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# Spin(7) orientifolds and 2d $\mathcal{N} = (0, 1)$ triality

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Abstract: We present a new, geometric perspective on the recently proposed triality of 2d $\mathcal{N} = (0, 1)$ gauge theories, based on its engineering in terms of D1-branes probing Spin(7) orientifolds. In this context, triality translates into the fact that multiple gauge theories correspond to the same underlying orientifold. We show how Spin(7) orientifolds based on a particular involution, which we call the universal involution, give rise to precisely the original version of $\mathcal{N} = (0, 1)$ triality. Interestingly, our work also shows that the space of possibilities is significantly richer. Indeed, general Spin(7) orientifolds extend triality to theories that can be regarded as consisting of coupled $\mathcal{N} = (0, 2)$ and $(0, 1)$ sectors. The geometric construction of 2d gauge theories in terms of D1-branes at singularities therefore leads to extensions of triality that interpolate between the pure $\mathcal{N} = (0, 2)$ and $(0, 1)$ cases.

Keywords: Duality in Gauge Field Theories, Field Theories in Lower Dimensions, Supersymmetric Gauge Theory

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## Contents



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## 1 Introduction

2d $\mathcal{N} = (0,1)$ quantum field theories are extremely interesting, since they are barely supersymmetric and live at the borderline between non-SUSY theories and others with higher amounts of SUSY, for which powerful tools such as holomorphy become applicable. Due to the reduced SUSY, they enjoy a broad range of interesting dynamics. While there has been recent progress in their understanding, they remain relatively unexplored.

In [1], it was discovered that 2d $\mathcal{N} = (0,2)$ theories exhibit IR dualities reminiscent of Seiberg duality in 4d $\mathcal{N} = 1$ gauge theories [2]. This low-energy equivalence was dubbed *triality* since, in its simplest incarnation, three SQCD-like theories become equivalent at low energies. Recently, an IR *triality* between 2d $\mathcal{N} = (0,1)$ theories with SO and USp gauge groups was proposed in [3]. Evidence supporting the proposal includes matching of anomalies and elliptic genera. This new triality can be regarded as a relative of its $\mathcal{N} = (0,2)$ counterpart.

The geometric engineering of 2d $\mathcal{N} = (0,1)$ gauge theories on D1-branes probing singularities was initiated in [4], where a new class of backgrounds denoted Spin(7) *orientifolds* was introduced. These orientifolds are quotients of Calabi-Yau (CY) 4-folds by a combination of an anti-holomorphic involution leading to a Spin(7) cone and worldsheet parity. They provide a beautiful correspondence between the perspective of $\mathcal{N} = (0,1)$ theories as real slices of $\mathcal{N} = (0,2)$ theories and Joyce's geometric construction of Spin(7) manifolds starting from CY 4-folds. This geometric perspective provides a new approach for studying 2d $\mathcal{N} = (0,1)$ theories.

For branes at singularities, a single geometry often corresponds to multiple gauge theories. Such non-uniqueness is the manifestation of gauge theory dualities in this context. Examples of this phenomenon abound in different dimensions. The various 4d $\mathcal{N} = 1$ gauge theories on D3-branes over the same CY 3-fold are related by Seiberg duality [2, 5, 6]. The triality of 2d $\mathcal{N} = (0,2)$ gauge theories on D1-branes over CY 4-folds and the quadrality of 0d $\mathcal{N} = 1$ gauge theories on D$(-1)$-branes over CY 5-folds can be similarly understood [7, 8]. These ideas were further extended to the $(m+1)$-dualities of the $m$-graded quivers that describe the open string sector of the topological B-model on CY $(m+2)$-folds for arbitrary $m \geq 0$ [9–11]. In this paper, we will show that the engineering of 2d $\mathcal{N} = (0,1)$ gauge theories in terms of D1-branes probing Spin(7) orientifolds leads to a similar perspective on $\mathcal{N} = (0,1)$ triality.

The paper is organized as follows. In section 2 we review $\mathcal{N} = (0,2)$ and $\mathcal{N} = (0,1)$ trialities in their original formulations and comment on their generalizations to quivers. We discuss Spin(7) orientifolds and the corresponding 2d $\mathcal{N} = (0,1)$ field theories arising on D1-branes probing them in section 3. In section 4 we explain how the basic $\mathcal{N} = (0,1)$ triality arises from the *universal involution*. In section 5 we investigate how (generalizations of) $\mathcal{N} = (0,1)$ triality arise in the case of Spin(7) orientifolds based on more general involutions. We present our conclusions in section 6. There are, also, three appendices that may help the reader to follow the discussion in the main text. In appendix A we review the $\mathcal{N} = (0,1)$ formalism for 2d gauge theories, and in appendix B we list the possible contributions to 2d gauge anomalies for the groups and representations that we will encounter in the main



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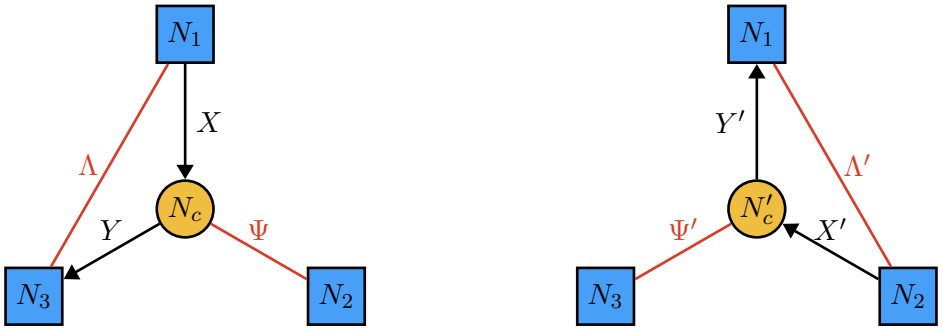


**Figure 1**. 2d $\mathcal{N} = (0, 2)$ SQCD and its triality dual. The central nodes have ranks given in (2.1).

text. Finally, in appendix C we give all the necessary details for the phases of $Q^{1,1,1}/\mathbb{Z}_2$ involved in the triality web introduced in section 5.2.

## 2 $\mathcal{N} = (0, 2)$ and $\mathcal{N} = (0, 1)$ triality

In this section, we review the trialities of 2d $\mathcal{N} = (0, 2)$ [1] and $\mathcal{N} = (0, 1)$ [3] gauge theories. Discussing $\mathcal{N} = (0, 2)$ triality first is not only useful for setting the stage since both trialities share various features, but it is also convenient since Spin(7) orientifolds connect them.

### 2.1 $\mathcal{N} = (0, 2)$ triality

Here we present a quick review of 2d $\mathcal{N} = (0, 2)$ triality. A detailed discussion can be found in [1]. Additional developments, including connections to 4d, its realization in terms of D1-branes at $CY_4$ singularities, brane brick models and mirror symmetry, appear in [7, 12–15].

Without loss of generality, we can focus on the quiver shown in figure 1a, which can be regarded as 2d $\mathcal{N} = (0, 2)$ SQCD. The yellow node represents the $SU(N_c)$ gauge group that undergoes triality, while the blue nodes are flavor $SU(N_i)$ groups, $i = 1, \ldots, 3$.[1] We have absorbed the multiplicities of flavor fields in the ranks of the flavor nodes. In $\mathcal{N} = (0, 2)$ quivers, we adopt the convention that the head and tail of the arrow associated to a chiral field correspond to fundamental and antifundamental representations, respectively. A Fermi field connecting the flavor nodes 1 and 3 has been included to make the original and dual theories more similar.

The triality dual is shown in figure 1b. The rank of the central node in both theories is determined by anomaly cancellation to be

$$N_c = \frac{N_1 + N_3 - N_2}{2}\,, \qquad N_c' = \frac{N_2 + N_1 - N_3}{2}\,. \tag{2.1}$$

The transformation of the rank can also be written as

$$N_c' = N_1 - N_c\,. \tag{2.2}$$

Both theories in figure 1 have $J$-/$E$- terms associated to the triangular loops in the quivers.

[1] More generally, as in theories arising on D1-branes probing $CY_4$ singularities, such groups can have additional matter charged under them and be gauged.



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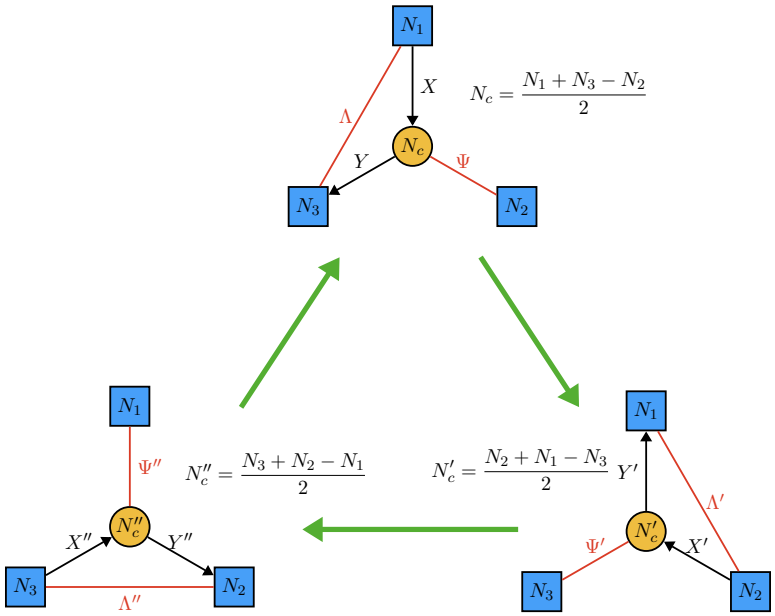


**Figure 2**. Triality loop for 2d $\mathcal{N} = (0, 2)$ SQCD.

Taking the dual theory as the new starting point and acting on it with triality, we obtain the theory shown on the bottom left of figure 2. Applying triality a third time takes us back to the original theory. We can therefore think about this second dual as connected to the original theory by inverse triality.[2] The triality among these three theories can be viewed as a cyclic permutation of $N_1$, $N_2$ and $N_3$.

We will later use $\mathcal{N} = (0, 2)$ gauge theories engineered on D1-branes probing CY 4-folds as starting points of orientifold constructions. Such theories have U($N$) gauge groups. A U($N_c$) version of $\mathcal{N} = (0, 2)$ triality was also introduced in [1]. It only differs from the SU($N_c$) triality depicted in figure 2 by the presence of additional Fermi fields in the determinant representation of the gauge group, which are necessary for the cancellation of the Abelian anomaly. It is expected that Abelian anomalies of gauge theories on D1-branes are cancelled via a generalized Green-Schwarz mechanism (see [16, 17] for 4d $\mathcal{N} = 1$ and 2d $\mathcal{N} = (0, 2)$ theories realized on D-branes probing orbifolds/orientifolds singularities). For this reason, the determinant Fermi fields are not present in such theories and triality reduces to the one considered in this section.

$\mathcal{N} = (0, 2)$ triality can be extended to general quivers (see e.g. [1, 7, 13–15, 18]). It acts as a local operation on the dualized node, with the part of the quiver that is not connected to it acting as a spectator. The transformation of such a theory under triality on a gauge node $k$ can be summarized as follows. The rank of node $k$ changes according to

$$N_k' = \sum_{j \neq k} n_{jk}^{\chi} N_j - N_k \,, \tag{2.3}$$

[2]The distinction between triality and inverse triality is just a convention.



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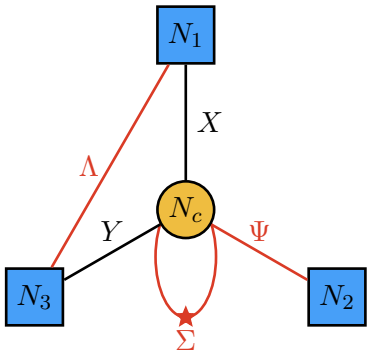


| | $X$ | $Y$ | $\Psi$ | $\Lambda$ | $\Sigma$ |
|---|---|---|---|---|---|
| $\mathrm{SO}(N_c)$ | vec. | vec. | vec. | · | sym. |
| $\mathrm{SO}(N_1)$ | vec. | · | · | vec. | · |
| $\mathrm{SO}(N_2)$ | · | · | vec. | · | · |
| $\mathrm{SO}(N_3)$ | · | vec. | · | vec. | · |

**Figure 3**. 2d $\mathcal{N} = (0,1)$ SQCD. $N_c$ is given in eq. (2.4).

where $n_{jk}^{\chi}$ is the number of chiral fields from node $j$ to node $k$. All other ranks remain the same. The field content around node $k$ changes according to the following rules:

(R.1) **Dual Flavors**. Replace each of $(\to k)$, $(\leftarrow k)$, $(\text{—} k)$ by $(\leftarrow k)$, $(\text{—} k)$, $(\to k)$, respectively.

(R.2) **Chiral-Chiral Mesons**. For each subquiver $i \to k \to j$, add a new chiral field $i \to j$.

(R.3) **Chiral-Fermi Mesons**. For each subquiver $i \to k \text{—} j$, add a new Fermi field $i \text{—} j$.

(R.4) Remove all chiral-Fermi massive pairs generated in the previous steps.

For a detailed discussion of the transformation of $J$- and $E$-terms, see e.g., [9].

### 2.2 $\mathcal{N} = (0,1)$ triality

A similar triality for 2d $\mathcal{N} = (0,1)$ gauge theories was introduced in [3]. The primary example in which the proposal was investigated is 2d $\mathcal{N} = (0,1)$ SQCD with $\mathrm{SO}(N_c)$ gauge group, whose quiver diagram is shown in figure 3.[3] The theory has $N_1 + N_3$ scalar multiplets in the vector representation of $\mathrm{SO}(N_c)$. These scalar fields are further divided into two sets, $X$ and $Y$, transforming under $\mathrm{SO}(N_1)$ and $\mathrm{SO}(N_3)$ flavor groups, respectively. A bifundamental Fermi multiplet $\Lambda$ connects $\mathrm{SO}(N_1)$ and $\mathrm{SO}(N_3)$.[4] There are also $N_2$ Fermi multiplets $\Psi$ in the vector representation of $\mathrm{SO}(N_c)$ and a Fermi multiplet $\Sigma$ in the symmetric representation of $\mathrm{SO}(N_c)$.

Anomaly cancellation for the $\mathrm{SO}(N_c)$ gauge group requires that[5]

$$N_c = \frac{N_1 + N_3 - N_2}{2} \,. \tag{2.4}$$

[3]When drawing $\mathcal{N} = (0,1)$ quivers, black and red lines correspond to real $\mathcal{N} = (0,1)$ scalar and Fermi fields, respectively. In addition, we indicate symmetric and antisymmetric representations with star and diamond symbols, respectively.

[4]We will use the term bifundamental in the case of matter fields that connect pairs of nodes, even when one or both of them is either SO or USp.

[5]The anomaly contributions of $\mathcal{N} = (0,1)$ multiplets in various representations are listed in appendix B.



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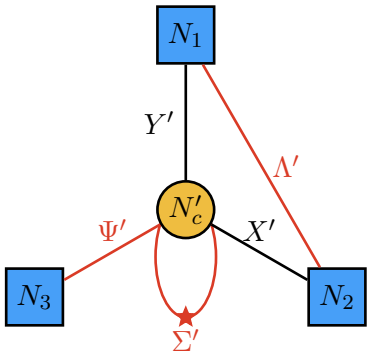


| | $X'$ | $Y'$ | $\Psi'$ | $\Lambda'$ | $\Sigma'$ |
|---|---|---|---|---|---|
| $\mathrm{SO}(N_c)$ | vec. | vec. | vec. | · | sym. |
| $\mathrm{SO}(N_1)$ | · | vec. | · | vec. | · |
| $\mathrm{SO}(N_2)$ | vec. | · | · | vec. | · |
| $\mathrm{SO}(N_3)$ | · | · | vec. | · | · |

**Figure 4**. 2d $\mathcal{N}=(0,1)$ triality dual of the theory in figure 3. $N_c'$ is given in eq. (2.6).

The theory also has the following superpotential consistent with its symmetries

$$W^{(0,1)} = \sum_{\alpha,\beta=1}^{N_c} \Sigma_{\alpha\beta} \left( \sum_{a=1}^{N_1} X_\alpha^a X_\beta^a + \sum_{b=1}^{N_3} Y_\alpha^b Y_\beta^b - \delta_{\alpha\beta} \right) + \sum_{a=1}^{N_1} \sum_{b=1}^{N_3} \sum_{\alpha=1}^{N_c} \Lambda_{ab} X_\alpha^a Y_\alpha^b \,. \tag{2.5}$$

Figure 4 shows the dual under triality. The transformation is rather similar to the $\mathcal{N}=(0,2)$ triality discussed in the previous section. Once again, in this simple example, the structure of the dual theory is identical to the original one up to a cyclic permutation of $N_1$, $N_2$ and $N_3$. For the flavors, scalar multiplets $X$, $Y$ and Fermi multiplets $\Psi$ are replaced by scalar multiplets $Y'$, Fermi multiplets $\Psi'$, and scalar multiplets $X'$, respectively. The new theory also contains a Fermi field $\Sigma'$ in the symmetric representation of the gauge group.

The gauge group is $\mathrm{SO}(N_c')$, with the rank determined by anomaly cancellation

$$N_c' = \frac{N_2 + N_1 - N_3}{2} \,, \tag{2.6}$$

which can be expressed as

$$N_c' = N_1 - N_c \,. \tag{2.7}$$

Very much like Rule (R.3) of the previous section, the new Fermi $\Lambda'$ in the bifundamental representation of $\mathrm{SO}(N_1) \times \mathrm{SO}(N_2)$ can be regarded as a scalar-Fermi meson in terms of the fields in the initial theory, i.e. $\Lambda' = X\Psi$. Similarly, we can interpret the disappearance of the original Fermi $\Lambda$ between figures 3 and 4 as the result of it becoming massive via its superpotential coupling to the scalar-scalar meson $XY$, which is analogous to the chiral-chiral mesons of Rule (R.2). An interesting difference with respect to $\mathcal{N}=(0,2)$ SQCD follows from the fact that SO representations are real. Equivalently, the quivers under consideration are not oriented. It is therefore natural to ask why, in addition to $\Lambda' = X\Psi$, figure 4 does not *simultaneously* have another scalar-Fermi meson $Y\Psi$ in the bifundamental representation of $\mathrm{SO}(N_2) \times \mathrm{SO}(N_3)$. Its absence can be interpreted as descending from $\mathcal{N}=(0,2)$ triality, in which the orientation of chiral fields prevent the formation of such a gauge invariant. Additional thoughts on the connection between $\mathcal{N}=(0,2)$ and $\mathcal{N}=(0,1)$



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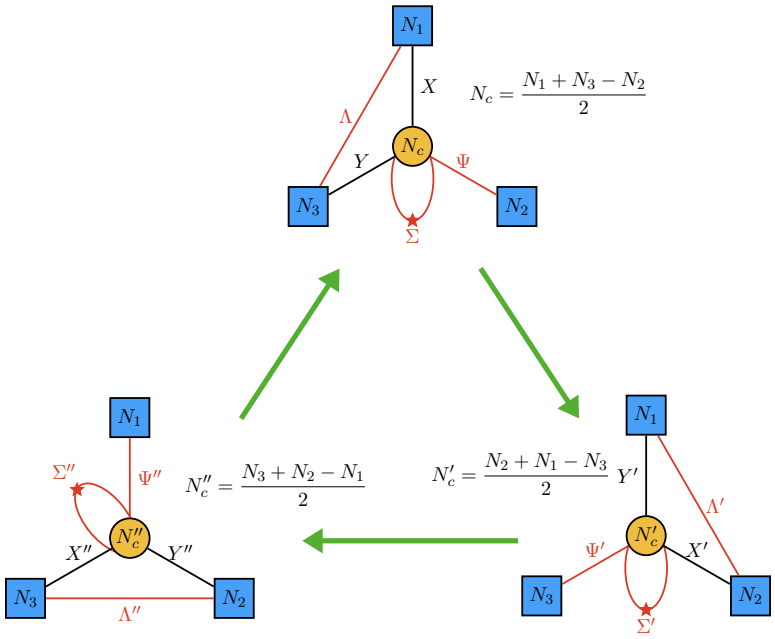


**Figure 5**. Triality loop for 2d $\mathcal{N} = (0, 1)$ SQCD.

trialities will be presented in section 2.3. Also related to this issue, in the coming section, we will discuss scalar-Fermi mesons in more general quivers.

The superpotential is identical to (2.5) upon replacing all fields by the primed counterparts and permuting $N_1$, $N_2$ and $N_3$.

Acting with triality again gives rise to the theory shown on the bottom left of figure 5. A third triality takes us back to the original theory.

There is also a symplectic version of $\mathcal{N} = (0, 1)$ triality [3]. The corresponding SQCD has $\mathrm{USp}(N_c)$ gauge group and $\mathrm{USp}(N_1) \times \mathrm{USp}(N_2) \times \mathrm{USp}(N_3)$ global symmetry.[6] The matter content is almost the same as in the $\mathrm{SO}(N_c)$ SQCD quiver shown in figure 3, with the exception that the Fermi field $\Sigma$ instead transforms in the antisymmetric representation of $\mathrm{USp}(N_c)$. The rank of the gauge group is $N_c = \frac{N_1+N_3-N_2}{2}$ to cancel gauge anomalies. In this case, the triality loop is identical to the one shown in figure 5.

Evidence for the $\mathcal{N} = (0, 1)$ triality proposal includes matching of anomalies and elliptic genera [3]. In the coming sections, we will provide further support for this idea, by realizing 2d $\mathcal{N} = (0, 1)$ theories via Spin(7) orientifolds.

### 2.3 $\mathcal{N} = (0, 1)$ triality for quiver gauge theories

Let us consider the extension of $\mathcal{N} = (0, 1)$ triality to general quivers. To do so, it is useful to first draw some lessons from Seiberg duality and $\mathcal{N} = (0, 2)$ triality. In both cases, incoming chiral fields at the dualized gauge group play a special role.[7] They control the

[6] Differently from [3], we adopt the convention $\mathrm{USp}(2) \simeq \mathrm{SU}(2)$ in order to be consistent with the notation of the orientifold theories we construct later.

[7] This is a general phenomenon that applies e.g. to the order $(m + 1)$ dualities of $m$-graded quivers [9]. Seiberg duality and $\mathcal{N} = (0, 2)$ triality correspond to the $m = 1$ and 2 cases, respectively.



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rank of the dual gauge group and, for triality, determine which mesons are formed. Since $\mathcal{N} = (0,1)$ quivers are unoriented, how to split the scalar fields terminating on a dualized node into two sets analogous to "incoming" and "outgoing" flavors is not clear. This issue was hinted in our discussion in the previous section.

In [3], a generalization of triality to a simple class of quiver theories with $\mathrm{SO}(N_{c_1}) \times \mathrm{SO}(N_{c_2}) \times \ldots$ gauge group (or the symplectic counterpart) was briefly discussed. Theories in this family are obtained by combining various $\mathcal{N} = (0,1)$ SQCD building blocks, which are glued by identifying any of the three global nodes of a given theory with the gauge node of another one. Locally, the resulting theories have the same structure of basic SQCD. Namely, every gauge node is connected to three other nodes, to two of them via scalar fields and to the remaining one via Fermi fields. Due to this simple structure, the dualization of any of the gauge groups is unambiguous and proceeds as in basic triality. For every node, the two possible choices of scalar fields acting as "incoming" or "outgoing" corresponds to acting with triality or inverse triality.

For general quivers, in which a given node can be connected to multiple others, how to separate the flavor scalar fields at every gauge group into two sets is an open question. All the theories that we will construct later using Spin(7) orientifolds are indeed beyond the above special class. However, this ambiguity is resolved in them by inheriting the separation of flavors from the parent $\mathcal{N} = (0,2)$ theories.

## 3 Spin(7) orientifolds

In this section we review the construction of Spin(7) orientifolds introduced in [4] and the 2d $\mathcal{N} = (0,1)$ field theories arising on D1-branes probing them. We focus the overview on a few key points relevant for subsequent sections, and refer the reader to this reference for additional details.

Our starting point is a toric CY 4-fold singularity $\mathbf{M}_8$. When probed by a stack of D1-branes at the singular point, the worldvolume theory corresponds to an $\mathcal{N} = (0,2)$ quiver gauge field theory. When $\mathbf{M}_8$ is toric, the structure of gauge groups, matter content and interactions of these theories is nicely encoded by brane brick models [13, 19, 20] (see [21] for an early related construction). Nevertheless, for our purposes it suffices to use the quiver description, supplemented by the explicit expression of the interaction terms ($J$- and $E$-terms).

We then perform an orientifold quotient by the action $\Omega\sigma$, where $\Omega$ is worldsheet parity and $\sigma$ is an anti-holomorphic involution of $\mathbf{M}_8$ leaving a specific 4-form, that we call $\Omega^{(4)}$, invariant. Such 4-form is constructed from the CY holomorphic 4-form $\Omega^{(4,0)}$ and the Kähler form $J^{(1,1)}$ as

$$\Omega^{(4)} = \mathrm{Re}\left(\Omega^{(4,0)}\right) + \frac{1}{2} J^{(1,1)} \wedge J^{(1,1)} \,. \tag{3.1}$$

If the quotient did not involve worldsheet parity, this quotient corresponds to Joyce's construction of Spin(7) geometries, with $\Omega^{(4)}$ defining the invariant Cayley 4-form of such varieties. To keep this connection in mind, the above orientifold quotients were dubbed Spin(7) *orientifolds* in [4].



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This orientifold quotient has a natural counterpart on the D1-brane systems, and naturally realizes a "real projection" of the 2d $\mathcal{N} = (0, 2)$ theories in [3], resulting in a 2d $\mathcal{N} = (0, 1)$ gauge field theory. Its structure is determined by a set of rules analogous to those of orientifold field theories in higher dimensions (see e.g. [22] in the 4d context), and which were explicitly determined in [4]. Morally, it corresponds to identifying the gauge factors and matter fields of the parent $\mathcal{N} = (0, 2)$ theory under an involution symmetry $\tilde{\sigma}$ of the quiver, compatible with the set of interactions.

To describe the orientifold action on the field theory in more detail, we label the different nodes by an index $i$, and their orientifold images by $i'$ (with $i' = i$ corresponding to nodes mapped to themselves under the orientifold action), and denote $X_{ij}$ and $\Lambda_{ij}$ the bifundamental $\mathcal{N} = (0, 2)$ chiral or Fermi multiplets charged under the gauge factors $i$ and $j$ (with $j = i$ corresponding to adjoints). The results of [4] are:

1a. Two gauge factors $\mathrm{U}(N)_i$, $\mathrm{U}(N)_{i'}$ mapped to each other under the orientifold action (namely $i \neq i'$) are identified and give rise to a single $\mathrm{U}(N)$ factor in the orientifold theory.

1b. On the other hand, a gauge factor $\mathrm{U}(N)_i$ mapped to itself (namely, $i' = i$) is projected down to $\mathrm{SO}(N)$ or $\mathrm{USp}(N)$.

2a. Two *different* chiral or Fermi bifundamental fields $X_{ij}$ and $X_{i'j'}$, mapped to each other under the orientifold action, become identified[8] and lead to a single (chiral or Fermi) bifundamental field. This holds even in special cases for the gauge factors, such as $i' = i$, or simultaneously $i' = i$ and $j' = j$, and for the special case of fields in the adjoint, $j = i$, $j' = i'$.

2b. Two *different* chiral or Fermi bifundamental fields $X_{ii'}$ and $Y_{i'i}$, related each to the (conjugate of the) other under the orientifold action, give rise to one field in the two-index symmetric and one field in the two-index antisymmetric representation of the corresponding SO / USp $i^{\text{th}}$ gauge factor in the orientifold quotient. The rule holds also in the case of adjoint fields, namely $i' = i$.

3a. A bifundamental field $X_{ij}$ that is mapped to itself by the orientifold action gives rise to a real $\mathcal{N} = (0, 1)$ field transforming under the bifundamental of $G_i \times G_j$, where $G_i$ and $G_j$ are the same type of SO or USp gauge group.

3b. A bifundamental Fermi field $\Lambda_{ii'}$ can only be mapped to itself (resp. minus itself) in the case of a holomorphic transformation, and gives rise to a complex Fermi superfield in the symmetric (resp. antisymmetric) representation of the resulting $\mathrm{U}(n)_i$ group.

3c. Closely related to Rule 3b, an adjoint complex Fermi field $\Lambda_{ii}$ that is mapped to itself (resp. minus itself) via a holomorphic transformation, gives rise to a complex Fermi

[8] In the presence of multiple sets of fields in these representations, the mapping may include a non-trivial action on the flavor index, encoded in a matrix $\eta$. As explained in [4], the choice can impact on the orientifold projection of the relevant gauge factors. We will encounter a non-trivial use of this freedom in the example in section 5.1.



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field in the symmetric/antisymmetric (resp. antisymmetric/symmetric) representation of SO / USp.

3d. An adjoint complex scalar or Fermi field that is mapped to itself gives rise to two real scalar or Fermi fields, one symmetric and one antisymmetric.

4a. A real Fermi $\Lambda^R_{ii}$ which transforms into $\Lambda^R_{i'i'}$, with $i' \neq i$, are projected down to a single real Fermi $\Lambda^R_{ii}$.

4b. A real Fermi $\Lambda^R_{ii}$ mapped to itself (resp. minus itself), with $i' \neq i$, gives rise to a symmetric (resp. antisymmetric) real Fermi for an SO (resp. USp) projection of the node $i$.

These rules suffice to construct large classes of examples of 2d $\mathcal{N} = (0,1)$ field theories, in particular the explicit examples in coming sections.

Note that the $\mathcal{N} = (0,1)$ theory obtained upon orientifolding the parent $\mathcal{N} = (0,2)$ may have non-abelian gauge anomalies. In such cases, the models require the introduction of extra flavor branes (namely, D5- or D9-branes extending in the non-compact dimensions of the CY 4-fold) for consistency. As already remarked in [4], very often the orientifolded theories happen to be non-anomalous, and hence do not require flavor branes. This will be the case in our examples later on.

**The universal involution.** We would like to conclude this overview by recalling from [4] that any $\mathcal{N} = (0,2)$ quiver gauge theory from D1-branes at toric CY 4-fold singularities admits a universal anti-holomorphic involution. It corresponds to mapping each gauge factor to itself (maintaining all with the same SO or USp projection), and mapping every $\mathcal{N} = (0,2)$ chiral or Fermi field to itself anti-holomorphically.

To be more explicit, let us introduce a set of matrices $\gamma_{\Omega_i}$ implementing the action of the orientifold on the gauge degrees of freedom of the $i^{\text{th}}$ node.[9] Then, the orientifold projections for the universal involution read

$$X_{ij} \to \gamma_{\Omega_i} \bar{X}_{ij} \gamma^{-1}_{\Omega_j}\,, \qquad \Lambda_{ij} \to \gamma_{\Omega_i} \bar{\Lambda}_{ij} \gamma^{-1}_{\Omega_j}\,, \tag{3.2}$$

where, by $X_{ij}$ or $\Lambda_{ij}$ we mean any chiral or Fermi field present in the gauge theory. In addition, the $\mathcal{N} = (0,1)$ adjoint Fermi fields coming from $\mathcal{N} = (0,2)$ vector multiplets transform as

$$\Lambda^R_i \to \gamma_{\Omega_i} \Lambda^{R\,T}_{i'} \gamma^{-1}_{\Omega_i}\,. \tag{3.3}$$

There is relative sign between this projection and the one for gauge fields, which implies that an SO or USp projection of the gauge group is correlated with a projection of $\Lambda^R_i$ into a symmetric or antisymmetric representation, respectively. These projections are consistent with the invariance of the $\mathcal{N} = (0,1)$ superpotential. Modding out by this orientifold action, the resulting $\mathcal{N} = (0,1)$ field theory is determined by applying the above rules.

[9] Actually, the matrices $\gamma_{\Omega_i}$ are a useful ingredient in implementing the orientifold projection, even in examples beyond the universal involution, as we will exploit in explicit examples in later sections.



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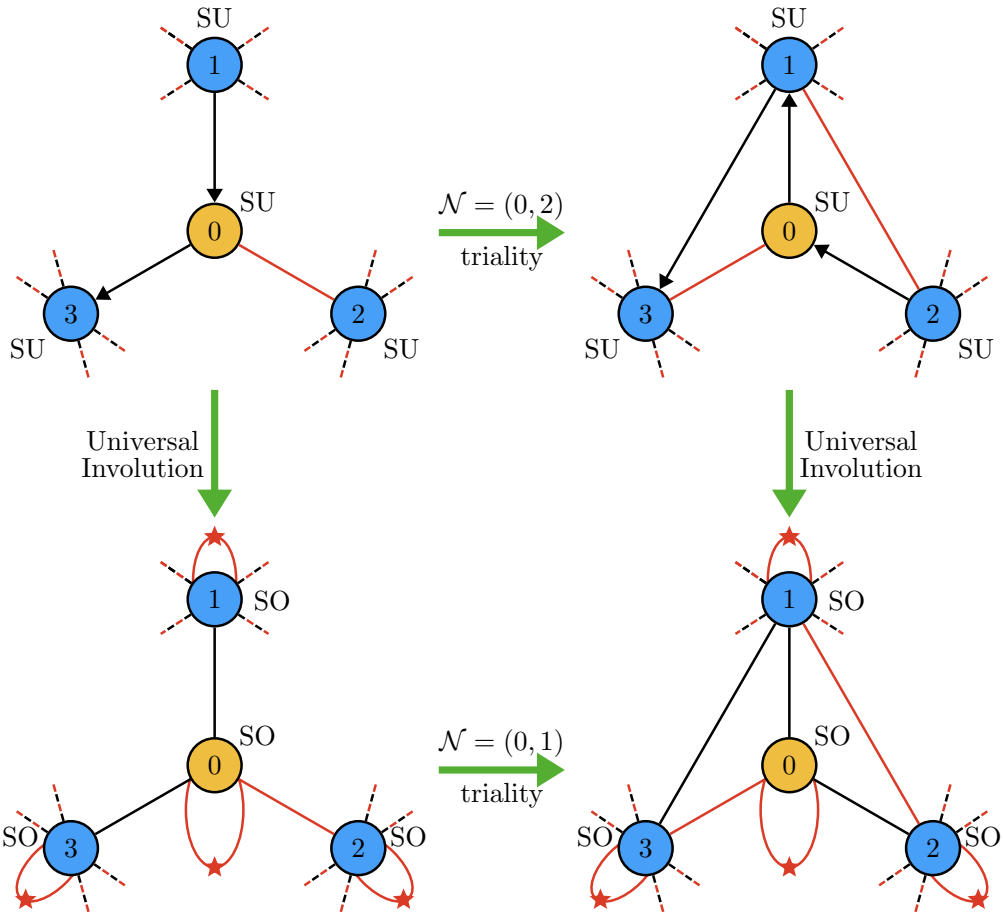


**Figure 6**. The universal involution on $\mathcal{N} = (0,2)$ triality results in $\mathcal{N} = (0,1)$ triality.

From the geometric perspective, this universal involution corresponds to the conjugation of all generators of the toric CY 4-fold. The action on the holomorphic 4-form is $\Omega^{(4,0)} \to \bar{\Omega}^{(0,4)}$, suitable for the realization of an Spin(7) orientifold. The following section focuses on models obtained via the universal involution.

## 4 $\mathcal{N} = (0,1)$ triality and the universal involution

Let us consider what happens when the universal involution is applied to two gauge theories associated to the same CY 4-fold, which are therefore related by $\mathcal{N} = (0,2)$ triality. Remarkably, we obtain two theories connected by precisely $\mathcal{N} = (0,1)$ triality. By construction, the two theories correspond to the same underlying Spin(7) orientifold, realizing the general idea of $\mathcal{N} = (0,1)$ triality arising from the non-uniqueness of the map between Spin(7) orientifolds and gauge theories.

We illustrate this projection in figure 6, which shows the neighborhood of the quiver around a dualized node 0.[10] As in the previous section, nodes 1, 2 and 3 represent possibly

[10]The universal involution with USp projection is analogous.



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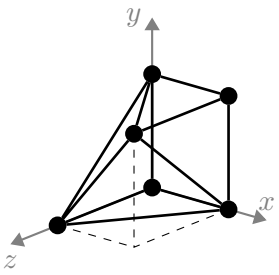


**Figure 7**. Toric diagram for $H_4$.

multiple nodes which, in turn, might be connected to node 0 by different multiplicities of fields. The red and black dashed lines represent the rest of the quiver, which might include fields stretching between nodes 1, 2 and 3. If triality generates massive fields, they can be integrated out.

An explicit example of a triality pairs associated to the universal involution will be presented in section 4.1. However, in section 5, we will show how more general orientifold actions lead to interesting generalizations of the basic $\mathcal{N} = (0,1)$ triality. The general strategy will be to focus on parent $CY_4$ geometries with more than one $\mathcal{N} = (0,2)$ triality dual toric phases[11] (see e.g. [7, 15]) and to consider anti-holomorphic involutions leading to the same Spin(7) orientifold.

### 4.1 The universal involution of $H_4$

As explained above, the universal involution works for every $CY_4$. Therefore, it is sufficient to present one example to illustrate the main features of the construction. Let us consider the $CY_4$ with toric diagram shown in figure 7, which is often referred to as $H_4$. Below we consider two toric phases for D1-branes probing $H_4$ and construct the $\mathcal{N} = (0,1)$ theories that correspond to them via the universal involution.

#### 4.1.1 Phase A

Figure 8 shows the quiver diagram for one of the toric phases of $H_4$, which we denote phase A. This theory was first introduced in [20].

The corresponding $J$- and $E$-terms are

| | $J$ | $E$ |
|---|---|---|
| $\Lambda^1_{11}$ : | $X_{14}X_{41} - X_{13}X_{32}Z_{21}$ | $Y_{13}X_{34}Z_{41} - X_{12}Y_{21}$ |
| $\Lambda^2_{11}$ : | $X_{14}Y_{41} - Y_{13}X_{32}Z_{21}$ | $X_{12}X_{21} - X_{13}X_{34}Z_{41}$ |
| $\Lambda^3_{11}$ : | $X_{14}Z_{41} - X_{12}Z_{21}$ | $X_{13}X_{32}Y_{21} - Y_{13}X_{34}X_{41}$ |
| $\Lambda^1_{13}$ : | $X_{32}X_{21} - X_{34}X_{41}$ | $Y_{13}X_{33} - X_{14}Z_{41}Y_{13}$ |
| $\Lambda^2_{13}$ : | $X_{32}Y_{21} - X_{34}Y_{41}$ | $X_{12}Z_{21}X_{13} - X_{13}X_{33}$ |
| $\Lambda^1_{42}$ : | $X_{21}X_{14} - Z_{21}X_{13}X_{34}$ | $Z_{41}Y_{13}X_{32} - Y_{41}X_{12}$ |

[11] We refer to a toric phase as one associated to a brane brick model [19], for which the connection to the underlying $CY_4$ is considerably simplified.



