FLOER FIELD THEORY
FOR COPRIME RANK AND DEGREE

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ABSTRACT. We construct partial category-valued field theories in 2+1-dimensions using Lagrangian Floer theory in moduli spaces of central-curvature unitary connections with fixed determinant of rank r and degree d where r, d are coprime positive integers. These theories associate to a closed, connected, oriented surface the Fukaya category of the moduli space, and to a connected bordism between two surfaces a functor between extended Fukaya categories. We obtain the latter by combining Cerf theory with holomorphic quilt invariants. These functors satisfy the natural composition law.

CONTENTS

1. INTRODUCTION

Floer's instanton homology [16] associates to any homology three-sphere a $\mathbb{Z}/8\mathbb{Z}$-graded group that is a version of the Morse homology of the Chern-Simons functional on the space of $SU(2)$-connections. This homology forms a natural receptacle for a *relative instanton* invariant of four-manifolds with boundary, defined by counting solutions to the anti-self-dual Yang-Mills equations on the four-manifold obtained by attaching a cylindrical end. These invariants satisfy a gluing law for four-manifolds with a common boundary component. In particular, the Donaldson invariants of a four-manifold that splits along a homology three-sphere can be computed from the relative invariants of the two parts [11]. The guiding principle of topological field theory now asks for an invariant of three-manifolds with boundary that satisfies a gluing law such that composition gives rise to the instanton Floer homology for closed three-manifolds. A natural algebraic framework for such an invariant is that of category-valued field theories: functor-valued invariants of bordisms satisfying a composition law. The most well-known example of such a theory is associated

with the *Wess-Zumino-Witten* model of conformal field theory; it associates to a two-dimensional bordism a functor between categories of representations of affine Lie algebras. In particular, the functor associated to the pair of pants gives these categories the structure of a tensor product [23]. This theory extends to a $1 + 1 + 1$-dimensional theory which associates to any closed three-manifold the Witten-Reshetikhin-Turaev invariant. The field theory associated to the anti-self-dual Yang-Mills equations is hoped to be a $2 + 1 + 1$-dimensional theory, including a category-valued $2 + 1$-dimensional theory. A strategy towards defining such an invariant was proposed by Donaldson and Fukaya [17] who, inspired by the Atiyah-Floer conjecture, suggested that the category associated to the 2-manifold should be a category of Lagrangian submanifolds of the moduli space of flat $SU(2)$-bundles.

These moduli spaces, however, are singular due to unavoidable reducible connections on the necessarily trivial bundles. In order to obtain smooth moduli spaces of flat connections, one can consider nontrivial $SO(3)$-bundles. Counting anti-self-dual connections on such bundles give rise to instanton Floer homology for closed 3-manifolds with $b_1 > 0$. These were some of the motivating examples for the definition of the Fukaya category. Fukaya also proposed in [17] that a three-manifold with boundary should give rise to an object of the dual of the Fukaya category. Such an instanton Floer homology with Lagrangian boundary conditions was also proposed by Salamon in [45] and is rigorously constructed in Salamon-Wehrheim [46]. More generally, if we fix coprime positive integers r, d, then there is a unique isomorphism class of $U(r)$-bundles of degree d on each surface X. The moduli space of central-curvature connections with fixed determinant is a smooth symplectic manifold $M(X)$, described in more detail in Section 3. To (various types of) symplectic manifolds one can associate a Fukaya category or its cohomological category, the Donaldson category whose objects are Lagrangian submanifolds and morphism spaces are Floer homology groups. We use this as starting point for the construction of a $2 + 1$-dimensional connected category-valued field theory. The version of the Fukaya category needed is an *extended Fukaya category* $\text{Fuk}^\#(M(X))$ of $M(X)$ whose objects are sequences of correspondences

(1) $$\underline{L} = (L_1, \dots, L_m), \quad L_j \subset N_{j-1}^- \times N_j, \quad j = 1, \dots, m$$

from a point $N_0 = \text{pt}$ to $N_j = M(X)$ as in [55]. The composition maps use counts of *quilted disks* in Ma'u-Wehrheim-Woodward [35].

In this paper we construct a functor from the category of (compact, connected, connected oriented 2-manifolds, 3-bordisms) to the category of (A_∞ categories, homotopy classes classes of A_∞ functors). The theory associates to each compact oriented 2-manifold X the extended Fukaya category $\text{Fuk}^\#(M(X))$ of $M(X)$. Theorem 4.2.1 below proves the existence of functors

$$\Phi(Y) : \text{Fuk}^\#(M(X_-)) \to \text{Fuk}^\#(M(X_+))$$

between these Fukaya categories given by counting holomorphic quilts with boundary and seams in spaces of central-curvature connections over 3-bordisms Y from X_- to X_+. These functors satisfy a natural composition law whenever a bordism is formed

by gluing bordisms Y, Y' along a common boundary:

$$\Phi(Y \circ Y') = \Phi(Y') \circ \Phi(Y).$$

Rather than using anti-self-duality equations, we apply a dimensional reduction. Thus we wish to associate to a bordism Y a Lagrangian correspondence

$$L(Y) \subset M(X_-)^- \times M(X_+)$$

arising from the moduli space of central-curvature fixed-determinant connections on a bundle over Y. While such moduli spaces on an arbitrary bordism are not smooth, the moduli spaces for elementary bordisms (cylinders or handle attachments) are smooth. Thus a Cerf decomposition of the bordism into elementary pieces provides a sequence of Lagrangian correspondences. A categorification functor constructed from pseudoholomorphic quilts in [35] can be used to convert each Lagrangian correspondence to a functor. To show independence of the composition of these functors from the choice of decomposition of the bordism we prove that Cerf moves between the decompositions are reflected by an equivalence (embedded geometric composition) between the associated Lagrangian correspondences, up to a possible grading shift. Finally, equivalent sequences of correspondences give rise to isomorphic functors by the "strip-shrinking" analysis developed in [56]. Because we need to restrict to situations in which there are no reducible connections to obtain smooth symplectic manifolds, not all the axioms of a topological field theory are satisfied. For instance, the surfaces must be connected, which precludes the product axiom. Moreover, the invariant for closed three-manifolds arising from this Floer field theory is trivial, since the moduli spaces associated to two-spheres are empty. However, via a connect sum construction we associate to a closed three-manifold a relatively $\mathbb{Z}/4\mathbb{Z}$-graded quilted Floer homology group (Definition 4.4.1 below) which, by an unproven version of the Atiyah-Floer conjecture, should agree with a $U(r)$ instanton Floer homology of the connect sum. Since the preprint was first circulated other approaches and extensions have appeared in Manolescu-Woodward [32] and Cazassus [8], [9] who use Floer theory in an extended moduli space, and Horton [22], who uses traceless character varieties.

The structure of the paper is the following. Precise definitions of the involved categories can be found in Section 2, leading to a rigorous formulation of this construction framework. This strategy has already been applied to obtain various other gauge theoretic $2 + 1$ field theories. For example, a sequel to this paper [54] uses similar $U(r)$ moduli spaces to develop invariants for tangles that conjecturally correspond to Floer homology invariants arising from singular instantons developed by Kronheimer-Mrowka [26]. A first application of the results of this paper to symplectic mapping class groups of representation varieties was given by Smith [50].

The present paper is an updated and more detailed version of a paper the authors have circulated since 2007. The authors have unreconciled differences over the exposition in the paper, and explain their points of view at https://math.berkeley.edu/~katrin/wwpapers/ resp. http://christwoodwardmath.blogspot.com/. The publication in the current form is the result of a mediation.

2. Field theories via connected Cerf theory

In this section we review a version of the theory of Cerf describing the decomposition of connected bordisms into elementary bordisms between connected surfaces, and the "Cerf moves" between different decompositions. Then we explain how to build a field theory by assigning morphisms to elementary bordisms so that certain "Cerf relations" are satisfied. We begin by fixing the notation for field theories.

2.1. **Field theories.** Our language for topological field theories for bordisms adapts that in, for example, Lurie [31], rephrasing the earlier definition of Atiyah. Let n be a non-negative integer and $X_\pm$ compact oriented n-manifolds. Since the theory of connected bordisms is trivial for $n = 0, 1$, we take $n \geq 2$.

Definition 2.1.1. (Connected bordism category)
(a) A *bordism* from X_- to X_+ is a compact, oriented $n + 1$-manifold Y with boundary equipped with an orientation-preserving diffeomorphism

$$\phi : \partial Y \to \overline{X}_- \sqcup X_+.$$

Here $\overline{X}_-$ denotes the manifold X_- equipped with the opposite orientation.
(b) The *connected bordism category* Bor_{n+1}^0 is the category whose
   (i) objects are compact, connected, oriented n-dimensional smooth manifolds X;
   (ii) morphisms from X_- to X_+ are $n + 1$-dimensional connected bordisms (Y, ϕ) from X_- to X_+ modulo the equivalence given as follows: Set two bordisms $(Y, \phi), (Y', \phi')$ from X_- to X_+ to be equivalent if there exists an orientation-preserving diffeomorphism ψ that extends the given diffeomorphism on their boundaries

$$\psi : Y \to Y', \quad \phi' \circ \psi|_{\partial Y} = \phi : \partial Y \to \overline{X}_- \sqcup X_+;$$

   (iii) composition of morphisms is given by gluing bordisms together: Given two bordisms (Y_1, ϕ_1) from X_0 to X_1 and (Y_2, ϕ_2) from X_1 to X_2, we may glue them together to a bordism $(Y_1, \phi_1) \cup_{X_1} (Y_2, \phi_2)$ from X_0 to X_2. For an explicit construction choose collar neighborhoods

$$\kappa_1 : (-\epsilon, 0] \times X_1 \to Y_1, \quad \kappa_2 : [0, \epsilon) \times X_1 \to Y_2$$

and define

$$Y_1 \circ Y_2 := Y_1 \sqcup Y_2 \sqcup ((-\epsilon, \epsilon) \times X_1) / \sim$$

where $\sim$ is the obvious equivalence given by κ_1, κ_2. The resulting bordism $Y_1 \circ Y_2$ is well-defined up to equivalence, since any two choices of collar neighborhoods are isotopic by [38, Thm.1.4]. Given two morphisms $[Y_1] = [(Y_1, \phi_1)]$ and $[Y_2] = [(Y_2, \phi_2)]$ we define $[Y_1] \circ [Y_2] = [Y_1 \circ Y_2]$ with the boundary identification induced from ϕ_1 and ϕ_2, independent of the choice of collar neighborhood.
   (iv) The identity morphism for a manifold X is represented by the trivial bordism $[0, 1] \times X$ with the obvious identifications on the boundary.

Definition 2.1.2. (Connected field theories) Let $\mathcal{C}$ be a category. A $\mathcal{C}$-*valued con-nected field theory* in $n + 1$ dimensions is a functor Φ from Bor^0_{n+1} to $\mathcal{C}$.

For example, a *connected A_∞ -category-valued field theory* is a field theory taking values in the category of (A_∞ categories, homotopy classes of A_∞ functors). In Section 3.1 below we will construct a connected A_∞ -category-valued field theory by composing a symplectic-valued field theory and the categorification functor from [35], which associates A_∞ functors to Lagrangian correspondences.

Field theories usually allow for disconnected manifolds and bordisms. In this case one would take $\mathcal{C}$ to be a symmetric monoidal category and require the product axiom $\Phi(X_0 \sqcup X_1) = \Phi(X_0) \otimes \Phi(X_1)$ for disjoint unions $X_0 \sqcup X_1$. However, in our examples the Fukaya categories associated to 2-manifolds are well defined only in the connected case: Otherwise the underlying symplectic space, a moduli space of bundles over a disconnected surface, may be singular. Hence we have restricted to the connected bordism category.

2.2. **Cerf decompositions of bordisms.**

In this section we describe the decom-position of bordisms into elementary pieces in the connected bordism category. In the following, let X_-, X_+ be compact, connected, oriented manifolds of dimension $n \geq 1$, and let $(Y, \phi : \partial Y \to \overline{X}_- \sqcup X_+)$ be a compact, connected, oriented bordism from X_- to X_+.

Definition 2.2.1. (Elementary and cylindrical bordisms)

(a) A *Morse datum* for (Y, ϕ) consists of a pair $(f, \underline{b})$ of a Morse function $f : Y \to \mathbb{R}$ and an ordered tuple

(2) $$\underline{b} = (b_0 < b_1 < \ldots < b_m) \in \mathbb{R}^{m+1}$$

such that
   (i) the minima and maxima of f are

$$\phi^{-1}(X_-) = f^{-1}(b_0), \quad \phi^{-1}(X_+) = f^{-1}(b_m);$$

   (ii) each level set $f^{-1}(b)$ for $b \in \mathbb{R}$ is connected;
   (iii) f has distinct values at the (isolated) critical points, i.e. it induces a bijection $\mathrm{Crit} f \to f(\mathrm{Crit} f)$ between critical points and critical values;
   (iv) $b_0, \ldots, b_m \in \mathbb{R} \setminus f(\mathrm{Crit} f)$ are regular values of f such that each interval $(b_{i-1}, b_i), i = 1, \ldots, m$ contains at most one critical value of f:

$$\#\mathrm{Crit}(f) \cap f^{-1}(b_{i-1}, b_i) \leq 1.$$

A Morse function $f : Y \to \mathbb{R}$ is *adapted* to Y if the first condition above (ai) holds.

(b) A connected bordism (Y, ϕ) is an *elementary bordism* if Y admits a Morse datum $(f, \underline{b} = (\min f, \max f))$, that is f is a Morse function with at most one critical point.

(c) (Y, ϕ) is a *cylindrical bordism* if Y admits a Morse datum $(f, \underline{b} = (\min f, \max f))$, where f is a Morse function with no critical point.

(d) A morphism $[(Y, \phi)]$ of Bor_{n+1}^0 is an *elementary resp. cylindrical morphism* if one (and hence all) of its representatives is an elementary resp. cylindrical bordism.

Definition 2.2.2. (Cerf decompositions)

(a) A *Cerf decomposition* of the bordism (Y, ϕ) is a decomposition

$$Y = Y_1 \cup_{X_1} Y_2 \cup_{X_2} \ldots \cup_{X_{m-1}} Y_m$$

into a sequence $(Y_i \subset Y)_{i=1,\ldots m}$ of elementary bordisms embedded in Y that are disjoint from each other and ∂Y except for

$$Y_1 \cap \partial Y = \phi^{-1}(X_-), \quad Y_m \cap \partial Y = \phi^{-1}(X_+), \quad X_i := Y_i \cap Y_{i+1}$$

that are also connected submanifolds in Y of codimension 1. As a consequence we have

$$\partial Y_i \cong \overline{X}_{i-1} \sqcup X_i, \quad i = 1, \ldots, m, \quad X_0 = \phi^{-1}(X_-), \quad X_m = \phi^{-1}(X_+).$$

(b) A *Cerf decomposition* of the morphism $[Y]$ in the connected bordism category Bor_{n+1} is a sequence $([Y_i])_{i=1,\ldots m}$ of elementary morphisms that compose to

$$[(Y, \phi)] = [(Y_1, \phi_1)] \circ [(Y_2, \phi_2)] \circ \ldots \circ [(Y_m, \phi_m)].$$

(c) Two Cerf decompositions

$$[(Y, \phi)] = [(Y_1, \phi_1)] \circ \ldots \circ [(Y_m, \phi_m)] = [(Y_1, \phi_1)'] \circ \ldots \circ [(Y_m, \phi_m)']$$

are *equivalent* if there exist orientation-preserving diffeomorphisms

$$\delta_0 = \mathrm{Id}_{X_0}, \quad \delta_1 : X_1 \to X_1', \ldots, \delta_{m-1} : X_{m-1} \to X_{m-1}', \quad \delta_m = \mathrm{Id}_{X_m}$$

such that for each $i = 1, \ldots, m$,

$$[(Y_i, \phi_i)] = [(Y_i', (\delta_{i-1} \sqcup \delta_i) \circ \phi_i)].$$

Remark 2.2.3. (Cerf decomposition via Morse datum) Any Morse datum $(f, \underline{b})$ for the bordism (Y, ϕ) induces a Cerf decomposition

$$Y = Y_1 \cup_{X_1} \ldots \cup_{X_{m-1}} Y_m, \quad (Y_i := f^{-1}([b_{i-1}, b_i]))_{i=1,\ldots m}$$

into elementary bordisms between the connected level sets $X_i = f^{-1}(b_i)$. Moreover, any Cerf decomposition of a representative $Y \in [Y]$ induces a Cerf decomposition of the morphism $[Y]$. On the other hand, any Cerf decomposition of a bordism or morphism arises from a Morse datum.

Remark 2.2.4. (Handle attachments) The pieces Y_i of a Cerf decomposition have simple topological descriptions as follows: If the elementary bordism Y_i contains no critical point then it is in fact a cylindrical bordism. In that case Y_i is diffeomorphic to the *cylinder* $X_i \times [b_{i-1}, b_i]$, and X_{i-1} is diffeomorphic to X_i.

Suppose Y_i contains a single critical point with index $k \in \{1, \ldots, n\}$. In that case Y_i is obtained (up to homotopy) from the incoming manifold X_{i-1} by attaching a handle $B^k \times B^{n-k}$. Here B^k is a k-ball and the handle $B^k \times B^{n-k}$ is attached via an *attaching map* $\alpha : S^{k-1} \times B^{n-k} \to X_{i-1}$:

$$Y_i \cong X_{i-1} \sqcup (B^k \times B^{n-k})/(x \sim \alpha(x), x \in S^{k-1} \times B^{n-k}).$$

The image of $S^{k-1} \times \{0\}$ in X_{i-1} is an *attaching cycle* and the image of $S^{k-1} \times B^{n-k}$ is viewed as a thickening of the attaching cycle; we often omit the thickenings from the description.

Conversely, Y_i can be obtained from the outgoing manifold X_i up to homotopy by attaching a handle of opposite index to an attaching cycle in X_i. Concrete attaching cycles can be specified by choosing a metric on Y_i. Then the attaching cycles in X_{i-1} resp. X_i are given by the intersection with the stable resp. unstable manifold for the upward gradient flow of f from the unique critical point in Y_i, see Figure 1. As explained in Milnor [38], the notion of attaching cycle can also be defined without a metric via a *gradient-like vector field*, that is, a vector field $v \in \mathrm{Vect}(Y_i)$ such that $D_v f > 0$ everywhere except at the critical points. This notion is sometimes useful in order to, for example, show that any Cerf decomposition can be re-arranged so that the indices of the critical points are in increasing order, see [38, Theorem 8.1].
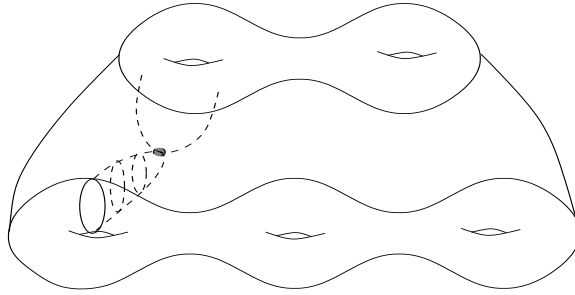

FIGURE 1. Stable and unstable manifolds inside an elementary bordism

Lemma 2.2.5. *For $n \geq 2$ any bordism Y as above admits a Morse datum.*

Proof. By Milnor [38, Theorem 8.1], there exists an adapted Morse function $f : Y \to \mathbb{R}$ such that f is *self-indexing* in the sense that the critical points of index i have critical value i, for each $i = 0, \ldots, n+1$, and furthermore there are no critical points of index 0 or $n + 1$. After a small perturbation, we may assume that the critical values of f are distinct, by Milnor [38, Chapter 4], but still with the property that the order on critical values is the same as that on index:

$$(3) \qquad \forall y, y' \in \mathrm{Crit}(f), \quad (i(y) < i(y')) \implies (f(y) < f(y)').$$

This ordering property (3) implies that the fibers of f are connected. Indeed, each level set $f^{-1}(b)$ is obtained by attaching handles to lower level sets $f^{-1}(b'), b' < b$; the level sets $f^{-1}(b)$ become disconnected either because of a critical point $y \in \mathrm{Crit}(f)$ of index 0, which does not exist by assumption, or by attaching a handle of index n with disconnecting attaching cycle. Once a level set $f^{-1}(b)$ is disconnected, it can be become connected again only by attaching a handle of index one, with the points of the attaching cycle in different components of $f^{-1}(b)$. But since the Morse function f is self-indexing and $n \geq 2$, the n-handles are attached after the 1-handles. The existence of a disconnecting n-handle would imply that X_+ is disconnected, a

contradiction. Given such a Morse function f, let

$$b_0 := \min f, \quad b_m := \max f, \quad m \geq \#\mathrm{Crit} f.$$

There evidently always exists a choice of $b_1 < \ldots < b_{m-1}$ satisfying condition (aiv), hence making $(f, \underline{b})$ a Morse datum. □

Note that our definition of a Cerf decomposition differs from the standard handle decomposition in that we allow the elementary bordisms Y_i to be cylindrical bordisms and we do not keep track of the attaching cycles. This definition also simplifies the moves between different decompositions: Since we do not fix a metric or require the Smale condition of stable and unstable manifolds intersecting transversally, we need not consider the handle slide move discussed by Kirby [25, p. 40]. On the other hand, we use much finer decompositions than Heegaard splittings which are commonly used to define topological invariants via Floer theory. The latter are specific to dimension 3 and for bordisms defined as follows.

