arXiv:1602.03018v2 [gr-qc] 29 Dec 2016

A class of solutions to the Einstein equations with AVTD behavior in generalized wave gauges

Ellery Ames*, Florian Beyer†, James Isenberg‡, and Philippe G. LeFloch§

December 2016

Abstract

We establish the existence of smooth vacuum Gowdy solutions, which are asymptotically velocity term dominated (AVTD) and have T^3-spatial topology, in an infinite dimensional family of generalized wave gauges. These results show that the AVTD property, which has so far been known to hold for solutions in areal coordinates only, is stable to perturbations of the coordinate systems. Our proof is based on an analysis of the singular initial value problem for the Einstein vacuum equations in the generalized wave gauge formalism, and provides a framework which we anticipate to be useful for more general spacetimes.

Contents

*Department of Mathematical Sciences, Chalmers University of Technology, SE-412 96 Göteborg, Sweden. Email: ellery.ames@chalmers.se

†Department of Mathematics and Statistics, University of Otago, P.O. Box 56, Dunedin 9054, New Zealand. Email: fbeyer@maths.otago.ac.nz.

‡Department of Mathematics, University of Oregon, Eugene, OR 97403, USA. Email: isenberg@uoregon.edu.

§Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, Université Pierre et Marie Curie (Paris 6), 4 Place Jussieu, 75252 Paris, France. Email: contact@philippelefloch.org.

1 Introduction

One of the most compelling issues in mathematical relativity concerns the nature of the boundaries[1] of spacetimes that are evolved as solutions of Einstein's equations from specified initial data sets. The work of Choquet-Bruhat and Geroch [14, 10] shows that for every initial data set which satisfies the Einstein constraint equations, there is a unique "maximal development", which is a globally hyperbolic spacetime solution of the full Einstein system, is consistent with the specified initial data, and contains all such spacetimes (up to diffeomorphism). The work of Penrose and Hawking [25, 17] shows that many of the maximal development spacetimes are geodesically incomplete, and therefore have non-trivial boundaries. In some cases (e.g., the Schwarzschild solution) one cannot extend across the boundary, and it is characterized by unbounded curvature (and consequently unbounded tidal forces). In other cases (e.g., the Taub-NUT solutions) the boundary is a Cauchy horizon, and one can extend the spacetime smoothly into a region characterized by closed causal paths. The Strong Cosmic Censorship (SCC) conjecture (see the recent review in [18] for references) claims that, in generic spacetimes, one cannot smoothly extend solutions beyond the maximal development.

While the issue of strong cosmic censorship is wide open for the general class of solutions of Einstein's equations, a model version of the conjecture has been proven for the family of Gowdy spacetimes (which we describe below in Section 3). In the proof of this result [28], the verification that generic Gowdy solutions exhibit *asymptotically velocity-term dominated* (AVTD) behavior plays an important role.[2] Roughly speaking, a solution[3] (M, g, ψ) of the Einstein equations has AVTD behavior if there exists a system of coordinates for M such that, as one approaches the boundary of the spacetime, (M, g, ψ) asymptotically approaches a spacetime $(M, \hat{g}, \hat{\psi})$ which satisfies a system of equations (the VTD equations) which is the same as Einstein's equations but with most[4] of the terms containing spatial derivatives dropped. This property is very useful for studying SCC in a family of spacetimes $\mathcal{A}$ which is generically AVTD because it is often easier to calculate asymptotic curvature behavior in solutions of the VTD equations than in solutions of the Einstein equations. Thus, to prove a model SCC theorem for $\mathcal{A}$, it is sufficient to first show that generic solutions of Einstein's equations in $\mathcal{A}$ are AVTD, and then show that generic solutions of the VTD equations corresponding to $\mathcal{A}$ have unbounded curvature. It follows that generic solutions of the Einstein equations contained in $\mathcal{A}$ cannot be extended.

As noted above, to verify that a given spacetime is AVTD, one needs to find *some* coordinate system in terms of which the asymptotic condition described above holds. For the Gowdy spacetimes, the *areal coordinate* system is geometrically natural, and AVTD behavior has been verified using areal coordinates [23, 26, 28]. Does it follow that the vacuum Gowdy spacetimes are manifestly AVTD in terms of other coordinate systems as well? This is the question we address in this paper.

Theorem 4.1 below establishes the existence of a wide class of smooth AVTD Gowdy solutions to the vacuum Einstein equations in an infinite dimensional family of coordinates which contains the areal coordinates. While this result does not determine that Gowdy spacetimes are manifestly AVTD in terms of any choice of coordinates, it does show for the first time that the AVTD property of Gowdy spacetimes is not limited to areal coordinates, and that this property is in a sense stable to coordinate perturbations. The family of coordinates which we consider are generated by a certain class of gauge source functions using the generalized wave gauge formulation of the Einstein equations.

Generalized wave coordinates (which we define below, in Section 3) are important for two reasons: First, unlike areal coordinates, which are well-defined only for families of spacetimes with two commuting Killing fields, generalized wave coordinates are defined for all spacetimes. Second, in terms of wave coordinates, the Einstein equations take a manifestly hyperbolic form.

[1]These are often referred to as spacetime "singularities"; however, in view of the ambiguity of the term "singularity", we avoid that term here.

[2]The role of AVTD behavior in proving model SCC theorems is especially evident in the proof of SCC for the polarized Gowdy spacetimes; see [21, 12]. The idea of this role stems from the original work in [13].

[3]Here, M is the spacetime manifold, g is the spacetime metric, and ψ represents any non-gravitational fields. We presume that this solution is the maximal development of a set of initial data.

[4]But not necessarily all; see e.g., [21] or [1].

Such a form is essential for carrying out the analyses we use here to verify AVTD behavior.

The Gowdy family of solutions is a useful laboratory for studying a variety of issues in mathematical relativity because the Einstein system of equations for the Gowdy family, while retaining the nonlinearities, the constraints, and the gauge freedom which mark the Einstein system generally, is relatively accessible to analysis. We already know (using areal coordinates) that generic T^3 Gowdy spacetimes are AVTD. The motivation for the present study is to learn about this property in systems of coordinates which can be used in wider families of solutions, such as $U(1)$-symmetric vacuum solutions (currently work in progress by us) or solutions with matter fields [9], as well as solutions with no symmetries.

Based on numerical studies [15, 5, 4, 6] we do not expect the general class of solutions of the vacuum Einstein equations (with no symmetries imposed) to exhibit AVTD behavior in any choice of coordinates. However, such studies do suggest that polarized $U(1)$-symmetric solutions[1] of the vacuum Einstein equations as well as general solutions of the Einstein-scalar field equations do exhibit AVTD behavior. The existence of *analytic* spacetimes in these families of solutions which show AVTD behavior has been confirmed using analytic Fuchsian techniques [22, 11, 3]. A major part of the motivation for the present study is to develop the tools needed to show that *non-analytic* spacetimes (smooth, or with less regularity) in these families also show AVTD behavior. This is crucial because the use of the AVTD property for studying model versions of the strong cosmic censorship conjecture for a given family of spacetimes requires that AVTD behavior be ascertained in non-analytic as well as analytic spacetimes in the chosen family.

The mathematical basis for our work here is our analysis in [1] of the *Singular Initial Value Problem* (SIVP) for quasilinear symmetric-hyperbolic (non-analytic) PDE systems. The basic idea, we recall, is the following: We seek solutions $u : \Sigma^n \times \mathbb{R}^+ \to \mathbb{R}^d$ which satisfy the system of equations

$$\mathcal{E}[u, x, t] = 0, \tag{1.1}$$

where $\mathcal{E}$ is a function of u and its (first) derivatives[2], as well as a function of spacetime. In the standard Cauchy initial value problem for this system, we seek a solution of (1.1) which agrees with a specified set of initial data $u_{[t_0]} : \Sigma^n \times \{t_0\} \to \mathbb{R}^d$, for some $t_0 \in \mathbb{R}^+$. By contrast, in working with the singular initial value problem, we seek a solution of (1.1) for which $t = 0$ marks the boundary of the maximal development of some (unspecified) initial data set, and which asymptotically agrees with a specified "leading order term" $u_* : \Sigma^n \times (0, \delta] \to \mathbb{R}^d$ for some $\delta > 0$.

We discuss in Section 2 sets of conditions on the PDE system (1.1) and on the leading order term u_* which guarantee that indeed there is a (unique) solution u of $\mathcal{E}[u, x, t] = 0$ which asymptotically approaches u_* at the desired rate. These SIVP well-posedness results,[3] stated in Theorem 2.10, are adapted from [1]; note however that our statement of SIVP well-posedness here is somewhat simpler than in [1], partly as a consequence of our introducing the convenient notion of *function operators* (see Section 2 below) and partly because we focus here on the smooth category rather than working with less regular solutions. With some additional effort, one can show that the same techniques do also cover solutions with finite differentiability. In any case, a key feature of our work is that our results do not require real-analyticity.

The SIVP approach is especially well-suited for proving that spacetimes admit AVTD behavior, since both focus on the asymptotic behavior of solutions of PDE systems. In particular, to show that a solution of (1.1) has AVTD behavior, it is sufficient to choose the leading order term u_* so that it either satisfies a set of VTD equations corresponding to (1.1) or asymptotically approaches a solution of such equations, and then show that the singular initial value problem for $\mathcal{E}[u, x, t] = 0$ with leading order term u_* is well-posed.

In previous studies of AVTD behavior in Gowdy spacetimes [26, 7, 8], areal coordinates have been chosen from the start, and the analysis has been carried out in terms of metric components

[1]The metrics of the $U(1)$-symmetric solutions which are labelled *polarized* can be written as warped products of $(2 + 1)$-dimensional Lorentz-signature metrics with the circle orbits of the (spatially-acting) isometry group.

[2]For convenience, we presume that the system has been cast into first-order form.

[3]A well-posedness theorem for the standard Cauchy problem implies continuity of the map from initial data sets to local solutions, as well as local existence and uniqueness. We label a singular initial value problem well-posed so long as local existence and uniqueness hold; we are not concerned with continuity.

and PDEs defined by a fixed areal coordinate basis. Here, working with generalized wave coordinates, we must proceed differently. Since generalized wave coordinates are defined *dynamically*, through solutions of a wave-type system of equations, we work with a coupled system which combines the Einstein equations (in generalized wave coordinates) with these dynamic equations for the coordinates. It is for this combined system that we seek to set up a singular initial value problem, which we use to verify the existence of Gowdy solutions which are AVTD with respect to generalized wave coordinates.

Since the singular initial value problem plays a central role in our work here, we present a brief review of it in Section 2. Included in this section is a review of the weighted function spaces we use in our analysis here, along with a well-posedness theorem for singular initial value problems of the sort which arise in this work. Next, in Section 3, we discuss generalized wave coordinates and their use as a tool for working with all solutions of Einstein's equations, we introduce the Gowdy solutions and their representations in various coordinate systems, and we discuss particular versions of generalized wave coordinates which are useful in generating Gowdy solutions which are manifestly AVTD in terms of those coordinates. Also in this section we examine Kasner spacetimes, and use them to help determine the asymptotic form Gowdy solutions (metric and coordinates) should take, if they are to show AVTD behavior. This allows us to specify the appropriate form for the leading order term for our SIVP. Our main result is presented in Section 4 where we verify that the SIVP we have set up is indeed well-posed, and hence there are Gowdy solutions which are manifestly AVTD in terms of generalized wave coordinates. In Section 5 we discuss the solution space of our main theorem and thereby relate the AVTD solutions obtained in the previous section in generalized wave coordinates to AVTD solutions obtained in the more conventional areal coordinates.

2 Review of the singular initial value problem

As noted above, the idea of the singular initial value problem for a given PDE system is to find solutions to that system which have prescribed asymptotic behavior in the neighborhood of a designated "boundary" or "singularity". To be able to carefully define asymptotic convergence and state conditions for the well-posedness of the SIVP, we briefly review a class of time-weighted function spaces and a set of function operators on these spaces; details regarding these spaces and their properties appear in [1]. We use these spaces and operators to define the sorts of equations for which the SIVP is well-posed, and we then state a well-posedness theorem for the singular initial value problem in a form which is most useful for the present work. In particular, the theorem stated here is restricted to the smooth (but not real-analytic) setting, and to operators which are rational function operators. Also, for simplicity we presume that the spatial sections of the spacetime manifold on which we work are topologically T^n.

Function spaces and function operators

To define the family of time-weighted Sobolev spaces, we choose $\mu : T^n \to \mathbb{R}^d$ to be a smooth function, we define the $d \times d$-matrix

$$\mathcal{R}[\mu](t,x) := \operatorname{diag}\left(t^{-\mu_1(x)}, \dots, t^{-\mu_d(x)}\right), \tag{2.1}$$

and then for functions $w : (0,\delta] \times T^n \mapsto \mathbb{R}^d$ in $C^\infty((0,\delta] \times T^n)$ we specify the norm

$$\begin{aligned} ||w||_{\delta,\mu,q} &:= \sup_{t\in(0,\delta]} ||\mathcal{R}[\mu]w||_{H^q(T^n)} \\ &= \sup_{t\in(0,\delta]} \left(\sum_{|\alpha|=0}^{q} \int_{T^n} |\partial^\alpha(\mathcal{R}[\mu]w)|^2 dx \right)^{1/2}. \end{aligned} \tag{2.2}$$

Here $H^q(T^n)$ denotes the usual Sobolev space of order q on the n-torus T^n, α denotes a partial derivative multi-index, and the standard Lebesgue measure is used for the integration. Based on this norm, we define the function space $X_{\delta,\mu,q}(T^n)$ to be the completion of the set of functions $w \in C^\infty((0,\delta] \times T^n)$ for which the above norm is finite. Since the spatial sections are understood to be T^n, for convenience we generally drop the T^n argument, denoting these spaces as $X_{\delta,\mu,q}$. We let $B_{\delta,\mu,q,r}$ denote a closed ball of radius r about 0 in $X_{\delta,\mu,q}$. To handle functions which are infinitely differentiable and for which we control all derivatives (the "smooth case"), we also define the space $X_{\delta,\mu,\infty} := \cap_{q=0}^{\infty} X_{\delta,\mu,q}$. In the following, we refer to the quantity μ as an *exponent vector*, or if, $d = 1$, as *exponent scalar*. If we have two exponent vectors ν and μ of the same dimension, we write $\nu > \mu$ if each component of ν is strictly larger than the corresponding component of μ at each spatial point. If μ is an exponent vector and γ an exponent scalar, then $\mu + \gamma$ refers to the exponent vector with components $\mu_i + \gamma$.

In working with $d \times d$-matrix-valued functions (such as S^0 in Eq. (2.10) below), we use analogous norms and function spaces. In these cases, we consider d-vector-valued exponents ξ as above and then define the space $X_{\delta,\xi,q}$ of $d \times d$-matrix-valued functions S in the same way as above, but with $\mathcal{R}[\mu]w$ in Eq. (2.2) replaced by $\mathcal{R}[\xi] \cdot S$ (where $\cdot$ denotes the matrix product). We note that this definition is a special case of that used in [1, 2]; it is sufficient for our purposes here.

We next introduce a class of maps which we label as *function operators*. Though not discussed in previous work [1, 2], these function operators and their properties are very useful for discussing the regularity and asymptotic time behavior of the coefficients (S^0, S^a, N) and source terms (f) appearing in the PDEs Eq. (2.10) we consider here. Formally, we define a function operator to be a map g from functions $w : (0,\delta] \times T^n \to \mathbb{R}^d$ to functions $g(w) : (0,\delta] \times T^n \to \mathbb{R}^m$, where d and m are positive integers. A particularly important class of such objects may be constructed as follows. Let γ be a specified continuous function

$$\gamma : (0,\delta] \times T^n \times U \to \mathbb{R}^m, \quad (t,x,u) \mapsto \gamma(t,x,u), \tag{2.3}$$

where U is an open subset of $\mathbb{R}^d$. Associated to γ is the function operator g which maps functions $w : (0,\delta] \times T^n \to \mathbb{R}^d$ whose range is a subset of U, to functions $g(w)$ defined by

$$g(w) : (0,\delta] \times T^n \to \mathbb{R}^m, \quad g(w)(t,x) := \gamma(t,x,w(t,x)). \tag{2.4}$$

For our work here we need precise control of the domain and range of such maps. To attain this, we require that the domain and range of the function operators we use here both be subsets of time-weighted Sobolev spaces of the sort defined above. Specifically, for fixed dimension index n (referring to T^n), for exponent d-vector μ (possibly zero), for exponent m-vector ν, and for differentiability index q (possibly ∞) we define the indexed classes $\mathcal{G}_{[\delta;\mu,\nu,q]}$ of function operators as follows: First, for $\mu = 0$ and for finite q, we have

Definition 2.1 ($\mathcal{G}_{[\delta;0,\nu,q]}$)**.** *Fix positive integers n, d, m and $q > n/2$, and fix $\delta > 0$. For any real number $s_0 > 0$ or $s_0 = \infty$, let*

$$H_{\delta,q,s_0} := \left\{ w : (0,\delta] \times T^n \to \mathbb{R}^d \text{ in } X_{\delta,0,q} \;\middle|\; \sup_{t\in(0,\delta]} \|w(t)\|_{L^\infty(T^n)} \le s_0 \right\}. \tag{2.5}$$

Let ν be an exponent m-vector. For any $w \in H_{\delta,q,s_0}$, we call a map $w \mapsto g(w)$ a $(0,\nu,q)$-operator (an element of $\mathcal{G}_{[\delta;0,\nu,q]}$) provided that the following hold:

(i) There exists a constant $s_0 > 0$ ($s_0 = \infty$ is allowed) such that for each $\delta' \in (0,\delta]$ and for all $w \in H_{\delta',q,s_0}$, the image $g(w)$ is a well-defined function $g(w) : (0,\delta'] \times T^n \to \mathbb{R}^m$ contained in $X_{\delta',\nu,q}$.

(ii) There exists a constant $C > 0$ such that for each $\delta' \in (0,\delta]$ and for each $q' = q$ and $q' = q-1$, and for all $w, \tilde{w} \in H_{\delta',q,s_0}$, the following local Lipschitz estimate holds

$$\|g(w) - g(\tilde{w})\|_{\delta',\nu,q'} \le C\,(1 + \|w\|_{\delta',0,q'} + \|\tilde{w}\|_{\delta',0,q'})\,\|w - \tilde{w}\|_{\delta',0,q'}. \tag{2.6}$$

Before continuing on to define $\mathcal{G}_{[\delta;\mu,\nu,q]}$ and $\mathcal{G}_{[\delta;\mu,\nu,\infty]}$, we note the following:

1. As mentioned above, in some cases (not all), a function operator may be specified by choosing a continuous function $\gamma : (0, \delta] \times T^n \times U \to \mathbb{R}^m$ and defining the map $w \mapsto g(w)$ as in Eq. (2.4). However, in doing so we must be able to choose the constant s_0 in Definition 2.1 so that the ranges of the functions $w \in H_{\delta,q,s_0}$ are contained in the open set U. If this can be done and if Condition (i) holds for $\delta' = \delta$, then it automatically holds for every $\delta' \in (0, \delta]$. We shall often make use of this fact without further notice.

2. If $w \in B_{\delta,0,q,s_0/C_{q,n}}$ where $C_{q,n}$ is the Sobolev embedding constant for $H^q(T^n)$, then

$$\sup_{t\in(0,\delta]} \|w(t)\|_{L^\infty(T^n)} \le C_{q,n} \sup_{t\in(0,\delta]} \|w(t)\|_{H^q(T^n)} \le s_0. \tag{2.7}$$

   Hence, $w \in H_{\delta,q,s_0}$ and therefore $B_{\delta,0,q,s_0/C_{q,n}} \subset H_{\delta,q,s_0}$.

