

TWO CONSEQUENCES OF DAVIES'S HARDY INEQUALITY

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In memory of M. Z. Solomyak, on the occasion of his 90th birthday

1. INTRODUCTION

In this short note we would like to show that one can use Davies's Hardy inequality to rederive well-known results of Lieb [8] and Rozenblum [10]. Throughout the following we fix an open set $\Omega \subset \mathbb{R}^d$ and define, for $\omega \in \mathbb{S}^{d-1}$,

$$\delta(x) := \left(d |\mathbb{S}^{d-1}|^{-1} \int_{\mathbb{S}^{d-1}} d_\omega(x)^{-2} d\omega \right)^{-1/2} \quad \text{where} \quad d_\omega(x) := \inf\{|t| : x + t\omega \notin \Omega\}$$

(with the convention that $\inf \emptyset = 0$). Then Davies's Hardy inequality [1] states that

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega} \delta^{-2} |u|^2 dx \quad \text{for all } u \in H_0^1(\Omega). \quad (1)$$

The following simple lemma is key to our argument.

Lemma 1. *For any $x \in \Omega$ and any $\rho > 0$,*

$$|\Omega \cap B_\rho(x)| \geq (1 - d^{-1} \rho^2 \delta(x)^{-2}) |B_\rho(x)|.$$

Proof. We have

$$|\Omega \cap B_\rho(x)| = \int_{\mathbb{S}^{d-1}} \int_0^\rho \mathbb{1}_\Omega(x + t\omega) t^{d-1} dt d\omega$$

and clearly, for any $\omega \in \mathbb{S}^{d-1}$ with $d_\omega(x) > \rho$, we have $x + t\omega \in \Omega$ for all $t \in (0, \rho)$. Thus,

$$|\Omega \cap B_\rho(x)| \geq |\{\omega \in \mathbb{S}^{d-1} : d_\omega(x) > \rho\}| d^{-1} \rho^d. \quad (2)$$

On the other hand, clearly,

$$\rho^{-2} |\{\omega \in \mathbb{S}^{d-1} : d_\omega(x) \leq \rho\}| \leq \int_{\mathbb{S}^{d-1}} d_\omega(x)^{-2} d\omega = d^{-1} |\mathbb{S}^{d-1}| \delta(x)^{-2},$$

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In March 2022, Yi Huang made us aware of the fact that Theorem 2 and its derivation from Davies's Hardy inequality are known and appear as [2, Theorem 1.5.11]. Moreover, he pointed out a typographical error in the definition of $\delta(x)$, which propagated through the paper and which is corrected here. We thank Yi Huang for these comments.

or, equivalently,

$$|\{\omega \in \mathbb{S}^{d-1} : d_\omega(x) > \rho\}| \geq (1 - d^{-1}\rho^2\delta(x)^{-2}) |\mathbb{S}^{d-1}|.$$

Inserting this bound into (2) implies the lemma. $\square$

2. A THEOREM OF LIEB

Let $-\Delta_\Omega^D$ be the Dirichlet Laplacian in $L^2(\Omega)$ and

$$\lambda_\Omega := \inf \operatorname{spec}(-\Delta_\Omega^D) = \inf \left\{ \int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_\Omega |u|^2 dx = 1 \right\}. \quad (3)$$

It is well-known that if Ω is mean-convex, then λ_Ω is bounded from below by a constant times the inverse square of the radius of the largest ball contained in Ω and that this is not true for general open Ω . It is a theorem of Lieb [8] that this remains true for general open Ω , provided ‘the largest ball contained in Ω ’ is replaced by ‘a ball that intersects Ω significantly’. Here we give a simple alternative proof of this result using (1) (albeit with a worse constant)¹.

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be open. Then for any $\rho > 0$,*

$$\lambda_\Omega \geq \frac{d}{4\rho^2} \left(1 - \sup_{x \in \Omega} \frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} \right).$$

Clearly, this theorem implies for all $0 < \theta < 1$,

$$\lambda_\Omega \geq \frac{d(1-\theta)}{4\rho_\theta^2}, \quad \text{where} \quad \rho_\theta := \inf \left\{ \rho > 0 : \sup_{x \in \Omega} \frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} \leq \theta \right\}.$$

Proof. Inserting (1) into (3), we obtain

$$\lambda_\Omega \geq \frac{1}{4} \inf \left\{ \int_\Omega \delta^{-2} |u|^2 dx : u \in H_0^1(\Omega), \int_\Omega |u|^2 dx = 1 \right\} \geq \frac{1}{4} \inf_\Omega \delta^{-2}.$$

Inserting the lower bound on δ^{-2} from Lemma 1 we obtain the theorem. $\square$

Remarks. (1) The theorem remains valid for the principal eigenvalue of the p -Laplacian with $1 < p < \infty$. This follows from the validity of the analogue of (1) for $1 < p < \infty$. Lieb’s proof works also in the case $p = 1$.

(2) If λ is an eigenvalue of $-\Delta_\Omega$, then there is an $x \in \Omega$ such that for all $\rho > 0$, $\lambda \geq d(4\rho^2)^{-1}(1 - |\Omega \cap B_\rho(x)|/|B_\rho(x)|)$. This follows from the same method of proof, by noting that in this case the inequality $\lambda \geq (1/4) \int_\Omega \delta^{-2} |u_0|^2 dx$ for a normalized eigenfunction u_0 implies that there is an $x \in \Omega$ with $\lambda \geq 1/(4\delta(x)^2)$.

(3) Lieb’s result was improved upon in [9] in the sense that the overlap between Ω and $B_\rho(x)$ is quantified in terms of capacity instead of measure. It would be interesting to investigate whether there is a strengthening of (1) that implies this result.