Definition 2.2.6. (Heegaard splittings of bordisms) Let Y be a compact connected oriented bordism between compact connected oriented surfaces $X_\pm$.

(a) Y is a *compression body* if it admits a Morse function such that all critical points have the same index (namely 1 or 2):

$$\exists f : Y \to \mathbb{R} \text{ Morse}, \quad (y_1, y_2 \in \mathrm{Crit}(f)) \implies (i(y_1) = i(y_2)).$$

Equivalently, Y is obtained from $\phi^{-1}(X_-)$ by adding only handles of the same index, or from $\phi^{-1}(X_+)$ by adding only handles of the opposite index.

(b) A *Heegaard splitting* of Y is a decomposition $Y = Y_- \cup_X Y_+$ into compression bodies Y_-, Y_+ with a common boundary X, such that Y_- contains $\phi^{-1}(X_-)$ and Y_+ contains $\phi^{-1}(X_+)$, and both are obtained from these boundary components by adding handles of index 1. We call X the *Heegaard surface* of the splitting.

Remark 2.2.7. (Heegaard splittings via Morse functions) Let $f : Y \to \mathbb{R}$ be an adapted Morse function such that all critical points of index 1 have values less than those of the critical points of index 2. Pick a value $c \in \mathbb{R}$ that separates the critical values of index 1 from those of index 2:

$$\forall y_1, y_2 \in \mathrm{Crit}(f), \quad (f(y_1) < f(y_2)) \iff (i(y_1) < i(y_2)).$$

Then

$$Y = Y_- \cup Y_+, \quad Y_- = f^{-1}(-\infty, c], \quad Y_+ = f^{-1}[c, \infty)$$

form a Heegaard splitting of Y. Note that any such f also satisfies (ii) in Definition 2.2.1 automatically, and can be perturbed to satisfy (iii). The function f can then be completed to a Morse datum $(f, \underline{b})$ that induces a special Cerf decomposition of Y, whose elementary bordisms are ordered by index.

Remark 2.2.8. Given two representatives Y and Y' of the same morphism in Bor^0_{n+1} and a diffeomorphism $\psi : Y \to Y'$, any Morse datum $(f, \underline{b})$ for Y' induces a Morse datum $(\psi^* f, \underline{b})$ for Y by pullback. The Cerf decompositions of Y induced by $(f, \underline{b})$ and $(\psi^* f, \underline{b})$ are then equivalent via the collection $(\psi|_{Y_i})_{i=1,\ldots,m}$ of diffeomorphisms of the elementary bordisms.

The existence of Morse data in Lemma 2.2.5 implies that every morphism in the bordism category has a Cerf decomposition. The subsequent Cerf Theorem 2.2.11 implies that Cerf decompositions are unique up to the Cerf moves, which will be defined in the following. For simplicity of notation, we drop the boundary identifications from the notation.

Definition 2.2.9. (Cerf moves) Let Y be a bordism and $[Y] = [Y_1] \circ \ldots \circ [Y_m]$ a Cerf decomposition. A *Cerf move* is one of the following operations on $([Y_i])_{i=1,\ldots,m}$.

(a) A *critical point cancellation* is the move

$$\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y_j \cup Y_{j+1}] \ldots,$$

where for some $j \in \{1, \ldots, m-1\}$ the two consecutive elementary bordisms Y_j, Y_{j+1} compose to a cylindrical bordism $Y_j \cup Y_{j+1} \subset Y$. More precisely, in this situation, critical point cancellation is the move from $([Y_i])_{i=1,\ldots,m}$ to $([Y_i'])_{i=1,\ldots,m'}$ with $m' = m - 1$ given by

$$[Y_i'] = [Y_i], \ i < j, \quad [Y_j'] = [Y_j \cup Y_{j+1}], \quad [Y_i'] = [Y_{i+1}], \ i > j.$$

A *critical point creation* is the same move with the roles of $([Y_i])_{i=1,\ldots,m}$ and $([Y_i'])_{i=1,\ldots,m'}$ interchanged.

(b) A *critical point switch* is the move

$$\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y_j'] \circ [Y_{j+1}'] \ldots,$$

where for some $j \in \{1, \ldots, m-1\}$ the composition $[Y_j] \circ [Y_{j+1}]$ equals $[Y_j'] \circ [Y_{j+1}']$, and the two Cerf decompositions $[Y_j] \circ [Y_{j+1}] = [Y_j'] \circ [Y_{j+1}']$ of the same morphism are given by Morse data $(f, \underline{b})$ and $(f', \underline{b}')$ on representatives with unique critical points $y_{j(+1)} \in Y_{j(+1)}$ and $y_{j(+1)}' \in Y_{j(+1)}'$ in each part, whose attaching cycles (for some choice of a metric) switch in the following sense: The attaching cycles of y_j and y_{j+1} in X_j and those of y_j' and y_{j+1}' in X_j' are disjoint, while in $X_{j-1} = X_{j-1}'$ the attaching cycle of y_j is homotopic to that of y_{j+1}', and the attaching cycle of y_{j+1} is homotopic to that of y_j'; and analogously for the intersections of stable manifolds with $X_{j+1} = X_{j+1}'$. More precisely, in this situation, critical point switch is the move from $([Y_i])_{i=1,\ldots,m}$ to $([Y_i'])_{i=1,\ldots,m'}$ with $m' = m$, $[Y_i'] = [Y_i]$ for $i < j$ and $i > j + 1$, and $[Y_j'] \circ [Y_{j+1}'] = [Y_j] \circ [Y_{j+1}]$ as above.

(c) A *cylinder cancellation* is the move

$$\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y_j \cup Y_{j+1}] \ldots,$$

where for some $j \in \{1, \ldots, m-1\}$ one of the two consecutive elementary morphisms $[Y_j]$, $[Y_{j+1}]$ is cylindrical. Then the composition $[Y_j] \circ [Y_{j+1}]$ is an elementary morphism as well. More precisely, in this situation, critical point cancellation is the move from $([Y_i])_{i=1,\ldots,m}$ to $([Y_i'])_{i=1,\ldots,m'}$ with $m' = m - 1$,

$$[Y_i'] = [Y_i], \ i < j, \quad [Y_j'] = [Y_j] \circ [Y_{j+1}], \quad [Y_i'] = [Y_{i+1}]$$

for $i > j$. A *cylinder creation* is the same move with the roles of $([Y_i])_{i=1,\ldots,m}$ and $([Y_i'])_{i=1,\ldots,m'}$ interchanged.

Remark 2.2.10. (Stabilizations of Heegaard splittings versus Cerf moves) In dimension $n = 2$ we can compare critical point creation to the stabilization of Heegaard splittings. A *stabilization* of a Heegaard splitting is obtained by connect sum with the standard Heegaard splitting of a sphere $S^3 = H_- \cup H_+$ into solid tori H_-, H_+. More precisely, given a Heegaard splitting $Y = Y_- \cup_X Y_+$, its stabilization is obtained by pulling the Heegaard splitting $Y \# S^3 = Y'_- \cup Y'_+$ into compression bodies $Y'_\pm = Y_\pm \# H_\pm$ back to $Y \cong Y \# S^3$. Equivalently, let $[-1, 1] \times X = Y''_- \cup Y''_+$ be the decomposition of the cylindrical bordism consisting of two elementary bordisms, each carrying a Morse function with index 1 resp. 2. Then $Y'_\pm$ is obtained from $Y_\pm$ by attaching $Y''_\pm$ at X, since attaching a one-handle is equivalent to connected sum with a torus. Thus if the Heegaard splitting $Y = Y_- \cup Y_+$ is induced by a Cerf decomposition then the stabilization is obtained from a critical point creation. Conversely any critical point creation can be viewed as a connected sum as above, and so induces a stabilization of the corresponding Heegaard splitting.
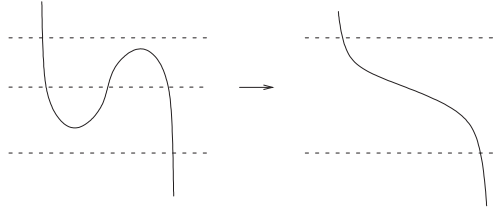

FIGURE 2. Critical point cancellation

Theorem 2.2.11 (Connected Cerf theory). *Let Y be a bordism as before of dimension $n + 1 \geq 3$, and fix a Cerf decomposition $[Y] = [Y_1] \circ \ldots \circ [Y_m]$. Then any other Cerf decomposition of $[Y]$ can be obtained up to equivalence from $([Y_i])_{i=1,\ldots,m}$ by a finite sequence of Cerf moves.*

Remarks on the Proof of Theorem 2.2.11: The statement without the connectedness conditions can be proved using theorems of Thom and Mather [34], see also [33]. A generic homotopy $\tilde{f} : Y \times [0, 1] \to \mathbb{R}$ between two Morse functions f_0, f_1 with distinct critical values has only a finite number of cusp singularities, corresponding to the critical point cancellations and creations, and a finite number of times $s \in [0, 1]$ such that the critical values of $\tilde{f}_s := \tilde{f}(\cdot, s)$ are not distinct, and at such times two critical values cross. A homotopy between Morse functions with connected fibers does not necessarily have connected fibers. However, Gay and Kirby [18, Theorem 2] show that there exists a generic homotopy between any two Morse functions with connected fibers. Let $c_1, \ldots, c_m \in (0, 1)$ be the times for which either the critical values of $\tilde{f}_s$ coincide or $\tilde{f}_s$ is not Morse.

Away from the critical values the Cerf decompositions are equivalent by diffeomorphisms. Indeed, choose ϵ small and smoothly varying $b_1(s), \ldots, b_{m-1}(s)$ separating the critical values of $\tilde{f}_s$ for $s \in [c_i + \epsilon, c_{i+1} - \epsilon]$. The inverse images of the level sets $\tilde{f}_s^{-1}(b_i(s))$ flow out smooth submanifolds of $Y \times [c_i + \epsilon, c_{i+1} - \epsilon]$ denoted $\tilde{f}^{-1}(b_i)$, by

the implicit function theorem. Choose a vector field $v \in \mathrm{Vect}(Y \times [c_i + \epsilon, c_{i+1} - \epsilon])$ tangent to the level sets $\tilde{f}^{-1}(b_i)$ and satisfying $(D_{y,s}\pi_2)_* v = \partial_s$ for any $(y, s) \in Y \times [c_i + \epsilon, c_{i+1} - \epsilon]$, where π_2 is projection onto the second factor. Such a vector field v_0 exists on each level set $\tilde{f}^{-1}(b_i)$ since the $b_i(s)$ are regular values:

$$(4) \qquad T_{y,s}\tilde{f}^{-1}(b_i) \cap (T_y Y \times \{0\}) = T_y f^{-1}(b_i(s)), \quad D\pi_2(T_{y,s}\tilde{f}^{-1}(b_i)) = \mathbb{R}.$$

Next v_0 extends to a neighborhood of each level set $\tilde{f}^{-1}(b_i)$ by the tubular neighborhood theorem. One may then extend v_0 to a vector field on $Y \times [c_i + \epsilon, c_{i+1} - \epsilon])$ using interpolation with the vector field $\partial_s \in \mathrm{Vect}(Y \times [c_i + \epsilon, c_{i+1} - \epsilon])$. That is, let $\rho \in C^\infty(Y \times [c_i + \epsilon, c_{i+1} - \epsilon])$ be a bump function equal to one on a neighborhood of each $\tilde{f}^{-1}(b_i)$ and vanishing outside of a small neighborhood of the union of $\tilde{f}^{-1}(b_i)$. Define $v = \rho v_0 + (1 - \rho)\partial_s$. The flow ψ_s of v preserves the level sets of $b_0, \ldots, b_m$ and so defines diffeomorphisms of the pieces of the Cerf decomposition of Y for $\tilde{f}_s$:

$$\psi_{s_2 - s_1}(\tilde{f}_{s_1}^{-1}(b_i(s_1), b_{i+1}(s_1))) = \tilde{f}_{s_2}^{-1}(b_i(s_2), b_{i+1}(s_2)).$$

Hence the functions $\tilde{f}_s$ and values $b_i(s)$ for $s \in [c_i + \epsilon, c_{i+1} - \epsilon]$ define an equivalent Cerf decomposition of $[Y]$.

On the other hand, the Cerf decompositions for $c_i - \epsilon, c_i + \epsilon$ are equal for all but one or two pieces by the same argument in the previous paragraph. For those pieces, one either has a critical point switch move or critical point cancellation by the local model for the cusp singularities [34, p.157]. $\qquad \square$

Remark 2.2.12. (Alternative approaches in dimension three) Alternatively in dimension three, one may show that given a sequence of Cerf moves with possibly disconnected fibers, one may modify the sequence so that one obtains a sequence of Cerf moves preserving connectedness, but so that the sequence is not necessarily associated to a homotopy. This is the approach taken in Juhasz [24, p. 1434-1437][1]. Finally, one may reduce the theorem to a relative case of the Reidemeister-Singer theorem (that any two Heegaard splittings are related by a sequence of stabilizations and de-stabilizations). By Motto's thesis, [39, Chapter 4] any two Heegaard splittings of a connected bordism Y are related by a sequence of stabilizations and de-stabilizations. In order to deduce Theorem 2.2.11, it suffices to modify the given Cerf decomposition to one corresponding to a Heegaard splitting by a sequence of critical point switches. That this is possible follows from Milnor [38, Theorem 4.4,4.1,4.2 Extension].

The Cerf theorem above implies that for any category $\mathcal{C}$, in order to construct a connected $\mathcal{C}$-valued field theory in the sense of Definition 2.1.2, it suffices to construct the functors on elementary bordisms and check that the Cerf moves correspond to composition identities in $\mathcal{C}$. For objects M_1, M_2, M_3 of $\mathcal{C}$ we denote by $\circ : \mathrm{Hom}(M_1, M_2) \times \mathrm{Hom}(M_2, M_3) \to \mathrm{Hom}(M_1, M_3)$ the composition map. For any object M of $\mathcal{C}$ denote by $1_M \in \mathrm{Hom}(M, M)$ the identity.

Theorem 2.2.13. (Field theories via morphisms for elementary bordisms) *Suppose we are given for some $n \geq 2$ a partial functor Φ from Bor_{n+1}^0 to $\mathcal{C}$ that associates*

---

[1]We thank T. Perutz for pointing out this reference to us.

(a) to each compact oriented n-manifold X, an object $\Phi(X) \in \mathrm{Obj}(\mathcal{C})$,
(b) to each elementary morphism $[Y]$ from X_- to X_+, a morphism $\Phi([Y])$ from $\Phi(X_-)$ to $\Phi(X_+)$,
(c) to the trivial morphism $[[0,1] \times X]$ the identity morphism $1_{\Phi(X)}$ of $\Phi(X)$;

and satisfy the following Cerf relations for any pair of elementary morphisms $[Y_1]$ from X_0 to X_1 and $[Y_2]$ from X_1 to X_2 :

(a) If $[Y_1] \circ [Y_2]$ is a cylindrical morphism, then

$$\Phi([Y_1]) \circ \Phi([Y_2]) = \Phi([Y_1] \circ [Y_2]).$$

(b) If $[Y_1], [Y_2]$ are related by critical point switch to two other elementary morphisms $[Y_1'], [Y_2']$ from X_0 to X_1' and from X_1' to X_2, then

$$\Phi([Y_1]) \circ \Phi([Y_2]) = \Phi([Y_1']) \circ \Phi([Y_2']).$$

(c) If one of $[Y_1], [Y_2]$ is cylindrical, then

$$\Phi([Y_1]) \circ \Phi([Y_2]) = \Phi([Y_1] \circ [Y_2]).$$

Then there exists a unique extension of Φ to a $n+1$-dimensional connected $\mathcal{C}$-valued field theory $\overline{\Phi} : \mathrm{Bor}_{n+1}^0 \to \mathcal{C}$.

3. Central curvature connections

In this section we show that assigning to each closed manifold resp. elementary bordism a moduli space of connections, considered as a symplectic manifold resp. Lagrangian correspondence, gives rise to a symplectic-valued field theory. The general idea is well-known to experts, especially in the context of quantum Chern-Simons theory where according to Witten's suggestion the invariants of three-dimensional bordisms arise by "quantizing" these Lagrangian correspondences [59].

We begin by reviewing the construction of a symplectic category in Section 3.1. Section 3.2 provides general background on moduli spaces of central curvature connections with fixed determinant. Then Sections 3.3 and 3.4 construct a partial symplectic-valued field theory in the sense of Theorem 2.2.13.

3.1. **The symplectic category.** The moduli spaces of connections over compact oriented surfaces will be symplectic manifolds, whose dimension varies with the genus of the surfaces. Thus the morphisms between these symplectic manifolds, associated to elementary bordisms, can not in general be symplectomorphisms. However, elementary bordisms define the more general morphisms between symplectic manifolds introduced by Weinstein [58] as follows.

Definition 3.1.1. (Geometric composition of Lagrangian correspondences) For $j = 0, 1, 2$ let M_j be a symplectic manifold with symplectic form ω_{M_j}.

(a) A *Lagrangian correspondence* from M_0 to M_1 is a Lagrangian submanifold $L \subset M_0^- \times M_1$ with respect to the symplectic structure $(-\omega_{M_0}) \oplus \omega_{M_1}$.
(b) The *geometric composition* of Lagrangian correspondences

$$L_1 \subset M_0^- \times M_1, \quad L_2 \subset M_1^- \times M_2$$

is the point set

$$(5) \qquad L_1 \circ L_2 := \pi_{M_0 \times M_2}\big((L_1 \times L_2) \cap (M_0 \times \Delta_{M_1} \times M_2)\big) \subset M_0 \times M_2.$$

  (c) A geometric composition is called *transverse* if the intersection in (5) is transverse (and hence smooth) and *embedded* if the projection $\pi_{M_0 \times M_2}$ restricts to an injection of the smooth intersection. In that case the image of the intersection is a smooth Lagrangian correspondence $L_1 \circ L_2 \subset M_0^- \times M_2$.

In general, even after a generic perturbation, the fiber product $L_1 \times_{M_1} L_2$ is at most immersed by projection onto $M_0 \times M_2$. In order to be able to work more easily with holomorphic curves, however, we wish to have embedded Lagrangians. In [52] and [35] we thus constructed a symplectic category for Lagrangian correspondences using a more algebraic composition, as follows.