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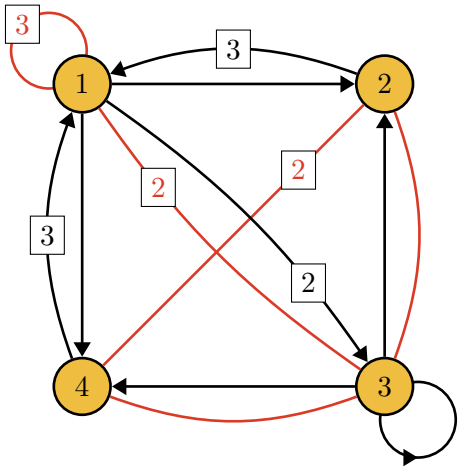


**Figure 8**. Quiver diagram for phase A of $H_4$.

$$\begin{array}{lll} \Lambda^2_{42}: & Y_{21}X_{14} - Z_{21}Y_{13}X_{34} & X_{41}X_{12} - Z_{41}X_{13}X_{32} \\ \Lambda_{23}: & X_{33}X_{32} - X_{32}Z_{21}X_{12} & Y_{21}X_{13} - X_{21}Y_{13} \\ \Lambda_{43}: & X_{33}X_{34} - X_{34}Z_{41}X_{14} & X_{41}Y_{13} - Y_{41}X_{13}\,. \end{array} \tag{4.1}$$

The $\mathcal{N} = (0,1)$ superpotential is then

$$\begin{aligned} W^{(0,1)} = W^{(0,2)} &+ \Lambda^{4R}_{11}(X^\dagger_{12}X_{12} + X^\dagger_{14}X_{14} + X^\dagger_{21}X_{21} + Y^\dagger_{21}Y_{21} + Z^\dagger_{21}Z_{21} + \\ &+ X^\dagger_{41}X_{41} + Y^\dagger_{41}Y_{41} + Z^\dagger_{41}Z_{41} + X^\dagger_{13}X_{13} + Y^\dagger_{13}Y_{13}) + \\ &+ \Lambda^R_{22}(X^\dagger_{12}X_{12} + X^\dagger_{32}X_{32} + X^\dagger_{21}X_{21} + Y^\dagger_{21}Y_{21} + Z^\dagger_{21}Z_{21}) + \\ &+ \Lambda^R_{33}(X^\dagger_{33}X_{33} + X^\dagger_{32}X_{32} + X^\dagger_{34}X_{34} + X^\dagger_{13}X_{13} + Y^\dagger_{13}Y_{13}) + \\ &+ \Lambda^R_{44}(X^\dagger_{14}X_{14} + X^\dagger_{34}X_{34} + X^\dagger_{41}X_{41} + Y^\dagger_{41}Y_{41} + Z^\dagger_{41}Z_{41})\,. \end{aligned} \tag{4.2}$$

The generators of $H_4$, which arises as the moduli space of the gauge theory, can be determined for instance using the Hilbert Series (HS) [19, 23, 24] (see also [4]). We list them in table 1, together with their expressions as mesons in terms of chiral fields in phase A.

The generators satisfy the following relations

$$\begin{aligned} \mathcal{I} = \langle & M_1M_4 = M_2M_3\,, M_1M_7 = M_2M_6\,, M_1M_7 = M_3M_5\,, M_2M_7 = M_4M_5\,, \\ & M_3M_7 = M_4M_6\,, M_1M_8 = M_5M_6\,, M_2M_8 = M_5M_7\,, M_3M_8 = M_6M_7\,, \\ & M_4M_8 = M_7^2 \rangle\,. \end{aligned} \tag{4.3}$$

The universal involution acts on the fields of any theory as (3.2). This results in the expected map of the generators

$$\begin{array}{llll} M_1 \to \bar{M}_1\,, & M_2 \to \bar{M}_2\,, & M_3 \to \bar{M}_3\,, & M_4 \to \bar{M}_4\,, \\ M_5 \to \bar{M}_5\,, & M_6 \to \bar{M}_6\,, & M_7 \to \bar{M}_7\,, & M_8 \to \bar{M}_8\,. \end{array} \tag{4.4}$$



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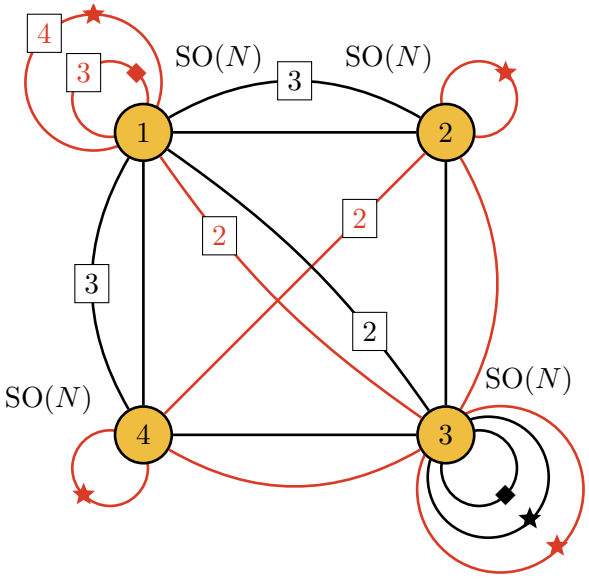


**Figure 9**. Quiver diagram for the Spin(7) orientifold of phase A of $H_4$ using the universal involution.

| Meson | Chiral superfields |
|---|---|
| $M_1$ | $X_{33} = X_{14}Z_{41} = Z_{21}X_{12}$ |
| $M_2$ | $Y_{21}X_{12} = Z_{41}Y_{13}X_{34}$ |
| $M_3$ | $X_{14}Y_{41} = Z_{21}Y_{13}X_{32}$ |
| $M_4$ | $X_{32}Y_{21}Y_{13} = X_{34}Y_{41}Y_{13}$ |
| $M_5$ | $X_{21}X_{12} = Z_{41}X_{13}X_{34}$ |
| $M_6$ | $X_{14}X_{41} = Z_{21}X_{13}X_{32}$ |
| $M_7$ | $X_{32}Y_{21}X_{13} = X_{32}X_{21}Y_{13} = X_{34}Y_{41}X_{13} = X_{34}X_{41}Y_{13}$ |
| $M_8$ | $X_{32}X_{21}X_{13} = X_{34}X_{41}X_{13}$ |

**Table 1**. Generators of $H_4$ in terms of fields in phase A.

The quiver for the 2d $\mathcal{N} = (0,1)$ orientifold theory is shown in figure 9. It is rather straightforward to write the projected superpotential but, for brevity, we will omit it here and in the examples that follow.

### 4.1.2 Phase B

Let us now consider the so-called phase $B$ of $H_4$ [20]. Its quiver diagram is shown in figure 10.

The $J$- and $E$-terms are

$$\begin{array}{cll} & J & E \\ \Lambda_{21}: & X_{13}X_{34}Y_{42} - Y_{13}X_{34}X_{42} & X_{21}X_{14}X_{41} - X_{23}X_{32}X_{21} \\ \Lambda^1_{12}: & X_{23}X_{34}Y_{42}X_{21} - X_{21}Y_{13}X_{34}X_{41} & X_{13}X_{32} - X_{14}X_{42} \end{array}$$



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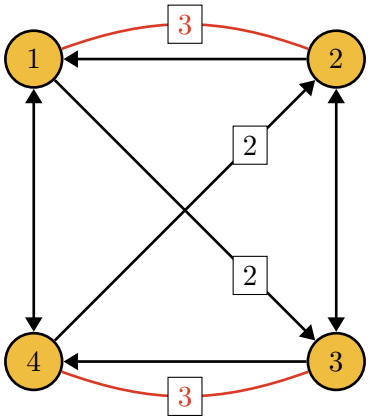


**Figure 10**. Quiver diagram for phase B of $H_4$.

$$\begin{array}{lll}
\Lambda^2_{12}: & X_{21}X_{13}X_{34}X_{41} - X_{23}X_{34}X_{42}X_{21} & Y_{13}X_{32} - X_{14}Y_{42} \\
\Lambda_{34}: & Y_{42}X_{21}X_{13} - X_{42}X_{21}Y_{13} & X_{34}X_{41}X_{14} - X_{32}X_{23}X_{34} \\
\Lambda^1_{43}: & X_{34}Y_{42}X_{21}X_{14} - X_{32}X_{21}Y_{13}X_{34} & X_{42}X_{23} - X_{41}X_{13} \\
\Lambda^2_{43}: & X_{32}X_{21}X_{13}X_{34} - X_{34}X_{42}X_{21}X_{14} & Y_{42}X_{23} - X_{41}Y_{13}\,.
\end{array} \tag{4.5}$$

The corresponding $W^{(0,1)}$ is

$$\begin{aligned}
W^{(0,1)} = W^{(0,2)} &+ \Lambda^R_{11}(X_{21}X^\dagger_{21} + X_{41}X^\dagger_{41} + X_{14}X^\dagger_{14} + X_{13}X^\dagger_{13} + Y_{13}Y^\dagger_{13}) + \\
&+ \Lambda^R_{22}(X_{23}X^\dagger_{23} + X_{21}X^\dagger_{21} + X_{42}X^\dagger_{42} + X_{32}X^\dagger_{32} + Y_{42}Y^\dagger_{42}) + \\
&+ \Lambda^R_{33}(X_{23}X^\dagger_{23} + X_{32}X^\dagger_{32} + X_{34}X^\dagger_{34} + X_{13}X^\dagger_{13} + Y_{13}Y^\dagger_{13}) + \\
&+ \Lambda^R_{44}(X_{42}X^\dagger_{42} + X_{41}X^\dagger_{41} + X_{34}X^\dagger_{34} + X_{14}X^\dagger_{14} + Y_{42}Y^\dagger_{42})\,.
\end{aligned} \tag{4.6}$$

Table 2 lists the generators of $H_4$, this time expressed in terms of chiral fields in phase B. They satisfy the same relations we presented in (4.3) when discussing Phase A.

Once again, we consider the universal involution, which acts on the fields of phase B as in (3.2). This, in turn, maps the generators as in (4.4).

Figure 11, shows the resulting quiver for the orientifold theory. By construction, this gauge theory corresponds to the same Spin(7) orientifold as the one constructed from phase A in the previous section. In section 4.1.3, we will elaborate on the connection between both theories.

### 4.1.3 Triality between the orientifolded theories

Let us now elaborate on the connection between the two theories that we have constructed via the universal involution. Both of them correspond to the same Spin(7) orientifold of $H_4$. The parent theories, phases A and B of $H_4$, are related by $\mathcal{N} = (0,2)$ triality on either node 2 or 4 of phase A (equivalently, by inverse triality on the same nodes of phase B). This leads to a similar connection between the two orientifolded theories, this time via



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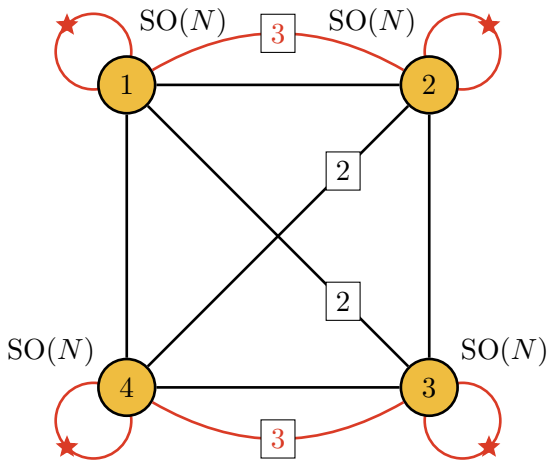


**Figure 11**. Quiver diagram for the Spin(7) orientifold of phase B of $H_4$ using the universal involution.

| Meson | Chiral superfields |
|---|---|
| $M_1$ | $X_{23}X_{32} = X_{41}X_{14}$ |
| $M_2$ | $X_{34}Y_{42}X_{23} = X_{34}X_{41}Y_{13}$ |
| $M_3$ | $X_{21}X_{14}Y_{42} = X_{21}Y_{13}X_{32}$ |
| $M_4$ | $X_{34}Y_{42}X_{21}Y_{13}$ |
| $M_5$ | $X_{34}X_{42}X_{23} = X_{34}X_{41}X_{13}$ |
| $M_6$ | $X_{21}X_{14}X_{42} = X_{21}X_{13}X_{32}$ |
| $M_7$ | $X_{42}X_{21}Y_{13}X_{34} = Y_{42}X_{21}X_{13}X_{34}$ |
| $M_8$ | $X_{42}X_{21}X_{13}X_{34}$ |

**Table 2**. Generators of $H_4$ in terms of fields in phase B.

$\mathcal{N} = (0, 1)$ triality on node 2 or 4. Figure 12 summarizes the interplay between triality and orientifolding. This was expected, given our general discussion of the universal involution in section 4.

It is important to emphasize that it is possible for two Spin(7) orientifolds to correspond to the same geometric involution while differing in the choice of vector structure. In practical terms, the appearance of the choices of vector structure in orientifolds arises when, for a given geometry, there are different $\mathbb{Z}_2$ symmetries on the underlying quiver gauge theory, which differ in the action on the quiver nodes. Such a discrete choice generalizes beyond orbifold singularities, and it was studied in detail in [4], in anticipation of the application of Spin(7) orientifolds to triality that we carry out in this paper. In order for equivalent orientifold geometric involutions to actually produce dual theories, it is necessary that they also agree on the choice of vector structure they implicitly define. This is the case for all the examples considered in this paper.

Finally, it is interesting to note that, as we discussed in section 2.3, in orientifold theories the number of "incoming flavors" at the dualized node is inherited from the parent.



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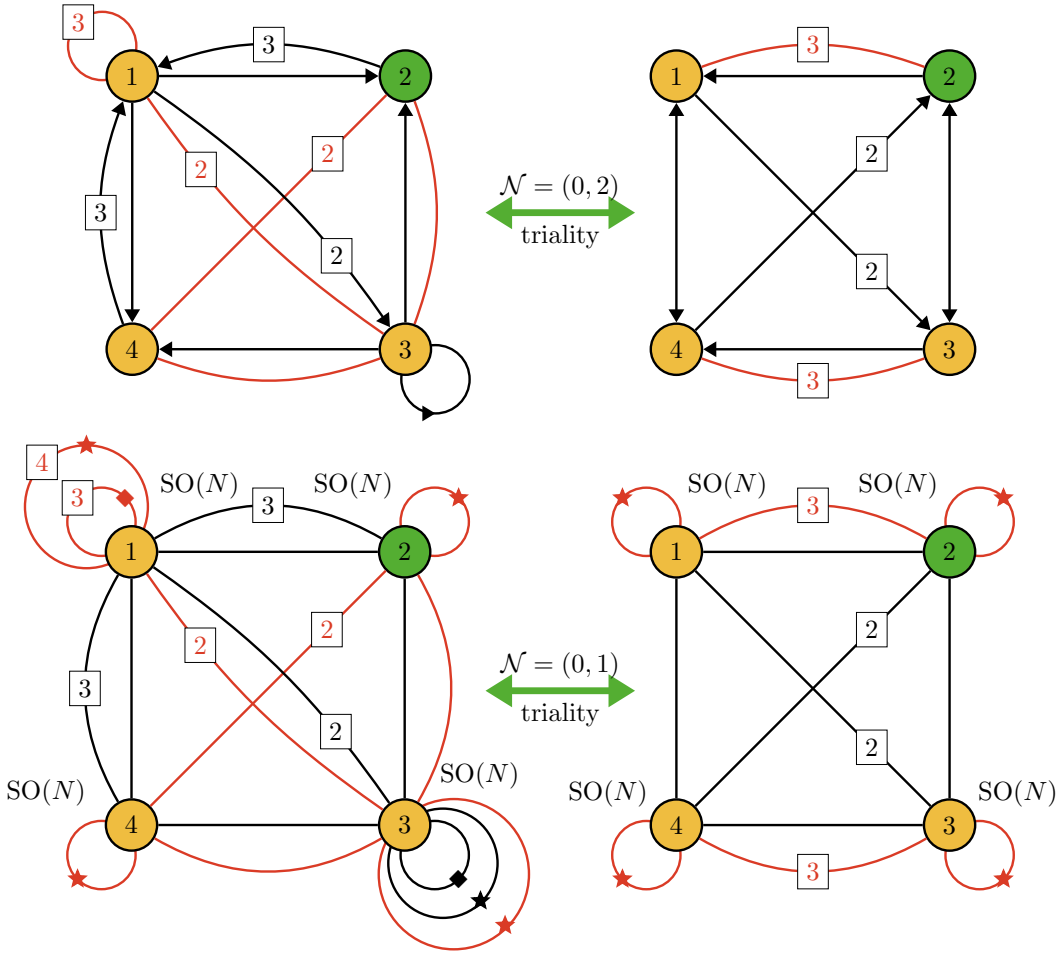


**Figure 12**. Phases A and B of $H_4$ are connected by $\mathcal{N} = (0, 2)$ triality on node 2 (shown in green). Upon orientifolding with the universal involution, the resulting theories are similarly connected by $\mathcal{N} = (0, 1)$ triality.

## 5 Beyond the universal involution

In this section, we present theories that are obtained from $\mathcal{N} = (0, 2)$ triality dual parents by Spin(7) orientifolds that do not correspond to the universal involution. We will see that they lead to interesting generalizations of the basic $\mathcal{N} = (0, 1)$ triality.[12]

### 5.1 $Q^{1,1,1}$

Let us now consider the cone over $Q^{1,1,1}$, or $Q^{1,1,1}$ for short, whose toric diagram is shown in figure 13. The $\mathcal{N} = (0, 2)$ gauge theories, brane brick models and the triality web relating

[12] We will rightfully continue referring to the resulting equivalences between theories as trialities, due to their connections to the basic trialities of SQCD-type theories. It is reasonable to expect that we can indeed perform these transformations three times on the same quiver node. However, the three transformations, can sometimes fall outside our analysis, provided they actually exist. This is due to our restriction to the class of theories obtained as Spin(7) orientifolds of toric phases.



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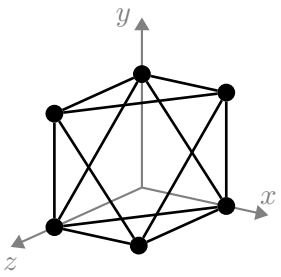


**Figure 13**. Toric diagram for $Q^{1,1,1}$.

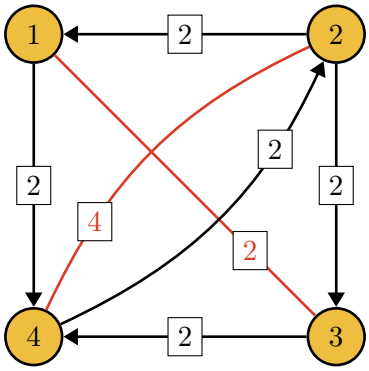


**Figure 14**. Quiver diagram for phase A of $Q^{1,1,1}$.

the toric phases for this geometry have been studied at length [7, 19, 25]. However, none of its Spin(7) orientifolds has been presented in the literature. Below, we construct an orientifold based on a non-universal involution.

### 5.1.1 Phase A

The toric phases for $Q^{1,1,1}$ were studied in [7]. Figure 14 shows the quiver for the so-called phase A.

The $J$- and $E$-terms are

$$\begin{array}{lll} & J & E \\ \Lambda^1_{24}: & Y_{42}X_{23}Y_{34}X_{42} - Y_{42}X_{21}X_{14}X_{42} & Y_{23}X_{34} - Y_{21}Y_{14} \\ \Lambda^2_{24}: & Y_{42}Y_{21}Y_{14}X_{42} - Y_{42}Y_{23}X_{34}X_{42} & X_{23}Y_{34} - X_{21}X_{14} \\ \Lambda^3_{24}: & Y_{42}X_{14}Y_{21}X_{42} - Y_{42}X_{23}X_{34}X_{42} & X_{21}Y_{14} - Y_{23}Y_{34} \\ \Lambda^4_{24}: & Y_{42}X_{21}Y_{14}X_{42} - Y_{42}Y_{23}Y_{34}X_{42} & X_{23}X_{34} - Y_{21}X_{14} \\ \Lambda^1_{31}: & Y_{14}X_{42}X_{23} - X_{14}X_{42}Y_{23} & X_{34}Y_{42}X_{21} - Y_{34}Y_{42}Y_{21} \\ \Lambda^2_{31}: & X_{14}Y_{42}Y_{23} - Y_{14}Y_{42}X_{23} & X_{34}X_{42}X_{21} - Y_{34}X_{42}Y_{21}\,. \end{array} \tag{5.1}$$

Finding the corresponding $W^{(0,1)}$ is a simple exercise, but we omit it here for brevity. Table 3 lists the generators for $Q^{1,1,1}$ written in terms of the gauge theory.



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| Field | Chiral superfields |
|---|---|
| $M_1$ | $Y_{42}Y_{23}X_{34} = Y_{42}Y_{21}Y_{14}$ |
| $M_2$ | $X_{42}Y_{23}X_{34} = X_{42}Y_{21}Y_{14}$ |
| $M_3$ | $Y_{42}X_{23}X_{34} = Y_{42}Y_{21}X_{14}$ |
| $M_4$ | $X_{42}X_{23}X_{34} = X_{42}Y_{21}X_{14}$ |
| $M_5$ | $Y_{42}Y_{23}Y_{34} = Y_{42}X_{21}Y_{14}$ |
| $M_6$ | $X_{42}Y_{23}Y_{34} = X_{42}X_{21}Y_{14}$ |
| $M_7$ | $Y_{42}X_{23}Y_{34} = Y_{42}X_{21}X_{14}$ |
| $M_8$ | $X_{42}X_{23}Y_{34} = X_{42}X_{21}X_{14}$ |

**Table 3**. Generators of $Q^{1,1,1}$ in terms of fields in phase A.

The generators satisfy the following relations

$$\begin{aligned}\mathcal{I} = \langle & M_1M_7 = M_3M_5\,, M_3M_8 = M_4M_7\,, M_1M_4 = M_2M_3\,, M_5M_8 = M_7M_6\,,\\ & M_1M_8 = M_2M_7\,, M_3M_6 = M_4M_5\,, M_1M_8 = M_4M_5\,, M_1M_6 = M_5M_2\,,\\ & M_2M_8 = M_4M_6\rangle\ .\end{aligned} \tag{5.2}$$

Let us now consider the involution that maps all the four gauge groups to themselves and has the following action on chiral fields

$$\begin{aligned} &Y_{42} \to -\gamma_{\Omega_4}\bar{X}_{42}\gamma_{\Omega_2}^{-1}\,, \quad X_{42} \to \gamma_{\Omega_4}\bar{Y}_{42}\gamma_{\Omega_2}^{-1}\,, \quad X_{34} \to \gamma_{\Omega_3}\bar{Y}_{34}\gamma_{\Omega_4}^{-1}\,, \quad Y_{34} \to -\gamma_{\Omega_3}\bar{X}_{34}\gamma_{\Omega_4}^{-1}\,,\\ &X_{21} \to -\gamma_{\Omega_2}\bar{Y}_{21}\gamma_{\Omega_1}^{-1}\,, \quad Y_{21} \to \gamma_{\Omega_2}\bar{X}_{21}\gamma_{\Omega_1}^{-1}\,, \quad Y_{23} \to \gamma_{\Omega_2}\bar{Y}_{23}\gamma_{\Omega_3}^{-1}\,, \quad X_{23} \to \gamma_{\Omega_2}\bar{X}_{23}\gamma_{\Omega_3}^{-1}\,,\\ &\qquad\qquad Y_{14} \to \gamma_{\Omega_1}\bar{Y}_{14}\gamma_{\Omega_4}^{-1}\,, \quad X_{14} \to \gamma_{\Omega_1}\bar{X}_{14}\gamma_{\Omega_4}^{-1}\,, \end{aligned} \tag{5.3}$$

where we have used the $\gamma_{\Omega_i}$ matrices mentioned in footnote 9.

Invariance of $W^{(0,1)}$ further implies that the involution acts on Fermi fields as follows

$$\begin{aligned} &\Lambda^1_{24} \to -\gamma_{\Omega_2}\bar{\Lambda}^3_{24}\gamma_{\Omega_4}^{-1}\,, \quad \Lambda^2_{24} \to -\gamma_{\Omega_2}\bar{\Lambda}^4_{24}\gamma_{\Omega_4}^{-1}\,, \quad \Lambda^3_{24} \to \gamma_{\Omega_2}\bar{\Lambda}^1_{24}\gamma_{\Omega_4}^{-1}\,,\\ &\Lambda^4_{24} \to \gamma_{\Omega_2}\bar{\Lambda}^2_{24}\gamma_{\Omega_4}^{-1}\,, \quad \Lambda^1_{31} \to -\gamma_{\Omega_3}\bar{\Lambda}^2_{31}\gamma_{\Omega_1}^{-1}\,, \quad \Lambda^2_{31} \to \gamma_{\Omega_3}\Lambda^1_{31}\gamma_{\Omega_1}^{-1}\,, \end{aligned} \tag{5.4}$$

and

$$\Lambda^R_{11} \to \gamma_{\Omega_1}\Lambda^{R\,T}_{11}\gamma_{\Omega_1}^{-1}\,, \quad \Lambda^R_{22} \to \gamma_{\Omega_2}\Lambda^{R\,T}_{22}\gamma_{\Omega_2}^{-1}\,, \quad \Lambda^R_{33} \to \gamma_{\Omega_3}\Lambda^{R\,T}_{33}\gamma_{\Omega_3}^{-1}\,, \quad \Lambda^R_{44} \to \gamma_{\Omega_4}\Lambda^{R\,T}_{44}\gamma_{\Omega_4}^{-1}\,. \tag{5.5}$$

Interestingly, the involution in (5.3) and (5.4) involves a non-trivial action on flavor indices (see e.g. the action on pairs of fields such as $(X_{21}, Y_{21})$). As briefly mentioned in section 3, this leads to a constraint on the matrices $\gamma_{\Omega_i}$ that encode the action of the orientifold group on the gauge groups, which reads

$$\gamma_{\Omega_1} = \gamma_{\Omega_4} \neq \gamma_{\Omega_2} = \gamma_{\Omega_3}\,. \tag{5.6}$$

This constraint follows for requiring that the involution squares to the identity. For a detailed discussion of this constraint and additional explicit examples, we refer the interested reader to our previous work [4].



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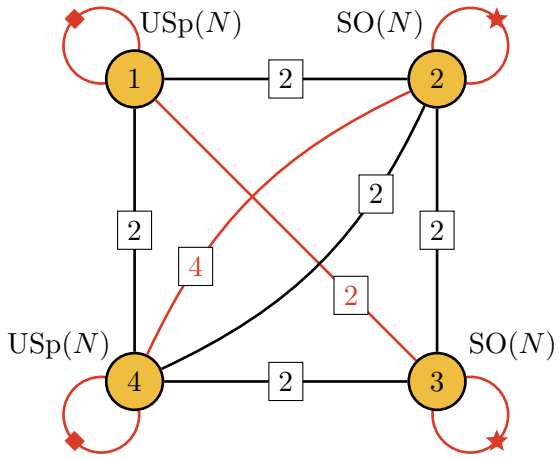


**Figure 15**. Quiver diagram for the Spin(7) orientifold of phase A of $Q^{1,1,1}$ using the involution in eqs. (5.3), (5.4) and (5.5), together with our choice of $\gamma_{\Omega_i}$ matrices.

For concreteness, we will focus on the following solution to the constraint

$$\begin{aligned} \gamma_{\Omega_1} &= \gamma_{\Omega_4} = J\,, \\ \gamma_{\Omega_2} &= \gamma_{\Omega_3} = \mathbb{1}_N\,, \end{aligned} \tag{5.7}$$

where $J = i\epsilon_{N/2}$ is the symplectic matrix, and $\mathbb{1}_N$ is the identity matrix.

Using table 3, the involution in (5.3) translates into the following action at the level of the geometry

$$\begin{aligned} &M_1 \to -\bar{M}_6\,, \quad M_2 \to \bar{M}_5\,, \quad M_3 \to -\bar{M}_8\,, \quad M_4 \to \bar{M}_7\,, \\ &M_5 \to \bar{M}_2\,, \quad M_6 \to -\bar{M}_1\,, \quad M_7 \to \bar{M}_4\,, \quad M_8 \to -\bar{M}_3\,, \end{aligned} \tag{5.8}$$

which is clearly not the universal involution.

Figure 15 shows the quiver for the orientifold theory, which is free of gauge anomalies.

### 5.1.2 Phase S

Figure 16 shows the quiver for phase S of $Q^{1,1,1}$ [7].

The $J$- and $E$-terms are

| | $J$ | $E$ |
|---|---|---|
| $\Lambda^1_{23}$ : | $X_{34}Y_{42} - Y_{34}W_{42}$ | $Y_{24}X_{43} - X_{24}Z_{43}$ |
| $\Lambda^2_{23}$ : | $X_{34}X_{42} - Y_{34}Z_{42}$ | $X_{24}W_{43} - Y_{24}Y_{43}$ |
| $\Lambda^1_{31}$ : | $X_{14}Z_{43} - Y_{14}W_{43}$ | $X_{34}X_{41} - Y_{34}Z_{41}$ |
| $\Lambda^2_{31}$ : | $X_{14}X_{43} - Y_{14}Y_{43}$ | $Y_{34}W_{41} - X_{34}Y_{41}$ |
| $\Lambda^1_{12}$ : | $Y_{24}Z_{41} - X_{24}W_{41}$ | $X_{14}X_{42} - Y_{14}Y_{42}$ |
| $\Lambda^2_{12}$ : | $Y_{24}X_{41} - X_{24}Y_{41}$ | $Y_{14}W_{42} - X_{14}Z_{42}$ |
| $\Lambda^1_{44}$ : | $Y_{43}Y_{34} - X_{41}X_{14}$ | $W_{41}Y_{14} - Z_{42}Y_{24}$ |



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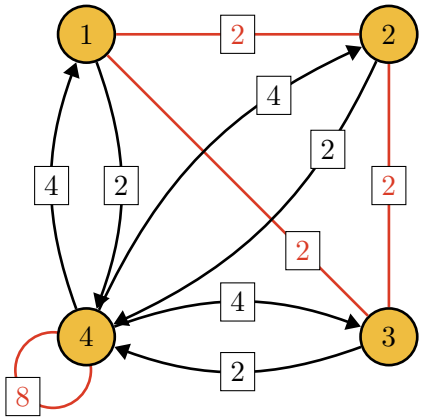


**Figure 16**. Quiver diagram for phase S of $Q^{1,1,1}$.

$$\begin{aligned}
\Lambda^2_{44}: &\quad W_{41}Y_{14} - Z_{43}X_{34} \qquad X_{41}X_{14} - Y_{42}X_{24}\\
\Lambda^3_{44}: &\quad Y_{41}Y_{14} - X_{42}Y_{24} \qquad W_{42}X_{24} - Y_{43}X_{34}\\
\Lambda^4_{44}: &\quad W_{42}X_{24} - Z_{41}X_{14} \qquad X_{42}Y_{24} - Z_{43}Y_{34}\\
\Lambda^5_{44}: &\quad Z_{42}X_{24} - X_{43}X_{34} \qquad Y_{24}Y_{42} - Y_{41}X_{14}\\
\Lambda^6_{44}: &\quad W_{43}Y_{34} - Y_{42}Y_{24} \qquad X_{24}Z_{42} - Z_{41}Y_{14}\\
\Lambda^7_{44}: &\quad W_{43}X_{34} - W_{41}X_{14} \qquad X_{41}Y_{14} - X_{42}X_{24}\\
\Lambda^8_{44}: &\quad X_{41}Y_{14} - X_{43}Y_{34} \qquad W_{41}X_{14} - W_{42}Y_{24}\,.
\end{aligned} \tag{5.9}$$

Table 4 shows the generators of $Q^{1,1,1}$ in terms of the gauge theory. They satisfy the same relations given in (5.2).

Let us consider the involution that maps all gauge groups to themselves and acts on chiral fields as follows

$$\begin{aligned}
&Z_{42}\to\gamma_{\Omega_4}\bar{X}_{42}\gamma^{-1}_{\Omega_2}\,, \quad X_{42}\to-\gamma_{\Omega_4}\bar{Z}_{42}\gamma^{-1}_{\Omega_2}\,, \quad Y_{24}\to-\gamma_{\Omega_2}\bar{X}_{24}\gamma^{-1}_{\Omega_4}\,, \quad X_{24}\to\gamma_{\Omega_2}\bar{Y}_{24}\gamma^{-1}_{\Omega_4}\,,\\
&Z_{43}\to-\gamma_{\Omega_4}\bar{X}_{43}\gamma^{-1}_{\Omega_3}\,, \quad X_{43}\to\gamma_{\Omega_4}\bar{Z}_{43}\gamma^{-1}_{\Omega_3}\,, \quad X_{34}\to\gamma_{\Omega_3}\bar{Y}_{34}\gamma^{-1}_{\Omega_4}\,, \quad Y_{34}\to-\gamma_{\Omega_3}\bar{X}_{34}\gamma^{-1}_{\Omega_4}\,,\\
&W_{41}\to-\gamma_{\Omega_4}\bar{X}_{41}\gamma^{-1}_{\Omega_1}\,, \quad X_{41}\to-\gamma_{\Omega_4}\bar{W}_{41}\gamma^{-1}_{\Omega_1}\,, \quad Z_{41}\to\gamma_{\Omega_4}\bar{Y}_{41}\gamma^{-1}_{\Omega_1}\,, \quad Y_{41}\to\gamma_{\Omega_4}\bar{Z}_{41}\gamma^{-1}_{\Omega_1}\,,\\
&W_{42}\to\gamma_{\Omega_4}\bar{Y}_{42}\gamma^{-1}_{\Omega_2}\,, \quad Y_{42}\to-\gamma_{\Omega_4}\bar{W}_{42}\gamma^{-1}_{\Omega_2}\,, \quad W_{43}\to-\gamma_{\Omega_4}\bar{Y}_{43}\gamma^{-1}_{\Omega_3}\,, \quad Y_{43}\to\gamma_{\Omega_4}\bar{W}_{43}\gamma^{-1}_{\Omega_3}\,,\\
&\qquad\qquad Y_{14}\to\gamma_{\Omega_1}\bar{Y}_{14}\gamma^{-1}_{\Omega_4}\,, \quad X_{14}\to\gamma_{\Omega_1}\bar{X}_{14}\gamma^{-1}_{\Omega_4}\,.
\end{aligned} \tag{5.10}$$

As we will explain shortly, we have chosen this involution in order to connect to the orientifold of phase A that we constructed in the previous section.

Invariance of $W^{(0,1)}$ implies the following action on Fermi fields

$$\begin{aligned}
&\Lambda^1_{23}\to\gamma_{\Omega_2}\bar{\Lambda}^1_{23}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^2_{23}\to\gamma_{\Omega_2}\bar{\Lambda}^2_{23}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^1_{31}\to-\gamma_{\Omega_3}\bar{\Lambda}^2_{31}\gamma^{-1}_{\Omega_1}\,, \quad \Lambda^2_{31}\to\gamma_{\Omega_3}\bar{\Lambda}^1_{31}\gamma^{-1}_{\Omega_1}\,,\\
&\Lambda^1_{12}\to-\gamma_{\Omega_1}\bar{\Lambda}^2_{12}\gamma^{-1}_{\Omega_2}\,, \quad \Lambda^2_{12}\to\gamma_{\Omega_1}\bar{\Lambda}^1_{12}\gamma^{-1}_{\Omega_2}\,, \quad \Lambda^1_{44}\to-\gamma_{\Omega_4}\bar{\Lambda}^7_{44}\gamma^{-1}_{\Omega_4}\,, \quad \Lambda^7_{44}\to-\gamma_{\Omega_4}\bar{\Lambda}^1_{44}\gamma^{-1}_{\Omega_4}\,,
\end{aligned}$$



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| Field | Chiral superfields |
|---|---|
| $M_1$ | $Z_{42}Y_{24} = Z_{43}X_{34} = W_{41}Y_{14}$ |
| $M_2$ | $Z_{42}X_{24} = X_{43}X_{34} = Z_{41}Y_{14}$ |
| $M_3$ | $W_{42}Y_{24} = W_{43}X_{34} = W_{41}X_{14}$ |
| $M_4$ | $W_{42}X_{24} = Y_{43}X_{34} = Z_{41}X_{14}$ |
| $M_5$ | $X_{42}Y_{24} = Z_{43}Y_{34} = Y_{41}Y_{14}$ |
| $M_6$ | $X_{42}X_{24} = X_{43}Y_{34} = X_{41}Y_{14}$ |
| $M_7$ | $Y_{42}Y_{24} = W_{43}Y_{34} = Y_{41}X_{14}$ |
| $M_8$ | $Y_{42}X_{24} = Y_{43}Y_{34} = X_{41}X_{14}$ |

**Table 4**. Generators of $Q^{1,1,1}$ in terms of fields in phase S.

$$\Lambda^2_{44} \to -\gamma_{\Omega_4}\bar{\Lambda}^8_{44}\gamma^{-1}_{\Omega_4}\,, \quad \Lambda^8_{44} \to -\gamma_{\Omega_4}\bar{\Lambda}^2_{44}\gamma^{-1}_{\Omega_4}\,, \quad \Lambda^3_{44} \to -\gamma_{\Omega_4}\Lambda^{6\,T}_{44}\gamma^{-1}_{\Omega_4}\,, \quad \Lambda^6_{44} \to -\gamma_{\Omega_4}\Lambda^{3\,T}_{44}\gamma^{-1}_{\Omega_4}\,,$$
$$\Lambda^4_{44} \to \gamma_{\Omega_4}\Lambda^{5\,T}_{44}\gamma^{-1}_{\Omega_4}\,, \quad \Lambda^5_{44} \to \gamma_{\Omega_4}\Lambda^{4\,T}_{44}\gamma^{-1}_{\Omega_4}\,, \tag{5.11}$$

and

$$\Lambda^R_{11} \to \gamma_{\Omega_1}\Lambda^{R\,T}_{11}\gamma^{-1}_{\Omega_1}\,, \quad \Lambda^R_{22} \to \gamma_{\Omega_2}\Lambda^{R\,T}_{22}\gamma^{-1}_{\Omega_2}\,, \quad \Lambda^R_{33} \to \gamma_{\Omega_3}\Lambda^{R\,T}_{33}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^R_{44} \to \gamma_{\Omega_4}\Lambda^{R\,T}_{44}\gamma^{-1}_{\Omega_4}\,. \tag{5.12}$$

The involution on bifundamental fields leads to the same constraints on the $\gamma_{\Omega_i}$ matrices as in (5.6). As for phase A, we pick

$$\begin{aligned} \gamma_{\Omega_1} &= \gamma_{\Omega_4} = J\,, \\ \gamma_{\Omega_2} &= \gamma_{\Omega_3} = \mathbb{1}_N\,. \end{aligned} \tag{5.13}$$

Using table 4, (5.10) translates into the following action on the generators

$$\begin{aligned} &M_1 \to -\bar{M}_6\,, \quad M_2 \to \bar{M}_5\,, \quad M_3 \to -\bar{M}_8\,, \quad M_4 \to \bar{M}_7\,, \\ &M_5 \to \bar{M}_2\,, \quad M_6 \to -\bar{M}_1\,, \quad M_7 \to \bar{M}_4\,, \quad M_8 \to -\bar{M}_3\,, \end{aligned} \tag{5.14}$$

which is the same geometric involution that we found for phase A in (5.8). Therefore, the involutions considered on these two phases correspond to the same Spin(7) orientifold of $Q^{1,1,1}$. Figure 17 shows the quiver for the orientifold theory, which is free of gauge anomalies.

### 5.1.3 Triality between the orientifolded theories

Figure 18 summarizes the connections between the theories considered in this section. Again, we observe that the two theories we constructed for the same Spin(7) orientifold are related by $\mathcal{N} = (0,1)$ triality. More precisely, they are related by a simple generalization of the basic triality reviewed in section 2.2. First, in this case, triality is applied to quivers with multiple gauge nodes. More importantly, some of the nodes that act as flavor groups are of a different type (in this example, USp) than the dualized node. As in previous examples, the orientifold construction leads to a clear prescription on how to treat scalar flavors, which is inherited from the parent theories.



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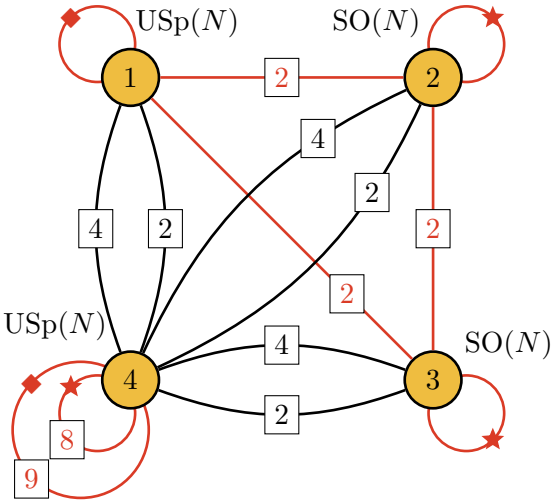


**Figure 17**. Quiver diagram for the Spin(7) orientifold of phase S of $Q^{1,1,1}$ using the involution in eqs. (5.10), (5.11) and (5.12), together with our choice of $\gamma_{\Omega_i}$ matrices.

