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# Topology of the Nodal Set of Random Equivariant Spherical Harmonics on $\mathbb{S}^3$

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We show that real and imaginary parts of equivariant spherical harmonics on $\mathbb{S}^3$ have almost surely a single nodal component. Moreover, if the degree of the spherical harmonic is $N$ and the equivariance degree is $m$, then the expected genus is proportional to $m\left(\frac{N^2-m^2}{2}+N\right)$. Hence, if $\frac{m}{N}=c$ for fixed $0<c<1$, then the genus has order $N^3$.

## 1 Introduction and Statements of the Results

In a recent article [8], the authors proved that nodal sets of real or imaginary parts of equivariant (but non-invariant) eigenfunctions of Laplacians $\Delta_{KK}$ of generic "Kaluza–Klein" metrics $g_{KK}$ on unit tangent (or cotangent) bundles $\pi : M \to X$ over Riemann surfaces $(X,g)$ have a single connected component. The generic condition is 0 being a regular value for the eigenfunctions. The unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ with its standard metric and Laplacian has the standard Hopf fibration $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ and is an example of a Kaluza–Klein metric. It is a double cover $\mathbb{S}^3 \to SO(3) \simeq U(\mathbb{S}^2)$ of the unit tangent bundle of $\mathbb{S}^2$. Recently, the nodal sets of random wave on 3D Euclidean space have been the subject of numerical investigations by Barnett *et al.* [1], which exhibit a surprising feature: only a small number of nodal components are visible in the computer graphics (Figure 1).

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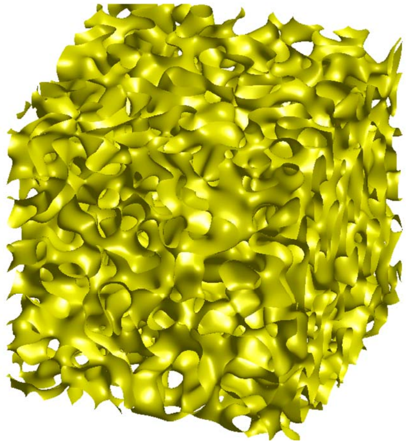

**Fig. 1.** Nodal surface ([1]).

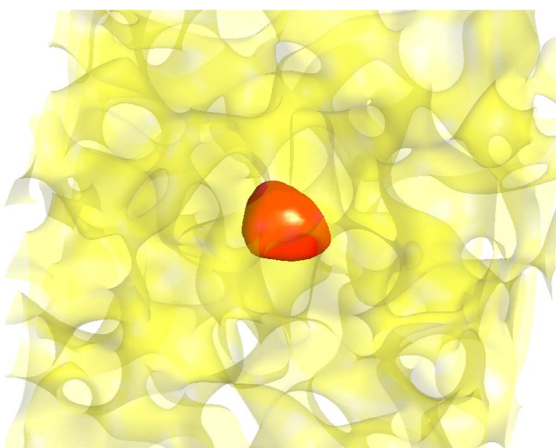

**Fig. 2.** Nodal surface plus one extra component ([1]).

The results of Nazarov–Sodin [10] show that in fact there must be $cN^3$ distinct nodal components for some (very small) $c > 0$, but the other components are evidently too small to be seen in the computer graphics. For this reason, it is conjectured that with probability one, the random spherical harmonic of fixed degree $N$ on $\mathbb{S}^3$ has one *giant component* and many much smaller components. In Figure 2, one does see a 2nd small (red) component.

Sarnak [11] posed the problem of finding its expected genus and has proposed that the expected genus is of the order of magnitude of $N^3$ [11]. Milnor has proved that the maximal genus of the zero set of a polynomial of degree $N$ is of this order of magnitude, so Sarnak's proposal is that the nodal sets of random spherical harmonics on $\mathbb{S}^3$ are rather like Harnack curves (real algebraic curves of degree $N$ and of roughly maximal genus).