Definition 3.1.2. (Algebraic composition of Lagrangian correspondences)
  (a) A *generalized Lagrangian correspondence* $\underline{L}$ from M_- to M_+ (both symplectic manifolds) of length $m \geq 0$ consists of
      (i) a sequence $N_0, \ldots, N_m$ of symplectic manifolds with $N_0 = M_-$ and $N_m = M_+$; and
      (ii) a sequence $\underline{L} = (L_1, \ldots, L_m)$ of compact Lagrangian correspondences with $L_j \subset N_{j-1}^- \times N_j$ for $j = 1, \ldots, m$.
      Here we allow empty symplectic manifolds or Lagrangian correspondences.
  (b) The *algebraic composition* of generalized Lagrangian correspondences $\underline{L}$ and $\underline{L}'$ is given by concatenation $\underline{L} \# \underline{L}' = (L_1, \ldots, L_m, L_1', \ldots, L_{m'}')$.

Definition 3.1.3. (Symplectic category) Let $\mathrm{Symp}^\#$ be the category whose
  (a) objects are symplectic manifolds (M, ω) (in particular, we include the empty manifold $\emptyset$);
  (b) morphisms $\mathrm{Hom}(M_-, M_+)$ of $\mathrm{Symp}^\#$ are compact oriented generalized Lagrangian correspondences $\underline{L}$ from M_- to M_+ equipped with gradings and relative spin structure modulo the *composition equivalence* relation $\sim$ generated by

$$(6) \qquad \big(\ldots, L_j, L_{j+1}, \ldots\big) \sim \big(\ldots, L_j \circ L_{j+1}, \ldots\big)$$

  for all sequences and j such that $L_j \circ L_{j+1}$ is transverse and embedded; we also set $\Delta_M \sim \emptyset$ where $\Delta_M \subset M^- \times M$ is the diagonal and $\emptyset$ is the empty sequence.
  (c) composition of morphisms $[\underline{L}] \in \mathrm{Hom}(M, M')$ and $[\underline{L}'] \in \mathrm{Hom}(M', M'')$ is defined by

$$[\underline{L}] \circ [\underline{L}'] := [\underline{L} \# \underline{L}'] \in \mathrm{Hom}(M, M'').$$

An identity $1_M \in \mathrm{Hom}(M, M)$ is given by the equivalence class of the empty sequence $[\emptyset]$ of length zero (since concatenating with the empty set is the identity on sequences) or equivalently, the equivalence class $1_M := [\Delta_M]$ of the diagonal.

Remark 3.1.4. Let Symp be the category of (symplectic manifolds, symplectomorphisms). There is a canonical functor

$$\text{Symp} \to \text{Symp}^{\#}, \quad \begin{pmatrix} M \mapsto M \\ \phi \mapsto [(\text{graph}(\phi))] \end{pmatrix}.$$

That is, the identity maps to the diagonal correspondence and the composition of graphs is the graph of the composition of two symplectomorphisms. Indeed, the geometric composition

(7) $$\text{graph}(\varphi_2) \circ \text{graph}(\varphi_1) = \text{graph}(\varphi_1 \circ \varphi_2)$$

(reversal of order due to conventions) of graphs of symplectomorphisms φ_1, φ_2 is always transverse and embedded.

Definition 3.1.5. (Symplectomorphism equivalences) Let $\underline{L}, \underline{L}'$ be generalized Lagrangian correspondences from M_- to M_+ of the same length m. A *symplectomorphism equivalence* from $\underline{L}$ to $\underline{L}'$ is a collection of symplectomorphisms $(\varphi_j : N_j \to N_j')_{j=0,\dots,m}$ such that

$$\varphi_0 = \text{Id}_{M_-}, \quad \varphi_m = \text{Id}_{M_+}, \quad L_j' = (\varphi_{j-1} \times \varphi_j)(L_j), \quad j = 1, \dots, m.$$

Lemma 3.1.6. (Composition equivalences via symplectomorphism equivalences) *Any symplectomorphism equivalence from $\underline{L}$ to $\underline{L}'$ induces a composition equivalence $\underline{L} \sim \underline{L}'$.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccccccc}
M_- = N_0 & \xrightarrow{L_1} & N_1 & \xrightarrow{L_2} & \cdots & \xrightarrow{L_m} & N_m = M_+ \\
\varphi_0 = \text{Id} \downarrow & & \varphi_1 \downarrow & & \varphi_{m-1} \downarrow & & \varphi_m = \text{Id} \downarrow \\
M_- = N_0' & \xrightarrow{L_1'} & N_1' & \xrightarrow{L_2'} & \cdots & \xrightarrow{L_m'} & N_m' = M_+
\end{array}
$$

Each diagonal morphism can be written in two ways as the composition of a Lagrangian correspondence with the graph of a symplectomorphism,

$$L_j \circ \text{graph}(\varphi_j) = \text{graph}(\varphi_{j-1}) \circ L_j'.$$

A composition equivalence from $\underline{L}$ to $\underline{L}'$ is defined by first replacing L_m with $(\text{graph}(\varphi_{m-1}), L_m')$, then iteratively replacing $(L_j, \text{graph}(\varphi_j))$ with $(\text{graph}(\varphi_{j-1}), L_j')$ for $j = m-1$ to $j = 2$, and eventually replacing $(L_1, \text{graph}(\varphi_1))$ with L_1'. $\square$

For the purposes of Floer theory, we will need our symplectic manifolds and correspondences to carry additional structures and satisfy additional hypotheses.

Definition 3.1.7. (Monotone symplectic manifolds) For any monotonicity constant $\tau > 0$ we introduce the following admissible classes of symplectic manifolds and generalized Lagrangian correspondences.

(a) A symplectic manifold (M, ω) is *monotone* with monotonicity constant τ if $\tau c_1(M) = [\omega]$ in $H^2(M)$.

(b) A symplectic manifold (M, ω) is τ-*admissible* if it is compact, monotone with monotonicity constant τ, and has even minimal Chern number:

$$\langle c_1(M), H_2(M, \mathbb{Z}) \rangle \subset 2\mathbb{Z}.$$

(c) A generalized Lagrangian correspondence $\underline{L} = (L_1, \ldots, L_m)$ from M_- to M_+ is *admissible* if each Lagrangian correspondence in the sequence L_i is simply-connected, compact, oriented, and relatively spin.

Let $\mathrm{Symp}_{\tau,N}^{\#}$ denote the category whose

(i) objects are τ-admissible symplectic manifolds M equipped with N-fold Maslov covers $\mathrm{Lag}^N(M) \to \mathrm{Lag}(M)$ in the sense of Seidel [48];
(ii) morphisms from M_0 to M_1 are equivalence classes $[\underline{L}]$ of admissible generalized Lagrangian correspondences from M_0 to M_1, each correspondence in the sequence equipped with relative spin structures and N-gradings, and where the composition equivalence relation $\sim$ is that generated by (6) restricted to such correspondences; and
(iii) composition of morphisms $\underline{L}_{01}, \underline{L}_{12}$ is defined as concatenation $\underline{L}_{01} \# \underline{L}_{12}$ as before.

Remark 3.1.8.   (a) (Other possible assumptions on correspondences) The condition in (c) can be replaced with other conditions that guarantee monotonicity, or just requiring monotonicity itself, but in practice in this paper we just check simply-connectedness.
(b) (Inclusion of monotone symplectic categories in the symplectic category) There is a canonical functor $\mathrm{Symp}_{\tau,N}^{\#} \to \mathrm{Symp}^{\#}$ induced by inclusion of objects

$$\mathrm{Ob}(\mathrm{Symp}_{\tau,N}^{\#}) \hookrightarrow \mathrm{Ob}(\mathrm{Symp}^{\#})$$

and on morphisms

(8) $$\mathrm{Hom}(\mathrm{Symp}_{\tau,N}^{\#}) \to \mathrm{Hom}(\mathrm{Symp}^{\#}), \quad [\underline{L}]_{\tau,N} \to [\underline{L}]$$

mapping equivalence classes $[\underline{L}]_\tau$ of generalized admissible Lagrangian correspondences to equivalence classes $[\underline{L}]$ of generalized Lagrangian correspondences. However, the map on morphisms (8) may not be an inclusion since two admissible generalized Lagrangian correspondences may be equivalent through a non-admissible generalized Lagrangian correspondence.

3.2. Moduli spaces of central curvature connections with fixed determinant.
This section introduces the moduli spaces of central curvature connections with fixed determinant on compact manifolds of any dimension, for a general class of structure groups.

3.2.1. *Central curvature connections and gauge transformations.* We first introduce some notation for connections and gauge transformations. Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. Let Z denote the center of G and $\mathfrak{z}$ its tangent space in $\mathfrak{g}$. Let X be a compact oriented manifold of dimension $\dim(X) = n$ with (possibly empty) boundary. Let $\pi : P \to X$ be a principal G-bundle.

Definition 3.2.1.   (a) (Associated vector bundles) For any finite-dimensional real G-representation V we denote by $P(V) = (P \times V)/G$ the *associated vector bundle*, where G acts on $P \times V$ by $g(p,v) = (pg^{-1}, gv)$. Denote by

$$\Omega(X, P(V)) := \bigoplus_{k=0}^{n} \Omega^k(X, P(V))$$

the space of forms with values in $P(V)$.
   (b) (Adjoint bundles) In particular, $P(\mathfrak{g}) = (P \times \mathfrak{g})/G$ is the *adjoint bundle* associated to the adjoint representation of G on $\mathfrak{g}$.
   (c) (Splittings of the adjoint bundles) Any invariant subspace $\mathfrak{h} \subset \mathfrak{g}$ induces an inclusion $P(\mathfrak{h}) \subset P(\mathfrak{g})$. The splitting $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}$ into the semisimple and central parts of $\mathfrak{g}$ induces a splitting of the adjoint bundle,

$$(9) \qquad\qquad P(\mathfrak{g}) = P([\mathfrak{g}, \mathfrak{g}]) \oplus P(\mathfrak{z}).$$

In the case $G = U(r)$ to which we will specialize later, the center is given by the diagonal matrices $Z = U(1)\mathrm{Id}$, and the splitting is

$$\mathfrak{u}(r) = \mathfrak{su}(r) \oplus \mathfrak{u}(1)\mathrm{Id}.$$

   (d) (Affine space of connections) Let $\mathcal{A}(P)$ be the space of connections on P,

$$\mathcal{A}(P) = \left\{ \alpha \in \Omega^1(P, \mathfrak{g}) \;\middle|\; \begin{array}{ll} \alpha(\xi_P) = \xi & \forall \xi \in \mathfrak{g}, \\ \alpha(vg) = \mathrm{Ad}(g^{-1})\alpha(v) & \forall v \in TP, g \in G \end{array} \right\}.$$

Here $\xi_P \in \mathrm{Vect}(P)$ denotes the vector field generated by the action of $\xi \in \mathfrak{g}$.
   (e) (Basic forms) For any non-negative integer k the space $\Omega^k(X, P(\mathfrak{g}))$ of k-forms with values in $P(\mathfrak{g})$ is isomorphic via π^* to the space $\Omega^k(P, \mathfrak{g})_{\mathrm{basic}}$ of basic (that is, equivariant and horizontal) k-forms. With this notation $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(X, P(\mathfrak{g}))$. That is, $\Omega^1(X, P(\mathfrak{g}))$ acts on $A(P)$ faithfully transitively by $\alpha \mapsto \alpha + \pi^*b$ for $\alpha \in \mathcal{A}(P)$, $b \in \Omega^1(X, P(\mathfrak{g}))$.
   (f) (Curvature) The curvature of $\alpha \in \mathcal{A}(P)$ is the two form $F_\alpha \in \Omega^2(X, P(\mathfrak{g}))$ defined by

$$\pi^* F_\alpha = \mathrm{d}\alpha + \tfrac{1}{2}[\alpha \wedge \alpha] \in \Omega^2(P, \mathfrak{g})_{\mathrm{basic}}.$$

   (g) (Covariant derivative) The covariant derivative in the adjoint representation is

$$\mathrm{d}_\alpha : \Omega^*(X, P(\mathfrak{g})) \to \Omega^{*+1}(X, P(\mathfrak{g})), \quad \pi^* \mathrm{d}_\alpha \beta = \mathrm{d}\pi^* \beta + [\alpha \wedge \pi^* \beta].$$

   (h) (Bianchi identity) $\mathrm{d}_\alpha F_\alpha = 0$.
   (i) (Central curvature connections) A connection α is *central curvature* if F_α takes values in $P(\mathfrak{z}) \subset P(\mathfrak{g})$, that is, $F_\alpha^{[\mathfrak{g},\mathfrak{g}]} = 0$.
   (j) (Group of gauge transformations) Let

$$\mathcal{G}(P) = \left\{ \phi : P \to P \;\middle|\; \pi \circ \phi = \pi, \; \phi(pg) = \phi(p)g \;\; \forall\, p \in P, \; g \in G \right\}$$

denote the group of gauge transformations, that is, G-equivariant automorphisms of P.

(k) (Action of gauge transformations on connections) The group $\mathcal{G}(P)$ acts on the left on $\mathcal{A}(P)$ by

$$\mathcal{G}(P) \times \mathcal{A}(P) \to \mathcal{A}(P), \quad (\phi, \alpha) \mapsto (\phi^{-1})^* \alpha.$$

(l) (Infinitesimal gauge transformations) The Lie algebra of $\mathcal{G}(P)$ can be identified with $\Omega^0(X, P(\mathfrak{g}))$ by associating the vector field $p \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} p \exp(t\xi)$ to $\xi \in \Omega^1(P, \mathfrak{g})_{\mathrm{basic}} = \Omega^0(X, P(\mathfrak{g}))$. With this notation, the infinitesimal action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is given by

(10) $$\Omega^0(X, P(\mathfrak{g})) \times \mathcal{A}(P) \to \Omega^1(X, P(\mathfrak{g})), \quad (\xi, \alpha) \mapsto \mathrm{d}_\alpha \xi.$$

(m) (Action of gauge transformations on covariant derivatives) The action of $\phi \in \mathcal{G}(P)$ on P induces an action on $P(V)$ for any G-representation V, denoted by ϕ_V. The covariant derivative d_α and curvature F_α transform as

$$\mathrm{d}_{\phi\alpha} = \phi_{\mathfrak{g}} \mathrm{d}_\alpha \phi_{\mathfrak{g}}^{-1}, \quad F_{\phi\alpha} = \phi_{\mathfrak{g}} F_\alpha.$$

Hence the subset of central curvature connections is invariant under gauge transformations.

Remark 3.2.2.   (a) (Splitting of the covariant derivative) Using the splitting (9) we write

$$\mathrm{d}_\alpha = \mathrm{d}_\alpha^{\mathfrak{z}} \oplus \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]} \qquad \text{and} \qquad F_\alpha = F_\alpha^{\mathfrak{z}} \oplus F_\alpha^{[\mathfrak{g},\mathfrak{g}]}.$$

(b) (Basic inner product) Let $\langle \cdot, \cdot \rangle$ denote the basic invariant inner product on $\mathfrak{g}$, see [42, p.49]. For G simple, this is the unique inner product such that the norm-square of the highest coroot is 2. For example, in the case that $G = U(r)$, the basic inner product is $\langle \xi, \zeta \rangle = -\operatorname{Tr}(\xi\zeta)$ for $\xi, \zeta \in \mathfrak{u}(r)$.

(c) (Adjoint of the covariant derivative) A choice of metric on X induces a Hodge star operator

$$* : \Omega^k(X, P(\mathfrak{g})) \mapsto \Omega^{n-k}(X, P(\mathfrak{g})), \quad k = 0, \dots, n.$$

Together with the inner product on $\mathfrak{g}$ this induces a metric on each $\Omega^k(X, P(\mathfrak{g}))$. The formal adjoint of the covariant derivative is

$$\mathrm{d}_\alpha^* : \Omega^\bullet(X, P(\mathfrak{g})) \to \Omega^{\bullet-1}(X, P(\mathfrak{g})), \quad \beta \mapsto -(-1)^{(n-\bullet)(\bullet-1)} * \mathrm{d}_\alpha * \beta.$$

3.2.2. *Linear theory.* In this subsection we discuss the cohomology of the covariant derivative associated to a central curvature connection. We review the well-known fact that in certain dimensions the cohomology of a compact oriented manifold with boundary X restricts to a Lagrangian subspace in the middle dimensional cohomology on the boundary ∂X.

Definition 3.2.3. Let $P \to X$ be a principal G-bundle.

(a) (Cohomology of a central curvature connection) Let $\alpha \in \mathcal{A}(P)$ be a central curvature connection. Since its curvature F_α is central we have $\mathrm{d}_\alpha^2 = 0$. Define cohomology groups

$$H(X; \mathrm{d}_\alpha) = \bigoplus_{k=0}^n H^k(X; \mathrm{d}_\alpha), \quad H^k(X; \mathrm{d}_\alpha) = \frac{\ker(\mathrm{d}_\alpha | \Omega^k(X, P(\mathfrak{g})))}{\operatorname{im}(\mathrm{d}_\alpha | \Omega^{k-1}(X, P(\mathfrak{g})))}.$$

(b) (Relative cohomology of a central curvature connection) Denote restriction to the boundary by

$$\rho : \Omega(X, P(\mathfrak{g})) \to \Omega(\partial X; P(\mathfrak{g})|_{\partial X}). \tag{11}$$

The map ρ is a cochain map and so induces a map of cohomologies $H(X; \mathrm{d}_\alpha) \to H(\partial X; \mathrm{d}_{\alpha|\partial X})$. Let $H(X, \partial X; \mathrm{d}_\alpha)$ be the relative cohomology groups consisting of forms whose tangential part to the boundary vanishes,

$$H(X, \partial X; \mathrm{d}_\alpha) = \ker(\mathrm{d}_\alpha| \ker \rho)/ \operatorname{im}(\mathrm{d}_\alpha| \ker \rho).$$

(c) (Hodge isomorphisms) The Hodge isomorphisms on manifolds with boundary (see e.g. [19, Section 4.1]) give

$$H(X; \mathrm{d}_\alpha) \cong \ker(\mathrm{d}_\alpha) \cap \operatorname{im}(\mathrm{d}_\alpha)^\perp = \ker(\mathrm{d}_\alpha \oplus \mathrm{d}_\alpha^* \oplus \rho*), \tag{12}$$

$$H(X, \partial X; \mathrm{d}_\alpha) \cong \ker(\mathrm{d}_\alpha \oplus \rho) \cap \operatorname{im}(\mathrm{d}_\alpha|_{\ker \rho})^\perp = \ker(\mathrm{d}_\alpha \oplus \mathrm{d}_\alpha^* \oplus \rho).$$

Using the first identification we have a Hodge decomposition

$$\Omega(X, P(\mathfrak{g})) = \operatorname{im}(\mathrm{d}_\alpha) \oplus H(X; \mathrm{d}_\alpha) \oplus \operatorname{im}(\mathrm{d}_\alpha^*| \ker \rho*). \tag{13}$$

Here the direct sum holds in any L_2^s-Sobolev completion for $s \geq 0$ (that is we use the norm $\bar{H}^s$ of [21] measuring s fractional derivatives in L^2), as a consequence of the following elliptic estimate from e.g. [21, Section 20.1]. Let Π_{H^k} denote the L^2-orthogonal projection onto $H^k(X; \mathrm{d}_\alpha) \subset \Omega^k(X, P(\mathfrak{g}))$. There is a constant C depending on α such that for all $\eta \in \Omega^k(X, P(\mathfrak{g}))$

$$\|\eta\|_{L_2^{s+1}(X)} \leq C\big(\|(\mathrm{d}_\alpha + \mathrm{d}_\alpha^*)\eta\|_{L_2^s(X)} + \|\rho * \eta\|_{L_2^{s+1/2}(\partial X)} + \|\Pi_{H^k}\eta\|_{L_2(X)}\big). \tag{14}$$

(d) (Non-degeneracy of the duality pairings) Using these Hodge isomorphisms, the Hodge star induces a duality isomorphism

$$H^*(X; \mathrm{d}_\alpha) \to H^{n-*}(X, \partial X; \mathrm{d}_\alpha), \quad \beta \mapsto *\beta. \tag{15}$$

This duality implies non-degeneracy of the pairing

$$H^*(X; \mathrm{d}_\alpha) \times H^{n-*}(X, \partial X; \mathrm{d}_\alpha) \to \mathbb{R}, \quad (\gamma, \beta) \mapsto \int_X \langle \gamma \wedge \beta \rangle. \tag{16}$$

Replacing X with ∂X we have a non-degenerate pairing on $H(\partial X; \mathrm{d}_{\alpha|\partial X})$.

