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Algorithms for Intersection Graphs of t-Intervals and t-Pseudodisks

Chandra Chekuri* Tanmay Inamdar†

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Abstract. Intersection graphs of planar geometric objects such as intervals, disks, rectangles and pseudodisks are well-studied. Motivated by various applications, Butman et al. (ACM Trans. Algorithms, 2010) considered algorithmic questions in intersection graphs of t-intervals. A t-interval is a union of t intervals—these graphs are also referred to as multiple-interval graphs. Subsequent work by Kammer et al. (APPROX-RANDOM 2010) considered intersection graphs of t-disks (union of t disks), and other geometric objects. In this paper we revisit some of these algorithmic questions via more recent developments in computational geometry. For the *minimum-weight dominating set problem* in t-interval graphs, we obtain a polynomial-time $O(t \log t)$-approximation algorithm, improving upon the previously known polynomial-time t^2-approximation by Butman et al. (op. cit.). In the same class of graphs we show that it is NP-hard to obtain a $(t-1-\epsilon)$-approximation for any fixed $t \geq 3$ and $\epsilon > 0$.

The approximation ratio for dominating set extends to the intersection graphs of a collection of t-pseudodisks (nicely intersecting t-tuples of closed Jordan domains). We obtain an $\Omega(1/t)$-approximation for the *maximum-weight independent set* in the

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intersection graph of t-pseudodisks in polynomial time. Our results are obtained via simple reductions to existing algorithms by appropriately bounding the union complexity of the objects under consideration.

1 Introduction

A number of interesting optimization problems can be modeled as packing and covering problems involving geometric objects in the plane such as intervals, disks, rectangles, triangles, convex polygons and pseudodisks[1]. Some of these problems can be studied via properties of the intersection graph[2] defined by the objects under consideration. Geometric intersection graphs are interesting for both theoretical and practical reasons. For instance, the well-known Koebe–Andreev–Thurston theorem shows that every planar graph can be represented as the intersection graph of interior-disjoint disks in the plane (see Sections 13.6 and 13.7 of [34]).

Interval graphs are another well-studied class of geometric intersection graphs that are defined by a finite set of intervals on the real line. Algorithmic problems on interval graphs have been motivated by applications in scheduling, resource allocation, and computational biology.

Several papers [8, 9, 10, 26] have studied geometric intersection graphs in the setting where each composite object is now the union of a set of base geometric objects. To make the discussion concrete we first discuss t-interval graphs. For an integer parameter $t \geq 1$, a *t-interval* is the union of t intervals. A *t-interval graph* is the intersection graph of a set of t-intervals. These graphs are also called multiple-interval graphs. They have been well-studied from graph theoretic and algorithmic points of view. For instance, every graph with maximum degree Δ can be represented as a t-interval graph for $t = \lceil (\Delta + 1)/2 \rceil$ [21]. This demonstrates the modeling power obtained by considering unions of simple geometric objects. Butman et al. [10], building on [8, 9] (which primarily studied the maximum independent set problem), considered several optimization problems in t-interval graphs such as minimum vertex cover, dominating set and maximum clique. Unlike the case of interval graphs, where these problems are tractable, the corresponding problems in t-interval graphs are NP-hard even for small values of t and unweighted instances; this can be seen from the preceding comment on the modeling power of t-interval graphs, and the NP-hardness of several problems on bounded-degree graphs. Butman et al. describe polynomial-time approximation algorithms for these problems, and the approximation ratios they obtain depend on t. We refer the reader to [10] and references therein for a more detailed discussion on applications of multiple-interval graphs. In subsequent work, Kammer et al. [26, 25] studied (among other models) intersection graphs of t-disks (a *t-disk* is a union of t disks) and *t-fat objects*[3]. They obtained approximation algorithms for optimization problems on these graphs such as maximum independent set, minimum vertex cover and

[1]A *collection of pseudodisks* is a set of closed Jordan domains in which the boundaries of any two in the set intersect in at most two points, counting a point of tangency as a double intersection. Note that the property is defined by the entire set. See also Def. 2.4.

[2]The intersection graph of a set of sets has a node for each set and two nodes are adjacent if the corresponding sets intersect (their intersection is not empty).

[3]There are several equivalent definitions of *fatness* in the plane. We say that a geometric object O is α-fat for some $\alpha \geq 1$, if the ratio of the radius of the smallest enclosing disk of O to the radius of the largest disk enclosed by O, is at

minimum dominating set. We also refer the reader to [38] for connections between geometric intersection graphs and approximation algorithms.

We now formally define the three problems that are central to this paper. Let $G = (V, E)$ be an undirected graph with non-negative rational weights on the nodes. $S \subseteq V$ is said to be a *dominating set* if for any node $u \in V$, $u \in S$ or u has a neighbor in S. The *Minimum-Weight Dominating Set (MWDS)* problem is to find a dominating set of minimum weight in a given weighted graph. A subset of nodes, $I \subseteq V$, is said to be an *independent set* if no two nodes in I are adjacent in G. The *Maximum-Weight Independent Set (MWIS)* problem is to find an independent set of maximum weight in a given weighted graph. Consider a set system $(X, \mathcal{S})$, where X is a finite ground set, and $\mathcal{S}$ is a set of subsets of X. Each set in $\mathcal{S}$ has a non-negative rational weight. The *Minimum-Weight Set Cover (MWSC)* problem is to find a set $\mathcal{R}^* \subseteq \mathcal{S}$ of minimum weight such that $X = \bigcup_{S_i \in \mathcal{S}} S_i$. We denote the unweighted versions (that is, all weights are unit) of the MWDS, MWIS, and MWSC problems by MDS, MIS, and MSC, respectively.

Convention 1.1 (rational weights)**.** In all our results, the weights are rational numbers.

Remark 1.2. The focus in this paper is on polynomial-time approximation algorithms for a class of NP-hard optimization problems. As common in this literature, we use the term "approximation algorithm" to refer to a *deterministic polynomial-time approximation algorithm* unless we explicitly mention the running time or additional aspects such as the use of randomness or specific oracles.

