A Convex Optimization Approach to Chance-Constrained Linear Stochastic Drift Counteraction Optimal Control

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Abstract— **In this paper, we propose a convex optimization approach to chance-constrained drift counteraction optimal control (DCOC) problems for linear systems with additive stochastic disturbances. Chance-constrained DCOC aims to compute an optimal control law to maximize the time duration before the probability of violating a prescribed set of constraints can no longer be maintained to be below a specified risk level. While conventional approaches to this problem involve solving a mixed-integer programming problem, we show that an optimal solution to the problem can also be found by solving a convex second-order cone programming problem without integer variables. We illustrate the application of chance-constrained DCOC to an automotive adaptive cruise control example.**

I. Introduction

Drift counteraction optimal control (DCOC) addresses a class of optimal control problems in which the objective is to maximize a cumulative yield function before system trajectory exits a prescribed set (called operating region) [1]. In this paper, we focus on DCOC problems where the time before occurrence of any constraint violation needs to be maximized. In continuous time, such problems are also known as "exit time" problems, and they have been studied extensively in the literature [2]–[4]. A conventional approach is to reduce the problem to a non-smooth Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) [5]. Compared to computing numerical solutions to such a PDE, the use of a discrete-time setting provides a more computationally tractable approach [1].

Discrete-time DCOC has been studied in both deterministic and stochastic settings. Approaches based on dynamic programming (DP) were developed in [1], [6], [7]. Although DP-based algorithms are capable of treating DCOC problems with general objectives represented by a variety of yield functions, such approaches are prone to the *curse of dimensionality* and become computationally prohibitive for reasonably high order systems [8]. For exit time problems, alternative approaches based on mixed-integer programming (MIP) were proposed in [9], [10]. The resulting MIP problems are computationally challenging in scenarios where longer planning horizons are needed due to two reasons: the number of integer variables grows linearly with the planning horizon length, and the worst-case complexity of the MIP problems is combinatorially related to the number of integer decision variables [11].

In the deterministic setting, approaches based on optimization with purely continuous variables were developed to reduce computational complexity. For linear systems with the prescribed operating region described by affine constraints, a continuous relaxation of the mixed-integer linear programming (MILP) approach has been introduced in [9]. By replacing integer variables with continuous ones, the relaxed linear programming (LP) problem can be solved efficiently. However, an optimal solution to the LP problem is only sub-optimal in terms of time-before-exit. Inspired by [12], we proposed a continuous nonlinear programming (NLP)-based approach in our recent work [13]. With an exponentially decreasing weighting scheme, our NLP-based approach encourages "late exit" by heavily penalizing early occurrence of constraint violations. It is shown that an optimal solution to the NLP problem achieves the maximum time-before-exit for the original DCOC problem.

For systems with stochastic disturbances, the stochastic DCOC problem of maximizing the expectation of time (or of a general cumulative yield function) before constraint violations has been considered from the perspective of mixed-integer nonlinear programming (MINLP) in [1], [7]. In particular, for linear systems with additive stochastic disturbances, an MILP-based model predictive control (MPC) formulation is developed in [10] based on a scenario tree approach inspired by [14]. In the inner loop of the stochastic MPC, an LP relaxation to the MILP problem was proposed as a fall back strategy in case the MILP problem fails to be solved effectively within allocated time [10]. Although the relaxed problem can always provide a feasible solution, the connections between the optimal solution to the MILP problem and the relaxed LP problem have not been studied.

In this paper, we propose a chance-constrained DCOC formulation for linear systems with additive stochastic disturbances. Instead of maximizing the average time-before-exit, the objective is to maximize the time duration before the probability of system trajectory exiting the prescribed set can no longer be maintained below a specified risk level $1-\beta$. The resulting optimal control problem is essentially a joint chance-constrained programming (JCCP) problem, which can be approximated by a deterministically constrained problem using suitable probability inequalities [15]. In this paper, Boole's inequality is first exploited to approximate the joint chance constraint as a set of individual chance constraints through risk allocation, and Chebyshev's inequality is then used to provide a conservative reformulation of the individual chance constraints into deterministic second-order cone constraints. Following the strategy in [13], we further convert the resulting deterministically constrained MIP problem to a second-order cone programming (SOCP) problem with only continuous variables, which can be efficiently solved by an off-the-shelf SOCP solver.

This paper is organized as follows. In Section II, we introduce the chance-constrained DCOC formulation for linear systems with additive stochastic disturbances. In Section III, we derive safe approximations to the chance-constrained DCOC problem using suitable probability inequalities and

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present the resulting deterministically constrained optimization problem. In Section IV, we reduce the problem in Section III to a convex SOCP problem with only continuous variables. In Section V, we illustrate the practical application of the proposed DCOC formulation and approach in an automotive adaptive cruise control example. The paper is concluded in Section VI.

II. Chance-constrained Stochastic Drift Counteraction Optimal Control

Consider a linear system subject to additive stochastic disturbances described by the following discrete-time model:

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state at the discrete time instant k, $u_k \in \mathbb{R}^{n_u}$ is the control input at k, and $w_k \in \mathbb{R}^{n_w}$ represents a stochastic disturbance input at k. We make the following assumptions on the disturbance w_k:

Assumption 1: The disturbance inputs $\{w_k\}_{k\in\mathbb{Z}_{\geq 0}}$ are independent random variables. Their distributions do not depend on initial state x_0 or control inputs $\{u_k\}_{k\in\mathbb{Z}_{\geq 0}}$, and their expectations $\mu_{w,k}$ and covariances $\Sigma_{w,k}$ are known and finite.