If $\dim(X) = 2k+1$ is odd, the pairing in the last item restricts to a non-degenerate pairing on the middle dimensional homology $H^k(\partial X; \mathrm{d}_{\alpha|\partial X})$. The pairing is skew-symmetric, hence symplectic, if k is odd. In that case, the following Lemma shows that the restriction of $H(X; \mathrm{d}_\alpha)$ to the boundary gives rise to a Lagrangian subspace.

Lemma 3.2.4. (Lagrangian restriction of odd cohomology) *Let* $\alpha \in \mathcal{A}(P)$ *be a central curvature connection. The image* $\rho(H(X; \mathrm{d}_\alpha))$ *of* $H(X; \mathrm{d}_\alpha)$ *in* $H(\partial X; \mathrm{d}_{\alpha|\partial X})$ *is maximally isotropic. In particular, if* $\dim(X) = 2k + 1$ *for some integer* k *then* $\rho(H^k(X; \mathrm{d}_\alpha)) \subset H^k(\partial X; \mathrm{d}_{\alpha|\partial X})$ *is maximally isotropic, and if* k *is odd,*

$$\dim\big(\rho(H^k(X; \mathrm{d}_\alpha))\big) = \tfrac{1}{2} \dim\big(H^k(\partial X; \mathrm{d}_{\alpha|\partial X})\big). \tag{17}$$

Proof. Stokes' theorem implies that the image $\rho(H(X; \mathrm{d}_\alpha))$ is isotropic: For $\beta_1, \beta_2 \in \ker(\mathrm{d}_\alpha)$

$$
\begin{aligned}
\int_{\partial X} \langle \rho(\beta_1) \wedge \rho(\beta_2) \rangle &= \int_X \mathrm{d}\langle \beta_1 \wedge \beta_2 \rangle \\
&= \int_X \langle \mathrm{d}_\alpha \beta_1 \wedge \beta_2 \rangle + (-1)^{|\beta_1|} \langle \beta_1 \wedge \mathrm{d}_\alpha \beta_2 \rangle = 0.
\end{aligned}
$$

To see that the image is maximal isotropic, first note that restriction to the boundary induces a long exact sequence

(18) $\qquad \ldots \to H^\bullet(X, \partial X; \mathrm{d}_\alpha) \to H^\bullet(X; \mathrm{d}_\alpha) \to H^\bullet(\partial X; \mathrm{d}_{\alpha|_{\partial X}}) \to \ldots$

Now consider the commutative diagram formed from the long exact sequence (18) and its dual:

$$
\begin{array}{ccccc}
H^\bullet(X; \mathrm{d}_\alpha) & \longrightarrow & H^\bullet(\partial X; \mathrm{d}_{\alpha|_{\partial X}}) & \xrightarrow{\ c\ } & H^{\bullet+1}(X, \partial X; \mathrm{d}_\alpha) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\cdots & & & & \cdots. \\
H^{n-\bullet}(X, \partial X; \mathrm{d}_\alpha)^\vee & \longrightarrow & H^{n-1-\bullet}(\partial X; \mathrm{d}_{\alpha|_{\partial X}})^\vee & \longrightarrow & H^{n-1-\bullet}(X; \mathrm{d}_\alpha)^\vee
\end{array}
$$

Here the vertical arrows are given by (15) and the pairing (16). To check commutativity, use Stokes' theorem and the fact that the connecting morphism c above is given by $\beta \mapsto \mathrm{d}_\alpha \tilde{\beta}$ for any extension $\tilde{\beta}|_{\partial X} = \beta$.

Now suppose $\beta \in H^\bullet(\partial X; \mathrm{d}_{\alpha|_{\partial X}})$ lies in the annihilator of the image of $H^{n-1-\bullet}(X; \mathrm{d}_\alpha)$. By definition $\beta \in H^\bullet(\partial X; \mathrm{d}_{\alpha|_{\partial X}})$ maps to 0 in $H^{n-1-\bullet}(X; \mathrm{d}_\alpha)^\vee$. Since the vertical maps are isomorphisms and the diagram is commutative, the image of β in $H^{\bullet+1}(X, \partial X; \mathrm{d}_\alpha)$ vanishes, so by exactness of the top sequence β lies in the image of $H^\bullet(X; \mathrm{d}_\alpha)$. The last claim follows from the fact that maximally isotropic subspaces of symplectic vector spaces are half-dimensional. For analogous results on Dirac operators on manifolds with boundary see [4, Theorem 22.24]. $\qquad \square$

Corollary 3.2.5. *If X has dimension 3 then $\rho(H^1(X; \mathrm{d}_\alpha)) \subset H^1(\partial X; \mathrm{d}_{\alpha|_{\partial X}})$ is a Lagrangian subspace. Furthermore, if $H^1(X, \partial X; \mathrm{d}_\alpha) = 0$ then $\rho : H^1(X; \mathrm{d}_\alpha) \to H^1(\partial X; \mathrm{d}_{\alpha|_{\partial X}})$ is a Lagrangian embedding.*

Proof. The first statement follows immediately from Lemma 3.2.4 with $k = 1$. The injectivity of ρ in the second statement follows from the long exact sequence (18). $\quad \square$

3.2.3. *Moduli spaces.* This subsection introduces the moduli space of connections with central curvature and fixed determinant. As before, G is a compact connected group and X a compact oriented manifold with (possibly empty) boundary.

Definition 3.2.6.

(a) (Commutator subgroup) Since G is compact and connected, by Gotô's theorem, the commutator mapping $(a_1, b_1, \ldots, a_g, b_g) \to \prod_{j=1}^g [a_j, b_j]$ is surjective onto the commutator subgroup

$$
G_0 := [G, G] = \{ [g_1, g_2] \, | \, g_1, g_2 \in G \}.
$$

In particular, $[U(r), U(r)] = SU(r)$. More generally, G_0 is the subgroup whose Lie algebra equals to the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ of the Lie algebra $\mathfrak{g}$.

(b) (Determinant homomorphism) The group homomorphism $G \to G/G_0$ induces for any principal G-bundle P a principal G/G_0-bundle

$$\det(P) := P/G_0 \cong P \times_G (G/G_0) \xrightarrow{\pi'} X.$$

In the special case $G = U(r)$ the bundle $\det(P)$ is the principal $U(1)$-bundle induced by the determinant $\det : U(r) \to U(1)$.

(c) (Determinant of a connection) The homomorphism of Lie algebras

$$\pi_{\mathfrak{z}} : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{z}$$

induces a map $\det : \mathcal{A}(P) \to \mathcal{A}(\det(P))$. Indeed for any connection $\alpha \in \mathcal{A}(P)$ the form $\pi_{\mathfrak{z}} \circ \alpha \in \Omega^1(P, \mathfrak{z})$ is a basic form for the G_0-bundle $P \to \det(P)$ and descends to a connection form on $\det(P)$. Given a connection $\delta \in \mathcal{A}(\det(P))$ denote by

$$\mathcal{A}_\delta(P) = \left\{ \alpha \in \mathcal{A}(P) \, \big| \, F_\alpha^{[\mathfrak{g}, \mathfrak{g}]} = 0, \det(\alpha) = \delta \right\}$$

its inverse image in the space of central curvature connections.

(d) (Gauge transformations fixing the determinant) The group of gauge transformations with trivial determinant $\mathcal{G}_0(P)$ is defined to be the kernel of the homomorphism $\mathcal{G}(P) \to \mathcal{G}(\det(P))$. It acts on $\mathcal{A}_\delta(P)$ for any $\delta \in \mathcal{A}(\det(P))$.

(e) (Moduli spaces with fixed determinant) Denote by

$$M_\delta(P) = \mathcal{A}_\delta(P)/\mathcal{G}_0(P)$$

its quotient, the moduli space of connections with central curvature and fixed determinant.

Remark 3.2.7. (Independence of the moduli spaces from the choice of determinant) The spaces $M_\delta(P)$ as δ ranges over connections on $\det(P)$ are identified as follows. The vector space $\Omega^1(X, \mathfrak{z})$ acts on $\mathcal{A}(P)$ by $\alpha \mapsto \alpha + \pi^* a$ for $a \in \Omega^1(X, \mathfrak{z})$. The curvature and determinant transform as

$$F_{\alpha + \pi^* a} = F_\alpha + \mathrm{d}a, \quad \det(\alpha + \pi^* a) = \det(\alpha) + \pi'^* a.$$

Indeed, the defining equation is

$$\pi_{G_0}^* \det(\alpha + \pi^* a) = \pi_{\mathfrak{z}} \circ (\alpha + \pi^* a) = \pi_{\mathfrak{z}} \circ \alpha + \pi_{G_0}^* \pi'^* a = \pi_{G_0}^* \big(\det(\alpha) + \pi'^* a \big).$$

Each $a \in \Omega^1(X, \mathfrak{z})$ induces an identification of moduli spaces

$$M_\delta(P) \to M_{\delta + \pi'^* a}(P).$$

This shift provides unique identifications of $M_{\delta_1}(P), M_{\delta_2}(P)$ for all $\delta_1, \delta_2 \in \mathcal{A}(\det(P))$. Indeed, $\mathcal{A}(\det(P))$ is an affine space over $\Omega^1(X, \mathfrak{z})$ via $\delta \mapsto \delta + \pi'^* a$ for $a \in \Omega^1(X, \mathfrak{z})$. We will hence from now on refer to

$$M(P) := M_\delta(P) = \mathcal{A}_\delta(P)/\mathcal{G}_0(P)$$

as *the* moduli spaces of central curvature connections with fixed determinant with only minor abuse of language.

Proposition 3.2.8. (Condition for smoothness of the moduli space) *Let $\alpha \in \mathcal{A}(P)$ be a central curvature connection. If $H^0(X; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = H^2(X; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = 0$ then $M(P)$ is a finite-dimensional orbifold at $[\alpha]$ with tangent space isomorphic to $H^1(X; d_\alpha^{[\mathfrak{g},\mathfrak{g}]})$. If in addition $G = U(r)$, then $M(P)$ is a finite-dimensional manifold in a neighborhood of $[\alpha]$.*

Proof. Fix $\delta \in \mathcal{A}(\det(P))$. Any $\alpha \in \mathcal{A}_\delta(P)$ splits into semi-simple and central part $\alpha = \alpha_1 + \pi_{G_0}^* \delta$, where $\alpha_1 \in \Omega^1(P, [\mathfrak{g}, \mathfrak{g}])$ satisfies

$$(19) \qquad \pi^* F_\alpha^{[\mathfrak{g},\mathfrak{g}]} = d\alpha_1 + \frac{1}{2}[\alpha_1, \alpha_1] = 0.$$

By standard arguments (as for flat connections), any solution of (19) of Sobolev class L_2^s with $2s > n$ is gauge equivalent to a smooth solution. Hence we can think of $M(P)$ as the quotient of solutions of (19) of class L_2^s by the L_2^{s+1}-closure of $\mathcal{G}_0(P)$. In the first step, we will show that the equation $d\alpha_1 + \frac{1}{2}[\alpha_1, \alpha_1] = 0$ for $\alpha_1 \in \Omega^1(P, [\mathfrak{g}, \mathfrak{g}])_s$ cuts out a smooth Banach submanifold whose tangent space at α_1 is the kernel of

$$d_\alpha^{[\mathfrak{g},\mathfrak{g}]} = d_{\alpha_1} : \Omega^1(X, P([\mathfrak{g}, \mathfrak{g}]))_s \to \Omega^2(X, P([\mathfrak{g}, \mathfrak{g}]))_{s-1}.$$

Here we denote by subscripts such as $\Omega(X, P([\mathfrak{g}, \mathfrak{g}]))_s$ the L_2^s-completion of spaces of smooth forms such as $\Omega(X, P([\mathfrak{g}, \mathfrak{g}]))$, and moreover choose $s \geq 2$. Then by the vanishing of $H^2(X; d_\alpha^{[\mathfrak{g},\mathfrak{g}]})$, the Hodge estimate (14) becomes

$$\|b\|_{L_2^{s-1}(X)} \leq C\big(\|d_{\alpha_1} b\|_{L_2^{s-2}(X)} + \|d_{\alpha_1}^* b\|_{L_2^{s-2}(X)} + \| * b|_{\partial X}\|_{L_2^{s-3/2}(\partial X)}\big).$$

for all $b \in \Omega^2(X, P([\mathfrak{g}, \mathfrak{g}]))_{s-1}$. A similar estimate, possibly with a different constant, holds with d_{α_1} replaced by $d_{\alpha_1 + \pi^* a}$ for sufficiently small $\|a\|_{L_2^s(X)}$. Hence, applying this new estimate to $b = F_{\alpha_1} + \pi^* a$ and using the Bianchi identity,

$$(20) \qquad F_{\alpha_1 + \pi^* a}^{[\mathfrak{g},\mathfrak{g}]} = 0 \quad \Longleftrightarrow \quad (d_{\alpha_1}^* \oplus \rho*) F_{\alpha_1 + \pi^* a}^{[\mathfrak{g},\mathfrak{g}]} = 0.$$

It follows that $\mathcal{A}_\delta(P)$ near α is the set of sums $\alpha + \pi^* a$ where a is a zero of the map

$$(21) \qquad \Omega^1(X, P([\mathfrak{g}, \mathfrak{g}]))_s \to \text{im}(d_{\alpha_1}^* \oplus \rho*), \quad a \mapsto (d_{\alpha_1}^* \oplus \rho*) F_{\alpha_1 + \pi^* a}^{[\mathfrak{g},\mathfrak{g}]}.$$

Here the target

$$(d_{\alpha_1}^* \oplus \rho*)\Omega^2(X, P([\mathfrak{g}, \mathfrak{g}]))_{s-1} \subset \Omega^1(X, P([\mathfrak{g}, \mathfrak{g}]))_{s-2} \times \Omega^{n-2}(\partial X, P([\mathfrak{g}, \mathfrak{g}])|\partial X)_{s-\frac{3}{2}}$$

is closed. Indeed, note that $\text{im}(d_{\alpha_1})^\perp = \ker(d_{\alpha_1}^* \oplus \rho*)$. This implies

$$\text{im}(d_{\alpha_1}^* \oplus \rho*) = \text{im}((d_{\alpha_1}^* \oplus \rho*)d_{\alpha_1}) \cong \text{im}((d_{\alpha_1} \oplus d_{\alpha_1}^* \oplus \rho*)d_{\alpha_1}),$$

where the latter isomorphism holds since $d_{\alpha_1}^2 = 0$. Now $\text{im}(d_{\alpha_1})$ is a closed subspace of $\Omega^2(X, P([\mathfrak{g}, \mathfrak{g}]))_{s-1}$, by (13), and the Hodge estimate implies that the image of $\text{im}(d_{\alpha_1})$ under $d_{\alpha_1} \oplus d_{\alpha_1}^* \oplus \rho*$ is also closed, by [36, Lemma A.1.1].

With this setup, the linearized operator $(d_{\alpha_1}^* \oplus \rho*)d_{\alpha_1}$ of (21) is surjective and has kernel that of d_{α_1} since $\ker(d_{\alpha_1}^* \oplus \rho*) = \text{im}(d_{\alpha_1})^\perp$. Hence the implicit function theorem provides a smooth map from the formal tangent space $\ker(d_{\alpha_1})$ to its

complement in (13)

$$\ker(\mathrm{d}_{\alpha_1}) \to \mathrm{im}(\mathrm{d}^*_{\alpha_1} | \ker \rho*), \quad a \mapsto b(a)$$

such that the map

$$\ker(\mathrm{d}_{\alpha_1}) \to \mathcal{A}_\delta(P), \quad a \mapsto \alpha + \pi^*(a + b(a))$$

is a local chart for $\mathcal{A}_\delta(P)$.

To construct the orbifold structure on the quotient, we show that $\alpha + \ker(\mathrm{d}^*_{\alpha_1} \oplus \rho*)$ is a local slice for the action of $\mathcal{G}_0(P)$. The assumption $H^0(\mathrm{d}^{[\mathfrak{g},\mathfrak{g}]}_\alpha) = 0$ ensures that the local slice conditions

$$\mathrm{d}^*_{\alpha_1}((a + b(a)) = 0, \quad *((a + b(a))|_{\partial X} = 0$$

are transverse to $\ker(\mathrm{d}_{\alpha_1})$. To see that the stabilizers are finite, note that any automorphism of a bundle with connection is determined by its restriction to a point, and so the stabilizer embeds into G. The stabilizer is discrete by vanishing of H^0, and so is a finite subgroup of G. Standard arguments (e.g. [12, Lemma 4.2.4]) show that $M(P) = \mathcal{A}_\delta(P)/\mathcal{G}_0(P)$ is Hausdorff. Hence $M(P)$ is a smooth orbifold.

If all the stabilizers are the central-valued gauge transformations, the quotient is in fact a manifold. Consider the subgroup $\mathcal{G}_0^{\mathrm{central}}(P) \subset \mathcal{G}_0(P)$ given by the central automorphisms of the form $g(p) = pz(p)$ for some $z : P \to Z$. If all stabilizers of α are contained in $\mathcal{G}_0^{\mathrm{central}}(P)$, then $M(P)$ is a smooth manifold near $[\alpha]$. The subgroup $\mathcal{G}_0^{\mathrm{central}}(P)$ acts trivially on $\mathcal{A}_\delta(P)$. On the other hand, $\mathcal{G}_0^{\mathrm{central}}(P)$ is finite since (by connectedness of X and G) the map $z \equiv z^{ss} \in Z^{ss} = Z \cap G_0$ is constant. Hence we can also realize $M(P)$ as the quotient

$$M(P) = \mathcal{A}_\delta(P)/(\mathcal{G}_0(P)/\mathcal{G}_0^{\mathrm{central}}(P)).$$

Since slices exist, this quotient has a natural manifold structure on the locus where $\mathcal{G}_0(P)/\mathcal{G}_0^{\mathrm{central}}(P)$ acts freely.