Our assumption that the *weights are rational* (as opposed to real) is necessitated by the fact that our algorithms rely on solving linear programming relaxations. The emphasis in this paper is on the approximation ratio achievable via the LP relaxation approach. For these reasons we do not emphasize the precise polynomial running times of the algorithms.

1.1 Motivation and our contribution

In this paper we utilize powerful techniques from computational geometry [11, 12, 35] to provide algorithmic results for t-interval graphs, t-disks and other geometric objects, in a unified fashion. For some problems we obtain substantially improved approximation bounds that are near-optimal. Our results extend to t-pseudodisks while techniques in earlier work that exploited properties of intervals [10], or fatness properties of the underlying objects [26], do not apply.

Before stating our results in full generality, we discuss the following geometric covering problem to illustrate some of the basic ideas. Given a set of points on the line, and a set of weighted intervals, find a minimum-weight subset of the given intervals that cover all the points. This is a special case of MWSC, and can be solved efficiently via dynamic programming or via mathematical programming. The natural LP relaxation in the interval case happens to yield an integer polytope—the incidence matrix between intervals and points is totally unimodular (TUM) (it has the consecutive ones property) [33]. Now consider the same problem where we need to cover a given set of points by (weighted) t-intervals. Approximation algorithms

most α. We say that $\mathcal{R}$ is a *set of fat objects,* if there exists a constant α such that every object in $\mathcal{R}$ is α-fat.

for this problem were first given by Hochbaum and Levin [23] (in the more general setting of multicover)—they derived a t-approximation for this problem by reducing it, via the natural LP relaxation, to the case of $t = 1$. As far as we are aware, this was the best known approximation to this problem until our work. A natural question is whether the approximation ratio of t can be improved. In this paper we show that an $O(\log t)$-approximation can be obtained via tools from computational geometry such as shallow-cell complexity and quasi-uniform sampling. It is an easy observation that an MWSC instance in which each set has at most t elements can be captured as a special case of covering points by t-intervals. Thus, for large values of t, via known hardness results for MSC [30, 18], one obtains NP-hardness of $(c \log t)$-approximation for some universal constant $c > 0$ for covering points by t-intervals. The geometric machinery allows us to derive an $O(\log t)$-approximation in the much more general setting of covering points by t-pseudodisks.

We state our results for t-pseudodisks which capture several shapes of interest, including intervals and disks. The geometric approach applies in greater generality but here we confine our attention to pseudodisks.

- An $O(\log t)$-approximation for minimum-weight cover of points by t-pseudodisks, in other words, the instances of MWSC defined by points and t-pseudodisks in the plane (Theorem 3.9).

- An $O(t \log t)$-approximation for MWDS in t-pseudodisk graphs (Theorem 3.7). Even for t-intervals, the best known previous approximation factor was t^2 [10]. We observe that, via a simple reduction from Hypergraph Vertex Cover, it is NP-hard to approximate MDS within a factor better than $(t - 1 - \epsilon)$ for any fixed $\epsilon > 0$ in t-interval graphs. We also show UG-hardness[4] for reaching the slightly improved approximation ratio of $(t - \epsilon)$ (Corollary 3.13).

- An $\Omega(1/t)$-approximation for MWIS in t-pseudodisk graphs (Theorem 4.4). A $1/(2t)$-approximation has been known for t-intervals [9]. For more general shapes such as disks and fat objects, the approximation bounds in [25] depend on t and on the fatness parameters. (See [25] for the actual definition.) Our approach also works for packing weighted t-pseudodisks into capacitated points (Theorem 4.5).

Our results are obtained via simple reductions to existing algorithms in geometric packing and covering, however, the known algorithms use fairly sophisticated ideas. Consequently, in some cases, the constants in the approximation guarantees we obtain may be worse compared to the known results for special cases. On the other hand, our results are applicable to a much more general class of geometric objects. It may be possible to improve the constants for various special cases by examining the details more closely.

We describe the necessary geometric background in the next subsection.

[4] UG refers to Khot's "Unique Games" problem [28, 29].

1.2 Background from geometric approximation via LP relaxations

The approximability of MWDS and MWIS in general graphs is well understood and essentially tight upper and lower bounds are known. For MWDS there is an approximation ratio of $(1 + \ln n)$ where n is the number of nodes [36, 37], and moreover, it is NP-hard to obtain an $((1 - o(1)) \ln n)$-approximation [30, 18] even in the unweighted setting. In general graphs, it is NP-hard to approximate MIS to within a factor of $1/n^{1-\epsilon}$ for any fixed $\epsilon > 0$ [22, 39], and the best known approximation ratio is $\Omega((\log^c n)/n)$ for some small integer constant c (see [19, 6]). MWDS and MWSC are closely related and their approximability is essentially the same.

The preceding results rule out constant-factor approximations for MWDS and MWIS in graphs, even in the unweighted case. However, in various geometric settings it is possible to obtain substantially improved algorithms including approximation schemes (PTASes and QPTASes) [12, 20, 11, 2, 32, 1] and constant-factor approximations [12, 11, 20]. In this paper we are interested in LP-based approximations for MWSC and MWIS that have been established via techniques relying on the *union complexity* of the underlying geometric objects. Union complexity measures the worst-case representation size of the union of a given set of objects of a particular type or shape. In the setting of planar objects, the typical measure is the number of vertices in the arrangement that appear on the boundary of the union. It is well known that many geometric objects such as intervals (on a line), disks, and squares (in the plane) have linear union complexity. In fact, this holds for an even larger class of sets of geometric objects, namely collections of pseudodisks [3, 27].

Bounds on union complexity have been used to obtain constant-factor and sublogarithmic approximations for geometric MWSC and its variants (see [15, 13, 35, 11], and [31] for a survey). Chan and Har-Peled [12] showed that upper bounds on union complexity can also be used to obtain improved approximations for the MWIS problem. They give an LP rounding algorithm with approximation guarantee $\Omega(n/u(n))$ for computing an MWIS of n objects where $u(n)$ is an upper bound on the worst-case union complexity of n objects under consideration. We use this result to give an $\Omega(1/t)$-approximation for computing MWIS of t-pseudodisks. This implies $\Omega(1/t)$-approximation for MWIS of t-intervals, t-disks and t-squares; however, these results have already been known [8, 26] but these previous results are based on using certain "fatness" properties of the underlying geometric objects, and hence do not apply to arbitrary pseudodisks.