3. Elements of $\mathcal{G}_{[\delta;0,\nu,q]}$ are uniformly bounded in the following sense: Let w be an arbitrary function in $B_{\delta,0,q,\tilde{s}_0}$ for some sufficiently small $\tilde{s}_0 > 0$. It follows from the above remark that the map $w \mapsto g(w)$ is well-defined, and that

$$\|g(w)\|_{\delta,\nu,q} \le \|g(0)\|_{\delta,\nu,q} + C\|w\|_{\delta,0,q} \le \|g(0)\|_{\delta,\nu,q} + C\tilde{s}_0.$$

Definition 2.2 ($\mathcal{G}_{[\delta;\mu,\nu,q]}$ and $\mathcal{G}_{[\delta;\mu,\nu,\infty]}$). *Fix positive integers n, d, m and $q > n/2$, and fix $\delta > 0$. Let ν be an exponent m-vector and let μ be an exponent d-vector. We call the map $w \mapsto g(w)$ a (μ, ν, q)-operator (an element of $\mathcal{G}_{[\delta;\mu,\nu,q]}$) if the map $w \mapsto g(\mathcal{R}[-\mu]w)$ is a $(0, \nu, q)$-operator. We call the map $w \mapsto g(w)$ a (μ, ν, ∞)-operator (an element of $\mathcal{G}_{[\delta;\mu,\nu,\infty]}$) if there exists an integer $p > n/2$ such that $w \mapsto g(w)$ is a (μ, ν, q)-operator for each $q \ge p$, with a common constant s_0 for all $q \ge p$.*

In the "smooth case" $q = \infty$, notice that we do *not* make any assumptions regarding the dependence on q of the constant C in Condition (ii) above.

It is useful for our analysis below to state a technical result which permits us in certain circumstances to evaluate a given function operator on a function which is not a-priori known to be in the domain of that function operator (cf. Condition (i) of Definition 2.1).

Lemma 2.3. *Fix positive integers n, d and m, and fix $\delta > 0$. Let ν be an exponent m-vector and let μ be an exponent d-vector. Suppose that $g \in \mathcal{G}_{[\delta;\mu-\epsilon,\nu,\infty]}$ for all values of $\epsilon \in [0, \epsilon_0]$ where $\epsilon_0 > 0$ is some (possibly very small) constant. If w is any d-vector valued function in $X_{\delta,\mu,\infty}$, then g is well-defined and $g(w) \in X_{\hat{\delta},\nu,\infty}$ for some $\hat{\delta} \in (0, \delta]$.*

We now wish to define certain special classes of function operators. All are constructed via the model described above in Eq. (2.4); the special classes are defined by the form of the function γ. For fixed choices of positive integers n, d and N, and for fixed $\delta > 0$, if $\gamma : (0, \delta] \times T^n \times \mathbb{R}^d \to \mathbb{R}$ takes the following polynomial form

$$\gamma(t, x, u) = \sum_{i_1,\ldots,i_d=0}^{N} \gamma_{i_1,\ldots,i_d}(t, x)\, u_1^{i_1} \cdots u_d^{i_d} \tag{2.8}$$

for some collection of coefficient functions $\gamma_{i_1,\ldots,i_d}(t, x)$ in $X_{\delta,\tilde{\nu}_{i_1,\ldots,i_d},\infty}$ (here $\tilde{\nu}_{i_1,\ldots,i_d}$ is a set of scalar exponents) then we call the function operator $w \mapsto \gamma(w)$ associated to this polynomial γ a *scalar polynomial function operator*. If the function operator is constructed in this way, but with γ being d-vector-valued or $d \times d$-matrix-valued, then the result is labeled a *vector (or matrix) polynomial function operator*. Equivalently, a vector or matrix polynomial function operator is one such that each component is a scalar function polynomial operator.

Finally, taking quotients of polynomial function operators, we define the following class of function operators (which play a major role in our analysis here):

Definition 2.4. *Suppose that h_0 is a scalar-valued function in $X_{\delta,\eta,\infty}$ (for some scalar exponent η) such that $1/h_0 \in X_{\delta,-\eta,\infty}$. Let $w \mapsto P_1(w)$ and $w \mapsto P_2(w)$ be a pair of scalar polynomial function operators such that $w \mapsto P_2(w)$ is a (μ, ζ, ∞)-operator for a scalar exponent $\zeta > 0$. Then*

$$w \mapsto H(w) := \frac{P_1(w)}{(1 + P_2(w))h_0} \tag{2.9}$$

is called a scalar rational function operator. If $w \mapsto F(w)$ is a d-vector-valued (or $d \times d$-matrix-valued) function operator such that each component $w \mapsto F_j(w)$ (or $w \mapsto F_{ij}(w)$) is a scalar rational function operator, then $w \mapsto F(w)$ is labeled a vector (or matrix) rational function operator.

Lemma 2.5. *Suppose that $w \mapsto H(w)$ is a scalar rational function operator specified by Eq. (2.9) for some choice of P_1, P_2 and h_0 as in Definition 2.4. Assume in addition that $w \mapsto P_1(w)$ is a (μ, ν, q)-operator for another scalar exponent ν. Then $w \mapsto H(w)$ is a $(\mu, \nu - \eta, q)$-operator.*

This lemma can be proved with tools from the discussion in [1] and can be easily extended to vector and matrix rational function operators.

Class of equations

Our results in this paper rely on working with singular initial value problems for which the partial differential equation system can be cast into the following first-order form:

$$\begin{aligned} S^0(t,x,u(t,x))Du(t,x) + \sum_{a=1}^{n} S^a(t,x,u(t,x))t\partial_a u(t,x) + N(t,x,u(t,x))u(t,x) \\ = f(t,x,u(t,x)). \end{aligned} \tag{2.10}$$

Here $u : (0,\delta] \times T^n \to \mathbb{R}^d$ is the vector-valued function for which the SIVP is to be solved, each of the $n+2$ maps $S^0, \dots, S^n$ and N is a specified $d \times d$ matrix-valued function of the spacetime coordinates $(t,x) \in (0,\delta] \times T^n$ and of the unknown u (but not of the derivatives of u), while $f = f(t,x,u)$ is a $\mathbb{R}^d$–valued function of (t,x,u). All matrices $S^0, \dots, S^n$ are assumed to be symmetric. We set $D := t\,\partial_t = t\frac{\partial}{\partial t} = x^0 \frac{\partial}{\partial x^0}$, while $\partial_a := \frac{\partial}{\partial x^a}$ for[1] $a = 1, \dots, n$. We note that while one could incorporate the term $N(t,x,u)u$ into the source term $f(t,x,u)$, for stating the conditions we need to impose on the coefficients of the PDE (2.10) for well-posedness of the SIVP as well as other requirements, it is convenient to keep these terms separate. We also note that this is the form we have used in our previous studies [1] (for $n = 1$) and [2] (for general n). For convenience, we define the differential operator $\widehat{L}$ as follows:

$$\widehat{L}(u)[v] := \sum_{j=0}^{n} S^j(u)\, t\partial_j v + N(u)v. \tag{2.11}$$

Thus the PDE (2.10) can be written as

$$\widehat{L}(u)[u] = f(u), \tag{2.12}$$

where $f(u)$ denotes the right-hand side of (2.10).

If, for a class of initial data sets, S^0 is a positive-definite matrix (in the sense of eigenvalues) at each spacetime point (t,x) with $t \neq 0$, then the system Eq. (2.10) is symmetric hyperbolic, and the corresponding Cauchy problem for initial data chosen at $t_0 > 0$ is well-posed. To study the singular initial value problem for Eq. (2.10), we prescribe a leading order term u_* and seek a solution $u = u_* + w$ for Eq. (2.10) with w in some specified function space with prescribed $t \searrow 0$ behavior of w. Substituting $u = u_* + w$ into Eq. (2.10), one obtains a PDE system for w which takes the form

$$\widehat{L}(u_* + w)[w] = \mathcal{F}(u_*)[w] := f(u_* + w) - \widehat{L}(u_* + w)[u_*]. \tag{2.13}$$

The operator $\mathcal{F}(u_*)[w]$ is often referred to as the *reduced source term operator*. For a fixed u_*, the problem of existence and uniqueness for the singular initial value problem is now equivalent to establishing the existence and uniqueness of a solution w to Eq. (2.13) in the specified function space. The key definition for studying this issue is the following:

[1] In all of what follows, indices $i, j, \dots$ run over $0, 1, \dots, n$, while indices $a, b, \dots$ take the values $1, \dots, n$.

Definition 2.6. *The PDE system Eq.* (2.10) *is a quasilinear symmetric hyperbolic Fuchsian system around a specified leading order term* $u_* \in C^\infty((0,\delta] \times T^n)$ *for parameters* $\delta > 0$ *and* μ *if there exists a positive-definite and symmetric matrix-valued function* $S_0^0(u_*) \in C^\infty(T^n)$ *and a matrix-valued function* $N_0(u_*) \in C^\infty(T^n)$*, such that all of the following function operators are* (μ, ζ, ∞)*-operators for some* $\zeta > 0$*:*

$$w \mapsto N(u_* + w) - N_0(u_*), \tag{2.14}$$

$$w \mapsto S_1^0(u_* + w) := S^0(u_* + w) - S_0^0(u_*), \tag{2.15}$$

$$w \mapsto tS^a(u_* + w), \tag{2.16}$$

$$w \mapsto \mathcal{R}[\mu]\mathcal{F}(u_*)[w]. \tag{2.17}$$

If all of the function operators are rational in the sense of Definition 2.4, then the PDE system is labeled a quasilinear symmetric hyperbolic rational-Fuchsian system. If the functions $S^0(t,x,u)$*,* $S^a(t,x,u)$*,* $N(t,x,u)$ *and* $f(t,x,u)$ *appearing* (2.10) *are all explicitly smooth, then the system is labeled a smooth quasilinear symmetric hyperbolic Fuchsian system.*

While Definition 2.6 appears to be different from the one given in [1, 2], it is essentially the same. The definition given here does not involve the splitting of $tS^a(u_* + w)$ that is carried out in Definition 2.2 of [1]; that splitting, however, is not really needed to state the (equivalent) conditions imposed on S^0, S^a, N, f and u_* in order to define a quasilinear symmetric hyperbolic Fuchsian system. We do add qualifications here – smoothness and rationality of the function operators. However, in our work below, these qualifications hold *only* if stated explicitly. We note that most of the results we prove here can be extended to finitely differential operators and to function operators which are not rational; to simplify the discussion, we impose these restrictions in our applications below.

We notice that, as a consequence of the requirement in this definition that the function operators defined in Eqs. (2.14) and (2.15) be (μ, ζ, ∞)-operators, it follows that for each choice of the remainder w in the specified space, the (t,x)-dependent functions $S^0(t,x,u_*(t,x)+w(t,x))$ and $N(t,x,u_*(t,x) + w(t,x))$ are $O(1)$ in the limit $t \searrow 0$. This is a relatively strong restriction. Indeed in practice, to satisfy this condition it may be necessary to multiply the whole system of equations by some power of t. Moreover, there are some example cases in which this condition can only be satisfied if one multiplies the whole system by a *matrix* of time weights; as a consequence, the symmetry of the coefficient matrices may be destroyed. Such examples suggest that our definition of quasilinear symmetric hyperbolic Fuchsian systems may be too restrictive for some purposes. However for the application discussed in this paper, Definition 2.6 is sufficient.

Well-posedness of the singular initial value problem for Fuchsian systems

The main existence and uniqueness result for the SIVP for Fuchsian systems relies on additional structural conditions on the matrix functions appearing in Eq. (2.10). To state these conditions, we use the following definition:

Definition 2.7 (Block diagonality with respect to μ)**.** *Suppose that* $M : (0,\delta] \times T^n \to \mathbb{R}^{d\times d}$ *is any smooth* $d \times d$*-matrix-valued function, and that* μ *is some* d*-vector-valued exponent.* M *is called block diagonal with respect to* μ *provided that*

$$M(t,x)\mathcal{R}[\mu](t,x) - \mathcal{R}[\mu](t,x)M(t,x) = 0,$$

(recall the definition of $\mathcal{R}[\mu]$ *given in Eq.* (2.1)*) for all* $(t,x) \in (0,\delta] \times T^n$.

The following simple algebraic result motivates this terminology.

Lemma 2.8. *Let* μ *be a* d*-vector-valued exponent which is ordered, in the sense that*

$$\mu(x) = \Big(\underbrace{\mu^{(1)}(x), \ldots, \mu^{(1)}(x)}_{d_1\text{-times}}, \underbrace{\mu^{(2)}(x), \ldots, \mu^{(2)}(x)}_{d_2\text{-times}}, \ldots, \underbrace{\mu^{(l)}(x), \ldots, \mu^{(l)}(x)}_{d_l\text{-times}}\Big), \tag{2.18}$$

where

- $l \in \{1, \dots, d\}$,
- $\mu^{(i)} \neq \mu^{(j)}$ *for all* $i \neq j \in \{1, \dots, l\}$,
- $d_1, \dots, d_l$ *are positive integers with* $d_1 + d_2 + \dots + d_l = d$.

Then any smooth $d \times d$*-matrix-valued function* M *is block diagonal with respect to* μ *if and only if* M *is of the form*

$$M(t,x) = \operatorname{diag}\Big(M^{(1)}(t,x), \dots, M^{(l)}(t,x)\Big), \tag{2.19}$$

where each $M^{(i)}(t,x)$ *is a smooth* $d_i \times d_i$*-matrix-valued function. Moreover, if* ν *is any other* d*-vector-valued exponent with the same ordering as* μ*, in the sense that*

$$\nu(x) = \Big(\underbrace{\nu^{(1)}(x), \dots, \nu^{(1)}(x)}_{d_1\text{-times}}, \underbrace{\nu^{(2)}(x), \dots, \nu^{(2)}(x)}_{d_2\text{-times}} \dots, \underbrace{\nu^{(l)}(x), \dots, \nu^{(l)}(x)}_{d_l\text{-times}}\Big),$$

for the same integers $d_1, \dots, d_l$*, then* M *is also block diagonal with respect to* ν.

We now use this notion of block diagonality to characterize the SIVP for Eq. (2.10) with a specified leading order term u_*.

Definition 2.9. *Fixing a finite integer* $q > n/2 + 2$ *and a constant* $\delta > 0$*, suppose that* u_* *is a given leading order term and* μ *is an exponent vector. The system (2.10) is called block diagonal with respect to* μ *if, for all* $u = u_* + w$ *with* $w \in X_{\delta,\mu,q}$ *for which* $S^j(u)$ *and* $N(u)$ *are defined, these matrix-valued functions are block diagonal with respect to* μ.

This diagonality condition is essential for deriving the energy estimates which are needed for the proof of the SIVP well-posedness theorem below. It ensures that both of the matrices $S^j(u)$ and $\mathcal{R}[\mu]S^j(u)\mathcal{R}[-\mu]$ are symmetric. Moreover, it guarantees that the differential operator $\widehat{L}(u)[u]$ (see Eq. (2.11)) only couples those components of the unknown function u which decay in t at the same rate.

To proceed, we assume that the system Eq. (2.10) is block diagonal with respect to μ (see Definition 2.9) and that μ is ordered (as in Eq. (2.18)) and hence, according to Lemma 2.8, all matrices in $\widehat{L}(u)[u]$ have the same block diagonal structure as in Eq. (2.19). In particular, the matrix

$$\mathcal{N} = \mathcal{N}(u_*) := \left(S_0^0(u_*)\right)^{-1} N_0(u_*), \tag{2.20}$$

is block diagonal with respect to μ in the sense of Definition 2.7 because it has the same block structure as do all matrices in $\widehat{L}(u)[u]$. Here we note that since Definition 2.6 requires that $S_0^0(u_*)$ be invertible, it follows that $\mathcal{N}$ is well-defined. We use

$$\Lambda := (\lambda_1, \dots, \lambda_d), \tag{2.21}$$

to denote the list of (in general complex-valued) eigenvalues of $\mathcal{N}$, sorted so that the eigenvalues corresponding to each block of $\mathcal{N}$ are listed sequentially.

With these prerequisites established, we state a well-posedness theorem for the singular initial value problem for PDE systems of the type we consider in this work.

Theorem 2.10. *Suppose, for some choice of an ordered exponent vector* μ*, a positive real number* δ*, and a leading order term* u_**, that Eq.* (2.10) *is a smooth quasilinear symmetric hyperbolic rational-Fuchsian system around* u_**, as specified in Definition 2.6. Suppose that Eq.* (2.10) *is block diagonal with respect to* μ *and that*

$$\mu > -\operatorname{Re}\Lambda, \tag{2.22}$$

where Λ *is defined in Eq.* (2.21)*. Then there exists a unique solution* u *to Eq.* (2.10) *with remainder* $w := u - u_*$ *belonging to* $X_{\widetilde{\delta},\mu,\infty}$ *for some* $\widetilde{\delta} \in (0, \delta]$*. Moreover,* $Dw \in X_{\widetilde{\delta},\mu,\infty}$.

The proof of this theorem essentially follows that of Theorem 2.21 in [1]. As noted above, the statement of Theorem 2.10 here is considerably simpler than that of Theorem 2.21 in [1], because the requirement here that the PDE system Eq. (2.10) be *rational* automatically implies the extra technical conditions which appear in the latter case.

3 T^3-Gowdy spacetimes and generalized wave coordinates

In this section, we begin by describing what generalized wave coordinates are, and how they are used in studying general solutions of Einstein's equations. We next introduce the T^3-Gowdy spacetimes, writing them both in areal coordinates and in a general form more suited for studies involving other gauge choices. We then apply generalized wave coordinates to the Gowdy spacetimes. In doing this, we use the Kasner solutions (a subset of the T^3-Gowdy spacetimes) to aid us in choosing generators of generalized wave coordinates which lead to explicit AVTD behavior.

Generalized wave coordinate gauges

The idea of the generalized wave coordinate gauges is to cast the vacuum Einstein equations $G_{ij} = 0$ into an explicit (coordinate-dependent) form which is manifestly a (quasilinear) hyperbolic PDE system for the spacetime metric[1] g_{ij}. The fact that such coordinates can be chosen for any globally hyperbolic spacetime satisfying Einstein's equations depends upon the following readily-verified key observations:

I) Let $\mathcal{F}_i$ be any specified set of four smooth spacetime functions, let ${C_{ij}}^k$ (satisfying the condition ${C_{[ij]}}^k = 0$) be any chosen set of twenty-four smooth spacetime functions, and let $\Gamma_{kmi} := \frac{1}{2}\left(\partial_k g_{mi} + \partial_i g_{mk} - \partial_m g_{ki}\right)$ and $\Gamma_m := g^{ki}\Gamma_{kmi}$ denote the indicated Levi-Civita connections quantities. The vacuum Einstein equations are equivalent to the (coordinate-dependent) system

$$\begin{aligned} -\frac{1}{2}g^{kl}\partial_k\partial_l g_{ij} + \nabla_{(i}\mathcal{F}_{j)} + g^{kl}g^{mn}\left(\Gamma_{kmi}\Gamma_{lnj} + \Gamma_{kmi}\Gamma_{ljn} + \Gamma_{kmj}\Gamma_{lin}\right) \\ + {C_{ij}}^k(\mathcal{F}_k - \Gamma_k) = 0 \end{aligned} \tag{3.1}$$

if and only if $\mathcal{F}_k - \Gamma_k = 0$. For any fixed choice of $\mathcal{F}_k$, Eq. (3.1) is a quasilinear hyperbolic system for g_{ij}.

II) For any set of initial data consisting of a Riemannian metric γ and a symmetric tensor K satisfying the Einstein constraint equations $G_{0j} = 0$, for any spacetime metric g which is compatible with this choice of initial data[2], and for any choice of the four spacetime functions $\mathcal{F}_i$, there exists a system of spacetime coordinates in terms of which the quantity

$$\mathcal{D}_i := \mathcal{F}_i - \Gamma_i \tag{3.2}$$

vanishes at $t = t_0$ (corresponding to the spacelike slice on which g induces (γ, K)).

III) If, at $t = t_0$, the spacetime metric g_{ij} satisfies the evolution equations Eq. (3.1) and induces initial data satisfying the constraints, and if coordinates have been chosen so that $\mathcal{D}_i = 0$ at $t = t_0$, then it follows that $\partial_t\mathcal{D}_i = 0$ at $t = t_0$. This can be seen from the following relation

$$G^{i0} = -\frac{1}{2}g^{00}g^{ij}\partial_t\mathcal{D}_j, \tag{3.3}$$

which is satisfied at $t = t_0$ if the metric g_{ij} satisfies the Einstein evolution equations, and if coordinates have been chosen such that $\mathcal{D}_i = 0$.

