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# Nondefinability of Rings of Integers in Most Algebraic Fields

Philip Dittmann and Arno Fehm

**Abstract** We show that the set of algebraic extensions $F$ of $\mathbb{Q}$ in which $\mathbb{Z}$ or the ring of integers $\mathcal{O}_F$ are definable is meager in the set of all algebraic extensions.

It is proved in Theorem 1.1 and Corollary 5.7 of Eisentraeger et al. [5] that the set of subfields $F$ of $\overline{\mathbb{Q}}$ in which one of $\mathbb{Z}$, $\mathbb{Q}\setminus\mathbb{Z}$, $\mathcal{O}_F$, $F\setminus\mathcal{O}_F$ is existentially definable is a meager subset of the space $\mathcal{E}$ of all subfields $E$ of $\overline{\mathbb{Q}}$, in the topology induced from $2^{\overline{\mathbb{Q}}}$. In this short note, we explain how a stronger statement can be deduced from known results from field arithmetic (which, in particular, studies certain properties of algebraic extensions of $\mathbb{Q}$) and model theory (which studies definable subsets in structures with certain properties).

Recall that a field $F$ is *PAC* if every geometrically irreducible $F$-variety has an $F$-rational point, *$\omega$-free* if every finite embedding problem for the absolute Galois group $G_F$ is solvable, and *Hilbertian* if $\mathbb{A}^1(F)$ is not thin; that is, for every finitely many absolutely irreducible $f_1,\dots,f_n \in F[X,Y]$ monic of degree at least 2 in $Y$, and $0 \neq g \in F[X]$, there exists $x \in F$ such that $g(x) \neq 0$ and $f_1 \cdots f_n(x,Y)$ has no zero in $F$ (see Chapters 11, 27, and 12 and Section 13.5 of Fried and Jarden [7]).

**Proposition 1** *The set of subfields $F$ of $\overline{\mathbb{Q}}$ which are $\omega$-free and PAC is comeager in $\mathcal{E}$.*

**Proof** We claim that both the set $\mathcal{P}$ of PAC fields in $\mathcal{E}$ and the set $\mathcal{H}$ of Hilbertian fields in $\mathcal{E}$ are dense $G_\delta$-sets and therefore comeager. Since the union of two meager sets is meager, and Hilbertian PAC fields are $\omega$-free (see Jarden [9, Theorem 5.10.3]), this then implies the claim.

The set $\mathcal{P}$ is dense in $\mathcal{E}$, since for any finite extensions $\mathbb{Q} \subseteq K \subseteq L$, Jarden's PAC Nullstellensatz (see [7, Theorem 18.6.1]) gives a PAC field $K \subseteq F \subseteq \overline{\mathbb{Q}}$ with

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$F \cap L = K$. Moreover, $\mathcal{P}$ is the intersection of the countably many open sets

$$U_f = \{F \in \mathcal{E} : f \notin F[X,Y]\} \cup \bigcup_{x,y \in \overline{\mathbb{Q}}, f(x,y)=0} \{F \in \mathcal{E} : x, y \in F\}$$

for $f \in \overline{\mathbb{Q}}[X,Y]$ irreducible, and hence a $G_\delta$-set.

The set $\mathcal{H}$ is dense in $\mathcal{E}$, since every number field is Hilbertian (this is Hilbert's irreducibility theorem; see Serre [14, Theorem 3.4.1] or [7, Theorem 13.3.5]). Moreover, $\mathcal{H}$ is the intersection of the countably many open sets

$$V_{f_1,\dots,f_n,g} = \mathcal{E} \setminus \{F \in \mathcal{E} : f_1, \dots, f_n, g \in F[X,Y]\} \cup \bigcup_{x \in \overline{\mathbb{Q}}, g(x) \neq 0} \bigcap_{i=1}^{n} \bigcap_{y \in \overline{\mathbb{Q}}, f_i(x,y)=0} \{F \in \mathcal{E} : x \in F, y \notin F\},$$

where $n > 0$, $f_1, \dots, f_n \in \overline{\mathbb{Q}}[X,Y]$ monic of degree at least 2 in $Y$ and irreducible, and $0 \neq g \in \overline{\mathbb{Q}}[X]$. □

**Remark 2** By [7, Theorem 11.2.3], it would suffice to take $U_f$ with $f \in \mathbb{Q}[X,Y]$. The fact that the set of Hilbertian PAC fields $F \subseteq \overline{\mathbb{Q}}$ is dense in $\mathcal{E}$ could also be deduced directly by applying Jarden [8, Theorem 2.7] instead of the PAC Nullstellensatz.