Shallow-cell complexity (SCC) of a set system provides quantitative bounds on a certain hereditary sparsity property [11]. A formal definion is given later in the paper. Upper bounds on SCC have been used to obtain improved approximations for MWSC and MWDS in geometric settings as well as some combinatorial settings. The general approach here is to round a feasible LP solution using a framework called *quasi-uniform sampling* introduced by Varadarajan [35] for geometric settings. This framework was refined and improved by Chan et al. [11]. We use this framework, and some known results on SCC for disks and pseudodisks [20, 5], to derive our results for MWDS and MWSC on t-pseudodisks.

Organization. Section 2 introduces some relevant notation and definitions. Section 3 describes algorithms for covering problems MWDS and MWSC. Section 4 describes algorithms for MWIS and a generalization.

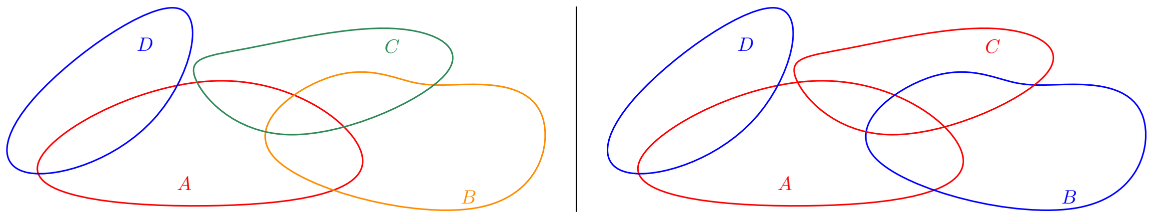

Figure 1: Left: $\mathcal{S}' = \{A, B, C, D\}$ is a set of closed Jordan domains that intersect nicely, and hence is a collection of pseudodisks. Right: $\mathcal{S} = \{A \cup C, B \cup D\}$ is a collection of 2-pseudodisks.

2 Preliminaries

Let C be a family of sets we call "basic objects." Given C, a t-object is defined as the union of at most t basic objects, where t is a positive integer. We do *not* assume that the t basic objects that define a specific t-object are pairwise non-intersecting; such an assumption may be helpful in some settings but can be restrictive in others. Without loss of generality, we assume that each t-object is a union of exactly t basic objects. We are typically interested in the case when the basic objects come from a specific class of geometric shapes such as intervals, disks, and pseudodisks (closed Jordan domains).

Definition 2.1 (closed Jordan domain)**.** A subset of the plane is a *closed Jordan domain* if it is a compact subset of $\mathbb{R}^2$ whose boundary is a simple closed curve.

Definition 2.2 (nice intersection)**.** We say that two closed Jordan domains intersect *nicely* if their boundaries are either (i) disjoint or (ii) properly intersect exactly twice or (iii) are tangent exactly once.

Definition 2.3 (t-pseudodisk)**.** For $t \geq 1$, a t-pseudodisk is the union of at most t closed Jordan domains that pairwise intersect nicely.

Definition 2.4 (collection of pseudodisks and t-pseudodisks)**.** Let $\mathcal{S}'$ be a set of closed Jordan domains in the plane. We call $\mathcal{S}'$ a *collection of pseudodisks* [5] if the boundaries of any two distinct pseudodisks in $\mathcal{S}'$ intersect nicely. Let $\mathcal{S}'$ be a collection of pseudodisks, and let $\mathcal{S}$ be a set of objects where each object in $\mathcal{S}$ is the union of at most t objects from $\mathcal{S}'$. We say that $\mathcal{S}$ is a *collection of t-pseudodisks.*

Remark 2.5. When speaking of a collection $\mathcal{S}$ of t-pseudodisks, we understand that there is an underlying collection $\mathcal{S}'$ of pseudodisks.

The definition allows two different t-pseudodisks in $\mathcal{S}$ to share a common pseudodisk from $\mathcal{S}'$.

Consider a t-object O_i. We use $o_i^{(1)}, o_i^{(2)}, \ldots, o_i^{(t)}$ to denote the objects whose union equals O_i. Slightly abusing the notation, we also denote by O_i the set $\{o_i^{(1)}, o_i^{(2)}, \ldots, o_i^{(t)}\}$. Next, we formally define the notion of union complexity.

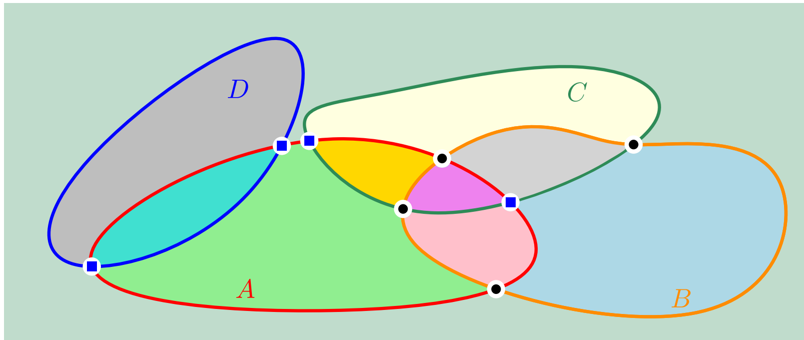

Figure 2: The vertices, elementary arcs, and faces of dimension two formed by the collection $\mathcal{S}' = \{A, B, C, D\}$ of pseudodisks are shown as points (filled squares/circles), arcs, and solid regions, respectively. For $\mathcal{R}' = \{A, C, D\}$ the vertices appearing on $\mathcal{U}(\mathcal{R}')$ are shown as blue squares.