IV) For any choice of the spacetime metric g which satisfies Eq. (3.1) for given functions $\mathcal{F}_i$ and ${C_{ij}}^k$, the Bianchi identities on g imply that the quantity $\mathcal{D}_i$ satisfies the PDE system

$$\nabla^i\nabla_i\mathcal{D}_j + {R_j}^l\mathcal{D}_l + \left(2\nabla_i {C^i}_j{}^k - \nabla_j {C_l}^{lk}\right)\mathcal{D}_k + \left(2{C^i}_j{}^k - {C_l}^{lk}\delta^i_j\right)\nabla_i\mathcal{D}_k = 0, \tag{3.4}$$

where ∇ is the covariant derivative compatible with the metric g, and ${R_j}^l$ indicates the corresponding Ricci curvature. For fixed g and ${C_l}^{lk}$, this is a linear hyperbolic system for $\mathcal{D}_i$, with each of the lower-order terms containing either $\mathcal{D}_k$ or $\nabla_i\mathcal{D}_k$. Hence, for initial data $\mathcal{D}_k(t_0, x) = 0$ and $\nabla_i\mathcal{D}_k(t_0, x) = 0$, the unique solution to this system is $\mathcal{D}_k(t, x) = 0$ over the whole spacetime.

[1] In this section, we use mid-alphabet latin letters as spacetime indices.

[2] g is compatible with (γ, K) in the sense that it induces γ as the first fundamental form and K as the second fundamental form on a spacelike slice of the spacetime.

With these four observations established, we may show that the Cauchy problem for the Einstein equations is well-posed as follows. We choose a smooth set of initial data (γ, K) satisfying the constraints, and we choose the smooth spacetime functions $\mathcal{F}_i$ and $C_{ij}{}^k$. Using observations II and III, we know that there are coordinate choices which result in initial data $(g_{ij}(t_0, x), \partial_t g_{ij}(t_0, x))$ for the system Eq. (3.1) having $\mathcal{D}_k(t_0, x) = 0$ and $\partial_t \mathcal{D}_k(t_0, x) = 0$. We may then treat $(g_{ij}(t_0, x), \partial_t g_{ij}(t_0, x), \mathcal{D}_k(t_0, x) = 0, \partial_t \mathcal{D}_k = 0)$ as initial data for the combined hyperbolic system consisting of Eq. (3.1) coupled to Eq. (3.4). It follows from observations I and IV that there is locally a unique solution to this initial value problem, and that the solution has $\mathcal{D}_k(t, x) = 0$ over the whole spacetime. It then follows from I that the resulting spacetime metric $g_{ij}(t, x)$ is a solution of Einstein's equations.

We observe that in the above discussion the coordinate chart with coordinate functions $(t = x^0, x)$ only appears implicitly. However it follows from the definition of the Christoffel symbols that the condition $0 = \mathcal{D}_k(t, x) = \mathcal{F}_k - \Gamma_k$ can be rewritten as $g^{ij}\partial_i\partial_j x^k = g^{kl}\mathcal{F}_l$, which is a system of wave equations for the coordinate functions (x^k). Hence, the coupled hyperbolic system (3.1)-(3.4) may be viewed as a system of evolution equations for the metric together with the coordinates. In particular, if we express this wave equation explicitly in terms of an arbitrary local reference chart with coordinate functions (y^k), it becomes an explicitly hyperbolic PDE system for the transition map $x^k(y)$:

$$\begin{aligned} \Box_{g_{(y)}} x^i(y) &:= g^{jk}_{(y)}(y) \left(\partial_{y^j}\partial_{y^k} x^i(y) - (\Gamma_{(y)})^l_{jk}(y)\partial_{y^l} x^i(y)\right) \\ &= -g^{ik}_{(x)}(x(y))\mathcal{F}_k(x(y)). \end{aligned} \tag{3.5}$$

Here, $g^{jk}_{(y)}$ and $g^{ik}_{(x)}$ are the components of the contravariant metric with respect to the y- and x-coordinates respectively. The functions $(\Gamma_{(y)})^l_{jk}$ are the Christoffel symbols of the metric g with respect to the y-coordinates.

This general setup for proving the well-posedness of Einstein's equations has been known since the work of Y. Choquet-Bruhat [14]. Her work uses $\mathcal{F}_i = 0$, a condition which results in what has been called "harmonic coordinates", or equivalently "wave coordinates". Allowing more general choice of the functions $\mathcal{F}_i$, one has "generalized wave coordinates".[1] Since the functions $\mathcal{F}_i$ largely control the choice of coordinates, they are often labeled as the *gauge source functions*.

Generalized wave coordinates are an important alternative to areal coordinates in studies of AVTD behavior via the singular initial value problem because, as seen above, for solutions which are not real analytic, it is important in working with the SIVP that the PDE system of interest be manifestly hyperbolic.

T^3-Gowdy spacetimes

A $3 + 1$ dimensional spacetime is labeled a *Gowdy spacetime* [16] if (i) it is a solution of the vacuum Einstein equations, (ii) it admits a spatially-acting T^2 isometry group, and (iii) the twist quantities of the Killing fields generating the isometry group vanish.[2] The only spacetime manifolds consistent with these conditions are $\mathbb{R} \times T^3, \mathbb{R} \times S^3$ and $\mathbb{R} \times S^2 \times S^1$, along with various quotients of these. We restrict our attention here to the Gowdy spacetimes on $\mathbb{R} \times T^3$.

The Gowdy spacetimes (especially those on $\mathbb{R} \times T^3$) have been used extensively to study model versions of general spacetime conjectures. It has been shown that Strong Cosmic Censorship holds for these spacetimes [28], that T^3-Gowdy spacetimes generically exhibit AVTD behavior [28], that T^3-Gowdy spacetimes admit foliations by constant mean curvature hypersurfaces [20], and that these spacetimes can be covered globally by areal coordinates [24]. Most of these studies have been carried out using the areal coordinate form of the T^3-Gowdy spacetime metrics, which can

[1] In this paper, we use "coordinate choice" and "gauge choice" interchangeably.

[2] If X and Y are used to label the one-forms corresponding to the (commuting) Killing fields generating the isometry group, then the twist quantities vanish if and only if the four-forms $X \wedge Y \wedge dX$ and $X \wedge Y \wedge dY$ both vanish.

be written generally as follows[1]

$$g = \frac{1}{\sqrt{t}} e^{\lambda/2}(-dt^2 + dx^2) + t\left(e^P dy^2 + 2e^P Q dydz + (e^P Q^2 + e^{-P})dz^2\right), \tag{3.6}$$

where $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are the Killing fields, and where P, Q and λ are functions of the coordinates x and t.

Without any restrictions on the coordinate gauge–except that the Killing fields be $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$–the form taken by the Gowdy metrics involves many more terms: one generally has

$$g = g_{00}dt^2 + 2g_{01}dtdx + g_{11}dx^2 + g_{AB}d\xi^A d\xi^B + g_{0A}dtd\xi^A + g_{1A}dxd\xi^A, \tag{3.7}$$

where $\xi^2 = y$ and $\xi^3 = z$, where the indices A and B each take the values 2 and 3, and where all of the metric components $g_{00}, g_{01}, g_{11}, g_{ab}, g_{0A}$ and g_{1A} are functions of t and x (not of y and z). For our study here, we are not concerned with showing that *all* Gowdy metrics exhibit AVTD behavior in *every* possible coordinate system. Hence, to simplify our analysis (without too much loss of generality) we find it useful to impose the following restrictions on the metric components:

$$g_{02} \equiv g_{03} \equiv g_{12} \equiv g_{13} \equiv 0. \tag{3.8}$$

As we see below, these conditions are preserved by the Einstein evolution equations for Gowdy metrics in the coordinate gauge choices which we introduce below in Section 3. Presuming Eq. (3.8), we may write the Gowdy metric in the following form:

$$\begin{aligned} g =& g_{00}(t,x)dt^2 + 2g_{01}(t,x)dtdx + g_{11}(t,x)dx^2 \\ &+ R(t,x)\left(E(t,x)(dy + Q(t,x)dz)^2 + \frac{1}{E(t,x)}dz^2\right). \end{aligned} \tag{3.9}$$

We note that this form is consistent with areal coordinates — if one chooses $R(t,x) = t, E = e^P, g_{00} = -g_{11} = -\frac{1}{\sqrt{t}}e^{\lambda/2}$ and $g_{01} = 0$, then this is the areal coordinate form of the Gowdy metric — it is, however, more general.

Generalized wave coordinate choices for T^3-Gowdy spacetimes

While any specification of the gauge source functions $\mathcal{F}_i$ produces solutions in generalized wave coordinates, we focus here on certain choices which are manifestly compatible with the goal of finding Gowdy solutions which show AVTD behavior. To determine these choices, being mindful of the central role of Kasner solutions in AVTD behavior, it is useful to recall the explicit form of the Kasner spacetimes.

The family of Kasner spacetimes consists of the set of all globally hyperbolic solutions of the vacuum Einstein equations which are spatially homogeneous with isometry group T^3 (also known as "Bianchi Type I"), and generally non-isotropic. The members of the Kasner family are characterized by a single parameter $k \in \mathbb{R}$ (known as the *asymptotic velocity*), in terms of which the Kasner metrics can be written explicitly[2] in the form (on $M^{1+3} = \mathbb{R}^+ \times T^3$)

$$g = t^{\frac{k^2-1}{2}}\left(-dt^2 + dx^2\right) + t^{1-k}dy^2 + t^{1+k}dz^2. \tag{3.10}$$

We note that for all choices of the asymptotic velocity except for $k = \pm 1$ (the flat Kasners), these spacetimes are singular (with unbounded curvature) at $t = 0$. We also note that these coordinates are areal. Finally, we note that the Kasner spacetimes are a sub-family of the Gowdy spacetimes,

[1] We note that there are other areal coordinate representations of the T^3-Gowdy spacetime metrics that have appeared in the literature. These are all minor variations, of little consequence.

[2] This is not the standard form used for the Kasner spacetimes; one usually sees them written in the form $g = -d\tau^2 + \tau^{2p_1}dx^2 + \tau^{2p_2}dy^2 + \tau^{2p_3}dz^2$, with the constraints $p_1 + p_2 + p_3 = 1$ and $(p_1)^2 + (p_2)^2 + (p_3)^2 = 1$. One passes from the expression (3.10) to this form using the coordinate transformation $\tau = \frac{4}{k^2+3}t^{\frac{k^2+3}{4}}$ and the parameter transformation $p_1 = (k^2-1)/(k^2+3), \quad p_2 = 2(1-k)/(k^2+3), \quad p_3 = 2(1+k)/(k^2+3)$.

characterized by the presence of an extra Killing field ∂_x. In particular, the Kasner spacetimes can be written in the form Eq. (3.9), for some choice of the functions $g_{00}, g_{01}, g_{11}, R, E$, and Q.

It is straightforward to calculate the Christoffel quantities Γ_i for the Kasner spacetimes Eq. (3.10); one obtains

$$\Gamma_0 = -1/t, \quad \Gamma_1 = \Gamma_2 = \Gamma_3 = 0. \tag{3.11}$$

These results are the same for *all* Kasner spacetimes, with no dependence on the parameter k. Recalling that a spacetime exhibits AVTD behavior in terms of a given coordinate system if the geometry seen locally by each observer asymptotically approaches that of a Kasner spacetime, the expressions for the contracted Christoffel quantities in Eq. (3.11) motivate our choice of the leading order terms for the gauge source functions $\mathcal{F}_j$ of generalized wave coordinates for the Gowdy spacetimes. In view of the coupling of the gauge source functions $\mathcal{F}_j$ to the metric fields (which follows from the definition Eq. (3.2) of $\mathcal{D}_i$, together with the requirement that these quantities vanish), we are led to choose

$$\begin{aligned} &\mathcal{F}_0(t,x,g) = -\frac{1}{t} + F_0(t,x,g), \quad \mathcal{F}_1(t,x,g) = F_{10}(x) + F_1(t,x,g), \\ &\mathcal{F}_2(t,x,g) = \mathcal{F}_3(t,x,g) = 0, \end{aligned} \tag{3.12}$$

where F_0 is $O(t^{-1+\xi_0})$ and F_1 is $O(t^{\xi_1})$ near $t = 0$ for $\xi_0, \xi_1 > 0$ (we provide more precise conditions for F_0 and F_1 below) and where F_{10} is a smooth function (independent of t). This function, which vanishes for the Kasner spacetimes, must satisfy a constraint Eq. (4.3) involving the asymptotic metric fields.

It is important to note that the evolution of the metric which corresponds to the choice of gauge functions of the form Eq. (3.12), together with a suitable choice of the functions ${C_{ij}}^k$ (cf. Eq. (3.1)), preserves the conditions Eq. (3.8) along with the metric form Eq. (3.9) for Gowdy-symmetric metrics. As well, we readily verify that $\Gamma_2 \equiv \Gamma_3 \equiv 0$ holds for any metric Eq. (3.9)

The choice of gauge source functions Eq. (3.12) is *not* the most general choice that could be made for studying Gowdy solutions with AVTD behavior manifest in wave coordinates. One could, in particular, allow $\mathcal{F}_2$ and $\mathcal{F}_3$ to be non-zero, so long as they decay sufficiently quickly. We are not concerned, however, with full generality, and the choice Eq. (3.12) does simplify the analysis. Among other features, it helps to locate the singularity at $t = 0$.

Why not simplify further, and include the requirement that $g_{01} = 0$ among the restrictions Eq. (3.8) imposed on the metric? If this were to done, then to preserve this restriction we would need to set F_0, F_1, and F_{01} in Eq. (3.12) to zero, hence drastically reducing the range of gauge choices. To avoid this, we allow g_{01} to be nonzero. It follows that a key part of verifying AVTD behavior in the solutions we consider here is to show that g_{01} decays sufficiently rapidly.

Besides motivating the choice of gauge source functions for our analysis here, the Kasner metric functions also motivate our choice of the leading order terms for the metric fields $g_{00}, g_{01}, g_{11}, R, E$, and Q. The choice we make for these leading order terms, which encapsulate the asymptotic behavior of the metric coefficients, is spelled out in the hypothesis of Theorem 4.1 below[1]. Recalling that the coordinates in the generalized wave gauge formulation are evolved by an inhomogeneous wave equation (with inhomogeneity $\mathcal{F}_i$), we leave the coefficients of this t-dependence as free functions in order to introduce, together with Condition (iii) of Theorem 4.1, the largest possible family of coordinates which is consistent with these asymptotics.

To close this section we motivate the label *asymptotic wave gauge* for the coordinate gauges considered here and defined by the form Eq. (3.12). Suppose that g_{ij} is a solution of Einstein's equations in the generalized wave gauge formalism with gauge source functions of the form Eq. (3.12). Consider any time function t_h which satisfies the wave equation

$$\Box_g t_h = 0 \tag{3.13}$$

[1]This same Kasner-motivated choice of leading order terms occurs in the areal coordinate representation of the Gowdy spacetimes, where (for example in [27]) the leading order terms are chosen to be $\lambda = -k^2 \log t$, $P = -k \log t$, $Q = 0$. As well, this matching is done in studying AVTD behavior in the (half)-polarized $U(1)$-symmetric spacetimes. In the representation of [22], after the coordinate transformation $t = e^{-\tau}$ is carried out, one has $\phi = \frac{1+k}{2} \log t$, $\Lambda = \frac{k^2+2k-3}{4} \log t$, $x = \frac{t^{(k^2+2k-3)/2}-1}{t^{(k^2+2k-3)/2}+1}$, $\beta_a = 0$ and z such that $e^{4z} = 1 - x^2$. It is straightforward to show that Eq. (3.11) is asymptotically satisfied in this case.

with respect to g. Such a time function is called a *wave time function* (or harmonic time function), in accord with the wave gauge ($\mathcal{F}_i \equiv 0$) discussed above in Section 3. By looking for solutions t_h depending only on the time function t generated by the generalized wave gauge source functions Eq. (3.12), and presuming that the shift decays sufficiently fast, we determine that the solutions to the ordinary differential equation implied by Eq. (3.13) show that t_h is related logarithmically to t. Since this holds for *any* gauge source functions Eq. (3.12) at least asymptotically, (and presuming that the shift variable decays sufficiently fast close to $t = 0$), we call any set of gauge source functions Eq. (3.12) *asymptotic wave gauge source functions*.

4 AVTD T^3-Gowdy vacuum solutions in asymptotic wave gauges

The main result of this paper is that there is a fairly wide class of T^3-Gowdy spacetimes which exhibit AVTD behavior in a fairly wide class of generalized wave coordinates. While not precluding the possibility that such behavior is found in even larger classes of such spacetimes and such coordinates, Theorem 4.1 (our main result) carefully states what we mean by these "fairly wide classes" in terms of the free choice of certain functions and certain numbers which parametrize the asymptotic data for these spacetimes and for the gauge source functions. We present the detailed statement of Theorem 4.1 in Subsection 4 of this paper, along with clarifying comments. In Subsection 4, we outline the main steps of the proof of Theorem 4.1. Then in Subsection 4, we carry out the portion of the proof which uses a singular initial value problem formulation to construct these spacetimes and their coordinates, in Subsection 4, we show that these spacetimes are solutions of the vacuum Einstein equations, and in Subsection 4 we complete the proof by verifying that these solutions do indeed exhibit AVTD behavior. Certain of the technical calculations needed for the proof are included in the Appendices.

Main result

We state our main result, Theorem 4.1, by first listing the choices of parametrizing functions–which we collectively label $\mathcal{P}$–one makes to specify a particular Gowdy solution which is AVTD in terms of a particular set of wave coordinates. These parametrizing functions are all defined as smooth functions either on the circle (with coordinate x), or on an interval cross the circle (with coordinates (t, x)). We note here a change in notation from that used above in our review of the Singular Initial Value Problem. In that review, in Section 2, the exponent vector for a remainder function w is denoted by μ. Here in Section 4, it is useful to instead denote this exponent vector by $\kappa + \mu$, where κ is the exponent vector for the leading order term, and where $\mu > 0$.

Theorem 4.1 (Existence of AVTD Gowdy vacuum solutions in asymptotic wave gauges)**.** *Let the space $\mathcal{P}$ consist of the following functions:*

(i) Asymptotic velocity: A function $k \in C^\infty(T^1)$ such that $0 < k(x) < 3/4$ for all $x \in T^1$.

(ii) Asymptotic metric data: A set of functions $g_{11*}, R_*, E_*, Q_*, Q_{**} \in C^\infty(T^1)$, with $R_*, E_*, g_{11*} > 0$, collectively satisfying the constraint[1]

$$\int_0^{2\pi} \left(-k(x) \frac{E'_*(x)}{E_*(x)} + 2k(x)E_*^2(x)Q_{**}(x)Q'_*(x) + \frac{3 - k^2(x)}{2} \frac{R'_*(x)}{R_*(x)} \right) dx = 0, \qquad (4.1)$$

along with a positive constant $g_{00**} > 0$.

(iii) Asymptotic gauge source function data: A pair of functions $F_0 \in X_{\delta,-1+\xi_0,\infty} \cap C^\infty((0, \delta] \times T^1)$ and $F_1 \in X_{\delta,\xi_1,\infty} \cap C^\infty((0, \delta] \times T^1)$, for some $\delta > 0$ and for a pair of exponent functions ξ_0, ξ_1 with $\xi_0(x) > \max\{0, 2k(x) - 1\}$ and with $\xi_1(x) > 0$ for all $x \in T^1$.

[1] The origin of this constraint is explained below; see Eq. (4.2), together with the discussion immediately following.

For any given choice of an element in $\mathcal{P}$ (i.e., for any choice of the functions and constants listed above), construct the functions

$$g_{00*}(x) := -g_{00**} e^{\int_0^x \left(-k(\xi)\frac{E'_*(\xi)}{E_*(\xi)} + 2k(\xi)E_*^2(\xi)Q_{**}(\xi)Q'_*(\xi) + \frac{3-k^2(\xi)}{2}\frac{R'_*(\xi)}{R_*(\xi)}\right)d\xi}, \tag{4.2}$$

and

$$F_{10}(x) := -\frac{g'_{00*}(x)}{2g_{00*}(x)} + \frac{g'_{11*}(x)}{2g_{11*}(x)} - \frac{R'_*(x)}{R_*(x)}, \tag{4.3}$$

for all $x \in T^1$. (The function $g_{00*}(x)$ is well-defined on T^1, as a consequence of Eq. (4.1).)