We consider a prescribed set $X \subset \mathbb{R}^{n_x}$ and an admissible control input set $U \subset \mathbb{R}^{n_u}$. In this paper, X and U are assumed to be polytopes defined by the following affine constraints:

$$X \triangleq \{x \in \mathbb{R}^{n_x} : G^T x \leq g\}, \quad U \triangleq \{u \in \mathbb{R}^{n_u} : S^T u \leq s\}. \tag{2}$$

For system (1) subject to stochastic disturbances, enforcing constraints deterministically is in general challenging, especially in the presence of disturbances with unbounded support. The "time-before-exit," defined as the last time instant before the state trajectory exits set X in [13] for a deterministic DCOC problem, now becomes a random variable whose realization depends on the realizations of stochastic disturbances $\{w_k\}_{k\in\mathbb{Z}_{\geq 0}}$. When formulating the stochastic DCOC problem, we introduce the concept of "β-level time-before-exit," which is defined as the last time instant κ_β such that the probability of the time-before-exit being greater than κ_β is greater than a pre-specified confidence level $\beta \in (0.5, 1)$. The objective of this paper is to define a control policy $u_k = \pi_k(w_k)$ that maximizes such a "β-level time-before-exit." Formally, the stochastic DCOC problem treated in this paper is stated as follows:

$$\max_{\{\pi_k\}_{k=0}^{N-1}, \kappa_\beta} \kappa_\beta, \tag{3a}$$

$$\text{subject to } u_k = \pi_k(w_k) \in U, \tag{3b}$$

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \tag{3c}$$

$$\mathbb{P}\left[\bar{\kappa}(x_0, \{u_k\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N-1}) \geq \kappa_\beta\right] \geq \beta, \tag{3d}$$

where $x_0 \in X$ is a given initial condition, and the random variable "time-before-exit" $\bar{\kappa}$ is defined as

$$\bar{\kappa}(x_0, \{u_k\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N-1}) = \max\{k \in \mathbb{Z}_{[0,N]} : \forall t \in \mathbb{Z}_{[1,k]}, x_t \in X\}. \tag{4}$$

Inspired by results in robust optimal control [16], [17], under the assumption that the disturbances $\{w_k\}_{k\in\mathbb{Z}_{\geq 0}}$ are measured onboard, we consider the following affine disturbance feedback control law:

$$u_k = \pi_k(w_k) = K_k w_k + h_k, \quad K_k \in \mathbb{R}^{n_u \times n_w}, \quad h_k \in \mathbb{R}^{n_u}. \tag{5}$$

When the above affine feedback law is used, the control input u_k becomes a random variable, which makes it difficult to enforce the control input constraints in (3b) deterministically. We address this by considering the following modified definition of "time-before-exit," where the control constraints (3b) are incorporated:

$$\kappa(x_0, \{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N-1}) = \max\{k \in \mathbb{Z}_{[0,N]} : \forall t \in \mathbb{Z}_{[1,k]}, x_t \in X, \forall t \in \mathbb{Z}_{[0,k-1]}, u_t \in U\}. \tag{6}$$

Under the affine disturbance feedback policy (5), we restate the stochastic DCOC problem as follows:

$$\max_{\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}, \kappa_\beta} \kappa_\beta, \tag{7a}$$

$$\text{subject to } u_k = K_k w_k + h_k, \tag{7b}$$

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \tag{7c}$$

$$\mathbb{P}\left[\kappa(x_0, \{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N-1}) \geq \kappa_\beta\right] \geq \beta. \tag{7d}$$

The following result shows that any feasible solution to the modified DCOC problem (7) leads to a feasible solution of the original problem (3) through a projection operation:

Lemma 1: Consider a projection operator $p : \mathbb{R}^{n_u} \to U$ which maps an arbitrary control input $u \in \mathbb{R}^{n_u}$ to its closest point in the admissible set U,

$$p(u) = \mathrm{argmin}_{v\in U} ||v - u||.$$

Denote a feasible solution to (7) as $(\{K_k^*\}_{k=0}^{N-1}, \{h_k^*\}_{k=0}^{N-1}, \kappa_\beta^*)$. Then the control policy $u_k = \pi_k(w_k) = p(K_k^* w_k + h_k^*)$ and κ_β^* provides a feasible solution to (3).

Proof: For a given realization of disturbance trajectory $\{w_k\}_{k=0}^{N-1}$, denote the control input trajectory under the disturbance feedback control law as $\{u_k^* = K_k^* w_k + h_k^*\}_{k=0}^{N-1}$, and the resulting state trajectory as $\{x_k^*\}_{k=0}^{N}$. Using projection map p, $u_k = p(K_k^* w_k + h_k^*) \in U$ guarantees that (3b) is satisfied. Denote the state trajectory associated with $\{u_k\}_{k=0}^{N-1}$ as $\{x_k\}_{k=0}^{N}$. Note that the projection map is an identity map when $u \in U$. Therefore, suppose $u_k^* \in U$ holds for $k \in \mathbb{Z}_{[0,\kappa_\beta^*-1]}$, then $u_k = u_k^*$ holds for $k \in \mathbb{Z}_{[0,\kappa_\beta^*-1]}$, and consequently, $x_k = x_k^*$ for $k \in \mathbb{Z}_{[1,\kappa_\beta^*]}$. Since $(\{K_k^*\}_{k=0}^{N-1}, \{h_k^*\}_{k=0}^{N-1}, \kappa_\beta^*)$ is a feasible solution to (7), we have $\mathbb{P}\left[\kappa(x_0, \{K_k^*\}_{k=0}^{N-1}, \{h_k^*\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N-1}) \geq \kappa_\beta^*\right] \geq \beta$. Furthermore, we have:

$$\begin{aligned}
&\beta \leq \mathbb{P}\left[\kappa(x_0, \{K_k^*\}_{k=0}^{N-1}, \{h_k^*\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N-1}) \geq \kappa_\beta^*\right] \\
&= \mathbb{P}\left[\bigcap_{k=1}^{\kappa_\beta^*} (G^T x_k^* \leq g, S^T u_{k-1}^* \leq s)\right] \leq \mathbb{P}\left[\bigcap_{k=1}^{\kappa_\beta^*} (G^T x_k \leq g, S^T u_{k-1} \leq s)\right] \\
&\leq \mathbb{P}\left[\bigcap_{k=1}^{\kappa_\beta^*} (G^T x_k \leq g)\right] = \mathbb{P}\left[\bar{\kappa}(x_0, \{u_k\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N-1}) \geq \kappa_\beta^*\right],
\end{aligned}$$

where we have used the fact that any control input sequence $\{u_k^*\}_{k=0}^{N-1}$ satisfying $S^T u_k^* \leq s$ for all $k \in \mathbb{Z}_{[0,\kappa_\beta^*-1]}$ leads to $u_k = u_k^*$ and $x_k = x_k^*$ to derive the inequality in the second line. Therefore, $(\{\pi_k\}_{k=0}^{N-1}, \kappa_\beta^*)$ where $\pi_k(w_k) = p(K_k^* w_k + h_k^*)$ is a feasible solution to (3). ■

In what follows, we deal with the modified stochastic DCOC problem (7).

III. Deterministically Constrained Safe Approximations

In this section, we convert the chance-constrained formulation of stochastic DCOC problem, (7), into a deterministically constrained optimization problem based on risk allocation using Boole's inequality. In particular, we show that any globally optimal solution to the converted deterministically constrained problem is guaranteed to be a feasible solution to the original chance-constrained problem (7). In this case, the deterministically constrained problem is called a safe approximation to the original problem [15]. We consider two models for the disturbance inputs w_k: Gaussian disturbance and general (non-Gaussian) disturbance, and derive their corresponding safe approximations.

A. Gaussian disturbance model for w_k

Lemma 2: Denote by $(\{K_k^*\}_{k=0}^{N-1}, \{h_k^*\}_{k=0}^{N-1}, \kappa_\beta^*)$ a globally optimal solution to (7). Then, a globally optimal solution to the following deterministically constrained optimization problem (8), $(\{K_k^{(8)}\}_{k=0}^{N-1}, \{h_k^{(8)}\}_{k=0}^{N-1}, \kappa_\beta^{(8)}, \{\alpha_{jk}^{(8)}\}, \{\beta_{ik}^{(8)}\})$, provides a feasible solution to (7), which satisfies $\kappa_\beta^{(8)} \leq \kappa_\beta^*$.

$$\max_{\substack{\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}, \kappa_\beta, \\ \{\alpha_{ik}\}, \{\beta_{jk}\}}} \kappa_\beta, \tag{8a}$$

$$\text{s.t.} \quad \begin{aligned} & G_i^T \mu_{x,k} + \Phi^{-1}(\alpha_{ik}) \sqrt{G_i^T \Sigma_{x,k} G_i} - g_i \leq 0, \\ & k \in \mathbb{Z}_{[1,\kappa_\beta]}, i \in \mathbb{Z}_{[1,n_g]}, \end{aligned} \tag{8b}$$

$$\begin{aligned} & S_j^T \mu_{u,k} + \Phi^{-1}(\beta_{jk}) \sqrt{S_j^T \Sigma_{u,k} S_j} - s_j \leq 0, \\ & k \in \mathbb{Z}_{[0,\kappa_\beta - 1]}, j \in \mathbb{Z}_{[1,n_s]}, \end{aligned} \tag{8c}$$

$$\sum_{k=1}^{\kappa_\beta} \left(\sum_{i=1}^{n_g} (1 - \alpha_{ik}) + \sum_{j=1}^{n_s} (1 - \beta_{j(k-1)}) \right) \leq 1 - \beta, \tag{8d}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable, $\{\alpha_{ik}\}$ and $\{\beta_{jk}\}$ represent $\{\alpha_{ik}\}_{k \in \mathbb{Z}_{[1,\kappa_\beta]}}^{i \in \mathbb{Z}_{[1,n_g]}}$ and $\{\beta_{jk}\}_{k \in \mathbb{Z}_{[0,\kappa_\beta - 1]}}^{j \in \mathbb{Z}_{[1,n_s]}}$, respectively, for simplicity, and $\mu_{x,k}, \Sigma_{x,k}, \mu_{u,k}, \Sigma_{u,k}$ are defined in (12) and (14).