¹Davies’s result [1, Theorem 1.5.11] and Lieb’s [8] are superior if the supremum in Theorem 2 becomes very small.

3. A THEOREM OF ROZENBLUM

We denote by $N_{\leq}(\lambda, -\Delta_{\Omega}^D)$ the total spectral multiplicity of $-\Delta_{\Omega}^D$ in the interval $[0, \lambda]$. It is well-known [10] that for Ω of finite measure, one has Weyl asymptotics $N_{\leq}(\lambda, -\Delta_{\Omega}^D) \sim (2\pi)^{-d} \omega_d |\Omega| \lambda^{d/2}$ as $\lambda \rightarrow \infty$, as well as a universal bound $N_{\leq}(\lambda, -\Delta_{\Omega}^D) \leq C_d |\Omega| \lambda^{d/2}$ for all $\lambda > 0$. A theorem of Rozenblum [10] implies, in particular, that sets Ω that satisfy the reverse inequality $N(\lambda, -\Delta_{\Omega}^D) \geq \varepsilon |\Omega| \lambda^{d/2}$ for some $\lambda > 0$ have a substantial ‘well-structured’ component at spatial scale $\lambda^{-1/2}$.

Theorem 3. *For any $\theta \in (0, 1]$ there are constants $c_1(\theta), c_2(\theta, d) > 0$ with the following property. For any open set $\Omega \subset \mathbb{R}^d$ and any $\lambda > 0$ there are disjoint balls $B^{(1)}, \dots, B^{(M)} \subset \mathbb{R}^d$ of radius $c_1 \lambda^{-1/2}$ such that*

$$|\Omega \cap B^{(m)}| \geq (1 - \theta) |B^{(m)}| \quad \text{for all } m = 1, \dots, M$$

and

$$M \geq c_2 N_{\leq}(\lambda, -\Delta_{\Omega}^D).$$

Note that choosing $\lambda = \lambda_{\Omega}$ we obtain again Theorem 2, up to constants.

Proof. We begin by giving the proof in dimension $d \geq 3$, where we have

$$N_{\leq}(\lambda, -\Delta_{\Omega}^D) \leq L_d \int_{\Omega} \left(\lambda - \frac{1}{4\delta(x)^2} \right)_+^{\frac{d}{2}} dx. \quad (4)$$

This appears in [6], but a weaker version with $1/4$ replaced by a smaller constant follows easily by (2) and the CLR inequality (see [4] for references).

Let $E := \{x \in \Omega : \delta(x) \geq (4\lambda)^{-1/2}\}$. Then, by Lemma 1,

$$|\Omega \cap B_{\rho}(x)| \geq (1 - 4d^{-1}\rho^2\lambda) |B_{\rho}(x)| \quad \text{for all } x \in E \text{ and all } \rho > 0.$$

For $\rho = (\theta d / (4\lambda))^{1/2}$ the claimed density condition is satisfied for each such ball.

Let $B_{\rho}(x_m)$ be a maximal disjoint subcollection of $B_{\rho}(x)$, $x \in E$. Then $E \subset \bigcup_m B_{2\rho}(x_m)$ (since for any $x \in E$ there is an x_m such that $B_{\rho}(x)$ intersects $B_{\rho}(x_m)$, so $|x - x_m| < 2\rho$, so $x \in B_{2\rho}(x_m)$). In case there are infinitely many x_m we are done. If there are finitely many x_m , say M , then

$$\begin{aligned} \int_{\Omega} \left(\lambda - \frac{1}{4\delta(x)^2} \right)_+^{\frac{d}{2}} dx &= \int_E \left(\lambda - \frac{1}{4\delta(x)^2} \right)_+^{\frac{d}{2}} dx \leq \lambda^{\frac{d}{2}} |E| \leq \lambda^{\frac{d}{2}} \sum_m |B_{2\rho}(x_m)| \\ &= \omega_d 2^d \lambda^{\frac{d}{2}} \rho^d M = d^{\frac{d}{2}} \omega_d \theta^{\frac{d}{2}} M. \end{aligned}$$

Together with (4) this gives the claimed lower bound on M for $d \geq 3$.

For $d = 2$ (the case $d = 1$ is easy) we bound $N_{\leq}(\lambda, -\Delta_{\Omega}^D) \leq \lambda^{-\gamma} \text{Tr}(-\Delta_{\Omega}^D - 2\lambda)_{-}^{\gamma}$ for any $\gamma > 0$ and use the fact [6] that

$$\text{Tr}(-\Delta_{\Omega}^D - \mu)_{-}^{\gamma} \leq L_{\gamma,2} \int_{\Omega} \left(\mu - \frac{1}{4\delta(x)^2} \right)_+^{\gamma+1} dx.$$

The claimed bound now follows similarly as before. $\square$

- Remarks.* (1) In Rozenblum's formulation, the balls are required to be centered on $(c\lambda^{-1/2})\mathbb{Z}^d$. This can also be achieved by a minor modification of our proof.
- (2) In fact, Rozenblum proves a stronger theorem where the overlap between Ω and $B_\rho(x)$ is quantified in terms of capacity instead of measure. It would be interesting to investigate whether there is a corresponding strengthening of (4).
- (3) A related result for Schrödinger operators was proved in [3].
- (4) Theorem 3 might be useful in the problem of maximizing $\mathrm{Tr}(-\Delta_\Omega - \lambda)_-^\gamma$ among sets Ω of given measure; see [7, 5] for partial results for $\gamma \geq 1$.

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