Then there exists a $\hat{\delta} > 0$, an exponent vector $\mu = (\mu_1, \ldots, \mu_6) > 0$ and a unique smooth Gowdy symmetric Lorentzian metric g which satisfies Einstein's vacuum equations and which, for the coordinate gauge choice determined by the gauge source functions

$$\begin{aligned} &\mathcal{F}_0(t,x) = -\frac{1}{t} + F_0(t,x), \quad \mathcal{F}_1(t,x) = F_{10}(x) + F_1(t,x), \\ &\mathcal{F}_2(t,x) = \mathcal{F}_3(t,x) = 0, \end{aligned} \tag{4.4}$$

has metric components taking the following form:

$$g_{00}(t,x) = g_{00*}(x)t^{(k^2(x)-1)/2} + w_{00}(t,x), \tag{4.5}$$
$$g_{11}(t,x) = g_{11*}(x)t^{(k^2(x)-1)/2} + w_{11}(t,x), \tag{4.6}$$
$$g_{01}(t,x) = w_{01}(t,x), \tag{4.7}$$
$$g_{02} \equiv g_{03} \equiv g_{12} \equiv g_{13} \equiv 0, \tag{4.8}$$
$$g_{22}(t,x) = R(t,x)E(t,x), \tag{4.9}$$
$$g_{23}(t,x) = R(t,x)E(t,x)(Q_*(x) + Q(t,x)), \tag{4.10}$$
$$g_{33}(t,x) = R(t,x)E(t,x)(Q_*(x) + Q(t,x))^2 + R(t,x)/E(t,x), \tag{4.11}$$

and

$$R(t,x) = R_*(x)t + w_R(t,x), \tag{4.12}$$
$$E(t,x) = E_*(x)t^{-k(x)} + w_E(t,x), \tag{4.13}$$
$$Q(t,x) = Q_{**}(x)t^{2k(x)} + w_Q(t,x). \tag{4.14}$$

The remainders satisfy the fall-off conditions

$$w_{00} \in X_{\hat{\delta},(k^2-1)/2+\mu_1,\infty}, w_{11} \in X_{\hat{\delta},(k^2-1)/2+\mu_2,\infty}, w_{01} \in X_{\hat{\delta},(k^2+1)/2+\mu_3,\infty}, \tag{4.15}$$

and

$$w_R \in X_{\hat{\delta},1+\mu_4,\infty}, w_E \in X_{\hat{\delta},-k+\mu_5,\infty}, w_Q \in X_{\hat{\delta},2k+\mu_6,\infty}. \tag{4.16}$$

The same respective spaces describe time derivatives $D^l w_{00}, D^l w_{11}, D^l w_{01}, D^l w_R, D^l w_E$, and $D^l w_Q$ of arbitrary order $l \geq 0$. This metric g is AVTD with respect to the coordinates generated by the gauge choice Eq. (4.4).

Before carrying out the proof of this theorem (in Subsections 4, 4 and 4 below), we make a few comments:

Remark 4.2. Theorem 4.1 shows that for each choice of an element of $\mathcal{P}$–i.e., for each choice of the asymptotic data listed in (i)-(ii)-(iii) above–there is a vacuum solution to the Einstein equations which has Gowdy symmetry and which exhibits AVTD behavior in one of a large family of wave coordinate gauges. The number of free functions comprising the asymptotic parametrizing data $\mathcal{P}$ for specifying these spacetimes and their coordinate systems is larger than that needed to specify AVTD Gowdy spacetimes in areal coordinates, which are discussed in [26, 29, 7]. The areal coordinate case corresponds to the special case of Theorem 4.1 if one specifies $g_{00**} = 1$,

$R_* = 1$, $F_{10} = 0$ and $F_0 \equiv F_1 \equiv 0$, and where all other data functions are subject to the standard areal Gowdy constraint

$$\int_0^{2\pi} \left(-k(x) \frac{E_*'(x)}{E_*(x)} + 2k(x) E_*^2(x) Q_{**}(x) Q_*'(x) \right) dx = 0.$$

The two constraints Eqs. (4.2) and (4.3) then imply that

$$-g_{00*}(x) = g_{11*}(x) = e^{\int_0^x \left(-k(\xi) \frac{E_*'(\xi)}{E_*(\xi)} + 2k(\xi) E_*^2(\xi) Q_{**}(\xi) Q_*'(\xi) \right) d\xi}.$$

These expressions take the usual areal coordinate form if we identify $E_*(x) = e^{P_{**}(x)}$ and $g_{00*}(x) = e^{\lambda_{**}(x)/2}$. One finds that the corresponding solution described by our theorem has the property $R \equiv t$, $g_{00} \equiv -g_{11}$ and $g_{01} \equiv 0$.

A larger subset of the solutions obtained from Theorem 4.1 have coordinates which are "asymptotically areal" in the sense that the area function of the 2-surfaces of symmetry *approaches* the time coordinate (or a constant multiple thereof). This subset of solutions is therefore determined by taking the asymptotic data function $R_*(x)$ to be unity (or to be some other positive constant).

We emphasize, however that Theorem 4.1 also establishes the local existence of AVTD solutions in coordinates which are neither areal nor asymptotically areal. To the authors' knowledge this is the first result to this effect. Since Ringström [28] has considered generic Gowdy solutions and his results have been obtained using areal coordinates, one might guess that the non-areal solutions we find here are in fact diffeomorphic to Gowdy solutions in areal coordinates. In Section 5 we discuss, in particular, the relationship between areal coordinates and the general class of coordinates gauges considered in our theorem. The question as to whether *every* solution obtained via Theorem 4.1 is diffeomorphic to a solution known to be AVTD in terms of areal coordinates remains open, however. In any case, the most important consequence of our theorem is that the inherently coordinate-dependent notion of AVTD behavior in solutions of Einstein' equations is stable under changes of the coordinates, at least if the restrictions of Theorem 4.1 are imposed.

Remark 4.3. We observe that the asymptotic data for the gauge source functions, as described in condition (iii) of the hypothesis of Theorem 4.1 depend only on the spacetime coordinates and are, in particular, independent of the metric fields. This simplification is purely for the convenience of presentation. In fact, this restriction can be relaxed so long as the gauge source functions satisfy the more general restriction, listed as condition (v) of the hypothesis of Proposition 4.5. The reason we stick with the simpler version in Theorem 4.1 is that it is cumbersome to express the more general condition without the "first-order variables" introduced in Section 4 below.

Remark 4.4. In studies of AVTD behavior in Gowdy spacetimes in areal coordinates, the restriction on the asymptotic velocity k generally imposed has been $0 < k < 1$. Here, in Theorem 4.1, we require $0 < k < 3/4$. We believe that this is not a real difference, and that this restriction could be loosened. Indeed, in some of the earlier studies of Gowdy spacetimes in areal gauge [26, 29], a similar restriction on k is imposed. In these works, this restriction is loosened using a successively improved sequence of leading order terms. We believe that the same procedure could be applied here. However, since the analysis is significantly more complicated in the wave gauge formalism, we refrain from verifying this.

In the polarized ($Q_* = Q_{**} = 0$) and half-polarized ($Q_* = \text{const}$) cases, no additional arguments are necessary to make this restriction on k disappear. This is so because certain problematic terms in the Einstein evolution equations are then identically zero. In these special cases, k is allowed to be an arbitrary function.

Outline of the proof of the main result

The proof of Theorem 4.1 consists of the following three main tasks: 1) Showing that for any choice of a set of asymptotic data in $\mathcal{P}$, the singular initial value problem corresponding to Eq. (3.1) for the metrics with Gowdy symmetry is well-posed. 2) Showing that for any such choice of asymptotic data, it is also true that the singular initial value problem for the constraint-violation quantities $\mathcal{D}_i$ (see Eq. (3.2)) is well-posed, with solutions that necessarily vanish. 3) Showing that for any such chosen asymptotic data, the Gowdy solution whose existence and wave-coordinate representation follows from the first two tasks must exhibit AVTD behavior in those coordinates. We now outline, in a bit more detail, the concrete steps that must be carried through in order to accomplish these tasks and thereby prove Theorem 4.1. We label these steps in accord with these three main tasks.

Step 1a: The starting point for the proof, is the substitution into Eq. (3.1) of the various expressions Eq. (4.5)-Eq. (4.14) for the metric in terms of asymptotic data and remainder quantities, and Eq. (4.4) for the gauge source functions in terms of asymptotic data and remainder terms. This produces a second-order system (parametrized by asymptotic data) for the remainder terms. Since our formulation of the singular initial value problem (see Section 2) works rather with first-order PDE systems, we proceed by introducing new functions corresponding to the first derivatives of the remainder terms, thereby producing a (triple in size) first-order system. In doing this, we verify that the system is symmetric hyperbolic.

Step 1b: If we combine the first-order PDE system for the remainder terms (together with their first derivatives) obtained in Step 1a with the leading order terms corresponding to the choice of asymptotic data (an element of $\mathcal{P}$), we obtain a singular initial value problem. In this step, we verify that this singular initial value problem satisfies the hypotheses of Theorem 2.10, and therefore is well-posed. The statement of this verification appears in Proposition 4.5 below.

Step 1c: Using the existence and uniqueness results obtained in Step 1b for the first-order PDE system, we determine in this step that existence and uniqueness hold for solutions of the original second-order system. It follows that for the chosen set of asymptotic data (an element of $\mathcal{P}$), there exists a unique spacetime and a unique set of wave coordinates (in a neighborhood of the singularity) such that the components of the metric in terms of these coordinates satisfy Eq. (3.1). These results are stated in Proposition 4.11 below.

It is not true *a priori* that the spacetime whose existence and uniqueness are verified in Step 1c is a solution of the vacuum Einstein equations. To show this, it is sufficient to prove that for any choice of the asymptotic data consistent with the hypothesis of Theorem 4.1, the constraint violation quantities $\mathcal{D}_i$ vanish. As noted above, such a result follows if (i) the asymptotic data for $\mathcal{D}_i$ vanish, and (ii) the singular initial value problem for $\mathcal{D}_i$ corresponding to (a first-order version of) Eq. (3.4) has a unique solution. There is a subtlety involved in doing this which we explain in detail later. It turns out that we are only able to find sufficient conditions for both (i) and (ii) if we tighten the conditions on the metric and gauge source function asymptotic data. The definition of $\mathcal{P}$ in the hypothesis for Theorem 4.1 includes this tightening. As part of this process, the next step of the proof is a somewhat technical lemma:

Step 2a: Using the conditions on the asymptotic data imposed by conditions (i)-(iii) in Theorem 4.1, we prove in this step that the "shift quantity" g_{01} decays rapidly (at a rate described in Proposition 4.12) as t approaches the singularity (marked by $t = 0$).

Step 2b: We set up a singular initial value problem for $\mathcal{D}_i$ based on a first-order version of Eq. (3.4), together with asymptotic data for $\mathcal{D}_i$ and its first time derivative. We show that it follows from conditions (i)-(iii) in Theorem 4.1 that this asymptotic data vanishes. We then use the decay rate established in Step 2a to verify (based on Theorem 2.10) that this singular initial value problem is well-posed. The consequent existence and uniqueness for solutions of this problem then implies that since $\mathcal{D}_i = 0$ is a solution, it is necessarily the only solution. We conclude that the spacetimes whose existence is guaranteed in Step 1c must be solutions of the vacuum Einstein equations.

Step 3: To complete the proof of Theorem 4.1, we show in this step that for each choice of a set of asymptotic data (contained in the set $\mathcal{P}$), the Gowdy spacetime constructed via the singular initial value problem using this data, and represented in wave coordinates also generated from this data, must asymptotically approach a solution of a truncated "VTD" version of the Einstein equations. It follows that each such spacetime exhibits AVTD behavior in terms of these wave coordinates.

Construction of the spacetimes and coordinates

We begin carrying out the details of the proof in this subsection. Here, we focus on constructing the spacetimes along with the generalized wave coordinates, in terms of which the spacetime metric fields are represented. In doing this, we follow the steps of the outline presented above.

Carrying out Step 1a: We consider the Einstein evolution equations, Eq. (3.1), with the choice of gauge source functions specified in Eq. (3.12). It is useful to set ${C_{ij}}^k(t,x) = 0$ for all i, j, k except for the following

$$ {C_{00}}^0(t,x) = \frac{\gamma_0(x)}{t}, \quad {C_{01}}^1(t,x) = {C_{10}}^1(t,x) = \frac{\gamma_1(x)}{t}, \tag{4.17} $$

for as yet unspecified smooth functions γ_0 and γ_1. The resulting PDE system takes the form

$$ \sum_{k,l=0}^{1} g^{kl} \partial_{x^k} \partial_{x^l} g_{ij} = 2\widehat{H}_{ij}, \tag{4.18} $$

with

$$ \widehat{H}_{ij} := \nabla_{(i}\mathcal{F}_{j)} + g^{kl} g^{mn} \left(\Gamma_{kmi}\Gamma_{lnj} + \Gamma_{kmi}\Gamma_{ljn} + \Gamma_{kmj}\Gamma_{lin} \right) + {C_{ij}}^k \mathcal{D}_k. \tag{4.19} $$

We first argue that the metric restrictions

$$ g_{02} \equiv g_{03} \equiv g_{12} \equiv g_{13} \equiv 0, \tag{4.20} $$

(cf. Eq. (4.8)) are preserved by the Einstein evolution equations with T^2 isometry (and with $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial x^3}$ as Killing fields). We verify this by showing that the quantities $\widehat{H}_{ij}$ for $ij = 02, 03, 12,$ and 13 all vanish if we substitute the conditions (4.20) into the formula (4.19) for $\widehat{H}_{ij}$. We note that conditions enforce the vanishing of the T^2-symmetry twist quantities, and therefore guarantee that the spacetimes under study are indeed Gowdy spacetimes. We also note that (4.20) does not further restrict our work here to a subfamily of the Gowdy spacetimes.

We proceed to work with the Einstein equations for the remaining metric components g_{00}, g_{01}, g_{11}, g_{22}, g_{23} and g_{33}. The latter three are parametrized as in Eqs. (4.9) – (4.11). We presume that Q_* is a given smooth function and hence work with the following vector consisting of six unknown functions:

$$ u(t,x) = \left(g_{00}(t,x), g_{11}(t,x), g_{01}(t,x), R(t,x), E(t,x), Q(t,x) \right)^T. \tag{4.21} $$

It is straightforward to show that the system of wave equations for the metric components g_{ij}, Eqs. (4.18) – (4.19), implies a similar system of wave equations for the components of this unknown vector u of the form Eq. (A.1) with $d = 6$ and $n = 1$; i.e.,

$$ \sum_{k,l=0}^{1} g^{kl} \partial_{x^k} \partial_{x^l} u = 2H, \tag{4.22} $$

where the vector H can be computed explicitly from previous expressions.

We now wish to convert this second-order system into a first-order symmetric hyperbolic system. This is achieved by introducing the 18-dimensional vector U as in Eqs. (A.3) and (A.4); i.e., we set

$$ U^i_{-1} := u^i, \quad U^i_0 := Du^i - \alpha u^i, \quad U^i_1 := t\partial_x u^i, \quad U^i := (U^i_{-1}, U^i_0, U^i_1)^T, \tag{4.23} $$

for $i = 1, \dots, 6$, with α a constant to be fixed below, and we define

$$ U := (U^1, \dots, U^6)^T. \tag{4.24} $$

As discussed in Appendix A, the second-order system for u above implies a first-order system for the extended vector U of the form Eqs. (A.5) – (A.9); i.e.,

$$ S^0 DU + S^1 t\partial_x U + \tilde{N} U = \tilde{f}[U], \tag{4.25} $$

with block-diagonal matrices

$$S^0 = \mathrm{diag}(\mathbf{s}^0, \ldots, \mathbf{s}^0), \quad S^1 = \mathrm{diag}(\mathbf{s}^1, \ldots, \mathbf{s}^1), \quad \tilde{N} = \mathrm{diag}(\tilde{\mathbf{n}}, \ldots, \tilde{\mathbf{n}}). \tag{4.26}$$

The general form of the blocks $\mathbf{s}^i$ is given in Eq. (A.7). Recall that g^{ij} are the components of the inverse matrix of (g_{ij}). In our present application we find

$$\mathbf{s}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -g^{11}/g^{00} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -g_{00}/g_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -U^1_{-1}/U^2_{-1} \end{pmatrix}, \tag{4.27}$$

and

$$\begin{aligned} \mathbf{s}^1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2g^{01}/g^{00} & g^{11}/g^{00} \\ 0 & g^{11}/g^{00} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2g_{01}/g_{11} & g_{00}/g_{11} \\ 0 & g_{00}/g_{11} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2U^3_{-1}/U^2_{-1} & U^1_{-1}/U^2_{-1} \\ 0 & U^1_{-1}/U^2_{-1} & 0 \end{pmatrix}, \end{aligned} \tag{4.28}$$

while

$$\begin{aligned} \tilde{\mathbf{n}} &= \begin{pmatrix} -\alpha & -1 & 0 \\ -(1-\alpha)\alpha & -1+\alpha & 0 \\ 0 & 0 & (1+\alpha)g^{11}/g^{00} \end{pmatrix} \\ &= \begin{pmatrix} -\alpha & -1 & 0 \\ -(1-\alpha)\alpha & -1+\alpha & 0 \\ 0 & 0 & (1+\alpha)g_{00}/g_{11} \end{pmatrix} \\ &= \begin{pmatrix} -\alpha & -1 & 0 \\ -(1-\alpha)\alpha & -1+\alpha & 0 \\ 0 & 0 & (1+\alpha)U^1_{-1}/U^2_{-1} \end{pmatrix}. \end{aligned} \tag{4.29}$$

Moreover, we have $(i = 1, \ldots, 6)$,

$$\tilde{f}[U]^i = \left(0, \frac{2t^2\Xi}{U^2_{-1}} H^i + 2\alpha \frac{U^3_{-1}U^i_1}{U^2_{-1}}, 0\right)^T, \tag{4.30}$$

cf. Eq. (A.9), where H^i are the components of the 6-dimensional vector H in Eq. (4.22), and where

$$\Xi(t,x) = g_{00}g_{11} - g_{01}^2 = U^1_{-1}U^2_{-1} - (U^3_{-1})^2.$$

We verify by inspection that this first-order PDE system is symmetric hyperbolic. In the remainder of the paper, we refer to this system as the *first-order evolution system.*

Carrying out Step 1b: The aim is now to construct solutions of the first-order system Eqs. (4.25) – (4.30) with the leading-order behavior asserted in Theorem 4.1. To do this, it is sufficient to show that for a choice of a leading order term U_* for U which is compatible with the conditions in Theorem 4.1, and with a choice of the function space for the remainder term which is also compatible with Theorem 4.1, the resulting singular initial value problem satisfies the conditions of Theorem 2.10 and is consequently well-posed.

Mindful of conditions Eq. (4.5)-(4.14), we choose the leading order term for the components of the vector u_* (see Eq. (4.21)) in the form

$$\begin{aligned} u_*(t,x) = \Big(&g_{00*}(x)t^{(k^2(x)-1)/2}, g_{11*}(x)t^{(k^2(x)-1)/2}, 0, \\ &R_*(x)t, E_*(x)t^{-k(x)}, Q_{**}(x)t^{2k(x)}\Big)^T \end{aligned} \tag{4.31}$$

for asymptotic data g_{00*}, g_{11*}, R_*, E_* and Q_{**} in $C^\infty(T^1)$ and for an asymptotic velocity $k \in C^\infty(T^1)$. We note that at this stage, we do not require that the asymptotic data functions satisfy the conditions in the hypothesis of Theorem 4.1. For the vector U used in the formulation of the first-order system (see Eqs. (4.23)-(4.24)), we choose the leading order term U_* to take the form

$$U_* := (U_*^1, \ldots, U_*^6)^T, \quad U_*^i := (u_*^i, Du_*^i - \alpha u_*^i, 0)^T, \tag{4.32}$$

for $i = 1, \ldots, 6$.

To set up the function spaces for the remainder term, we next define the $\mathbb{R}^6$-vector

$$\kappa := (\kappa_1, \ldots, \kappa_6) = \big((k^2-1)/2, (k^2-1)/2, (k^2-1)/2, 1, -k, 2k\big), \tag{4.33}$$

from which we construct the $\mathbb{R}^{18}$-vector

$$\hat{\kappa} := (\kappa_1, \kappa_1, 1+\alpha; \ldots; \kappa_6, \kappa_6, 1+\alpha). \tag{4.34}$$

As well, we choose an $\mathbb{R}^6$-vector $\mu > 0$ with components μ_i , from which we define

$$\hat{\mu} = (\mu_1, \mu_1, \mu_1 - (1+\alpha) + \kappa_1; \ldots; \mu_6, \mu_6, \mu_6 - (1+\alpha) + \kappa_6). \tag{4.35}$$

With these constructions, we formulate a singular initial value problem for the first-order system Eqs. (4.25) – (4.30), seeking solutions of the form

$$U = U_* + W$$

with $W \in X_{\delta,\hat{\kappa}+\hat{\mu},\infty}$.