Proof: Given a fixed number $\kappa_\beta \in \mathbb{Z}_{[0,N]}$, the probability constraint (7d) can be written as

$$\mathbb{P}\left[\bigcap_{k=1}^{\kappa_\beta} \left(G^T x_k \leq g, S^T u_{k-1} \leq s \right) \right] \geq \beta,$$

$$\mathbb{P}\left[\bigcup_{k=1}^{\kappa_\beta} \left[\left(\bigcup_{i=1}^{n_g} G_i^T x_k > g_i \right) \bigcup \left(\bigcup_{j=1}^{n_s} S_j^T u_{k-1} > s_j \right) \right] \right] \leq 1 - \beta. \tag{9}$$

Using Boole's inequality, a sufficient condition for (9) to hold is that there exists a set of probability values $\{\alpha_{ik}\}, \{\beta_{jk}\}$ such that the following chance constraints and probability inequality are satisfied,

$$\mathbb{P}(G_i^T x_k \leq g_i) \geq \alpha_{ik}, i \in \mathbb{Z}_{[1,n_g]}, k \in \mathbb{Z}_{[1,\kappa_\beta]}, \tag{10a}$$

$$\mathbb{P}(S_j^T u_k \leq s_j) \geq \beta_{jk}, i \in \mathbb{Z}_{[1,n_g]}, k \in \mathbb{Z}_{[0,\kappa_\beta - 1]}, \tag{10b}$$

$$\sum_{k=1}^{\kappa_\beta} \left(\sum_{i=1}^{n_g} (1 - \alpha_{ik}) + \sum_{j=1}^{n_s} (1 - \beta_{j(k-1)}) \right) \leq 1 - \beta. \tag{10c}$$

Given x_0, the state transition equation for the closed-loop system can be derived from (1) and (5),

$$x_k = \prod_{i=0}^{k-1} A_i x_0 + \sum_{i=0}^{k-1} \left(\prod_{j=i+1}^{k-1} A_j \right) B_i h_i + \sum_{i=0}^{k-1} \left(\prod_{j=i+1}^{k-1} A_j \right) (B_i K_i + D_i) w_i, \tag{11}$$

where x_k is affine in the disturbance inputs $\{w_i\}_{i=0}^{k-1}$. When $\{w_i\}_{i=0}^{k-1}$ follows independent Gaussian distributions, x_k also follows a Gaussian distribution whose expectation $\mu_{x,k}$ and covariance $\Sigma_{x,k}$ are computed as follows,

$$\mu_{x,k} = \left(\prod_{i=0}^{k-1} A_i \right) x_0 + \sum_{i=0}^{k-1} E_i^{(k)} B_i h_i + \sum_{i=0}^{k-1} E_i^{(k)} (B_i K_i + D_i) \mu_{w,i}, \tag{12a}$$

$$\Sigma_{x,k} = \sum_{i=0}^{k-1} \left[E_i^{(k)} (B_i K_i + D_i) \right] \Sigma_{w,i} \left[E_i^{(k)} (B_i K_i + D_i) \right]^T, \tag{12b}$$

$$E_i^{(k)} = \begin{cases} \prod_{j=i+1}^{k-1} A_j, & 0 \leq i < k-1, \\ I_{n_x} (\text{identity matrix}), & i = k-1. \end{cases} \tag{12c}$$

For $k \in \mathbb{Z}_{[0,\kappa_\beta]}$, The constraint (10a) can be written as

$$\alpha_{ik} \leq \mathbb{P}(G_i^T x_k \leq g_i) = \mathbb{P}\left(\frac{G_i^T x_k - G_i^T \mu_{x,k}}{\sqrt{G_i^T \Sigma_{x,k} G_i}} \leq \frac{g_i - G_i^T \mu_{x,k}}{\sqrt{G_i^T \Sigma_{x,k} G_i}} \right)$$

$$\iff G_i^T \mu_{x,k} + \Phi^{-1}(\alpha_{ik}) \sqrt{G_i^T \Sigma_{x,k} G_i} - g_i \leq 0. \tag{13}$$

Similarly, since u_k is an affine function of w_k, u_k also follows a Gaussian distribution, $u_k \sim \mathcal{N}(\mu_{u,k}, \Sigma_{u,k})$, where

$$\mu_{u,k} = K_k \mu_{w,k} + h_k, \quad \Sigma_{u,k} = K_k \Sigma_{w,k} K_k^T. \tag{14}$$

For $k \in \mathbb{Z}_{[0,\kappa_\beta - 1]}$, the constraint (10b) can be written as

$$S_j^T \mu_{u,k} + \Phi^{-1}(\beta_{jk}) \sqrt{S_j^T \Sigma_{u,k} S_j} - s_j \leq 0. \tag{15}$$

Therefore, satisfaction of (8b), (8c) and (8d) is a sufficient condition for (7d). Consequently, $(\{K_k^{(8)}\}_{k=0}^{N-1}, \{h_k^{(8)}\}_{k=0}^{N-1}, \kappa_\beta^{(8)})$ is a feasible solution to (7), and $\kappa_\beta^* \geq \kappa_\beta^{(8)}$. ■

B. General disturbance model for w_k

Without assuming w_k to be Gaussian, Chebyshev's inequality can be used to provide a distributionally robust but also more conservative safe approximation to (7).

Assumption 2: $\forall x_0 \in X$, there exists $\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}$ such that $\forall k \in \mathbb{Z}_{[1,\kappa_\beta]}, G_i^T \mu_{x,k} \leq g_i, S_j^T \mu_{u,k-1} \leq s_j$.