2.1 Union complexity

The following definitions are adapted from [3, pp. 13-14]. Let $\mathcal{S}'$ be a collection of pseudodisks. Let $\mathcal{U}(\mathcal{S}') = \bigcup_{S \in \mathcal{S}'} S$ denote their union and let $\partial\mathcal{U}(\mathcal{S}')$ denote the boundary of the union. Each *face* of $\partial\mathcal{U}(\mathcal{S}')$ is a maximal connected (relatively open) subset of $\partial\mathcal{U}(\mathcal{S}')$ that lies on the boundaries of a fixed subset $\mathcal{R}' \subseteq \mathcal{S}'$, and avoids all other pseudodisks in in $\mathcal{S}'$. The faces of dimension 0 and 1 are called *vertices* and *elementary arcs*, respectively. We say that $\mathcal{S}'$ has *union complexity* $u(\cdot)$ for a function $u : \mathbb{Z}_+ \to \mathbb{Z}_+$ if the following condition is true: for any subset $\mathcal{R}' \subseteq \mathcal{S}'$, the number of vertices on $\partial\mathcal{U}(\mathcal{R}')$ is at most $u(|\mathcal{R}'|)$.

In the rest of the paper we assume that $u(n) \geq n$ for any $n \geq 1$ and that u is non-decreasing. These are natural assumptions for the settings we consider.

For example, a set of disks—and more generally, a collection of pseudodisks—in the plane have linear union complexity [3, 27]. The union complexity of n *fat triangles* is $O(n \log^* n)$ [4]. The union complexity of n axis-aligned rectangles in the plane can be $\Omega(n^2)$.

3 Minimum-Weight Dominating Set and Set Cover

As in Butman et al. [10], we consider the following slight generalization of the MWDS that is called the Red-Blue Dominating Set Problem. The input is an undirected bipartite intersection graph $G = (\mathcal{R} \sqcup \mathcal{B}, E)$, where $\mathcal{R} = \{R_1, \ldots, R_N\}$ and $\mathcal{B} = \{B_1, \ldots, B_M\}$ are sets of red and blue t-objects, respectively. There is an edge between $R_i \in \mathcal{R}$ and $B_j \in \mathcal{B}$ if $R_i \cap B_j \neq \emptyset$. Since each node corresponds to a (red or blue) t-object, we will use the term 'node' and 't-object' interchangeably. Each $R_i \in \mathcal{R}$ has a non-negative weight w_i. The goal is to find a subset $\mathcal{R}' \subseteq \mathcal{R}$ of minimum weight such that every blue node B_j has a neighbor in the set $\mathcal{R}'$. Note that the standard MWDS problem is equivalent to the setting where $\mathcal{R} = \mathcal{B}$.

3.1 LP relaxation and rounding

Let $\mathcal{I} = (\mathcal{R}, \mathcal{B})$ denote the given instance of the (generalized) MWDS. In an integer programming formulation we have a $\{0,1\}$ variable x_i corresponding to a red t-object R_i which is intended to be assigned 1 if R_i is selected in the solution, and is assigned 0 otherwise. We relax the integrality constraints and describe the natural LP relaxation below.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{R_i \in \mathcal{R}} w_i x_i \\
\text{subject to} \quad & \sum_{R_i : B_j \cap R_i \neq \emptyset} x_i \geq 1, \qquad \forall B_j \in \mathcal{B} && (3.1) \\
& x_i \in [0,1], \quad \forall R_i \in \mathcal{R} && (3.2)
\end{aligned}
$$

The LP-relaxation can be solved in polynomial time assuming that we are given the intersection graph defined by $\mathcal{I}$. Let x be an optimal LP solution for the given instance $\mathcal{I} = (\mathcal{R}, \mathcal{B})$. Since x is feasible, for every blue t-object B_j, we have $\sum_{R_i : B_j \cap R_i \neq \emptyset} x_i \geq 1$. For each $B_j \in \mathcal{B}$, let b'_j denote an object maximizing the quantity $\sum_{R_i : b'_j \cap R_i \neq \emptyset} x_i$, where the ties are broken arbitrarily. Let $\mathcal{B}' := \{b'_j : B_j \in \mathcal{B}\}$. We emphasize that $\mathcal{B}'$ is a set of (1-)objects, whereas $\mathcal{R}$ is a set of t-objects. Note that for any $b'_j \in \mathcal{B}'$, $\sum_{R_i : b'_j \cap R_i \neq \emptyset} x_i \geq 1/t$.

For any $R_i \in \mathcal{R}$, let $x'_i = \min\{t x_i, 1\}$, and let x' denote the resulting solution. Then, for any $b'_j \in \mathcal{B}'$, $\sum_{R_i : b'_j \cap R_i \neq \emptyset} x'_i \geq 1$. The following observation follows from the definition of x'.

Observation 3.1. *The solution x' is feasible for the instance $\mathcal{I}' = (\mathcal{R}, \mathcal{B}')$, and we have that $\sum_{R_i \in \mathcal{R}} w_i x'_i \leq t \cdot \sum_{R_i \in \mathcal{R}} w_i x_i$.*

The preceding step to reduce to $\mathcal{I}'$ is essentially the same as in [10].

3.1.1 Shallow-cell complexity

Definition 3.2 (Shallow-cell complexity)**.** Let $A \in \{0,1\}^{M \times N}$ denote an $M \times N$ (0,1)-matrix. Let $1 \leq k \leq n \leq N$. Let S be any set of n columns and let A_S be the matrix restricted to the columns of S. If the number of distinct rows in A_S with at most k ones is bounded by $f(n,k)$ for any choice of n, k and S, then A is said to have the shallow-cell complexity $f(n,k)$.

Now, we describe how to round x' to an integral solution via quasi-uniform sampling technique. The standard LP relaxation for an MWSC instance $(X, \mathcal{S}')$ is as follows.

$$\begin{aligned} \text{minimize} \quad & \sum_{S_i \in \mathcal{S}'} w_i x_i \\ \text{subject to} \quad & \sum_{S_i \ni j} x_i \geq 1, \qquad \forall j \in X && (3.3) \\ & x_i \in [0,1], \quad \forall S_i \in \mathcal{S}' && (3.4) \end{aligned}$$

Let $A \in \{0,1\}^{M \times N}$ denote the constraint matrix in the LP relaxation above. This is the incidence matrix of the set system $\{S_i \mid 1 \leq i \leq N\}$: each row of A corresponds to an element, and each column corresponds to a set. The entry $A_{ij} = 1$ if the element j is contained in S_i, otherwise $A_{ij} = 0$.