Our construction of $\hat{\kappa}$ and $\hat{\mu}$ and their use in defining the function spaces $X_{\delta,\hat{\kappa}+\hat{\mu},\infty}$ in which the remainder term lives are motivated by the following considerations. We write the exponent of the remainder term function space as the sum $\hat{\kappa}+\hat{\mu}$ since $\hat{\kappa}$ represents the t-powers of the leading order term and hence the remainder is of "higher order" as required if $\hat{\mu} > 0$. This, however, leaves the 1-components of $\hat{\kappa}$ undetermined since $U_{*,1}^i = 0$. We make the particular choice of the components in Eq. (4.33) for $\hat{\kappa}$ because, as we explain below in detail, one finds that $\hat{\kappa}$ agrees with the vector of eigenvalues Λ (see Eq. (2.21)) for our system. It follows then from the eigenvalue condition Eq. (2.22) of the well-posedness result (Theorem 2.10) that $\hat{\mu} > 0$. The particular form of the 1-components of $\hat{\mu}$ in Eq. (4.35) is thus a consequence of the block diagonal condition of Theorem 2.10 which, as we see below, requires that the -1-, 0- and 1-components of $\hat{\kappa} + \hat{\mu}$ are the same.

Proposition 4.5 (Existence of solutions of the singular initial value problem of the first-order evolution system)**.** *Let the space $\mathcal{Q}$ consist of the following functions:*

(i) A function $k \in C^\infty(T^1)$ such that $0 < k(x) < 3/4$ for all $x \in T^1$.

(ii) Functions $\xi_0, \xi_1 \in C^\infty(T^1)$ such that $\xi_0(x) > \max\{0, 2k(x)-1\}$ and $\xi_1(x) > 0$ for all $x \in T^1$.

(iii) An exponent vector μ such that

$$\begin{array}{rcccl} \max\{0,2k(x)-1\} & < & \mu_4(x) & < & \min\{1,\xi_0(x)\}, \\ \max\{0,2k(x)-1\} & < & \mu_5(x) & < & \min\{\xi_0(x), 2k(x), 2(1-k(x))\}, \\ \max\{0,2k(x)-1\} & < & \mu_1(x) & < & \min\{\mu_4(x),\mu_5(x)\}, \\ 0 & < & \mu_6(x) & < & \min\{\mu_5(x), \mu_1(x)+1-2k(x)\}, \\ \mu_1(x) & = & \mu_2(x) & = & \mu_3(x), \end{array}$$

for all $x \in T^1$.

(iv) Functions $g_{00*}, g_{11*}, R_*, E_*, Q_*, Q_{**} \in C^\infty(T^1)$ such that $-g_{00_*}$, g_{11_*}, R_*, and E_* are strictly positive.

(v) Smooth asymptotic gauge source function data $F_{10}(x)$, $F_0(t,x,u)$ and $F_1(t,x,u)$ such that the corresponding function operators (defined below in Remark 4.8) $W \mapsto F_0[W]$, $W \mapsto DF_0[W]$, and $W \mapsto \partial_x F_0[W]$ are rational $(\hat{\kappa}+\hat{\mu}, -1+\xi_0, \infty)$-operators and $W \mapsto \hat{F}_1[W]$, $W \mapsto D\hat{F}_1[W]$ and $W \mapsto \partial_x \hat{F}_1[W]$ are rational $(\hat{\kappa}+\hat{\mu}, \xi_1, \infty)$-operators.

Moreover, choose

$$\gamma_0(x) = \frac{1}{2}\left(3 + k(x)^2\right), \quad \gamma_1(x) = \frac{1}{4}\left(1 + k(x)^2\right) \tag{4.36}$$

in Eq. (4.17)*. If the constant* α *in Eqs.* (4.23) *and* (4.32) *is sufficiently negative, then there exists a unique solution* U *of the first-order system Eqs.* (4.25) – (4.30) *of the form*

$$U = U_* + W \quad \text{for some} \quad W \in X_{\tilde{\delta},\hat{\kappa}+\hat{\mu},\infty},$$

for some $\tilde{\delta} > 0$*, where* $\hat{\kappa}$*,* $\hat{\mu}$ *and* U_* *are given by Eqs.* (4.32) – (4.35)*. Moreover, the remainder* W *is differentiable in time and we have* $DW \in X_{\tilde{\delta},\hat{\kappa}+\hat{\mu},\infty}$*.*

Before proving this Proposition, we note the following:

Remark 4.6. Comparing $\mathcal{Q}$ and $\mathcal{P}$, we see that the hypothesis for Proposition 4.5 is significantly more general than that of Theorem 4.1. This is not surprising, since this proposition is concerned only with obtaining solutions to the evolution equations, while Theorem 4.1 is concerned with solutions of the *full* Einstein equations: the constraints as well as the evolution equations.

Remark 4.7. While Theorem 4.1 simply asserts the existence of an exponent vector μ for which the results hold, in this proposition Condition (iii) provides estimates for μ. One finds that these estimates are the source of the restriction $0 < k < 3/4$ appearing in Condition (i) here, as well in the hypothesis of Theorem 4.1 (see Remark 4.4). In particular, it is the second inequality in Condition (iii) which implies the necessary restriction $2k - 1 < 2(1 - k)$; i.e., $k < 3/4$.

We emphasize that both the upper and the lower bounds in the inequalities for μ in Condition (iii) are meaningful. On the one hand, the strongest uniqueness statement is obtained by choosing the components of μ as small as possible, thus giving rise to the "biggest" space for the remainder quantities. On the other hand, a large choice of μ close to the upper bound yields the most precise description of the actual behavior of the remainder quantities at $t = 0$.

As seen below, some of these upper bounds are not fully optimal yet. We note that the order of the inequalities in Condition (iii) corresponds to the order in which components of μ can be picked which satisfy the inequalities.

Remark 4.8. Here, we define the function operators appearing in Condition (v) (note Remark 4.3 above). Given the function operator $W \mapsto F_0[W]$ (the same holds for $W \mapsto F_1[W]$), we define the map

$$W \mapsto DF_0[W], \quad W \mapsto DF_0[W]$$

by specifying the function $DF_0[W](t, x)$ for any sufficiently regular function W as follows: (i) We apply the D-derivative to the function $F_0[W](t, x)$, and (ii) we replace DW_{-1} everywhere by $W_0 + \alpha W_{-1}$ in agreement with the definition of the first-order variables; see Eq. (4.23). Since gauge source functions are only allowed to depend the coordinates and on the metric, but in particular not on its derivatives, the map $W \mapsto DF_0[W]$ constructed like this is indeed a function operator. This would not be the case if there were terms including DW_0 or DW_1 after taking the D-derivative. The map $W \mapsto \partial_x F_0[W]$ is defined in the same way and for the same reason is a function operator.

Remark 4.9. In the polarized case ($Q_* = Q_{**} = 0$) the inequalities for μ_6 and thereby the non-trivial lower bounds for μ_4, μ_5 and μ_1 disappear. Moreover, the condition $\mu_5 < 2(1 - k)$ vanishes. As a consequence the asymptotic velocity k is allowed to be an arbitrary real function. As well, there is no restriction $\xi_0 > 2k - 1$. In the half-polarized case ($Q_* = 0$), the restriction $\mu_5 < 2(1 - k)$ disappears and hence k is allowed to be any positive function. The lower bound for ξ_0 however remains.

Proof of Proposition 4.5: We begin by rewriting the first-order evolution system Eqs. (4.25) – (4.30) in a form which is consistent with the criteria for establishing well-posedness in Theorem 2.10. To this end, we replace the matrix $\tilde{\mathbf{n}}$ in Eq. (4.29) by its leading-order expression

$$\tilde{\mathbf{n}} = \begin{pmatrix} -\alpha & -1 & 0 \\ -(1-\alpha)\alpha & -1+\alpha & 0 \\ 0 & 0 & (1+\alpha)\frac{g_{00*}}{g_{11*}} \end{pmatrix}$$

and we absorb the higher-order terms into the source term operator $\tilde{f}[U]$, whose components now become

$$\tilde{f}[U]^i = \left(0, \frac{2t^2\Xi}{U_{-1}^2}H^i + 2\alpha\frac{U_{-1}^3U_1^i}{U_{-1}^2}, -(1+\alpha)B[W]U_1^i\right)^T,$$

where

$$W \mapsto B[W] = \frac{g_{00*}t^{(k^2-1)/2} + W_{-1}^1}{g_{11*}t^{(k^2-1)/2} + W_{-1}^2} - \frac{g_{00*}}{g_{11*}}.$$

We then define the reduced source term operator in Eq. (2.13) as

$$W \mapsto \tilde{\mathcal{F}}(U_*)[W] = \tilde{f}[U_* + W] - \widehat{L}(U_* + W)[U_*] \tag{4.37}$$

where

$$\widehat{L}(U_* + W)[U_*] = S^0[U_* + W]DU_* + S^1[U_* + W]t\partial_x U_* + \tilde{N}U_*$$

and where the matrix $\tilde{N}$ is determined by the new matrix $\tilde{\mathbf{n}}$ via Eq. (4.26). We obtain

$$\begin{aligned}\widehat{L}(U_* + W)[U_*] = \Big(&0, \frac{1}{4}(3 - 4k^2 + k^4)t^{(k^2-1)/2}g_{00*}, 0; 0, \frac{1}{4}(3 - 4k^2 + k^4)t^{(k^2-1)/2}g_{11*}, 0;\\ &0,0,0;0,0,0;0,E_*k(1+k)t^{-k},0;0,2k(2k-1)Q_{**}t^{2k},0\Big)^T\\ &+ S^1[U_* + W]t\partial_x U_*,\end{aligned}$$

where

$$(S^1[U_* + W]t\partial_x U_*)^i = \left(0, -2t\frac{U_{-1}^3}{U_{-1}^2}\partial_x U_{0*}^i, t\frac{U_{-1}^1}{U_{-1}^2}\partial_x U_{0*}^i\right)^T$$

for each $i = 1, \ldots, 6$.

The idea, now (following the discussion in Section 2), is to establish that a modified version of the reduced source term operator $\tilde{\mathcal{F}}$ defined above in Eq. (4.37), which we label $\mathcal{F}$, has suitable regularity properties, and then show that it follows from Theorem 2.10 that the equation

$$\widehat{L}(U_* + W)[W] = \mathcal{F}(U_*)[W] \tag{4.38}$$

has unique solutions. We establish the existence of this modified reduced source term operator in the following lemma:

Lemma 4.10. *Let U_* be given by Eq. (4.32), let κ, $\hat{\kappa}$, μ, and $\hat{\mu}$ be given by Eqs. (4.33) – (4.35), and suppose in addition that Conditions (i) – (v) of Proposition 4.5 hold. Choose the $\mathbb{R}^{18\times18}$ matrix*

$$N = \mathrm{diag}(N_{01}, N_{22}, N_{33}, N_{44}),$$

where

$$N_{01} = \begin{pmatrix} -\alpha & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2}ad_1 & d_2 & 0 & \frac{abg_{00*}}{4g_{11*}} & \frac{bg_{00*}}{2g_{11*}} & 0 & 0 & 0 & -\frac{bg_{00*}}{g_{11*}}\\ 0 & 0 & \frac{(\alpha+1)g_{00*}}{g_{11*}} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{4}a^2 & -k^2+\alpha+1 & 0 & 0 & 0 & -2\\ 0 & 0 & 0 & 0 & 0 & \frac{(\alpha+1)g_{00*}}{g_{11*}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1 & 0\\ 0 & 0 & e_1 & 0 & 0 & -\frac{e_2g_{00*}}{g_{11*}} & \frac{1}{2}ac_1 & c_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(\alpha+1)g_{00*}}{g_{11*}} \end{pmatrix},$$

with

$$a=-k^2+2\alpha+1,\quad b=k^2-2\gamma_0-1,$$
$$c_1=-k^2+\alpha+2\gamma_1,\quad c_2=-\frac{3}{2}k^2+\alpha+2\gamma_1+\frac{1}{2},$$
$$d_1=-k^2+\alpha+\gamma_0-1,\quad d_2=-\frac{3}{2}k^2+\alpha+\gamma_0-\frac{1}{2},$$
$$e_1=\frac{1}{4}\left(k^2-4\gamma_1-5\right),\quad e_2=-\frac{1}{4}\left(k^2-4\gamma_1-1\right),$$

and

$$N_{22}=\begin{pmatrix}-\alpha & -1 & 0\\ (\alpha-1)^2 & \alpha-2 & 0\\ 0 & 0 & \frac{(\alpha+1)g_{00*}}{g_{11*}}\end{pmatrix},$$
$$N_{33}=\begin{pmatrix}-\alpha & -1 & 0\\ (\alpha+k)^2 & \alpha+2k & 0\\ 0 & 0 & \frac{(\alpha+1)g_{00*}}{g_{11*}}\end{pmatrix},$$
$$N_{44}=\begin{pmatrix}-\alpha & -1 & 0\\ \alpha(\alpha-2k) & \alpha-2k & 0\\ 0 & 0 & \frac{(\alpha+1)g_{00*}}{g_{11*}}\end{pmatrix}.$$

Then there exists an exponent vector $\hat\nu>\hat\kappa+\hat\mu$ and a rational $(\hat\kappa+\hat\mu,\hat\nu,\infty)$-operator $W\mapsto \mathcal{F}(U_*)[W]$ such that

$$W\mapsto -\tilde N\cdot W+\tilde{\mathcal{F}}(U_*)[W]=-N\cdot W+\mathcal{F}(U_*)[W].$$

The proof of this lemma–in particular, the claimed regularity of the operator $W\mapsto\mathcal{F}(U_*)[W]$–follows directly from a computer-aided algebraic computation of this operator from the above definition. The details of the computer algebra code used are given in Appendix B. It is important to note that this proof is fully rigorous; numerical approximations do not play a role. We also note that while we could state explicit estimates for the exponent $\hat\nu$, such estimates are not needed to complete the proof of Proposition 4.5, which we complete here.

Using the results of this lemma, we rewrite the first-order system Eqs. (4.25) – (4.30) in the Fuchsian form (see Eq. (2.10))

$$S^0DW+S^1t\partial_xW+NW=\mathcal{F}(U_*)[W]. \tag{4.39}$$

We need to verify that this system is indeed a smooth quasilinear symmetric hyperbolic Fuchsian system according to Definition 2.6. It is clear that $U_*\in C^\infty((0,\delta]\times T^n)\cap X_{\delta,\hat\kappa,\infty}$ and that all objects in the equations depend smoothly on their arguments on the relevant domains. Moreover, all function operators are rational. The matrix $S^0_0(U_*)$ can be constructed from S^0 in Eq. (4.26) by replacing $-U^1_{-1}/U^2_{-1}$ in Eq. (4.27) by $-g_{00*}/g_{11*}$. It follows from Condition (iv) of Proposition 4.5 that this matrix is symmetric (in fact diagonal) and positive definite. Using the techniques in Appendix B, it is then straightforward to show that $W\mapsto S^0(W)-S^0_0(U_*)$ is a $(\hat\kappa+\hat\mu,\zeta,\infty)$-operator for some $\zeta>0$ (see Eq. (2.15) of Definition 2.6). In the same way, we can show that $W\mapsto tS^1(W)$ is a $(\hat\kappa+\hat\mu,\zeta,\infty)$-operator (see Eq. (2.16) of Definition 2.6). This together with Lemma 4.10 establishes that our evolution system Eq. (4.39) is indeed a smooth quasilinear symmetric hyperbolic rational-Fuchsian system.

We are now ready to apply Theorem 2.10 and hence prove Proposition 4.5. From the above constructions of the exponent vector $\hat\mu$ and of the matrices S^0, S^1 and N in Lemma 4.10, it is clear that our system is block diagonal with respect to $\hat\mu$ and it is clear that $\hat\mu$ is ordered. The ordered vector of eigenvalues Λ of the matrix $\mathcal{N}$ (see Eqs. (2.20) and (2.21)) is found to be

$$\begin{aligned}\Lambda=\Big((1-k^2)/2,\gamma_0-1-k^2,-1-\alpha;(1-k^2)/2,(1-k^2)/2,-1-\alpha;\\ (1-k^2)/2,2\gamma_1-k^2,-1-\alpha;-1,-1,-1-\alpha;k,k,-1-\alpha;\\ -2k,-2k,-1-\alpha\Big).\end{aligned} \tag{4.40}$$

If we therefore choose γ_0 and γ_1 as in Eq. (4.36), it follows that

$$\begin{aligned}\Lambda = \Big(&(1-k^2)/2, (1-k^2)/2, -1-\alpha; (1-k^2)/2, (1-k^2)/2, -1-\alpha;\\ &(1-k^2)/2, (1-k^2)/2, -1-\alpha; -1, -1, -1-\alpha; k, k, -1-\alpha;\\ &-2k, -2k, -1-\alpha\Big).\end{aligned} \tag{4.41}$$

The condition Eq. (2.22) in Theorem 2.10 is therefore satisfied for *every* exponent vector $\hat{\mu} > 0$ so long as α has been chosen sufficiently negative. This completes the proof of Proposition 4.5.

Carrying out Step 1c: Suppose that $U = U_* + W$ is any smooth solution of the first-order system Eqs. (4.25) – (4.30) with U_* given by Eqs. (4.31)–(4.32) with smooth data, and with $W, DW \in X_{\tilde{\delta},\hat{\kappa}+\hat{\mu},\infty}$ for $\hat{\kappa}$ given by Eqs. (4.33)–(4.34) and for any $\hat{\mu} > 0$. We assume here that $k(x) \in (0,1)$ for all $x \in T^1$. We can then define

$$u := (U^1_{-1}, \dots, U^6_{-1})^T, \quad u_* := (U^1_{-1*}, \dots, U^6_{-1*})^T, \quad w := (W^1_{-1}, \dots, W^6_{-1})^T.$$

Clearly, we have $u = u_* + w$ with $w, Dw \in X_{\tilde{\delta},\kappa+\mu,\infty}$; in fact, $u \in C^\infty((0,\tilde{\delta}] \times T^1)$. In Appendix A we argue that this vector u is a solution of the original second-order system Eq. (4.22) if and only if the six quantities

$$C^i_1 := U^i_1 - t\partial_x U^i_{-1} \tag{4.42}$$

vanish identically (see Eq. (A.11)). We also argue in Appendix A that since U satisfies the first-order system Eqs. (4.25) – (4.30), these quantities must satisfy the subsidiary system

$$DC^i_1 - (1+\alpha)C^i_1 = 0; \tag{4.43}$$

cf. Eq. (A.12). This subsidiary system yields six decoupled linear homogeneous Fuchsian ordinary differential equations for which we can formulate a suitable singular initial value problem and then apply Theorem 2.10 to this problem. Theorem 2.10 and the homogeneity of Eqs. (4.43) imply that for each choice of the index i, the unique solution of Eq. (4.43) contained in the space $X_{\delta,1+\alpha,\infty}$ is $C^i_1 \equiv 0$. The quantities C^i_1 given by the vector U by Eq. (4.42) are elements of the space $X_{\tilde{\delta},\kappa_i+\mu_i,\infty}$ (recall that $\mu_i < 1$ is a consequence of Condition (iii) in Proposition 4.5). If we therefore choose the free constant α to be sufficiently negative (recall that this is consistent with the hypothesis of Proposition 4.5), we can achieve that $1 + \alpha < \kappa_i + \mu_i$ for all i. The unique solution of Eq. (4.43) in $X_{\tilde{\delta},\kappa_i+\mu_i,\infty}$ is therefore indeed $C^i_1 \equiv 0$. We have thus derived the following statement.

