q of period dividing n. To see that the period of h is exactly n we compute c_k:

$$c_k = h^{(k)}(1,1,\cdots,1) = h(1,1,\cdots,1,2,3,\cdots,k+1) = \begin{cases} 3 & \text{if } k = 1; \\ 2 & \text{if } 2 \leq k \leq n. \end{cases} \tag{5.3}$$

Also, $h^{(k)}$ is invariant under μ_*^n, for all $k \geq 0$, since $h^{(k)}(\mu_*^{sn}(\mathbf{x})) = h^{(k+sn)}(\mathbf{x}) = h^{(k)}(\mathbf{x})$. So, for generic a_*, the frieze variety $X(Q, a_*)$ has n components

$$X(Q, a_*) = X_0 \cup X_1 \cup \cdots \cup X_{n-1};$$

the first component X_0, containing the μ_*^n-orbit of $a_* = (1, 1, \cdots, 1)$, is given by the n polynomials $F_{01}, \cdots, F_{0,n-1}, F_{0n} = F_{00}$ where

$$F_{01} = Num(h'(\mathbf{x}) - c_1) = x_0 x_{n-1} + x_1 x_n + 1 - 3x_0 x_n,$$

by Proposition 4.2 and (5.3); and for $2 \leq k \leq n$

$$F_{0k} = Num(h^{(k)}(\mathbf{x}) - c_k) = x_{k-2} + x_k - 2x_{k-1},$$

by Proposition 4.2 ,(5.2) and (5.3).

REMARK 5.2. Note that, for any point in X_0, the equations $F_{0k} = 0$ for $2 \leq k \leq n$ express the coordinates $x_2, x_3, \cdots, x_n$ as linear combinations of x_0, x_1. Thus, points in X_0 are determined by their first two coordinates. Since X_0 contains the 3 points $(1, 1, \cdots), (1, 2, \cdots), (n+1, 2n+3, \cdots)$ and their negatives (where we ignore all but the first two coordinates) and does not contain $(0, 0, \cdots)$, linear algebra in $\mathbb{C}^2$ shows X_0 cannot be a union of two straight lines. So, X_0 is an irreducible degree 2 curve containing the μ_*^n-orbit of the point $a_* = (1, 1, \cdots, 1)$. Since no two distinct curves can have an infinite intersection, X_0 is the Zariski closure of this set.

By Proposition 4.2, X_1 is given by the n polynomial equations $F_{1,t} = 0$ for $0 < t \leq n$

$$F_{1n} = Num(h(\mathbf{x}) - c_1) = x_{n-2} + x_n - 3x_{n-1}$$
$$F_{11} = Num(h'(\mathbf{x}) - c_2) = x_0 x_{n-1} + x_1 x_n + 1 - 2x_0 x_n$$
$$F_{1k} = F_{0k}, \text{ for } k = 2, \cdots, n-1.$$

The other polynomials F_{jt} are similar. As in the case of $\tilde{A}_2$, all polynomials F_{jk} are given by the single Laurent polynomial h and its $n-1$ mutations $h^{(k)}$ for $1 \leq k < n$.

6. Symmetry

One easy observation [2] is that, if a permutation σ of $\{1, 2 \cdots, n\}$ leaves the quiver Q invariant, then the frieze variety $X(Q)$ satisfies $x_k = x_{\sigma(k)}$ for all k. In terms of invariant rational functions, $h = x_{\sigma(k)}/x_k$ is invariant under μ_*^2 since μ_* inverts h.

A similar result holds true if a mutation of Q has symmetry. For example,

$$Q: \quad 1 \Longleftarrow 2 \Longleftarrow 3$$

becomes symmetric after one mutation:

$$\mu_1(Q): \quad 1' \Longrightarrow 2 \Longleftarrow 3$$

This implies that

$$h = \frac{x_1'}{x_3} = \frac{x_2^2 + 1}{x_1 x_3}$$

is invariant under μ_*^2. Since $c_0 = h(1, 1, 1) = 2$ and $c_1 = 1/c_0 = 1/2$, the frieze variety of Q is $X(Q) = X_0 \cup X_1$, where X_0 is the hypersurface given by $F_0 = 0$, where

$$F_0 = Num(h(\mathbf{x}) - c_0) = x_2^2 + 1 - 2x_1 x_3$$

and X_1 containing $\mu_*(1,1,1) = (2,5,26)$ is the hypersurface given by $F_1 = 0$, where

$$F_1 = Num\left(h(\mathbf{x}) - \frac{1}{2}\right) = Num\,(2h(\mathbf{x}) - 1) = 2x_2^2 + 2 - x_1x_3.$$

Remark 6.1. The hypersurface X_0 contains the μ_*^2-orbit of the point $(1,1,\cdots,1)$. Since this set is not contained in any curve by Theorem 1.2, there cannot be a smaller variety containing this set. So, X_0 is the Zariski closure of the μ_*^2 orbit of $(1,1,\cdots,1)$ and similarly for X_1.