In the case of the unitary group, the condition of the previous paragraph is automatically satisfied. Indeed the vanishing of $H^0(X; \mathrm{d}^{[\mathfrak{g},\mathfrak{g}]}_\alpha)$ implies that there are no non-central automorphism of α in $\mathcal{G}_0(P)$. For suppose that $g \in \mathcal{G}_0(P) \setminus \mathcal{G}_0^{\mathrm{central}}(P)$ is in the stabilizer of α. Consider the induced connection α on $P(\mathbb{C}^r)$. The gauge transformation g induces an automorphism of $P(\mathbb{C}^r)$ whose action on some fiber has at least two different eigenvalues; by parallel transport (independent of the choice of path since the holonomy commutes with g) we obtain a splitting

$$(22) \qquad P(\mathbb{C}^r) = E_1 \oplus E_2, \quad \mathrm{rank}(E_1) = r_1 > 0, \quad \mathrm{rank}(E_2) = r_2 > 0.$$

Hence there is a one-parameter family

$$(23) \qquad \left(e^{it/r_1} \mathrm{Id}_{E_1} \oplus e^{-it/r_2} \mathrm{Id}_{E_2}\right)_{t \in \mathbb{R}} \in \mathrm{Aut}(P(\mathbb{C}^r)), \quad t \in \mathbb{R}$$

of automorphisms of α on $P(\mathbb{C}^r)$. Since P is the frame bundle of $P(\mathbb{C}^r)$, this would imply a one-parameter family of non-central automorphisms of α on P, contradicting $H^0(X; d^{[\mathfrak{g},\mathfrak{g}]}_\alpha) = 0$. $\qquad \square$

The moduli spaces $M(P)$ can be described in terms of spaces of representations of the fundamental group, up to a twist which is determined by the determinant bundle:

Definition 3.2.9. (Adjoint moduli spaces) Let $\mathrm{Ad}(G) = G/Z$ and $Z^{\mathrm{ss}} = Z \cap G_0$. (In case $G = U(r)$ this means $\mathrm{Ad}(G) = PSU(r) = SU(r)/e^{2\pi i \mathbb{Z}/r}\mathrm{Id}$ and $Z^{\mathrm{ss}} = e^{2\pi i \mathbb{Z}/r}\mathrm{Id}$.) Denote by

$$M_{\mathrm{Ad}}(X) := \mathrm{Hom}(\pi_1(X), \mathrm{Ad}(G))/\mathrm{Ad}(G).$$

the moduli space of representations of $\pi_1(X)$ in $\mathrm{Ad}(G)$, up to conjugacy. As explained in e.g. Atiyah-Bott [2], the space $M_{\mathrm{Ad}}(X)$ is the union of the moduli spaces $M_{\mathrm{Ad}}(P)$ of flat connections on P as P ranges over $\mathrm{Ad}(G)$-bundles.

Lemma 3.2.10. (Relation to moduli of flat bundles) *Let $P \to X$ be a principal G-bundle. The moduli space $M(P)$ has the structure of a topological principal $\mathrm{Hom}(\pi_1(X), Z^{\mathrm{ss}})$-bundle over a component $M_{\mathrm{Ad}}(P)$ of $M_{\mathrm{Ad}}(X)$. In particular $M(P)$ is compact.*

Proof. Recall that the space of central curvature connections with fixed determinant is

$$\mathcal{A}_\delta(P) = \left\{ \alpha \in \Omega^1(P, \mathfrak{g})^G \;\middle|\; \begin{array}{l} \alpha(\xi_P) = \xi \; \forall \xi \in \mathfrak{g} \\ \det(\alpha) = \delta \\ F_\alpha^{[\mathfrak{g}, \mathfrak{g}]} = 0 \end{array} \right\},$$

where $\Omega^1(\ldots)^G$ denotes the equivariant forms and $\xi_P \in \mathrm{Vect}(P)$ is the vector field generated by $\xi \in \mathfrak{g}$. The exact sequence of groups

$$1 \to Z \to G \to G/Z =: \mathrm{Ad}(G) \to 1$$

induces a splitting of Lie algebras $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$. The $\mathfrak{z}$-component of any $\alpha \in \mathcal{A}_\delta(P)$ is uniquely determined by $\det(\alpha) = \delta$. Hence the projection to the $[\mathfrak{g}, \mathfrak{g}]$-component induces a homeomorphism between $\mathcal{A}_\delta(P)$ and the space

$$\left\{ \alpha \in \Omega^1(P, [\mathfrak{g}, \mathfrak{g}])^G \;\middle|\; \begin{array}{l} \alpha(\xi_P) = \xi \; \forall \xi \in [\mathfrak{g}, \mathfrak{g}] \\ \alpha(\xi_P) = 0 \; \forall \xi \in \mathfrak{z} \\ d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \end{array} \right\}$$

of flat, equivariant $[\mathfrak{g}, \mathfrak{g}]$-forms on P that are horizontal with respect to $\mathfrak{z}$. The latter forms descend to $\Omega^1(P/Z, [\mathfrak{g}, \mathfrak{g}])$. Hence we obtain a homeomorphism between $\mathcal{A}_\delta(P)$ and the space of flat $\mathrm{Ad}(G)$-connections on P/Z,

$$\mathcal{A}_{\mathrm{flat}}(P/Z) = \left\{ \alpha \in \Omega^1(P/Z, [\mathfrak{g}, \mathfrak{g}])^{G/Z} \;\middle|\; \begin{array}{l} \alpha(\xi_{P/Z}) = \xi \; \forall \xi \in [\mathfrak{g}, \mathfrak{g}] \\ F_\alpha = 0 \end{array} \right\}.$$

While $\mathcal{G}_0(P)$ acts on both these spaces, the larger group of gauge transformations $\mathcal{G}(P/Z)$ acts on $\mathcal{A}_{\mathrm{flat}}(P/Z)$. Consider the short exact sequence

$$1 \to Z^{\mathrm{ss}} \to [G, G] \to G/Z \to 1.$$

There is an isomorphism

$$\mathcal{G}(P/Z)/\mathcal{G}_0(P) \cong \mathrm{Hom}(\pi_1(X), Z^{\mathrm{ss}})$$

given by viewing the group of gauge transformations as sections of the bundle $P \times_G (G/Z)$ resp. $P \times_G [G,G]$. On the other hand, any gauge transformation in $\mathcal{G}(P/Z)$ that fixes a flat connection in $\mathcal{A}_{\text{flat}}(P/Z)$ automatically lifts to an element of $\mathcal{G}_0(P)$. Hence the projection

$$M(P) \to M_{\text{Ad}}(P) := \mathcal{A}_{\text{flat}}(P/Z)/\mathcal{G}(P/Z)$$

is a $\text{Hom}(\pi_1(X), Z^{\text{ss}})$-principal bundle. Finally, $M_{\text{Ad}}(P)$ is homeomorphic to a component of the representation space $M_{\text{Ad}}(X)$, given by those representations which lift to connections on P. Compactness follows since $\text{Ad}(G), Z^{\text{ss}}$ are compact and $\pi_1(X)$ is finitely generated. $\qquad\square$

3.2.4. *Moduli spaces for compositions of bordisms.* In this section we study moduli spaces for bordisms of bundles and the associated gluing law. Suppose that (Y, ϕ) is a compact oriented connected bordism between compact oriented connected manifolds $X_\pm$. Let (Q, ψ) be a bundle bordism between bundles $P_\pm \to X_\pm$. That is, Q is a bordism from P_- to P_+ with the structure of a G-bundle over Y equipped with an identification $\psi: \partial Q \to \overline{P}_- \cup P_+$ that is an isomorphism of G-bundles.

Definition 3.2.11. (a) (Restriction to the boundary) Denote the pullback map on connections by $\rho: \mathcal{A}(Q) \to \mathcal{A}(P_-) \times \mathcal{A}(P_+)$. The map ρ preserves the central curvature and fixed determinant conditions (with respect to appropriate restrictions) and is gauge equivariant. Hence ρ induces a map, also denoted

$$\rho: M(Q) \to M(P_-) \times M(P_+).$$

(b) (Correspondences associated to bordisms) Denote by $L(Q)$ the image of $M(Q)$ in $M(P_-) \times M(P_+)$,

$$L(Q) = \rho(M(Q)) \subset M(P_-) \times M(P_+).$$

Thus $L(Q)$ is a topological correspondence from $M(P_-)$ to $M(P_+)$, that is, a subspace of the product. In our applications, $L(Q)$ will be a Lagrangian correspondence between symplectic manifolds $M(P_-), M(P_+)$.

Remark 3.2.12. (Compatibility with maps to representation varieties) Let $x_\pm \in X_\pm, y \in Y$ be base points. The inclusion of the boundary $X_\pm \to Y$ induces a map $\pi_1(X_\pm, x_\pm) \to \pi_1(Y, y)$ depending on a choice of path from $x_\pm$ to y up to conjugacy. The map ρ is compatible with the bundle structure described in Lemma 3.2.10 in the sense that the diagram

$$
\begin{array}{ccc}
M(Q) & \longrightarrow & M(P_-) \times M(P_+) \\
\downarrow & & \downarrow \\
M_{\text{Ad}}(Y) & \longrightarrow & M_{\text{Ad}}(X_-) \times M_{\text{Ad}}(X_+)
\end{array}
$$

commutes and the top arrow is $\text{Hom}(\pi_1(Y), Z^{\text{ss}})$-equivariant. The group $\text{Hom}(\pi_1(Y), Z^{\text{ss}})$ acts on the right side of the diagram via the restriction homomorphism to

$$\text{Hom}(\pi_1(Y), Z^{\text{ss}}) \to \text{Hom}(\pi_1(X_-), Z^{\text{ss}}) \times \text{Hom}(\pi_1(X_+), Z^{\text{ss}})$$

Hence $L(Q)$ is contained in the image of the bottom arrow.

Proposition 3.2.13. (Sufficient condition for the correspondence associated to a bordism to be embedded) *Suppose that one of the push-forward maps $\pi_1(X_+) \to \pi_1(Y)$, $\pi_1(X_-) \to \pi_1(Y)$ is surjective. Then $\rho : M(Q) \to L(Q)$ is a bijection. Furthermore, suppose that*

(a) for any $[\alpha] \in M(P_\pm)$ we have

$$H^0(X_\pm; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}, \quad H^2(X_\pm; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}.$$

(b) for any $[\alpha] \in M(Q)$ we have

$$H^0(Y; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}, \quad H^2(Y; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}, \quad H^1(Y, \partial Y; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}.$$

Then $\rho : M(Q) \to L(Q) \subset M(P_-) \times M(P_+)$ is an embedding.

Proof. Suppose that one of $\pi_1(X_\pm) \to \pi_1(Y)$ is surjective. Then the maps on the corresponding representation varieties for $\mathrm{Ad}(G)$ and Z^{ss} in the diagram above are injective. Since the maps on the fiber and the base are injective, ρ is injective. To prove the second statement, note that the assumptions imply that the moduli spaces $M(Q), M(P_-), M(P_+)$ are smooth by Lemma 3.2.8. The linearization of restriction $M(Q) \to M(P_-) \times M(P_+)$ is then restriction on cohomology,

$$H^1(Y; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^1(X_-; d_{\alpha_-}^{[\mathfrak{g},\mathfrak{g}]}) \times H^1(X_+; d_{\alpha_+}^{[\mathfrak{g},\mathfrak{g}]}).$$

This is injective by the assumption $H^1(Y, \partial Y; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}$. Since any injective immersion of a compact space is an embedding, this completes the proof. $\square$

The notion of composition of bordisms extends naturally to composition of bundle bordisms. Let $Q_0 \to Y_0$ be a bundle bordism from $P_0 \to X_0$ to $P_1 \to X_1$ and $Q_1 \to Y_1$ a bundle bordism from $P_1 \to X_1$ to $P_2 \to X_2$, the composition $Q_0 \circ Q_1$ is defined using equivariant collar neighborhoods of P_1 in Q_0 and Q_1, and is independent up to bundle isomorphism of the choices.

Remark 3.2.14. (Correspondences for a composition of bundle bordisms) Pullback of connections under $\pi : Q_0 \sqcup Q_1 \to Q_0 \circ Q_1$ induces a map

(24) $$\pi^* : M(Q_0 \circ Q_1) \to M(Q_0) \times_{M(P_1)} M(Q_1)$$

where the fiber product is defined using the restriction maps $M(Q_j) \to M(P_1), j = 0, 1$. The map π^* fits into a commutative diagram

$$\begin{array}{ccc}
M(Q_0 \circ Q_1) & \xrightarrow{\pi^*} & M(Q_0) \times_{M(P_1)} M(Q_1) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{pr}_{M(P_0) \times M(P_2)} \circ (\rho \times \rho)} \\
L(Q_0 \circ Q_1) & \xrightarrow{\mathrm{id}_{M(P_0) \times M(P_2)}} & L(Q_0) \circ L(Q_1)
\end{array} \quad .$$

The bottom map has image in $L(Q_0) \circ L(Q_1)$ by the following argument: Let $[\alpha] \in M(Q_0 \circ Q_1)$ be a lift of $([\eta_0], [\eta_2]) \in L(Q_0 \circ Q_1) \subset M(P_0) \times M(P_2)$. Restricting to P_1 gives an element $[\eta_1] = [\alpha|_{P_1}] \in M(P_1)$ satisfying

$$([\eta_0], [\eta_1]) = \rho([\alpha|_{Q_0}]) \in L(Q_0), \quad ([\eta_1], [\eta_2]) = \rho([\alpha|_{Q_1}])) \in L(Q_1).$$

Hence $([\eta_0], [\eta_2])$ lies in

$$L(Q_0) \circ L(Q_1) = \big\{ (\zeta_0, \zeta_2) \,|\, \exists \zeta_1 \in M(P_1) : (\zeta_0, \zeta_1) \in L(Q_0), (\zeta_1, \zeta_2) \in L(Q_1) \big\}.$$

This ends the remark.

Proposition 3.2.13 gives conditions for the left arrow in the commutative diagram above to be a diffeomorphism; the following Proposition does the same for the top arrow; and the right arrow will be discussed in more specific cases in Section 3.4. The following composition property of the moduli spaces is a nonabelian version of the Mayer-Vietoris principle. For simplicity of notation we restrict to the case of gluing two connected components, although the same discussion holds e.g. for gluing along two boundary components of the same connected bundle bordism.

Proposition 3.2.15. (Conditions for composition of bordisms to give compositions of correspondences) *Let $Q_0 \to Y_0$ be a bundle bordism from $P_0 \to X_0$ to $P_1 \to X_1$ and $Q_1 \to Y_1$ a bundle bordism from $P_1 \to X_1$ to $P_2 \to X_2$. Suppose that the following conditions hold:*

(a) for any $[\alpha] \in M(P_1)$ we have $H^0(X_1; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}$, $H^2(X_1; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = 0$;
(b) for any $[\alpha] \in M(Q_j)$ for $j = 0$ or $j = 1$ we have

$$H^0(Y_j; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}, \quad H^2(Y_j; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\},$$

and the restriction map on stabilizers

$$\{ g \in \mathcal{G}_0(Q_j) \,|\, g^*\alpha = \alpha \} \to \{ g \in \mathcal{G}_0(P_1) \,|\, g^*\alpha|_{P_1} = \alpha|_{P_1} \}$$

is surjective for $j = 0$ or $j = 1$;
(c) for any $[\alpha] \in M(Q_0 \circ Q_1)$ we have

$$H^0(Y_0 \circ Y_1; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}, \quad H^2(Y_0 \circ Y_1; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\},$$

and the difference of the restriction maps on homology

$$H^1(Y_0; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \times H^1(Y_1; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^1(X_1; \mathrm{d}_{\alpha|X_1}^{[\mathfrak{g},\mathfrak{g}]}), (\eta_0, \eta_1) \mapsto \eta_0|_{X_1} - \eta_1|_{X_1}$$

is surjective;
(d) the restriction map $\rho : \mathcal{G}_0(Q_j) \to \mathcal{G}_0(P_1)$ is surjective for $j = 0$ or $j = 1$.
Then all the spaces in (24) are smooth and the map π^* is a diffeomorphism.

Proof. First we check that π^* is a bijection. To see that π^* is injective suppose that α, α' are connections on $Q_0 \circ Q_1$ mapping to the same element of $M(Q_0) \times M(Q_1)$ under π^*. Then there exists

$$g = (g_0, g_1) \in \mathcal{G}_0(Q_0) \times \mathcal{G}_0(Q_1), \quad g^*(\pi^*\alpha) = \pi^*\alpha'.$$

Since $(g_1|_{P_1})^*\alpha|_{P_1} = (g_0|_{P_1})^*\alpha|_{P_1}$ the product $(g_1|_{P_1})^{-1}(g_0|_{P_1})$ is an automorphism of $\alpha|_{P_1}$:

$$(25) \qquad\qquad (g_1|_{P_1})^{-1}(g_0|_{P_1})(\alpha|_{P_1}) = \alpha|_{P_1}.$$

By assumption (b), we may assume without loss of generality that (25) extends to an automorphism h of $\pi^*\alpha|_{Q_1}$. Then $g_0|_{P_1} = hg_1|_{P_1}$ and so g_0 and hg_1 glue together to a gauge transformation $\tilde{g}$ of $Q_0 \circ Q_1$ of class $W^{1,p}$ for any $2 < p < \infty$. Since $\tilde{g}^*\alpha = \alpha'$, and α, α' are smooth, $\tilde{g}$ is also smooth.

To see that π^* is surjective, let (α_0, α_1) represent an element of $M(Q_0) \times_{M(P_1)} M(Q_1)$. The restrictions $\alpha_0|_{P_1}, \alpha_1|_{P_1}$ are gauge equivalent by some $g_1 \in \mathcal{G}(P_1)$. By assumption (d), we may assume without loss of generality that g_1 extends over Q_1, and so after gauge transformation of (α_0, α_1) that $\alpha_0|_{P_1} = \alpha_1|_{P_1}$. We may also suppose that the determinant connection (δ_0, δ_1) is the pull-back $\pi^*\delta$ of a smooth determinant connection δ on $Q_0 \circ Q_1$. Then, after another gauge transformation on Q_1, we may assume that the normal components of α_0 and α_1 agree on P_1. Then the curvature equation implies that $(\alpha_0, \alpha_1) = \pi^*\alpha$ for some smooth connection α on $Q_0 \circ Q_1$.

Finally we check that π^* is a diffeomorphism. Let α represent an element of $M(Q_0 \circ Q_1)$ and $(\alpha_0, \alpha_1) = \pi^*\alpha$. By vanishing of H^0 and H^2 for $\alpha_0, \alpha_1, \alpha$, and $\alpha|_{P_1}$, the moduli spaces $M(Q_0), M(Q_1), M(Q_0 \circ Q_1)$, and $M(P_1)$ are smooth at the given points. The fiber product $M(Q_0) \times_{M(P_1)} M(Q_1)$ is smooth because the surjectivity of the difference map in assumption (c) implies transversality. To see that the linearization of π^* is an isomorphism, note that the Mayer-Vietoris long exact sequence gives a short exact sequence in first cohomology (since $H^0(X_1; \mathrm{d}_{\alpha|_{P_1}}) = \{0\}$ and $H^2(Y_0 \circ Y_1; \mathrm{d}_\alpha) = \{0\}$)

$$0 \to H^1(Y_0 \circ Y_1; \mathrm{d}_\alpha) \to H^1(Y_0; \mathrm{d}_{\alpha_0}) \oplus H^1(Y_1; \mathrm{d}_{\alpha_1}) \to H^1(X_1; \mathrm{d}_{\alpha|_{P_1}}) \to 0.$$

The induced isomorphism

$$H^1(Y_0 \circ Y_1; \mathrm{d}_\alpha) = H^1(Y_0; \mathrm{d}_{\alpha_0}) \times_{H^1(X_1; \mathrm{d}_{\alpha|_{P_1}})} H^1(Y_1; \mathrm{d}_{\alpha_1})$$

is equal to the linearization of (24), since the latter is the tangent space to the fiber product. It follows that (24) is a diffeomorphism. $\qquad\square$

3.3. **Moduli spaces for surfaces.** In this Section we make the first step towards constructing a symplectic-valued field theory via Theorem 2.2.13 by associating to any bundle any compact connected oriented surface a moduli space of constant curvature connections.

Remark 3.3.1. Let X be a compact, connected, oriented surface without boundary and $P \to X$ a G-bundle.