**Proposition 3** *In an $\omega$-free PAC field $F$, every definable subring $R \subseteq F$ is a field.*

**Proof** An integral domain $R$ is partially ordered by the relation

$$a \preceq b \iff a = b \vee (a \mid b \wedge b \nmid a).$$

If $R$ is not a field, then the powers of a nonzero nonunit form an infinite chain with respect to $\preceq$, which shows that $R$ has the strict order property (see Shelah [15, Definition 2.1]; cf. the argument in Poizat [13, Chapter 1.2 Lemma 1]). The strict order property implies the strong order property SOP (see [15, Definition 2.2, Claim 2.3(1)]), which in turn implies the 3-strong order property $SOP_3$ (see [15, Definition 2.5, Claim 2.6]). However, $\omega$-free PAC fields do not have $SOP_3$ by a result of Chatzidakis (see Chatzidakis [2, Theorem 3.10]), and hence neither does any structure definable in them. □

**Remark 4**

1. The same conclusion holds if the PAC field $F$ is "bounded" (rather than $\omega$-free), for example, if $G_F$ is finitely generated, since then its theory is even *simple* (see Chatzidakis and Pillay [3, Corollary 4.8]); in particular, it does not have $SOP_3$ (see [15, Claim 2.7]).
2. Moreover, a PAC field of characteristic zero also has no definable proper sub-*fields* (see Junker and Koenigsmann [11, Lemma 6.1 and Proposition 4.1]).
3. It is known that $\omega$-free PAC fields satisfy not even the weaker property $SOP_1$ (rather than $SOP_3$) (see Chernikov and Ramsey [4, Corollary 6.8] and Kaplan and Ramsey [12, Section 9.3]).

**Corollary 5** *The set of subfields $F$ of $\overline{\mathbb{Q}}$ in which $\mathbb{Z}$ or $\mathcal{O}_F$ are definable is meager in $\mathcal{E}$.*

**Remark 6** The same arguments go through for separable algebraic extensions of $\mathbb{F}_p(t)$ instead of $\mathbb{Q}$. If one is interested only in $\mathbb{Z}$ not being *existentially* definable, then one could apply the much more elementary Fehm [6, Theorem 2] and Anscombe [1, Theorem 1], which work more generally for *large* fields, instead of Proposition 3.

**Remark 7** By combining Proposition 1 and Remark 4(2), we also obtain the following strengthening of [5, Corollary 5.8]. For every number field $K$, the set of fields $F \subseteq \overline{\mathbb{Q}}$ containing $K$ in which $K$ is definable is meager in $\mathcal{E}$.

**Remark 8** Similarly, we obtain the following strengthening of [5, Corollary 5.14]. If $\bar{\mathcal{E}}$ denotes the space $\mathcal{E}$ modulo isomorphism of fields, then the set of isomorphism classes of fields $F \subseteq \overline{\mathbb{Q}}$ in which $\mathbb{Z}$, $\mathcal{O}_F$, or some some fixed number field $K$ are definable is meager in $\bar{\mathcal{E}}$. Indeed, as the sets $\mathcal{P}$ and $\mathcal{H}$ (notation from the proof of Proposition 1) are dense $G_\delta$-sets invariant under isomorphism, and the quotient map $\mathcal{E} \to \bar{\mathcal{E}}$ is continuous and closed, also the images of $\mathcal{P}$ and $\mathcal{H}$ are dense $G_\delta$-sets, and therefore comeager in $\bar{\mathcal{E}}$.