The purpose of this note is to link the results of [8] to Sarnak's proposal. Our main result is that the proposal is true for real/imaginary parts of random *equivariant spherical harmonics* of degree $N$ on $\mathbb{S}^3$. Equivariant spherical harmonics are complex-valued functions on $\mathbb{S}^3$, defined as follows: let $\mathcal{H}_N^m(\mathbb{S}^3)$ denote the space of complex-valued spherical harmonics of degree $N$, which are equivariant of degree $m$ with respect to the $\mathbb{S}^1$ action defining the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$. To be explicit, any $\psi_N^m \in \mathcal{H}_N^m(\mathbb{S}^3)$ satisfies

$$-\Delta_{\mathbb{S}^3}\psi_N^m = N(N+1)\psi_N^m \text{ and}$$

$$\psi_N^m((e^{i\theta}z_1, e^{i\theta}z_2)) = e^{im\theta}\psi_N^m((z_1, z_2)),$$

for all $\theta \in [0, 2\pi]$ and for all $z_1, z_2 \in \mathbb{C}$ such that $|z_1|^2 + |z_2|^2 = 1$ (see §2.1), where $\Delta_{\mathbb{S}^3}$ is the Laplace–Beltrami operator on $\mathbb{S}^3$. In §2.3, we describe $\mathcal{H}_N^m(\mathbb{S}^3)$ in terms of harmonic homogeneous $\mathbb{C}$-valued polynomials on $\mathbb{C}^2$. For short, we say that $\psi_N^m$ is equivariant of degree $(N, m)$. Since $L^2(\mathbb{S}^3, \mathbb{C}) = \bigoplus_{N=0}^{\infty} V_N \otimes V_N^*$ where $V_N$ is the $N$th irreducible representation of $\mathbb{S}^3$, fixing the weight $m$ of the $\mathbb{S}^1$ action is the same as fixing one line in $V_N^*$, so that $\dim_{\mathbb{C}} \mathcal{H}_N^m = \dim V_N = N+1$ (for details, see Proposition 2.2).

Our main results pertain to the nodal sets of the real, resp. imaginary, parts of

$$\psi_N^m = u_N^m + iv_N^m. \tag{1.1}$$

It is very useful to work however with the complex-valued $\psi_N^m$. The real, resp. imaginary, part of an equivariant spherical harmonic $\psi_N^m$ belongs to a subspace $\Re\mathcal{H}_N^m$ of the real vector space $\mathbb{R}V_N$ of real-valued spherical harmonics on $\mathbb{S}^3$ of degree $N$. The $L^2$ inner product on $\mathbb{R}V_N$ with respect to the standard volume form $dV$ of $\mathbb{S}^3$ induces a Gaussian measure, the usual "random spherical harmonics" of [10] (for instance). It restricts to the space $\Re\mathcal{H}_N^m$ to define a Gaussian measure $\Re\gamma_N^m$ on this subspace of $\mathbb{R}V_N$. The Hermitian inner product on $\mathcal{H}_N^m \subset L^2(\mathbb{S}^3, \mathbb{C})$ induces a complex Gaussian measure $\gamma_N^m$ on $\mathcal{H}_N^m$ (see Definition 4.1). The Gaussian measures $\gamma_N^m$ and $\Re\gamma_N^m$ are compatible in the sense that the real resp. imaginary parts of a complex Gaussian random $\psi_N^m$ are independent Gaussian random real spherical harmonics (see Section 4.3 and Lemma 4.2.)

A Gaussian measure is determined by its covariance function. The principal covariance function of this article is the Schwartz kernel $\Pi_N^m(x,y)$ of the orthogonal projection

$$\Pi_N^m : L^2(\mathbb{S}^3,\mathbb{C}) \to \mathcal{H}_N^m. \tag{1.2}$$

We denote by $g(X)$ the genus of a surface $X$, and we define

$$\mathbb{E}_{N,m}\, g(\mathcal{Z}_{u_N^m}) \tag{1.3}$$

to be the expected genus of the nodal set of the real part $u_N^m$ of an equivariant eigenfunction of degree $(N,m)$ with respect to $\Re\gamma_N^m$.