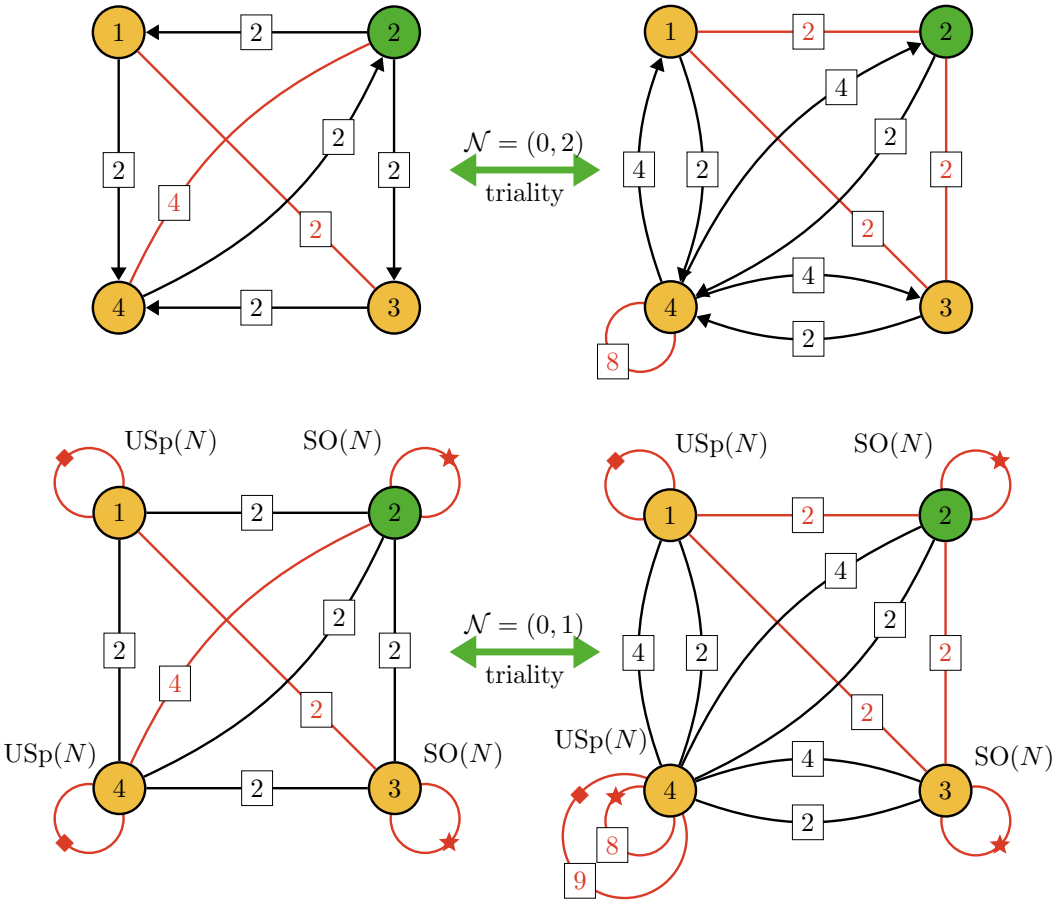


**Figure 18**. Phases A and S of $Q^{1,1,1}$ are connected by $\mathcal{N} = (0,2)$ triality on node 2 (shown in green). The orientifolded theories are similarly connected by $\mathcal{N} = (0,1)$ triality.



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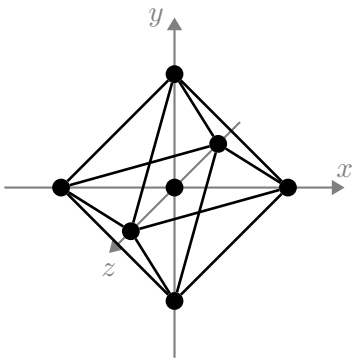


**Figure 19**. Toric diagram for $Q^{1,1,1}/\mathbb{Z}_2$.

### 5.2 Theories with unitary gauge groups: $Q^{1,1,1}/\mathbb{Z}_2$

All $\mathcal{N} = (0,1)$ triality examples we constructed so far contain only SO($N$) and USp($N$) gauge groups. Namely, the anti-holomorphic involutions of the parent $\mathcal{N} = (0,2)$ theories, universal or not, map all gauge groups to themselves. In this section we will construct Spin(7) orientifolds giving rise to gauge theories that include U($N$) gauge groups. To do so, we focus on $Q^{1,1,1}/\mathbb{Z}_2$, whose toric diagram is shown in figure 19.[13] This $CY_4$ has a rich family of 14 toric phases. They were classified in [15], whose nomenclature we will follow. We will restrict to a subset consisting of 5 of these toric phases. In order to streamline our discussion, several details about these theories are collected in appendix C.

Let us first consider phase D, whose quiver diagram is shown in figure 20. We provide a 3d representation of the quiver in order to make the action of the anti-holomorphic involution that we will use to construct a Spin(7) orientifold more manifest.

The $J$- and $E$-terms for this theory are

| | $J$ | $E$ |
|---|---|---|
| $\Lambda^1_{13}$ : | $W_{34}X_{41} - Y_{41}Z_{34}$ | $X_{18}X_{85}Y_{53} - X_{53}X_{85}Y_{18}$ |
| $\Lambda^2_{13}$ : | $X_{41}Y_{34} - X_{34}Y_{41}$ | $X_{53}Y_{18}Y_{85} - X_{18}Y_{53}Y_{85}$ |
| $\Lambda^1_{37}$ : | $X_{72}Y_{53}Y_{25} - X_{53}Y_{72}Y_{25}$ | $X_{47}Z_{34} - X_{34}Y_{47}$ |
| $\Lambda^2_{37}$ : | $X_{72}X_{25}Y_{53} - X_{53}X_{25}Y_{72}$ | $Y_{34}Y_{47} - W_{34}X_{47}$ |
| $\Lambda^1_{86}$ : | $X_{64}Y_{18}Y_{41} - X_{18}Y_{41}Y_{64}$ | $X_{56}Y_{85} - X_{85}Z_{56}$ |
| $\Lambda^2_{86}$ : | $X_{41}X_{64}Y_{18} - X_{18}X_{41}Y_{64}$ | $W_{56}X_{85} - Y_{56}Y_{85}$ |
| $\Lambda^1_{62}$ : | $Y_{25}Z_{56} - W_{56}X_{25}$ | $X_{47}X_{64}Y_{72} - X_{72}X_{47}Y_{64}$ |
| $\Lambda^2_{62}$ : | $X_{25}Y_{56} - X_{56}Y_{25}$ | $X_{64}Y_{72}Y_{47} - X_{72}Y_{47}Y_{64}$ |
| $\Lambda^1_{45}$ : | $W_{56}Y_{64} - W_{34}Y_{53}$ | $X_{18}X_{41}X_{85} - X_{72}X_{47}X_{25}$ |
| $\Lambda^2_{45}$ : | $W_{56}X_{64} - W_{34}X_{53}$ | $X_{47}X_{25}Y_{72} - X_{41}X_{85}Y_{18}$ |
| $\Lambda^3_{45}$ : | $Y_{64}Z_{56} - Y_{53}Z_{34}$ | $X_{72}X_{47}Y_{25} - X_{18}X_{85}Y_{41}$ |
| $\Lambda^4_{45}$ : | $X_{64}Z_{56} - X_{53}Z_{34}$ | $X_{85}Y_{18}Y_{41} - X_{47}Y_{72}Y_{25}$ |

[13] More precisely, this is the $\mathbb{Z}_2$ orbifold of the real cone over $Q^{1,1,1}$.



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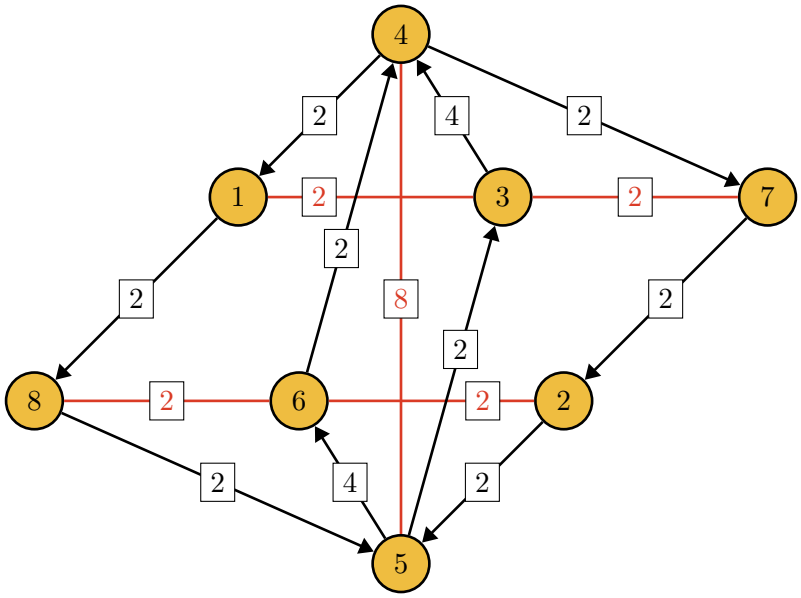


**Figure 20**. Quiver diagram for phase D of $Q^{1,1,1}/\mathbb{Z}_2$.

$$\begin{aligned}
\Lambda^5_{45}\ : &\quad Y_{56}Y_{64} - Y_{34}Y_{53} &&\quad X_{72}X_{25}Y_{47} - X_{18}X_{41}Y_{85}\\
\Lambda^6_{45}\ : &\quad X_{64}Y_{56} - X_{53}Y_{34} &&\quad X_{41}Y_{18}Y_{85} - X_{25}Y_{72}Y_{47}\\
\Lambda^7_{45}\ : &\quad X_{56}Y_{64} - X_{34}Y_{53} &&\quad X_{18}Y_{41}Y_{85} - X_{72}Y_{47}Y_{25}\\
\Lambda^8_{45}\ : &\quad X_{56}X_{64} - X_{34}X_{53} &&\quad Y_{72}Y_{47}Y_{25} - Y_{18}Y_{41}Y_{85}\,.
\end{aligned} \tag{5.15}$$

The generators of $Q^{1,1,1}/\mathbb{Z}_2$ in terms of the chiral fields in phase D are listed in table 7. Note that the generators and their relations are common to all the phases, but their realizations in terms of chiral superfields in each of them are different. Let us consider an anti-holomorphic involution of phase D which acts on figure 20 as a reflection with respect to the vertical plane that contains nodes 3, 4, 5 and 6. The nodes on the plane map to themselves, while the following pairs $1 \leftrightarrow 7$ and $2 \leftrightarrow 8$ get identified. This leads to the anticipated mixture of SO / USp and U gauge groups.

The involution on chiral fields is

$$\begin{aligned}
&X_{18} \to \gamma_{\Omega_7}\bar{Y}_{72}\gamma^{-1}_{\Omega_2}\,, && Y_{18} \to \gamma_{\Omega_7}\bar{X}_{72}\gamma^{-1}_{\Omega_2}\,, && X_{72} \to \gamma_{\Omega_1}\bar{Y}_{18}\gamma^{-1}_{\Omega_8}\,, && Y_{72} \to \gamma_{\Omega_1}\bar{X}_{18}\gamma^{-1}_{\Omega_8}\,,\\
&X_{34} \to \gamma_{\Omega_3}\bar{X}_{34}\gamma^{-1}_{\Omega_4}\,, && Y_{34} \to \gamma_{\Omega_3}\bar{Z}_{34}\gamma^{-1}_{\Omega_4}\,, && Z_{34} \to \gamma_{\Omega_3}\bar{Y}_{34}\gamma^{-1}_{\Omega_4}\,, && W_{34} \to \gamma_{\Omega_3}\bar{W}_{34}\gamma^{-1}_{\Omega_4}\,,\\
&X_{41} \to \gamma_{\Omega_4}\bar{X}_{47}\gamma^{-1}_{\Omega_7}\,, && Y_{41} \to \gamma_{\Omega_4}\bar{Y}_{47}\gamma^{-1}_{\Omega_7}\,, && X_{47} \to \gamma_{\Omega_4}\bar{X}_{41}\gamma^{-1}_{\Omega_1}\,, && Y_{47} \to \gamma_{\Omega_4}\bar{Y}_{41}\gamma^{-1}_{\Omega_1}\,,\\
&X_{53} \to \gamma_{\Omega_5}\bar{Y}_{53}\gamma^{-1}_{\Omega_3}\,, && Y_{53} \to \gamma_{\Omega_5}\bar{X}_{53}\gamma^{-1}_{\Omega_3}\,, && X_{56} \to \gamma_{\Omega_5}\bar{X}_{56}\gamma^{-1}_{\Omega_6}\,, && Y_{56} \to \gamma_{\Omega_5}\bar{Z}_{56}\gamma^{-1}_{\Omega_6}\,,\\
&Z_{56} \to \gamma_{\Omega_5}\bar{Y}_{56}\gamma^{-1}_{\Omega_6}\,, && W_{56} \to \gamma_{\Omega_5}\bar{W}_{56}\gamma^{-1}_{\Omega_6}\,, && X_{64} \to \gamma_{\Omega_6}\bar{Y}_{64}\gamma^{-1}_{\Omega_4}\,, && Y_{64} \to \gamma_{\Omega_6}\bar{X}_{64}\gamma^{-1}_{\Omega_4}\,,\\
&X_{25} \to \gamma_{\Omega_8}\bar{X}_{85}\gamma^{-1}_{\Omega_5}\,, && Y_{25} \to \gamma_{\Omega_8}\bar{Y}_{85}\gamma^{-1}_{\Omega_5}\,, && X_{85} \to \gamma_{\Omega_2}\bar{X}_{25}\gamma^{-1}_{\Omega_5}\,, && Y_{85} \to \gamma_{\Omega_2}\bar{Y}_{25}\gamma^{-1}_{\Omega_5}\,.
\end{aligned} \tag{5.16}$$



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Invariance of $W^{(0,1)}$ implies the following action on Fermi fields

$$\begin{aligned}
&\Lambda^1_{13} \to -\gamma_{\Omega_3}\Lambda^2_{37}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^2_{13} \to \gamma_{\Omega_3}\Lambda^1_{37}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^1_{37} \to \gamma_{\Omega_1}\Lambda^2_{13}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^2_{37} \to -\gamma_{\Omega_3}\Lambda^1_{37}\gamma^{-1}_{\Omega_7}\,,\\
&\Lambda^1_{86} \to -\gamma_{\Omega_6}\Lambda^2_{62}\gamma^{-1}_{\Omega_8}\,, \quad \Lambda^2_{86} \to -\gamma_{\Omega_6}\Lambda^1_{62}\gamma^{-1}_{\Omega_8}\,, \quad \Lambda^1_{62} \to -\gamma_{\Omega_8}\Lambda^2_{86}\gamma^{-1}_{\Omega_6}\,, \quad \Lambda^2_{62} \to -\gamma_{\Omega_8}\Lambda^1_{86}\gamma^{-1}_{\Omega_6}\,,\\
&\Lambda^1_{45} \to \gamma_{\Omega_4}\bar{\Lambda}^2_{45}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^2_{45} \to \gamma_{\Omega_4}\bar{\Lambda}^1_{45}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^3_{45} \to \gamma_{\Omega_4}\bar{\Lambda}^6_{45}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^4_{45} \to \gamma_{\Omega_4}\bar{\Lambda}^5_{45}\gamma^{-1}_{\Omega_5}\,,\\
&\Lambda^5_{45} \to \gamma_{\Omega_5}\bar{\Lambda}^4_{45}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^6_{45} \to \gamma_{\Omega_5}\bar{\Lambda}^3_{45}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^7_{45} \to \gamma_{\Omega_5}\bar{\Lambda}^8_{45}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^8_{45} \to \gamma_{\Omega_5}\bar{\Lambda}^7_{45}\gamma^{-1}_{\Omega_5}\,,
\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
&\Lambda^R_{11} \to \gamma_{\Omega_7}\Lambda^{R\,T}_{77}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^R_{22} \to \gamma_{\Omega_8}\Lambda^{R\,T}_{88}\gamma^{-1}_{\Omega_8}\,, \quad \Lambda^R_{33} \to \gamma_{\Omega_3}\Lambda^{R\,T}_{33}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^R_{44} \to \gamma_{\Omega_4}\Lambda^{R\,T}_{44}\gamma^{-1}_{\Omega_4}\,,\\
&\Lambda^R_{55} \to \gamma_{\Omega_5}\Lambda^{R\,T}_{55}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^R_{66} \to \gamma_{\Omega_6}\Lambda^{R\,T}_{66}\gamma^{-1}_{\Omega_6}\,, \quad \Lambda^R_{77} \to \gamma_{\Omega_1}\Lambda^{R\,T}_{11}\gamma^{-1}_{\Omega_1}\,, \quad \Lambda^R_{88} \to \gamma_{\Omega_2}\Lambda^{R\,T}_{22}\gamma^{-1}_{\Omega_2}\,.
\end{aligned} \tag{5.18}$$

Using table 7, we find the corresponding geometric involution on the generators of $Q^{1,1,1}/\mathbb{Z}_2$

$$\begin{aligned}
&M_1 \to \bar{M}_{27}\,, \quad M_2 \to \bar{M}_{24}\,, \quad M_3 \to \bar{M}_{21}\,, \quad M_4 \to \bar{M}_{18}\,, \quad M_5 \to \bar{M}_{12}\,,\\
&M_6 \to \bar{M}_{6}\,, \quad M_7 \to \bar{M}_{26}\,, \quad M_8 \to \bar{M}_{23}\,, \quad M_9 \to \bar{M}_{20}\,, \quad M_{10} \to \bar{M}_{17}\,,\\
&M_{11} \to \bar{M}_{11}\,, \quad M_{12} \to \bar{M}_{5}\,, \quad M_{13} \to \bar{M}_{25}\,, \quad M_{14} \to \bar{M}_{22}\,, \quad M_{15} \to \bar{M}_{19}\,,\\
&M_{16} \to \bar{M}_{16}\,, \quad M_{17} \to \bar{M}_{10}\,, \quad M_{18} \to \bar{M}_{4}\,, \quad M_{19} \to \bar{M}_{15}\,, \quad M_{20} \to \bar{M}_{9}\,,\\
&M_{21} \to \bar{M}_{3}\,, \quad M_{22} \to \bar{M}_{14}\,, \quad M_{23} \to \bar{M}_{8}\,, \quad M_{24} \to \bar{M}_{2}\,, \quad M_{25} \to \bar{M}_{13}\,,\\
&M_{26} \to \bar{M}_{7}\,, \quad M_{27} \to \bar{M}_{1}\,.
\end{aligned} \tag{5.19}$$

The orientifolded theory has gauge group $\mathrm{U}(N)_1 \times \mathrm{U}(N)_2 \times \prod_{i=3}^{6} G_i(N)$. The involution of fields connecting nodes 3, 4, 5 and 6 gives rise to the constraint

$$\gamma_{\Omega_3} = \gamma_{\Omega_4} = \gamma_{\Omega_5} = \gamma_{\Omega_6}\,. \tag{5.20}$$

Let us set the four matrices equal to $\mathbb{1}_N$, i.e. project the corresponding gauge groups to $\mathrm{SO}(N)$. Figure 21 shows the quiver for the resulting theory, which is free of gauge anomalies.

Let us pause for a moment to think about a possible interpretation on this theory. We note that it has two distinct types of nodes. First, we have $\mathrm{U}(N)$ nodes with adjoint Fermi fields, which can be combined into $\mathcal{N} = (0,2)$ vector multiplets. Second, there are $\mathrm{SO}(N)$ nodes with symmetric Fermi fields which, contrary to the previous case, are inherently $\mathcal{N} = (0,1)$. This is because the adjoint of $\mathrm{SO}(N)$ is instead the antisymmetric representation. We can similarly consider whether it is possible to combine the bifundamental fields into $\mathcal{N} = (0,2)$ multiplets, which may or may not be broken by the superpotential. In this example, all bifundamental fields come in pairs so, leaving the superpotential aside, they can form $\mathcal{N} = (0,2)$ multiplets. Broadly speaking, we can therefore regard this theory as consisting of coupled $\mathcal{N} = (0,1)$ and $\mathcal{N} = (0,2)$ sectors.[14] This discussion extends to the other orientifolds of $Q^{1,1,1}/\mathbb{Z}_2$ considered in this section and is a generic phenomenon.

[14] A similar interpretation in terms of coupled $\mathcal{N} = (0,1)$ and $\mathcal{N} = (0,2)$ sectors was proposed in the analysis of non-compact models in [3].



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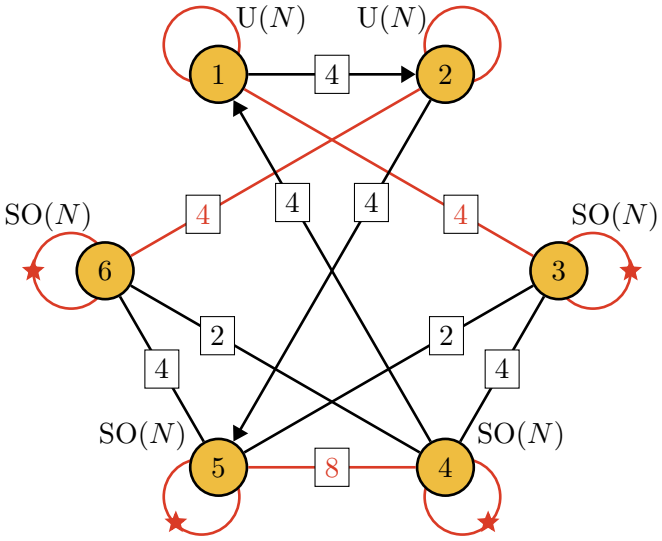


**Figure 21**. Quiver diagram for the Spin(7) orientifold of phase D of $Q^{1,1,1}/\mathbb{Z}_2$ using the involution in eqs. (5.16), (5.17) and (5.18), together with our choice of $\gamma_{\Omega_i}$ matrices.

Interestingly, we will see below that Spin(7) orientifolds produce theories in which triality acts on either of these two sectors.

In order to find other $\mathcal{N} = (0,1)$ theories associated with the same Spin(7) orientifold, one needs to find the field-theoretic involutions of other toric phases of $Q^{1,1,1}/\mathbb{Z}_2$ leading to $\mathrm{U}(N)^2 \times \mathrm{SO}(N)^4$ gauge theories, whose geometric involution is the same as (5.19). Scanning the 14 toric phases of $Q^{1,1,1}/\mathbb{Z}_2$, we found that only 5 of them (including phase D) admit $\mathcal{N} = (0,1)$ orientifolds with $\mathrm{U}(N)^2 \times \mathrm{SO}(N)^4$ gauge symmetry. Let us first present the $\mathcal{N} = (0,2)$ triality web for these 5 phases in figure 22, which can be regarded as a portion of the whole triality web for $Q^{1,1,1}/\mathbb{Z}_2$ in [15].

Colored arrows connecting different phases indicate $\mathcal{N} = (0,2)$ triality transformations between them. Furthermore, the quiver node on which triality acts is shown in the same color as the corresponding arrow. Note that from phase D to phase H there are two triality steps, where the intermediate stage is the so-called phase C in [15]. However, since phase C does not give rise to a $\mathrm{U}(N)^2 \times \mathrm{SO}(N)^4$ orientifold, we do not show its quiver here.

Similarly to phase D, we consider the anti-holomorphic involutions of phases E, H, J and L which act on their quivers shown in figure 22 as reflections with respect to the vertical plane that contains nodes 3, 4, 5 and 6. Then, the nodes on the plane map to themselves, while the pairs $1 \leftrightarrow 7$ and $2 \leftrightarrow 8$ get identified. In all these cases, we choose the $\gamma_{\Omega_i}$ matrices as for phase D, so they have $\mathrm{U}(N)^2 \times \mathrm{SO}(N)^4$ gauge group. The construction of the $\mathcal{N} = (0,1)$ theories associated with the Spin(7) orientifold for these phases is detailed in appendix C. The crucial point is that they all correspond to the same Spin(7) orientifold of $Q^{1,1,1}/\mathbb{Z}_2$, since they are all associated to the same geometric involution as that of phase D, given in (5.19).

From a field theory perspective, we find that the orientifolds of phases D, E, J and L are connected by $\mathcal{N} = (0,1)$ triality transformation on various $\mathrm{SO}(N)$ gauge groups (with



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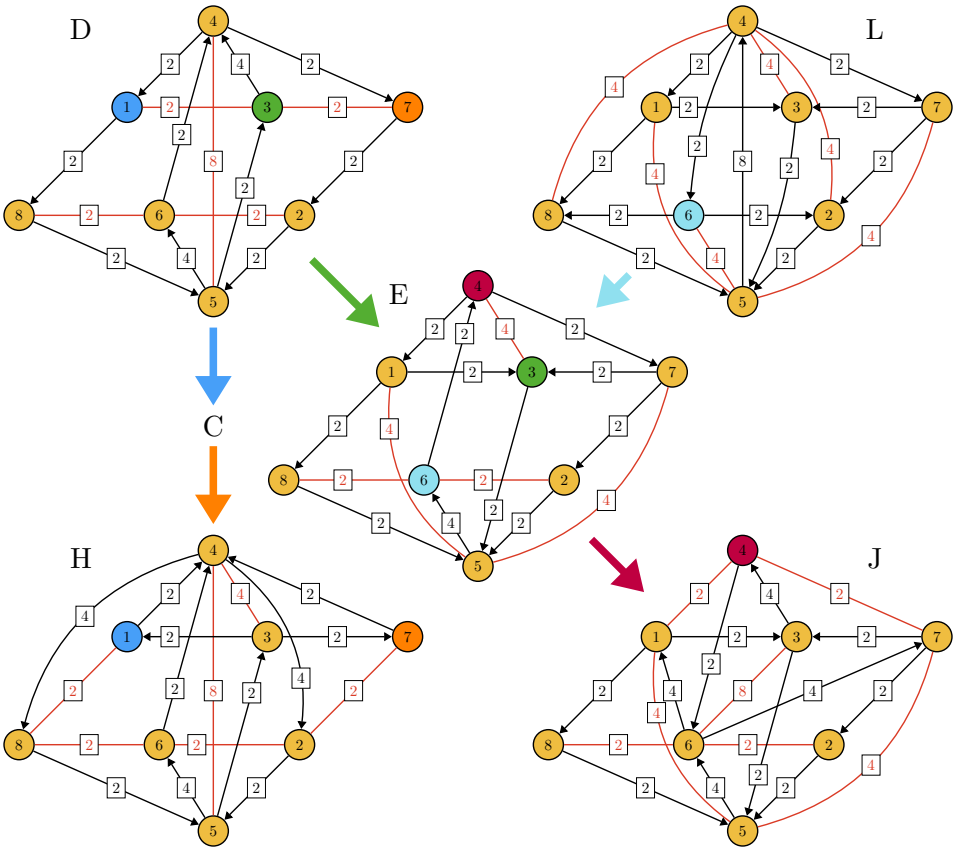


**Figure 22**. $\mathcal{N} = (0, 2)$ triality web for phases D, E, H, J and L of $Q^{1,1,1}/\mathbb{Z}_2$.

the obvious generalization to more general flavor groups). These are the first examples of $\mathcal{N} = (0, 1)$ triality in the presence of U($N$) gauge groups. Interestingly, the orientifolds of phases D and H are not connected by the usual $\mathcal{N} = (0, 1)$ triality on an SO($N$) node, but by triality on node 1, which is of U($N$) type. This transformation locally follows the rules of $\mathcal{N} = (0, 2)$ triality. Such U($N$) triality in $\mathcal{N} = (0, 1)$ gauge theories is a new phenomenon which, to the best of our knowledge, has not appeared in the literature before. Following our earlier discussion, it can be nicely interpreted as $\mathcal{N} = (0, 2)$ triality in the presence of an $\mathcal{N} = (0, 1)$ sector. In our Spin(7) orientifold construction, the U($N$) triality has a clear origin: the two $\mathcal{N} = (0, 2)$ trialities that connect phases D and H passing through phase C are projected onto a single U($N$) triality connecting the orientifolds of phases D and H. In the case of nodes that are not mapped to themselves, an even number of trialities in the parent is necessary in order to get a new phase that is also symmetric under the involution. Figure 23 summarizes the web of trialities for the Spin(7) orientifolds under consideration.

## 6 Conclusions

D1-branes probing singularities provide a powerful framework for engineering 2d gauge theories. In our previous work [4], these constructions were extended to $\mathcal{N} = (0, 1)$ theories with the introduction of Spin(7) orientifolds.



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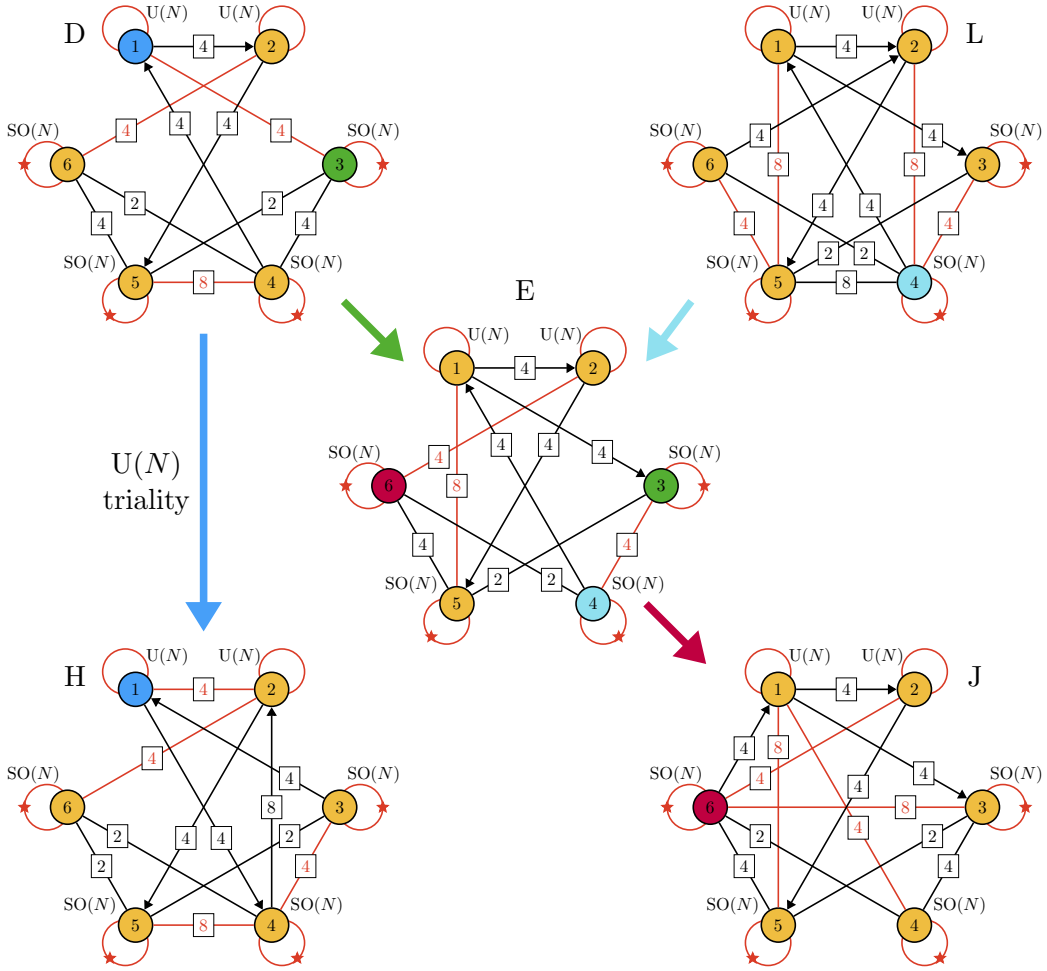


**Figure 23**. Triality web for the $\mathcal{N} = (0,1)$ theories associated with the Spin(7) orientifolds under consideration for phases D, E, H, J and L of $Q^{1,1,1}/\mathbb{Z}_2$.

In this paper we introduced a new, geometric, perspective on the triality of 2d $\mathcal{N} = (0,1)$ gauge theories, by showing that it arises from the non-uniqueness of the correspondence between Spin(7) orientifolds and the gauge theories on D1-brane probes.

Let us reflect on how 2d trialities with different amounts of SUSY are manifested in D1-branes at singularities. $\mathcal{N} = (0,2)$ triality similarly arises from the fact that multiple gauge theories can be associated to the same underlying $\mathrm{CY}_4$ [7]. We explained that Spin(7) orientifolds based on the universal involution give rise to exactly the $\mathcal{N} = (0,1)$ triality of [3]. But our work shows that the space of possibilities is far richer. Indeed, general Spin(7) orientifolds extend triality to theories that can be regarded as consisting of coupled $\mathcal{N} = (0,2)$ and $(0,1)$ sectors. The geometric construction of these theories therefore leads to extensions of triality that interpolate between the pure $\mathcal{N} = (0,2)$ and $(0,1)$ cases.

On the practical side, Spin(7) orientifolds also give a precise prescription for how scalar flavors transform under triality in general quivers, which is inherited from the transformation of the corresponding chiral flavors in the parent.



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## A 2d $\mathcal{N} = (0, 1)$ formalism

In this appendix we review the 2d $\mathcal{N} = (0, 1)$ field theory formalism as we did in [4]. Let us introduce the 2d $\mathcal{N} = (0, 1)$ superspace $(x^0, x^1, \theta^+)$, on which we can define three types of supermultiplets:

- vector multiplet:

$$\begin{aligned} V_+ &= \theta^+(A_0(x) + A_1(x))\,, \\ V_- &= A_0(x) - A_1(x) + \theta^+\lambda_-(x)\,. \end{aligned} \tag{A.1}$$

  It contains a gauge boson $A_\pm$ and a left-moving Majorana-Weyl fermion $\lambda_-$ in the adjoint representation.

- Scalar multiplet:

$$\Phi(x, \theta) = \phi(x) + \theta^+\psi_+(x)\,. \tag{A.2}$$

  It has a real scalar field $\phi$ and a right-moving Majorana-Weyl fermion $\psi_+$.

- Fermi multiplet:

$$\Lambda(x, \theta) = \psi_-(x) + \theta^+F(x)\,. \tag{A.3}$$

  It has a left-moving Majorana-Weyl spinor as its only on-shell degree of freedom. Here $F$ is an auxiliary field.

The kinetic terms for matter fields and their gauge couplings are given by

$$\mathcal{L}_s + \mathcal{L}_F = \int d\theta^+ \left(\frac{i}{2}\sum_i(\mathcal{D}_+\Phi_i\mathcal{D}_-\Phi_i) - \frac{1}{2}\sum_a(\Lambda_a\mathcal{D}_+\Lambda_a)\right), \tag{A.4}$$

where $\mathcal{D}_\pm$ are super-covariant derivatives [3].

We need also to introduce the $\mathcal{N} = (0, 1)$ analog of the $\mathcal{N} = (0, 2)$ $J$-term interaction, which is given by

$$\mathcal{L}_J \equiv \int d\theta^+ W^{(0,1)} = \int d\theta^+ \sum_a(\Lambda_a J^a(\Phi_i))\,, \tag{A.5}$$

where $J^a(\Phi_i)$ are real functions of scalar fields. We refer to $W^{(0,1)}$ as the *superpotential*. Both the field content and gauge symmetry (i.e., the quiver for the theories considered in this paper) and $W^{(0,1)}$ are necessary for fully specifying an $\mathcal{N}=(0,1)$ gauge theory.

After integrating out the auxiliary fields $F_a$, $\mathcal{L}_J$ produces various interactions, including Yukawa-like couplings

$$\sum_a \lambda_{-a} \frac{\partial J^a}{\partial \phi_i} \psi_{+i} \,, \tag{A.6}$$

as well as a scalar potential

$$\frac{1}{2} \sum_a (J^a(\phi_i))^2 \,. \tag{A.7}$$

### A.1 $\mathcal{N}=(0,2)$ gauge theories in $\mathcal{N}=(0,1)$ superspace

For the construction of Spin(7) manifolds, it is useful to express $\mathcal{N}=(0,2)$ gauge theories in $\mathcal{N}=(0,1)$ language. Here, we briefly sketch the decomposition, referring to [4] for details:

1. An $\mathcal{N}=(0,2)$ vector multiplet $V_i^{(0,2)}$ decomposes into an $\mathcal{N}=(0,1)$ vector multiplet $V_i$ and an $\mathcal{N}=(0,1)$ Fermi multiplet $\Lambda_i^R$.

2. An $\mathcal{N}=(0,2)$ chiral multiplet $\Phi_m^{(0,2)}$ decomposes into two $\mathcal{N}=(0,1)$ scalar multiplets $\Phi_m^a$ with $a=1,2$. It can then be further re-expressed in an $\mathcal{N}=(0,1)$ complex scalar multiplet $\Phi_m$.

3. An $\mathcal{N}=(0,2)$ Fermi multiplet $\Lambda_m^{(0,2)}$ decomposes into two $\mathcal{N}=(0,1)$ Fermi multiplets $\Lambda_m^a$, with $a=1,2$, that form an $\mathcal{N}=(0,1)$ complex Fermi multiplet $\Lambda_m$.

The $J$- and $E$-terms of the $\mathcal{N}=(0,2)$ gauge theory become part of $W^{(0,1)}$ upon the decomposition of Fermi and chiral multiplets in $\mathcal{N}=(0,1)$ language. The interactions between $\mathcal{N}=(0,2)$ vector and chiral multiplets also contribute to $W^{(0,1)}$ couplings between scalar multiplets and $\mathcal{N}=(0,1)$ Fermi multiplets $\Lambda_i^R$ coming from the $\mathcal{N}=(0,2)$ vector multiplets. The full $\mathcal{N}=(0,1)$ superpotential reads

$$W^{(0,1)} = \sum_a \int d\theta^+ [\Lambda_a (J^a(\Phi_m) + E^{\dagger a}(\Phi_m^\dagger)) + \Lambda^{\dagger a}(E_a(\Phi_m) + J_a^\dagger(\Phi_m^\dagger))] + \sum_i \sum_n \Lambda_i^R \Phi_n^\dagger \Phi_n \,, \tag{A.8}$$

where $n$ runs over all complex scalar multiplets transforming under a given gauge group $i$.

## B Anomalies

Here, we list the possible contributions to 2d gauge anomalies coming from fields in the representations considered in this paper. Generically, 2d anomalies are obtained by a 1-loop diagram as shown in figure 24, where left- and right-moving fermions running in the loop contribute oppositely.

In the case of gauge groups, anomalies must vanish for consistency of the theory at the quantum level. This leads to important constraints in our construction of 2d $\mathcal{N}=(0,1)$ theories, which may require the introduction of extra flavors to cancel anomalies.



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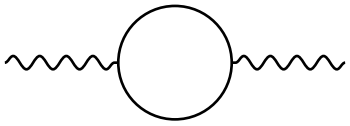

**Figure 24**. Generic 1-loop diagram associated with 2d anomalies.

| SU($N$) | fundamental | adjoint | antisymmetric | symmetric |
|---|---|---|---|---|
| vector multiplet | $\times$ | $-N$ | $\times$ | $\times$ |
| Fermi multiplet | $-\frac{1}{2}$ | $-N$ | $\frac{-N+2}{2}$ | $\frac{-N-2}{2}$ |
| scalar multiplet | $\frac{1}{2}$ | $N$ | $\frac{N-2}{2}$ | $\frac{N+2}{2}$ |

**Table 5**. Anomaly contributions of the 2d $\mathcal{N} = (0,1)$ multiplets in various representations of SU($N$). Since anomalies are quadratic in 2d, the same contributions apply for the conjugate representations.

Unlike gauge symmetries, global symmetries may indeed be anomalous. They are also preserved by RG flows, so they are useful for testing dualities between two or more theories. Examples of using global anomalies to check dualities in 2d $\mathcal{N} = (0,1)$ theories can be found in [3].