Proposition 4.11. *Let $U = U_* + W$ be any solution of the first-order system Eqs.* (4.25) – (4.30) *with U_* given by Eqs.* (4.31)–(4.32) *by smooth data, and with $W, DW \in X_{\tilde{\delta},\hat{\kappa}+\hat{\mu},\infty}$ for $\hat{\kappa}$ given by Eqs.* (4.33)–(4.34) *and for any $\hat{\mu} > 0$. We further assume that $k(x) \in (0,1)$ for all $x \in T^1$. Then*

$$u := (U^1_{-1}, \dots, U^6_{-1})^T$$

is a solution in $C^\infty((0,\tilde{\delta}] \times T^1)$ of the second-order system Eq. (4.22) *of the form $u = u_* + w$ with*

$$u_* := (U^1_{-1*}, \dots, U^6_{-1*})^T, \quad w := (W^1_{-1}, \dots, W^6_{-1})^T$$

where $w, Dw \in X_{\tilde{\delta},\kappa+\mu,\infty}$. Moreover, for each $i = 1, \dots, d$, we have

$$U^i_1 = W^i_1 = t\partial_x u^i,$$

and hence

$$W^i_1 = t\partial_x u^i_* + \tilde{W}^i_1, \tag{4.44}$$

with $\tilde{W}^i_1 \in X_{\tilde{\delta},\kappa_i+1+\tilde{\mu}_i,\infty}$ for some $\tilde{\mu}_i > 0$.

It is evident that Proposition 4.11 in particular applies to all solutions U of Proposition 4.5. Note, however, that some of the assumptions needed for Proposition 4.5–for example the restriction $k \in (0, 3/4)$–are not necessary here.

Verifying that the spacetimes are solutions

In Section 4, we have used the singular initial value problem to show that for a choice of asymptotic data contained in $\mathcal{P}$, one can develop a spacetime which satisfies the system Eq. (3.1) and matches this choice of asymptotic data. There is no guarantee, however, that this spacetime is a vacuum solution of the Einstein equations. In this subsection, as the second major part of the proof of Theorem 4.1, we show that indeed this spacetime is a vacuum solution. As noted above in the outline of the proof, before carrying through this verification that we have a proof, it useful to establish certain estimates for the shift, $g_{01} = U^3_{-1}$.

Carrying out Step 2a: We state and establish the desired estimates for the shift in the following result:

Proposition 4.12. *Suppose that in addition to the hypothesis of Proposition 4.5, the function F_{10} in Condition (v) satisfies*

$$F_{10} = \frac{1}{2}\left(-2k\frac{E'_*}{E_*} + 4kE_*^2 Q_{**} Q'_* + \left(1-k^2\right)\frac{R'_*}{R_*} - 3\frac{g'_{00*}}{g_{00*}} + \frac{g'_{11*}}{g_{11*}}\right). \tag{4.45}$$

Then the solution U whose existence is asserted by Proposition 4.5 has the property that there exists an exponent scalar $\gamma > 0$ such that the shift quantities U^3_{-1}, U^3_0, U^3_1 are contained in $X_{\tilde{\delta},(k^2-1)/2+1+\gamma,\infty}$.

The proof of this proposition proceeds as follows. Presuming that the hypothesis of Proposition 4.5 and Eq. (4.45) hold, we denote the solution of the first-order system asserted by Proposition 4.5 by $\hat{U}$. It follows from Eq. (4.44) that we can write $\hat{U}$ as $\hat{U}_* + \hat{W}$ with $\hat{U}_*$ taking the form

$$\hat{U}_* = (\hat{U}^1_*, \ldots, \hat{U}^6_*)^T, \quad \hat{U}^i_* = (u^i_*, Du^i_* - \alpha u^i_*, t\partial_x u^i_*)^T, \tag{4.46}$$

where u_* is given by Eq. (4.31). It further follows that $\hat{W}$ and $D\hat{W}$ are contained in $X_{\tilde{\delta},\hat{\kappa}+\hat{\mu},\infty}$ with

$$\hat{\kappa} := (\kappa_1, \kappa_1, \kappa_1; \ldots; \kappa_6, \kappa_6, \kappa_6) \tag{4.47}$$

where $\kappa_1, \ldots, \kappa_6$ are given by Eq. (4.33), and with

$$\begin{aligned}\hat{\mu} = (&\hat{\mu}_1, \hat{\mu}_1, \hat{\mu}_1 + 1; \hat{\mu}_2, \hat{\mu}_2, \hat{\mu}_2 + 1; \hat{\mu}_3, \hat{\mu}_3, \hat{\mu}_3 + 1;\\ &\hat{\mu}_4, \hat{\mu}_4, \hat{\mu}_4 + 1; \hat{\mu}_5, \hat{\mu}_5, \hat{\mu}_5 + 1; \hat{\mu}_6, \hat{\mu}_6, \hat{\mu}_6 + 1),\end{aligned} \tag{4.48}$$

where the quantities $\hat{\mu}_i$ are presumed to satisfy the inequalities in Condition (iii) in Proposition 4.5 with μ_i replaced by $\hat{\mu}_i$.

The basic idea is now to solve Eqs. (4.25) – (4.30) with the same data as in Proposition 4.5, but now *only for the shift quantities* U^3_{-1}, U^3_0, U^3_1, and to incorporate the PDEs for these quantities into a singular initial value problem with improved exponents. In doing this, we note that the less than optimal exponent for the shift quantities in Proposition 4.5 is a consequence of the restrictive block diagonal condition which is needed for the complete system. If instead we presume that all components of U are known—i.e., if we set $U = \hat{U}$ except for the components U^3_{-1}, U^3_0 and U^3_1 and if we then throw away all of the evolution equations from Eqs. (4.25) – (4.30) except for the ones for U^3_{-1}, U^3_0 and U^3_1—then the block diagonal condition becomes less restrictive, as we see below. This reduced system of PDEs can be rewritten as a first-order evolution system for the "unknowns" U^3_{-1}, U^3_0, and U^3_1 only, with all of the matrices and coefficients determined by the other components of $\hat{U}$:

$$\mathbf{s}^0 D\begin{pmatrix}U^3_{-1}\\U^3_0\\U^3_1\end{pmatrix} + \mathbf{s}^1 t\partial_x\begin{pmatrix}U^3_{-1}\\U^3_0\\U^3_1\end{pmatrix} + \tilde{\mathbf{n}}\begin{pmatrix}U^3_{-1}\\U^3_0\\U^3_1\end{pmatrix} = \tilde{g}[U^3_{-1}, U^3_0, U^3_1].$$

Here $\tilde{g}$ is some source term (which we note is quite lengthy). We now consider the singular initial value problem for these equations for U^3_{-1}, U^3_0, and U^3_1 with vanishing leading order terms

$$U^3_{-1} = W^3_{-1} \in X_{\tilde{\delta},\kappa_3+\mu_3,\infty}, \quad U^3_0 = W^3_0 \in X_{\tilde{\delta},\kappa_3+\mu_3,\infty}, \quad U^3_1 = W^3_1 \in X_{\tilde{\delta},\kappa_3+\mu_3,\infty},$$

where $\mu_3 > 0$ is thus far unspecified. Clearly, $U^3_{-1} = \hat{U}^3_{-1}$, $U^3_0 = \hat{U}^3_0$, $U^3_1 = \hat{U}^3_1$ is a solution of this singular initial value problem if $\mu_3 \leq \hat{\mu}_3$. Using only the available information concerning the components of $\hat{U}$ which is implied by Eqs. (4.46) – (4.48) together with Eq. (4.45), we can show (as a consequence of Theorem 2.10) that this singular initial value problem has a unique solution, provided that

$$0 < \mu_3 < 1 + \min\{\hat{\mu}_1, \hat{\mu}_6, \xi_1\}.$$

We notice that if we were to not assume Eq. (4.45), then we would find the same statement (as above) for

$$0 < \mu_3 < 1.$$

We may, as a first step, choose $\mu_3 \leq \hat{\mu}_3$ (which is always smaller than 1), and then use uniqueness to conclude that the particular choice $U^3_{-1} = \hat{U}^3_{-1}$, $U^3_0 = \hat{U}^3_0$, $U^3_1 = \hat{U}^3_1$ is the only solution of this singular initial value problem, as expected. We may then choose μ_3 to be a bit larger than one, which implies that this solution indeed has the property asserted by Proposition 4.12.

Carrying out Step 2b: To verify that the spacetimes constructed above in Section 4 are solutions of the Einstein equations, it is sufficient to show that, in terms of a chosen coordinate system, the quantities $\mathcal{D}_i$ vanish on these spacetimes. We show this here by setting up a singular initial value problem for $\mathcal{D}_i$ with vanishing leading-order data.

We start by using the definition Eq. (3.2) together with expressions Eqs. (4.8) – (4.11) for the Gowdy symmetric metric and expressions Eq. (4.4) for the gauge source functions to obtain the following formulas for the quantities $\mathcal{D}_i$:

$$\begin{aligned}
\mathcal{D}_0 &= -\frac{1}{t} + F_0 + \frac{R_t}{R} + \frac{g_{01}g_{00,x} - g_{00}g_{01,x} + \frac{1}{2}g_{00}g_{11,t} - \frac{1}{2}g_{11}g_{00,t}}{g_{00}g_{11} - g_{01}^2} \\
\mathcal{D}_1 &= F_1 + F_{10} + \frac{R_x}{R} + \frac{g_{01}g_{11,t} - g_{11}g_{01,t} + \frac{1}{2}g_{11}g_{00,x} - \frac{1}{2}g_{00}g_{11,x}}{g_{00}g_{11} - g_{01}^2} \\
\mathcal{D}_2 &= \mathcal{D}_3 = 0.
\end{aligned} \tag{4.49}$$

If we then differentiate these formulas with respect to t, replacing second time derivatives of g_{ij} by means of the Einstein evolution equations Eq. (4.18) and Eq. (4.19), we obtain corresponding (lengthy) formulas for $D\mathcal{D}_0$ and $D\mathcal{D}_1$. Based on these formulas, we now verify that the leading order terms in the quantities $\mathcal{D}_0$, $\mathcal{D}_1$, $D\mathcal{D}_0$, and $D\mathcal{D}_1$ (which we refer to collectively as the "gauge-violation quantities") all vanish, so long as the asymptotic data satisfy a certain asymptotic constraint condition, Eq. (4.51).[1]

Lemma 4.13. *Suppose that in addition to the hypothesis of Proposition 4.5, the function F_{10} in Condition (v) of Proposition 4.5 satisfies Eq. (4.45). Let U be the solution of the first-order evolution equations asserted by Proposition 4.5. Then, there exists an exponent scalar $\gamma > 0$ such that the corresponding constraint violation quantities satisfy*

$$\mathcal{D}_0, D\mathcal{D}_0 \in X_{\tilde{\delta},-1+\gamma,\infty}. \tag{4.50}$$

If in addition to the above conditions, the asymptotic data satisfies

$$F_{10} = -\frac{g'_{00*}}{2g_{00*}} + \frac{g'_{11*}}{2g_{11*}} - \frac{R'_*}{R_*}, \tag{4.51}$$

then

$$\mathcal{D}_1, D\mathcal{D}_1 \in X_{\tilde{\delta},\gamma,\infty}. \tag{4.52}$$

Before proving this lemma, we note the following:

Remark 4.14. It follows immediately from Eq. (4.50) and Eq. (4.52) that the leading order terms of the constraint-violation quantities vanish. We stress that we obtain these conclusions *only* if the asymptotic data satisfy both Eq. (4.45) and Eq. (4.51). In particular, if Eq. (4.45) holds but

[1] We note that this asymptotic constraint condition is included in the hypothesis of our main result, Theorem 4.1.

Eq. (4.51) is violated, we can show that $\mathcal{D}_1 \in X_{\tilde{\delta},0,\infty}$. Hence Eq. (4.51) can be interpreted as the condition which makes $\mathcal{D}_1$ vanish in leading order at $t=0$.

We can write the two asymptotic constraints Eqs. (4.45) and Eq. (4.51) in the following form:

$$\begin{aligned}\frac{g'_{00*}}{g_{00*}} &= -k\frac{E'_*}{E_*} + 2kE_*^2 Q_{**}Q'_* + \frac{3-k^2}{2}\frac{R'_*}{R_*},\\ \frac{g'_{11*}}{g_{11*}} &= \frac{g'_{00*}}{g_{00*}} + 2\frac{R'_*}{R_*} + 2F_{10}.\end{aligned}$$

The first two of these equations is the origin of the integral constraint Eq. (4.1) for the asymptotic data $g_{00*}, E_*, Q_*, Q_{**}$ in Theorem 4.1 and for Eqs. (4.2). The second equation is equivalent to Eq. (4.3). We remark that if one uses the more common parametrization of the asymptotic data $E_* = e^{P_{**}}$ and $g_{00*} = -e^{\Lambda_{**}/2}$, if one imposes the "conformal gauge condition" $g_{00*} = -g_{11*}$ (which is usually part of the areal gauge assumption) and and if one sets $F_{10}=0$, then these conditions imply

$$R'_* = 0, \quad \Lambda'_{**} = -2k(P'_{**} - 2e^{2P_{**}}Q_{**}Q'_*). \tag{4.53}$$

These formulas are familiar for the singular initial value problem of Gowdy solutions in areal gauge [26, 7].

Proof of Lemma 4.13: We presume that the hypothesis of Proposition 4.5 and Eq. (4.45) both hold. As a consequence of Proposition 4.12 and Eq. (4.44), we can argue (as in the proof of Proposition 4.12) that U can be written as $U_* + W$, with U_* given by

$$U_* = (U_*^1, \ldots, U_*^6)^T, \quad U_*^i = (u_*^i, Du_*^i - \alpha u_*^i, t\partial_x u_*^i)^T, \tag{4.54}$$

where u_* is given by Eq. (4.31). Moreover, it follows that W and DW are contained in $X_{\tilde{\delta},\hat{\kappa}+\hat{\mu},\infty}$, with

$$\begin{aligned}\hat{\kappa} := (&\kappa_1,\kappa_1,\kappa_1;\kappa_1,\kappa_1,\kappa_1;\kappa_2,\kappa_2,\kappa_2;\kappa_3+1,\kappa_3+1,\kappa_3+1;\\ &\kappa_4,\kappa_4,\kappa_4;\kappa_5,\kappa_5,\kappa_5;\kappa_6,\kappa_6,\kappa_6),\end{aligned} \tag{4.55}$$

with κ given by Eq. (4.33), and with

$$\begin{aligned}\hat{\mu} = (&\mu_1,\mu_1,\mu_1+1;\mu_2,\mu_2,\mu_2+1;\mu_3,\mu_3,\mu_3+1;\\ &\mu_4,\mu_4,\mu_4+1;\mu_5,\mu_5,\mu_5+1;\mu_6,\mu_6,\mu_6+1).\end{aligned} \tag{4.56}$$

All of the quantities μ_i except for μ_3 are assumed to satisfy the inequalities in Condition (iii) in Proposition 4.5, while μ_3 is some (sufficiently small) positive exponent (see Proposition 4.12). Using techniques similar to those we have applied above to derive expansions of operator functions, we verify (i) that $\mathcal{D}_0 \in X_{\tilde{\delta},-1+\gamma,\infty}$ with

$$\gamma = \min\{\xi_0, \mu_1\},$$

and (ii) that

$$\mathcal{D}_1 - \left(F_{10} + \frac{g'_{00*}}{2g_{00*}} - \frac{g'_{11*}}{2g_{11*}} + \frac{R'_*}{R_*}\right)$$

is contained in $X_{\tilde{\delta},\gamma,\infty}$ for

$$\gamma = \min\{\xi_1, \mu_1, \mu_3\}.$$

Similar arguments apply to the more complicated expressions of $D\mathcal{D}_0$ and $D\mathcal{D}_1$, thereby completing the proof of Lemma 4.13.

Having now derived the function spaces Eqs. (4.50) and (4.52) for the constraint violation quantities (presuming that the asymptotic constraints hold), our next step is to show that the constraint violation quantities must be *identically zero*. We know that the constraint violation quantities associated with a solution of the evolution equations must satisfy the constraint propagation system Eq. (3.4) with Eqs. (4.17) and (4.36). This system takes the form Eq. (A.1); i.e,

$$\sum_{k,l=0}^{1} g^{kl}\partial_{x^k}\partial_{x^l}\mathcal{D}_i = 2H_i \tag{4.57}$$

where H_i is determined by Eq. (3.4). We wish to replace this second-order PDE system with a first-order system (so that we can apply our results concerning the well-posedness of singular initial value problems); we do this using the ideas discussed in Appendix A. More specifically, we combine $\mathcal{D}_0$ and $\mathcal{D}_1$) into a vector

$$v = (\mathcal{D}_0, \mathcal{D}_1)^T,$$

we label the first derivatives of components of v in the form

$$V^i_{-1} := v^i, \quad V^i_0 := Dv^i - \alpha v^i, \quad V^i_1 := t\partial_x v^i, \quad V^i := (V^i_{-1}, V^i_0, V^i_1)^T, \tag{4.58}$$

for $i = 1, 2$, where α is a constant to be fixed below (possibly different from the constant α discussed above), and we combine these to form the six-dimensional vector

$$V := (V^1, V^2)^T. \tag{4.59}$$

One readily verifies that the second-order system for v implies a first-order system for V of the form Eqs. (A.5)–(A.9); i.e.,

$$S^0(t,x)DV(t,x) + S^1(t,x)t\partial_x V(t,x) + N(t,x)V(t,x) = 0, \tag{4.60}$$

where

$$S^0 = \mathrm{diag}(\mathbf{s}^0, \mathbf{s}^0), \quad S^1 = \mathrm{diag}(\mathbf{s}^1, \mathbf{s}^1),$$

with $\mathbf{s}^0$ and $\mathbf{s}^1$ given by Eqs. (4.27) and (4.28). The special form of the third term in Eq. (4.60) is a consequence of linear homogeneity.

To show that the singular initial value problem for V based on Eq. (4.60) is well-posed, we need to verify a certain fall-off rate for the matrix N. To do this, we first note that it follows from its construction (based on Eq. (4.57)) that N is fully determined by the components of the first-order vector U corresponding to the given solution of the evolution equations. More specifically, presuming that the hypothesis of Proposition 4.5 and Eq. (4.45) both hold, we argue (as in the proof of Lemma 4.13) that U is of the form $U_* + W$ with U_* given Eq. (4.54) and u_* given by Eq. (4.31). Moreover, W and DW are in $X_{\tilde{\delta},\hat{\kappa}+\hat{\mu},\infty}$ with $\hat{\kappa}$ given by Eq. (4.55) and κ by Eq. (4.33), and with $\hat{\mu}$ given by Eq. (4.56) where all quantities μ_i, except for μ_3, are assumed to satisfy the inequalities stated in Condition (iii) of Proposition 4.5, while μ_3 is some (sufficiently small) positive exponent. It follows that

$$N - N_0 \in X_{\tilde{\delta},\zeta,\infty}$$

for some $\zeta > 0$ where

$$N_0 := \begin{pmatrix} -\alpha & -1 & 0 & 0 & 0 & 0 \\ (\alpha+1)^2 & \alpha+2 & 0 & 0 & 0 & \frac{g_{00*}}{g_{11*}} \\ 0 & 0 & (1+\alpha)\frac{g_{00*}}{g_{11*}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & -1 & 0 \\ 0 & 0 & -2 & \alpha(\alpha+1) & \alpha+1 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1+\alpha)\frac{g_{00*}}{g_{11*}} \end{pmatrix}; \tag{4.61}$$

cf. Eq. (2.14) of Definition 2.6. Noting that the eigenvalues of the matrix $(S^0_0)^{-1}N_0$ are

$$\Lambda = (1, 1, -1-\alpha; 0, 1, -1-\alpha), \tag{4.62}$$

we determine that it follows from Theorem 2.10 that *if* we can show that the vector field V satisfies the regularity condition

$$V \in X_{\tilde{\delta},(\gamma,\gamma,\gamma,\gamma,\gamma,\gamma),\infty} \tag{4.63}$$

then the singular initial value problem for V based on Eq. (4.60) has a unique solution for any $\gamma > 0$; here the particular structure of the exponent in Eq. (4.63) is a consequence of the block diagonal condition. Since $V \equiv 0$ solves this singular initial value problem, it follows (presuming Eq. (4.63)) that this is the only solution of Eq. (4.60) in the space Eq. (4.63).