**Remark 9** We sketch how a strengthening of [5, Theorem 5.11] can also be obtained. The set of computable and decidable fields $F \subseteq \overline{\mathbb{Q}}$ in which neither $\mathbb{Z}$ nor $\mathcal{O}_F$ are definable is dense in $\mathcal{E}$. Indeed, given finite extensions $\mathbb{Q} \subseteq K \subseteq L$, let $e$ be the minimal number of generators of the Galois group of the Galois closure $\hat{L}$ of $L/K$. By slightly adapting the proof of Jarden and Shlapentokh [10, Proposition 2.5], one finds a computable and decidable PAC field $K \subseteq F \subseteq \overline{\mathbb{Q}}$ with absolute Galois group free profinite on $e$ generators and $F \cap \hat{L} = K$, and Remark 4(1) applies to $F$.

## References


[1] Anscombe, S., "Existentially generated subfields of large fields," *Journal of Algebra*, vol. 517 (2019), pp. 78–94. Zbl 1431.12009. MR 3869267. DOI 10.1016/j.jalgebra.2018.09.021. 591

[2] Chatzidakis, Z., "Amalgamation of types in pseudo-algebraically closed fields and applications," *Journal of Mathematical Logic* vol. 19 (2019), art. 1950006. MR 4014886. DOI 10.1142/S0219061319500065. 590

[3] Chatzidakis, Z., and A. Pillay, "Generic structures and simple theories," *Annals of Pure and Applied Logic*, vol. 95 (1998), pp. 71–92, 1998. Zbl 0929.03043. MR 1650667. DOI 10.1016/S0168-0072(98)00021-9. 590

[4] Chernikov, A., and N. Ramsey, "On model-theoretic tree properties," *Journal of Mathematical Logic*, vol. 16 (2016), art. 1650009. MR 3580894. DOI 10.1142/S0219061316500094. 590

[5] Eisentraeger, K., R. Miller, C. Springer, and L. Westrick, "A topological approach to undefinability in algebraic extensions of $\mathbb{Q}$," preprint, arXiv:2010.09551v1 [math.NT]. 589, 591

[6] Fehm, A., "Subfields of ample fields: Rational maps and definability," *Journal of Algebra*, vol. 323 (2010), pp. 1738–44. Zbl 1258.14028. MR 2588135. DOI 10.1016/j.jalgebra.2009.11.037. 591

[7] Fried, M. D., and M. Jarden, *Field Arithmetic*, 3rd ed., Springer, Berlin, 2008. MR 2445111. 589, 590

[8] Jarden, M., "Large normal extension of Hilbertian fields," *Mathematische Zeitschrift*, vol. 224 (1997), pp. 555–65. Zbl 0873.12001. MR 1452049. DOI 10.1007/PL00004298. 590

[9] Jarden, M., *Algebraic Patching*, Springer, Heidelberg, 2011. Zbl 1235.12002. MR 2768285. DOI 10.1007/978-3-642-15128-6. 589

[10] Jarden, M., and A. Shlapentokh, "Decidable algebraic fields," *Journal of Symbolic Logic*, vol. 82 (2017), 474–88. Zbl 1387.12008. MR 3663413. DOI 10.1017/jsl.2017.10. 591

[11] Junker, M., and J. Koenigsmann, "Schlanke Körper (slim fields)," *Journal of Symbolic Logic*, vol. 75 (2010), pp. 481–500. Zbl 1218.03027. MR 2648152. DOI 10.2178/jsl/1268917491. 590

[12] Kaplan, I., and N. Ramsey, "On Kim-independence," *Journal of the European Mathematical Society*, vol. 22 (2020), pp. 1423–74. MR 4081726. DOI 10.4171/jems/948. 590

[13] Poizat, B., *Stable Groups*, American Mathematical Society, Providence, 2001. MR 1827833. DOI 10.1090/surv/087. 590

[14] Serre, J.-P., *Topics in Galois Theory*, Jones and Bartlett, Boston, 1992. MR 1162313. 590

[15] Shelah, S., "Toward classifying unstable theories," *Annals of Pure and Applied Logic*, vol. 80 (1996), pp. 229–55. Zbl 0874.03043. MR 1402297. DOI 10.1016/0168-0072(95)00066-6. 590

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*Dittmann*
Fakultät Mathematik, Institut für Algebra, Technische Universität Dresden, Dresden, Germany; philip.dittmann@tu-dresden.de

*Fehm*
Fakultät Mathematik, Institut für Algebra, Technische Universität Dresden, Dresden, Germany; arno.fehm@tu-dresden.de