Generically, the gauge theories on D1-branes probing Spin(7) orientifolds that we construct in this paper have non-vanishing Abelian gauge anomalies. However, similarly to the discussion in [20, 25], we expect that such anomalies are canceled by the bulk fields in the closed string sector via a generalized Green-Schwarz (GS) mechanism (see [16, 17] for derivations in 4d $\mathcal{N} = 1$ and 2d $\mathcal{N} = (0,2)$ theories realized at orbifolds/orientifold singularities). For this reason, we mainly focus on non-Abelian anomalies.

Let us consider pure non-Abelian $G^2$ gauge or global anomalies, where $G$ is SU($N$), SO($N$) or USp($N$) group. The corresponding anomaly is given by

$$\mathrm{Tr}[\gamma^3 J_G J_G]\,, \tag{B.1}$$

where $\gamma^3$ is the chirality matrix in 2d and $J_G$ is the current associated to $G$. The resulting anomaly from a field in representation $\rho$ of $G$ can be computed in terms of the Dynkin index $T(\rho)$:

$$T(\rho) = C_2(\rho)\frac{d(\rho)}{d(\text{adjoint})}\,, \tag{B.2}$$

where $C_2(\rho)$ is the quadratic Casimir for representation $\rho$. In table 5 we present anomaly contributions for superfields in the most common representations of SU($N$). In table 6, we present anomaly contributions for various representations of SO($N$) and USp($N$) groups, computed using Dynkin indices listed in [26].

## C Details on $Q^{1,1,1}/\mathbb{Z}_2$

In section 5.2, we introduced a web of trialities that contains a Spin(7) orientifold of phase D of $Q^{1,1,1}/\mathbb{Z}_2$ and summarized it in figure 23. In this appendix, we collect all the relevant



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| SO($N$) | fundamental | antisymmetric (adjoint) | symmetric |
|---|---|---|---|
| vector multiplet | $\times$ | $-N+2$ | $\times$ |
| Fermi multiplet | $-1$ | $-N+2$ | $-N-2$ |
| scalar multiplet | 1 | $N-2$ | $N+2$ |
| USp($N$) | fundamental | antisymmetric | symmetric (adjoint) |
| vector multiplet | $\times$ | $\times$ | $-N-2$ |
| Fermi multiplet | $-1$ | $-N+2$ | $-N-2$ |
| scalar multiplet | 1 | $N-2$ | $N+2$ |

**Table 6**. Anomaly contributions of the 2d $\mathcal{N}=(0,1)$ multiplets in various representations of SO($N$) and USp($N$).

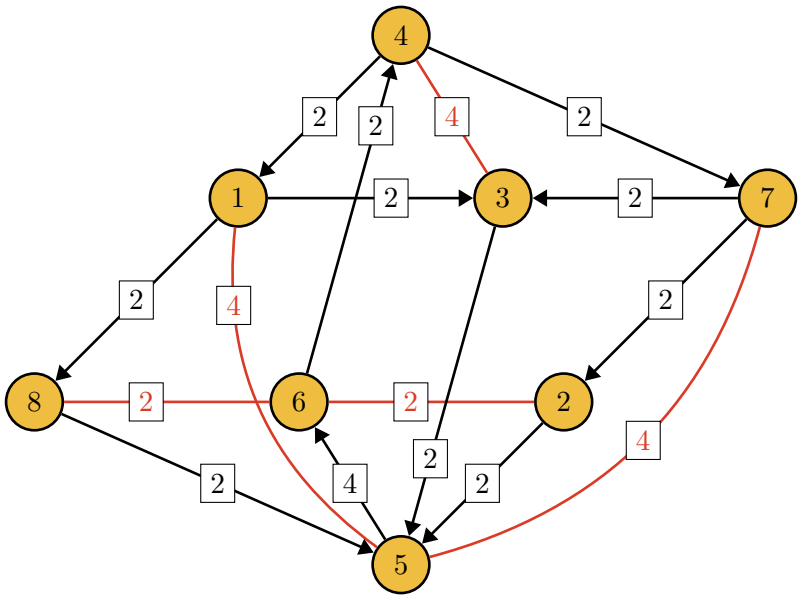


**Figure 25**. Quiver diagram for phase E of $Q^{1,1,1}/\mathbb{Z}_2$.

information for the other theories in this web.

### C.1 Phase E

The quiver for phase E is shown in figure 25.

The corresponding $J$- and $E$-terms are given by

$$\begin{array}{cll} & J & E \\ \Lambda^1_{86}: & X_{64}Y_{41}Y_{18}-Y_{64}Y_{41}X_{18} & Y_{85}X_{56}-X_{85}Z_{56} \\ \Lambda^2_{86}: & Y_{64}X_{41}X_{18}-X_{64}X_{41}Y_{18} & Y_{85}W_{56}-X_{85}Y_{56} \\ \Lambda^1_{26}: & Y_{64}Y_{47}X_{72}-X_{64}Y_{47}Y_{72} & Y_{25}X_{56}-X_{25}W_{56} \\ \Lambda^2_{26}: & X_{64}X_{47}Y_{72}-Y_{64}X_{47}X_{72} & Y_{25}Z_{56}-X_{25}Y_{56} \end{array}$$



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$$
\begin{aligned}
\Lambda^1_{75}&: \quad Y_{56}Y_{64}X_{47} - W_{56}Y_{64}Y_{47} \qquad X_{73}X_{35} - X_{72}X_{25}\\
\Lambda^2_{75}&: \quad Z_{56}X_{64}X_{47} - X_{56}X_{64}Y_{47} \qquad Y_{73}Y_{35} - Y_{72}Y_{25}\\
\Lambda^3_{75}&: \quad X_{56}Y_{64}Y_{47} - Z_{56}Y_{64}X_{47} \qquad Y_{73}X_{35} - X_{72}Y_{25}\\
\Lambda^4_{75}&: \quad W_{56}X_{64}Y_{47} - X_{64}Y_{56}X_{47} \qquad X_{73}Y_{35} - Y_{72}X_{25}\\
\Lambda^1_{43}&: \quad Y_{35}Y_{56}X_{64} - X_{35}Y_{56}Y_{64} \qquad X_{47}X_{73} - X_{41}X_{13}\\
\Lambda^2_{43}&: \quad Y_{35}X_{56}X_{64} - X_{35}X_{56}Y_{64} \qquad Y_{47}Y_{73} - Y_{41}Y_{13}\\
\Lambda^3_{43}&: \quad X_{35}W_{56}Y_{64} - Y_{35}W_{56}X_{64} \qquad Y_{47}X_{73} - X_{41}Y_{13}\\
\Lambda^4_{43}&: \quad X_{35}Z_{56}Y_{64} - Y_{35}Z_{56}X_{64} \qquad X_{47}Y_{73} - Y_{41}X_{13}\\
\Lambda^1_{15}&: \quad Z_{56}Y_{64}Y_{41} - Y_{56}Y_{64}X_{41} \qquad X_{13}X_{35} - X_{18}X_{85}\\
\Lambda^2_{15}&: \quad X_{56}X_{64}Y_{41} - W_{56}X_{64}X_{41} \qquad Y_{13}Y_{35} - Y_{18}Y_{85}\\
\Lambda^3_{15}&: \quad W_{56}Y_{64}X_{41} - X_{56}Y_{64}Y_{41} \qquad Y_{13}X_{35} - X_{18}Y_{85}\\
\Lambda^4_{15}&: \quad X_{64}Y_{56}X_{41} - X_{64}Z_{56}Y_{41} \qquad X_{13}Y_{35} - Y_{18}X_{85}\,.
\end{aligned}
\tag{C.1}
$$

Finally, the generators of the moduli space expressed in terms of the chiral fields are listed in table 8.

**$\mathbf{U(N)^2 \times SO(N)^4}$ orientifold** Let us consider an anti-holomorphic involution of phase E which acts on the nodes in figure 25 as $1 \leftrightarrow 7$, $2 \leftrightarrow 8$ and maps all other nodes mapped to themselves. Chiral fields transform according to

$$
\begin{aligned}
&X_{41} \to \gamma_{\Omega_4}\bar{X}_{47}\gamma^{-1}_{\Omega_7}\,, \quad Y_{41} \to \gamma_{\Omega_4}\bar{Y}_{47}\gamma^{-1}_{\Omega_7}\,, \quad X_{72} \to \gamma_{\Omega_1}\bar{Y}_{18}\gamma^{-1}_{\Omega_8}\,, \quad Y_{72} \to \gamma_{\Omega_1}\bar{X}_{18}\gamma^{-1}_{\Omega_8}\,,\\
&X_{18} \to \gamma_{\Omega_7}\bar{Y}_{72}\gamma^{-1}_{\Omega_2}\,, \quad Y_{18} \to \gamma_{\Omega_7}\bar{X}_{72}\gamma^{-1}_{\Omega_2}\,, \quad X_{64} \to \gamma_{\Omega_6}\bar{Y}_{64}\gamma^{-1}_{\Omega_4}\,, \quad Y_{64} \to \gamma_{\Omega_6}\bar{X}_{64}\gamma^{-1}_{\Omega_4}\,,\\
&X_{25} \to \gamma_{\Omega_8}\bar{X}_{85}\gamma^{-1}_{\Omega_5}\,, \quad Y_{25} \to \gamma_{\Omega_8}\bar{Y}_{85}\gamma^{-1}_{\Omega_5}\,, \quad X_{85} \to \gamma_{\Omega_2}\bar{X}_{25}\gamma^{-1}_{\Omega_5}\,, \quad Y_{85} \to \gamma_{\Omega_2}\bar{Y}_{25}\gamma^{-1}_{\Omega_5}\,,\\
&X_{35} \to \gamma_{\Omega_3}\bar{Y}_{35}\gamma^{-1}_{\Omega_5}\,, \quad Y_{35} \to \gamma_{\Omega_3}\bar{X}_{35}\gamma^{-1}_{\Omega_5}\,, \quad X_{47} \to \gamma_{\Omega_4}\bar{X}_{41}\gamma^{-1}_{\Omega_1}\,, \quad Y_{47} \to \gamma_{\Omega_4}\bar{Y}_{41}\gamma^{-1}_{\Omega_1}\,,\\
&X_{56} \to \gamma_{\Omega_5}\bar{X}_{56}\gamma^{-1}_{\Omega_6}\,, \quad Y_{56} \to \gamma_{\Omega_5}\bar{Y}_{56}\gamma^{-1}_{\Omega_6}\,, \quad Z_{56} \to \gamma_{\Omega_5}\bar{W}_{56}\gamma^{-1}_{\Omega_6}\,, \quad W_{56} \to \gamma_{\Omega_5}\bar{Z}_{56}\gamma^{-1}_{\Omega_6}\,,\\
&X_{13} \to \gamma_{\Omega_7}\bar{X}_{73}\gamma^{-1}_{\Omega_3}\,, \quad Y_{13} \to \gamma_{\Omega_7}\bar{Y}_{73}\gamma^{-1}_{\Omega_3}\,, \quad X_{73} \to \gamma_{\Omega_1}\bar{X}_{13}\gamma^{-1}_{\Omega_3}\,, \quad Y_{73} \to \gamma_{\Omega_1}\bar{Y}_{13}\gamma^{-1}_{\Omega_3}\,.
\end{aligned}
\tag{C.2}
$$

Requiring the invariance of $W^{0,1}$, the Fermi fields transform as

$$
\begin{aligned}
&\Lambda^1_{86} \to \gamma_{\Omega_2}\bar{\Lambda}^1_{26}\gamma^{-1}_{\Omega_6}\,, \quad \Lambda^2_{86} \to \gamma_{\Omega_2}\bar{\Lambda}^2_{26}\gamma^{-1}_{\Omega_6}\,, \quad \Lambda^1_{26} \to \gamma_{\Omega_8}\bar{\Lambda}^1_{86}\gamma^{-1}_{\Omega_6}\,, \quad \Lambda^2_{26} \to \gamma_{\Omega_8}\bar{\Lambda}^2_{86}\gamma^{-1}_{\Omega_6}\,,\\
&\Lambda^1_{75} \to \gamma_{\Omega_1}\bar{\Lambda}^4_{15}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^2_{75} \to \gamma_{\Omega_1}\bar{\Lambda}^3_{15}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^3_{75} \to \gamma_{\Omega_1}\bar{\Lambda}^2_{15}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^4_{75} \to \gamma_{\Omega_1}\bar{\Lambda}^1_{15}\gamma^{-1}_{\Omega_5}\,,\\
&\Lambda^1_{43} \to -\gamma_{\Omega_4}\bar{\Lambda}^2_{43}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^2_{43} \to -\gamma_{\Omega_4}\bar{\Lambda}^1_{43}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^3_{43} \to -\gamma_{\Omega_4}\bar{\Lambda}^4_{43}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^4_{43} \to -\gamma_{\Omega_4}\bar{\Lambda}^3_{43}\gamma^{-1}_{\Omega_3}\,,\\
&\Lambda^1_{15} \to \gamma_{\Omega_7}\bar{\Lambda}^4_{75}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^2_{15} \to \gamma_{\Omega_7}\bar{\Lambda}^3_{75}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^3_{15} \to \gamma_{\Omega_7}\bar{\Lambda}^2_{75}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^4_{15} \to \gamma_{\Omega_7}\bar{\Lambda}^1_{75}\gamma^{-1}_{\Omega_5}\,,
\end{aligned}
\tag{C.3}
$$

and

$$
\begin{aligned}
&\Lambda^R_{11} \to \gamma_{\Omega_7}\Lambda^{R\,T}_{77}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^R_{22} \to \gamma_{\Omega_8}\Lambda^{R\,T}_{88}\gamma^{-1}_{\Omega_8}\,, \quad \Lambda^R_{33} \to \gamma_{\Omega_3}\Lambda^{R\,T}_{33}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^R_{44} \to \gamma_{\Omega_4}\Lambda^{R\,T}_{44}\gamma^{-1}_{\Omega_4}\,,\\
&\Lambda^R_{55} \to \gamma_{\Omega_5}\Lambda^{R\,T}_{55}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^R_{66} \to \gamma_{\Omega_6}\Lambda^{R\,T}_{66}\gamma^{-1}_{\Omega_6}\,, \quad \Lambda^R_{77} \to \gamma_{\Omega_1}\Lambda^{R\,T}_{11}\gamma^{-1}_{\Omega_1}\,, \quad \Lambda^R_{88} \to \gamma_{\Omega_2}\Lambda^{R\,T}_{22}\gamma^{-1}_{\Omega_2}\,.
\end{aligned}
\tag{C.4}
$$



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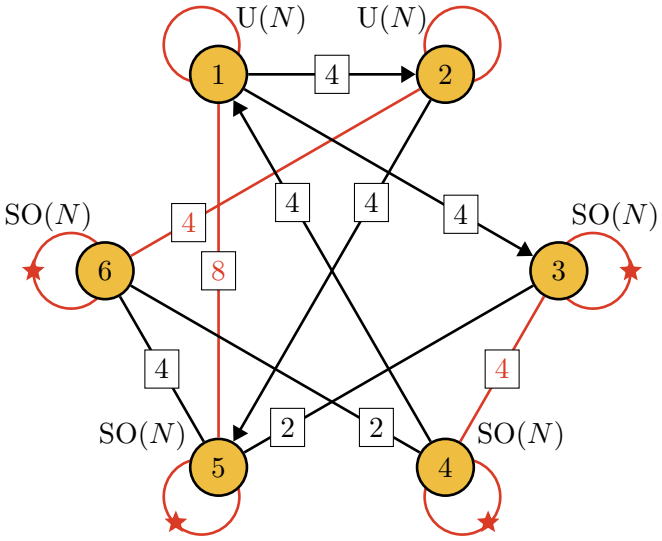


**Figure 26**. Quiver diagram for the Spin(7) orientifold of phase E of $Q^{1,1,1}/\mathbb{Z}_2$ using the involution in eqs. (C.2), (C.3) and (C.4), together with our choice of $\gamma_{\Omega_i}$ matrices.

Using table 8, the corresponding geometric involution acting on the generators reads

$$\begin{array}{lllll} M_1 \to \bar{M}_{27}\,, & M_2 \to \bar{M}_{24}\,, & M_3 \to \bar{M}_{21}\,, & M_4 \to \bar{M}_{18}\,, & M_5 \to \bar{M}_{12}\,, \\ M_6 \to \bar{M}_6\,, & M_7 \to \bar{M}_{26}\,, & M_8 \to \bar{M}_{23}\,, & M_9 \to \bar{M}_{20}\,, & M_{10} \to \bar{M}_{17}\,, \\ M_{11} \to \bar{M}_{11}\,, & M_{12} \to \bar{M}_5\,, & M_{13} \to \bar{M}_{25}\,, & M_{14} \to \bar{M}_{22}\,, & M_{15} \to \bar{M}_{19}\,, \\ M_{16} \to \bar{M}_{16}\,, & M_{17} \to \bar{M}_{10}\,, & M_{18} \to \bar{M}_4\,, & M_{19} \to \bar{M}_{15}\,, & M_{20} \to \bar{M}_9\,, \\ M_{21} \to \bar{M}_3\,, & M_{22} \to \bar{M}_{14}\,, & M_{23} \to \bar{M}_8\,, & M_{24} \to \bar{M}_2\,, & M_{25} \to \bar{M}_{13}\,, \\ M_{26} \to \bar{M}_7\,, & M_{27} \to \bar{M}_1\,. & & & \end{array} \tag{C.5}$$

This geometric involution is the same of (5.19).

The $\gamma_{\Omega_i}$ matrices are constrained as in (5.20). As for phase D, we choose

$$\gamma_{\Omega_3} = \gamma_{\Omega_4} = \gamma_{\Omega_5} = \gamma_{\Omega_6} = \mathbb{1}_N\,. \tag{C.6}$$

The resulting orientifold is shown in figure 26.

### C.2 Phase H

The quiver for phase H is shown in figure 27.

The $J$- and $E$-terms are

| | $J$ | $E$ |
|---|---|---|
| $\Lambda^1_{26}$ : | $Y_{64}Z_{42} - X_{64}W_{42}$ | $Y_{25}X_{56} - X_{25}Z_{56}$ |
| $\Lambda^7_{26}$ : | $Y_{64}X_{42} - Y_{42}X_{64}$ | $X_{25}W_{56} - Y_{25}Y_{56}$ |
| $\Lambda^1_{27}$ : | $X_{74}W_{42} - Y_{74}Y_{42}$ | $Y_{25}X_{53}X_{37} - X_{25}X_{53}Y_{37}$ |
| $\Lambda^7_{27}$ : | $X_{74}Z_{42} - Y_{74}X_{42}$ | $X_{25}Y_{53}Y_{37} - Y_{25}Y_{53}X_{37}$ |



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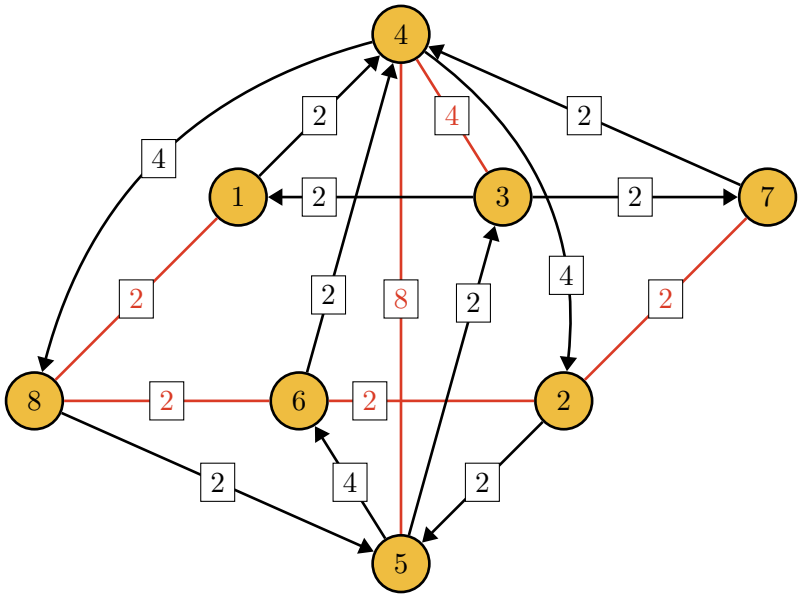


**Figure 27**. Quiver diagram for phase H of $Q^{1,1,1}/\mathbb{Z}_2$.

$$
\begin{aligned}
\Lambda^1_{68} &: X_{85}W_{56} - Y_{85}Z_{56} & X_{64}Y_{48} - Y_{64}X_{48} \\
\Lambda^7_{68} &: X_{85}Y_{56} - Y_{85}X_{56} & Y_{64}Z_{48} - X_{64}W_{48} \\
\Lambda^1_{81} &: X_{14}W_{48} - Y_{14}Y_{48} & X_{85}X_{53}Y_{31} - Y_{85}X_{53}X_{31} \\
\Lambda^7_{81} &: X_{14}Z_{48} - Y_{14}X_{48} & Y_{85}Y_{53}X_{31} - X_{85}Y_{53}Y_{31} \\
\Lambda^1_{54} &: W_{48}Y_{85} - W_{42}Y_{25} & X_{53}X_{31}X_{14} - X_{56}X_{64} \\
\Lambda^7_{54} &: Z_{48}Y_{85} - Z_{42}Y_{25} & X_{56}Y_{64} - Y_{53}X_{31}X_{14} \\
\Lambda^3_{54} &: W_{48}X_{85} - Y_{42}Y_{25} & Y_{56}X_{64} - X_{53}Y_{31}X_{14} \\
\Lambda^4_{54} &: Z_{48}X_{85} - X_{42}Y_{25} & Y_{53}X_{37}Y_{74} - Y_{56}Y_{64} \\
\Lambda^5_{54} &: Y_{48}Y_{85} - W_{42}X_{25} & Z_{56}X_{64} - X_{53}X_{31}Y_{14} \\
\Lambda^6_{54} &: X_{48}Y_{85} - Z_{42}X_{25} & Y_{53}Y_{37}X_{74} - Z_{56}Y_{64} \\
\Lambda^2_{54} &: Y_{48}X_{85} - Y_{42}X_{25} & X_{53}Y_{37}Y_{74} - W_{56}X_{64} \\
\Lambda^8_{54} &: X_{48}X_{85} - X_{42}X_{25} & W_{56}Y_{64} - Y_{53}Y_{37}Y_{74} \\
\Lambda^1_{43} &: Y_{31}Y_{14} - Y_{37}Y_{74} & Y_{48}X_{85}X_{53} - X_{48}X_{85}Y_{53} \\
\Lambda^7_{43} &: X_{31}Y_{14} - Y_{37}X_{74} & X_{48}Y_{85}Y_{53} - W_{42}X_{25}X_{53} \\
\Lambda^3_{43} &: Y_{31}X_{14} - X_{37}Y_{74} & Z_{48}X_{85}Y_{53} - Y_{42}Y_{25}X_{53} \\
\Lambda^4_{43} &: X_{31}X_{14} - X_{37}X_{74} & W_{42}Y_{25}X_{53} - Z_{42}Y_{25}Y_{53}\,.
\end{aligned}
\tag{C.7}
$$

The generators of the moduli space expressed in terms of the chiral fields are listed in table 9.

**$\mathbf{U(N)^2 \times SO(N)^4}$ orientifold** Let us consider an anti-holomorphic involution of phase H which acts on the nodes in figure 27 as $1 \leftrightarrow 7$ and $2 \leftrightarrow 8$ and maps all other nodes



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mapped to themselves. Chiral fields transform according to

$$\begin{aligned}
&Y_{64} \to \gamma_{\Omega_6} \bar{X}_{64} \gamma_{\Omega_4}^{-2}\,, \quad X_{64} \to \gamma_{\Omega_6} \bar{Y}_{64} \gamma_{\Omega_4}^{-2}\,, \quad Z_{42} \to \gamma_{\Omega_4} \bar{Y}_{48} \gamma_{\Omega_8}^{-2}\,, \quad Y_{48} \to \gamma_{\Omega_4} \bar{Z}_{42} \gamma_{\Omega_2}^{-2}\,,\\
&W_{42} \to \gamma_{\Omega_4} \bar{X}_{48} \gamma_{\Omega_8}^{-2}\,, \quad X_{48} \to \gamma_{\Omega_4} \bar{W}_{42} \gamma_{\Omega_2}^{-2}\,, \quad Y_{25} \to \gamma_{\Omega_8} \bar{X}_{85} \gamma_{\Omega_5}^{-2}\,, \quad X_{85} \to \gamma_{\Omega_2} \bar{Y}_{25} \gamma_{\Omega_5}^{-2}\,,\\
&X_{56} \to \gamma_{\Omega_5} \bar{W}_{56} \gamma_{\Omega_6}^{-2}\,, \quad W_{56} \to \gamma_{\Omega_5} \bar{X}_{56} \gamma_{\Omega_6}^{-2}\,, \quad X_{25} \to \gamma_{\Omega_8} \bar{Y}_{85} \gamma_{\Omega_5}^{-2}\,, \quad Y_{85} \to \gamma_{\Omega_2} \bar{X}_{25} \gamma_{\Omega_5}^{-2}\,,\\
&Z_{56} \to \gamma_{\Omega_5} \bar{Z}_{56} \gamma_{\Omega_6}^{-2}\,, \quad X_{42} \to \gamma_{\Omega_4} \bar{W}_{48} \gamma_{\Omega_8}^{-2}\,, \quad W_{48} \to \gamma_{\Omega_4} \bar{X}_{42} \gamma_{\Omega_2}^{-2}\,, \quad Y_{42} \to \gamma_{\Omega_4} \bar{Z}_{48} \gamma_{\Omega_8}^{-2}\,,\\
&Z_{48} \to \gamma_{\Omega_4} \bar{Y}_{42} \gamma_{\Omega_2}^{-2}\,, \quad Y_{56} \to \gamma_{\Omega_5} \bar{Y}_{56} \gamma_{\Omega_6}^{-2}\,, \quad X_{74} \to \gamma_{\Omega_1} \bar{Y}_{14} \gamma_{\Omega_4}^{-2}\,, \quad Y_{14} \to \gamma_{\Omega_7} \bar{X}_{74} \gamma_{\Omega_4}^{-2}\,,\\
&Y_{74} \to \gamma_{\Omega_1} \bar{X}_{14} \gamma_{\Omega_4}^{-2}\,, \quad X_{14} \to \gamma_{\Omega_7} \bar{Y}_{74} \gamma_{\Omega_4}^{-2}\,, \quad X_{53} \to \gamma_{\Omega_5} \bar{Y}_{53} \gamma_{\Omega_3}^{-2}\,, \quad Y_{53} \to \gamma_{\Omega_5} \bar{X}_{53} \gamma_{\Omega_3}^{-2}\,,\\
&X_{37} \to \gamma_{\Omega_3} \bar{Y}_{31} \gamma_{\Omega_1}^{-2}\,, \quad Y_{31} \to \gamma_{\Omega_3} \bar{X}_{37} \gamma_{\Omega_7}^{-2}\,, \quad Y_{37} \to \gamma_{\Omega_3} \bar{X}_{31} \gamma_{\Omega_1}^{-2}\,, \quad X_{31} \to \gamma_{\Omega_3} \bar{Y}_{37} \gamma_{\Omega_7}^{-2}\,.
\end{aligned} \tag{C.8}$$

Requiring the invariance of $W^{(0,1)}$, the Fermi fields transform as

$$\begin{aligned}
&\Lambda^2_{26} \to \gamma_{\Omega_6} \Lambda^1_{68} \gamma_{\Omega_8}^{-2}\,, \quad \Lambda^8_{26} \to -\gamma_{\Omega_6} \Lambda^2_{68} \gamma_{\Omega_8}^{-2}\,, \quad \Lambda^2_{27} \to -\gamma_{\Omega_8} \bar{\Lambda}^2_{81} \gamma_{\Omega_1}^{-2}\,, \quad \Lambda^8_{27} \to -\gamma_{\Omega_8} \bar{\Lambda}^1_{81} \gamma_{\Omega_1}^{-2}\,,\\
&\Lambda^2_{68} \to \gamma_{\Omega_2} \Lambda^1_{26} \gamma_{\Omega_6}^{-2}\,, \quad \Lambda^8_{68} \to -\gamma_{\Omega_2} \Lambda^2_{26} \gamma_{\Omega_6}^{-2}\,, \quad \Lambda^2_{81} \to -\gamma_{\Omega_2} \bar{\Lambda}^2_{27} \gamma_{\Omega_7}^{-2}\,, \quad \Lambda^8_{81} \to -\gamma_{\Omega_2} \bar{\Lambda}^1_{27} \gamma_{\Omega_7}^{-2}\,,\\
&\Lambda^2_{54} \to -\gamma_{\Omega_5} \bar{\Lambda}^8_{54} \gamma_{\Omega_4}^{-2}\,, \quad \Lambda^8_{54} \to -\gamma_{\Omega_5} \bar{\Lambda}^7_{54} \gamma_{\Omega_4}^{-2}\,, \quad \Lambda^5_{54} \to -\gamma_{\Omega_5} \bar{\Lambda}^4_{54} \gamma_{\Omega_4}^{-2}\,, \quad \Lambda^6_{54} \to -\gamma_{\Omega_5} \bar{\Lambda}^3_{54} \gamma_{\Omega_4}^{-2}\,,\\
&\Lambda^3_{54} \to -\gamma_{\Omega_5} \bar{\Lambda}^6_{54} \gamma_{\Omega_4}^{-2}\,, \quad \Lambda^4_{54} \to -\gamma_{\Omega_5} \bar{\Lambda}^5_{54} \gamma_{\Omega_4}^{-2}\,, \quad \Lambda^1_{54} \to -\gamma_{\Omega_5} \bar{\Lambda}^2_{54} \gamma_{\Omega_4}^{-2}\,, \quad \Lambda^7_{54} \to -\gamma_{\Omega_5} \bar{\Lambda}^1_{54} \gamma_{\Omega_4}^{-2}\,,\\
&\Lambda^2_{43} \to -\gamma_{\Omega_4} \bar{\Lambda}^4_{43} \gamma_{\Omega_3}^{-2}\,, \quad \Lambda^8_{43} \to -\gamma_{\Omega_4} \bar{\Lambda}^2_{43} \gamma_{\Omega_3}^{-2}\,, \quad \Lambda^5_{43} \to -\gamma_{\Omega_4} \bar{\Lambda}^3_{43} \gamma_{\Omega_3}^{-2}\,, \quad \Lambda^6_{43} \to -\gamma_{\Omega_4} \bar{\Lambda}^1_{43} \gamma_{\Omega_3}^{-2}\,,
\end{aligned} \tag{C.9}$$

and

$$\begin{aligned}
&\Lambda^R_{11} \to \gamma_{\Omega_7} \Lambda^{R\,T}_{77} \gamma_{\Omega_7}^{-1}\,, \quad \Lambda^R_{22} \to \gamma_{\Omega_8} \Lambda^{R\,T}_{88} \gamma_{\Omega_8}^{-1}\,, \quad \Lambda^R_{33} \to \gamma_{\Omega_3} \Lambda^{R\,T}_{33} \gamma_{\Omega_3}^{-1}\,, \quad \Lambda^R_{44} \to \gamma_{\Omega_4} \Lambda^{R\,T}_{44} \gamma_{\Omega_4}^{-1}\,,\\
&\Lambda^R_{55} \to \gamma_{\Omega_5} \Lambda^{R\,T}_{55} \gamma_{\Omega_5}^{-1}\,, \quad \Lambda^R_{66} \to \gamma_{\Omega_6} \Lambda^{R\,T}_{66} \gamma_{\Omega_6}^{-1}\,, \quad \Lambda^R_{77} \to \gamma_{\Omega_1} \Lambda^{R\,T}_{11} \gamma_{\Omega_1}^{-1}\,, \quad \Lambda^R_{88} \to \gamma_{\Omega_2} \Lambda^{R\,T}_{22} \gamma_{\Omega_2}^{-1}\,.
\end{aligned} \tag{C.10}$$

Using table 9, the corresponding geometric involution acting on the generators reads

$$\begin{aligned}
&M_1 \to \bar{M}_{27}\,, \quad M_2 \to \bar{M}_{24}\,, \quad M_3 \to \bar{M}_{21}\,, \quad M_4 \to \bar{M}_{18}\,, \quad M_5 \to \bar{M}_{12}\,,\\
&M_6 \to \bar{M}_{6}\,, \quad M_7 \to \bar{M}_{26}\,, \quad M_8 \to \bar{M}_{23}\,, \quad M_9 \to \bar{M}_{20}\,, \quad M_{10} \to \bar{M}_{17}\,,\\
&M_{11} \to \bar{M}_{11}\,, \quad M_{12} \to \bar{M}_{5}\,, \quad M_{13} \to \bar{M}_{25}\,, \quad M_{14} \to \bar{M}_{22}\,, \quad M_{15} \to \bar{M}_{19}\,,\\
&M_{16} \to \bar{M}_{16}\,, \quad M_{17} \to \bar{M}_{10}\,, \quad M_{18} \to \bar{M}_{4}\,, \quad M_{19} \to \bar{M}_{15}\,, \quad M_{20} \to \bar{M}_{9}\,,\\
&M_{21} \to \bar{M}_{3}\,, \quad M_{22} \to \bar{M}_{14}\,, \quad M_{23} \to \bar{M}_{8}\,, \quad M_{24} \to \bar{M}_{2}\,, \quad M_{25} \to \bar{M}_{13}\,,\\
&M_{26} \to \bar{M}_{7}\,, \quad M_{27} \to \bar{M}_{1}\,.
\end{aligned} \tag{C.11}$$

Once again, this is the same involution of phase D in (5.19).

The $\gamma_{\Omega_i}$ matrices are constrained as in (5.20). As for phase D, we choose

$$\gamma_{\Omega_3} = \gamma_{\Omega_4} = \gamma_{\Omega_5} = \gamma_{\Omega_6} = \mathbb{1}_N\,. \tag{C.12}$$

Figure 28 shows the quiver for the resulting orientifold of phase H.

### C.3 Phase J

The quiver for phase J is shown in figure 29.



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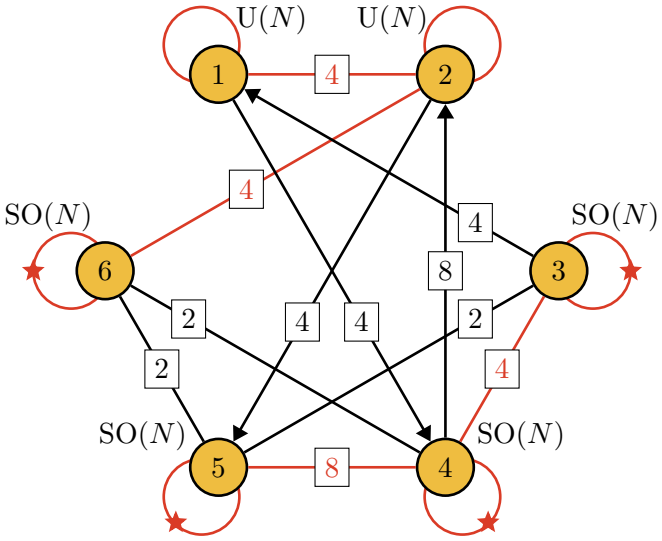


**Figure 28**. Quiver diagram for the Spin(7) orientifold of phase H of $Q^{1,1,1}/\mathbb{Z}_2$ using the involution in eqs. (C.8), (C.9) and (C.10), together with our choice of $\gamma_{\Omega_i}$ matrices.

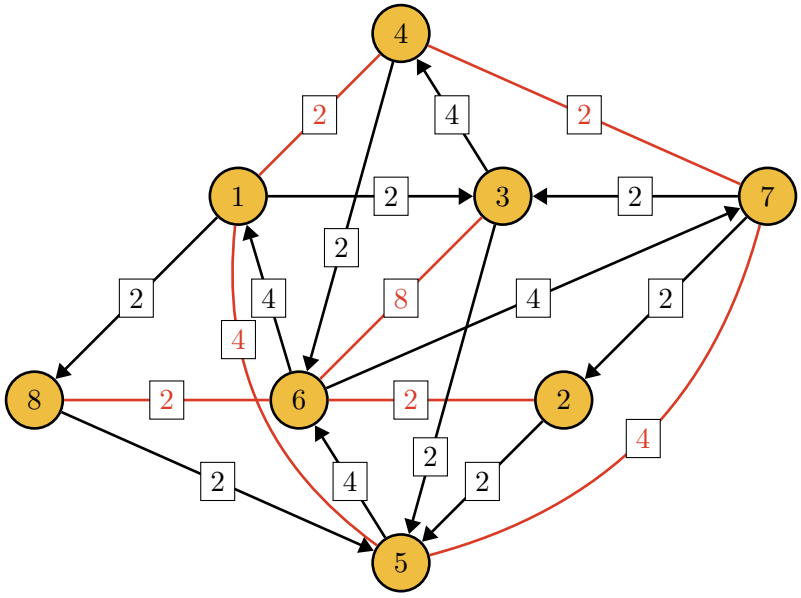


**Figure 29**. Quiver diagram for phase J of $Q^{1,1,1}/\mathbb{Z}_2$.

The $J$- and $E$-terms are

| | $J$ | $E$ |
|---|---|---|
| $\Lambda^1_{14}$ : | $Y_{46}Y_{61} - X_{46}W_{61}$ | $Y_{13}X_{34} - X_{13}Y_{34}$ |
| $\Lambda^2_{14}$ : | $Y_{46}X_{61} - X_{46}Z_{61}$ | $X_{13}W_{34} - Y_{13}Z_{34}$ |
| $\Lambda^1_{15}$ : | $Y_{56}W_{61} - W_{56}Z_{61}$ | $X_{18}X_{85} - X_{13}X_{35}$ |
| $\Lambda^2_{15}$ : | $Z_{56}Z_{61} - X_{56}W_{61}$ | $X_{18}Y_{85} - Y_{13}X_{35}$ |



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$$
\begin{aligned}
\Lambda^3_{15}: &\quad Y_{56}Y_{61} - W_{56}X_{61} \qquad X_{13}Y_{35} - Y_{18}X_{85}\\
\Lambda^4_{15}: &\quad X_{56}Y_{61} - Z_{56}X_{61} \qquad Y_{18}Y_{85} - Y_{13}Y_{35}\\
\Lambda^1_{86}: &\quad Y_{61}Y_{18} - W_{61}X_{18} \qquad X_{85}Y_{56} - Y_{85}X_{56}\\
\Lambda^2_{86}: &\quad X_{61}Y_{18} - Z_{61}X_{18} \qquad Y_{85}Z_{56} - X_{85}W_{56}\\
\Lambda^1_{47}: &\quad X_{73}W_{34} - Y_{73}Y_{34} \qquad X_{46}Z_{67} - Y_{46}X_{67}\\
\Lambda^2_{47}: &\quad X_{73}Z_{34} - Y_{73}X_{34} \qquad Y_{46}Y_{67} - X_{46}W_{67}\\
\Lambda^1_{63}: &\quad Y_{35}W_{56} - W_{34}Y_{46} \qquad X_{61}X_{13} - X_{67}X_{73}\\
\Lambda^2_{63}: &\quad Y_{35}Z_{56} - Z_{34}Y_{46} \qquad Y_{67}X_{73} - X_{61}Y_{13}\\
\Lambda^3_{63}: &\quad Y_{35}Y_{56} - Y_{34}Y_{46} \qquad X_{67}Y_{73} - Y_{61}X_{13}\\
\Lambda^4_{63}: &\quad Y_{35}X_{56} - X_{34}Y_{46} \qquad Y_{61}Y_{13} - Y_{67}Y_{73}\\
\Lambda^5_{63}: &\quad X_{35}W_{56} - W_{34}X_{46} \qquad Z_{67}X_{73} - Z_{61}X_{13}\\
\Lambda^6_{63}: &\quad X_{35}Z_{56} - Z_{34}X_{46} \qquad Z_{61}Y_{13} - W_{67}X_{73}\\
\Lambda^7_{63}: &\quad X_{35}Y_{56} - Y_{34}X_{46} \qquad W_{61}X_{13} - Z_{67}Y_{73}\\
\Lambda^8_{63}: &\quad X_{35}X_{56} - X_{34}X_{46} \qquad W_{67}Y_{73} - W_{61}Y_{13}\\
\Lambda^1_{62}: &\quad X_{25}W_{56} - Y_{25}Y_{56} \qquad X_{67}Y_{72} - Z_{67}X_{72}\\
\Lambda^2_{62}: &\quad X_{25}Z_{56} - Y_{25}X_{56} \qquad W_{67}X_{72} - Y_{67}Y_{72}\\
\Lambda^1_{57}: &\quad Y_{72}Y_{25} - Y_{73}Y_{35} \qquad Y_{56}X_{67} - X_{56}Y_{67}\\
\Lambda^2_{57}: &\quad Y_{72}X_{25} - X_{73}Y_{35} \qquad Z_{56}Y_{67} - W_{56}X_{67}\\
\Lambda^3_{57}: &\quad X_{72}Y_{25} - Y_{73}X_{35} \qquad X_{56}W_{67} - Y_{56}Z_{67}\\
\Lambda^4_{57}: &\quad X_{72}X_{25} - X_{73}X_{35} \qquad W_{56}Z_{67} - Z_{56}W_{67}\,.
\end{aligned}
\tag{C.13}
$$

Once again, the generators can be found in table 10.