As noted in Section 4 (following the preview of Step 1c), in fact there is a mismatch between the regularity for V provided by Lemma 4.13– as stated explicitly in Eq. (4.50) and Eq. (4.52)– and that needed for the singular initial value problem to be well-posed, as stated in Eq. (4.63). To compare these, we note that the regularity provided by Lemma 4.13 can be stated as

$$V \in X_{\tilde{\delta},(-1+\gamma,-1+\gamma,-1+\gamma,\gamma,\gamma,\gamma),\infty}. \tag{4.64}$$

To show that in fact the conditions hypothesized in Lemma 4.13 *are* sufficient to guarantee the regularity Eq. (4.63), we use arguments very similar to those used in Proposition 4.12 (in Step 2a) to prove the required enhanced regularity of the shift. Specifically, presuming that the hypothesis of Lemma 4.13 and Eq. (4.51) hold, we readily determine that Eq. (4.52) implies estimates for $\mathcal{D}_1$ and $D\mathcal{D}_1$ which are sufficient for Eq. (4.63). The required estimates for $\mathcal{D}_0$ and $D\mathcal{D}_0$ are not so immediate. To obtain them, we choose *any* function $\mathcal{D}_0$ which is consistent with the above stated regularity (we do not, however, choose $\mathcal{D}_0 \equiv 0$ since this is one of the things we are aiming to show) and we work with Eq. (4.60) as an evolution system *only* for $V^0 = (V^0_{-1}, V^0_0, V^0_1)^T$; i.e., we delete the evolution equations for the now *given* quantity $V^1 = (V^1_{-1}, V^1_0, V^1_1)^T$, but keep the evolution equations for the now *unknown* quantity $V^0 = (V^0_{-1}, V^0_0, V^0_1)^T$.

For this smaller system here with a hence less restrictive block diagonal condition we are led to conclude that this singular initial value problem has a unique solution

$$V^0 = (V^0_{-1}, V^0_0, V^0_1) \in X_{\tilde{\delta},(-1+\eta,-1+\eta,-1+\eta),\infty}$$

provided $0 < \eta < 1 + \gamma$. In analogy with the arguments in the proof of Proposition 4.12, one yields the sought improved estimates. We thus have verified that indeed Eq. (4.63) holds. The argument leading to the vanishing of $\mathcal{D}_i$ follows, and we have the following result:

Proposition 4.15. *The constraint violation quantities in Lemma 4.13 vanish identically, i.e.,*

$$\mathcal{D}_0 \equiv \mathcal{D}_1 \equiv 0$$

on the whole existence interval $(0, \tilde{\delta}]$ *of the solution* U.

As noted above, the vanishing of the constraint violation quantities implies that the spacetimes built in Steps 1a-1c are solutions of the vacuum Einstein equations.

Verifying that the spacetimes exhibit AVTD behavior

To complete the proof of our main result, Theorem 4.1, it remains to show that these Gowdy spacetimes exhibit AVTD behavior in terms of the general (wave-type) coordinates employed in the constructions described in Steps 1a-1c. We do this here.

Carrying out Step 3: The concept of solutions of Einstein's equations exhibiting AVTD behavior has been formalized in [21, 19, 22] through the introduction of a "velocity term dominated" (VTD) PDE system. The VTD system consists of both evolution and constraint equations and is constructed, with respect to a given system of coordinates, by dropping the spatial derivative terms in the Einstein evolution equations and in the Hamiltonian constraint. A solution of the full Einstein equations is said to be AVTD with respect to the chosen system of coordinates if it approaches, in a suitable norm, a solution to the VTD system (or its leading order).

We recall the usual procedure in the literature for establishing the existence of AVTD solutions. The VTD evolution system forms a spatially parameterized system of ODEs. It may be possible to find explicit solutions to this system, although knowledge of the leading order behavior is sufficient to establish the VTD property. One establishes the existence of solutions with AVTD behavior by first setting up a singular initial value problem for the evolution equations, where the leading order term is chosen to be in agreement with the VTD solution. In a subsequent step, one formulates a singular initial value problem for the Hamiltonian and Momentum constraint violation quantities. It follows that provided certain constraints on the spatially-varying asymptotic data hold, one obtains unique solutions to the full Einstein system. Moreover, it follows that if the singular initial value problem takes the Fuchsian form Definition 2.6, then by definition these solutions must be AVTD.

To facilitate this discussion, it is useful to introduce a bit of terminology concerning systems Eq. (2.10). By the corresponding *truncated system* we mean the first-order system formed from Eq. (2.10) by dropping the spatial derivative terms $\sum_{a=1}^{n} S^a(U)t\partial_a U$. The following corollary of Theorem 2.10 concerns existence of solutions to the singular initial value problem for such a truncated system.

Corollary 4.16 (of Theorem 2.10)**.** *Suppose that for a system Eq.* (2.10) *the conditions of Theorem 2.10 have been met for some leading order term $U_*(t,x)$, with asymptotic data (parametrized by a set of quantities δ and μ) satisfying certain constraints $\mathcal{C}$. Then the corresponding truncated system also satisfies the conditions of Theorem 2.10 with the same leading order term $U_*(t,x)$, and with the same (μ and δ parametrized) asymptotic data satisfying $\mathcal{C}$. Thus there exists a family of solutions with leading order term $U_*(t,x)$, parametrized by the same set of asymptotic data, to the corresponding truncated system.*

This corollary follows from the definition of a Fuchsian system, Definition 2.6. For such a Fuchsian system the function operators $W \mapsto tS^a(W)$, which are the coefficients of the spatial derivative terms, are (by definition) (μ, ζ, ∞)-operators for some exponent vector $\zeta > 0$. As such, the spatial derivative terms are guaranteed to be higher order in t than the terms which match the decay of W, and thus these terms do not constrain the singular decay rate of the solutions obtained in Theorem 2.10. The singular initial value problem for the truncated system can be seen as just a special case of Theorem 2.10, with ζ approaching infinity.

As we discuss now, the AVTD property of the solutions under consideration in Theorem 4.1 is almost a consequence of Corollary 4.16. To see this, we consider the family of solutions constructed as discussed in Theorem 4.1, with functions F_0 and F_1 (cf. Eq. (4.4)) in function spaces parametrized by ξ_0, ξ_1 and satisfying Condition (iii). For any such choice of gauge, this family (which we label as $\mathcal{S}_{\xi,F}$) is parametrized by the set of asymptotic data, $\mathcal{P}$, satisfying the relations Conditions (i) - (ii) of that theorem. In particular, these solutions satisfy the evolution equations Eqs. (4.25) – (4.30), and the hypotheses of Proposition 4.5. An application of Corollary 4.16 verifies the existence of a corresponding family of solutions, which we denote by $\widetilde{\mathcal{S}}_{\xi,F}$, to the corresponding truncated system with the same functions F_0 and F_1 in function spaces parametrized by ξ_0, ξ_1 and which is parametrized by the same set of asymptotic data $\mathcal{P}$. This argument shows that each of the solutions to the full Einstein system obtained in Theorem 4.1 approaches a corresponding solution of the first-order truncated evolution equations.

We now argue that the first-order truncated system is almost equivalent to the VTD system associated to the Einstein equations. One might worry that the system has been truncated at first-order, not second-order, and hence the spatial derivatives are still there in the form of the first-order fields U_1^i. One finds that in the truncated system the equations for the U_1^i decouples from the other equations and forms a homogeneous system of ODE. It follows from the uniqueness of the solutions in Corollary 4.16 that $U_1^i = 0$ is the only solution, and as a consequence this system is equivalent to the first-order system formed from the VTD equations.

In our application there is an additional subtlety due to the (non-standard) definition of Q in Eqs. (4.10) and (4.11). As a result of this definition, the truncated system corresponding to Eqs. (4.25) – (4.30) (with, in addition, $U_1^i = 0$ in accord with the argument above) differs from the first-order VTD system by terms proportional to $Q'_*(x)$ and $Q''_*(x)$. This simply reflects that fact that in our choice of the variable Q we have already "accounted for" part of the VTD leading order term, and moreover, it is straightforward to check that the truncated system with these terms removed has the same existence properties as the full truncated system.

In summary we have established that for any fixed set of asymptotic data and gauge source functions consistent with the constraints and restrictions of Theorem 4.1 the two singular initial value problems, (i) for the full Einstein equations (asserted by Theorem 4.1), and, (ii) for the VTD equations, each have a solution. Because both solutions have the same asymptotic data and their remainders are controlled by the same t-dependent norms, their difference approaches zero in the sense of the function spaces in Theorem 4.1. We have therefore established that the solutions given in Theorem 4.1 are AVTD.

Does this demonstration that the solutions $\mathcal{S}_{\xi,F}$ exhibit AVTD behavior include the Einstein constraint equations as well as the evolution equations? In fact it does; this follows from Eq. (3.3), which relates the Einstein constraints to the vanishing of the generalized wave gauge constraint violation quantities. It follows from this relation that the vanishing of $\partial_t\mathcal{D}_0$ and $\partial_t\mathcal{D}_1$ to leading order is equivalent to the constraints vanishing at leading order. For areal coordinates, this equivalence is manifest in Eq. (4.53).

5 Main solution space and relationship between coordinate systems

It is well-established that in terms of areal coordinates, T^3-Gowdy solutions generically exhibit AVTD behavior. Since the main result of this work is the demonstration that there are Gowdy solutions which exhibit AVTD behavior in terms of generalized wave coordinates as well, it is useful to examine the relationship between AVTD behavior as seen in alternative coordinate systems, and how such features change under coordinate transformations from one system to another. We do this analysis here; for brevity, we omit some of the technical details.

We recall that it follows from Theorem 4.1 that for each choice of data k, g_{11*}, g_{00**}, R_*, E_*, Q_*, Q_{**}, and for each choice of the gauge source functions of the form Eq. (3.12) which are consistent with the restrictions of the theorem, there is a unique metric g which solves Einstein's vacuum equations and which is given in the unique coordinate representation Eqs. (4.5) – (4.14). For the present discussion, we consider any two such metrics $g^{(1)}$ and $g^{(2)}$ of Theorem 4.1 to be the same — and hence we write $g^{(1)} = g^{(2)}$ — if and only if they are determined by the same data and the same gauge source functions. We consider two sets of data and gauge source functions as the same if and only if they are the same in the sense of functions, respectively. We stress that in the discussion here we are intentionally *not* considering diffeomorphism-equivalence classes of solutions of Theorem 4.1.

Let $\mathcal{S}$ be the set of all solutions obtained from the theorem in this sense. Let $\mathcal{S}^{\mathrm{A}} \subset \mathcal{S}$ be the subset of *areal solutions*; i.e., the subset of $\mathcal{S}$ which is determined by the special data $g_{00**} = 1$, $R_* = 1$, $F_{10} = 0$ and $F_0 \equiv F_1 \equiv 0$, and where all other data functions are subject to the standard areal Gowdy constraint

$$\int_0^{2\pi} \left(-k(x)\frac{E_*'(x)}{E_*(x)} + 2k(x)E_*^2(x)Q_{**}(x)Q_*'(x) \right) dx = 0.$$

The two constraints Eqs. (4.2) and (4.3) then imply that

$$-g_{00*}(x) = g_{11*}(x) = e^{\int_0^x \left(-k(\xi)\frac{E_*'(\xi)}{E_*(\xi)} + 2k(\xi)E_*^2(\xi)Q_{**}(\xi)Q_*'(\xi) \right) d\xi}.$$

Comparing Theorem 4.1 with areal-coordinate AVTD results [26, 7], we conclude that all elements in $\mathcal{S}^{\mathrm{A}}$ have the property $R \equiv t$, $g_{01} \equiv 0$ and $g_{00} \equiv -g_{11}$, and hence these metrics are indeed represented in areal coordinates.

In order to distinguish areal coordinates in the following discussion from any other coordinate system consistent with Theorem 4.1 we refer to the former as $(t^{\mathrm{A}}, x^{\mathrm{A}}, y^{\mathrm{A}}, z^{\mathrm{A}})$. We shall demonstrate now that the following type of coordinate transformations plays an important role for Theorem 4.1:

$$\begin{aligned}
t^{\mathrm{A}}(t,x,y,z) &= t^{\mathrm{A}}(t,x) = (\tau(x) + f_0(t,x))t, \\
x^{\mathrm{A}}(t,x,y,z) &= x^{\mathrm{A}}(t,x) = x + h_0(x) + (h_1(x) + f_1(t,x))t^2, \\
y^{\mathrm{A}}(t,x,y,z) &= y, \quad z^{\mathrm{A}}(t,x,y,z) = z,
\end{aligned} \tag{5.1}$$

for so far unspecified smooth 2π-periodic (with respect to x) functions $\tau(x)$, $h_0(x)$, $h_1(x)$, $f_0(t,x)$ and $f_1(t,x)$ which have the property that $\tau(x) > 0$ and $h_0'(x) > -1$ for all $x \in T^1$, and that

f_0 and f_1 are in $X_{\delta,\eta,\infty} \cap C^\infty((0,\delta] \times T^1)$ for some $\eta > 0$. If $\delta > 0$ is sufficiently small as we always assume, the map $(t, x, y, z) \mapsto (t^A, x^A, y^A, z^A)$ is invertible on $(0, \delta] \times T^3$ and hence indeed a coordinate transformation. Each such coordinate transformation maps any element $g^A \in \mathcal{S}^A$ to some g; that is, it transforms any metric from its representation in areal coordinates (t^A, x^A, y^A, z^A) to its representation in some other coordinates (t, x, y, z). We can show under suitable further technical assumptions on f_0 and f_1 that

$$g_{00}(t,x) = -g^A_{00*}(x + h_0(x))\tau^2(x)(\tau(x))^{(k^2(x+h_0(x))-1)/2} \cdot \cdot\, t^{(k^2(x+h_0(x))-1)/2}(1 + \ldots), \tag{5.2}$$

$$g_{11}(t,x) = g^A_{00*}(x + h_0(x))(1 + h_0'(x))^2(\tau(x))^{(k^2(x+h_0(x))-1)/2} \cdot \cdot\, t^{(k^2(x+h_0(x))-1)/2}(1 + \ldots), \tag{5.3}$$

$$g_{01}(t,x) = g^A_{00*}(x + h_0(x))\,(2h_1(x)(1 + h_0'(x)) - \tau(x)\tau'(x)) \cdot \cdot\, (\tau(x))^{(k^2(x+h_0(x))-1)/2} t^{(k^2(x+h_0(x))+1)/2}(1 + \ldots), \tag{5.4}$$

$$g_{02} \equiv g_{03} \equiv g_{12} \equiv g_{13} \equiv 0, \tag{5.5}$$

and

$$R(t,x) = t\,\tau(x)(1 + \ldots), \tag{5.6}$$

$$E(t,x) = E^A_*(x + h_0(x))\tau(x)^{-k(x+h_0(x))} t^{-k(x+h_0(x))}(1 + \ldots), \tag{5.7}$$

$$Q(t,x) = Q^A_{**}(x + h_0(x))\tau(x)^{2k(x+h_0(x))} t^{2k(x+h_0(x))}(1 + \ldots), \tag{5.8}$$

where the data which determine the original areal solution g^A are labelled with ^{A}. Here we write

$$H_1(t,x) = H_2(t,x) + \ldots$$

for two arbitrary functions H_1 and H_2 provided $H_1 - H_2$ is a function in $X_{\delta,\epsilon,\infty} \cap C^\infty((0,\delta] \times T^1)$ for some $\epsilon > 0$. In order to make the following discussion fully rigorous we would need to give precise estimates of the higher-order terms represented by "..." above in terms of η. It is not difficult to obtain those, but for brevity we do not discuss them here. One can show that if we choose η "sufficiently large" then everything in the following is justified rigorously.

Further calculations, similar to those which led to Eqs. (5.2) – (5.8), allow us to find

$$\Gamma_0 = -\frac{1}{t} + \ldots, \tag{5.9}$$

$$\Gamma_1 = -\frac{2h_1(1 + h_0')}{\tau^2} - \frac{\tau'}{\tau} + \frac{h_0''}{1 + h_0'} + \ldots. \tag{5.10}$$

Now, if $g^A \in \mathcal{S}^A$ and hence the metric represented by g^A is a solution of the vacuum equation, the same is true for the image metric of the coordinate transformation above which we continue to refer to as g. Nevertheless, this g is *not* always in $\mathcal{S}$. In particular we observe that if the leading term in Eq. (5.4) does not vanish, then Eqs. (4.7) and (4.15) of our theorem are violated. However, g must be a solution of Theorem 4.1 and hence be an element of $\mathcal{S}$ if, (i), the asymptotic data for g implied by the leading terms of Eqs. (5.2) – (5.8) satisfy the constraints Eqs. (4.1) – (4.3) of Theorem 4.1, and if, (ii), η is sufficiently large so that Condition (iii) of Theorem 4.1 is met. This is a consequence of uniqueness. It turns out that this is the case if and only if, (i),

$$2h_1(x)(1 + h_0'(x)) - \tau(x)\tau'(x) = 0, \tag{5.11}$$

i.e., the leading term in Eq. (5.4) indeed vanishes, and (ii), η is sufficiently large.

Next, let Ξ denote the set of all coordinate transformations of the form above which is consistent with (5.11) and for which η is sufficiently large. As we have seen, any element ϕ of Ξ defines a map

$$\Phi_{(\phi)} : \mathcal{S}^A \to \mathcal{S}_{(\phi)}, \quad g^A \mapsto g$$

given by Eqs. (5.2) – (5.8), where

$$\mathcal{S}_{(\phi)} := \Phi_{(\phi)}(\mathcal{S}^{\mathrm{A}}) \subset \mathcal{S}.$$

We can show that for each $\phi \in \Xi$, this map $\Phi_{(\phi)}$ is bijective. It is obvious that $\mathcal{S}_{(\phi)}$ is a proper subset of $\mathcal{S}$ and that

$$\cup_{\phi\in\Xi}\mathcal{S}_{(\phi)} \subset \mathcal{S}.$$

An interesting question, which arises from this but which we shall not fully answer in this paper, is whether

$$\cup_{\phi\in\Xi}\,\mathcal{S}_{(\phi)} = \mathcal{S}. \tag{5.12}$$

If the answer is no, then there exists at least one solution guaranteed by Theorem 4.1 which cannot be obtained from an areal solution by means of a coordinate transformation $\phi \in \Xi$. Does $\mathcal{S}$ possibly contain solutions which are *geometrically* distinct from areal solutions? Or, could the equality in Eq. (5.12) fail just because the class of coordinate transformations Ξ is not general enough?

In order to approach such questions, we need to study whether it is possible to construct a coordinate transformation which maps an arbitrary solution g in $\mathcal{S}$ to an areal solution g^{A} in $\mathcal{S}^{\mathrm{A}}$. Here we can exploit the fact that in the generalized wave formalism this coordinate transformation map must be a solution of the following system of wave equations (cf. Eq. (3.5)):

$$\begin{aligned}\Box_g t^{\mathrm{A}}(t,x) &= -\mathcal{F}^{\mathrm{A}\,0}(t^{\mathrm{A}}(t,x),x^{\mathrm{A}}(t,x)) = \frac{1}{t^{\mathrm{A}}(t,x)} g^{\mathrm{A}\,00}(t^{\mathrm{A}}(t,x),x^{\mathrm{A}}(t,x))\\ &= \frac{1}{t^{\mathrm{A}}(t,x)}\left(g^{00}(t,x)\left(\frac{\partial t^{\mathrm{A}}}{\partial t}\right)^2 + 2g^{01}(t,x)\frac{\partial t^{\mathrm{A}}}{\partial x}\frac{\partial t^{\mathrm{A}}}{\partial t} + g^{11}(t,x)\left(\frac{\partial t^{\mathrm{A}}}{\partial x}\right)^2\right),\\ \Box_g x^{\mathrm{A}}(t,x) &= 0.\end{aligned}$$

The idea would be to formulate a singular initial value problem for this system with the leading-order behavior given by Eq. (5.1) and Eq. (5.11). If this turned out to be successful and certain further technical details were met we would be able to decide whether Eq. (5.12) is true.

Finally, another consequence of Eq. (5.1) and Eq. (5.11). It suggests that the assumption that the shift g_{01} be $o(t^{(k^2+1)/2})$ which we were forced to make in the course of the proof of our main theorem is possibly of purely technical nature. Namely, Eq. (5.4) shows that a metric with a shift which does not satisfy this assumption can easily be generated via the coordinate transformation Eq. (5.1) simply by violating Eq. (5.11).