   (a) (Symplectic structure on the affine space of connections) The affine space $\mathcal{A}(P)$ carries a canonical weakly symplectic $\mathcal{G}(P)$-invariant two-form ω that induces the Hodge pairing (16). At a point $\alpha \in \mathcal{A}(P)$ the two-form is given by

$$(26) \qquad \omega_\alpha : \Omega^1(X, P(\mathfrak{g})) \times \Omega^1(X, P(\mathfrak{g})) \to \mathbb{R}, \quad (a_1, a_2) \mapsto \int_X \langle a_1 \wedge a_2 \rangle.$$

Here $\langle a_1 \wedge a_2 \rangle$ denotes the form in $\Omega^2(X)$ obtained by combining wedge product and the inner product on $P(\mathfrak{g})$. Weakly symplectic means weakly non-degenerate and closed, where

(i) weakly non-degenerate means that for every non-zero $a_1 \in \Omega^1(X, P(\mathfrak{g}))$ there exists an $a_2 \in \Omega^1(X, P(\mathfrak{g}))$ so that $\omega_\alpha(a_1, a_2) \neq 0$;

(ii) closed means $\mathrm{d}\omega = 0$ where the de Rham operator is defined on a Sobolev completion as in [28, Section 5.3].

The action of $\mathcal{G}_0(P)$ is Hamiltonian with moment map

(27) $$\mathcal{A}(P) \mapsto \Omega^0(X, P([\mathfrak{g}, \mathfrak{g}]))^\vee, \quad \alpha \mapsto \int_X \langle F_\alpha^{[\mathfrak{g},\mathfrak{g}]} \wedge \cdot \rangle$$

in the sense that

(i) ω is $\mathcal{G}_0(P)$-invariant,

(ii) the map (27) is equivariant with respect to the coadjoint action of $\mathcal{G}_0(P)$ on $\Omega^0(X, P([\mathfrak{g}, \mathfrak{g}]))^\vee$, and

(iii) the infinitesimal action (10) of $\xi \in \Omega^0(X, P([\mathfrak{g}, \mathfrak{g}]))$ is the Hamiltonian vector field of the function obtained from (27) by pairing: For all $\eta \in \Omega^1(X, P(\mathfrak{g}))$, we have

(28) $$\omega(\mathrm{d}_\alpha \xi, \eta) = \int_X \langle \mathrm{d}_\alpha \xi \wedge \eta \rangle = \int_X \langle \xi \wedge -\mathrm{d}_\alpha \eta \rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_X \langle F_{\alpha+t\pi^*\eta}^{[\mathfrak{g},\mathfrak{g}]} \wedge \xi \rangle.$$

(b) (Moduli of connections as a symplectic quotient [2]) The connections with fixed determinant $\det(\alpha) = \delta$ form a symplectic submanifold $\det^{-1}(\delta)$ of $\mathcal{A}(P)$. Indeed, the tangent space $\Omega^1(X, P([\mathfrak{g}, \mathfrak{g}])$ of $\det^{-1}(\delta)$ is a symplectic subspace of $\Omega^1(X, P(\mathfrak{g}))$. The action of $\mathcal{G}_0(P)$ preserves $\det^{-1}(\delta)$. Hence the moduli space of fixed determinant central curvature connections

$$M(P) = \{\alpha \in \mathcal{A}(P) | \det(\alpha) = \delta, \ F_\alpha^{[\mathfrak{g},\mathfrak{g}]} = 0\}/\mathcal{G}_0(P) = \mathcal{A}_\delta(P)/\mathcal{G}_0(P)$$

can be viewed as a symplectic quotient.

Using this point of view we establish the existence of a symplectic structure on $M(P)$. Recall, see e.g. [42], that the *dual Coxeter number* c associated to simple G is the positive integer $c := \langle \rho, \alpha_0 \rangle + 1$ where α_0 is the highest root and ρ the half-sum of positive roots. In particular for $G = SU(r)$ the dual Coxeter number is r.

Proposition 3.3.2. (Symplectic nature of the moduli space) *Let X be a compact, connected, oriented surface without boundary and $P \to X$ a G-bundle. Suppose that for any $[\alpha] \in M(P)$ we have $H^0(X; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}$. Then $M(P)$ is a compact symplectic orbifold of dimension $(2g(X) - 2) \dim G_0$, where $g(X)$ is the genus of X. If furthermore $[G, G]$ is simple, then $M(P)$ is monotone with monotonicity constant $1/2c$.*

Proof. Lemma 3.2.10 proves compactness, and the smoothness assertions were proven in Proposition 3.2.8 since $H^2(X; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \cong H^0(X; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ by Poincaré duality. The dimension formula follows from Riemann-Roch,

$$\dim T_{[\alpha]} M(P) = \dim H^1(X; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = (2g(X) - 2) \dim G_0.$$

To establish the symplectic structure, note that after Sobolev completion (taking Sobolev class L_2^s with $s > 1$) we may take $\mathcal{A}(P)$ to be an affine Banach space, equipped with the form ω above. The action

$$\mathcal{G}_0(P) \times \mathcal{A}(P) \to \mathcal{A}(P), \quad (\phi, \alpha) \mapsto (\phi^{-1})^* \alpha$$

extends to the Sobolev completion of $\mathcal{G}_0(P)$ of class L_2^{s+1} on connections of class L_2^s. Hence the moment map equation (28) holds for $\xi \in \Omega^0(X, P([\mathfrak{g}, \mathfrak{g}])$ of class L_2^{s+1} and $\eta \in \Omega^1(X, P([\mathfrak{g}, \mathfrak{g}]))$ of class L_2^s. Recall that ω is translationally invariant, hence closed in the sense of forms on Banach manifolds as in [28, Section 5.3]. Next, we restrict ω to $\mathcal{A}_\delta(P) \subset \mathcal{A}(P)$. The latter is a Banach submanifold for $s \geq 2$ by Proposition 3.2.8. Since the de Rham operator commutes with pull-back,

$$\omega|_{\mathcal{A}_\delta(P)} \in \Omega^2(\mathcal{A}_\delta(P)), \quad \mathrm{d}\omega|_{\mathcal{A}_\delta(P)} = 0$$

is again a closed two-form. Finally, note that $\omega|_{\mathcal{A}_\delta(P)}$ is $\mathcal{G}_0(P)$-invariant and vanishes on the vertical vectors $\mathrm{d}_\alpha \xi, \xi \in \Omega^0(X, P([\mathfrak{g}, \mathfrak{g}]))$ by (28). Hence it is the pull-back of a two-form on the quotient $M(P)$. The two-form form $\omega_{M(P)}$ on the quotient is closed because the de Rham operator commutes with pull-back and the map $\mathcal{A}_\delta(P) \to M(P)$ is a submersion. Non-degeneracy follows from the identity

$$(29) \qquad \int_X \langle a \wedge *a \rangle = \|a\|_{L_2}^2, \quad \forall [a] \in H^1(X; \mathrm{d}_\alpha) \cong T_{[\alpha]} M(P)$$

for a harmonic representative a of the class $[a]$. The assertion on monotonicity is a variation on the Drezet-Narasimhan theorem [14, Theorem F], see also [29] and [27]. A symplectic proof is given in [37]. There it is shown, using the description in Remark 3.3.7 below, that the first Chern class $c_1(M(P))$ is $2c$ times the symplectic class $[\omega_{M(P)}]$. $\qquad\qquad\square$

Now restricting to $G = U(r)$ we have the following main result of this section.

Theorem 3.3.3. (Properties of moduli spaces of connections on a surface) *Let P be a principal $U(r)$-bundle of degree d coprime to r over a compact, connected, oriented surface X without boundary.*

(a) *If X has genus $g(X) \geq 1$, then the moduli space $M(P)$ is a nonempty compact symplectic manifold of dimension*

$$\dim(M(P)) = (2g(X) - 2)(r^2 - 1)$$

with even Chern numbers

$$\langle c_1(M(P)), H_2(M(P)) \rangle \subset 2\mathbb{Z}$$

and satisfying the monotonicity relation

$$c_1(M(P)) = 2r[\omega_{M(P)}].$$

(b) *If X has genus $g(X) = 0$, then $M(P) = \emptyset$.*
(c) *If X has genus $g(X) = 1$, then $M(P) = \mathrm{pt}$ is a point.*

Moreover, $M(P)$ is always connected and simply-connected.

Remark 3.3.4. (Non-coprime case) Suppose we are in the situation of Theorem 3.3.3, except that r, d do have a common divisor. Then for $g(X) \geq 1$ there exist connections $\alpha \in \mathcal{A}_\delta(P)$ with non-discrete automorphism groups, i.e. $H^0(X; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \neq 0$. For $g(X) = 0$ we have $M(P) = \emptyset$ iff $d/r \in \mathbb{Z}$, and otherwise $M(P)$ is a point.

Proof of Theorem 3.3.3 and Remark 3.3.4. The smoothness assertions are standard, see for example [40], but we give a proof for convenience. The smooth manifold structure follows from Proposition 3.2.8 and vanishing $H^0(X; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ (and hence of $H^2(X; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ by Poincaré duality) for all $\alpha \in \mathcal{A}_\delta(P)$. To show the latter, suppose that $\alpha \in \mathcal{A}_\delta(P)$ is fixed by an infinitesimal gauge transformation $\xi \in \Omega^0(X, P(\mathfrak{g}))$ with trivial determinant, that is, $\mathrm{d}_\alpha \xi = 0$. This infinitesimal automorphism acts on the bundle $E := P(\mathbb{C}^r)$ with at least two distinct eigenvalues. A choice of group of eigenvalues into two distinct groups defines a splitting of the corresponding vector bundle

$$(30) \qquad E = E_1 \oplus E_2, \quad \mathrm{rank}(E_1) = k, \quad \mathrm{rank}(E_2) = r - k, \quad 0 < k < r.$$

The adjoint bundle $E^\vee \otimes E$ contains the direct summands $E_j^\vee \otimes E_j$ and the identity section $\mathrm{Id}_E = \mathrm{Id}_{E_1} \oplus \mathrm{Id}_{E_2}$ is the sum of identities in the summands. The curvature also splits $F_\alpha = F_{\alpha|_{E_1}} \oplus F_{\alpha|_{E_2}}$ and is a multiple of the identity $F_\alpha = \mathrm{Id}_E \eta$ for some $\eta \in \Omega^2(X)$. This implies $F_{\alpha|_{E_1}} = \mathrm{Id}_{E_1} \eta$. Hence the first Chern number of E_1 equals

$$\langle c_1(E_1), [X] \rangle = \frac{\mathrm{rk}(E_1)}{r} \langle c_1(E), [X] \rangle = \frac{kd}{r}.$$

Since the first Chern number of E_1 should be an integer and $k < r$, this implies that r and d cannot be coprime. That is, if r, d are coprime then no such splitting and hence no infinitesimal automorphism can occur.

Now the symplectic structure and monotonicity follow from Proposition 3.3.2 together with dual Coxeter number $c = r$ for $U(r)$. The assertion on the minimal Chern number is proved in [14]. The claims on (non)emptiness, connectedness and simply-connectedness follow from the stratification of $\mathcal{A}(P)$ established in [2], see especially [2, Theorem 9.12] with $G_0 = SU(r)$ simply-connected. If $X \cong T^2$ has genus $g = 1$, then $\dim M(P) = 0$ by the dimension formula and connectedness implies that $M(P)$ is a point. See also [47, 5] where moduli spaces of bundles on elliptic curves are investigated more extensively.

Alternatively, the (non)emptiness for general r, d can be seen from the description in Remark 3.3.7 below:

- If X has genus zero, then $d/r \notin \mathbb{Z}$ implies $M(P) = \emptyset$. Indeed the product of commutators in 3.3.7 must be $z = \exp(2\pi i d/r)\mathrm{Id}_{U(r)} \neq \mathrm{Id}_{U(r)}$ which is impossible. On the other hand, $M(P)$ is a single point for $d/r \in \mathbb{Z}$.
- If X has positive genus $g \geq 1$, then $M(P)$ is nonempty by e.g. Gotô's theorem [20].

Finally, if $r = r'm$ and $d = d'm$ have a common factor $m > 1$ and $g(X) \geq 1$, then by the above the moduli space $M(P')$ for a $U(r')$-bundle P' of degree d' is nonempty. Any representative of $[\alpha'] \in M(P')$ induces a connection on the corresponding vector bundle $E' = P'(\mathbb{C}^{r'})$. Then α' also induces a central curvature connection on $E =$

$\oplus_{i=1}^{m} E'$. From this one obtains a connection on the corresponding principal $U(r)$-bundle, which is isomorphic to P. The splitting of the bundle implies that the connection has a non-discrete automorphism group. $\qquad\square$

Remark 3.3.5. (Holomorphic description of the moduli spaces) The moduli spaces $M(P)$ have a holomorphic description, for $G = U(r)$ due to a famous theorem of Narasimhan-Seshadri [40], generalized to arbitrary groups in Ramanathan's thesis [43], [44]. However, we never use the holomorphic description.

Lemma 3.3.6. (Symplectomorphisms of moduli spaces induced by bundle isomorphisms) *Let r, d be coprime integers.*

  (a) *For any two $U(r)$-bundles $P_0 \to X_0, P_1 \to X_1$ of degree d, any bundle isomorphism $\psi : P_0 \to P_1$ covering an orientation-preserving diffeomorphism from X_0 to X_1 induces a symplectomorphism $\psi^* : M(P_1) \to M(P_0)$ given by pull-back of representatives.*
  (b) *Any two such bundle isomorphisms from P_0 to P_1 covering the same diffeomorphism from X_0 to X_1 induce the same symplectomorphism.*
  (c) *If $P_0 \to X, P_1 \to X$ are $U(r)$-bundles of degree d over the same surface, then the moduli spaces $M(P_0), M(P_1)$ are canonically symplectomorphic.*

Proof. For manifolds of dimension at most three, bundles are classified up to isomorphism by their first Chern class. For connected surfaces, the Chern class is determined by the degree. Hence if $\pi_j : P_j \to X_j, j = 0, 1$ are bundles of the same degree then there exists an isomorphism $\phi : P_0 \to P_1$ with $\pi_1 \circ \phi = \pi_0$. This induces a map from $\mathcal{A}(P_1)$ to $\mathcal{A}(P_0)$, given on the level of tangent spaces by the pull-back of basic forms $\phi^* : \Omega^1(P_1, \mathfrak{g})_{\text{basic}} \to \Omega^1(P_0, \mathfrak{g})_{\text{basic}}$. This pull-back acts symplectically: for $a, b \in \Omega^1(P_1, \mathfrak{g})_{\text{basic}}$ we have

$$\int_{X_1} \pi_{1,*}\langle a \wedge b \rangle = \int_{X_0} \pi_{0,*}\langle \phi^* a \wedge \phi^* b \rangle$$

since ϕ covers an orientation-preserving diffeomorphism from X_0 to X_1. Moreover pull-back by ϕ maps $\mathcal{A}_\delta(P_1)$ to $\mathcal{A}_{\phi^*\delta}(P_0)$ and is equivariant with respect to the gauge actions of $\mathcal{G}_0(P_1)$ and $\mathcal{G}_0(P_0) = \phi^* \mathcal{G}_0(P_1)$. Hence ϕ^* descends to a symplectomorphism $M(P_1) \to M(P_0)$. Any two such isomorphisms covering the same diffeomorphism differ by a gauge transformation of P_1. Since any gauge transformation of P_1 induces the identity on $M(P_1)$ and $M(P_1)$ is independent of the choice of determinant connection (see Remark 3.2.7), the identification $M(P_1) \to M(P_0)$ depends only on the choice of diffeomorphism from X_0 to X_1. $\qquad\square$

Remark 3.3.7. (Moduli spaces as representations of fundamental group of the punctured surface) Atiyah-Bott [2] provide a description of $M(P)$ in terms of representations, which makes Lemma 3.2.10 more precise in the case of bundles over surfaces. In that case, $M(P)$ can be identified with the moduli space of G_0-representations of the fundamental group of the punctured surface $X - \{x\}$, whose value on a small loop around x is equal to a certain central element $z \in G_0$. Here z is determined by the choice of the bundle $P \to X$. For example in the case that $G = U(r)$ and $\langle c_1(P), [X] \rangle = d$ we have $z = \exp(2\pi i d/r)\text{Id}_{U(r)}$. More explicitly, the fundamental

group of $X - \{x\}$ can be described as the free group on $2g$ generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ with the product of commutators $\prod_{j=1}^{g}[\alpha_j, \beta_j]$ being the class of the loop around the base point x. Thus we have a homeomorphism

$$(31) \qquad M(P) \cong \left\{ (a_1, b_1, \ldots, a_g, b_g) \in G_0^{2g} \;\middle|\; \prod_{j=1}^{g}[a_j, b_j] = z \right\}/G_0.$$

In particular, for $X \cong S^2$ we have

$$M(P) = \begin{cases} \emptyset & z \neq \mathrm{Id}_G \\ \mathrm{pt} & z = \mathrm{Id}_G. \end{cases}$$

The map (31) is given by choosing a determinant connection whose curvature is concentrated near a base point, and mapping each $[\alpha]$ to the representation of $\pi_1(X - \{x\})$ given by the holonomies.

3.4. **Moduli spaces for three-dimensional elementary bordisms.** In this Section we make the second step towards constructing a symplectic-valued field theory in dimension $2 + 1$ via Theorem 2.2.13, by defining a partial functor on elementary bordisms.

Definition 3.4.1. (Correspondences for bordisms) Fix $G = U(r)$ for some positive integer r and let $P_\pm \to X_\pm$ be principal $U(r)$-bundles of the same degree d coprime to r over surfaces $X_\pm$ as in Section 3.3. Let (Q, ψ) be a bundle bordism from $P_- \to X_-$ to $P_+ \to X_+$ as in Section 3.2.4; in particular Y is a 3-dimensional bordism from X_- to X_+. The image of $M(Q)$ under pullback as in Definition 3.2.11 is

$$L(Q) := \rho(M(Q)) \subset M(P_-)^- \times M(P_+).$$

Remark 3.4.2. (Low genus cases)
   (a) If either X_+ or X_- has genus 0, then the corresponding moduli space $M(P_\pm)$ is empty by Theorem 3.3.3, and therefore so is $L(Q)$.
   (b) If either X_+ or $X_- = T^2$ has genus 1, then the corresponding moduli space $M(P_\pm)$ is a point. Furthermore, by a special case of the following Theorem, $L(Q)$ is a Lagrangian correspondence from a point to $M(P_\mp)$ and can also be viewed as Lagrangian submanifold of $M(P_\mp)$.

Theorem 3.4.3. (Lagrangians for elementary bordisms) *If Y is a compression body as in Definition 2.2.6 and $Q \to Y$ is a principal G-bundle then $\rho : M(Q) \to L(Q) \subset M(P_-)^- \times M(P_+)$ is a Lagrangian embedding. If moreover Y is an elementary bordism as in Definition 2.2.1 (b), then $L(Q)$ is simply-connected and spin.*

Proof. By Theorem 3.3.3, $H^0(X_\pm; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \cong H^2(X_\pm; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}$ for any $[\alpha] \in M(P_\pm)$. Lemma 3.4.4 (a) below shows that the assumptions of Proposition 3.2.13 are satisfied so that ρ is an embedding. The image is Lagrangian since the image of $T_{[\alpha]}M(Q) \cong H^1(Y; \mathrm{d}_\alpha)$ in $T_{([\alpha|_{P_-}], [\alpha|_{P_+}])}(M(P_-)^- \times M(P_+)) \cong H^1(\partial Y; \mathrm{d}_{\alpha|_{\partial Y}})$ is a Lagrangian subspace by Corollary 3.2.5.