The following crucial definition is from [11].

Definition 3.3 (Shallow-cell complexity of MWSC instance)**.** Let A be the constraint matrix of the MWSC instance $\mathcal{I}$. We define the shallow-cell complexity of $\mathcal{I}$ to be the shallow-cell complexity of A.

Using bounds on the shallow-cell complexity of an MWSC instance, it is possible to round a feasible LP solution using a technique known as quasi-uniform sampling [11, 35].

Theorem 3.4 (Chan et al. [11], Varadarajan [35])**.** *Consider an MWSC instance with shallow-cell complexity $f(n,k) = n\phi(n) \cdot k^c$, where $\phi(n) = O(n)$, and $c \geq 0$ is a constant. Then, there exists a polynomial-time algorithm to round a given fractional solution to the LP relaxation for this instance to an integral solution whose cost is within an $O(\max\{\log \phi(N), 1\})$ factor of the cost of the fractional solution, where the constant hidden in the big-Oh notation depends on the exponent c, and N is the number of sets in the given instance.*

Usually, $\phi(n)$ is a function of n such that $\phi(n) = O(n)$. However, the same guarantee holds even when ϕ is independent of n . When we apply this theorem, we will set $\phi(n) = O(t^4)$, which is independent of n.

Now, we consider an instance $\mathcal{I}' = (\mathcal{R}, \mathcal{B}')$ of the MWDS problem obtained in the previous section. Recall that $\mathcal{R}$ is a set of t-objects and $\mathcal{B}$ is a set of 1-objects. Now, let $\mathcal{R}' = \{r_i^{(k)} \in R_i : R_i \in \mathcal{R}\}$ denote the set of constituent red 1-objects from $\mathcal{R}$. Let $\mathcal{I}'' = (\mathcal{R}', \mathcal{B}')$ denote the MWDS instance thus obtained. Notice that an MWDS instance can also be thought of as an MWSC instance. We first prove the following simple lemma.

Lemma 3.5. *Let $\mathcal{I}', \mathcal{I}''$ be MWDS instances as defined above. If the shallow-cell complexity of $\mathcal{I}''$ is $f(n,k)$, then the shallow-cell complexity of the corresponding instance $\mathcal{I}'$ is $g(n,k) \leq f(nt,kt)$ for any $1 \leq k \leq n \leq N$.*

Proof. We prove this fact from the definition of the shallow-cell complexity. Let $A^{\mathcal{I}'}$ denote the constraint matrix corresponding to the instance $\mathcal{I}'$. Fix some positive integers n, k such that $1 \leq k \leq n \leq |\mathcal{R}|$, and fix a set $S \subseteq \mathcal{R}$ of columns (i. e., t-objects), where $|S| = n$. Let $A_S^{\mathcal{I}'}$

denote the constraint matrix restricted to the columns corresponding to S. Let P denote the set of distinct rows (i. e., blue objects) in $A_S^{\mathcal{I}'}$ with at most k ones. We seek to bound $|P|$.

Let $S' = \{r_i^{(k)} \in R_i : R_i \in S\}$ be the corresponding constituent 1-objects. Note that $S' \subseteq \mathcal{R}'$, and $|S'| \leq nt$. Let $A^{\mathcal{I}''}$ denote the constraint matrix corresponding to the instance $\mathcal{I}''$ and let $A_{S'}^{\mathcal{I}''}$ denote $A^{\mathcal{I}''}$ restricted to the columns of S'. Let P' denote the set of rows in $A_{S'}^{\mathcal{I}''}$ with at most kt ones. Note that, if a row has at most k ones in $A_S^{\mathcal{I}'}$, then it corresponds to exactly one row in P'. Therefore, $|P| \leq |P'| \leq f(nt, kt)$, where the last inequality follows from the definition of the shallow-cell complexity of the instance $\mathcal{I}''$. ◻

Now, we state the known bounds on the shallow-cell complexity of an MWDS instance defined by red and blue sets of pseudodisks.

Theorem 3.6 (Aronov et al. [5]). *The shallow-cell complexity of an MWDS instance defined by a collection of pseudodisks is at most $f(n,k) = O(nk^3)$.*

Combining Theorem 3.4, Theorem 3.6, and Lemma 3.5, we obtain the following result.

Theorem 3.7. *There exists a polynomial-time $O(t \log t)$-approximation algorithm for the Red-Blue Dominating Set problem defined by a collection of t-pseudodisks.*

Proof. Let $\mathcal{I} = (\mathcal{R}, \mathcal{B})$ be the original instance and let x be an optimal LP solution. Define instance $\mathcal{I}' = (\mathcal{R}, \mathcal{B}')$ and the corresponding LP solution x' as before. From Observation 3.1, we have that $\sum_{R_i \in \mathcal{R}} w_i x_i' \leq t \cdot \sum_{R_i \in \mathcal{R}} w_i x_i$. By Lemma 3.5 the shallow-cell complexity of $\mathcal{I}'$ is $g(n,k) \leq f(nt, kt)$, where $f(n,k) = O(nk^3)$ by Theorem 3.6. Therefore, $g(n,k) \leq O(nt^4k^3)$. Now, using the algorithm from Theorem 3.4 with $\phi(n) = O(t^4)$, we can round x' to an integral solution of cost at most $O(\log t) \cdot \sum_{R_i \in \mathcal{R}} w_i x_i' \leq O(t \log t) \cdot \sum_{R_i \in \mathcal{R}} w_i x_i$, where the inequality follows from Observation 3.1. ◻

Remark 3.8. Suppose we have an instance of $(\mathcal{R}, \mathcal{B})$ of generalized MWDS where each red object is a t_R-pseudodisk and each blue object is a t_B-pseudodisk. Then the preceding analysis can be extended to obtain an $O(t_B \log t_R)$-approximation. We note that for the analogous version of intervals, a $(t_B \cdot t_R)$-approximation was obtained in [10].