Lemma 3: Denote $(\{K_k^*\}_{k=0}^{N-1}, \{h_k^*\}_{k=0}^{N-1}, \kappa_\beta^*)$ as a globally optimal solution to (7). A globally optimal solution to the following optimization problem, $(\{K_k^{(16)}\}_{k=0}^{N-1}, \{h_k^{(16)}\}_{k=0}^{N-1}, \kappa_\beta^{(16)}, \{\alpha_{ik}^{(16)}\}, \{\beta_{jk}^{(16)}\})$, provides a feasible solution to (7), which satisfies $\kappa_\beta^{(16)} \leq \kappa_\beta^*$.

$$\max_{\substack{\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}, \kappa_\beta, \\ \{\alpha_{ik}\}, \{\beta_{jk}\}}} \kappa_\beta, \tag{16a}$$

$$\text{s.t.} \quad G_i^T \mu_{x,k} + \sqrt{\frac{G_i^T \Sigma_{x,k} G_i}{1 - \alpha_{ik}}} - g_i \leq 0, k \in \mathbb{Z}_{[1,\kappa_\beta]}, i \in \mathbb{Z}_{[1,n_g]}, \tag{16b}$$

$$S_j^T \mu_{u,k} + \sqrt{\frac{S_j^T \Sigma_{u,k} S_j}{1 - \beta_{jk}}} - s_j \leq 0, k \in \mathbb{Z}_{[0,\kappa_\beta - 1]}, j \in \mathbb{Z}_{[1,n_s]}, \tag{16c}$$

$$\sum_{k=1}^{\kappa_\beta} \left(\sum_{i=1}^{n_g} (1 - \alpha_{ik}) + \sum_{j=1}^{n_s} (1 - \beta_{j(k-1)}) \right) \leq 1 - \beta. \tag{16d}$$

Proof: The proof follows a similar procedure as in the proof of Lemma 2. Using (8d) and Boole's inequality, it is shown that (10) is a sufficient condition for (7d). Since a general disturbance model for w_k is considered, it is difficult to know the distribution of x_k and u_k a priori. However, the derivations of expectations and variances of x_k and u_k in (12) and (14) still hold, respectively. Therefore, we use Chebyshev's inequality to derive deterministic constraints that safely approximate the chance constraints. Under Assumption 2, $G_i^T \mu_{x,k} \le g_i$, using Chebyshev's inequality,

$$\begin{aligned}&\mathbb{P}(G_i^T x_k \le g_i) = 1 - \mathbb{P}(G_i^T x_k \ge g_i)\\ &\ge 1 - \mathbb{P}\left[|G_i^T x_k - G_i^T \mu_{x,k}| \ge g_i - G_i^T \mu_{x,k}\right] \ge 1 - \frac{G_i^T \Sigma_{x,k} G_i}{(g_i - G_i^T \mu_{x,k})^2}.\end{aligned}$$

Therefore, a sufficient condition for (10a) can be written as the following deterministic constraint,

$$1 - \frac{G_i^T \Sigma_{x,k} G_i}{(g_i - G_i^T \mu_{x,k})^2} \ge \alpha_{ik} \implies G_i^T \mu_{x,k} + \sqrt{\frac{G_i^T \Sigma_{x,k} G_i}{1-\alpha_{ik}}} - g_i \le 0. \tag{17}$$

Similarly, a sufficient condition for (10b) can be written as a deterministic constraint:

$$S_j^T \mu_{u,k} + \sqrt{\frac{S_j^T \Sigma_{u,k} S_j}{1-\beta_{jk}}} - s_j \le 0. \tag{18}$$

Thus, (16b), (16c) and (16d) provide a sufficient condition for (7d) to hold. Consequently, $(\{K_k^{(16)}\}_{k=0}^{N-1}, \{h_k^{(16)}\}_{k=0}^{N-1}, \kappa_\beta^{(16)})$ is a feasible solution to (7), and $\kappa_\beta^* \ge \kappa_\beta^{(16)}$. ■

IV. A Second-order Cone Programming Approach

We have shown that the deterministically constrained problems (8) and (16) are safe approximations to the original chance-constrained stochastic DCOC problem (7). In this section, we propose an SOCP-based approach to solving these problems. To simplify the exposition, we use $\mathscr{G}_{ik}(\mu_{x,k}, \Sigma_{x,k}) \le 0$ to represent either (8b) or (16b), and $\mathscr{S}_{jk}(\mu_{u,k}, \Sigma_{u,k}) \le 0$ to represent either (8c) or (16c). Furthermore, we remove $\{\{\alpha_{ik}\}, \{\beta_{jk}\}\}$ from optimization variables and treat them as prescribed confidence level values. Assigning prescribed values to $\{\{\alpha_{ik}\}, \{\beta_{jk}\}\}$ can significantly simplify the computation. In particular, given prescribed values of $\{\{\alpha_{ik}\}, \{\beta_{jk}\}\}$, it can be easily seen that (8b), (16b), (8c) and (16c) reduce to second-order cone constraints on $(\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1})$ [18]. Indeed, in many applications, it is also beneficial to be able to assign $\{\{\alpha_{ik}\}, \{\beta_{jk}\}\}$ values based on priority levels among system constraints. For example, high confidence values can be assigned to the α_{ik}, β_{jk} that are associated with safety-critical constraints to improve system safety.

Given prescribed individual chance constraint confidence level parameters $\{\{\alpha_{ik}\}, \{\beta_{jk}\}\}$, (8) and (16) can be expressed as the following deterministically constrained optimization problem:

$$\begin{aligned}
&\max_{\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}, \kappa_\beta} \kappa_\beta, && (19a)\\
&\text{s.t. } \mathscr{G}_{ik}(\mu_{x,k}, \Sigma_{x,k}) \le 0, k \in \mathbb{Z}_{[1,\kappa_\beta]}, i \in \mathbb{Z}_{[1,n_g]}, && (19b)\\
&\mathscr{S}_{jk}(\mu_{u,k}, \Sigma_{u,k}) \le 0, k \in \mathbb{Z}_{[0,\kappa_\beta-1]}, j \in \mathbb{Z}_{[1,n_s]}, && (19c)\\
&(\mu_{x,k}, \Sigma_{x,k}) = H_{x,k}(\{K_i\}_{i=0}^{k-1}, \{h_i\}_{i=0}^{k-1}), k \in \mathbb{Z}_{[1,\kappa_\beta]}, && (19d)\\
&(\mu_{u,k}, \Sigma_{u,k}) = H_{u,k}(K_k, h_k), k \in \mathbb{Z}_{[0,\kappa_\beta-1]}, && (19e)
\end{aligned}$$

where $H_{x,k}(\cdot), H_{u,k}(\cdot)$ are the expressions for the expectations and covariances of x_k, u_k derived in (12) and (14), respectively. Note that the above problem (19) has a similar form as a deterministic DCOC problem [13] and therefore can be solved using techniques developed previously for deterministic DCOC problems.