Next, suppose that Y is elementary and without loss of generality X_- is the surface of lower genus. By Lemma 3.4.5 below, $L(Q)$ is a principal G_0-bundle over

the moduli space $M(P_-)$. The space $M(P_-)$ is simply-connected by Theorem 3.3.3. Since the fiber and base are simply-connected, so is $L(Q)$. Now let π denote the projection of $L(Q)$ onto $M(P_-)$. By Lemma 3.4.5 the tangent bundle of $L(Q)$ is the sum of the pull-back of $TM(P_-)$ with a vector bundle with fiber $\mathfrak{g}_0$ and structure group G_0. By Lemma 3.3.2 the canonical bundle of $M(P_-)$ admits a square root. Since G_0 is simply-connected, the representation $\mathfrak{g}_0$ is spin and hence so is the vertical part of $TL(Q)$. Hence $L(Q)$ is also spin. $\qquad\square$

Lemma 3.4.4. *Suppose that Y is a compression body as in Definition 2.2.6 between surfaces X_-, X_+ of positive genus, and the genus of X_+ is larger than or equal to the genus of X_-. Then the following holds.*

(a) *The map of fundamental groups $\pi_1(X_+) \to \pi_1(Y)$ is a surjection and the map $\pi_1(X_-) \to \pi_1(Y)$ is an injection.*

(b1) *For any connection $\alpha \in \mathcal{A}_\delta(Q)$, the map induced by restriction $H^0(Y, \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^0(X_+, \mathrm{d}_{\alpha|_{P_+}}^{[\mathfrak{g},\mathfrak{g}]})$ is an isomorphism and the map $H^0(Y, \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^0(X_-, \mathrm{d}_{\alpha|_{P_-}}^{[\mathfrak{g},\mathfrak{g}]})$ is an injection;*

(b2) *the map $H^1(Y, \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^1(X_+, \mathrm{d}_{\alpha|_{P_+}}^{[\mathfrak{g},\mathfrak{g}]})$ is injective and the map $H^1(Y, \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^1(X_-, \mathrm{d}_{\alpha|_{P_-}}^{[\mathfrak{g},\mathfrak{g}]})$ is surjective;*

(b3) *the map $H^2(Y, \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^2(X_-, \mathrm{d}_{\alpha|_{P_-}}^{[\mathfrak{g},\mathfrak{g}]})$ is an isomorphism and the map $H^2(Y, \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^2(X_+, \mathrm{d}_{\alpha|_{P_+}}^{[\mathfrak{g},\mathfrak{g}]})$ is a surjection.*

Proof. For simplicity of notation assume that $\partial Y = \overline{X}_- \sqcup X_+$, that is, the boundary identification ϕ is the identity. By assumption, Y supports a Morse function that is maximal on X_+, minimal on X_-, and only has critical points of index 1. Then Y deformation retracts onto the union of X_- and the stable manifolds of the critical points, which are intervals. Since X_- is connected, the attaching points of these intervals may be homotoped so that they are equal. Thus Y is homotopic to the wedge product of X_- and a collection of circles. By the Seifert-van Kampen theorem, $\pi_1(Y)$ is the free product of $\pi_1(X_-)$ and a copy of $\mathbb{Z}$ for each critical point. Hence $\pi_1(X_-) \to \pi_1(Y)$ is an injection. Similarly Y is obtained from X_+ by attaching 2-handles. Seifert-van Kampen in this case presents $\pi_1(Y)$ as the quotient of $\pi_1(X_+)$ by actions of $\mathbb{Z}$ for each critical point, given by the image of the attaching cycle in $\pi_1(X_+)$.

The assertions on zeroth cohomology follow from its interpretation as infinitesimal automorphisms of a flat connection, which are the Lie algebras of the centralizers of its holonomy groups, and part (a). The remaining assertions follow from the Mayer-Vietoris principle for the cohomology of d_α. To apply it, we consider Y as the union $U \cup V$ where U is a neighborhood of $X_\pm$ and V is a neighborhood of the union of stable manifolds of the critical points, so that $U \cap V$ is a neighborhood of the attaching cycles. Then $H^0(V; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^0(X_+ \cap V; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ is surjective, since any infinitesimal automorphism of a flat connection over the boundary of a disk extends over the interior. By the long exact sequence $H^1(Y; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^1(X_+; \mathrm{d}_{\alpha|_{P_+}}^{[\mathfrak{g},\mathfrak{g}]})$

is injective. On the other hand $H^1(X_- \cap V; d_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ vanishes, since $X_- \cap V$ is homotopic to a finite set of points and the Poincaré lemma holds for cohomology with twisted coefficients. Hence $H^1(Y; d_\alpha^{[\mathfrak{g},\mathfrak{g}]}; d_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ surjects onto $H^1(X_-; d_{\alpha|P_-}^{[\mathfrak{g},\mathfrak{g}]})$. Since $H^1(X_- \cap V; d_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ and $H^2(X_- \cap V; d_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ vanish, $H^2(Y; d_\alpha^{[\mathfrak{g},\mathfrak{g}]})$ is isomorphic to $H^2(X_-; d_{\alpha|P_-}^{[\mathfrak{g},\mathfrak{g}]}) \cong H^0(X_-; d_{\alpha|P_-}^{[\mathfrak{g},\mathfrak{g}]})$. Finally surjectivity onto $H^2(X_+; d_{\alpha|P_+}^{[\mathfrak{g},\mathfrak{g}]})$ follows from the exact sequence. $\square$

In order to understand the topology of $L(Q)$ we use an explicit description in terms of representations of the extended fundamental group as in Remark 3.3.7.

Lemma 3.4.5. *Suppose that Y is an elementary bundle bordism from a surface X_- of genus g to a surface X_+ of genus $g + 1$. Let Q be a G-bundle over Y. Then in the description of Remark 3.3.7, the Lagrangian $L(Q) \subset M(P_-)^- \times M(P_+)$ is the set of pairs of equivalence classes*

$$\big([a_1, b_1, \ldots, a_g, b_g], [a_1, b_1, \ldots, a_{g+1}, \mathrm{Id}]\big) \in G_0^{2g}/G_0 \times G_0^{2g+2}/G_0$$

satisfying the relation $\prod_{j=1}^g [a_j, b_j] = z$. In particular, the projection $\pi : L(Q) \to M(P_-)$ gives $L(Q)$ the structure of a G_0-bundle over $M(P_-)$, whose vertical tangent space is the associated bundle

$$\ker(D\pi) \cong \left\{ [a_1, b_1, \ldots, a_g, b_g, \xi] \in G_0^{2g} \times \mathfrak{g} \,\middle|\, \prod_{j=1}^g [a_j, b_j] = z \right\} /G_0.$$

Proof. By Lemma 3.2.10, the moduli space $M(Q)$ is a $\mathrm{Hom}(\pi_1(Y), Z^{\mathrm{ss}})$ fiber bundle over a component of $\mathrm{Hom}(\pi_1(Y), \mathrm{Ad}(G))/\mathrm{Ad}(G)$. Choose generators

$$\alpha_1, \beta_1, \ldots, \alpha_{g+1}, \beta_{g+1} \in \pi_1(X_+), \quad \prod_{j=1}^{g+1}[\alpha_j, \beta_j] = \mathrm{Id}_{\pi_1(X_+)}$$

such that the maps of fundamental groups in Lemma 3.4.4 (a) realize

  (a) $\pi_1(Y)$ as the quotient of $\pi_1(X_+)$ by the subgroup generated by β_{g+1}, and
  (b) $\pi_1(X_-)$ as the subgroup of Y generated by $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$.

Now $\mathrm{Hom}(\pi_1(Y), \mathrm{Ad}(G))/\mathrm{Ad}(G)$ and $\mathrm{Hom}(\pi_1(Y), Z)$ are the subsets of the spaces of representations of $\pi_1(X_+)$ given by requiring that the representation vanishes on β_{g+1}. It follows that $L(Q)$ is equal to the space of representations mapping β_{g+1} to some element h of Z.

Now we assume that Q is equipped with a determinant connection whose curvature vanishes except on a small neighborhood of a path connecting base points in $X_\pm$, disjoint from the stable and unstable manifolds. In particular, the curvature vanishes on a disk with boundary β_{g+1}. Since h is equal to the holonomy around the vanishing cycle β_{g+1}, h must equal the identity. $\square$

This finishes the proof of Theorem 3.4.3. Together with the subsequent Lemma this defines the functors for elementary bordisms for any fixed coprime r, d:

Lemma 3.4.6. (Existence and uniqueness of Lagrangians associated to elementary bordisms) *For d, r coprime positive integers:*

(a) *For any elementary bordism Y from X_- to X_+ there exists a $U(r)$-bundle $Q \to Y$ such that both $Q|_{X_-}$ and $Q|_{X_+}$ have degree d.*

(b) *The Lagrangian correspondence $L(Q) \subset M(P_-)^- \times M(P_+)$ is independent of the choice of Q under the canonical symplectomorphisms of Lemma 3.3.6.*

(c) *The Lagrangian correspondence associated to the trivial bordism $[0,1] \times X$ is the diagonal $L([0,1] \times P) = \Delta_{M(P)} \subset M(P)^- \times M(P)$.*

Proof. Let Y be an elementary bordism from X_- to X_+. A simple computation using e.g. cellular homology shows that since Y is elementary, $H^2(Y) \cong H^2(X_\pm) \cong \mathbb{Z}$. Taking a $U(r)$ bundle induced from a $U(1)$-bundle with first Chern class d shows that the bundle Q above exists. Since the homotopy groups $\pi_1(SU(r))$ and $\pi_2(SU(r))$ are trivial, the bundle Q is unique up to isomorphism. These arguments imply that $M(Q)$ is well-defined and $L(Q)$ is independent of the choice of Q. In the case of a product bordism $Q = [0,1] \times P$, we take δ to be the pull-back of a determinant connection on $P_\pm$. Since any connection is gauge-equivalent to one vanishing on the fibers of $[0,1] \times P \to P$, pull-back defines an isomorphism $M(Q) \cong M(P_\pm)$. Thus $M(Q)$ is embedded in $M(P_-) \times M(P_+)$ via the diagonal. $\square$

3.5. **Cerf moves for moduli spaces.** This Section provides the final steps in the construction of a symplectic-valued field theory in dimension $2+1$ via Theorem 2.2.13. As before, we fix $G = U(r)$ for some $r \in \mathbb{N}$ and fix a degree $d \in \mathbb{Z}$ coprime to r.

Theorem 3.5.1. (Field theory via moduli spaces) *For d, r coprime positive integers there exists a unique $2+1$-dimensional connected symplectic-valued field theory $\Phi : \mathrm{Bor}^0_{2+1} \to \mathrm{Symp}^\#$ that associates*

(a) *to each surface X the moduli space*

$$\Phi(X) = M(P) =: M(X)$$

constructed in Section 3.3 using a choice of $U(r)$-bundle $P \to X$ of degree d, and

(b) *to each elementary bordism $[Y]$ from X_- to X_+, the equivalence class*

$$\Phi(Y) = [L(Y)], \quad L(Y) := L(Q) \subset M(X_-)^- \times M(X_+)$$

constructed in Section 3.4 using a choice of $U(r)$-bundle $Q \to Y$ pulling back to the given bundles over $X_\pm$.

Remark 3.5.2. (Generalized correspondences for bordisms) Theorem 3.5.1 associates to any morphism $[Y]$ from X_- to X_+ in Bor^0_{2+1} an equivalence class of generalized Lagrangian correspondences as follows. Let

$$Y = Y_1 \cup_{X_1} Y_2 \cup_{X_2} \ldots \cup_{X_{r-1}} Y_m$$

be a Cerf decomposition into elementary bordisms. Associated to each piece Y_k is a Lagrangian correspondence $L(Y_k)$ from $M(X_{k-1})$ to $M(X_k)$, where by convention $M(X_0) = M(X_-)$ and $M(X_m) = M(X_+)$. Hence our construction associates a generalized Lagrangian correspondence from $M(X_-)$ to $M(X_+)$ to the Cerf decomposition of Y,

$$\underline{L}((Y_k)_{k=1,\ldots,m}) := \big(L(Y_1), \ldots, L(Y_m)\big).$$

Theorem 3.5.1 implies that the equivalence class

$$\Phi(Y) = [\underline{L}((Y_k)_{k=1,\ldots,m}))] = [(L(Y_1),\ldots,L(Y_m))] = \Phi(Y_1) \circ \ldots \circ \Phi(Y_m)$$

as an element in $\mathrm{Hom}_{\mathrm{Symp}^\#}(M(X_-), M(X_+))$ is independent of the choice of Cerf decomposition. This ends the remark.

We already proved in Theorem 3.4.3 that the Lagrangian correspondences involved are simply-connected and spin. To complete the proof of the theorem it suffices to show that the Cerf relations of Theorem 2.2.13 are satisfied for the Lagrangian correspondences constructed in the previous section.

Lemma 3.5.3. *For each of the Cerf moves in Theorem 2.2.11, the conditions in Proposition 3.2.15 hold.*

Proof. Note that by assumption H^0 and H^2 of $\mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}$ vanish on $Y_i, Y_{i+1}, X_{i-1}, X_i, X_{i+1}$. Recall $G_0 = [G,G] = SU(r)$ and $\mathcal{G}_0(P)$ is the corresponding group of gauge transformations. The restriction map $\mathcal{G}_0(Q) \to \mathcal{G}_0(P_-) \times \mathcal{G}_0(P_+)$ is surjective, since $\pi_1(G_0), \pi_2(G_0)$, vanish. It follows that there is no obstruction to extending over the $1, 2$ and 3-dimensional cells of the bordism. It remains to check the surjectivity of the difference of the restriction maps $H^1(Y_i \sqcup Y_{i+1}; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^1(X_i; \mathrm{d}_{\alpha|P_i}^{[\mathfrak{g},\mathfrak{g}]})$. By Mayer-Vietoris this surjectivity is equivalent to injectivity of $H^2(Y_i \cup Y_{i+1}; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) \to H^2(Y_i \sqcup Y_{i+1}; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]})$. This injectivity holds if $H^2(Y_i \cup Y_{i+1}; \mathrm{d}_\alpha^{[\mathfrak{g},\mathfrak{g}]}) = \{0\}$. We check this in each case.

Critical point cancellation: Suppose that two adjacent pieces Y_i, Y_{i+1} are replaced by a single piece $Y_i \cup Y_{i+1}$ diffeomorphic to a cylinder $[-1,1] \times X_{i-1}$. Then every point in $M(Y_i \cup Y_{i+1}) \cong M(X_{i-1})$ has vanishing H^2, as required.

Gluing in a cylinder: An elementary bordism Y_i and cylindrical bordism Y_{i+1} is replaced with another elementary bordism diffeomorphic to Y_i. Then every point in $M(Y_i \cup Y_{i+1}) \cong M(Y_i)$ has vanishing H^2, as required.

Reversing order of critical points: This move can be broken down into stages, where in the first stage two elementary bordisms are replaced by a compression body, and in the second stage the compression body is replaced by two other elementary bordisms. If the two critical points have the same index, then H^2 vanishes by Lemma 3.4.4. In the case of differing index, suppose that the two-handle is attached first. Then the surjectivity claim holds by Lemma 3.4.4, since X_i has larger genus than X_{i+1} and Y_i is a compression body. Thus H^2 vanishes. By Mayer-Vietoris the surjectivity property in Proposition 3.2.15 holds for the decomposition corresponding to attaching a one-handle first, as well. $\square$

Remark 3.5.4. (Holonomy description of Cerf moves) Suppose that X_{i-1} is a surface of genus g and X_i is a surface of genus $g+1$.

(a) (Critical point cancellation) By Remark 3.4.5 $L(Y_i)$ resp. the transpose of $L(Y_{i+1})$ may be identified with the set of pairs of orbits

$$([a_1, b_1, \ldots, a_g, b_g], [a'_1, b'_1, \ldots, a'_{g+1}, b'_{g+1}])$$

such that

$$b'_{g+1} = \text{Id}, \quad a_j = a'_j, \quad b_j = b'_j, \quad j = 1, \ldots, g$$

resp.

$$a'_{g+1} = \text{Id}, \quad a_j = a'_j, \quad b_j = b'_j, \quad j = 1, \ldots, g.$$

Thus the composition $L(Y_i) \circ L(Y_{i+1})$ is the diagonal.

(b) (Critical point switch) Suppose that the initial decomposition Y_i, Y_{i+1} corresponds to attaching a one-handle and then attaching a two-handle to X_{i-1}. The surface X_i is obtained from X'_{i+1} by attaching two one-handles, so that the attaching one-cycles in X_i correspond to disjoint generators of $\pi_1(X_i)$. These generators we may take to equal α_1 resp. α_{g+1}. Then

$$L(Y_i) = \left\{ \begin{array}{c} ([a_2, b_2, \ldots, a_g, b_g], [a'_1, b'_1, \ldots, a'_{g+1}, b'_{g+1}]) \\ a_1 = \text{Id}, \quad a_j = a'_j, \quad b_j = b'_j, \quad j = 2, \ldots, g \end{array} \right\}.$$

Furthermore

$$L(Y_{i+1}) = \left\{ \begin{array}{c} ([a'_1, b'_1, \ldots, a'_{g+1}, b'_{g+1}], [a''_1, b''_1, \ldots, a''_g, b''_g]) \\ a'_{g+1} = \text{Id}, \quad a'_j = a''_j, \quad b'_j = b''_j, \quad j = 1, \ldots, g \end{array} \right\}.$$

Thus $L(Y_i \cup Y_{i+1})$ is embedded into $M(X_{i-1})^- \times M(X_{i+1})$, and the composition $L(Y_i) \circ L(Y_{i+1})$ is transversal and equal to $L(Y_i \cup Y_{i+1})$. The arguments for the other order of attaching, and the other indices of critical points, are similar.

Applying Proposition 3.2.15 and Theorem 2.2.13 yields Theorem 3.5.1.

4. FUNCTORS FOR BORDISMS VIA QUILTS

In this section, we associate to any three-dimensional bordism as above a functor between the extended Fukaya categories of the moduli spaces of connections associated to the incoming and outgoing surfaces, using quilted Floer theory.

4.1. **Quilted Floer theory.** In the paper [55] we generalized Lagrangian Floer theory to the setting of Lagrangian correspondences.

Definition 4.1.1. (Quilted Floer homology) Suppose that

(a) $\underline{M} = (M_0, M_1, \ldots, M_m)$ is a sequence of compact simply-connected monotone symplectic manifolds with the same monotonicity constant τ and N-fold Maslov covers $\text{Lag}^N(M_k) \to \text{Lag}(M_k), k = 0, \ldots, m$;

(b) $\underline{L} = (L_{01}, L_{12}, \ldots, L_{(m-1)m})$ is a sequence of compact graded relatively spin Lagrangian correspondences from M_0 to M_1, M_1 to M_2 etc.

(c) $L_0 \subset M_0, L_m \subset M_m$ are compact simply-connected graded relatively spin Lagrangian submanifolds.

The *quilted Floer cohomology* $HF(\underline{L})$ of a generalized Lagrangian correspondence $\underline{L} = (L_1, L_2, \ldots, L_m)$ is the $\mathbb{Z}_N$-graded cohomology of a complex whose differential is a signed count of *quilted* holomorphic strips with boundary in L_0, L_m and seams in $L_{(j-1)j}, j = 1, \ldots, m$.

Remark 4.1.2. (Effect of geometric composition on quilted Floer cohomology) We proved in [56] that the quilted Floer cohomology groups behave well under composition: if for some j, the composition $L_{(j-1)(j+1)} := L_{(j-1)j} \circ L_{j(j+1)}$ is smooth and embedded into $M_{j-1}^- \times M_{j+1}$ then the quilted Floer cohomology group is unchanged up to isomorphism by replacing the pair $L_{(j-1)j}, L_{j(j+1)}$ with $L_{(j-1)(j+1)}$. An example is shown in Figure 3, for the case $m = 2$.
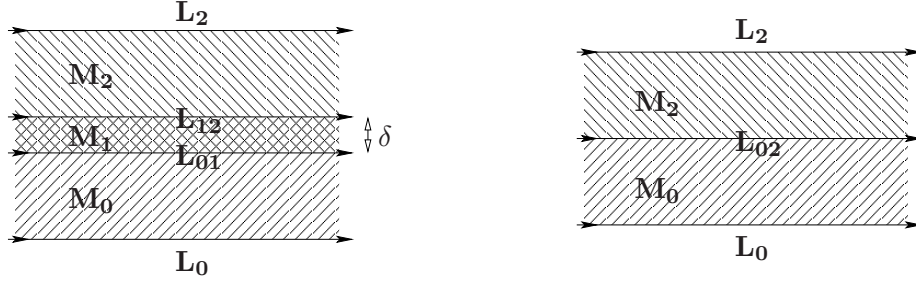

FIGURE 3. Shrinking the middle strip

In [35] we associated to any Lagrangian correspondence L_{01} from M_0 to M_1 a functor $\Phi(L_{01})$ from a version of the Fukaya category of M_0 to that of M_1:

Definition 4.1.3. Let M be a τ-admissible symplectic manifold as in Definition 3.1.7 equipped with an N-fold Maslov cover $\mathrm{Lag}^N(M) \to \mathrm{Lag}(M)$.