3.1.2 Geometric MWSC with t-objects.

Let $\mathcal{S}'$ denote a set of (1-)objects. Consider a Geometric MWSC instance $(X, \mathcal{S})$, where X is a set of points, and $\mathcal{S}$ is a set of N sets, each of which is obtained by taking the union of at most t (1-)objects from $\mathcal{S}'$. Each set in $\mathcal{S}$ has a non-negative weight. The goal is to find a minimum-weight set $\mathcal{R} \subseteq \mathcal{S}$ that *covers* the set of points X. Furthermore, suppose the set system $(X, \mathcal{S}')$ has shallow-cell complexity $f(n,k) = nk^c$, for some fixed constant c. This includes geometric objects such as intervals on a line and (pseudo-)disks in the plane. Then, using similar arguments as we did for MWDS, one can obtain a bound of $O(nk^ct^{c+1})$ on the shallow-cell complexity of the MWSC instance $(X, \mathcal{S})$. Then Theorem 3.4 implies an $O(\log t)$-approximation for the MWSC instance $(X, \mathcal{S})$.

When $\mathcal{S}'$ is a set of fat triangles, the shallow-cell complexity of the system $(X, \mathcal{S}')$ can be bounded by $f(n,k) = n\log^* n \cdot k^c$, for some fixed constant c [4, 11]. We can then bound the shallow-cell complexity of the MWSC instance $(X, \mathcal{S})$ by $f(nt, kt) = n\log^*(nt) \cdot t^{c+1}k^c$. Then, Theorem 3.4 implies an $O(\log(t^{c+1} \cdot \log^*(Nt))) = O(\log t + \log\log^*(N))$-approximation. Therefore, we obtain the following result.

Theorem 3.9. *There is a polynomial-time $O(\log t)$-approximation algorithm for MWSC instances defined by covering points by t-pseudodisks. For covering points by t-fat-triangles, there is an $O(\log t + \log\log^*(N))$-approximation where N is the number of triangles in the instance.*

The preceding result improves the t-approximation for covering points by t-intervals [23]. The approximation ratio of $O(\log t)$ is tight up to constant factors; it is NP-hard to obtain an $o(\log t)$-approximation for covering points by t-intervals. This follows via a reduction from MWSC [10].

3.2 Integrality of MWDS LP for intervals

Let $\mathcal{I} = (\mathcal{R}, \mathcal{B})$ be an MWDS instance where $\mathcal{R}$ and $\mathcal{B}$ are set of intervals on a line. In this subsection, we prove that the MWDS LP for intervals is integral. Butman et al. [10] proved this fact using a primal-dual algorithm that constructs an integral solution of the same cost as the LP. Here we give a simpler proof via the structure of the constraint matrix.

First, we preprocess blue intervals such that no blue interval is completely contained inside another blue interval. To understand this, suppose $B_1, B_2 \in \mathcal{B}$ are two intervals such that $B_1 \subset B_2$. Then, any feasible integral solution must include a red interval R that intersects B_1, and thus B_2. Furthermore, the LP constraint corresponding to B_2 is implied by the constraint corresponding to B_1. Therefore, the feasible regions of the LP's corresponding to $\mathcal{I}$ and the instance obtained by removing B_2 are the same. This justifies removing B_2 from $\mathcal{I}$. Repeating this removal step as along as one blue interval is contained in another blue interval, we obtain an instance $\mathcal{I}'$.

In the following theorem we show that the constraint matrix of the LP corresponding to $\mathcal{I}'$ satisfies the consecutive ones property, and thus it is totally unimodular. This implies that the LP corresponding to $\mathcal{I}'$ (and therefore $\mathcal{I}$) is integral (see, e.g., [33]).

Theorem 3.10. *Let $\mathcal{I} = (\mathcal{R}, \mathcal{B})$ be an MWDS instance, where $\mathcal{R}$ and $\mathcal{B}$ are sets of (1-)intervals. Then, the MWDS LP is integral.*

Proof. Let us denote the left (resp. right) endpoint of an interval I by $\mathsf{Left}(I)$ (resp. $\mathsf{Right}(I)$). Let A denote the constraint matrix corresponding to the preprocessed instance, where the rows (i. e., blue intervals) are sorted in the non-decreasing order of their left endpoints. We show that A has the consecutive ones property for any column.

Consider any column (i. e., a red interval) R. Let B_j and B_k denote the first (leftmost) and last (rightmost) blue intervals intersecting R in this order respectively. Note that $j \le k$. Now, suppose for contradiction that there is an interval B_ℓ with $j < \ell < k$ that does not intersect R. Let us consider two cases.

If $\mathsf{Right}(B_\ell) < \mathsf{Left}(R)$, then we have that $\mathsf{Left}(B_j) < \mathsf{Left}(B_\ell) \leq \mathsf{Right}(B_\ell) < \mathsf{Left}(R) \leq \mathsf{Right}(B_j)$. Here, the third and fourth inequalities follow from the assumptions that $R \cap B_\ell = \emptyset$ and $R \cap B_j \neq \emptyset$ respectively. However, this implies that $B_\ell \subset B_j$, which is a contradiction.

On the other hand, if $\mathsf{Right}(B_\ell) > \mathsf{Left}(R)$, then it must be the case that $\mathsf{Left}(B_\ell) > \mathsf{Right}(R)$; otherwise $R \cap B_\ell \neq \emptyset$, which is a contradiction. However, this implies that $\mathsf{Left}(R) \leq \mathsf{Right}(R) < \mathsf{Left}(B_\ell) \leq \mathsf{Left}(B_k) \leq \mathsf{Right}(B_k)$, where the third inequality follows from the fact that $\ell < k$. However, this contradicts the assumption that $B_k \cap R \neq \emptyset$.

Thus, the column corresponding to R satisfies the consecutive ones property. □

3.3 Hardness results for MWDS

We show hardness of approximation for MWDS in t-interval graphs via a simple reduction. We first consider the Red-Blue Dominating Set problem for t-intervals which illustrates the basic idea. We then modify it slightly to prove hardness for the MWDS problem.