In particular, a deterministic DCOC problem can be solved using an MIP-based approach. Denote a globally optimal solution to (19) as $(\{K_k^*\}_{k=0}^{N-1}, \{h_k^*\}_{k=0}^{N-1}, \kappa_\beta^*)$. It is shown in [9] that a globally optimal solution to the following MIP problem, $(\{K_k^m\}_{k=0}^{N-1}, \{h_k^m\}_{k=0}^{N-1}, \{\delta_k^m\}_{k=0}^{N})$, satisfies $\delta_k^m = 0$ for $k = 0, ..., \kappa_\beta^*$, and $\delta_k^m = 1$ for $k = \kappa_\beta^* + 1, ..., N$, and it provides a globally optimal solution $(\{K_k^m\}_{k=0}^{N-1}, \{h_k^m\}_{k=0}^{N-1}, \kappa_\beta^*)$ to (19).

$$\begin{aligned}
&\min_{\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}, \{\delta_k\}_{k=0}^{N}} \Sigma_{k=0}^{N} \delta_k, && (20a)\\
&\text{s.t. } \mathscr{G}_{ik}(\mu_{x,k}, \Sigma_{x,k}) \le M\delta_k, && (20b)\\
&\mathscr{S}_{jk}(\mu_{u,t}, \Sigma_{u,k}) \le M\delta_k, && (20c)\\
&(\mu_{x,k}, \Sigma_{x,k}) = H_{x,k}(\{K_i\}_{i=0}^{k-1}, \{h_i\}_{i=0}^{k-1}), && (20d)\\
&(\mu_{u,k}, \Sigma_{u,k}) = H_{u,k}(K_k, h_k), && (20e)\\
&\delta_k \in \{0, 1\}, && (20f)\\
&\delta_k \le \delta_{k+1}, && (20g)
\end{aligned}$$

where M is a sufficiently large positive number.

Moreover, it is shown in [13] that the above MIP formulation of our DCOC problem can be solved using a continuous optimization approach based on an exponentially decreasing weighting scheme. This continuous optimization-based approach has significant computational advantage over the MIP-based approach. We now present the application of this continuous optimization-based approach to our deterministically constrained formulation (19) of stochastic DCOC problem in the following theorem:

Theorem 1: Consider the following continuous nonlinear programming problem,

$$\begin{aligned}
&\min_{\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1}, \{\varepsilon_k\}_{k=0}^{N}} \Sigma_{k=0}^{N} \theta^{N-k} \varepsilon_k, && (21a)\\
&\text{s.t. } \mathscr{G}_{ik}(\mu_{x,k}, \Sigma_{x,k}) \le M\varepsilon_k, && (21b)\\
&\mathscr{S}_{jk}(\mu_{u,t}, \Sigma_{u,k}) \le M\varepsilon_k, && (21c)\\
&(\mu_{x,k}, \Sigma_{x,k}) = H_{x,k}(\{K_i\}_{i=0}^{k-1}, \{h_i\}_{i=0}^{k-1}), && (21d)\\
&(\mu_{u,k}, \Sigma_{u,k}) = H_{u,k}(K_k, h_k), && (21e)\\
&0 \le \varepsilon_k \le \varepsilon_{k+1}, && (21f)
\end{aligned}$$

where $\theta > 1$ is a weighting parameter.

Given $x_0 \in X$, there exists a number $\theta_0 > 1$ such that if $\theta \ge \theta_0$, then any globally optimal solution to (21), $(\{K_k^o\}_{k=0}^{N-1}, \{h_k^o\}_{k=0}^{N-1}, \{\varepsilon_k^o\}_{k=0}^{N})$, satisfies $\varepsilon_k^o = 0$ for all $k = 0, ..., \kappa_\beta^*$, and $\varepsilon_k^o > 0$ for all $k = \kappa_\beta^* + 1, ..., N$, where κ_β^* is the maximum time-before-exit obtained through a globally optimal solution to (19), $(\{K_k^*\}_{k=0}^{N-1}, \{h_k^*\}_{k=0}^{N-1}, \kappa_\beta^*)$. Furthermore, κ_β^* is a lower bound for $\kappa_\beta^{(3)}$, which corresponds to a globally optimal solution to (3), i.e., $\kappa_\beta^* \le \kappa_\beta^{(3)}$.