**$\mathbf{U(N)^2 \times SO(N)^4}$ orientifold** Let us consider an anti-holomorphic involution of phase J which acts on the nodes in figure 29 as $1 \leftrightarrow 7$ and $2 \leftrightarrow 8$ and maps all other nodes mapped to themselves. Chiral fields transform according to

$$
\begin{aligned}
&Y_{46} \to \gamma_{\Omega_4}\bar{X}_{46}\gamma^{-1}_{\Omega_6}\,, \quad X_{46} \to \gamma_{\Omega_4}\bar{Y}_{46}\gamma^{-1}_{\Omega_6}\,, \quad Y_{61} \to \gamma_{\Omega_6}\bar{Z}_{67}\gamma^{-1}_{\Omega_7}\,, \quad Z_{67} \to \gamma_{\Omega_6}\bar{Y}_{61}\gamma^{-1}_{\Omega_1}\,,\\
&W_{61} \to \gamma_{\Omega_6}\bar{X}_{67}\gamma^{-1}_{\Omega_7}\,, \quad X_{67} \to \gamma_{\Omega_6}\bar{W}_{61}\gamma^{-1}_{\Omega_1}\,, \quad Y_{13} \to \gamma_{\Omega_7}\bar{X}_{73}\gamma^{-1}_{\Omega_3}\,, \quad X_{73} \to \gamma_{\Omega_1}\bar{Y}_{13}\gamma^{-1}_{\Omega_3}\,,\\
&X_{13} \to \gamma_{\Omega_7}\bar{Y}_{73}\gamma^{-1}_{\Omega_3}\,, \quad Y_{73} \to \gamma_{\Omega_1}\bar{X}_{13}\gamma^{-1}_{\Omega_3}\,, \quad X_{34} \to \gamma_{\Omega_3}\bar{W}_{34}\gamma^{-1}_{\Omega_4}\,, \quad W_{34} \to \gamma_{\Omega_3}\bar{X}_{34}\gamma^{-1}_{\Omega_4}\,,\\
&Y_{34} \to \gamma_{\Omega_3}\bar{Y}_{34}\gamma^{-1}_{\Omega_4}\,, \quad X_{61} \to \gamma_{\Omega_6}\bar{W}_{67}\gamma^{-1}_{\Omega_7}\,, \quad W_{67} \to \gamma_{\Omega_6}\bar{X}_{61}\gamma^{-1}_{\Omega_1}\,, \quad Z_{61} \to \gamma_{\Omega_6}\bar{Y}_{67}\gamma^{-1}_{\Omega_7}\,,\\
&Y_{67} \to \gamma_{\Omega_6}\bar{Z}_{61}\gamma^{-1}_{\Omega_1}\,, \quad Z_{34} \to \gamma_{\Omega_3}\bar{Z}_{34}\gamma^{-1}_{\Omega_4}\,, \quad Y_{56} \to \gamma_{\Omega_5}\bar{Y}_{56}\gamma^{-1}_{\Omega_6}\,, \quad W_{56} \to \gamma_{\Omega_5}\bar{X}_{56}\gamma^{-1}_{\Omega_6}\,,\\
&X_{56} \to \gamma_{\Omega_5}\bar{W}_{56}\gamma^{-1}_{\Omega_6}\,, \quad X_{35} \to \gamma_{\Omega_3}\bar{Y}_{35}\gamma^{-1}_{\Omega_5}\,, \quad Y_{35} \to \gamma_{\Omega_3}\bar{X}_{35}\gamma^{-1}_{\Omega_5}\,, \quad X_{18} \to \gamma_{\Omega_7}\bar{Y}_{72}\gamma^{-1}_{\Omega_2}\,,\\
&Y_{72} \to \gamma_{\Omega_1}\bar{X}_{18}\gamma^{-1}_{\Omega_8}\,, \quad X_{85} \to \gamma_{\Omega_2}\bar{Y}_{25}\gamma^{-1}_{\Omega_5}\,, \quad Y_{25} \to \gamma_{\Omega_8}\bar{X}_{85}\gamma^{-1}_{\Omega_5}\,, \quad Z_{56} \to \gamma_{\Omega_5}\bar{Z}_{56}\gamma^{-1}_{\Omega_6}\,,\\
&Y_{18} \to \gamma_{\Omega_7}\bar{X}_{72}\gamma^{-1}_{\Omega_2}\,, \quad X_{72} \to \gamma_{\Omega_1}\bar{Y}_{18}\gamma^{-1}_{\Omega_8}\,, \quad Y_{85} \to \gamma_{\Omega_2}\bar{X}_{25}\gamma^{-1}_{\Omega_5}\,, \quad X_{25} \to \gamma_{\Omega_8}\bar{Y}_{85}\gamma^{-1}_{\Omega_5}\,,\\
&\qquad\qquad Y_{46} \to \gamma_{\Omega_4}\bar{X}_{46}\gamma^{-1}_{\Omega_4}\,, \quad X_{46} \to \gamma_{\Omega_4}\bar{Y}_{46}\gamma^{-1}_{\Omega_6}\,.
\end{aligned}
\tag{C.14}
$$



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Requiring the invariance of $W^{(0,1)}$, the Fermi fields transform as

$$\begin{aligned}
&\Lambda^1_{14} \to \gamma_{\Omega_4}\Lambda^1_{47}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^2_{14} \to -\gamma_{\Omega_4}\Lambda^2_{47}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^1_{15} \to \gamma_{\Omega_5}\Lambda^1_{57}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^2_{15} \to \gamma_{\Omega_5}\Lambda^2_{57}\gamma^{-1}_{\Omega_7}\,,\\
&\Lambda^3_{15} \to -\gamma_{\Omega_5}\Lambda^3_{57}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^4_{15} \to \gamma_{\Omega_5}\Lambda^4_{57}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^1_{86} \to -\gamma_{\Omega_6}\Lambda^1_{62}\gamma^{-1}_{\Omega_2}\,, \quad \Lambda^2_{86} \to \gamma_{\Omega_6}\Lambda^2_{62}\gamma^{-1}_{\Omega_2}\,,\\
&\Lambda^1_{47} \to \gamma_{\Omega_1}\Lambda^1_{14}\gamma^{-1}_{\Omega_4}\,, \quad \Lambda^2_{47} \to -\gamma_{\Omega_4}\Lambda^2_{47}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^1_{63} \to \gamma_{\Omega_6}\bar{\Lambda}^8_{63}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^2_{63} \to \gamma_{\Omega_6}\bar{\Lambda}^6_{63}\gamma^{-1}_{\Omega_3}\,,\\
&\Lambda^3_{63} \to \gamma_{\Omega_6}\bar{\Lambda}^7_{63}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^4_{63} \to \gamma_{\Omega_6}\bar{\Lambda}^5_{63}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^5_{63} \to \gamma_{\Omega_6}\bar{\Lambda}^4_{63}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^6_{63} \to \gamma_{\Omega_6}\bar{\Lambda}^2_{63}\gamma^{-1}_{\Omega_3}\,,\\
&\Lambda^7_{63} \to \gamma_{\Omega_6}\bar{\Lambda}^3_{63}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^8_{63} \to \gamma_{\Omega_6}\bar{\Lambda}^1_{63}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^1_{62} \to -\gamma_{\Omega_8}\Lambda^1_{86}\gamma^{-1}_{\Omega_6}\,, \quad \Lambda^2_{62} \to \gamma_{\Omega_8}\Lambda^2_{86}\gamma^{-1}_{\Omega_6}\,,\\
&\Lambda^1_{57} \to \gamma_{\Omega_1}\Lambda^1_{15}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^2_{57} \to \gamma_{\Omega_1}\Lambda^2_{15}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^3_{57} \to -\gamma_{\Omega_1}\Lambda^3_{15}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^4_{57} \to \gamma_{\Omega_1}\Lambda^4_{15}\gamma^{-1}_{\Omega_5}\,.
\end{aligned} \tag{C.15}$$

and

$$\begin{aligned}
&\Lambda^R_{11} \to \gamma_{\Omega_7}\Lambda^{R\,T}_{77}\gamma^{-1}_{\Omega_7}\,, \quad \Lambda^R_{22} \to \gamma_{\Omega_8}\Lambda^{R\,T}_{88}\gamma^{-1}_{\Omega_8}\,, \quad \Lambda^R_{33} \to \gamma_{\Omega_3}\Lambda^{R\,T}_{33}\gamma^{-1}_{\Omega_3}\,, \quad \Lambda^R_{44} \to \gamma_{\Omega_4}\Lambda^{R\,T}_{44}\gamma^{-1}_{\Omega_4}\,,\\
&\Lambda^R_{55} \to \gamma_{\Omega_5}\Lambda^{R\,T}_{55}\gamma^{-1}_{\Omega_5}\,, \quad \Lambda^R_{66} \to \gamma_{\Omega_6}\Lambda^{R\,T}_{66}\gamma^{-1}_{\Omega_6}\,, \quad \Lambda^R_{77} \to \gamma_{\Omega_1}\Lambda^{R\,T}_{11}\gamma^{-1}_{\Omega_1}\,, \quad \Lambda^R_{88} \to \gamma_{\Omega_2}\Lambda^{R\,T}_{22}\gamma^{-1}_{\Omega_2}\,.
\end{aligned} \tag{C.16}$$

Using table 10, the corresponding geometric involution acting on the generators reads

$$\begin{aligned}
&M_1 \to \bar{M}_{27}\,, \quad M_2 \to \bar{M}_{24}\,, \quad M_3 \to \bar{M}_{21}\,, \quad M_4 \to \bar{M}_{18}\,, \quad M_5 \to \bar{M}_{12}\,,\\
&M_6 \to \bar{M}_{6}\,, \quad M_7 \to \bar{M}_{26}\,, \quad M_8 \to \bar{M}_{23}\,, \quad M_9 \to \bar{M}_{20}\,, \quad M_{10} \to \bar{M}_{17}\,,\\
&M_{11} \to \bar{M}_{11}\,, \quad M_{12} \to \bar{M}_{5}\,, \quad M_{13} \to \bar{M}_{25}\,, \quad M_{14} \to \bar{M}_{22}\,, \quad M_{15} \to \bar{M}_{19}\,,\\
&M_{16} \to \bar{M}_{16}\,, \quad M_{17} \to \bar{M}_{10}\,, \quad M_{18} \to \bar{M}_{4}\,, \quad M_{19} \to \bar{M}_{15}\,, \quad M_{20} \to \bar{M}_{9}\,,\\
&M_{21} \to \bar{M}_{3}\,, \quad M_{22} \to \bar{M}_{14}\,, \quad M_{23} \to \bar{M}_{8}\,, \quad M_{24} \to \bar{M}_{2}\,, \quad M_{25} \to \bar{M}_{13}\,,\\
&M_{26} \to \bar{M}_{7}\,, \quad M_{27} \to \bar{M}_{1}\,. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad xs
\end{aligned} \tag{C.17}$$

Notice, again, that this is the same geometric action that we have found for phase D in (5.19).

The $\gamma_{\Omega_i}$ matrices are constrained as in (5.20). As for phase D, we choose

$$\gamma_{\Omega_3} = \gamma_{\Omega_4} = \gamma_{\Omega_5} = \gamma_{\Omega_6} = \mathbb{1}_N\,. \tag{C.18}$$

The resulting orientifold of phase J is shown in figure 30.

### C.4 Phase L

The last $\mathcal{N} = (0,2)$ quiver we consider is phase L, shown in figure 31.

The $J$- and $E$-terms are

| | $J$ | $E$ |
|---|---|---|
| $\Lambda^1_{15}$ : | $B_{54}Y_{41} - D_{54}X_{41}$ | $X_{13}X_{35} - X_{18}X_{85}$ |
| $\Lambda^2_{15}$ : | $C_{54}X_{41} - A_{54}Y_{41}$ | $X_{13}Y_{35} - Y_{18}X_{85}$ |
| $\Lambda^3_{15}$ : | $W_{54}X_{41} - Y_{54}Y_{41}$ | $Y_{13}X_{35} - X_{18}Y_{85}$ |
| $\Lambda^4_{15}$ : | $Z_{54}X_{41} - X_{54}Y_{41}$ | $Y_{18}Y_{85} - Y_{13}Y_{35}$ |
| $\Lambda^1_{34}$ : | $Y_{41}Y_{13} - Y_{47}Y_{73}$ | $X_{35}Y_{54} - Y_{35}X_{54}$ |
| $\Lambda^2_{34}$ : | $X_{41}Y_{13} - Y_{47}X_{73}$ | $Y_{35}Z_{54} - X_{35}W_{54}$ |



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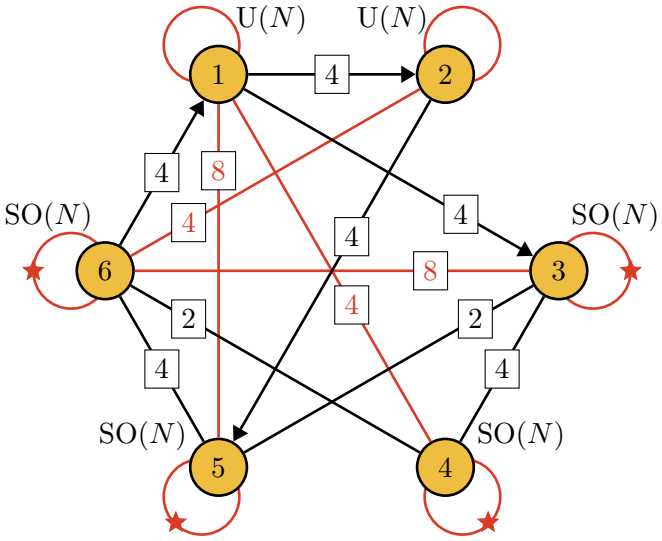


**Figure 30**. Quiver diagram for the Spin(7) orientifold of phase J of $Q^{1,1,1}/\mathbb{Z}_2$ using the involution in eqs. (C.14), (C.15) and (C.16), together with our choice of $\gamma_{\Omega_i}$ matrices.

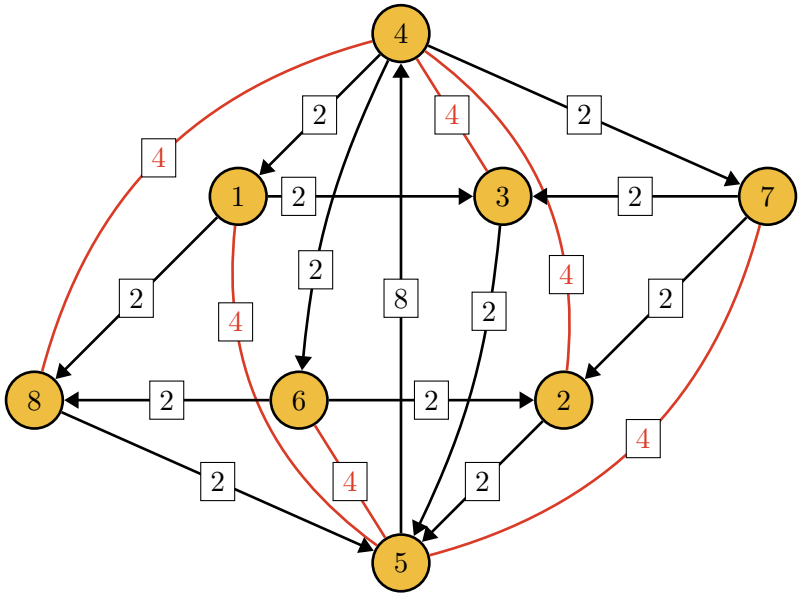


**Figure 31**. Quiver diagram for phase L of $Q^{1,1,1}/\mathbb{Z}_2$.

$$
\begin{aligned}
\Lambda^3_{34} &: \quad Y_{41}X_{13} - X_{47}Y_{73} \qquad Y_{35}A_{54} - X_{35}B_{54} \\
\Lambda^4_{34} &: \quad X_{41}X_{13} - X_{47}X_{73} \qquad X_{35}D_{54} - Y_{35}C_{54} \\
\Lambda^1_{48} &: \quad X_{85}D_{54} - Y_{85}W_{54} \qquad X_{46}X_{68} - X_{41}X_{18} \\
\Lambda^2_{48} &: \quad X_{85}C_{54} - Y_{85}Z_{54} \qquad X_{41}Y_{18} - Y_{46}X_{68} \\
\Lambda^3_{48} &: \quad X_{85}B_{54} - Y_{85}Y_{54} \qquad Y_{41}X_{18} - X_{46}Y_{68} \\
\Lambda^4_{48} &: \quad X_{85}A_{54} - Y_{85}X_{54} \qquad Y_{46}Y_{68} - Y_{41}Y_{18} \\
\Lambda^1_{42} &: \quad X_{25}D_{54} - Y_{25}B_{54} \qquad X_{47}X_{72} - X_{46}X_{62} \\
\Lambda^2_{42} &: \quad X_{25}C_{54} - Y_{25}A_{54} \qquad Y_{46}X_{62} - X_{47}Y_{72}
\end{aligned}
$$



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$$
\begin{array}{lll}
\Lambda^3_{42}: & X_{25}W_{54}-Y_{25}Y_{54} & X_{46}Y_{62}-Y_{47}X_{72}\\
\Lambda^4_{42}: & X_{25}Z_{54}-Y_{25}X_{54} & Y_{47}Y_{72}-Y_{46}Y_{62}\\
\Lambda^1_{75}: & W_{54}Y_{47}-D_{54}X_{47} & X_{72}X_{25}-X_{73}X_{35}\\
\Lambda^2_{75}: & Z_{54}Y_{47}-C_{54}X_{47} & X_{73}Y_{35}-Y_{72}X_{25}\\
\Lambda^3_{75}: & Y_{54}Y_{47}-B_{54}X_{47} & Y_{73}X_{35}-X_{72}Y_{25}\\
\Lambda^4_{75}: & X_{54}Y_{47}-A_{54}X_{47} & Y_{72}Y_{25}-Y_{73}Y_{35}\\
\Lambda^1_{56}: & Y_{68}Y_{85}-Y_{62}Y_{25} & X_{54}Y_{46}-Y_{54}X_{46}\\
\Lambda^2_{56}: & X_{68}Y_{85}-Y_{62}X_{25} & W_{54}X_{46}-Z_{54}Y_{46}\\
\Lambda^3_{56}: & Y_{68}X_{85}-X_{62}Y_{25} & B_{54}X_{46}-A_{54}Y_{46}\\
\Lambda^4_{56}: & X_{68}X_{85}-X_{62}X_{25} & C_{54}Y_{46}-D_{54}X_{46}\,.
\end{array}
\tag{C.19}
$$

**$\mathbf{U(N)^2\times SO(N)^4}$ orientifold.** We can, again, look for an involution that maps the nodes in figure 31 $1\leftrightarrow 7$ and $2\leftrightarrow 8$ and all the other nodes to themselves. The map on the fields is

$$
\begin{array}{llll}
B_{54}\to\gamma_{\Omega_5}\bar{Z}_{54}\gamma^{-1}_{\Omega_4}\,, & Z_{54}\to\gamma_{\Omega_5}\bar{B}_{54}\gamma^{-1}_{\Omega_4}\,, & D_{54}\to\gamma_{\Omega_5}\bar{C}_{54}\gamma^{-1}_{\Omega_4}\,, & C_{54}\to\gamma_{\Omega_5}\bar{D}_{54}\gamma^{-1}_{\Omega_4}\,,\\
Y_{41}\to\gamma_{\Omega_4}\bar{Y}_{47}\gamma^{-1}_{\Omega_7}\,, & Y_{47}\to\gamma_{\Omega_4}\bar{Y}_{41}\gamma^{-1}_{\Omega_1}\,, & X_{41}\to\gamma_{\Omega_4}\bar{X}_{47}\gamma^{-1}_{\Omega_7}\,, & X_{47}\to\gamma_{\Omega_4}\bar{X}_{41}\gamma^{-1}_{\Omega_1}\,,\\
X_{13}\to\gamma_{\Omega_7}\bar{X}_{73}\gamma^{-1}_{\Omega_3}\,, & X_{73}\to\gamma_{\Omega_1}\bar{X}_{13}\gamma^{-1}_{\Omega_3}\,, & X_{35}\to\gamma_{\Omega_3}\bar{Y}_{35}\gamma^{-1}_{\Omega_5}\,, & Y_{35}\to\gamma_{\Omega_3}\bar{X}_{35}\gamma^{-1}_{\Omega_5}\,,\\
X_{18}\to\gamma_{\Omega_7}\bar{Y}_{72}\gamma^{-1}_{\Omega_2}\,, & Y_{72}\to\gamma_{\Omega_1}\bar{X}_{18}\gamma^{-1}_{\Omega_8}\,, & X_{85}\to\gamma_{\Omega_2}\bar{X}_{25}\gamma^{-1}_{\Omega_5}\,, & X_{25}\to\gamma_{\Omega_8}\bar{X}_{85}\gamma^{-1}_{\Omega_5}\,,\\
A_{54}\to\gamma_{\Omega_5}\bar{W}_{54}\gamma^{-1}_{\Omega_4}\,, & W_{54}\to\gamma_{\Omega_5}\bar{A}_{54}\gamma^{-1}_{\Omega_4}\,, & Y_{18}\to\gamma_{\Omega_7}\bar{X}_{72}\gamma^{-1}_{\Omega_2}\,, & X_{72}\to\gamma_{\Omega_1}\bar{Y}_{18}\gamma^{-1}_{\Omega_8}\,,\\
Y_{13}\to\gamma_{\Omega_7}\bar{Y}_{73}\gamma^{-1}_{\Omega_3}\,, & Y_{73}\to\gamma_{\Omega_1}\bar{Y}_{13}\gamma^{-1}_{\Omega_3}\,, & Y_{85}\to\gamma_{\Omega_2}\bar{Y}_{25}\gamma^{-1}_{\Omega_5}\,, & Y_{25}\to\gamma_{\Omega_8}\bar{Y}_{85}\gamma^{-1}_{\Omega_5}\,,\\
Y_{54}\to\gamma_{\Omega_5}\bar{X}_{54}\gamma^{-1}_{\Omega_4}\,, & X_{54}\to\gamma_{\Omega_5}\bar{Y}_{54}\gamma^{-1}_{\Omega_4}\,, & X_{46}\to\gamma_{\Omega_4}\bar{Y}_{46}\gamma^{-1}_{\Omega_6}\,, & Y_{46}\to\gamma_{\Omega_4}\bar{X}_{46}\gamma^{-1}_{\Omega_6}\,,\\
Y_{68}\to\gamma_{\Omega_6}\bar{Y}_{62}\gamma^{-1}_{\Omega_2}\,, & Y_{62}\to\gamma_{\Omega_6}\bar{Y}_{68}\gamma^{-1}_{\Omega_8}\,, & X_{68}\to\gamma_{\Omega_6}\bar{X}_{62}\gamma^{-1}_{\Omega_2}\,, & X_{62}\to\gamma_{\Omega_6}\bar{X}_{68}\gamma^{-1}_{\Omega_8}\,.
\end{array}
\tag{C.20}
$$

Requiring the invariance of the $W^{(0,1)}$ superpotential, we obtain also the following transformations for the Fermi fields:

$$
\begin{array}{llll}
\Lambda^1_{15}\to\gamma_{\Omega_7}\bar{\Lambda}^2_{75}\gamma^{-1}_{\Omega_5}\,, & \Lambda^2_{15}\to-\gamma_{\Omega_7}\bar{\Lambda}^1_{75}\gamma^{-1}_{\Omega_5}\,, & \Lambda^3_{15}\to-\gamma_{\Omega_7}\bar{\Lambda}^4_{75}\gamma^{-1}_{\Omega_5}\,, & \Lambda^4_{15}\to-\gamma_{\Omega_7}\bar{\Lambda}^3_{75}\gamma^{-1}_{\Omega_5}\,,\\
\Lambda^1_{34}\to-\gamma_{\Omega_3}\bar{\Lambda}^1_{34}\gamma^{-1}_{\Omega_4}\,, & \Lambda^2_{34}\to-\gamma_{\Omega_3}\bar{\Lambda}^3_{34}\gamma^{-1}_{\Omega_4}\,, & \Lambda^3_{34}\to-\gamma_{\Omega_3}\bar{\Lambda}^2_{34}\gamma^{-1}_{\Omega_4}\,, & \Lambda^4_{34}\to-\gamma_{\Omega_3}\bar{\Lambda}^4_{34}\gamma^{-1}_{\Omega_4}\,,\\
\Lambda^1_{48}\to\gamma_{\Omega_4}\bar{\Lambda}^2_{42}\gamma^{-1}_{\Omega_2}\,, & \Lambda^2_{48}\to\gamma_{\Omega_4}\bar{\Lambda}^1_{42}\gamma^{-1}_{\Omega_2}\,, & \Lambda^3_{48}\to\gamma_{\Omega_4}\bar{\Lambda}^4_{42}\gamma^{-1}_{\Omega_2}\,, & \Lambda^4_{48}\to\gamma_{\Omega_4}\bar{\Lambda}^3_{42}\gamma^{-1}_{\Omega_2}\,,\\
\Lambda^1_{42}\to\gamma_{\Omega_4}\bar{\Lambda}^2_{48}\gamma^{-1}_{\Omega_8}\,, & \Lambda^2_{42}\to\gamma_{\Omega_4}\bar{\Lambda}^1_{48}\gamma^{-1}_{\Omega_8}\,, & \Lambda^3_{42}\to\gamma_{\Omega_4}\bar{\Lambda}^4_{48}\gamma^{-1}_{\Omega_8}\,, & \Lambda^4_{42}\to\gamma_{\Omega_4}\bar{\Lambda}^3_{48}\gamma^{-1}_{\Omega_8}\,,\\
\Lambda^1_{75}\to-\gamma_{\Omega_1}\bar{\Lambda}^2_{15}\gamma^{-1}_{\Omega_5}\,, & \Lambda^2_{75}\to\gamma_{\Omega_1}\bar{\Lambda}^1_{15}\gamma^{-1}_{\Omega_5}\,, & \Lambda^3_{75}\to-\gamma_{\Omega_1}\bar{\Lambda}^4_{15}\gamma^{-1}_{\Omega_5}\,, & \Lambda^4_{75}\to-\gamma_{\Omega_1}\bar{\Lambda}^3_{15}\gamma^{-1}_{\Omega_5}\,,\\
\Lambda^1_{56}\to-\gamma_{\Omega_5}\bar{\Lambda}^1_{56}\gamma^{-1}_{\Omega_6}\,, & \Lambda^2_{56}\to-\gamma_{\Omega_5}\bar{\Lambda}^3_{56}\gamma^{-1}_{\Omega_6}\,, & \Lambda^3_{56}\to-\gamma_{\Omega_5}\bar{\Lambda}^2_{56}\gamma^{-1}_{\Omega_6}\,, & \Lambda^4_{56}\to-\gamma_{\Omega_5}\bar{\Lambda}^4_{56}\gamma^{-1}_{\Omega_6}\,,
\end{array}
\tag{C.21}
$$

and

$$
\begin{array}{llll}
\Lambda^R_{11}\to\gamma_{\Omega_7}\Lambda^{R\,T}_{77}\gamma^{-1}_{\Omega_7}\,, & \Lambda^R_{22}\to\gamma_{\Omega_8}\Lambda^{R\,T}_{88}\gamma^{-1}_{\Omega_8}\,, & \Lambda^R_{33}\to\gamma_{\Omega_3}\Lambda^{R\,T}_{33}\gamma^{-1}_{\Omega_3}\,, & \Lambda^R_{44}\to\gamma_{\Omega_4}\Lambda^{R\,T}_{44}\gamma^{-1}_{\Omega_4}\,,\\
\Lambda^R_{55}\to\gamma_{\Omega_5}\Lambda^{R\,T}_{55}\gamma^{-1}_{\Omega_5}\,, & \Lambda^R_{66}\to\gamma_{\Omega_1}\Lambda^{R\,T}_{11}\gamma^{-1}_{\Omega_1}\,, & \Lambda^R_{77}\to\gamma_{\Omega_1}\Lambda^{R\,T}_{11}\gamma^{-1}_{\Omega_1}\,, & \Lambda^R_{88}\to\gamma_{\Omega_2}\Lambda^{R\,T}_{22}\gamma^{-1}_{\Omega_2}\,.
\end{array}
\tag{C.22}
$$

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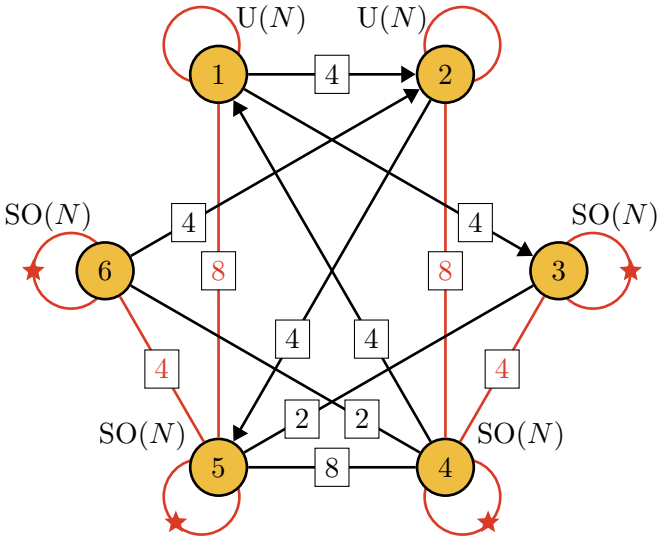


**Figure 32**. Quiver diagram for the Spin(7) orientifold of phase L of $Q^{1,1,1}/\mathbb{Z}_2$ using the involution in eqs. (C.20), (C.21) and (C.22), together with our choice of $\gamma_{\Omega_i}$ matrices.

Using table 11, the corresponding geometric involution acting on the generators reads

$$\begin{array}{lllll} M_1 \to \bar{M}_{27}\,, & M_2 \to \bar{M}_{24}\,, & M_3 \to \bar{M}_{21}\,, & M_4 \to \bar{M}_{18}\,, & M_5 \to \bar{M}_{12}\,, \\ M_6 \to \bar{M}_{6}\,, & M_7 \to \bar{M}_{26}\,, & M_8 \to \bar{M}_{23}\,, & M_9 \to \bar{M}_{20}\,, & M_{10} \to \bar{M}_{17}\,, \\ M_{11} \to \bar{M}_{11}\,, & M_{12} \to \bar{M}_{5}\,, & M_{13} \to \bar{M}_{25}\,, & M_{14} \to \bar{M}_{22}\,, & M_{15} \to \bar{M}_{19}\,, \\ M_{16} \to \bar{M}_{16}\,, & M_{17} \to \bar{M}_{10}\,, & M_{18} \to \bar{M}_{4}\,, & M_{19} \to \bar{M}_{15}\,, & M_{20} \to \bar{M}_{9}\,, \\ M_{21} \to \bar{M}_{3}\,, & M_{22} \to \bar{M}_{14}\,, & M_{23} \to \bar{M}_{8}\,, & M_{24} \to \bar{M}_{2}\,, & M_{25} \to \bar{M}_{13}\,, \\ M_{26} \to \bar{M}_{7}\,, & M_{27} \to \bar{M}_{1}\,. & & & \end{array} \tag{C.23}$$

This is, once again, the same geometric involution.

The $\gamma_{\Omega_i}$ matrices are constrained as in (5.20). As for phase D, we choose

$$\gamma_{\Omega_3} = \gamma_{\Omega_4} = \gamma_{\Omega_5} = \gamma_{\Omega_6} = \mathbb{1}_N\,. \tag{C.24}$$

The resulting orientifold is shown in figure 32.

### C.5 Generators of $Q^{1,1,1}/\mathbb{Z}_2$

In tables 7, 8, 9, 10 and 11 we list the generators of $Q^{1,1,1}/\mathbb{Z}_2$ in terms of the chiral fields of phases D, E, H, J and L.

The relations among the generators are the same for all phases, and they are:

$$\begin{aligned} \mathcal{I}=\big\langle & M_1M_3=M_2^2\,, M_1M_5=M_2M_4\,, M_2M_6=M_3M_5\,, M_1M_8=M_2M_7\,, M_2M_9=M_3M_8\,, M_1M_{10}=M_4M_7\,, \\ & M_3M_{12}=M_6M_9\,, M_1M_{13}=M_7^2\,, M_7M_{14}=M_8M_{13}\,, M_3M_{15}=M_9^2\,, M_8M_{15}=M_9M_{14}\,, M_{13}M_{15}=M_{14}^2\,, \\ & M_7M_{16}=M_{10}M_{13}\,, M_{13}M_{17}=M_{14}M_{16}\,, M_9M_{18}=M_{12}M_{15}\,, M_{14}M_{18}=M_{15}M_{17}\,, M_1M_{19}=M_4^2\,, \\ & M_4M_{20}=M_5M_{19}\,, M_3M_{21}=M_6^2\,, M_5M_{21}=M_6M_{20}\,, M_{19}M_{21}=M_{20}^2\,, M_4M_{22}=M_{10}M_{19}\,, \end{aligned}$$