**Theorem 1.1.** Let $(\mathcal{H}_N^m,\gamma_N^m)$ be the Gaussian space of equivariant spherical harmonics of degree $(N,m)$. In the following, we assume $m \neq 0$.

(i) With probability 1 (w.r.t. $\gamma_N^m$ or equivalently $\Re\gamma_N^m$), the nodal set $\mathcal{Z}_{u_N^m}$ of the real part $u_N^m = \Re\psi_N^m$ (resp. the nodal set $\mathcal{Z}_{v_N^m}$ of the imaginary part $v_N^m = \Im\psi_N^m$) of a random equivariant spherical harmonic $\psi_N^m \in \mathcal{H}_N^m$ has a single connected component that partitions $\mathbb{S}^3$ into two nodal domains.

(ii) The expected genus of the nodal component is given by

$$\mathbb{E}_{N,m} g(\mathcal{Z}_{u_N^m}) = \frac{1+\eta^2}{8\pi}|m|\left(\frac{N^2-m^2}{2}+N\right) - |m| + 1,$$

where

$$\eta = \frac{m/2}{\frac{N^2-m^2}{2}+N},$$

which has modulus less than or equal to $1/2$.

It follows that when $\delta < \frac{m}{N} < 1-\delta$ for some small fixed $\delta > 0$, the genus of $\mathcal{Z}_{u_N^m}$ is of order $N^3$.

The 1st statement is almost an application of the main result of [8], where the "genericity" assumption was only used to prove that each real (resp. imaginary) part of an equivariant eigenfunction has 0 as a regular value. Thus, to prove Theorem 1.1(i), it is only necessary to prove the Bertini-type theorem that $0 \in \mathbb{C}$ is a regular value of $\psi_N^m : \mathbb{S}^3 \to \mathbb{C}$ with $\gamma_N^m$-probability 1. This is done in Section 5.4. The 2nd statement has one topological simple part and one probabilistic part. The topological makes use of the identification of equivariant functions in $\mathcal{H}_N^m$ with sections $f_N^m e_L^m$ of the complex line bundle $L^m \to \mathbb{S}^2$, where $e_L$ denotes a local frame of $L$ over the affine chart (which we choose to be a holomorphic frame). The line bundle $L$ is the hyperplane line bundle over $\mathbb{S}^2 = \mathbb{CP}^1$. It is often denoted by $\mathcal{O}(1)$ (or, less commonly, $\mathcal{O}(H)$) in algebraic geometry, and its $m$th tensor power is denoted by $\mathcal{O}(m)$. We refer to Section 2.2 and to [7, pp. 145–166] for background on complex line bundles over Riemann surfaces; $L^m$ is denoted $\mathcal{O}(H^m)$ in [7].

**Lemma 1.2.** If 0 is a regular value of $\psi_N^m$, then the genus of $\mathcal{Z}_{\Re\psi_N^m}$ is given by

$$\frac{|m|(\#\{f_N^m = 0\} - 2)}{2} + 1, \tag{1.4}$$

where $\psi_N^m$ is the lift of the section $f_N^m(z, \bar{z})e_L^m$.

As mentioned in [8], the key point of the proof is to show that $\pi : \mathcal{Z}_{\Re\psi_N^m} \to X$ is a kind of "helicoid cover". That is, it is an $m$-fold cover over the complement of the zeros of $f_N^m$, while the inverse image of a zero is an $\mathbb{S}^1$ orbit. Thus, it is not a branched cover in the standard sense; rather $\pi$ is locally like the projection of a vertical helicoid onto the horizontal plane. (We thank J. Y. Welschinger for helpful discussions of the local picture.)

The 2nd part is the following Kac–Rice-type calculation.