Proof: By Theorem 1 in [13], there exists a number $\theta_0 > 1$ such that if $\theta \ge \theta_0$, $(\{K_k^o\}_{k=0}^{N-1}, \{h_k^o\}_{k=0}^{N-1}, \{\varepsilon_k^o\}_{k=0}^{N})$ satisfies $\varepsilon_k^o = 0, k = 0, ..., \kappa_\beta^*$, and $\varepsilon_k^o > 0$ for all $k \in \mathbb{Z}_{[\kappa_\beta^*+1,N]}$. For Gaussian (non-Gaussian) disturbances w_k, Lemma 2 (3) shows $\kappa_\beta^* \le \kappa_\beta^{(7)}$, where $\kappa_\beta^{(7)}$ corresponds to a globally optimal solution to (7), $(\{K_k^{(7)}\}_{k=0}^{N-1}, \{h_k^{(7)}\}_{k=0}^{N-1}, \kappa_\beta^{(7)})$. Using Lemma 1, $\kappa_\beta^{(7)} \le \kappa_\beta^{(3)}$, therefore, $\kappa_\beta^* \le \kappa_\beta^{(3)}$. ■

Based on Theorem 1, we can enlarge the time-before-exit of our original stochastic DCOC problem (3) by enlarging its lower bound κ_β^* through solving the continuous optimization problem (21). Note also that the deterministic constraints (21b) and (21c) are both convex second-order cone constraints of $(\{K_k\}_{k=0}^{N-1}, \{h_k\}_{k=0}^{N-1})$. Since $\{\varepsilon_k\}_{k=0}^{N-1}$ enters the problem linearly, (21) is a convex second-order cone programming problem with pure continuous variables.

V. Numerical Example

A case study of chance-constrained DCOC is developed based on an adaptive cruise control example to demonstrate the SOCP-based approach in (21). We consider a scenario where a passenger car is following a lead vehicle, the velocity of which is modeled as a stochastic disturbance (see Fig. 1). The objective is to compute an optimal disturbance feedback control law for the time rate of change in the follower vehicle's longitudinal acceleration, which maintains the system states within specified constraints for as long as possible.

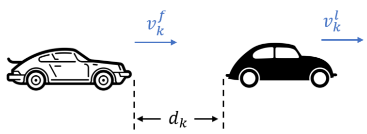

Fig. 1: Diagram for the ACC example.

The dynamics of the system can be represented by the following discrete-time linear system,

$$d_{k+1} = d_k + (v_k^l - v_k^f)\Delta t, \tag{22a}$$

$$v_{k+1}^f = v_k^f + a_k^f \Delta t, \tag{22b}$$

$$a_{k+1}^f = a_k^f + j_k \Delta t, \tag{22c}$$

where k represents the time instant, Δt [sec] refers to the sampling period, d [m] is the distance between the lead vehicle and the follower vehicle at time k, v_k^l, v_k^f [m/s] denote the longitudinal speeds of the lead and follower vehicle at k, respectively, a_k^f $[m \cdot s^{-2}]$ is the longitudinal acceleration of the follower vehicle at k, and j_k $[m \cdot s^{-3}]$ is the jerk of the follower vehicle, i.e. time rate of change of a_k^f at k.

In this example, we describe the behavior of the lead vehicle by modelling its velocity v_k^l as a Gaussian random variable, i.e. $v_k^l = w_k \sim \mathcal{N}(\mu_{w,k}, \Sigma_{w,k})$, and we assume the values of $\mu_{w,k}$ and $\Sigma_{w,k}$ are known. The state vector of the system is defined as $x_k = [d_k, v_k^f, a_k^f]^T$, and the control input is defined as $u_k = j_k$. We consider the following constraints imposed on state variables and control inputs:

1) The time-headway of the follower vehicle $T_{gap,k} = d_k / v_k^f$ needs to satisfy
$$T_{g,min} \le T_{gap,k} \le T_{g,max}, \tag{23}$$
where $T_{g,min} = 1.5$ reflects minimum distance requirement for safety, and $T_{g,max} = 2.5$ limits the distance to mitigate impact on traffic flow and avoid other vehicles cutting-in that can degrade safety and fuel efficiency.
2) The follower vehicle's longitudinal speed needs to be maintained within the speed limits,
$$60 \cdot 0.44704 \le v_k^f \le 70 \cdot 0.44704, \tag{24}$$
where 0.44704 is the converting factor from miles-per-hour (MPH) to meters-per-second (m/s).
3) The follower vehicle's acceleration is constrained due to fuel economy concerns,
$$-1 \le a_k^f \le 1. \tag{25}$$
4) Since ride comfort is affected by the time rate of change in acceleration [19], u_k needs to satisfy
$$-1 \le u_k \le 1. \tag{26}$$

The lead vehicle's longitudinal speeds $\{v_k^l\}_{k=0}^{N-1}$ are modelled as independent and identically distributed Gaussian variables and represent the disturbances to the system $\{w_k\}_{k=0}^{N-1}$. The Gaussian distribution of w_k is characterized by its expectation $\mu_{w,k} = 72 \cdot 0.44704$ and covariance $\Sigma_{w,k} = (5 \cdot 0.44704)^2$, which describes a scenario where the follower vehicle is following a lead vehicle driving slightly over speed limit. The initial condition of the system is specified as $d_0 = 65 \cdot 0.44704 \cdot 2.2$, $v_0^f = 65 \cdot 0.44704$, and $a_0^f = 0$. The confidence levels for each chance constraint $\{\alpha_{ik}, \beta_{jk}\}$ are defined as constant values over time according to the priority of their corresponding requirements. In particular, for constraints on the time-headway (23), $\alpha_{1k} = \alpha_{2k} = 0.85$, for constraints on the follower vehicle's speed (24), $\alpha_{3k} = \alpha_{4k} = 0.75$, for constraints on the follower vehicle's acceleration (25), $\alpha_{5k} = \alpha_{6k} = 0.95$, and for control constraints (26), $\beta_{1k} = \beta_{2k} = 0.95$. The sampling period is chosen as $\Delta t = 0.25$. Then the continuous NLP problem is formulated as (21) with $N = 40$ and $\theta = 1.2$.