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$$M_{19}M_{23}=M_{20}M_{22}\,,M_6M_{24}=M_{12}M_{21}\,,M_{20}M_{24}=M_{21}M_{23}\,,M_{10}M_{25}=M_{16}M_{22}\,,M_{13}M_{25}=M_{16}^2\,,$$
$$M_{19}M_{25}=M_{22}^2\,,M_{16}M_{26}=M_{17}M_{25}\,,M_{22}M_{26}=M_{23}M_{25}\,,M_{12}M_{27}=M_{18}M_{24}\,,M_{15}M_{27}=M_{18}^2\,,$$
$$M_{17}M_{27}=M_{18}M_{26}\,,M_{21}M_{27}=M_{24}^2\,,M_{23}M_{27}=M_{24}M_{26}\,,M_{25}M_{27}=M_{26}^2\,,M_1M_6=M_2M_5=M_3M_4\,,$$
$$M_1M_9=M_2M_8=M_3M_7\,,M_1M_{14}=M_2M_{13}=M_7M_8\,,M_2M_{15}=M_3M_{14}=M_8M_9\,,$$
$$M_7M_{15}=M_8M_{14}=M_9M_{13}\,,M_1M_{16}=M_4M_{13}=M_7M_{10}\,,M_3M_{18}=M_6M_{15}=M_9M_{12}\,,$$
$$M_{13}M_{18}=M_{14}M_{17}=M_{15}M_{16}\,,M_1M_{20}=M_2M_{19}=M_4M_5\,,M_2M_{21}=M_3M_{20}=M_5M_6\,,$$
$$M_4M_{21}=M_5M_{20}=M_6M_{19}\,,M_1M_{22}=M_4M_{10}=M_7M_{19}\,,M_3M_{24}=M_6M_{12}=M_9M_{21}\,,$$
$$M_{19}M_{24}=M_{20}M_{23}=M_{21}M_{22}\,,M_4M_{25}=M_{10}M_{22}=M_{16}M_{19}\,,M_7M_{25}=M_{10}M_{16}=M_{13}M_{22}\,,$$
$$M_{13}M_{26}=M_{14}M_{25}=M_{16}M_{17}\,,M_{19}M_{26}=M_{20}M_{25}=M_{22}M_{23}\,,M_6M_{27}=M_{12}M_{24}=M_{18}M_{21}\,,$$
$$M_9M_{27}=M_{12}M_{18}=M_{15}M_{24}\,,M_{14}M_{27}=M_{15}M_{26}=M_{17}M_{18}\,,M_{16}M_{27}=M_{17}M_{26}=M_{18}M_{25}\,,$$
$$M_{20}M_{27}=M_{21}M_{26}=M_{23}M_{24}\,,M_{22}M_{27}=M_{23}M_{26}=M_{24}M_{25}\,,M_1M_{11}=M_2M_{10}=M_4M_8=M_5M_7\,,$$
$$M_2M_{12}=M_3M_{11}=M_5M_9=M_6M_8\,,M_7M_{17}=M_8M_{16}=M_{10}M_{14}=M_{11}M_{13}\,,$$
$$M_8M_{18}=M_9M_{17}=M_{11}M_{15}=M_{12}M_{14}\,,M_4M_{23}=M_5M_{22}=M_{10}M_{20}=M_{11}M_{19}\,,$$
$$M_5M_{24}=M_6M_{23}=M_{11}M_{21}=M_{12}M_{20}\,,M_{10}M_{26}=M_{11}M_{25}=M_{16}M_{23}=M_{17}M_{22}\,,$$
$$M_{11}M_{27}=M_{12}M_{26}=M_{17}M_{24}=M_{18}M_{23}\,,M_1M_{15}=M_2M_{14}=M_3M_{13}=M_7M_9=M_8^2\,,$$
$$M_1M_{21}=M_2M_{20}=M_3M_{19}=M_4M_6=M_5^2\,,M_1M_{25}=M_4M_{16}=M_7M_{22}=M_{10}^2=M_{13}M_{19}\,,$$
$$M_3M_{27}=M_6M_{18}=M_9M_{24}=M_{12}^2=M_{15}M_{21}\,,M_{13}M_{27}=M_{14}M_{26}=M_{15}M_{25}=M_{16}M_{18}=M_{17}^2\,,$$
$$M_{19}M_{27}=M_{20}M_{26}=M_{21}M_{25}=M_{22}M_{24}=M_{23}^2\,,M_1M_{12}=M_2M_{11}=M_3M_{10}=M_4M_9=M_5M_8=M_6M_7\,,$$
$$M_1M_{17}=M_2M_{16}=M_4M_{14}=M_5M_{13}=M_7M_{11}=M_8M_{10}\,,M_2M_{18}=M_3M_{17}=M_5M_{15}=M_6M_{14}=M_8M_{12}=$$
$$=M_9M_{11}\,,M_7M_{18}=M_8M_{17}=M_9M_{16}=M_{10}M_{15}=M_{11}M_{14}=M_{12}M_{13}\,,M_1M_{23}=M_2M_{22}=M_4M_{11}=$$
$$=M_5M_{10}=M_7M_{20}=M_8M_{19}\,,M_2M_{24}=M_3M_{23}=M_5M_{12}=M_6M_{11}=M_8M_{21}=M_9M_{20}\,,M_4M_{24}=M_5M_{23}=$$
$$=M_6M_{22}=M_{10}M_{21}=M_{11}M_{20}=M_{12}M_{19}\,,M_4M_{26}=M_5M_{25}=M_{10}M_{23}=M_{11}M_{22}=M_{16}M_{20}=M_{17}M_{19}\,,$$
$$M_7M_{26}=M_8M_{25}=M_{10}M_{17}=M_{11}M_{16}=M_{13}M_{23}=M_{14}M_{22}\,,M_5M_{27}=M_6M_{26}=M_{11}M_{24}=M_{12}M_{23}=M_{17}M_{21}=$$
$$=M_{18}M_{20}\,,M_8M_{27}=M_9M_{26}=M_{11}M_{18}=M_{12}M_{17}=M_{14}M_{24}=M_{15}M_{23}\,,M_{10}M_{27}=M_{11}M_{26}=M_{12}M_{25}=$$
$$=M_{16}M_{24}=M_{17}M_{23}=M_{18}M_{22}\,,M_1M_{18}=M_2M_{17}=M_3M_{16}=M_4M_{15}=M_5M_{14}=M_6M_{13}=M_7M_{12}=M_8M_{11}=$$
$$=M_9M_{10}\,,M_1M_{24}=M_2M_{23}=M_3M_{22}=M_4M_{12}=M_5M_{11}=M_6M_{10}=M_7M_{21}=M_8M_{20}=M_9M_{19}\,,M_1M_{26}=$$
$$=M_2M_{25}=M_4M_{17}=M_5M_{16}=M_7M_{23}=M_8M_{22}=M_{10}M_{11}=M_{13}M_{20}=M_{14}M_{19}\,,M_2M_{27}=M_3M_{26}=$$
$$=M_5M_{18}=M_6M_{17}=M_8M_{24}=M_9M_{23}=M_{11}M_{12}=M_{14}M_{21}=M_{15}M_{20}\,,M_4M_{27}=M_5M_{26}=M_6M_{25}=M_{10}M_{24}=$$
$$=M_{11}M_{23}=M_{12}M_{22}=M_{16}M_{21}=M_{17}M_{20}=M_{18}M_{19}\,,M_7M_{27}=M_8M_{26}=M_9M_{25}=M_{10}M_{18}=M_{11}M_{17}=$$
$$=M_{12}M_{16}=M_{13}M_{24}=M_{14}M_{23}=M_{15}M_{22}\,,M_1M_{27}=M_2M_{26}=M_3M_{25}=M_4M_{18}=M_5M_{17}=M_6M_{16}=M_7M_{24}=$$
$$=M_8M_{23}=M_9M_{22}=M_{10}M_{12}=M_{11}^2=M_{13}M_{21}=M_{14}M_{20}=M_{15}M_{19}\,.\big\rangle \tag{C.25}$$

| Field | Chiral superfields |
|---|---|
| $M_1$ | $X_{72}X_{47}X_{64}Y_{56}Y_{25} = X_{72}X_{47}X_{53}Y_{34}Y_{25} = X_{18}X_{64}X_{85}Y_{41}Y_{56} =$ $= X_{18}X_{53}X_{85}Y_{34}Y_{41}$ |
| $M_2$ | $X_{72}X_{47}X_{64}X_{25}Y_{56} = X_{72}X_{47}X_{56}X_{64}Y_{25} = X_{72}X_{47}X_{53}X_{25}Y_{34} =$ $= X_{72}X_{34}X_{47}X_{53}Y_{25} = X_{18}X_{56}X_{64}X_{85}Y_{41} = X_{18}X_{41}X_{64}X_{85}Y_{56} =$ $= X_{18}X_{41}X_{53}X_{85}Y_{34} = X_{18}X_{34}X_{53}X_{85}Y_{41}$ |
| $M_3$ | $X_{72}X_{47}X_{56}X_{64}X_{25} = X_{72}X_{34}X_{47}X_{53}X_{25} = X_{18}X_{41}X_{56}X_{64}X_{85} =$ $= X_{18}X_{34}X_{41}X_{53}X_{85}$ |
| $M_4$ | $X_{64}X_{85}Y_{18}Y_{41}Y_{56} = X_{53}X_{85}Y_{18}Y_{34}Y_{41} = X_{47}X_{64}Y_{72}Y_{56}Y_{25} =$ $= X_{47}X_{53}Y_{72}Y_{34}Y_{25} = X_{72}X_{47}Y_{56}Y_{64}Y_{25} = X_{72}X_{47}Y_{34}Y_{53}Y_{25} =$ $= X_{18}X_{85}Y_{41}Y_{56}Y_{64} = X_{18}X_{85}Y_{34}Y_{41}Y_{53}$ |
| $M_5$ | $X_{56}X_{64}X_{85}Y_{18}Y_{41} = X_{47}X_{64}X_{25}Y_{72}Y_{56} = X_{47}X_{56}X_{64}Y_{72}Y_{25} =$ $= X_{47}X_{53}X_{25}Y_{72}Y_{34} = X_{41}X_{64}X_{85}Y_{18}Y_{56} = X_{41}X_{53}X_{85}Y_{18}Y_{34} =$ $= X_{34}X_{53}X_{85}Y_{18}Y_{41} = X_{34}X_{47}X_{53}Y_{72}Y_{25} = X_{72}X_{47}X_{25}Y_{56}Y_{64} =$ $= X_{72}X_{47}X_{25}Y_{34}Y_{53} = X_{72}X_{47}X_{56}Y_{64}Y_{25} = X_{72}X_{34}X_{47}Y_{53}Y_{25} =$ $= X_{18}X_{56}X_{85}Y_{41}Y_{64} = X_{18}X_{41}X_{85}Y_{56}Y_{64} = X_{18}X_{41}X_{85}Y_{34}Y_{53} =$ $= X_{18}X_{34}X_{85}Y_{41}Y_{53}$ |
| $M_6$ | $X_{47}X_{56}X_{64}X_{25}Y_{72} = X_{41}X_{56}X_{64}X_{85}Y_{18} = X_{34}X_{47}X_{53}X_{25}Y_{72} =$ $= X_{34}X_{41}X_{53}X_{85}Y_{18} = X_{72}X_{47}X_{56}X_{25}Y_{64} = X_{72}X_{34}X_{47}X_{25}Y_{53} =$ $= X_{18}X_{41}X_{56}X_{85}Y_{64} = X_{18}X_{34}X_{41}X_{85}Y_{53}$ |
| $M_2$ | $X_{72}X_{64}Y_{47}Y_{56}Y_{25} = X_{72}X_{53}Y_{34}Y_{47}Y_{25} = X_{18}X_{64}Y_{41}Y_{56}Y_{85} =$ $= X_{18}X_{53}Y_{34}Y_{41}Y_{85} = W_{56}X_{72}X_{47}X_{64}Y_{25} = W_{56}X_{18}X_{64}X_{85}Y_{41} =$ $= W_{34}X_{72}X_{47}X_{53}Y_{25} = W_{34}X_{18}X_{53}X_{85}Y_{41}$ |
| $M_8$ | $X_{72}X_{64}X_{25}Y_{47}Y_{56} = X_{72}X_{56}X_{64}Y_{47}Y_{25} = X_{72}X_{53}X_{25}Y_{34}Y_{47} =$ $= X_{72}X_{47}X_{64}Y_{25}Z_{56} = X_{72}X_{47}X_{53}Y_{25}Z_{34} = X_{72}X_{34}X_{53}Y_{47}Y_{25} =$ $= X_{18}X_{64}X_{85}Y_{41}Z_{56} = X_{18}X_{56}X_{64}Y_{41}Y_{85} = X_{18}X_{53}X_{85}Y_{41}Z_{34} =$ $= X_{18}X_{41}X_{64}Y_{56}Y_{85} = X_{18}X_{41}X_{53}Y_{34}Y_{85} = X_{18}X_{34}X_{53}Y_{41}Y_{85} =$ $= W_{56}X_{72}X_{47}X_{64}X_{25} = W_{56}X_{18}X_{41}X_{64}X_{85} = W_{34}X_{72}X_{47}X_{53}X_{25} =$ $= W_{34}X_{18}X_{41}X_{53}X_{85}$ |
| $M_9$ | $X_{72}X_{56}X_{64}X_{25}Y_{47} = X_{72}X_{47}X_{64}X_{25}Z_{56} = X_{72}X_{47}X_{53}X_{25}Z_{34} =$ $= X_{72}X_{34}X_{53}X_{25}Y_{47} = X_{18}X_{41}X_{64}X_{85}Z_{56} = X_{18}X_{41}X_{56}X_{64}Y_{85} =$ $= X_{18}X_{41}X_{53}X_{85}Z_{34} = X_{18}X_{34}X_{41}X_{53}Y_{85}$ |
| $M_{10}$ | $X_{64}Y_{72}Y_{47}Y_{56}Y_{25} = X_{64}Y_{18}Y_{41}Y_{56}Y_{85} = X_{53}Y_{72}Y_{34}Y_{47}Y_{25} =$ $= X_{53}Y_{18}Y_{34}Y_{41}Y_{85} = X_{72}Y_{47}Y_{56}Y_{64}Y_{25} = X_{72}Y_{34}Y_{47}Y_{53}Y_{25} =$ $= X_{18}Y_{41}Y_{56}Y_{64}Y_{85} = X_{18}Y_{34}Y_{41}Y_{53}Y_{85} = W_{56}X_{64}X_{85}Y_{18}Y_{41} =$ |

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| Field | Chiral superfields |
|---|---|
| | $= W_{56}X_{47}X_{64}Y_{72}Y_{25} = W_{56}X_{72}X_{47}Y_{64}Y_{25} = W_{56}X_{18}X_{85}Y_{41}Y_{64} =$ $= W_{34}X_{53}X_{85}Y_{18}Y_{41} = W_{34}X_{47}X_{53}Y_{72}Y_{25} = W_{34}X_{72}X_{47}Y_{53}Y_{25} =$ $= W_{34}X_{18}X_{85}Y_{41}Y_{53}$ |
| $M_{11}$ | $X_{64}X_{85}Y_{18}Y_{41}Z_{56} = X_{64}X_{25}Y_{72}Y_{47}Y_{56} = X_{56}X_{64}Y_{72}Y_{47}Y_{25} =$ $= X_{56}X_{64}Y_{18}Y_{41}Y_{85} = X_{53}X_{85}Y_{18}Y_{41}Z_{34} = X_{53}X_{25}Y_{72}Y_{34}Y_{47} =$ $= X_{47}X_{64}Y_{72}Y_{25}Z_{56} = X_{47}X_{53}Y_{72}Y_{25}Z_{34} = X_{41}X_{64}Y_{18}Y_{56}Y_{85} =$ $= X_{41}X_{53}Y_{18}Y_{34}Y_{85} = X_{34}X_{53}Y_{72}Y_{47}Y_{25} = X_{34}X_{53}Y_{18}Y_{41}Y_{85} =$ $= X_{72}X_{25}Y_{47}Y_{56}Y_{64} = X_{72}X_{25}Y_{34}Y_{47}Y_{53} = X_{72}X_{56}Y_{47}Y_{64}Y_{25} =$ $= X_{72}X_{47}Y_{64}Y_{25}Z_{56} = X_{72}X_{47}Y_{53}Y_{25}Z_{34} = X_{72}X_{34}Y_{47}Y_{53}Y_{25} =$ $= X_{18}X_{85}Y_{41}Y_{64}Z_{56} = X_{18}X_{85}Y_{41}Y_{53}Z_{34} = X_{18}X_{56}Y_{41}Y_{64}Y_{85} =$ $= X_{18}X_{41}Y_{56}Y_{64}Y_{85} = X_{18}X_{41}Y_{34}Y_{53}Y_{85} = X_{18}X_{34}Y_{41}Y_{53}Y_{85} =$ $= W_{56}X_{47}X_{64}X_{25}Y_{72} = W_{56}X_{41}X_{64}X_{85}Y_{18} = W_{56}X_{72}X_{47}X_{25}Y_{64} =$ $= W_{56}X_{18}X_{41}X_{85}Y_{64} = W_{34}X_{47}X_{53}X_{25}Y_{72} = W_{34}X_{41}X_{53}X_{85}Y_{18} =$ $= W_{34}X_{72}X_{47}X_{25}Y_{53} = W_{34}X_{18}X_{41}X_{85}Y_{53}$ |
| $M_{12}$ | $X_{56}X_{64}X_{25}Y_{72}Y_{47} = X_{47}X_{64}X_{25}Y_{72}Z_{56} = X_{47}X_{53}X_{25}Y_{72}Z_{34} =$ $= X_{41}X_{64}X_{85}Y_{18}Z_{56} = X_{41}X_{56}X_{64}Y_{18}Y_{85} = X_{41}X_{53}X_{85}Y_{18}Z_{34} =$ $= X_{34}X_{53}X_{25}Y_{72}Y_{47} = X_{34}X_{41}X_{53}Y_{18}Y_{85} = X_{72}X_{56}X_{25}Y_{47}Y_{64} =$ $= X_{72}X_{47}X_{25}Y_{64}Z_{56} = X_{72}X_{47}X_{25}Y_{53}Z_{34} = X_{72}X_{34}X_{25}Y_{47}Y_{53} =$ $= X_{18}X_{41}X_{85}Y_{64}Z_{56} = X_{18}X_{41}X_{85}Y_{53}Z_{34} = X_{18}X_{41}X_{56}Y_{64}Y_{85} =$ $= X_{18}X_{34}X_{41}Y_{53}Y_{85}$ |
| $M_{13}$ | $W_{56}X_{72}X_{64}Y_{47}Y_{25} = W_{56}X_{18}X_{64}Y_{41}Y_{85} = W_{34}X_{72}X_{53}Y_{47}Y_{25} =$ $= W_{34}X_{18}X_{53}Y_{41}Y_{85}$ |
| $M_{14}$ | $X_{72}X_{64}Y_{47}Y_{25}Z_{56} = X_{72}X_{53}Y_{47}Y_{25}Z_{34} = X_{18}X_{64}Y_{41}Y_{85}Z_{56} =$ $= X_{18}X_{53}Y_{41}Y_{85}Z_{34} = W_{56}X_{72}X_{64}X_{25}Y_{47} = W_{56}X_{18}X_{41}X_{64}Y_{85} =$ $= W_{34}X_{72}X_{53}X_{25}Y_{47} = W_{34}X_{18}X_{41}X_{53}Y_{85}$ |
| $M_{15}$ | $X_{72}X_{64}X_{25}Y_{47}Z_{56} = X_{72}X_{53}X_{25}Y_{47}Z_{34} = X_{18}X_{41}X_{64}Y_{85}Z_{56} =$ $= X_{18}X_{41}X_{53}Y_{85}Z_{34}$ |
| $M_{16}$ | $W_{56}X_{64}Y_{72}Y_{47}Y_{25} = W_{56}X_{64}Y_{18}Y_{41}Y_{85} = W_{56}X_{72}Y_{47}Y_{64}Y_{25} =$ $= W_{56}X_{18}Y_{41}Y_{64}Y_{85} = W_{34}X_{53}Y_{72}Y_{47}Y_{25} = W_{34}X_{53}Y_{18}Y_{41}Y_{85} =$ $= W_{34}X_{72}Y_{47}Y_{53}Y_{25} = W_{34}X_{18}Y_{41}Y_{53}Y_{85}$ |
| $M_{17}$ | $X_{64}Y_{72}Y_{47}Y_{25}Z_{56} = X_{64}Y_{18}Y_{41}Y_{85}Z_{56} = X_{53}Y_{72}Y_{47}Y_{25}Z_{34} =$ $= X_{53}Y_{18}Y_{41}Y_{85}Z_{34} = X_{72}Y_{47}Y_{64}Y_{25}Z_{56} = X_{72}Y_{47}Y_{53}Y_{25}Z_{34} =$ $= X_{18}Y_{41}Y_{64}Y_{85}Z_{56} = X_{18}Y_{41}Y_{53}Y_{85}Z_{34} = W_{56}X_{64}X_{25}Y_{72}Y_{47} =$ $= W_{56}X_{41}X_{64}Y_{18}Y_{85} = W_{56}X_{72}X_{25}Y_{47}Y_{64} = W_{56}X_{18}X_{41}Y_{64}Y_{85} =$ |

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| Field | Chiral superfields |
|---|---|
| | $= W_{34}X_{53}X_{25}Y_{72}Y_{47} = W_{34}X_{41}X_{53}Y_{18}Y_{85} = W_{34}X_{72}X_{25}Y_{47}Y_{53} =$ $= W_{34}X_{18}X_{41}Y_{53}Y_{85}$ |
| $M_{18}$ | $X_{64}X_{25}Y_{72}Y_{47}Z_{56} = X_{53}X_{25}Y_{72}Y_{47}Z_{34} = X_{41}X_{64}Y_{18}Y_{85}Z_{56} =$ $= X_{41}X_{53}Y_{18}Y_{85}Z_{34} = X_{72}X_{25}Y_{47}Y_{64}Z_{56} = X_{72}X_{25}Y_{47}Y_{53}Z_{34} =$ $= X_{18}X_{41}Y_{64}Y_{85}Z_{56} = X_{18}X_{41}Y_{53}Y_{85}Z_{34}$ |
| $M_{19}$ | $X_{85}Y_{18}Y_{41}Y_{56}Y_{64} = X_{85}Y_{18}Y_{34}Y_{41}Y_{53} = X_{47}Y_{72}Y_{56}Y_{64}Y_{25} =$ $= X_{47}Y_{72}Y_{34}Y_{53}Y_{25}$ |
| $M_{20}$ | $X_{56}X_{85}Y_{18}Y_{41}Y_{64} = X_{47}X_{25}Y_{72}Y_{56}Y_{64} = X_{47}X_{25}Y_{72}Y_{34}Y_{53} =$ $= X_{47}X_{56}Y_{72}Y_{64}Y_{25} = X_{41}X_{85}Y_{18}Y_{56}Y_{64} = X_{41}X_{85}Y_{18}Y_{34}Y_{53} =$ $= X_{34}X_{85}Y_{18}Y_{41}Y_{53} = X_{34}X_{47}Y_{72}Y_{53}Y_{25}$ |
| $M_{21}$ | $X_{47}X_{56}X_{25}Y_{72}Y_{64} = X_{41}X_{56}X_{85}Y_{18}Y_{64} = X_{34}X_{47}X_{25}Y_{72}Y_{53} =$ $= X_{34}X_{41}X_{85}Y_{18}Y_{53}$ |
| $M_{27}$ | $Y_{72}Y_{47}Y_{56}Y_{64}Y_{25} = Y_{72}Y_{34}Y_{47}Y_{53}Y_{25} = Y_{18}Y_{41}Y_{56}Y_{64}Y_{85} =$ $= Y_{18}Y_{34}Y_{41}Y_{53}Y_{85} = W_{56}X_{85}Y_{18}Y_{41}Y_{64} = W_{56}X_{47}Y_{72}Y_{64}Y_{25} =$ $= W_{34}X_{85}Y_{18}Y_{41}Y_{53} = W_{34}X_{47}Y_{72}Y_{53}Y_{25}$ |
| $M_{23}$ | $X_{85}Y_{18}Y_{41}Y_{64}Z_{56} = X_{85}Y_{18}Y_{41}Y_{53}Z_{34} = X_{25}Y_{72}Y_{47}Y_{56}Y_{64} =$ $= X_{25}Y_{72}Y_{34}Y_{47}Y_{53} = X_{56}Y_{72}Y_{47}Y_{64}Y_{25} = X_{56}Y_{18}Y_{41}Y_{64}Y_{85} =$ $= X_{47}Y_{72}Y_{64}Y_{25}Z_{56} = X_{47}Y_{72}Y_{53}Y_{25}Z_{34} = X_{41}Y_{18}Y_{56}Y_{64}Y_{85} =$ $= X_{41}Y_{18}Y_{34}Y_{53}Y_{85} = X_{34}Y_{72}Y_{47}Y_{53}Y_{25} = X_{34}Y_{18}Y_{41}Y_{53}Y_{85} =$ $= W_{56}X_{47}X_{25}Y_{72}Y_{64} = W_{56}X_{41}X_{85}Y_{18}Y_{64} = W_{34}X_{47}X_{25}Y_{72}Y_{53} =$ $= W_{34}X_{41}X_{85}Y_{18}Y_{53}$ |
| $M_{24}$ | $X_{56}X_{25}Y_{72}Y_{47}Y_{64} = X_{47}X_{25}Y_{72}Y_{64}Z_{56} = X_{47}X_{25}Y_{72}Y_{53}Z_{34} =$ $= X_{41}X_{85}Y_{18}Y_{64}Z_{56} = X_{41}X_{85}Y_{18}Y_{53}Z_{34} = X_{41}X_{56}Y_{18}Y_{64}Y_{85} =$ $= X_{34}X_{25}Y_{72}Y_{47}Y_{53} = X_{34}X_{41}Y_{18}Y_{53}Y_{85}$ |
| $M_{25}$ | $W_{56}Y_{72}Y_{47}Y_{64}Y_{25} = W_{56}Y_{18}Y_{41}Y_{64}Y_{85} = W_{34}Y_{72}Y_{47}Y_{53}Y_{25} =$ $= W_{34}Y_{18}Y_{41}Y_{53}Y_{85}$ |
| $M_{26}$ | $Y_{72}Y_{47}Y_{64}Y_{25}Z_{56} = Y_{72}Y_{47}Y_{53}Y_{25}Z_{34} = Y_{18}Y_{41}Y_{64}Y_{85}Z_{56} =$ $= Y_{18}Y_{41}Y_{53}Y_{85}Z_{34} = W_{56}X_{25}Y_{72}Y_{47}Y_{64} = W_{56}X_{41}Y_{18}Y_{64}Y_{85} =$ $= W_{34}X_{25}Y_{72}Y_{47}Y_{53} = W_{34}X_{41}Y_{18}Y_{53}Y_{85}$ |
| $M_{27}$ | $X_{25}Y_{72}Y_{47}Y_{64}Z_{56} = X_{25}Y_{72}Y_{47}Y_{53}Z_{34} = X_{41}Y_{18}Y_{64}Y_{85}Z_{56} =$ $= X_{41}Y_{18}Y_{53}Y_{85}Z_{34}$ |

**Table 7.** Generators of $Q^{1,1,1}/\mathbb{Z}_2$ in Phase D.



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| Field | Chiral superfields |
|---|---|
| $M_1$ | $W_{56}X_{35}X_{47}X_{64}Y_{73} = W_{56}X_{18}X_{64}X_{85}Y_{41} = W_{56}X_{13}X_{35}X_{64}Y_{41} = = W_{56}X_{72}X_{47}X_{64}Y_{25}$ |
| $M_2$ | $X_{35}X_{47}X_{64}Y_{73}Y_{56} = W_{56}X_{35}X_{64}Y_{73}Y_{47} = X_{18}X_{64}X_{85}Y_{41}Y_{56} = = X_{13}X_{35}X_{64}Y_{41}Y_{56} = W_{56}X_{18}X_{64}Y_{41}Y_{85} = X_{72}X_{47}X_{64}Y_{56}Y_{25} = = W_{56}X_{72}X_{64}Y_{47}Y_{25} = W_{56}X_{35}X_{64}Y_{13}Y_{41}$ |
| $M_3$ | $X_{35}X_{64}Y_{73}Y_{47}Y_{56} = X_{18}X_{64}Y_{41}Y_{56}Y_{85} = X_{72}X_{64}Y_{47}Y_{56}Y_{25} = = X_{35}X_{64}Y_{13}Y_{41}Y_{56}$ |
| $M_4$ | $W_{56}X_{47}X_{64}Y_{73}Y_{35} = W_{56}X_{35}X_{47}Y_{73}Y_{64} = W_{56}X_{13}X_{64}Y_{35}Y_{41} = = W_{56}X_{18}X_{85}Y_{41}Y_{64} = W_{56}X_{13}X_{35}Y_{41}Y_{64} = W_{56}X_{72}X_{47}Y_{64}Y_{25} = = W_{56}X_{47}X_{64}Y_{72}Y_{25} = W_{56}X_{64}X_{85}Y_{18}Y_{41}$ |
| $M_5$ | $X_{47}X_{64}Y_{73}Y_{35}Y_{56} = X_{35}X_{47}Y_{73}Y_{56}Y_{64} = W_{56}X_{64}Y_{73}Y_{35}Y_{47} = = W_{56}X_{35}Y_{73}Y_{47}Y_{64} = X_{13}X_{64}Y_{35}Y_{41}Y_{56} = X_{18}X_{85}Y_{41}Y_{56}Y_{64} = = X_{13}X_{35}Y_{41}Y_{56}Y_{64} = W_{56}X_{18}Y_{41}Y_{64}Y_{85} = X_{72}X_{47}Y_{56}Y_{64}Y_{25} = = X_{47}X_{64}Y_{72}Y_{56}Y_{25} = W_{56}X_{72}Y_{47}Y_{64}Y_{25} = W_{56}X_{64}Y_{72}Y_{47}Y_{25} = = W_{56}X_{64}Y_{13}Y_{35}Y_{41} = W_{56}X_{35}Y_{13}Y_{41}Y_{64} = X_{64}X_{85}Y_{18}Y_{41}Y_{56} = = W_{56}X_{64}Y_{18}Y_{41}Y_{85}$ |
| $M_6$ | $X_{64}Y_{73}Y_{35}Y_{47}Y_{56} = X_{35}Y_{73}Y_{47}Y_{56}Y_{64} = X_{18}Y_{41}Y_{56}Y_{64}Y_{85} = = X_{72}Y_{47}Y_{56}Y_{64}Y_{25} = X_{64}Y_{72}Y_{47}Y_{56}Y_{25} = X_{64}Y_{13}Y_{35}Y_{41}Y_{56} = = X_{35}Y_{13}Y_{41}Y_{56}Y_{64} = X_{64}Y_{18}Y_{41}Y_{56}Y_{85}$ |
| $M_2$ | $X_{35}X_{47}X_{56}X_{64}Y_{73} = W_{56}X_{18}X_{41}X_{64}X_{85} = W_{56}X_{13}X_{35}X_{41}X_{64} = = W_{56}X_{72}X_{47}X_{64}X_{25} = W_{56}X_{73}X_{35}X_{47}X_{64} = X_{18}X_{56}X_{64}X_{85}Y_{41} = = X_{13}X_{35}X_{56}X_{64}Y_{41} = X_{72}X_{47}X_{56}X_{64}Y_{25}$ |
| $M_8$ | $X_{18}X_{41}X_{64}X_{85}Y_{56} = X_{13}X_{35}X_{41}X_{64}Y_{56} = X_{72}X_{47}X_{64}X_{25}Y_{56} = = X_{73}X_{35}X_{47}X_{64}Y_{56} = X_{35}X_{47}X_{64}Y_{73}Z_{56} = X_{35}X_{56}X_{64}Y_{73}Y_{47} = = W_{56}X_{72}X_{64}X_{25}Y_{47} = W_{56}X_{73}X_{35}X_{64}Y_{47} = X_{18}X_{64}X_{85}Y_{41}Z_{56} = = X_{13}X_{35}X_{64}Y_{41}Z_{56} = W_{56}X_{18}X_{41}X_{64}Y_{85} = X_{18}X_{56}X_{64}Y_{41}Y_{85} = = X_{72}X_{47}X_{64}Y_{25}Z_{56} = X_{72}X_{56}X_{64}Y_{47}Y_{25} = W_{56}X_{35}X_{41}X_{64}Y_{13} = = X_{35}X_{56}X_{64}Y_{13}Y_{41}$ |
| $M_9$ | $X_{72}X_{64}X_{25}Y_{47}Y_{56} = X_{73}X_{35}X_{64}Y_{47}Y_{56} = X_{35}X_{64}Y_{73}Y_{47}Z_{56} = = X_{18}X_{41}X_{64}Y_{56}Y_{85} = X_{18}X_{64}Y_{41}Y_{85}Z_{56} = X_{72}X_{64}Y_{47}Y_{25}Z_{56} = = X_{35}X_{41}X_{64}Y_{13}Y_{56} = X_{35}X_{64}Y_{13}Y_{41}Z_{56}$ |
| $M_{10}$ | $X_{47}X_{56}X_{64}Y_{73}Y_{35} = X_{35}X_{47}X_{56}Y_{73}Y_{64} = W_{56}X_{13}X_{41}X_{64}Y_{35} = = W_{56}X_{73}X_{47}X_{64}Y_{35} = W_{56}X_{18}X_{41}X_{85}Y_{64} = W_{56}X_{13}X_{35}X_{41}Y_{64} = = W_{56}X_{72}X_{47}X_{25}Y_{64} = W_{56}X_{73}X_{35}X_{47}Y_{64} = W_{56}X_{47}X_{64}X_{25}Y_{72} =$ |

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| Field | Chiral superfields |
|---|---|
| | $= X_{13}X_{56}X_{64}Y_{35}Y_{41} = X_{18}X_{56}X_{85}Y_{41}Y_{64} = X_{13}X_{35}X_{56}Y_{41}Y_{64} =$ $= X_{72}X_{47}X_{56}Y_{64}Y_{25} = X_{47}X_{56}X_{64}Y_{72}Y_{25} = W_{56}X_{41}X_{64}X_{85}Y_{18} =$ $= X_{56}X_{64}X_{85}Y_{18}Y_{41}$ |
| $M_{11}$ | $X_{13}X_{41}X_{64}Y_{35}Y_{56} = X_{73}X_{47}X_{64}Y_{35}Y_{56} = X_{18}X_{41}X_{85}Y_{56}Y_{64} =$ $= X_{13}X_{35}X_{41}Y_{56}Y_{64} = X_{72}X_{47}X_{25}Y_{56}Y_{64} = X_{73}X_{35}X_{47}Y_{56}Y_{64} =$ $= X_{47}X_{64}X_{25}Y_{72}Y_{56} = X_{47}X_{64}Y_{73}Y_{35}Z_{56} = X_{35}X_{47}Y_{73}Y_{64}Z_{56} =$ $= X_{56}X_{64}Y_{73}Y_{35}Y_{47} = X_{35}X_{56}Y_{73}Y_{47}Y_{64} = W_{56}X_{73}X_{64}Y_{35}Y_{47} =$ $= W_{56}X_{72}X_{25}Y_{47}Y_{64} = W_{56}X_{73}X_{35}Y_{47}Y_{64} = W_{56}X_{64}X_{25}Y_{72}Y_{47} =$ $= X_{13}X_{64}Y_{35}Y_{41}Z_{56} = X_{18}X_{85}Y_{41}Y_{64}Z_{56} = X_{13}X_{35}Y_{41}Y_{64}Z_{56} =$ $= W_{56}X_{18}X_{41}Y_{64}Y_{85} = X_{18}X_{56}Y_{41}Y_{64}Y_{85} = X_{72}X_{47}Y_{64}Y_{25}Z_{56} =$ $= X_{47}X_{64}Y_{72}Y_{25}Z_{56} = X_{72}X_{56}Y_{47}Y_{64}Y_{25} = X_{56}X_{64}Y_{72}Y_{47}Y_{25} =$ $= W_{56}X_{41}X_{64}Y_{13}Y_{35} = W_{56}X_{35}X_{41}Y_{13}Y_{64} = X_{56}X_{64}Y_{13}Y_{35}Y_{41} =$ $= X_{35}X_{56}Y_{13}Y_{41}Y_{64} = X_{41}X_{64}X_{85}Y_{18}Y_{56} = X_{64}X_{85}Y_{18}Y_{41}Z_{56} =$ $= W_{56}X_{41}X_{64}Y_{18}Y_{85} = X_{56}X_{64}Y_{18}Y_{41}Y_{85}$ |
| $M_{12}$ | $X_{73}X_{64}Y_{35}Y_{47}Y_{56} = X_{72}X_{25}Y_{47}Y_{56}Y_{64} = X_{73}X_{35}Y_{47}Y_{56}Y_{64} =$ $= X_{64}X_{25}Y_{72}Y_{47}Y_{56} = X_{64}Y_{73}Y_{35}Y_{47}Z_{56} = X_{35}Y_{73}Y_{47}Y_{64}Z_{56} =$ $= X_{18}X_{41}Y_{56}Y_{64}Y_{85} = X_{18}Y_{41}Y_{64}Y_{85}Z_{56} = X_{72}Y_{47}Y_{64}Y_{25}Z_{56} =$ $= X_{64}Y_{72}Y_{47}Y_{25}Z_{56} = X_{41}X_{64}Y_{13}Y_{35}Y_{56} = X_{35}X_{41}Y_{13}Y_{56}Y_{64} =$ $= X_{64}Y_{13}Y_{35}Y_{41}Z_{56} = X_{35}Y_{13}Y_{41}Y_{64}Z_{56} = X_{41}X_{64}Y_{18}Y_{56}Y_{85} =$ $= X_{64}Y_{18}Y_{41}Y_{85}Z_{56}$ |
| $M_{13}$ | $X_{18}X_{41}X_{56}X_{64}X_{85} = X_{13}X_{35}X_{41}X_{56}X_{64} = X_{72}X_{47}X_{56}X_{64}X_{25} =$ $= X_{73}X_{35}X_{47}X_{56}X_{64}$ |
| $M_{14}$ | $X_{18}X_{41}X_{64}X_{85}Z_{56} = X_{13}X_{35}X_{41}X_{64}Z_{56} = X_{72}X_{47}X_{64}X_{25}Z_{56} =$ $= X_{73}X_{35}X_{47}X_{64}Z_{56} = X_{72}X_{56}X_{64}X_{25}Y_{47} = X_{73}X_{35}X_{56}X_{64}Y_{47} =$ $= X_{18}X_{41}X_{56}X_{64}Y_{85} = X_{35}X_{41}X_{56}X_{64}Y_{13}$ |
| $M_{15}$ | $X_{72}X_{64}X_{25}Y_{47}Z_{56} = X_{73}X_{35}X_{64}Y_{47}Z_{56} = X_{18}X_{41}X_{64}Y_{85}Z_{56} =$ $= X_{35}X_{41}X_{64}Y_{13}Z_{56}$ |
| $M_{16}$ | $X_{13}X_{41}X_{56}X_{64}Y_{35} = X_{73}X_{47}X_{56}X_{64}Y_{35} = X_{18}X_{41}X_{56}X_{85}Y_{64} =$ $= X_{13}X_{35}X_{41}X_{56}Y_{64} = X_{72}X_{47}X_{56}X_{25}Y_{64} = X_{73}X_{35}X_{47}X_{56}Y_{64} =$ $= X_{47}X_{56}X_{64}X_{25}Y_{72} = X_{41}X_{56}X_{64}X_{85}Y_{18}$ |
| $M_{17}$ | $X_{13}X_{41}X_{64}Y_{35}Z_{56} = X_{73}X_{47}X_{64}Y_{35}Z_{56} = X_{18}X_{41}X_{85}Y_{64}Z_{56} =$ $= X_{13}X_{35}X_{41}Y_{64}Z_{56} = X_{72}X_{47}X_{25}Y_{64}Z_{56} = X_{73}X_{35}X_{47}Y_{64}Z_{56} =$ $= X_{47}X_{64}X_{25}Y_{72}Z_{56} = X_{73}X_{56}X_{64}Y_{35}Y_{47} = X_{72}X_{56}X_{25}Y_{47}Y_{64} =$ $= X_{73}X_{35}X_{56}Y_{47}Y_{64} = X_{56}X_{64}X_{25}Y_{72}Y_{47} = X_{18}X_{41}X_{56}Y_{64}Y_{85} =$ |

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| Field | Chiral superfields |
|---|---|
| | $= X_{41}X_{56}X_{64}Y_{13}Y_{35} = X_{35}X_{41}X_{56}Y_{13}Y_{64} = X_{41}X_{64}X_{85}Y_{18}Z_{56} =$ $= X_{41}X_{56}X_{64}Y_{18}Y_{85}$ |
| $M_{18}$ | $X_{73}X_{64}Y_{35}Y_{47}Z_{56} = X_{72}X_{25}Y_{47}Y_{64}Z_{56} = X_{73}X_{35}Y_{47}Y_{64}Z_{56} =$ $= X_{64}X_{25}Y_{72}Y_{47}Z_{56} = X_{18}X_{41}Y_{64}Y_{85}Z_{56} = X_{41}X_{64}Y_{13}Y_{35}Z_{56} =$ $= X_{35}X_{41}Y_{13}Y_{64}Z_{56} = X_{41}X_{64}Y_{18}Y_{85}Z_{56}$ |
| $M_{19}$ | $W_{56}X_{47}Y_{73}Y_{35}Y_{64} = W_{56}X_{13}Y_{35}Y_{41}Y_{64} = W_{56}X_{47}Y_{72}Y_{64}Y_{25} =$ $= W_{56}X_{85}Y_{18}Y_{41}Y_{64}$ |
| $M_{20}$ | $X_{47}Y_{73}Y_{35}Y_{56}Y_{64} = W_{56}Y_{73}Y_{35}Y_{47}Y_{64} = X_{13}Y_{35}Y_{41}Y_{56}Y_{64} =$ $= X_{47}Y_{72}Y_{56}Y_{64}Y_{25} = W_{56}Y_{72}Y_{47}Y_{64}Y_{25} = W_{56}Y_{13}Y_{35}Y_{41}Y_{64} =$ $= X_{85}Y_{18}Y_{41}Y_{56}Y_{64} = W_{56}Y_{18}Y_{41}Y_{64}Y_{85}$ |
| $M_{21}$ | $Y_{73}Y_{35}Y_{47}Y_{56}Y_{64} = Y_{72}Y_{47}Y_{56}Y_{64}Y_{25} = Y_{13}Y_{35}Y_{41}Y_{56}Y_{64} =$ $= Y_{18}Y_{41}Y_{56}Y_{64}Y_{85}$ |
| $M_{27}$ | $X_{47}X_{56}Y_{73}Y_{35}Y_{64} = W_{56}X_{13}X_{41}Y_{35}Y_{64} = W_{56}X_{73}X_{47}Y_{35}Y_{64} =$ $= W_{56}X_{47}X_{25}Y_{72}Y_{64} = X_{13}X_{56}Y_{35}Y_{41}Y_{64} = X_{47}X_{56}Y_{72}Y_{64}Y_{25} =$ $= W_{56}X_{41}X_{85}Y_{18}Y_{64} = X_{56}X_{85}Y_{18}Y_{41}Y_{64}$ |
| $M_{23}$ | $X_{13}X_{41}Y_{35}Y_{56}Y_{64} = X_{73}X_{47}Y_{35}Y_{56}Y_{64} = X_{47}X_{25}Y_{72}Y_{56}Y_{64} =$ $= X_{47}Y_{73}Y_{35}Y_{64}Z_{56} = X_{56}Y_{73}Y_{35}Y_{47}Y_{64} = W_{56}X_{73}Y_{35}Y_{47}Y_{64} =$ $= W_{56}X_{25}Y_{72}Y_{47}Y_{64} = X_{13}Y_{35}Y_{41}Y_{64}Z_{56} = X_{47}Y_{72}Y_{64}Y_{25}Z_{56} =$ $= X_{56}Y_{72}Y_{47}Y_{64}Y_{25} = W_{56}X_{41}Y_{13}Y_{35}Y_{64} = X_{56}Y_{13}Y_{35}Y_{41}Y_{64} =$ $= X_{41}X_{85}Y_{18}Y_{56}Y_{64} = X_{85}Y_{18}Y_{41}Y_{64}Z_{56} = W_{56}X_{41}Y_{18}Y_{64}Y_{85} =$ $= X_{56}Y_{18}Y_{41}Y_{64}Y_{85}$ |
| $M_{24}$ | $X_{73}Y_{35}Y_{47}Y_{56}Y_{64} = X_{25}Y_{72}Y_{47}Y_{56}Y_{64} = Y_{73}Y_{35}Y_{47}Y_{64}Z_{56} =$ $= Y_{72}Y_{47}Y_{64}Y_{25}Z_{56} = X_{41}Y_{13}Y_{35}Y_{56}Y_{64} = Y_{13}Y_{35}Y_{41}Y_{64}Z_{56} =$ $= X_{41}Y_{18}Y_{56}Y_{64}Y_{85} = Y_{18}Y_{41}Y_{64}Y_{85}Z_{56}$ |
| $M_{25}$ | $X_{13}X_{41}X_{56}Y_{35}Y_{64} = X_{73}X_{47}X_{56}Y_{35}Y_{64} = X_{47}X_{56}X_{25}Y_{72}Y_{64} =$ $= X_{41}X_{56}X_{85}Y_{18}Y_{64}$ |
| $M_{26}$ | $X_{13}X_{41}Y_{35}Y_{64}Z_{56} = X_{73}X_{47}Y_{35}Y_{64}Z_{56} = X_{47}X_{25}Y_{72}Y_{64}Z_{56} =$ $= X_{73}X_{56}Y_{35}Y_{47}Y_{64} = X_{56}X_{25}Y_{72}Y_{47}Y_{64} = X_{41}X_{56}Y_{13}Y_{35}Y_{64} =$ $= X_{41}X_{85}Y_{18}Y_{64}Z_{56} = X_{41}X_{56}Y_{18}Y_{64}Y_{85}$ |
| $M_{27}$ | $X_{73}Y_{35}Y_{47}Y_{64}Z_{56} = X_{25}Y_{72}Y_{47}Y_{64}Z_{56} = X_{41}Y_{13}Y_{35}Y_{64}Z_{56} =$ $= X_{41}Y_{18}Y_{64}Y_{85}Z_{56}$ |