  (a) (Generalized Lagrangian submanifolds) A *generalized Lagrangian submanifold* of M is a generalized Lagrangian correspondence from a point to M, that is, a sequence $L_{-s(-s+1)}, \ldots, L_{(-1)0}$ of correspondences from $M_{-s} = \mathrm{pt}$ to $M_0 = M$. We say that a generalized Lagrangian correspondence satisfies a certain property (simply-connected, compact, etc.) if each correspondence in the sequence satisfies that property.
  (b) (extended Fukaya category) Let $\mathrm{Fuk}^\#(M)$ denote the A_∞ category whose
      (i) objects are compact, oriented generalized Lagrangian submanifolds of M equipped with gradings and relative spin structures;
      (ii) morphisms from an object $\underline{L}_0$ to an object $\underline{L}_1$ are quilted Floer homology cochains:

(32) $$\mathrm{Hom}(\underline{L}_0, \underline{L}_1) = CF(\underline{L}_0, \underline{L}_1)$$

      constructed in [55];
      (iii) composition and identities are defined by counting holomorphic quilts with strip-like ends and Lagrangian boundary and seam conditions as in [35].
  (c) (Functors for Lagrangian correspondences) Let M_0, M_1 be τ-admissible symplectic manifolds. For any compact, oriented, simply-connected relatively-spin graded correspondence $L_{01} \subset M_0^- \times M_1$ the functor

$$\Phi(L_{01}) : \mathrm{Fuk}^\#(M_0) \to \mathrm{Fuk}^\#(M_1)$$

is defined on objects by

$$(L_{-s(-s+1)}, \ldots, L_{(-1)0}) \mapsto (L_{-s(-s+1)}, \ldots, L_{(-1)0}, L_{01}).$$

On morphisms and at the level of cohomology $\Phi(L_{01})$ is defined by counting holomorphic quilts of the form in Figure 4.
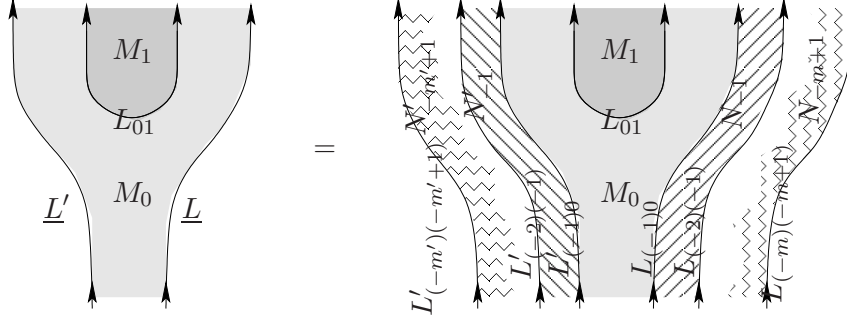

FIGURE 4. The Lagrangian correspondence functor $\Phi(L_{01})$ on morphisms

Remark 4.1.4. (Functors for geometric compositions) The composition result on quilted Floer homology generalizes to the categorical setting as follows [56]: Suppose that L_{01} and L_{12} are Lagrangian correspondences as above such that $L_{01} \circ L_{12}$ is smooth and embedded into $M_0^- \times M_2$, and simply-connected with spin structure induced from that of L_{01}, L_{12} and M_1. Then $\Phi(L_{12}) \circ \Phi(L_{01})$ is homotopic to $\Phi(L_{01} \circ L_{12})$. Note that the convention for composition of functors (first functor on the right) is opposite to the convention for composition of correspondences (first correspondence on the left).

Definition 4.1.5. (Category of A_∞ categories) For a fixed positive even integer N let $A_\infty(N)$ be the category whose objects are $\mathbb{Z}_N$-graded A_∞ categories $\mathcal{C}$ and whose morphisms are homotopy classes of A_∞ functors $\Phi : \mathcal{C}_0 \to \mathcal{C}_1$. Composition is composition of functors. Because our theory is only relatively graded, we also require that the categories are equipped with *shift functors*

$$S : \mathcal{C} \to \mathcal{C}, \quad \mathrm{Hom}(S(L_0), L_1)[1] = \mathrm{Hom}(L_0, L_1) = \mathrm{Hom}(L_0, S(L_1))[-1]$$

in the sense of Seidel [49, Section 3d, Section 12h], where $[k]$ denotes degree shift by k; and the functors $\Phi : \mathcal{C}_0 \to \mathcal{C}_1$ between categories are required to commute with the shift functors: $\Phi \circ S_0 = S_1 \circ \Phi$.

Remark 4.1.6.   (a) Any Fukaya category $\mathrm{Fuk}(M)$ has such a shift functor obtained on objects by shifting the grading on the Lagrangians $L \mapsto L[1]$ by Seidel [49, Section 12h] in the exact setting; the same construction holds for the extended Fukaya categories for monotone symplectic manifolds by taking $\underline{L}[1] = (L_0, \ldots, L_k[1])$ for $\underline{L} = (L_0, \ldots, L_k)$.
   (b) The functors associated to graded Lagrangian correspondences commute with the shift functors under the following assumption: The Maslov cover on any

product $X_0^- \times X_1$ are obtained from the Maslov covers on the factors as in Wehrheim-Woodward [55, Section 3] by $\mathrm{Sp}^N(2n_0 + 2n_1) \times_{\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1)} \mathrm{Lag}^N(X_0) \times \mathrm{Lag}^N(X_1)$, where Sp^N is the group of N-graded symplectomorphisms, with notation that of Seidel [48].

Remarks 4.1.4 and 4.1.6 imply the following:

Theorem 4.1.7. *For any $\tau > 0$ the maps*

$$M \mapsto \mathrm{Fuk}^\#(M), \quad [\underline{L}_{01}] \mapsto [\Phi(\underline{L}_{01})]$$

define a categorification functor $\mathrm{Fuk}^\# : \mathrm{Symp}_{\tau,N}^\# \to A_\infty(N)$.

Proof. In [35] we constructed the functors $\Phi(\underline{L}_{01})$ and proved that if $\underline{L}_{01}, \underline{L}'_{01}$ are related by a composition equivalence (6) then the functors $\Phi(\underline{L}_{01}), \Phi(\underline{L}'_{01})$ are homotopic. It follows that the homotopy class $[\Phi(\underline{L}_{01})]$ is independent of the choice of representative $\underline{L}_{01}$. $\square$

Definition 4.1.8. Two morphisms $[\Phi], [\Phi'] : \mathcal{C}_0 \to \mathcal{C}_1$ in $A_\infty(N)$ are *shift equivalent* if after some grading shift the functors become homotopy equivalent, that is, $\Phi S_0^k \simeq \Phi'$ for some integer k. Denote by $A_\infty(N)/S$ the category whose objects are those of $A_\infty(N)$ and whose morphisms spaces are shift equivalence classes of morphisms in $A_\infty(N)$. Similarly, let $\mathrm{Symp}_{\tau,N}^\#/S$ be the category whose objects are those of $\mathrm{Symp}_{\tau,N}^\#$ and whose morphisms are equivalences classes of generalized Lagrangian branes, where two branes are equivalent if they are related by composition equivalences and grading shifts $\underline{L} \to \underline{L}[k]$.

Remark 4.1.9. (Relative gradings) Given such an equivalence class of functors $[\Phi] \in \mathrm{Hom}_{A_\infty(N)/S}(\mathcal{C}_0, \mathcal{C}_1)$, and objects $L_0 \in \mathrm{Obj}(\mathcal{C}_0), L_1 \in \mathrm{Obj}(\mathcal{C}_1)$, the morphism space $\mathrm{Hom}(\Phi(L_0), L_1)$ is independent of all choices as a *relatively* $\mathbb{Z}_N$-graded group. That is, $\mathrm{Hom}(\Phi(L_0), L_1)$ admits a canonical decomposition into summands $\mathrm{Hom}^k(\Phi(L_0), L_1), k \in \mathbb{Z}_N$ up to a shift in grading. The categorification functor $\mathrm{Fuk}^\# : \mathrm{Symp}_{\tau,N}^\# \to A_\infty(N)$ of Theorem 4.1.7 induces a relatively graded version

$$\mathrm{Fuk}^\#/S : \quad \mathrm{Symp}_{\tau,N}^\#/S \to A_\infty(N)/S.$$

In particular, given a morphism $[L] \in \mathrm{Hom}_{\mathrm{Symp}_{\tau,N}^\#/S}(M_0, M_1)$, and objects $L_0 \in \mathrm{Obj}(\mathrm{Fuk}^\#(M_0)), L_0 \in \mathrm{Obj}(\mathrm{Fuk}^\#(M_1))$, the morphism space $\mathrm{Hom}(\Phi(\underline{L})(L_0), L_1)$ is a $\mathbb{Z}_N$-relatively-graded group.

4.2. **Floer field theory.** We can now prove the main result of the paper:

Theorem 4.2.1. (Floer field theory) *For any coprime positive integers r, d, the maps for surfaces X given by*

$$X \mapsto \Phi(X) := \mathrm{Fuk}^\#(M(X))$$

and for elementary morphisms $[Y]$ given by

$$[Y] \mapsto \Phi([Y]) := [\Phi(L(Y))]$$

extend to a connected field theory

$$\Phi : \mathrm{Bor}^0_{2+1} \to A_\infty(4)/S$$

with values in the category A_∞ of (small $\mathbb{Z}_4$-graded A_∞ categories with shift, homotopy classes of A_∞ functors up to grading shifts).

Proof. The desired functor is obtained by composing the functors Φ of Theorem 3.5.1 (assigning a generalized Lagrangian correspondence to any cobordism) with the functor $\mathrm{Fuk}^\#/S$ of Remark 4.1.9 with $\tau = 1/2r$ and $N = 4$ (assigning a functor to any such correspondence, modulo grading shifts). $\square$

Remark 4.2.2.   (a) (Group-valued invariants for bordisms) Let $L_\pm \subset M(X_\pm)$ be simply-connected compact oriented spin Lagrangian submanifolds. For a bordism Y from X_- to X_+ we denote by $[\underline{L}(Y)]$ the corresponding equivalence class of generalized Lagrangian correspondences from $M(X_-)$ to $M(X_+)$. The quilted Floer cohomology group

$$HF(Y; L_-, L_+) := H(\mathrm{Hom}(\Phi([\underline{L}(Y)])L_-, L_+)) = HF(L_-, \underline{L}(Y), L_+)$$

is a *topological invariant* of the bordism Y up to isomorphism. These Floer cohomology groups are relatively $\mathbb{Z}_4$-graded, since the minimal Chern number of the moduli space $M(X)$ of bundles is even by Proposition 3.1.7.

    (b) (Excision property) The functors $\Phi(Y)$ satisfy a gluing property for gluing: if $Y_{02} = Y_{01} \circ Y_{12}$ then

$$\Phi(Y_{02}) = \Phi(Y_{12}) \circ \Phi(Y_{01}).$$

This relation can be regarded as a generalization of Floer's excision property [6]. If X_1 has genus zero then $\Phi(Y)$ is the trivial functor, while if X_1 has genus one then

(33) $H(\mathrm{Hom}(\Phi(Y_{02})L_-, L_+)) \cong H(\mathrm{Hom}(\Phi(Y_{01})L_-, \mathrm{pt}) \otimes \mathrm{Hom}(\Phi(Y_{12}) \mathrm{pt}, L_+)).$

Equation (33) follows immediately from Theorem 3.3.3 since in this case $M(X_1)$ is empty resp. a point and so the Floer complex is the product of Floer complexes $\mathrm{Hom}(\Phi(Y_{01})L_-, \mathrm{pt}), \mathrm{Hom}(\Phi(Y_{12}) \mathrm{pt}, L_+))$ for $(\Phi(Y_{01})L_-, \mathrm{pt})$ and $(\Phi(Y_{12}), L_+)$.

4.3. Invariants for three-manifolds with circle-valued Morse functions.

In this section we define invariants for three-manifolds which admit circle-valued Morse functions. Let Y be a compact connected oriented three-manifold containing a non-separating embedded compact connected oriented surface X. By replacing X with two copies of itself we obtain a bordism $\overline{Y}$ from X to itself. Any adapted Morse function on $\overline{Y}$ gives rise to a circle-valued Morse function on Y.

Definition 4.3.1. The *cyclic Floer homology* of (Y, X) is the quilted Floer cohomology $HF(Y, X) := HF(\underline{L}(\overline{Y}), \Delta_{M(X)})$.

Remark 4.3.2.   (a) The results of Gay-Kirby [18, Theorem 1.2] imply that any two decompositions of Y into elementary bordisms are related by Cerf moves. It follows that the equivalence class of the generalized cyclic correspondence

$(\Delta_{M(X)}, \underline{L}(X))$ depends only on the homotopy class $[f : Y \to S^1]$ of the circle-valued Morse function. Furthermore, any homotopy class $[f : Y \to S^1]$ of maps to the circle inducing a surjection of fundamental groups has a representative $f : Y \to S^1$ that is a Morse function with connected fibers:

$$\#\pi_0(f^{-1}(b)) = 1, \quad \forall b \in S^1.$$

Such a Morse function gives rise to a presentation of Y as a cyclic composition of elementary bordisms $Y_1, \ldots, Y_m$.

(b) The homotopy class of the circle-valued Morse functions is the first cohomology class on Y, via the identification of S^1 with the first Eilenberg-Maclane space. Thus the isomorphism class of $HF(Y, X)$ depends only on the choice of cohomology class $[X]^\vee \in H^1(Y, \mathbb{Z})$ dual to $[X]$.

(c) If $Y = S^1 \times X$ is the trivial fiber bundle with fiber X then

$$HF(Y, X) = HF(\Delta_{M(X)}) \cong QH(M(X))$$

is the quantum cohomology of the moduli space $M(X)$. More generally, for any fiber bundle with monodromy $\psi : X \to X$ we may take for $\underline{L}(Y)$ the graph of a symplectomorphism $(\psi^{-1})^* : M(X) \to M(X)$. In this case

$$HF(Y, X) = HF(\text{graph}(\psi^{-1})^*) = HF^{\text{inst}}(Y)$$

is equivalent to the periodic instanton Floer cohomology $HF^{\text{inst}}(Y)$ by the proof of the Atiyah-Floer conjecture for fibrations by Dostoglou-Salamon [13]. Partial results towards a correspondence with instanton Floer homology in the circle-valued Morse case are given in Duncan [15] and Lipyanskiy [30].

4.4. **Invariants for closed 3-manifolds.** The invariant associated to a bundle over a closed three-manifold, considered as a bordism from the empty surface to itself is trivial. Indeed, such a three-manifold admits a Cerf decomposition where one of the surfaces is a sphere and the moduli space of bundles of coprime rank and degree is empty in genus zero. The following device suggested by Kronheimer and Mrowka [26] gives non-trivial invariants of closed three-manifolds, possibly with trivial first Betti number.

Definition 4.4.1. (a) (Torus-summed three-manifolds) Given a closed oriented three-manifold Y let $\overline{Y} := Y \# ([-1, 1] \times T^2)$ denote the *torus-summed* bordism obtained as connected sum of Y with a product bordism $[-1, 1] \times T^2$ between tori T^2.

(b) (Torus-summed Floer homology) The *torus-summed Lagrangian Floer homology* for rank r and coprime degree d of a closed three-manifold Y is

$$\overline{HF}(Y) := H(\text{Hom}(\Phi(\overline{Y}) \text{pt}, \text{pt})) = HF(L(Y_1), \ldots, L(Y_m))$$

for some choice of Cerf decomposition $\overline{Y} = Y_1 \cup_{X_1} \ldots \cup_{X_{r-1}} Y_m$.

Example 4.4.2. (Torus-summed Lagrangian Floer homology for connect sums of $S^2 \times S^1$) Suppose that $r = 2$, and for some positive integer n the three-manifold

$$Y = \underbrace{(S^2 \times S^1) \# \ldots \# (S^2 \times S^1)}_{n}$$

is the connected sum of n copies of $S^2 \times S^1$. Then $\overline{HF}(Y) = H(S^3)^n$. Indeed, consider a Morse function on Y that splits $Y \# ([-1,1] \times T^2)$ into compression bodies $Y_\pm$, each with n critical points of index one resp. two. Each compression body $Y_\pm$ is a bordism between a surface X of genus $n+1$ and a surface of genus 1. The Lagrangians

$$L(Y_-) = L(Y_+) \cong SU(2)^n \cong (S^3)^n$$

are identical and given in the holonomy description by

$$L(Y_\pm) = \{a_1 = \ldots = a_n = 1\} \subset \left\{ \prod_{i=1}^{n+1} [a_i, b_i] = -1 \right\}.$$

By definition $\overline{HF}(Y) = HF(L(Y_-), L(Y_+))$. As in Biran and Cornea [3, Proposition 6.1.4], if L is a Lagrangian product of spheres, then $HF(L, L)$ is either 0 or isomorphic to the ordinary cohomology $H(L)$; this is a consequence of the spectral sequence computing Floer cohomology starting from the singular cohomology, as in Oh [41], [7]. By Albers [1, Corollary 2.11], $HF(L, L; \mathbb{Z}_2)$ is non-trivial if the class $[L]_{\mathbb{Z}_2} \in H(M(X); \mathbb{Z}_2)$ of L is non-zero. The class of $L(Y_\pm) \cong (S^3)^n$ is non-zero in $H(M(X); \mathbb{Z}_2)$. Indeed, L intersects the Lagrangian $L' \cong (S^3)^n$, given by $b_1 = \ldots = b_n = 1$ transversally in a single point, namely the unique (up to conjugation) pair a_{n+1}, b_{n+1} with $[a_{n+1}, b_{n+1}] = 1$. Hence $HF(L, L; \mathbb{Z}_2)$ is non-zero. This implies that $HF(L, L)$ is non-zero by the universal coefficient theorem. Putting everything together implies the claim.

Remark 4.4.3.   (a) (Four-manifold invariants?) Naturally one expects these invariants to extend to invariants of four-manifolds, fitting into a bundle field theory in $2+1+1$ dimensions: a bifunctor from a bicategory of (2-manifolds, 3-bordisms, 4-bordisms) with bundles to an "A_∞ bicategory" of (A_∞ categories, A_∞ functors, A_∞ natural transformations) via rank r gauge theory. In particular, this theory should associate to a bundle bordism R between bordisms $Y_\pm$ a natural transformation of A_∞ functors $\Pi(R) : \Phi(Y_-) \to \Phi(Y_+)$.

   (b) (Surgery exact triangles) The surgery exact triangles for the theory (in the rank two case) are a consequence of a fibered generalization of a triangle for Dehn twists by Seidel, proved in [51].

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5. Description of changes in the third version

(a) *In Definition 4.1.8: "After some grade shift" After some grading shift?*

   Fixed in Definition 4.1.8 on page 40.

(b) *Remark 4.1.6 (b) This sentence is difficult to read, isn't there an "is" missing before "obtained"*

   Fixed in Remark 4.1.6 (b) on page 39.
(c) The retrieval information for reference [17] by Fukaya was added.
(d) The bibliographic information for reference [35] by Ma'u-Wehrheim-Woodward was updated.

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