More generally we have the following.

Proposition 6.2. *Suppose that i is a sink in the quiver Q and j is a source so that, for any other vertex k, the number n_k of arrows from k to i is equal to the number of arrows from j to k. Then, the frieze variety $X(Q)$ is contained in the union of two hypersurfaces X_0, X_1 given by the equations $F_0 = 0$ and $F_1 = 0$, where*

$$F_0 = 1 - 2x_ix_j + \prod_k x_k^{n_k}$$

$$F_1 = 2 - x_ix_j + 2\prod_k x_k^{n_k}.$$

Proof. After the mutation $\mu_* = \mu_n \circ \cdots \circ \mu_1$, we will have x'_* where

$$x'_i = \frac{1+\prod x_k^{n_k}}{x_i}, \quad x'_j = \frac{1+\prod (x'_k)^{n_k}}{x_j}$$

So, the rational function $h(\mathbf{x}) = x'_i/x_j$ will mutate to

$$h(\mu_*(\mathbf{x})) = \frac{x''_i}{x'_j} = \left(\frac{1+\prod(x'_k)^{n_k}}{x'_i}\right)\left(\frac{x_j}{1+\prod(x'_k)^{n_k}}\right) = \frac{x_j}{x'_i} = \frac{1}{h(\mathbf{x})}.$$

So, $h(\mathbf{x})$ is μ_*^2 invariant. Since $h(\mathbf{x}) = f(\mathbf{x})/g(\mathbf{x})$ where $f(\mathbf{x}) = 1 + \prod x_k^{n_k}$ and $g(\mathbf{x}) = x_ix_j$, $c_0 = h(1,1,\cdots,1) = 2$, $c_1 = 1/c_0 = \frac{1}{2}$, the numerator of $h(\mathbf{x}) - c_0$ is F_0 and the numerator of$h(\mathbf{x}) - c_1$ is F_1. By Proposition 4.2, the μ_*^2 orbit of $(1,1,\cdots,1)$ satisfies $F_0 = 0$ and the μ_*^2 orbit of $\mu_*(1,1,\cdots,1)$ satisfies $F_1 = 0$. ☐

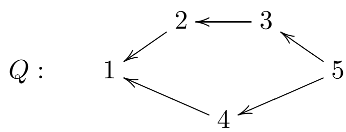
7. Questions and answers

We list a few questions from the first version of this paper and short answers to these questions following suggestions by Gordana Todorov. Details will be given in another paper.

(1) Question: In the tame case, does an invariant Laurent polynomial $h(\mathbf{x})$ always exist? Answer: Yes. The cluster character of a regular module in a tube of rank k will be an invariant Laurent polynomial of period k.
(2) Question: Can $h(\mathbf{x})$ be chosen to have positive integer coefficients? Answer: In the tame case yes. In the wild case we also believe the answer is yes since we believe that the only invariant rational functions are the ones given by symmetry of the quiver as in Proposition 6.2.
(3) Question: Can $h(\mathbf{x})$ be chosen such that all iterated Coxeter mutations $h(\mu_*^t(\mathbf{x}))$ have positive integer coefficients? Answer: In the tame case, yes. The answer also seems to be yes in the wild case if, as we suspect, the only invariant rational functions come from symmetry of the quiver

as observed in [2] or as given in Proposition 6.2. However we note that, in the latter case, $h(\mu_*(\mathbf{x})) = 1/h(\mathbf{x})$ is not Laurent.

(4) Question: Is the period of the invariant Laurent polynomial always equal to the number of components of $X(Q, a_*)$? Answer: No, a counterexample is given by the following quiver



Here there is a tube of rank 3 giving an invariant Laurent polynomial of period 3 and another tube of rank 2 giving an invariant Laurent polynomial of period 2. This suggest that there should be 6 components.

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