**Table 8.** Generators of $Q^{1,1,1}/\mathbb{Z}_2$ in Phase E.



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| Field | Chiral superfields |
|---|---|
| $M_1$ | $X_{42}X_{56}X_{64}X_{25} = X_{48}X_{56}X_{64}X_{85} = X_{14}X_{31}X_{42}X_{53}X_{25} =$ $= X_{74}X_{37}X_{42}X_{53}X_{25} = X_{14}X_{31}X_{48}X_{53}X_{85} = X_{74}X_{37}X_{48}X_{53}X_{85}$ |
| $M_2$ | $X_{37}X_{42}X_{53}X_{25}Y_{74} = X_{37}X_{48}X_{53}X_{85}Y_{74} = X_{56}X_{64}X_{25}Z_{42} =$ $= X_{14}X_{31}X_{53}X_{25}Z_{42} = X_{74}X_{37}X_{53}X_{25}Z_{42} = X_{14}X_{42}X_{53}X_{25}Y_{31} =$ $= X_{14}X_{48}X_{53}X_{85}Y_{31} = X_{42}X_{64}X_{25}Y_{56} = X_{48}X_{64}X_{85}Y_{56} =$ $= X_{48}X_{56}X_{64}Y_{85} = X_{14}X_{31}X_{48}X_{53}Y_{85} = X_{74}X_{37}X_{48}X_{53}Y_{85}$ |
| $M_3$ | $X_{37}X_{53}X_{25}Y_{74}Z_{42} = X_{14}X_{53}X_{25}Y_{31}Z_{42} = X_{64}X_{25}Y_{56}Z_{42} =$ $= X_{37}X_{48}X_{53}Y_{74}Y_{85} = X_{14}X_{48}X_{53}Y_{31}Y_{85} = X_{48}X_{64}Y_{56}Y_{85}$ |
| $M_4$ | $X_{56}X_{64}X_{85}Y_{48} = X_{14}X_{31}X_{53}X_{85}Y_{48} = X_{74}X_{37}X_{53}X_{85}Y_{48} =$ $= X_{56}X_{64}X_{25}Y_{42} = X_{14}X_{31}X_{53}X_{25}Y_{42} = X_{74}X_{37}X_{53}X_{25}Y_{42} =$ $= X_{42}X_{56}X_{25}Y_{64} = X_{48}X_{56}X_{85}Y_{64} = X_{14}X_{31}X_{42}X_{25}Y_{53} =$ $= X_{74}X_{37}X_{42}X_{25}Y_{53} = X_{14}X_{31}X_{48}X_{85}Y_{53} = X_{74}X_{37}X_{48}X_{85}Y_{53}$ |
| $M_5$ | $X_{37}X_{53}X_{85}Y_{74}Y_{48} = X_{37}X_{53}X_{25}Y_{74}Y_{42} = W_{42}X_{56}X_{64}X_{25} =$ $= W_{42}X_{14}X_{31}X_{53}X_{25} = W_{42}X_{74}X_{37}X_{53}X_{25} = X_{14}X_{53}X_{85}Y_{31}Y_{48} =$ $= X_{14}X_{53}X_{25}Y_{31}Y_{42} = X_{56}X_{25}Y_{64}Z_{42} = X_{37}X_{42}X_{25}Y_{74}Y_{53} =$ $= X_{37}X_{48}X_{85}Y_{74}Y_{53} = X_{14}X_{31}X_{25}Y_{53}Z_{42} = X_{74}X_{37}X_{25}Y_{53}Z_{42} =$ $= X_{14}X_{42}X_{25}Y_{31}Y_{53} = X_{14}X_{48}X_{85}Y_{31}Y_{53} = X_{64}X_{85}Y_{48}Y_{56} =$ $= X_{64}X_{25}Y_{42}Y_{56} = X_{42}X_{25}Y_{56}Y_{64} = X_{48}X_{85}Y_{56}Y_{64} =$ $= X_{56}X_{64}Y_{48}Y_{85} = X_{14}X_{31}X_{53}Y_{48}Y_{85} = X_{74}X_{37}X_{53}Y_{48}Y_{85} =$ $= X_{48}X_{56}Y_{64}Y_{85} = X_{14}X_{31}X_{48}Y_{53}Y_{85} = X_{74}X_{37}X_{48}Y_{53}Y_{85}$ |
| $M_6$ | $W_{42}X_{37}X_{53}X_{25}Y_{74} = W_{42}X_{14}X_{53}X_{25}Y_{31} = X_{37}X_{25}Y_{74}Y_{53}Z_{42} =$ $= X_{14}X_{25}Y_{31}Y_{53}Z_{42} = W_{42}X_{64}X_{25}Y_{56} = X_{25}Y_{56}Y_{64}Z_{42} =$ $= X_{37}X_{53}Y_{74}Y_{48}Y_{85} = X_{14}X_{53}Y_{31}Y_{48}Y_{85} = X_{37}X_{48}Y_{74}Y_{53}Y_{85} =$ $= X_{14}X_{48}Y_{31}Y_{53}Y_{85} = X_{64}Y_{48}Y_{56}Y_{85} = X_{48}Y_{56}Y_{64}Y_{85}$ |
| $M_2$ | $X_{31}X_{42}X_{53}X_{25}Y_{14} = X_{31}X_{48}X_{53}X_{85}Y_{14} = X_{56}X_{64}X_{85}Z_{48} =$ $= X_{14}X_{31}X_{53}X_{85}Z_{48} = X_{74}X_{37}X_{53}X_{85}Z_{48} = X_{74}X_{42}X_{53}X_{25}Y_{37} =$ $= X_{74}X_{48}X_{53}X_{85}Y_{37} = X_{42}X_{64}X_{25}Z_{56} = X_{48}X_{64}X_{85}Z_{56} =$ $= X_{42}X_{56}X_{64}Y_{25} = X_{14}X_{31}X_{42}X_{53}Y_{25} = X_{74}X_{37}X_{42}X_{53}Y_{25}$ |
| $M_8$ | $X_{37}X_{53}X_{85}Y_{74}Z_{48} = X_{31}X_{53}X_{25}Y_{14}Z_{42} = X_{42}X_{53}X_{25}Y_{74}Y_{37} =$ $= X_{48}X_{53}X_{85}Y_{74}Y_{37} = X_{74}X_{53}X_{25}Y_{37}Z_{42} = X_{42}X_{53}X_{25}Y_{14}Y_{31} =$ $= X_{48}X_{53}X_{85}Y_{14}Y_{31} = X_{14}X_{53}X_{85}Y_{31}Z_{48} = X_{64}X_{85}Y_{56}Z_{48} =$ $= X_{64}X_{25}Z_{42}Z_{56} = W_{56}X_{42}X_{64}X_{25} = W_{56}X_{48}X_{64}X_{85} =$ $= X_{31}X_{48}X_{53}Y_{14}Y_{85} = X_{56}X_{64}Y_{85}Z_{48} = X_{14}X_{31}X_{53}Y_{85}Z_{48} =$ $= X_{74}X_{37}X_{53}Y_{85}Z_{48} = X_{74}X_{48}X_{53}Y_{37}Y_{85} = X_{48}X_{64}Y_{85}Z_{56} =$ |

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| Field | Chiral superfields |
|---|---|
| | $= X_{37}X_{42}X_{53}Y_{74}Y_{25} = X_{56}X_{64}Y_{25}Z_{42} = X_{14}X_{31}X_{53}Y_{25}Z_{42} =$ $= X_{74}X_{37}X_{53}Y_{25}Z_{42} = X_{14}X_{42}X_{53}Y_{31}Y_{25} = X_{42}X_{64}Y_{56}Y_{25}$ |
| $M_9$ | $X_{53}X_{25}Y_{74}Y_{37}Z_{42} = X_{53}X_{25}Y_{14}Y_{31}Z_{42} = W_{56}X_{64}X_{25}Z_{42} =$ $= X_{37}X_{53}Y_{74}Y_{85}Z_{48} = X_{48}X_{53}Y_{74}Y_{37}Y_{85} = X_{48}X_{53}Y_{14}Y_{31}Y_{85} =$ $= X_{14}X_{53}Y_{31}Y_{85}Z_{48} = X_{64}Y_{56}Y_{85}Z_{48} = W_{56}X_{48}X_{64}Y_{85} =$ $= X_{37}X_{53}Y_{74}Y_{25}Z_{42} = X_{14}X_{53}Y_{31}Y_{25}Z_{42} = X_{64}Y_{56}Y_{25}Z_{42}$ |
| $M_{10}$ | $X_{31}X_{53}X_{85}Y_{14}Y_{48} = W_{48}X_{56}X_{64}X_{85} = W_{48}X_{14}X_{31}X_{53}X_{85} =$ $= W_{48}X_{74}X_{37}X_{53}X_{85} = X_{31}X_{53}X_{25}Y_{14}Y_{42} = X_{74}X_{53}X_{85}Y_{37}Y_{48} =$ $= X_{74}X_{53}X_{25}Y_{37}Y_{42} = X_{56}X_{85}Y_{64}Z_{48} = X_{31}X_{42}X_{25}Y_{14}Y_{53} =$ $= X_{31}X_{48}X_{85}Y_{14}Y_{53} = X_{14}X_{31}X_{85}Y_{53}Z_{48} = X_{74}X_{37}X_{85}Y_{53}Z_{48} =$ $= X_{74}X_{42}X_{25}Y_{37}Y_{53} = X_{74}X_{48}X_{85}Y_{37}Y_{53} = X_{64}X_{85}Y_{48}Z_{56} =$ $= X_{64}X_{25}Y_{42}Z_{56} = X_{42}X_{25}Y_{64}Z_{56} = X_{48}X_{85}Y_{64}Z_{56} =$ $= X_{56}X_{64}Y_{42}Y_{25} = X_{14}X_{31}X_{53}Y_{42}Y_{25} = X_{74}X_{37}X_{53}Y_{42}Y_{25} =$ $= X_{42}X_{56}Y_{64}Y_{25} = X_{14}X_{31}X_{42}Y_{53}Y_{25} = X_{74}X_{37}X_{42}Y_{53}Y_{25}$ |
| $M_{11}$ | $W_{48}X_{37}X_{53}X_{85}Y_{74} = W_{42}X_{31}X_{53}X_{25}Y_{14} = X_{53}X_{85}Y_{74}Y_{37}Y_{48} =$ $= X_{53}X_{25}Y_{74}Y_{37}Y_{42} = W_{42}X_{74}X_{53}X_{25}Y_{37} = X_{53}X_{85}Y_{14}Y_{31}Y_{48} =$ $= W_{48}X_{14}X_{53}X_{85}Y_{31} = X_{53}X_{25}Y_{14}Y_{31}Y_{42} = X_{37}X_{85}Y_{74}Y_{53}Z_{48} =$ $= X_{31}X_{25}Y_{14}Y_{53}Z_{42} = X_{42}X_{25}Y_{74}Y_{37}Y_{53} = X_{48}X_{85}Y_{74}Y_{37}Y_{53} =$ $= X_{74}X_{25}Y_{37}Y_{53}Z_{42} = X_{42}X_{25}Y_{14}Y_{31}Y_{53} = X_{48}X_{85}Y_{14}Y_{31}Y_{53} =$ $= X_{14}X_{85}Y_{31}Y_{53}Z_{48} = W_{48}X_{64}X_{85}Y_{56} = X_{85}Y_{56}Y_{64}Z_{48} =$ $= W_{42}X_{64}X_{25}Z_{56} = X_{25}Y_{64}Z_{42}Z_{56} = W_{56}X_{64}X_{85}Y_{48} =$ $= W_{56}X_{64}X_{25}Y_{42} = W_{56}X_{42}X_{25}Y_{64} = W_{56}X_{48}X_{85}Y_{64} =$ $= X_{31}X_{53}Y_{14}Y_{48}Y_{85} = W_{48}X_{56}X_{64}Y_{85} = W_{48}X_{14}X_{31}X_{53}Y_{85} =$ $= W_{48}X_{74}X_{37}X_{53}Y_{85} = X_{74}X_{53}Y_{37}Y_{48}Y_{85} = X_{56}Y_{64}Y_{85}Z_{48} =$ $= X_{31}X_{48}Y_{14}Y_{53}Y_{85} = X_{14}X_{31}Y_{53}Y_{85}Z_{48} = X_{74}X_{37}Y_{53}Y_{85}Z_{48} =$ $= X_{74}X_{48}Y_{37}Y_{53}Y_{85} = X_{64}Y_{48}Y_{85}Z_{56} = X_{48}Y_{64}Y_{85}Z_{56} =$ $= X_{37}X_{53}Y_{74}Y_{42}Y_{25} = W_{42}X_{56}X_{64}Y_{25} = W_{42}X_{14}X_{31}X_{53}Y_{25} =$ $= W_{42}X_{74}X_{37}X_{53}Y_{25} = X_{14}X_{53}Y_{31}Y_{42}Y_{25} = X_{56}Y_{64}Y_{25}Z_{42} =$ $= X_{37}X_{42}Y_{74}Y_{53}Y_{25} = X_{14}X_{31}Y_{53}Y_{25}Z_{42} = X_{74}X_{37}Y_{53}Y_{25}Z_{42} =$ $= X_{14}X_{42}Y_{31}Y_{53}Y_{25} = X_{64}Y_{42}Y_{56}Y_{25} = X_{42}Y_{56}Y_{64}Y_{25}$ |
| $M_{12}$ | $W_{42}X_{53}X_{25}Y_{74}Y_{37} = W_{42}X_{53}X_{25}Y_{14}Y_{31} = X_{25}Y_{74}Y_{37}Y_{53}Z_{42} =$ $= X_{25}Y_{14}Y_{31}Y_{53}Z_{42} = W_{42}W_{56}X_{64}X_{25} = W_{56}X_{25}Y_{64}Z_{42} =$ $= W_{48}X_{37}X_{53}Y_{74}Y_{85} = X_{53}Y_{74}Y_{37}Y_{48}Y_{85} = X_{53}Y_{14}Y_{31}Y_{48}Y_{85} =$ $= W_{48}X_{14}X_{53}Y_{31}Y_{85} = X_{37}Y_{74}Y_{53}Y_{85}Z_{48} = X_{48}Y_{74}Y_{37}Y_{53}Y_{85} =$ |



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| Field | Chiral superfields |
|---|---|
| | $= X_{48}Y_{14}Y_{31}Y_{53}Y_{85} = X_{14}Y_{31}Y_{53}Y_{85}Z_{48} = W_{48}X_{64}Y_{56}Y_{85} =$ $= Y_{56}Y_{64}Y_{85}Z_{48} = W_{56}X_{64}Y_{48}Y_{85} = W_{56}X_{48}Y_{64}Y_{85} =$ $= W_{42}X_{37}X_{53}Y_{74}Y_{25} = W_{42}X_{14}X_{53}Y_{31}Y_{25} = X_{37}Y_{74}Y_{53}Y_{25}Z_{42} =$ $= X_{14}Y_{31}Y_{53}Y_{25}Z_{42} = W_{42}X_{64}Y_{56}Y_{25} = Y_{56}Y_{64}Y_{25}Z_{42}$ |
| $M_{13}$ | $X_{31}X_{53}X_{85}Y_{14}Z_{48} = X_{74}X_{53}X_{85}Y_{37}Z_{48} = X_{64}X_{85}Z_{48}Z_{56} =$ $= X_{31}X_{42}X_{53}Y_{14}Y_{25} = X_{74}X_{42}X_{53}Y_{37}Y_{25} = X_{42}X_{64}Y_{25}Z_{56}$ |
| $M_{14}$ | $X_{53}X_{85}Y_{74}Y_{37}Z_{48} = X_{53}X_{85}Y_{14}Y_{31}Z_{48} = W_{56}X_{64}X_{85}Z_{48} =$ $= X_{31}X_{53}Y_{14}Y_{85}Z_{48} = X_{74}X_{53}Y_{37}Y_{85}Z_{48} = X_{64}Y_{85}Z_{48}Z_{56} =$ $= X_{31}X_{53}Y_{14}Y_{25}Z_{42} = X_{42}X_{53}Y_{74}Y_{37}Y_{25} = X_{74}X_{53}Y_{37}Y_{25}Z_{42} =$ $= X_{42}X_{53}Y_{14}Y_{31}Y_{25} = X_{64}Y_{25}Z_{42}Z_{56} = W_{56}X_{42}X_{64}Y_{25}$ |
| $M_{15}$ | $X_{53}Y_{74}Y_{37}Y_{85}Z_{48} = X_{53}Y_{14}Y_{31}Y_{85}Z_{48} = W_{56}X_{64}Y_{85}Z_{48} =$ $= X_{53}Y_{74}Y_{37}Y_{25}Z_{42} = X_{53}Y_{14}Y_{31}Y_{25}Z_{42} = W_{56}X_{64}Y_{25}Z_{42}$ |
| $M_{16}$ | $W_{48}X_{31}X_{53}X_{85}Y_{14} = W_{48}X_{74}X_{53}X_{85}Y_{37} = X_{31}X_{85}Y_{14}Y_{53}Z_{48} =$ $= X_{74}X_{85}Y_{37}Y_{53}Z_{48} = W_{48}X_{64}X_{85}Z_{56} = X_{85}Y_{64}Z_{48}Z_{56} =$ $= X_{31}X_{53}Y_{14}Y_{42}Y_{25} = X_{74}X_{53}Y_{37}Y_{42}Y_{25} = X_{31}X_{42}Y_{14}Y_{53}Y_{25} =$ $= X_{74}X_{42}Y_{37}Y_{53}Y_{25} = X_{64}Y_{42}Y_{25}Z_{56} = X_{42}Y_{64}Y_{25}Z_{56}$ |
| $M_{17}$ | $W_{48}X_{53}X_{85}Y_{74}Y_{37} = W_{48}X_{53}X_{85}Y_{14}Y_{31} = X_{85}Y_{74}Y_{37}Y_{53}Z_{48} =$ $= X_{85}Y_{14}Y_{31}Y_{53}Z_{48} = W_{48}W_{56}X_{64}X_{85} = W_{56}X_{85}Y_{64}Z_{48} =$ $= W_{48}X_{31}X_{53}Y_{14}Y_{85} = W_{48}X_{74}X_{53}Y_{37}Y_{85} = X_{31}Y_{14}Y_{53}Y_{85}Z_{48} =$ $= X_{74}Y_{37}Y_{53}Y_{85}Z_{48} = W_{48}X_{64}Y_{85}Z_{56} = Y_{64}Y_{85}Z_{48}Z_{56} =$ $= W_{42}X_{31}X_{53}Y_{14}Y_{25} = X_{53}Y_{74}Y_{37}Y_{42}Y_{25} = W_{42}X_{74}X_{53}Y_{37}Y_{25} =$ $= X_{53}Y_{14}Y_{31}Y_{42}Y_{25} = X_{31}Y_{14}Y_{53}Y_{25}Z_{42} = X_{42}Y_{74}Y_{37}Y_{53}Y_{25} =$ $= X_{74}Y_{37}Y_{53}Y_{25}Z_{42} = X_{42}Y_{14}Y_{31}Y_{53}Y_{25} = W_{42}X_{64}Y_{25}Z_{56} =$ $= Y_{64}Y_{25}Z_{42}Z_{56} = W_{56}X_{64}Y_{42}Y_{25} = W_{56}X_{42}Y_{64}Y_{25}$ |
| $M_{18}$ | $W_{48}X_{53}Y_{74}Y_{37}Y_{85} = W_{48}X_{53}Y_{14}Y_{31}Y_{85} = Y_{74}Y_{37}Y_{53}Y_{85}Z_{48} =$ $= Y_{14}Y_{31}Y_{53}Y_{85}Z_{48} = W_{48}W_{56}X_{64}Y_{85} = W_{56}Y_{64}Y_{85}Z_{48} =$ $= W_{42}X_{53}Y_{74}Y_{37}Y_{25} = W_{42}X_{53}Y_{14}Y_{31}Y_{25} = Y_{74}Y_{37}Y_{53}Y_{25}Z_{42} =$ $= Y_{14}Y_{31}Y_{53}Y_{25}Z_{42} = W_{42}W_{56}X_{64}Y_{25} = W_{56}Y_{64}Y_{25}Z_{42}$ |
| $M_{19}$ | $X_{56}X_{85}Y_{48}Y_{64} = X_{56}X_{25}Y_{42}Y_{64} = X_{14}X_{31}X_{85}Y_{48}Y_{53} =$ $= X_{74}X_{37}X_{85}Y_{48}Y_{53} = X_{14}X_{31}X_{25}Y_{42}Y_{53} = X_{74}X_{37}X_{25}Y_{42}Y_{53}$ |
| $M_{20}$ | $W_{42}X_{56}X_{25}Y_{64} = X_{37}X_{85}Y_{74}Y_{48}Y_{53} = X_{37}X_{25}Y_{74}Y_{42}Y_{53} =$ $= W_{42}X_{14}X_{31}X_{25}Y_{53} = W_{42}X_{74}X_{37}X_{25}Y_{53} = X_{14}X_{85}Y_{31}Y_{48}Y_{53} =$ $= X_{14}X_{25}Y_{31}Y_{42}Y_{53} = X_{85}Y_{48}Y_{56}Y_{64} = X_{25}Y_{42}Y_{56}Y_{64} =$ $= X_{56}Y_{48}Y_{64}Y_{85} = X_{14}X_{31}Y_{48}Y_{53}Y_{85} = X_{74}X_{37}Y_{48}Y_{53}Y_{85}$ |

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| Field | Chiral superfields |
|---|---|
| $M_{21}$ | $W_{42}X_{37}X_{25}Y_{74}Y_{53} = W_{42}X_{14}X_{25}Y_{31}Y_{53} = W_{42}X_{25}Y_{56}Y_{64} =$ $= X_{37}Y_{74}Y_{48}Y_{53}Y_{85} = X_{14}Y_{31}Y_{48}Y_{53}Y_{85} = Y_{48}Y_{56}Y_{64}Y_{85}$ |
| $M_{27}$ | $W_{48}X_{56}X_{85}Y_{64} = X_{31}X_{85}Y_{14}Y_{48}Y_{53} = W_{48}X_{14}X_{31}X_{85}Y_{53} =$ $= W_{48}X_{74}X_{37}X_{85}Y_{53} = X_{31}X_{25}Y_{14}Y_{42}Y_{53} = X_{74}X_{85}Y_{37}Y_{48}Y_{53} =$ $= X_{74}X_{25}Y_{37}Y_{42}Y_{53} = X_{85}Y_{48}Y_{64}Z_{56} = X_{25}Y_{42}Y_{64}Z_{56} =$ $= X_{56}Y_{42}Y_{64}Y_{25} = X_{14}X_{31}Y_{42}Y_{53}Y_{25} = X_{74}X_{37}Y_{42}Y_{53}Y_{25}$ |
| $M_{23}$ | $W_{48}X_{37}X_{85}Y_{74}Y_{53} = W_{42}X_{31}X_{25}Y_{14}Y_{53} = X_{85}Y_{74}Y_{37}Y_{48}Y_{53} =$ $= X_{25}Y_{74}Y_{37}Y_{42}Y_{53} = W_{42}X_{74}X_{25}Y_{37}Y_{53} = X_{85}Y_{14}Y_{31}Y_{48}Y_{53} =$ $= W_{48}X_{14}X_{85}Y_{31}Y_{53} = X_{25}Y_{14}Y_{31}Y_{42}Y_{53} = W_{48}X_{85}Y_{56}Y_{64} =$ $= W_{42}X_{25}Y_{64}Z_{56} = W_{56}X_{85}Y_{48}Y_{64} = W_{56}X_{25}Y_{42}Y_{64} =$ $= W_{48}X_{56}Y_{64}Y_{85} = X_{31}Y_{14}Y_{48}Y_{53}Y_{85} = W_{48}X_{14}X_{31}Y_{53}Y_{85} =$ $= W_{48}X_{74}X_{37}Y_{53}Y_{85} = X_{74}Y_{37}Y_{48}Y_{53}Y_{85} = Y_{48}Y_{64}Y_{85}Z_{56} =$ $= W_{42}X_{56}Y_{64}Y_{25} = X_{37}Y_{74}Y_{42}Y_{53}Y_{25} = W_{42}X_{14}X_{31}Y_{53}Y_{25} =$ $= W_{42}X_{74}X_{37}Y_{53}Y_{25} = X_{14}Y_{31}Y_{42}Y_{53}Y_{25} = Y_{42}Y_{56}Y_{64}Y_{25}$ |
| $M_{24}$ | $W_{42}X_{25}Y_{74}Y_{37}Y_{53} = W_{42}X_{25}Y_{14}Y_{31}Y_{53} = W_{42}W_{56}X_{25}Y_{64} =$ $= W_{48}X_{37}Y_{74}Y_{53}Y_{85} = Y_{74}Y_{37}Y_{48}Y_{53}Y_{85} = Y_{14}Y_{31}Y_{48}Y_{53}Y_{85} =$ $= W_{48}X_{14}Y_{31}Y_{53}Y_{85} = W_{48}Y_{56}Y_{64}Y_{85} = W_{56}Y_{48}Y_{64}Y_{85} =$ $= W_{42}X_{37}Y_{74}Y_{53}Y_{25} = W_{42}X_{14}Y_{31}Y_{53}Y_{25} = W_{42}Y_{56}Y_{64}Y_{25}$ |
| $M_{25}$ | $W_{48}X_{31}X_{85}Y_{14}Y_{53} = W_{48}X_{74}X_{85}Y_{37}Y_{53} = W_{48}X_{85}Y_{64}Z_{56} =$ $= X_{31}Y_{14}Y_{42}Y_{53}Y_{25} = X_{74}Y_{37}Y_{42}Y_{53}Y_{25} = Y_{42}Y_{64}Y_{25}Z_{56}$ |
| $M_{26}$ | $W_{48}X_{85}Y_{74}Y_{37}Y_{53} = W_{48}X_{85}Y_{14}Y_{31}Y_{53} = W_{48}W_{56}X_{85}Y_{64} =$ $= W_{48}X_{31}Y_{14}Y_{53}Y_{85} = W_{48}X_{74}Y_{37}Y_{53}Y_{85} = W_{48}Y_{64}Y_{85}Z_{56} =$ $= W_{42}X_{31}Y_{14}Y_{53}Y_{25} = Y_{74}Y_{37}Y_{42}Y_{53}Y_{25} = W_{42}X_{74}Y_{37}Y_{53}Y_{25} =$ $= Y_{14}Y_{31}Y_{42}Y_{53}Y_{25} = W_{42}Y_{64}Y_{25}Z_{56} = W_{56}Y_{42}Y_{64}Y_{25}$ |
| $M_{27}$ | $W_{48}Y_{74}Y_{37}Y_{53}Y_{85} = W_{48}Y_{14}Y_{31}Y_{53}Y_{85} = W_{48}W_{56}Y_{64}Y_{85} =$ $= W_{42}Y_{74}Y_{37}Y_{53}Y_{25} = W_{42}Y_{14}Y_{31}Y_{53}Y_{25} = W_{42}W_{56}Y_{64}Y_{25}$ |

**Table 9.** Generators of $Q^{1,1,1}/\mathbb{Z}_2$ in Phase H.

| Field | Chiral superfields |
|---|---|
| $M_1$ | $X_{18}X_{56}X_{61}X_{85} = X_{13}X_{34}X_{46}X_{61} = X_{13}X_{35}X_{56}X_{61} = X_{73}X_{34}X_{46}X_{67} =$ $= X_{72}X_{56}X_{67}X_{25} = X_{73}X_{35}X_{56}X_{67}$ |
| $M_2$ | $X_{13}X_{46}X_{61}Z_{34} = X_{73}X_{46}X_{67}Z_{34} = X_{72}X_{56}X_{67}Y_{25} = X_{34}X_{46}X_{67}Y_{73} =$ $= X_{35}X_{56}X_{67}Y_{73} = X_{18}X_{61}X_{85}Z_{56} = X_{13}X_{35}X_{61}Z_{56} = X_{72}X_{67}X_{25}Z_{56} =$ $= X_{73}X_{35}X_{67}Z_{56} = X_{18}X_{56}X_{85}Y_{61} = X_{13}X_{34}X_{46}Y_{61} = X_{13}X_{35}X_{56}Y_{61}$ |



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| Field | Chiral superfields |
|---|---|
| $M_3$ | $X_{46}X_{67}Y_{73}Z_{34} = X_{72}X_{67}Y_{25}Z_{56} = X_{35}X_{67}Y_{73}Z_{56} = X_{13}X_{46}Y_{61}Z_{34} =$ $= X_{18}X_{85}Y_{61}Z_{56} = X_{13}X_{35}Y_{61}Z_{56}$ |
| $M_4$ | $X_{13}X_{56}X_{61}Y_{35} = X_{73}X_{56}X_{67}Y_{35} = X_{56}X_{67}X_{25}Y_{72} = X_{13}X_{34}X_{61}Y_{46} =$ $= X_{73}X_{34}X_{67}Y_{46} = X_{73}X_{34}X_{46}Z_{67} = X_{72}X_{56}X_{25}Z_{67} = X_{73}X_{35}X_{56}Z_{67} =$ $= X_{18}X_{56}X_{85}Z_{61} = X_{13}X_{34}X_{46}Z_{61} = X_{13}X_{35}X_{56}Z_{61} = X_{56}X_{61}X_{85}Y_{18}$ |
| $M_5$ | $X_{56}X_{67}Y_{73}Y_{35} = X_{56}X_{67}Y_{72}Y_{25} = X_{13}X_{61}Y_{35}Z_{56} = X_{73}X_{67}Y_{35}Z_{56} =$ $= X_{67}X_{25}Y_{72}Z_{56} = X_{13}X_{61}Y_{46}Z_{34} = X_{73}X_{67}Y_{46}Z_{34} = X_{34}X_{67}Y_{73}Y_{46} =$ $= X_{73}X_{46}Z_{34}Z_{67} = X_{72}X_{56}Y_{25}Z_{67} = X_{34}X_{46}Y_{73}Z_{67} = X_{35}X_{56}Y_{73}Z_{67} =$ $= X_{72}X_{25}Z_{56}Z_{67} = X_{73}X_{35}Z_{56}Z_{67} = X_{13}X_{56}Y_{35}Y_{61} = X_{13}X_{34}Y_{46}Y_{61} =$ $= X_{13}X_{46}Z_{34}Z_{61} = X_{18}X_{85}Z_{56}Z_{61} = X_{13}X_{35}Z_{56}Z_{61} = W_{61}X_{18}X_{56}X_{85} =$ $= W_{61}X_{13}X_{34}X_{46} = W_{61}X_{13}X_{35}X_{56} = X_{61}X_{85}Y_{18}Z_{56} = X_{56}X_{85}Y_{18}Y_{61}$ |
| $M_6$ | $X_{67}Y_{73}Y_{35}Z_{56} = X_{67}Y_{72}Y_{25}Z_{56} = X_{67}Y_{73}Y_{46}Z_{34} = X_{46}Y_{73}Z_{34}Z_{67} =$ $= X_{72}Y_{25}Z_{56}Z_{67} = X_{35}Y_{73}Z_{56}Z_{67} = X_{13}Y_{35}Y_{61}Z_{56} = X_{13}Y_{46}Y_{61}Z_{34} =$ $= W_{61}X_{13}X_{46}Z_{34} = W_{61}X_{18}X_{85}Z_{56} = W_{61}X_{13}X_{35}Z_{56} = X_{85}Y_{18}Y_{61}Z_{56}$ |
| $M_2$ | $X_{13}X_{46}X_{61}Y_{34} = X_{73}X_{46}X_{67}Y_{34} = X_{18}X_{61}X_{85}Y_{56} = X_{13}X_{35}X_{61}Y_{56} =$ $= X_{72}X_{67}X_{25}Y_{56} = X_{73}X_{35}X_{67}Y_{56} = X_{18}X_{56}X_{61}Y_{85} = X_{73}X_{34}X_{46}Y_{67} =$ $= X_{72}X_{56}X_{25}Y_{67} = X_{73}X_{35}X_{56}Y_{67} = X_{34}X_{46}X_{61}Y_{13} = X_{35}X_{56}X_{61}Y_{13}$ |
| $M_8$ | $W_{34}X_{13}X_{46}X_{61} = W_{34}X_{73}X_{46}X_{67} = X_{46}X_{67}Y_{73}Y_{34} = X_{72}X_{67}Y_{56}Y_{25} =$ $= X_{35}X_{67}Y_{73}Y_{56} = W_{56}X_{18}X_{61}X_{85} = W_{56}X_{13}X_{35}X_{61} = W_{56}X_{72}X_{67}X_{25} =$ $= W_{56}X_{73}X_{35}X_{67} = X_{18}X_{61}Y_{85}Z_{56} = X_{73}X_{46}Y_{67}Z_{34} = X_{72}X_{56}Y_{67}Y_{25} =$ $= X_{34}X_{46}Y_{73}Y_{67} = X_{35}X_{56}Y_{73}Y_{67} = X_{72}X_{25}Y_{67}Z_{56} = X_{73}X_{35}Y_{67}Z_{56} =$ $= X_{13}X_{46}Y_{34}Y_{61} = X_{18}X_{85}Y_{56}Y_{61} = X_{13}X_{35}Y_{56}Y_{61} = X_{18}X_{56}Y_{61}Y_{85} =$ $= X_{46}X_{61}Y_{13}Z_{34} = X_{35}X_{61}Y_{13}Z_{56} = X_{34}X_{46}Y_{13}Y_{61} = X_{35}X_{56}Y_{13}Y_{61}$ |
| $M_9$ | $W_{34}X_{46}X_{67}Y_{73} = W_{56}X_{72}X_{67}Y_{25} = W_{56}X_{35}X_{67}Y_{73} = X_{46}Y_{73}Y_{67}Z_{34} =$ $= X_{72}Y_{67}Y_{25}Z_{56} = X_{35}Y_{73}Y_{67}Z_{56} = W_{34}X_{13}X_{46}Y_{61} = W_{56}X_{18}X_{85}Y_{61} =$ $= W_{56}X_{13}X_{35}Y_{61} = X_{18}Y_{61}Y_{85}Z_{56} = X_{46}Y_{13}Y_{61}Z_{34} = X_{35}Y_{13}Y_{61}Z_{56}$ |
| $M_{10}$ | $X_{13}X_{61}Y_{35}Y_{56} = X_{73}X_{67}Y_{35}Y_{56} = X_{67}X_{25}Y_{72}Y_{56} = X_{13}X_{61}Y_{34}Y_{46} =$ $= X_{73}X_{67}Y_{34}Y_{46} = X_{73}X_{56}Y_{35}Y_{67} = X_{56}X_{25}Y_{72}Y_{67} = X_{73}X_{34}Y_{46}Y_{67} =$ $= X_{73}X_{46}Y_{34}Z_{67} = X_{72}X_{25}Y_{56}Z_{67} = X_{73}X_{35}Y_{56}Z_{67} = W_{67}X_{73}X_{34}X_{46} =$ $= W_{67}X_{72}X_{56}X_{25} = W_{67}X_{73}X_{35}X_{56} = X_{13}X_{46}Y_{34}Z_{61} = X_{18}X_{85}Y_{56}Z_{61} =$ $= X_{13}X_{35}Y_{56}Z_{61} = X_{18}X_{56}Y_{85}Z_{61} = X_{56}X_{61}Y_{13}Y_{35} = X_{34}X_{61}Y_{13}Y_{46} =$ $= X_{34}X_{46}Y_{13}Z_{61} = X_{35}X_{56}Y_{13}Z_{61} = X_{61}X_{85}Y_{18}Y_{56} = X_{56}X_{61}Y_{18}Y_{85}$ |
| $M_{11}$ | $X_{67}Y_{73}Y_{35}Y_{56} = X_{67}Y_{72}Y_{56}Y_{25} = W_{56}X_{13}X_{61}Y_{35} = W_{56}X_{73}X_{67}Y_{35} =$ $= W_{56}X_{67}X_{25}Y_{72} = W_{34}X_{13}X_{61}Y_{46} = W_{34}X_{73}X_{67}Y_{46} = X_{67}Y_{73}Y_{34}Y_{46} =$ |

| Field | Chiral superfields |
|---|---|
| | $= X_{56}Y_{73}Y_{35}Y_{67} = X_{56}Y_{72}Y_{67}Y_{25} = X_{73}Y_{35}Y_{67}Z_{56} = X_{25}Y_{72}Y_{67}Z_{56} =$ $= X_{73}Y_{46}Y_{67}Z_{34} = X_{34}Y_{73}Y_{46}Y_{67} = W_{34}X_{73}X_{46}Z_{67} = X_{46}Y_{73}Y_{34}Z_{67} =$ $= X_{72}Y_{56}Y_{25}Z_{67} = X_{35}Y_{73}Y_{56}Z_{67} = W_{56}X_{72}X_{25}Z_{67} = W_{56}X_{73}X_{35}Z_{67} =$ $= W_{67}X_{73}X_{46}Z_{34} = W_{67}X_{72}X_{56}Y_{25} = W_{67}X_{34}X_{46}Y_{73} = W_{67}X_{35}X_{56}Y_{73} =$ $= W_{67}X_{72}X_{25}Z_{56} = W_{67}X_{73}X_{35}Z_{56} = X_{13}Y_{35}Y_{56}Y_{61} = X_{13}Y_{34}Y_{46}Y_{61} =$ $= W_{34}X_{13}X_{46}Z_{61} = W_{56}X_{18}X_{85}Z_{61} = W_{56}X_{13}X_{35}Z_{61} = X_{18}Y_{85}Z_{56}Z_{61} =$ $= W_{61}X_{13}X_{46}Y_{34} = W_{61}X_{18}X_{85}Y_{56} = W_{61}X_{13}X_{35}Y_{56} = W_{61}X_{18}X_{56}Y_{85} =$ $= X_{61}Y_{13}Y_{35}Z_{56} = X_{61}Y_{13}Y_{46}Z_{34} = X_{56}Y_{13}Y_{35}Y_{61} = X_{34}Y_{13}Y_{46}Y_{61} =$ $= X_{46}Y_{13}Z_{34}Z_{61} = X_{35}Y_{13}Z_{56}Z_{61} = W_{61}X_{34}X_{46}Y_{13} = W_{61}X_{35}X_{56}Y_{13} =$ $= W_{56}X_{61}X_{85}Y_{18} = X_{61}Y_{18}Y_{85}Z_{56} = X_{85}Y_{18}Y_{56}Y_{61} = X_{56}Y_{18}Y_{61}Y_{85}$ |
| $M_{12}$ | $W_{56}X_{67}Y_{73}Y_{35} = W_{56}X_{67}Y_{72}Y_{25} = W_{34}X_{67}Y_{73}Y_{46} = Y_{73}Y_{35}Y_{67}Z_{56} =$ $= Y_{72}Y_{67}Y_{25}Z_{56} = Y_{73}Y_{46}Y_{67}Z_{34} = W_{34}X_{46}Y_{73}Z_{67} = W_{56}X_{72}Y_{25}Z_{67} =$ $= W_{56}X_{35}Y_{73}Z_{67} = W_{67}X_{46}Y_{73}Z_{34} = W_{67}X_{72}Y_{25}Z_{56} = W_{67}X_{35}Y_{73}Z_{56} =$ $= W_{56}X_{13}Y_{35}Y_{61} = W_{34}X_{13}Y_{46}Y_{61} = W_{34}W_{61}X_{13}X_{46} = W_{56}W_{61}X_{18}X_{85} =$ $= W_{56}W_{61}X_{13}X_{35} = W_{61}X_{18}Y_{85}Z_{56} = Y_{13}Y_{35}Y_{61}Z_{56} = Y_{13}Y_{46}Y_{61}Z_{34} =$ $= W_{61}X_{46}Y_{13}Z_{34} = W_{61}X_{35}Y_{13}Z_{56} = W_{56}X_{85}Y_{18}Y_{61} = Y_{18}Y_{61}Y_{85}Z_{56}$ |
| $M_{13}$ | $X_{18}X_{61}Y_{56}Y_{85} = X_{73}X_{46}Y_{34}Y_{67} = X_{72}X_{25}Y_{56}Y_{67} = X_{73}X_{35}Y_{56}Y_{67} =$ $= X_{46}X_{61}Y_{13}Y_{34} = X_{35}X_{61}Y_{13}Y_{56}$ |
| $M_{14}$ | $W_{56}X_{18}X_{61}Y_{85} = W_{34}X_{73}X_{46}Y_{67} = X_{46}Y_{73}Y_{34}Y_{67} = X_{72}Y_{56}Y_{67}Y_{25} =$ $= X_{35}Y_{73}Y_{56}Y_{67} = W_{56}X_{72}X_{25}Y_{67} = W_{56}X_{73}X_{35}Y_{67} = X_{18}Y_{56}Y_{61}Y_{85} =$ $= W_{34}X_{46}X_{61}Y_{13} = W_{56}X_{35}X_{61}Y_{13} = X_{46}Y_{13}Y_{34}Y_{61} = X_{35}Y_{13}Y_{56}Y_{61}$ |
| $M_{15}$ | $W_{34}X_{46}Y_{73}Y_{67} = W_{56}X_{72}Y_{67}Y_{25} = W_{56}X_{35}Y_{73}Y_{67} = W_{56}X_{18}Y_{61}Y_{85} =$ $= W_{34}X_{46}Y_{13}Y_{61} = W_{56}X_{35}Y_{13}Y_{61}$ |
| $M_{16}$ | $X_{73}Y_{35}Y_{56}Y_{67} = X_{25}Y_{72}Y_{56}Y_{67} = X_{73}Y_{34}Y_{46}Y_{67} = W_{67}X_{73}X_{46}Y_{34} =$ $= W_{67}X_{72}X_{25}Y_{56} = W_{67}X_{73}X_{35}Y_{56} = X_{18}Y_{56}Y_{85}Z_{61} = X_{61}Y_{13}Y_{35}Y_{56} =$ $= X_{61}Y_{13}Y_{34}Y_{46} = X_{46}Y_{13}Y_{34}Z_{61} = X_{35}Y_{13}Y_{56}Z_{61} = X_{61}Y_{18}Y_{56}Y_{85}$ |
| $M_{17}$ | $Y_{73}Y_{35}Y_{56}Y_{67} = Y_{72}Y_{56}Y_{67}Y_{25} = W_{56}X_{73}Y_{35}Y_{67} = W_{56}X_{25}Y_{72}Y_{67} =$ $= W_{34}X_{73}Y_{46}Y_{67} = Y_{73}Y_{34}Y_{46}Y_{67} = W_{34}W_{67}X_{73}X_{46} = W_{67}X_{46}Y_{73}Y_{34} =$ $= W_{67}X_{72}Y_{56}Y_{25} = W_{67}X_{35}Y_{73}Y_{56} = W_{56}W_{67}X_{72}X_{25} = W_{56}W_{67}X_{73}X_{35} =$ $= W_{56}X_{18}Y_{85}Z_{61} = W_{61}X_{18}Y_{56}Y_{85} = W_{56}X_{61}Y_{13}Y_{35} = W_{34}X_{61}Y_{13}Y_{46} =$ $= Y_{13}Y_{35}Y_{56}Y_{61} = Y_{13}Y_{34}Y_{46}Y_{61} = W_{34}X_{46}Y_{13}Z_{61} = W_{56}X_{35}Y_{13}Z_{61} =$ $= W_{61}X_{46}Y_{13}Y_{34} = W_{61}X_{35}Y_{13}Y_{56} = W_{56}X_{61}Y_{18}Y_{85} = Y_{18}Y_{56}Y_{61}Y_{85}$ |