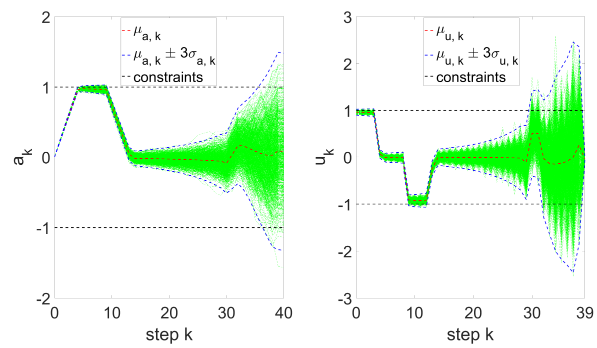

Fig. 2: Illustration of distributions of a_k^f and u_k.

The SOCP problem (21) is solved using YALMIP toolbox [20] and Gurobi solver [21] in MATLAB 2019b environment. The distributions of a_k^f and u_k are computed and plotted in Fig. 2, where the red dashed lines refer to the mean value trajectories, the blue dashed lines represent mean trajectory ± 3 standard deviations, the black dashed lines represent the constraints, and the green solid lines are 1000 simulated trajectories under the feedback control law with random realizations of the stochastic disturbances.

In this example, the time gap keeps increasing over time, and constraint (23) will eventually be violated since the lead vehicle exceeds the speed limit imposed on the follower vehicle. It is observed in Fig. 2 that the mean trajectory of acceleration stays positive for most time instants to increase the follower vehicle's longitudinal speed. Furthermore, before $k = 15$, the covariance of acceleration stays small such that accelerations (green trajectories) can take large values without growing outside the bounds (black dashed lines), which matches the fact that high confidence level values $\alpha_{5k} = \alpha_{6k} = 0.95$ are assigned to constraints (25). In Fig. 3, the probabilities of several constraints being satisfied are presented, where the blue dashed lines are the analytical predictions of probabilities of constraint satisfactions computed using cumulative distribution function of Gaussian

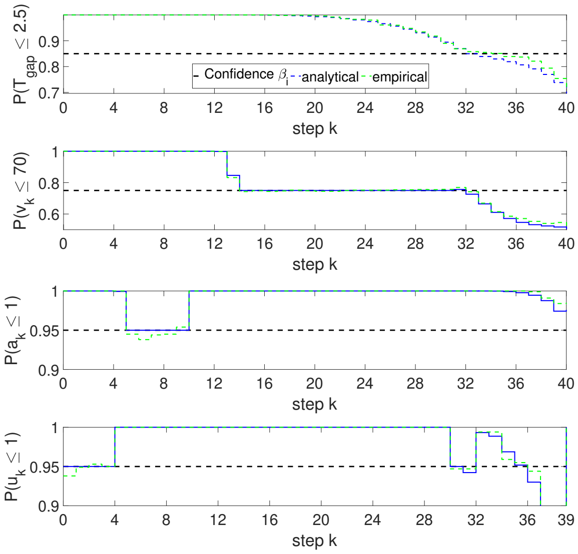

Fig. 3: Analytical predictions and empirical estimates of constraint satisfaction probabilities.

variables, the green dashed lines represent estimates of these probabilities using empirical data acquired from simulation trajectories generated by 1000 realizations of the stochastic disturbances, and the black dashed lines represent the prescribed confidence level values.

The safe approximation to the maximum "β-level time-before-exit" using the SOCP problem with confidence level values specified previously is $k = 32$. As illustrated in Fig. 3, the first chance constraint violation happens at $k = 33$ where $P(v_k \leq 70) < 0.75$. The MIP problem (20) is also solved in the same software environment for comparison to the SOCP-based approach. The safe approximation to the "β-level time-before-exit" given by the solution to the MIP problem is the same as the one found in the SOCP problem, $k = 32$. The computation times to compute one instance of the MIP problem (20) and the SOCP problem (21) with $\Delta t = 0.25$ and $N = 40$ are 11.5959 and 4.9263 seconds, respectively. To better depict the comparison between these two approaches, the computation times for one instance of the MIP and SOCP problems with horizon length $N = 30, 40, 50, 60, 70$ are plotted in Fig. 4. It is observed that the ratio between the computation times of solving MIP and SOCP problems grows with the length of the horizon, which demonstrates benefit in computational efficiency of the proposed SOCP-based approach.

VI. Conclusions

A convex optimization-based approach to DCOC for linear systems with additive stochastic disturbances has been proposed in this paper. A chance-constrained DCOC formulation is introduced and a safe approximation is defined under a disturbance-feedback control policy and exploiting risk allocation. Two approaches were then presented to further reduce the problem to a deterministically-constrained optimization problem either by modeling disturbances as Gaussian variables or by using Chebyshev's inequality. Inspired by recent results in deterministic DCOC, an SOCP-based approach without integer variables was developed to solve the resulting deterministically-constrained optimization problem and compared with a conventional MIP-based approach. Numerical examples demonstrate the effectiveness of our SOCP-based approach and its improvement in computational efficiency over the conventional MIP-based approach.

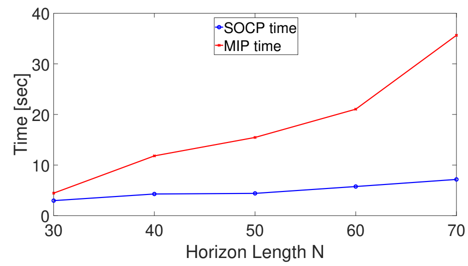

Fig. 4: Comparison of computation times.

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