Let $(X, \mathcal{S})$ be an f-uniform instance of MWSC. That is, any element in X is contained in exactly f sets of $\mathcal{S}$. We reduce this to the Red-Blue Dominating Set problem as follows. For every set $S_i \in \mathcal{S}$, we add a red interval $R_i = [y_i, z_i]$ on the line. The red intervals are non-overlapping; concretely, it suffices to choose $y_i = 2i$ and $z_i = 2i + 1$ for each i. We set the weight of R_i to be that of S_i. For any element $e_j \in X$, we add a exactly f blue points coinciding with points z_i, where $e_j \in S_i$. Note that a blue f-point is covered iff we select a red interval that covers one of its constituent f points. Thus, there is a one-to-one correspondence between feasible solutions to the original MWSC instance, and feasible solutions to the Red-Blue dominating set instance that is created. The following hardness results are known for the f-uniform MWSC problem even when sets have unit weights.

Theorem 3.11.

1. *It is* NP*-hard to obtain a $(f - 1 - \epsilon)$-approximation for f-uniform Set Cover, for any fixed $f \geq 3$ and $\epsilon > 0$ (Dinur et al. [16]).*

2. *Furthermore, it is* UG*-hard to obtain an $(f - \epsilon)$-approximation for any fixed $f \geq 2$ and $\epsilon > 0$ (Khot and Regev [29]).*

Therefore, from the preceding reduction, we get the following hardness results for the Red-Blue Dominating Set problem, even when all weights are unit.

Theorem 3.12. *It is* NP*-hard to obtain a $(t - 1 - \epsilon)$-approximation for the special case of Red-Blue Dominating Set problem where $\mathcal{R}$ is a set of intervals and $\mathcal{B}$ is a set of t-points. Moreover, in the same setting, it is* UG*-hard to obtain an $(t - \epsilon)$-approximation, for any $t \geq 2$ and $\epsilon > 0$.*

Corollary 3.13. *It is* NP*-hard to obtain a $(t - 1 - \epsilon)$-approximation for MWDS in t-interval graphs. Moreover, in the same setting, it is* UG*-hard to obtain an $(t - \epsilon)$-approximation, for any $t \geq 2$ and $\epsilon > 0$.*

Proof. We slightly modify the preceding reduction from t-uniform MWSC to the Red-Blue Dominating Set problem. For each red interval R_i we also add a green point G_i that coincides with y_i, the left end point of R_i, and we set its weight to 0. We set the weight of each blue t-point to be twice the total weight of all the sets in the MWSC instance (conceptually the weight is ∞). The MWDS instance now consits of the red intervals, the blue t-points, and the green points. The green points have zero weight and hence picking all of them will ensure that all green points and red intervals are dominated. The green points cannot cover any of the blue points. Since the blue t-points have very large weight, it is cheaper to pick all the red intervals than pick any of them. Thus we can restrict attention to dominating sets that include all the green points and a subset of the red intervals which together cover the blue t-points. Thus, t-uniform MWSC reduces in an approximation preserving fashion to MWDS in t-interval graphs. $\square$

4 Maximum-Weight Independent Set

We consider MWIS of t-objects. Here, we are given a set $\mathcal{S} = \{S_1, \ldots, S_n\}$, where each $S_i \in \mathcal{S}$ is a t-object, and has a non-negative weight w_i. In this section, we assume that we are given the geometric representation of the t-objects in $\mathcal{S}$. The goal is to find maximum-weight independent set $\mathcal{S}' \subseteq \mathcal{S}$ in the intersection graph of $\mathcal{S}$.

We confine attention to the case when the base objects are closed Jordan domains. Let $\mathcal{V}(\mathcal{S})$ denote the set of vertices on $\partial\mathcal{U}(\mathcal{S})$. First, we describe a natural LP relaxation for this problem (see for instance [12]).

$$
\begin{aligned}
\text{maximize} \quad & \sum_{S_i \in \mathcal{S}} w_i x_i \\
\text{subject to} \quad & \sum_{S_i \ni p} x_i \le 1, \qquad \forall p \in \mathcal{V}(\mathcal{S}) && (4.1) \\
& x_i \in [0,1], \qquad \forall S_i \in \mathcal{S} && (4.2)
\end{aligned}
$$

We will assume that the preceding LP relaxation can be solved in polynomial time for instances of interest. This is easy to see for geometric settings in which the vertices of the arrangement $\mathcal{S}$ can be explicitly computed in polynomial time.

We have a simple observation that relates the union complexities of t-objects and the corresponding (1-)objects.

Observation 4.1. *Let $\mathcal{S}$ be a set of t-objects, and suppose the set of underlying (1-)objects has union complexity $u(\cdot)$. Then, the union complexity of any k objects in $\mathcal{S}$ is at most $u(kt)$.*

Proof. Let $\mathcal{R} \subseteq \mathcal{S}$ be any subset of t-objects. Let $\mathcal{R}' = \{r_i^{(k)} \in R_i : R_i \in R\}$ denote the underlying set of (1-)objects. Note that $|R'| \le t \cdot |R|$. Since the underlying set of (1-)objects has union complexity $u(\cdot)$, the number of arcs on the boundary of R' is at most $u(|R'|) \le u(t \cdot |R|)$. $\square$

Chan and Har-Peled [12] describe an algorithm to round a feasible fractional solution to the LP relaxation which leads to the following guarantee.

Theorem 4.2 (Chan and Har-Peled [12])**.** *There is a polynomial-time $\Omega(n/u(n))$-approximation algorithm for MWIS in the intersection graph of a set of n closed Jordan domains with union complexity $u(\cdot)$.*

In particular, this implies an $\Omega(1)$-approximation for MWIS in the intersection graph of pseudodisks, since pseudodisks have linear union complexity [3, 27]. However, we cannot use the theorem directly, since a t-pseudodisk is not necessarily a closed Jordan domain. Although the result in [12] easily extends to our setting via Observation 4.1, for the sake of completeness, we briefly sketch the main reason. The following following technical lemma from [12] is the main ingredient in the proof of Theorem 4.2.