**Lemma 1.3.** With probability 1, $f_N^m$ has isolated non-degenerate zeros, and

$$\mathbb{E}\#\{f_N^m = 0\} = \frac{1+\eta^2}{4\pi}\left(\frac{N^2 - m^2}{2} + N\right),$$

where

$$\eta = \frac{m/2}{\frac{N^2-m^2}{2} + N}.$$

To determine the expected number of (complex) zeros of $f_N^m$, we use the Kac–Rice formula for the ensemble $(\mathcal{H}_N^m, \gamma_N^m)$, which consists of complex Gaussian random eigensections of the line bundle $L^m$. As usual, it gives an integral formula for the expected number of zeros in terms of the determinant of a matrix formed from values and derivatives of the covariance (two-point) function of the Gaussian random function. The covariance kernel is evaluated in Section 4.2 and turns out to be simple enough to yield the explicit formula of Lemma 1.3. An essential point is that the Euler characteristic calculation of Lemma 1.2 allows us to reduce the calculation of the expected Euler characteristic to a quantity that can be handled by the Kac–Rice formula.

We close the introduction with some remarks on the relation of Theorem 1.1 to the conjecture that nodal sets of random real spherical harmonics of degree $N$ on $\mathbb{S}^3$ contain a unique giant component. It is not clear that our methods and results apply to this conjecture. Rather, they demonstrate that the conjecture is true for the special spherical harmonics formed by real/imaginary parts of equivariant spherical harmonics, and indeed, there is just one component. A rather distant hope is to obtain results on the full space of random spherical harmonics by perturbation of the nodal sets of equivariant spherical harmonics. As $m$ varies, the latter comprise the union $\bigcup_{\substack{|m|\le N \\ 2|N-m}} \Re\mathcal{H}_N^m$ of $N+1$ real subspaces of dimension $N+1$. A random real spherical harmonic in $\mathcal{H}_N$ is a random linear combination of components in $\Re\mathcal{H}_N^m$. It is tempting to imagine that, locally, a random spherical harmonic in $\mathcal{H}_N$ might be a small perturbation of the real part of an equivariant spherical harmonic in $\mathcal{H}_N^m$, so that its nodal set locally resembles that of an equivariant one. The genus of the nodal set of $u_N^m = \Re\psi_N^m$ is so large that a small perturbation is unlikely to decrease the genus. An independent possibility is that the geometric analysis of nodal sets of $u_N^m \in \Re\mathcal{H}_N^m$ has a generalization in some form to nodal sets of general spherical harmonics of degree $N$.

## 2 Proof of Theorem 1.1(i) Modulo Bertini's Theorem

This section is devoted to the geometric setting of the article. Its main result is the proof of the 1st statement (i) of Theorem 1.1 modulo Bertini's theorem. As mentioned in the introduction, it is basically a corollary of the main results of [8] on eigenfunctions of Kaluza–Klein Laplacians on general $\mathbb{S}^1$ bundles over Riemann surface, which we state in Theorem 2.3 below. To derive statement (i) from Theorem 2.3, we need to show that equivariant spherical harmonics $\psi_N^m$ are Kaluza–Klein eigenfunctions on $\mathbb{S}^3$ and can also be regarded as sections of the complex line bundle $L^m \to \mathbb{S}^2$. All of the relevant geometric notions are reviewed in this section. To complete the proof of Theorem 1.1(i), we also need to prove that almost every $\psi_N^m$ has 0 as a regular value. We postpone the proof of the last statement, "Bertini's theorem", to Section 5.5.

### 2.1 Coordinates on $\mathbb{S}^3$ and Hopf fibration

We use two coordinate systems on $\mathbb{S}^3$:

$$(z_1, z_2) \mapsto \begin{pmatrix} \Re z_1 \\ \Im z_1 \\ \Re z_2 \\ \Im z_2 \end{pmatrix} \in \mathbb{S}^3 \subset \mathbb{R}^4, \tag{2.1}$$

where $z_1, z_2 \in \mathbb{C}$ and $|z_1|^2 + |z_2|^2 = 1$ or

$$(\alpha, \varphi, \theta) \mapsto \begin{pmatrix} \sin\alpha\cos(\theta+\varphi) \\ \sin\alpha\sin(\theta+\varphi) \\ \cos\alpha\cos(\theta-\varphi) \\ \cos\alpha\sin(\theta-\varphi) \end{pmatrix} \in \mathbb{S}^3 \subset \mathbb{R}^4, \tag{2.2}$$