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. OISE-1210144 while author EA was visiting the University of Otago on an EAPSI Fellowship in 2012. Portions of this paper were written during a visit of the author PLF at the University of Otago with the financial support of FB's "Divisional assistance grant" and during a visit of the author FB to the Université Pierre et Marie Curie with the support from the Agence Nationale de la Recherche via the grant 06-2–134423. This material is also based upon work supported by the National Science Foundation under Grant No. 0932078 000, while all authors were in residence at the Mathematical Science Research Institute in Berkeley, California, during the semester/year of 2013. The authors also gratefully acknowledge the support of the Institute Henri Poincaré during the 2015 Program of the Centre Émile Borel on Mathematical General Relativity. The author FB was also partly funded by a University of Otago Research Grant 2013. The author PLF also gratefully acknowledges support from the Simons Center for Geometry and Physics, Stony Brook University, in January 2015. JI is partially supported by NSF grant PHY-1306441.

References

[1] E. Ames, F. Beyer, J. Isenberg, and P. G. LeFloch. Quasilinear hyperbolic Fuchsian systems and AVTD behavior in T^2-symmetric vacuum spacetimes. *Ann. Henri Poincaré*, 14(6):1445–1523, 2013.

[2] E. Ames, F. Beyer, J. Isenberg, and P. G. LeFloch. Quasilinear symmetric hyperbolic Fuchsian systems in several space dimensions. In M. Agranovsky, M. Ben-Artzi, G. J. Galloway, L. Karp, V. Maz'ya, S. Reich, D. Shoikhet, G. Weinstein, and L. Zalcman, editors, *Complex Analysis and Dynamical Systems V*. American Mathematical Society, Providence, Rhode Island, 2013.

[3] L. Andersson and A. D. Rendall. Quiescent cosmological singularities. *Commun. Math. Phys.*, 218(3):479–511, 2001.

[4] B. K. Berger, D. Garfinkle, J. Isenberg, V. Moncrief, and M. Weaver. The singularity in generic gravitational collapse is spacelike, local and oscillatory. *Mod. Phys. Lett. A*, 13(19):1565–1574, 1998.

[5] B. K. Berger, J. Isenberg, and M. Weaver. Oscillatory approach to the singularity in vacuum spacetimes with T^2 isometry. *Phys. Rev. D*, 64(8), 2001.

[6] B. K. Berger and V. Moncrief. Evidence for an oscillatory singularity in generic $U(1)$ symmetric cosmologies on $T^3 \times R$. *Phys. Rev. D*, 58(6):064023, 1998.

[7] F. Beyer and P. G. LeFloch. Second-order hyperbolic Fuchsian systems and applications. *Class. Quantum Grav.*, 27(24):245012, 2010.

[8] F. Beyer and P. G. LeFloch. Second-order hyperbolic Fuchsian systems: Asymptotic behavior of geodesics in Gowdy spacetimes. *Phys. Rev. D*, 84(8):084036, 2011.

[9] F. Beyer and P. G. LeFloch. Dynamics of self-gravitating fluids in Gowdy-symmetric spacetimes near cosmological singularities. Preprint. arXiv:1512.07187.

[10] Y. Choquet-Bruhat and R. P. Geroch. Global aspects of the Cauchy problem in General Relativity. *Commun. Math. Phys.*, 14(4):329–335, 1969.

[11] Y. Choquet-Bruhat, J. Isenberg, and V. Moncrief. Topologically general $U(1)$ symmetric vacuum space-times with AVTD behavior. *Nuovo Ciment. B*, 119(7-9):625–638, 2004.

[12] P. T. Chruściel, J. Isenberg, and V. Moncrief. Strong cosmic censorship in polarised Gowdy spacetimes. *Class. Quantum Grav.*, 7(10):1671–1680, 1990.

[13] D. M. Eardley, E. Liang, and R. K. Sachs. Velocity-dominated singularities in irrotational dust cosmologies. *J. Math. Phys.*, 13(1):99, 1972.

[14] Y. Fourès-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Math.*, 88(1):141–225, 1952.

[15] D. Garfinkle. Numerical simulations of generic singularities. *Phys. Rev. Lett.*, 93(16), 2004.

[16] R. H. Gowdy. Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: Topologies and boundary conditions. *Ann. Phys.*, 83(1):203–241, 1974.

[17] S. W. Hawking and R. Penrose. The singularities of gravitational collapse and cosmology. *Proc. R. Soc. London A*, 314:529–548, 1970.

[18] J. Isenberg. On strong cosmic censorship. In L. Bieri and S.-T. Yau, editors, *Surveys in Differential Geometry (2015). One hundred years of General Relativity*, pages 17–36. International Press, Boston, 2015.

[19] J. Isenberg and S. Kichenassamy. Asymptotic behavior in polarized T^2-symmetric vacuum space–times. *J. Math. Phys.*, 40(1):340, 1999.

[20] J. Isenberg and V. Moncrief. The existence of constant mean curvature foliations of Gowdy 3-torus spacetimes. *Commun. Math. Phys.*, 86(4):485–493, 1982.

[21] J. Isenberg and V. Moncrief. Asymptotic behavior of the gravitational field and the nature of singularities in Gowdy spacetimes. *Ann. Phys.*, 199(1):84–122, 1990.

[22] J. Isenberg and V. Moncrief. Asymptotic behaviour in polarized and half-polarized $U(1)$ symmetric vacuum spacetimes. *Class. Quantum Grav.*, 19(21):5361–5386, 2002.

[23] S. Kichenassamy and A. D. Rendall. Analytic description of singularities in Gowdy spacetimes. *Class. Quantum Grav.*, 15(5):1339–1355, 1999.

[24] V. Moncrief and D. M. Eardley. The global existence problem and cosmic censorship in General Relativity. *Gen. Rel. Grav.*, 13(9):887–892, 1981.

[25] R. Penrose. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.*, 14(3):57–59, 1965.

[26] A. D. Rendall. Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity. *Class. Quantum Grav.*, 17(16):3305–3316, 2000.

[27] H. Ringström. Existence of an asymptotic velocity and implications for the asymptotic behavior in the direction of the singularity in T^3-Gowdy. *Comm. Pure Appl. Math.*, 59(7):977–1041, 2006.

[28] H. Ringström. Strong cosmic censorship in T^3-Gowdy spacetimes. *Ann. Math.*, 170(3):1181–1240, 2009.

[29] F. Ståhl. Fuchsian analysis of $S^2 \times S^1$ and S^3 Gowdy spacetimes. *Class. Quantum Grav.*, 19(17):4483–4504, 2002.

A First-order reduction of second-order wave equations

In this portion of the appendix, we describe the reduction of certain quasilinear second-order PDE systems to first-order symmetric hyperbolic PDE systems. The second-order systems we consider here take the form

$$\sum_{i,j=0}^{n} g^{ij}(t,x,u(t,x),\partial_t u(t,x),\partial_{x^a} u(t,x))\partial_{x^i}\partial_{x^j} u(t,x) \\ = 2H(t,x,u(t,x),\partial_t u(t,x),\partial_{x^a} u(t,x)) \tag{A.1}$$

where x^i (for i running from 0 to n, with $x^0 = t$) are local coordinates on an $(n+1)$-dimensional manifold M, where $u(x^i) = u(t,x^a) = u(t,x)$ (for $a = 1,\dots,n$) is an unknown $\mathbb{R}^d$-valued function on M, where $g^{ij}(t,x,u(t,x))$ are components of the inverse of a Lorentz-signature metric on M, and where $H(t,x,u(t,x),\partial_t u(t,x),\partial_{x^a} u(t,x))$ is an $\mathbb{R}^d$-valued function of the indicated variables. We presume that g^{ij} and H are specified function of the indicated quantities, and the system Eq. (A.1) is to be solved for u.

Since we are in particular interested in systems with degeneracies at $t = 0$, we find it useful to multiply both sides of Eq. (A.1) by t^2, and then rewrite (A.1) in the form

$$D^2u - 2\sum_{a=1}^{n} G^{0a} t\partial_{x^a} Du - \sum_{a,b=1}^{n} G^{ab} t^2 \partial_{x^a}\partial_{x^b} u - Du = \frac{2t^2}{g^{00}} H, \tag{A.2}$$

where $D := t\partial_t$, $G^{ij} := -\frac{g^{ij}}{g^{00}}$, and $G := (G^{ab})$.

To obtain first-order form, we define the variables

$$\begin{aligned} U_{-1}^J &:= u^J, \quad U_0^J := Du^J - \alpha u^J, \quad U_a^J := t\partial_{x^a} u^J, \\ U^J &:= (U_{-1}^J, U_0^J, U_1^J, \ldots, U_n^J)^T, \end{aligned} \tag{A.3}$$

for $J = 1, \ldots, d$ and $a = 1, \ldots, n$, and we define the $(n+2) \cdot d$-vector

$$U := (U^1, \ldots, U^d)^T. \tag{A.4}$$

Here α is a constant, which is useful in the Fuchsian analysis of these equations. In terms of U, Eq. (A.2) implies the first-order system

$$S^0 DU + \sum_{a=1}^{n} S^a t\partial_{x^a} U + \tilde{N} U = \tilde{f}[U], \tag{A.5}$$

with

$$S^0 = \mathrm{diag}(\mathbf{s}^0, \ldots, \mathbf{s}^0), \quad S^a = \mathrm{diag}(\mathbf{s}^a, \ldots, \mathbf{s}^a), \quad \tilde{N} = \mathrm{diag}(\tilde{\mathbf{n}}, \ldots, \tilde{\mathbf{n}}), \tag{A.6}$$

where

$$\mathbf{s}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & G \end{pmatrix}, \quad \mathbf{s}^a = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & -2G^{0a} & -G^{1a} & \ldots & -G^{na} \\ 0 & -G^{1a} & & & \\ \vdots & \vdots & & \mathbf{0}_n & \\ 0 & -G^{na} & & & \end{pmatrix}, \tag{A.7}$$

and

$$\tilde{\mathbf{n}} = \begin{pmatrix} -\alpha & -1 & 0 & \ldots & 0 \\ -(1-\alpha)\alpha & -1+\alpha & 0 & \ldots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & -(1+\alpha)G & \\ 0 & 0 & & & \end{pmatrix}, \tag{A.8}$$

and with

$$\begin{aligned} \tilde{f}[U] = \Bigg(& 0, \frac{2t^2}{g^{00}} H^1 + 2\alpha \sum_{a=1}^{n} G^{0a} U_a^1, 0, \ldots, 0; \ldots; \\ & 0, \frac{2t^2}{g^{00}} H^d + 2\alpha \sum_{a=1}^{n} G^{0a} U_a^d, 0, \ldots, 0 \Bigg)^T. \end{aligned} \tag{A.9}$$

This system is symmetric hyperbolic so long as the matrix G, which generally depends on the solution, is positive definite. We note that in these matrix equations, we use $\mathbf{0}_m$ to denote the $m \times m$-zero matrix.

It is relatively straightforward to verify the equivalence of the first-order system Eq. (A.5)-(A.9) and the original second-order system Eq. (A.1). In one direction, it follows from the derivation of system (A.5)-(A.9) that if an $\mathbb{R}^d$-valued function u is $C^2((0,\delta) \times T^3)$ and satisfies the second-order system (A.1), then if we define U by Eq. (A.3)-(A.4), U must satisfy the first-order system Eq. (A.5)-(A.9).

Going in the other direction, we consider an $\mathbb{R}^{(n+2)d}$-valued function U which is $C^1((0,\delta)\times T^3)$ and satisfies Eq. (A.5)-(A.9). If we define

$$u := (U_{-1}^1, \ldots, U_{-1}^d)^T, \tag{A.10}$$

we find that it is generally not a solution of Eq. (A.2). However, if all of the quantities U_{-1}^J are C^2, and if the $n \cdot d$ functions

$$C_a^J := U_a^J - t\partial_{x^a} U_{-1}^J \tag{A.11}$$

vanish identically for all $a = 1, \ldots, n$ and $i = 1, \ldots, d$, then u is indeed a classical solution of the second-order system Eq. (A.2).

In fact, it is sufficient that we know that the C_a^J quantities vanish for a particular value $t_* \in (0, \delta)$. This follows immediately from the first-order ODE system

$$DC_a^J - (1 + \alpha)C_a^J = 0 \tag{A.12}$$

for C_a^J, which is implied by the first-order system Eq. (A.5)-(A.9). Clearly if C_a^J vanishes at $t = t_*$ and if satisfies the above linear homogenous ODE system, then it vanishes for all $t \in (0, \delta)$.

B Some technical results and our computer algebra code

Fix some $\delta > 0$. As in Subsection 2, we consider $\mathbb{R}^d$-valued functions u, which can be written as $u_* + w$ for some fixed $u_* \in C^\infty((0, \delta] \times T^n) \cap X_{\delta,\kappa,\infty}$ and arbitrary functions $w \in X_{\delta,\kappa+\mu,\infty}$ for exponent d-vectors κ and $\mu > 0$. Let two function operators $w \mapsto f(w)$ and $w \mapsto g(w)$ be given. For the following it is useful to introduce the notation $w \mapsto f(w) = g(w) + O(t^\nu)$ if the function operator $w \mapsto f(w) - g(w)$ is a $(\kappa + \mu, \nu, \infty)$-operator for some exponent ν.

We consider the following algebraic operations involving function operators. The proofs of the following statements can be derived straightforwardly from the ideas in [1].

Sum of two function operators. Let ν_1 and ν_2 be two exponent scalars. Let a scalar-valued $(\kappa+\mu, \nu_1, \infty)$-operator $w \mapsto g_1(w)$ and a scalar-valued $(\kappa+\mu, \nu_2, \infty)$-operator $w \mapsto g_2(w)$ be given. Then the map $w \mapsto g_1(w) + g_2(w)$ is a $(\kappa + \mu, \min\{\nu_1, \nu_2\}, \infty)$-operator[1]. Moreover, for any two other function operators

$$w \mapsto h_1(w) := g_1(w) + O(t^{\eta_1}) \quad \text{and} \quad w \mapsto h_2(w) := g_2(w) + O(t^{\eta_2}),$$

for exponent scalars η_1, η_2, we have

$$w \mapsto h_1(w) + h_2(w) = g_1(w) + g_2(w) + O(t^{\min\{\eta_1,\eta_2\}}).$$

Product of two function operators. Given the same function operators as before, the map $w \mapsto g_1(w)g_2(w)$ is a $(\kappa + \mu, \nu_1 + \nu_2, \infty)$-operator, and

$$w \mapsto h_1(w)h_2(w) = g_1(w)g_2(w) + O(t^{\min\{\nu_1+\eta_2,\nu_2+\eta_1,\eta_1+\eta_2\}}).$$

Inverse of a function operator. Suppose that $w \mapsto P(w)$ is a scalar-valued $(\kappa + \mu, \zeta, \infty)$-operator for some $\zeta > 0$. Then $w \mapsto 1/(1 + P(w))$ is a $(\kappa + \mu, 0, \infty)$-operator, and

$$w \mapsto \frac{1}{1 + P(w)} = 1 - P(w) + O(t^{2\zeta}).$$

Now let (i) η, γ, ν be exponent scalars with $\nu < \gamma < \eta$, (ii) h_0 be a function in $X_{\delta,\nu,\infty}$ such that $1/h_0 \in X_{\delta,-\nu,\infty}$, and, (iii) $w \mapsto g(w)$ be a $(\kappa + \mu, \gamma, \infty)$-operator. Suppose

$$w \mapsto P(w) = h_0 + g(w) + O(t^\eta). \tag{B.1}$$

Then, we have

$$w \mapsto \frac{1}{P(w)} = \frac{1}{h_0} - \frac{g(w)}{h_0^2} + O(t^{-\nu+\min\{2(\gamma-\nu),\eta-\nu\}}). \tag{B.2}$$

[1] With a slight abuse of notation we write $\min\{\nu_1, \nu_2\}$ for any smooth function $\nu(x)$ with has the property $\nu(x) < \min\{\nu_1(x), \nu_2(x)\}$ for every $x \in T^1$. Here we consider any two exponent scalars ν_1 and ν_2. Notice that we can always choose the difference between $\nu(x)$ and the actual (possibly non-differentiable) function $\min\{\nu_1(x), \nu_2(x)\}$ to be arbitrarily small.

In our applications here, all of the function operators are rational (see Definition 2.4) and hence are built using (possibly very many) terms each of which has a simple structure to which the algebraic rules above apply. Each term can be written as

$$w \mapsto \frac{H^{(1)}[w]}{H^{(2)}[w]}, \tag{B.3}$$

where both $w \mapsto H^{(1)}[w]$ and $w \mapsto H^{(2)}[w]$ are scalar polynomial function operators. More specifically, we can assume that there is a smooth function $P^{(1)}(t,x)$ in $X_{\delta,\nu,\infty}$ for some exponent scalar ν and non-negative integers $i_1,\dots i_d$ such that

$$H^{(1)}[w](t,x) = P^{(1)}(t,x)\cdot(u_{*1}(t,x)+w_1(t,x))^{i_1}\cdots(u_{*d}(t,x)+w_d(t,x))^{i_d}. \tag{B.4}$$

This function operator can be analyzed by (i) considering the map $w \mapsto u_{*i}+w_i$ as a $(\kappa+\mu,\kappa_i+\mu_i,\infty)$-operator if $u_{*i}(t,x)=0$ for all $(t,x)\in(0,\delta]\times T^n$ or as a $(\kappa+\mu,\kappa_i,\infty)$-operator if $u_{*i}(t,x)\neq 0$ for some $(t,x)\in(0,\delta]\times T^n$, and, (ii) applying the above algebraic rules. Similarly, we consider the "trivial" map $w\mapsto P^{(1)}$ as a $(\kappa+\mu,\nu,\infty)$-operator. For most of our applications, we want to prove that $H^{(1)}$ satisfies a "linear expansion" of the form

$$H^{(1)}[w](t,x) = H_0^{(1)}(t,x)+\sum_{i=1}^{d} H_i^{(1)}(t,x)w_i(t,x)+O(t^\gamma) \tag{B.5}$$

and we want to determine the functions $H_0^{(1)}(t,x)$, $H_1^{(1)}(t,x)$, $\dots$, $H_d^{(1)}(t,x)$ explicitly and estimate the exponent scalar γ in terms of κ and μ. In order to achieve this, we expand Eq. (B.4) using the algebraic rules above and "linearize" every product as follows

$$w \mapsto w_i w_j = O(t^{\kappa_i+\kappa_j+\mu_i+\mu_j}),$$

for each $i,j=1,\dots,d$. While this linearization is justified rigorously, it may not always give optimal results because in complicated expressions there may be important cancellations of nonlinear terms. In practice one may therefore end up with formally correct, but useless linear expansions.

Regarding the denominator $H^{(2)}[w]$ in Eq. (B.3) we proceed in basically the same way as for the numerator. In general, $H^{(2)}[w]$ is a finite sum of terms of the form Eq. (B.4) and hence we can use the same algebraic rules and algorithm as above to derive an "expansion" of the form Eq. (B.5), i.e.,

$$H^{(2)}[w](t,x) = H_0^{(2)}(t,x)+\sum_{i=1}^{d} H_i^{(2)}(t,x)w_i(t,x)+O(t^\gamma), \tag{B.6}$$

for some possibly different exponent γ. In doing this, the idea is to apply the above rule for the inverse of function operators and finally multiply the result with the numerator function operator Eq. (B.5) using again the same rules. Eventually one obtains a "linear expansion" of the same form as in Eq. (B.5), but now for the *full* function operator in Eq. (B.3)

$$\frac{H^{(1)}[w]}{H^{(2)}[w]}(t,x) = P_0(t,x)+\sum_{i=1}^{d} P_i(t,x)w_i(t,x)+O(t^\gamma), \tag{B.7}$$

for an in general again different exponent γ.

In practice, one needs to pay particular attention in applying the inverse rule above because it only holds under strict assumptions. Fortunately, we find that while there are very many different numerator function operators in our applications, only a few different denominator operators appear. Hence, we are able to check that the assumptions for the inverse rule hold explicitly for each of these operators.

Our computer algebra code. In practical applications we have to deal with function operators which consist of hundreds of terms of the form above. Each term can be processed by means of the simple algebraic rules discussed above. The analysis therefore becomes a very repetitive task which is performed very well by means of computer algebra. Indeed, we have implemented all the rules above and all the function operators which appear in our applications, using Mathematica. We stress that the results obtained in this way are fully rigorous and, in particular, no numerical approximation is used anywhere.