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| Field | Chiral superfields |
|---|---|
| $M_{18}$ | $W_{56}Y_{73}Y_{35}Y_{67} = W_{56}Y_{72}Y_{67}Y_{25} = W_{34}Y_{73}Y_{46}Y_{67} = W_{34}W_{67}X_{46}Y_{73} =$ $= W_{56}W_{67}X_{72}Y_{25} = W_{56}W_{67}X_{35}Y_{73} = W_{56}W_{61}X_{18}Y_{85} = W_{56}Y_{13}Y_{35}Y_{61} =$ $= W_{34}Y_{13}Y_{46}Y_{61} = W_{34}W_{61}X_{46}Y_{13} = W_{56}W_{61}X_{35}Y_{13} = W_{56}Y_{18}Y_{61}Y_{85}$ |
| $M_{19}$ | $X_{73}X_{56}Y_{35}Z_{67} = X_{56}X_{25}Y_{72}Z_{67} = X_{73}X_{34}Y_{46}Z_{67} = X_{13}X_{56}Y_{35}Z_{61} =$ $= X_{13}X_{34}Y_{46}Z_{61} = X_{56}X_{85}Y_{18}Z_{61}$ |
| $M_{20}$ | $X_{56}Y_{73}Y_{35}Z_{67} = X_{56}Y_{72}Y_{25}Z_{67} = X_{73}Y_{35}Z_{56}Z_{67} = X_{25}Y_{72}Z_{56}Z_{67} =$ $= X_{73}Y_{46}Z_{34}Z_{67} = X_{34}Y_{73}Y_{46}Z_{67} = X_{13}Y_{35}Z_{56}Z_{61} = X_{13}Y_{46}Z_{34}Z_{61} =$ $= W_{61}X_{13}X_{56}Y_{35} = W_{61}X_{13}X_{34}Y_{46} = X_{85}Y_{18}Z_{56}Z_{61} = W_{61}X_{56}X_{85}Y_{18}$ |
| $M_{21}$ | $Y_{73}Y_{35}Z_{56}Z_{67} = Y_{72}Y_{25}Z_{56}Z_{67} = Y_{73}Y_{46}Z_{34}Z_{67} = W_{61}X_{13}Y_{35}Z_{56} =$ $= W_{61}X_{13}Y_{46}Z_{34} = W_{61}X_{85}Y_{18}Z_{56}$ |
| $M_{27}$ | $X_{73}Y_{35}Y_{56}Z_{67} = X_{25}Y_{72}Y_{56}Z_{67} = X_{73}Y_{34}Y_{46}Z_{67} = W_{67}X_{73}X_{56}Y_{35} =$ $= W_{67}X_{56}X_{25}Y_{72} = W_{67}X_{73}X_{34}Y_{46} = X_{13}Y_{35}Y_{56}Z_{61} = X_{13}Y_{34}Y_{46}Z_{61} =$ $= X_{56}Y_{13}Y_{35}Z_{61} = X_{34}Y_{13}Y_{46}Z_{61} = X_{85}Y_{18}Y_{56}Z_{61} = X_{56}Y_{18}Y_{85}Z_{61}$ |
| $M_{23}$ | $Y_{73}Y_{35}Y_{56}Z_{67} = Y_{72}Y_{56}Y_{25}Z_{67} = W_{56}X_{73}Y_{35}Z_{67} = W_{56}X_{25}Y_{72}Z_{67} =$ $= W_{34}X_{73}Y_{46}Z_{67} = Y_{73}Y_{34}Y_{46}Z_{67} = W_{67}X_{56}Y_{73}Y_{35} = W_{67}X_{56}Y_{72}Y_{25} =$ $= W_{67}X_{73}Y_{35}Z_{56} = W_{67}X_{25}Y_{72}Z_{56} = W_{67}X_{73}Y_{46}Z_{34} = W_{67}X_{34}Y_{73}Y_{46} =$ $= W_{56}X_{13}Y_{35}Z_{61} = W_{34}X_{13}Y_{46}Z_{61} = W_{61}X_{13}Y_{35}Y_{56} = W_{61}X_{13}Y_{34}Y_{46} =$ $= Y_{13}Y_{35}Z_{56}Z_{61} = Y_{13}Y_{46}Z_{34}Z_{61} = W_{61}X_{56}Y_{13}Y_{35} = W_{61}X_{34}Y_{13}Y_{46} =$ $= W_{56}X_{85}Y_{18}Z_{61} = Y_{18}Y_{85}Z_{56}Z_{61} = W_{61}X_{85}Y_{18}Y_{56} = W_{61}X_{56}Y_{18}Y_{85}$ |
| $M_{24}$ | $W_{56}Y_{73}Y_{35}Z_{67} = W_{56}Y_{72}Y_{25}Z_{67} = W_{34}Y_{73}Y_{46}Z_{67} = W_{67}Y_{73}Y_{35}Z_{56} =$ $= W_{67}Y_{72}Y_{25}Z_{56} = W_{67}Y_{73}Y_{46}Z_{34} = W_{56}W_{61}X_{13}Y_{35} = W_{34}W_{61}X_{13}Y_{46} =$ $= W_{61}Y_{13}Y_{35}Z_{56} = W_{61}Y_{13}Y_{46}Z_{34} = W_{56}W_{61}X_{85}Y_{18} = W_{61}Y_{18}Y_{85}Z_{56}$ |
| $M_{25}$ | $W_{67}X_{73}Y_{35}Y_{56} = W_{67}X_{25}Y_{72}Y_{56} = W_{67}X_{73}Y_{34}Y_{46} = Y_{13}Y_{35}Y_{56}Z_{61} =$ $= Y_{13}Y_{34}Y_{46}Z_{61} = Y_{18}Y_{56}Y_{85}Z_{61}$ |
| $M_{26}$ | $W_{67}Y_{73}Y_{35}Y_{56} = W_{67}Y_{72}Y_{56}Y_{25} = W_{56}W_{67}X_{73}Y_{35} = W_{56}W_{67}X_{25}Y_{72} =$ $= W_{34}W_{67}X_{73}Y_{46} = W_{67}Y_{73}Y_{34}Y_{46} = W_{56}Y_{13}Y_{35}Z_{61} = W_{34}Y_{13}Y_{46}Z_{61} =$ $= W_{61}Y_{13}Y_{35}Y_{56} = W_{61}Y_{13}Y_{34}Y_{46} = W_{56}Y_{18}Y_{85}Z_{61} = W_{61}Y_{18}Y_{56}Y_{85}$ |
| $M_{27}$ | $W_{56}W_{67}Y_{73}Y_{35} = W_{56}W_{67}Y_{72}Y_{25} = W_{34}W_{67}Y_{73}Y_{46} = W_{56}W_{61}Y_{13}Y_{35} =$ $= W_{34}W_{61}Y_{13}Y_{46} = W_{56}W_{61}Y_{18}Y_{85}$ |

**Table 10.** Generators of $Q^{1,1,1}/\mathbb{Z}_2$ in Phase J.

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| Field | Chiral superfields |
|---|---|
| $M_1$ | $A_{54}X_{18}X_{41}Y_{85} = A_{54}X_{46}X_{68}Y_{85} = A_{54}X_{46}X_{25}Y_{62} = A_{54}X_{73}X_{35}Y_{47} =$ $= A_{54}X_{72}X_{25}Y_{47} = A_{54}X_{35}X_{41}Y_{13}$ |
| $M_2$ | $X_{18}X_{41}X_{54}Y_{85} = X_{46}X_{54}X_{68}Y_{85} = X_{46}X_{54}X_{25}Y_{62} = A_{54}X_{13}X_{35}X_{41} =$ $= A_{54}X_{18}X_{41}X_{85} = A_{54}X_{73}X_{35}X_{47} = A_{54}X_{72}X_{47}X_{25} = A_{54}X_{46}X_{62}X_{25} =$ $= A_{54}X_{46}X_{68}X_{85} = X_{73}X_{35}X_{54}Y_{47} = X_{72}X_{54}X_{25}Y_{47} = X_{35}X_{41}X_{54}Y_{13}$ |
| $M_3$ | $X_{13}X_{35}X_{41}X_{54} = X_{18}X_{41}X_{54}X_{85} = X_{73}X_{35}X_{47}X_{54} = X_{72}X_{47}X_{54}X_{25} =$ $= X_{46}X_{54}X_{62}X_{25} = X_{46}X_{54}X_{68}X_{85}$ |
| $M_4$ | $B_{54}X_{18}X_{41}Y_{85} = B_{54}X_{46}X_{68}Y_{85} = B_{54}X_{46}X_{25}Y_{62} = A_{54}X_{68}Y_{46}Y_{85} =$ $= A_{54}X_{25}Y_{46}Y_{62} = B_{54}X_{73}X_{35}Y_{47} = B_{54}X_{72}X_{25}Y_{47} = A_{54}X_{73}Y_{35}Y_{47} =$ $= A_{54}X_{25}Y_{72}Y_{47} = A_{54}X_{41}Y_{18}Y_{85} = B_{54}X_{35}X_{41}Y_{13} = A_{54}X_{41}Y_{13}Y_{35}$ |
| $M_5$ | $X_{18}X_{41}Y_{54}Y_{85} = X_{46}X_{68}Y_{54}Y_{85} = X_{46}X_{25}Y_{54}Y_{62} = B_{54}X_{13}X_{35}X_{41} =$ $= B_{54}X_{18}X_{41}X_{85} = B_{54}X_{73}X_{35}X_{47} = B_{54}X_{72}X_{47}X_{25} = B_{54}X_{46}X_{62}X_{25} =$ $= B_{54}X_{46}X_{68}X_{85} = X_{54}X_{68}Y_{46}Y_{85} = X_{54}X_{25}Y_{46}Y_{62} = A_{54}X_{62}X_{25}Y_{46} =$ $= A_{54}X_{68}X_{85}Y_{46} = X_{73}X_{35}Y_{47}Y_{54} = X_{72}X_{25}Y_{47}Y_{54} = A_{54}X_{13}X_{41}Y_{35} =$ $= A_{54}X_{73}X_{47}Y_{35} = X_{73}X_{54}Y_{35}Y_{47} = A_{54}X_{47}X_{25}Y_{72} = X_{54}X_{25}Y_{72}Y_{47} =$ $= X_{41}X_{54}Y_{18}Y_{85} = A_{54}X_{41}X_{85}Y_{18} = X_{35}X_{41}Y_{13}Y_{54} = X_{41}X_{54}Y_{13}Y_{35}$ |
| $M_6$ | $X_{13}X_{35}X_{41}Y_{54} = X_{18}X_{41}X_{85}Y_{54} = X_{73}X_{35}X_{47}Y_{54} = X_{72}X_{47}X_{25}Y_{54} =$ $= X_{46}X_{62}X_{25}Y_{54} = X_{46}X_{68}X_{85}Y_{54} = X_{54}X_{62}X_{25}Y_{46} = X_{54}X_{68}X_{85}Y_{46} =$ $= X_{13}X_{41}X_{54}Y_{35} = X_{73}X_{47}X_{54}Y_{35} = X_{47}X_{54}X_{25}Y_{72} = X_{41}X_{54}X_{85}Y_{18}$ |
| $M_2$ | $A_{54}X_{46}Y_{68}Y_{85} = A_{54}X_{46}Y_{62}Y_{25} = X_{18}X_{41}Y_{85}C_{54} = X_{46}X_{68}Y_{85}C_{54} =$ $= X_{46}X_{25}Y_{62}C_{54} = A_{54}X_{72}Y_{47}Y_{25} = X_{73}X_{35}Y_{47}C_{54} = X_{72}X_{25}Y_{47}C_{54} =$ $= A_{54}X_{18}Y_{41}Y_{85} = A_{54}X_{35}Y_{73}Y_{47} = X_{35}X_{41}Y_{13}C_{54} = A_{54}X_{35}Y_{13}Y_{41}$ |
| $M_8$ | $X_{46}X_{54}Y_{68}Y_{85} = X_{46}X_{54}Y_{62}Y_{25} = X_{18}X_{41}Y_{85}Z_{54} = X_{46}X_{68}Y_{85}Z_{54} =$ $= X_{46}X_{25}Y_{62}Z_{54} = A_{54}X_{72}X_{47}Y_{25} = A_{54}X_{46}X_{62}Y_{25} = A_{54}X_{46}X_{85}Y_{68} =$ $= X_{13}X_{35}X_{41}C_{54} = X_{18}X_{41}X_{85}C_{54} = X_{73}X_{35}X_{47}C_{54} = X_{72}X_{47}X_{25}C_{54} =$ $= X_{46}X_{62}X_{25}C_{54} = X_{46}X_{68}X_{85}C_{54} = X_{72}X_{54}Y_{47}Y_{25} = X_{73}X_{35}Y_{47}Z_{54} =$ $= X_{72}X_{25}Y_{47}Z_{54} = X_{18}X_{54}Y_{41}Y_{85} = A_{54}X_{13}X_{35}Y_{41} = A_{54}X_{18}X_{85}Y_{41} =$ $= A_{54}X_{35}X_{47}Y_{73} = X_{35}X_{54}Y_{73}Y_{47} = X_{35}X_{41}Y_{13}Z_{54} = X_{35}X_{54}Y_{13}Y_{41}$ |
| $M_9$ | $X_{72}X_{47}X_{54}Y_{25} = X_{46}X_{54}X_{62}Y_{25} = X_{46}X_{54}X_{85}Y_{68} = X_{13}X_{35}X_{41}Z_{54} =$ $= X_{18}X_{41}X_{85}Z_{54} = X_{73}X_{35}X_{47}Z_{54} = X_{72}X_{47}X_{25}Z_{54} = X_{46}X_{62}X_{25}Z_{54} =$ $= X_{46}X_{68}X_{85}Z_{54} = X_{13}X_{35}X_{54}Y_{41} = X_{18}X_{54}X_{85}Y_{41} = X_{35}X_{47}X_{54}Y_{73}$ |
| $M_{10}$ | $B_{54}X_{46}Y_{68}Y_{85} = B_{54}X_{46}Y_{62}Y_{25} = D_{54}X_{18}X_{41}Y_{85} = D_{54}X_{46}X_{68}Y_{85} =$ $= D_{54}X_{46}X_{25}Y_{62} = A_{54}Y_{46}Y_{68}Y_{85} = A_{54}Y_{46}Y_{62}Y_{25} = X_{68}Y_{46}Y_{85}C_{54} =$ $= X_{25}Y_{46}Y_{62}C_{54} = B_{54}X_{72}Y_{47}Y_{25} = D_{54}X_{73}X_{35}Y_{47} = D_{54}X_{72}X_{25}Y_{47} =$ |

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| Field | Chiral superfields |
|---|---|
| | $= B_{54}X_{18}Y_{41}Y_{85} = X_{73}Y_{35}Y_{47}C_{54} = A_{54}Y_{72}Y_{47}Y_{25} = X_{25}Y_{72}Y_{47}C_{54} =$ $= B_{54}X_{35}Y_{73}Y_{47} = A_{54}Y_{73}Y_{35}Y_{47} = X_{41}Y_{18}Y_{85}C_{54} = A_{54}Y_{18}Y_{41}Y_{85} =$ $= D_{54}X_{35}X_{41}Y_{13} = B_{54}X_{35}Y_{13}Y_{41} = X_{41}Y_{13}Y_{35}C_{54} = A_{54}Y_{13}Y_{35}Y_{41}$ |
| $M_{11}$ | $X_{46}Y_{54}Y_{68}Y_{85} = X_{46}Y_{54}Y_{62}Y_{25} = W_{54}X_{18}X_{41}Y_{85} = W_{54}X_{46}X_{68}Y_{85} =$ $= W_{54}X_{46}X_{25}Y_{62} = B_{54}X_{72}X_{47}Y_{25} = B_{54}X_{46}X_{62}Y_{25} = B_{54}X_{46}X_{85}Y_{68} =$ $= D_{54}X_{13}X_{35}X_{41} = D_{54}X_{18}X_{41}X_{85} = D_{54}X_{73}X_{35}X_{47} = D_{54}X_{72}X_{47}X_{25} =$ $= D_{54}X_{46}X_{62}X_{25} = D_{54}X_{46}X_{68}X_{85} = X_{54}Y_{46}Y_{68}Y_{85} = X_{54}Y_{46}Y_{62}Y_{25} =$ $= X_{68}Y_{46}Y_{85}Z_{54} = X_{25}Y_{46}Y_{62}Z_{54} = A_{54}X_{62}Y_{46}Y_{25} = A_{54}X_{85}Y_{46}Y_{68} =$ $= X_{62}X_{25}Y_{46}C_{54} = X_{68}X_{85}Y_{46}C_{54} = X_{72}Y_{47}Y_{54}Y_{25} = W_{54}X_{73}X_{35}Y_{47} =$ $= W_{54}X_{72}X_{25}Y_{47} = X_{18}Y_{41}Y_{54}Y_{85} = B_{54}X_{13}X_{35}Y_{41} = B_{54}X_{18}X_{85}Y_{41} =$ $= X_{13}X_{41}Y_{35}C_{54} = X_{73}X_{47}Y_{35}C_{54} = X_{73}Y_{35}Y_{47}Z_{54} = A_{54}X_{13}Y_{35}Y_{41} =$ $= A_{54}X_{47}Y_{72}Y_{25} = X_{47}X_{25}Y_{72}C_{54} = X_{54}Y_{72}Y_{47}Y_{25} = X_{25}Y_{72}Y_{47}Z_{54} =$ $= B_{54}X_{35}X_{47}Y_{73} = X_{35}Y_{73}Y_{47}Y_{54} = A_{54}X_{47}Y_{73}Y_{35} = X_{54}Y_{73}Y_{35}Y_{47} =$ $= X_{41}Y_{18}Y_{85}Z_{54} = X_{41}X_{85}Y_{18}C_{54} = X_{54}Y_{18}Y_{41}Y_{85} = A_{54}X_{85}Y_{18}Y_{41} =$ $= W_{54}X_{35}X_{41}Y_{13} = X_{35}Y_{13}Y_{41}Y_{54} = X_{41}Y_{13}Y_{35}Z_{54} = X_{54}Y_{13}Y_{35}Y_{41}$ |
| $M_{12}$ | $X_{72}X_{47}Y_{54}Y_{25} = X_{46}X_{62}Y_{54}Y_{25} = X_{46}X_{85}Y_{54}Y_{68} = W_{54}X_{13}X_{35}X_{41} =$ $= W_{54}X_{18}X_{41}X_{85} = W_{54}X_{73}X_{35}X_{47} = W_{54}X_{72}X_{47}X_{25} = W_{54}X_{46}X_{62}X_{25} =$ $= W_{54}X_{46}X_{68}X_{85} = X_{54}X_{62}Y_{46}Y_{25} = X_{54}X_{85}Y_{46}Y_{68} = X_{62}X_{25}Y_{46}Z_{54} =$ $= X_{68}X_{85}Y_{46}Z_{54} = X_{13}X_{35}Y_{41}Y_{54} = X_{18}X_{85}Y_{41}Y_{54} = X_{13}X_{41}Y_{35}Z_{54} =$ $= X_{73}X_{47}Y_{35}Z_{54} = X_{13}X_{54}Y_{35}Y_{41} = X_{47}X_{54}Y_{72}Y_{25} = X_{47}X_{25}Y_{72}Z_{54} =$ $= X_{35}X_{47}Y_{73}Y_{54} = X_{47}X_{54}Y_{73}Y_{35} = X_{41}X_{85}Y_{18}Z_{54} = X_{54}X_{85}Y_{18}Y_{41}$ |
| $M_{13}$ | $X_{46}Y_{68}Y_{85}C_{54} = X_{46}Y_{62}Y_{25}C_{54} = X_{72}Y_{47}Y_{25}C_{54} = X_{18}Y_{41}Y_{85}C_{54} =$ $= X_{35}Y_{73}Y_{47}C_{54} = X_{35}Y_{13}Y_{41}C_{54}$ |
| $M_{14}$ | $X_{46}Y_{68}Y_{85}Z_{54} = X_{46}Y_{62}Y_{25}Z_{54} = X_{72}X_{47}Y_{25}C_{54} = X_{46}X_{62}Y_{25}C_{54} =$ $= X_{46}X_{85}Y_{68}C_{54} = X_{72}Y_{47}Y_{25}Z_{54} = X_{18}Y_{41}Y_{85}Z_{54} = X_{13}X_{35}Y_{41}C_{54} =$ $= X_{18}X_{85}Y_{41}C_{54} = X_{35}X_{47}Y_{73}C_{54} = X_{35}Y_{73}Y_{47}Z_{54} = X_{35}Y_{13}Y_{41}Z_{54}$ |
| $M_{15}$ | $X_{72}X_{47}Y_{25}Z_{54} = X_{46}X_{62}Y_{25}Z_{54} = X_{46}X_{85}Y_{68}Z_{54} = X_{13}X_{35}Y_{41}Z_{54} =$ $= X_{18}X_{85}Y_{41}Z_{54} = X_{35}X_{47}Y_{73}Z_{54}$ |
| $M_{16}$ | $D_{54}X_{46}Y_{68}Y_{85} = D_{54}X_{46}Y_{62}Y_{25} = Y_{46}Y_{68}Y_{85}C_{54} = Y_{46}Y_{62}Y_{25}C_{54} =$ $= D_{54}X_{72}Y_{47}Y_{25} = D_{54}X_{18}Y_{41}Y_{85} = Y_{72}Y_{47}Y_{25}C_{54} = D_{54}X_{35}Y_{73}Y_{47} =$ $= Y_{73}Y_{35}Y_{47}C_{54} = Y_{18}Y_{41}Y_{85}C_{54} = D_{54}X_{35}Y_{13}Y_{41} = Y_{13}Y_{35}Y_{41}C_{54}$ |
| $M_{17}$ | $W_{54}X_{46}Y_{68}Y_{85} = W_{54}X_{46}Y_{62}Y_{25} = D_{54}X_{72}X_{47}Y_{25} = D_{54}X_{46}X_{62}Y_{25} =$ $= D_{54}X_{46}X_{85}Y_{68} = Y_{46}Y_{68}Y_{85}Z_{54} = Y_{46}Y_{62}Y_{25}Z_{54} = X_{62}Y_{46}Y_{25}C_{54} =$ $= X_{85}Y_{46}Y_{68}C_{54} = W_{54}X_{72}Y_{47}Y_{25} = W_{54}X_{18}Y_{41}Y_{85} = D_{54}X_{13}X_{35}Y_{41} =$ |

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| Field | Chiral superfields |
|---|---|
| | $= D_{54}X_{18}X_{85}Y_{41} = X_{13}Y_{35}Y_{41}C_{54} = X_{47}Y_{72}Y_{25}C_{54} = Y_{72}Y_{47}Y_{25}Z_{54} =$ $= D_{54}X_{35}X_{47}Y_{73} = W_{54}X_{35}Y_{73}Y_{47} = X_{47}Y_{73}Y_{35}C_{54} = Y_{73}Y_{35}Y_{47}Z_{54} =$ $= Y_{18}Y_{41}Y_{85}Z_{54} = X_{85}Y_{18}Y_{41}C_{54} = W_{54}X_{35}Y_{13}Y_{41} = Y_{13}Y_{35}Y_{41}Z_{54}$ |
| $M_{18}$ | $W_{54}X_{72}X_{47}Y_{25} = W_{54}X_{46}X_{62}Y_{25} = W_{54}X_{46}X_{85}Y_{68} = X_{62}Y_{46}Y_{25}Z_{54} =$ $= X_{85}Y_{46}Y_{68}Z_{54} = W_{54}X_{13}X_{35}Y_{41} = W_{54}X_{18}X_{85}Y_{41} = X_{13}Y_{35}Y_{41}Z_{54} =$ $= X_{47}Y_{72}Y_{25}Z_{54} = W_{54}X_{35}X_{47}Y_{73} = X_{47}Y_{73}Y_{35}Z_{54} = X_{85}Y_{18}Y_{41}Z_{54}$ |
| $M_{19}$ | $B_{54}X_{68}Y_{46}Y_{85} = B_{54}X_{25}Y_{46}Y_{62} = B_{54}X_{73}Y_{35}Y_{47} = B_{54}X_{25}Y_{72}Y_{47} =$ $= B_{54}X_{41}Y_{18}Y_{85} = B_{54}X_{41}Y_{13}Y_{35}$ |
| $M_{20}$ | $X_{68}Y_{46}Y_{54}Y_{85} = X_{25}Y_{46}Y_{54}Y_{62} = B_{54}X_{62}X_{25}Y_{46} = B_{54}X_{68}X_{85}Y_{46} =$ $= B_{54}X_{13}X_{41}Y_{35} = B_{54}X_{73}X_{47}Y_{35} = X_{73}Y_{35}Y_{47}Y_{54} = B_{54}X_{47}X_{25}Y_{72} =$ $= X_{25}Y_{72}Y_{47}Y_{54} = X_{41}Y_{18}Y_{54}Y_{85} = B_{54}X_{41}X_{85}Y_{18} = X_{41}Y_{13}Y_{35}Y_{54}$ |
| $M_{21}$ | $X_{62}X_{25}Y_{46}Y_{54} = X_{68}X_{85}Y_{46}Y_{54} = X_{13}X_{41}Y_{35}Y_{54} = X_{73}X_{47}Y_{35}Y_{54} =$ $= X_{47}X_{25}Y_{72}Y_{54} = X_{41}X_{85}Y_{18}Y_{54}$ |
| $M_{27}$ | $B_{54}Y_{46}Y_{68}Y_{85} = B_{54}Y_{46}Y_{62}Y_{25} = D_{54}X_{68}Y_{46}Y_{85} = D_{54}X_{25}Y_{46}Y_{62} =$ $= D_{54}X_{73}Y_{35}Y_{47} = B_{54}Y_{72}Y_{47}Y_{25} = D_{54}X_{25}Y_{72}Y_{47} = B_{54}Y_{73}Y_{35}Y_{47} =$ $= D_{54}X_{41}Y_{18}Y_{85} = B_{54}Y_{18}Y_{41}Y_{85} = D_{54}X_{41}Y_{13}Y_{35} = B_{54}Y_{13}Y_{35}Y_{41}$ |
| $M_{23}$ | $Y_{46}Y_{54}Y_{68}Y_{85} = Y_{46}Y_{54}Y_{62}Y_{25} = W_{54}X_{68}Y_{46}Y_{85} = W_{54}X_{25}Y_{46}Y_{62} =$ $= B_{54}X_{62}Y_{46}Y_{25} = B_{54}X_{85}Y_{46}Y_{68} = D_{54}X_{62}X_{25}Y_{46} = D_{54}X_{68}X_{85}Y_{46} =$ $= D_{54}X_{13}X_{41}Y_{35} = D_{54}X_{73}X_{47}Y_{35} = W_{54}X_{73}Y_{35}Y_{47} = B_{54}X_{13}Y_{35}Y_{41} =$ $= B_{54}X_{47}Y_{72}Y_{25} = D_{54}X_{47}X_{25}Y_{72} = Y_{72}Y_{47}Y_{54}Y_{25} = W_{54}X_{25}Y_{72}Y_{47} =$ $= B_{54}X_{47}Y_{73}Y_{35} = Y_{73}Y_{35}Y_{47}Y_{54} = W_{54}X_{41}Y_{18}Y_{85} = D_{54}X_{41}X_{85}Y_{18} =$ $= Y_{18}Y_{41}Y_{54}Y_{85} = B_{54}X_{85}Y_{18}Y_{41} = W_{54}X_{41}Y_{13}Y_{35} = Y_{13}Y_{35}Y_{41}Y_{54}$ |
| $M_{24}$ | $X_{62}Y_{46}Y_{54}Y_{25} = X_{85}Y_{46}Y_{54}Y_{68} = W_{54}X_{62}X_{25}Y_{46} = W_{54}X_{68}X_{85}Y_{46} =$ $= W_{54}X_{13}X_{41}Y_{35} = W_{54}X_{73}X_{47}Y_{35} = X_{13}Y_{35}Y_{41}Y_{54} = X_{47}Y_{72}Y_{54}Y_{25} =$ $= W_{54}X_{47}X_{25}Y_{72} = X_{47}Y_{73}Y_{35}Y_{54} = W_{54}X_{41}X_{85}Y_{18} = X_{85}Y_{18}Y_{41}Y_{54}$ |
| $M_{25}$ | $D_{54}Y_{46}Y_{68}Y_{85} = D_{54}Y_{46}Y_{62}Y_{25} = D_{54}Y_{72}Y_{47}Y_{25} = D_{54}Y_{73}Y_{35}Y_{47} =$ $= D_{54}Y_{18}Y_{41}Y_{85} = D_{54}Y_{13}Y_{35}Y_{41}$ |
| $M_{26}$ | $W_{54}Y_{46}Y_{68}Y_{85} = W_{54}Y_{46}Y_{62}Y_{25} = D_{54}X_{62}Y_{46}Y_{25} = D_{54}X_{85}Y_{46}Y_{68} =$ $= D_{54}X_{13}Y_{35}Y_{41} = D_{54}X_{47}Y_{72}Y_{25} = W_{54}Y_{72}Y_{47}Y_{25} = D_{54}X_{47}Y_{73}Y_{35} =$ $= W_{54}Y_{73}Y_{35}Y_{47} = W_{54}Y_{18}Y_{41}Y_{85} = D_{54}X_{85}Y_{18}Y_{41} = W_{54}Y_{13}Y_{35}Y_{41}$ |
| $M_{27}$ | $W_{54}X_{62}Y_{46}Y_{25} = W_{54}X_{85}Y_{46}Y_{68} = W_{54}X_{13}Y_{35}Y_{41} = W_{54}X_{47}Y_{72}Y_{25} =$ $= W_{54}X_{47}Y_{73}Y_{35} = W_{54}X_{85}Y_{18}Y_{41}$ |

**Table 11.** Generators of $Q^{1,1,1}/\mathbb{Z}_2$ in Phase L.



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## References

[1] A. Gadde, S. Gukov and P. Putrov, $(0,2)$ *trialities*, *JHEP* **03** (2014) 076 [arXiv:1310.0818] [INSPIRE].

[2] N. Seiberg, *Electric-magnetic duality in supersymmetric nonAbelian gauge theories*, *Nucl. Phys. B* **435** (1995) 129 [hep-th/9411149] [INSPIRE].

[3] S. Gukov, D. Pei and P. Putrov, *Trialities of minimally supersymmetric* 2*d gauge theories*, *JHEP* **04** (2020) 079 [arXiv:1910.13455] [INSPIRE].

[4] S. Franco, A. Mininno, A.M. Uranga and X. Yu, 2*d* $\mathcal{N}=(0,1)$ *Gauge Theories and* Spin(7) *Orientifolds*, arXiv:2110.03696 [INSPIRE].

[5] C.E. Beasley and M.R. Plesser, *Toric duality is Seiberg duality*, *JHEP* **12** (2001) 001 [hep-th/0109053] [INSPIRE].

[6] B. Feng, A. Hanany, Y.-H. He and A.M. Uranga, *Toric duality as Seiberg duality and brane diamonds*, *JHEP* **12** (2001) 035 [hep-th/0109063] [INSPIRE].

[7] S. Franco, S. Lee and R.-K. Seong, *Brane brick models and* 2*d* $(0,2)$ *triality*, *JHEP* **05** (2016) 020 [arXiv:1602.01834] [INSPIRE].

[8] S. Franco, S. Lee, R.-K. Seong and C. Vafa, *Quadrality for Supersymmetric Matrix Models*, *JHEP* **07** (2017) 053 [arXiv:1612.06859] [INSPIRE].

[9] S. Franco and G. Musiker, *Higher Cluster Categories and QFT Dualities*, *Phys. Rev. D* **98** (2018) 046021 [arXiv:1711.01270] [INSPIRE].

[10] C. Closset, S. Franco, J. Guo and A. Hasan, *Graded quivers and B-branes at Calabi-Yau singularities*, *JHEP* **03** (2019) 053 [arXiv:1811.07016] [INSPIRE].

[11] S. Franco, A. Hasan and X. Yu, *On the Classification of Duality Webs for Graded Quivers*, *JHEP* **06** (2020) 130 [arXiv:2001.08776] [INSPIRE].

[12] A. Gadde, S.S. Razamat and B. Willett, *On the reduction of* 4*d* $\mathcal{N}=1$ *theories on* $\mathbb{S}^2$, *JHEP* **11** (2015) 163 [arXiv:1506.08795] [INSPIRE].

[13] S. Franco, S. Lee, R.-K. Seong and C. Vafa, *Brane Brick Models in the Mirror*, *JHEP* **02** (2017) 106 [arXiv:1609.01723] [INSPIRE].

[14] S. Franco, S. Lee and R.-K. Seong, *Orbifold Reduction and* 2*d* $(0,2)$ *Gauge Theories*, *JHEP* **03** (2017) 016 [arXiv:1609.07144] [INSPIRE].

[15] S. Franco and A. Hasan, 3*d printing of* 2*d* $\mathcal{N}=(0,2)$ *gauge theories*, *JHEP* **05** (2018) 082 [arXiv:1801.00799] [INSPIRE].

[16] L.E. Ibáñez, R. Rabadán and A.M. Uranga, *Anomalous* U(1)*'s in type-I and type IIB* $D=4$, $N=1$ *string vacua*, *Nucl. Phys. B* **542** (1999) 112 [hep-th/9808139] [INSPIRE].

[17] K. Mohri, *D-branes and quotient singularities of Calabi-Yau fourfolds*, *Nucl. Phys. B* **521** (1998) 161 [hep-th/9707012] [INSPIRE].

[18] S. Franco and X. Yu, *$BFT_2$: a General Class of* 2*d*$\mathcal{N}=(0,2)$ *Theories,* 3*-Manifolds and Toric Geometry*, arXiv:2107.00667 [INSPIRE].



JHEP01(2022)058

[19] S. Franco, S. Lee and R.-K. Seong, *Brane Brick Models, Toric Calabi-Yau* 4*-Folds and* 2*d* (0, 2) *Quivers*, *JHEP* **02** (2016) 047 [arXiv:1510.01744] [INSPIRE].

[20] S. Franco, D. Ghim, S. Lee and R.-K. Seong, *Elliptic Genera of* 2*d* (0, 2) *Gauge Theories from Brane Brick Models*, *JHEP* **06** (2017) 068 [arXiv:1702.02948] [INSPIRE].

[21] H. Garcia-Compean and A.M. Uranga, *Brane box realization of chiral gauge theories in two-dimensions*, *Nucl. Phys. B* **539** (1999) 329 [hep-th/9806177] [INSPIRE].

[22] S. Franco, A. Hanany, D. Krefl, J. Park, A.M. Uranga and D. Vegh, *Dimers and orientifolds*, *JHEP* **09** (2007) 075 [arXiv:0707.0298] [INSPIRE].

[23] S. Benvenuti, B. Feng, A. Hanany and Y.-H. He, *Counting BPS Operators in Gauge Theories: Quivers, Syzygies and Plethystics*, *JHEP* **11** (2007) 050 [hep-th/0608050] [INSPIRE].

[24] B. Feng, A. Hanany and Y.-H. He, *Counting gauge invariants: The Plethystic program*, *JHEP* **03** (2007) 090 [hep-th/0701063] [INSPIRE].

[25] S. Franco, D. Ghim, S. Lee, R.-K. Seong and D. Yokoyama, 2*d* (0, 2) *Quiver Gauge Theories and D-branes*, *JHEP* **09** (2015) 072 [arXiv:1506.03818] [INSPIRE].

[26] N. Yamatsu, *Finite-Dimensional Lie Algebras and Their Representations for Unified Model Building*, arXiv:1511.08771 [INSPIRE].

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