where $\alpha \in [0, \pi/2]$, $\varphi \in [0, 2\pi]$, and $\theta \in [-\pi, \pi]$. In the 1st coordinate system (2.1), the action of $\mathbb{S}^1$ is given by

$$r_\vartheta(z_1, z_2) = (e^{i\vartheta} z_1, e^{i\vartheta} z_2) \tag{2.3}$$

and the Hopf map $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ is given by

$$\pi : (z_1, z_2) \mapsto \left(2z_1\overline{z_2}, |z_1|^2 - |z_2|^2\right) \in \mathbb{C} \times \mathbb{R}. \tag{2.4}$$

In the 2nd coordinate system, (2.3) is equivalent to

$$(\alpha, \varphi, \theta) \mapsto (\alpha, \varphi, \theta + \vartheta)$$

and the Hopf map (2.4) is

$$\pi : (\alpha, \varphi, \theta) \mapsto \begin{pmatrix} \sin(2\alpha)\cos(2\varphi) \\ \sin(2\alpha)\sin(2\varphi) \\ \cos(2\alpha) \end{pmatrix} \in \mathbb{S}^2 \subset \mathbb{R}^3.$$

### 2.2 $\mathbb{S}^3$ as a Kaluza–Klein 3-fold

The purpose of this section is to explain how $\mathbb{S}^3$ (equipped with its standard metric) is an example of a Kaluza–Klein metric in the sense of [8]. We also review the relation between equivariant Laplace eigenfunctions on $\mathbb{S}^3$ and associated eigensections of line bundles over $\mathbb{S}^2$.

The 3-manifolds $M^3$ studied in [8] are $\mathbb{S}^1$ bundles $M^3 \to X$ over Riemann surfaces $X$, in particular the unit tangent or cotangent bundles. A Kaluza–Klein metric on $M^3$ is a bundle-metric $g$ defined by a connection $\nabla$ on $TM^3$ and a Riemannian metric $h$ on $X$. The circular fibers are geodesics of the metric (in particular have a constant length), and the metric on the horizontal spaces is the lift of the metric on $X$. It is evident that the standard metric on $\mathbb{S}^3$ is Kaluza–Klein with respect to the Hopf fibration $\pi : \mathbb{S}^3 \to \mathbb{S}^2$. A harmonic analysis on spheres and its relation to the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ is elementary and well known (see e.g. [6]), so we only review two aspects of it:

(1) relating eigensections of the Bochner Laplacians on $L^m$ to equivariant eigenfunctions of $\Delta_{\mathbb{S}^3}$; (2) CR geometry of $\mathbb{S}^3$.

Associated to the principal $\mathbb{S}^1$ bundle $\mathbb{S}^3 \to \mathbb{S}^2$ are the line bundles $L^m = \mathbb{S}^3 \times_{\chi^m} \mathbb{C}$ where $\chi(e^{i\theta}) = e^{i\theta}$. For $m = 1$, $L = \mathcal{O}(1) \to \mathbb{CP}^1$ in the notation of algebraic geometry [7]; it is the spin-bundle, i.e., the square root of the anti-canonical bundle, $K_{\mathbb{CP}}^{-1} = (T^{1,0})^{\frac{1}{2}}\mathbb{CP}^1$. When $m = -1$, $L^{-1} = K_{\mathbb{CP}^1} \simeq T^{*1,0}\mathbb{CP}^1$, the canonical bundle. In a standard way, we view $\mathbb{S}^3 \subset L^*$ as the unit bundle with respect to the Fubini–Study metric. The connection in this setting is the Chern connection of the Fubini–Study metric; we refer to [7, 8] for background.

Sections $s$ of $L^m \to \mathbb{S}^2$ naturally lift to $L^*$ by

$$\hat{s}(z, \lambda) := \lambda^m(s(z)).$$

Thus, the restriction of the lift of $s \in C(\mathbb{S}^2, L^m)$ to $\mathbb{S}^3$ satisfies $\hat{s}(r_\theta x) = e^{im\theta}\hat{s}(x)$. We refer to lifts of sections of $L^m$ as "equivariant" functions on $\mathbb{S}^3$ and denote the space of such functions by $\mathcal{H}^m$.