Lemma 4.3 (Lemma 4.1 from [12])**.** *Let $\mathcal{S}$ be a set of closed Jordan domains, and let $\{x_i : S_i \in \mathcal{S}\}$ be a feasible solution to the MWIS LP relaxation. Furthermore, let $\mathcal{R} \subseteq \mathcal{S}$ be any subset. Then, there is a universal constant c such that $\sum_{(p,i,j)\in\mathcal{V}(\mathcal{R})} x_i x_j \le c \cdot u(\mathcal{E}(\mathcal{R}))$, where $\mathcal{E}(\mathcal{R}) = \sum_{S_i\in\mathcal{R}} x_i$, and (p, i, j) denotes a vertex p formed by the intersection of closed Jordan domains S_i and S_j in the arrangement $\mathcal{V}(\mathcal{R})$.*

Lemma 4.3 is the only place in the proof of Theorem 4.2 which uses the assumption that $\mathcal{S}$ is a set of closed Jordan domains. In other words, a bound on $\sum_{(p,i,j)\in\mathcal{V}(\mathcal{R})} x_i x_j$ in terms of the union complexity, as in the statement of the lemma, readily implies an $\Omega(n/u(n))$-approximation for the respective set of objects. We need the generalization of Lemma 4.3 which works when $\mathcal{S}$ is a collection of t-pseudodisks. The proof of the lemma is based on a simple yet clever random sampling argument and does not rely on the objects being closed Jordan domains. Thus, the lemma holds when $\mathcal{S}$ is a collection of t-pseudodisks; indeed the lemma holds for t-objects where each object is a closed Jordan domain. We can therefore infer an $\Omega(n/u(n))$ approximation for MWIS in the intersection graph of n t-pseudodisks, where $u(\cdot)$ denotes the union complexity of t-pseudodisks.

Finally, recall that pseudodisks have linear union complexity [3, 27]. Therefore, Observation 4.1, implies that a collection of n t-pseudodisks have $O(nt)$ union complexity. Therefore, we get the following theorem.

Theorem 4.4. *There is a polynomial-time $\Omega(1/t)$-approximation algorithm for MWIS in the intersection graph of t-pseudodisks.*

Packing t-objects. We consider a related problem called *Maximum-Weight Region Packing*. We are given a set $\mathcal{S}$ of t-objects, and a set of points P. Each t-object S_i has a weight w_i and each point $p \in P$ has a capacity $c(p)$, which is a positive integer. The goal is to find a maximum-weight set $\mathcal{S}'$ of t-objects, such that for each point $p \in P$, the number of regions of $\mathcal{S}'$ that contain p is at most $c(p)$. Note that MWIS is a special case when $P = \mathcal{V}(S)$ is the set of all points in arrangement and $c(p) = 1$ for each $p \in P$.

Extending the LP rounding algorithm of Chan and Har-Peled [12] for the MWIS problem, Ene et al. [17] gave an $\Omega((\frac{n}{u(n)})^{1/C})$-approximation for Maximum-Weight Region Packing problem, where C is the minimum capacity of any point. Using similar arguments as in MWIS, we obtain the following result.

Theorem 4.5. *There is a polynomial-time $\Omega(1/t^{1/C})$-approximation algorithm for Maximum-Weight Region Packing with t-pseudodisks.*

5 Concluding remarks

In geometric settings, quasi-uniform sampling has been used for Set Multicover [7], and other related ideas have been used for Partial Set Cover [24, 14]. Our results for MWDS and MWSC for t-objects can be extended to these settings. We briefly sketch the ideas.

The results in [24, 14] establish a generic high-level reduction: an α-approximation via the natural LP relaxation for MWSC can be translated into an $O(\alpha)$-approximation for Partial Set Cover for hereditary instances. The geometric instances considered here are hereditary, and hence our results for MWSC extend to Partial Set Cover.

Set Multicover is a generalization of Set Cover. In the geometric setting of interest here, each point p has integer demand $d_p \geq 1$, and the goal is to choose a minimum-weight subset of objects so that each point is contained in at least d_p objects. Bansal and Pruhs [7] extended the shallow-cell complexity based framework from Set Cover to Set Multicover. Since we prove our results by establishing shallow-cell complexity for t-objects, the results in [7] extend to Set Multicover with t-pseudodisks.

Finally, we note that even for the special case of MWDS of t-intervals, there is a gap between the upper and lower bounds on polynomial-time approximation ratios that we established in this paper: $O(t \log t)$ and $\Omega(t)$. Resolving this gap is an interesting open question. Obtaining smaller leading constants in the approximation ratios, and avoiding the somewhat complicated machinery of union complexity, for special cases such as t-intervals, is also of interest.

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AUTHORS

Chandra Chekuri
Professor
Department of Computer Science
University of Illinois at Urbana-Champaign
Urbana, Illinois, United States
chekuri@illinois.edu
https://chekuri.cs.illinois.edu/

Tanmay Inamdar[5]
Researcher
Department of Informatics,
University of Bergen
Bergen, Norway
Tanmay.Inamdar@uib.no

ABOUT THE AUTHORS

CHANDRA CHEKURI received his B. Tech in Computer Science and Engineering from Indian Institute of Technology, Madras (now Chennai) in 1993 and received his Ph. D. in Computer Science from Stanford University in 1998 under the supervision of the late Rajeev Motwani. After finishing his Ph. D. he worked eight formative years at Lucent Bell Labs before moving to the University of Illinois. He mainly works in algorithm design, combinatorial optimization, graphs, and mathematical programming.

TANMAY INAMDAR received his Ph. D. in Computer Science from The University of Iowa in 2020, where he was advised by Kasturi Varadarajan. Currently he is working as a Researcher at The University of Bergen in Norway. His research interests include Approximation Algorithms, Computational Geometry, and Parameterized Algorithms.

[5]This work was done when the author was a Ph. D. student at The University of Iowa, during a visit to University of Illinois, Urbana-Champaign.