The standard Laplacian $\Delta_{\mathbb{S}^3}$ is a Kaluza–Klein Laplacian, i.e., has the form,

$$\Delta_{\mathbb{S}^3} = \Delta_H + \frac{\partial^2}{\partial\theta^2},$$

where $\Delta_H$ is the horizontal Laplacian. The fact that the fiber Laplacian is $\frac{\partial^2}{\partial\theta^2}$ reflects the fact that $\mathbb{S}^1$ orbits are geodesics isometric to $\mathbb{R}/2\pi\mathbb{Z}$. It is obvious that $[\Delta_{\mathbb{S}^3}, \frac{\partial^2}{\partial\theta^2}] = 0$. Since $\mathbb{S}^1$ acts isometrically on $(\mathbb{S}^3, G)$, we may decompose into its weight spaces,

$$L^2(\mathbb{S}^3, \mathbb{C}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^m,$$

where $\mathcal{H}^m = \{F : \mathbb{S}^3 \to \mathbb{C} : F(r_\theta.x) = e^{im\theta}F(x)\}$. The inner product is always the standard one (with respect to Haar measure). The weight spaces are $\Delta^H$-invariant, i.e., $\Delta_H : \mathcal{H}^m \to \mathcal{H}^m$.

**Definition 2.1.** We define $\mathcal{H}^m_N$ to be the subspace of degree $N$ spherical harmonics in $\mathcal{H}^m$. We call a function belonging to $\mathcal{H}^m_N$ an equivariant spherical harmonic of degree $(N, m)$.

The lifting map gives a canonical identification $\mathcal{H}_m \cong L^2(\mathbb{S}^2, L^m)$. The Bochner Laplacian $\nabla_m^*\nabla_m$ corresponds to the horizontal Laplacian under this identification, i.e.,

$$\nabla_m^*\nabla_m\widehat{(f(dz)^m)} = \Delta_H(\widehat{f(dz)^m}).$$

Since $\mathbb{S}^3$ is a group, $L^2(\mathbb{S}^3) = \bigoplus_{N=0}^{\infty} V_N \otimes V_N$ where $V_N$ is an irreducible representation of $\mathbb{S}^3$ of dimension $N+1$. Moreover, $\Delta|_{V_N \otimes V_N} = N(N+2) = (N+1)^2 - 1$.

### 2.3 CR structure

Another viewpoint is that $\mathbb{S}^3 = \partial B \subset \mathbb{C}^2$, i.e., that $\mathbb{S}^3$ is the boundary of the unit ball, a strictly pseudo-convex domain in $\mathbb{C}^2$. A defining function for $\mathbb{S}^3 \subset \mathbb{C}^2$ is the usual Euclidean distance $r = |Z|$ from the origin. Here, $Z \in \mathbb{C}^2$. Let $\bar{\partial} r$ be the associated $(0,1)$ form. The theory of spherical harmonics on $\mathbb{S}^3$ has been related to the CR geometry and the representation theory of $SU(2) = \mathbb{S}^3$ in [6].

Spherical harmonics on $\mathbb{S}^3$ are restrictions of homogeneous harmonic polynomials on $\mathbb{C}^2$. In complex coordinates $Z = (z_1, z_2)$, the Euclidean Laplacian is

$$\Delta_{\mathbb{R}^4} = 4 \sum_{j=1}^{2} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}.$$

Let $\mathcal{H}_N^{(p,q)}$ denote the space of harmonic homogeneous polynomials of degree $N$ on $\mathbb{C}^2$ which are of degree $p$ in $z_j$s and of degree $q$ in the $\bar{z}_j$s; $N = p + q$. Then $\mathcal{H}^{(p,q)}$ is an irreducible representation of $SU(2)$ and the space of all spherical harmonics of degree $N$ admits the decomposition

$$\mathcal{H}_N = \bigoplus_{p+q=N} \mathcal{H}_N^{(p,q)}. \tag{2.5}$$

The representation of $U(2)$ on $\mathcal{H}_N^{(p,q)}$ is denoted by $\rho(q,-p)$. One has $\rho(q,-p)|_{SU(2)} = \rho(q',-p')|_{SU(2)} \iff p+q = p'+q'$ and $\dim \rho(q,-p) = p+q+1$. Hence, the decomposition (2.5) is another decomposition of $V_N \otimes V_N^*$ into irreducibles.

The orbits of the Hopf fibration of the action (2.3) define the characteristic directions of the CR manifold and lie in the null space of $\partial r|_{T\mathbb{S}^3}$. It follows that on polynomials $z_1^{p_1} z_2^{p_2} \bar{z}_1^{q_1} \bar{z}_2^{q_2}$ of type $(p,q)$ (i.e., $p_1 + p_2 = p$ and $q_1 + q_2 = q$), $\mathbb{S}^1$ acts by $e^{i(p-q)\theta}$. Thus, the equivariance degree of $\psi_N^{(p,q)} \in \mathcal{H}_N^{(p,q)}$ is $m = p - q$. Since $p + q = N$, $m = 2p - N$, i.e., the data $(p,q)$ are equivalent to specifying only $p$ or $q$ or $m$. In particular, $\mathbb{S}^1$ acts by $e^{iN\theta}$ on the space $\mathcal{H}_N^{N,0}$ of holomorphic polynomials $(p,q) = (N,0)$.

**Proposition 2.2.** $\mathcal{H}_N^{(p,q)} = \mathcal{H}_N^{2p-N}$. In particular, the dimension of $\mathcal{H}_N^m$ is $N+1$ if $|m| \leq N$ and $2|N-m$ and 0 otherwise.

### 2.4 Associated sections of line bundles

It is useful to simultaneously keep in mind the "upstairs" picture of equivariant eigenfunctions of $\Delta_{\mathbb{S}^3}$ and the "downstairs" picture of sections of complex line bundles, as described in Section 2.2.

We denote the eigensection corresponding to $\psi_N^m \in \mathcal{H}_N^m$ as $f_N^m e_L^m$ in a local holomorphic frame $e_L$ of $L$, so that $f_N^m$ is a locally defined function on $\mathbb{CP}^1$, i.e., is a function on the affine chart $\mathbb{C}$. This is a useful description in analyzing the complex zeros of $\psi_N^n$.

We denote by $\mathcal{Z}_{f_N^m}$ the zero set of the eigensection $f_N^m e_L^m$ on $\mathbb{S}^2$:

$$\mathcal{Z}_{f_N^m} = \{z \in \mathbb{S}^2 : f_N^m(z) = 0\}.$$

It is easy to see that the zero set $\mathcal{Z}_{\psi_N^m}$ of $\psi_N^m$ is the inverse image of $\mathcal{Z}_{f_N^m}$ under the natural projection $\pi$:

$$\mathcal{Z}_{\psi_N^m} = \pi^{-1}\mathcal{Z}_{f_N^m}.$$

We note that $c_1(L^m) = m$ (1st Chern number, the integral of the 1st Chern class). By the Hopf theorem, $c_1(L^m)$ is the sum over zeros of a smooth section $f_N^m$ with non-degenerate zeros of the index in $\mathbb{Z}_2$ of the zero. The index is the degree of the locally defined map $\frac{s(z)}{|s(z)|}$ from a small circle centered at the zero to $\mathbb{C}$ in a local trivialization ([3, Theorem 11.17]–[3, Proposition 12.8].) In particular, if $s$ is a holomorphic section, then the indices are all equal to 1 and $s$ has precisely $m$ zeros (counted with multiplicity). This only occurs in the case $m = N$. Otherwise, the sections in $\mathcal{H}_N^m$ are smooth eigensections, and they have more than $m$ zeros on average, as is shown in the next section. In general, it is not obvious whether or not the zero set of $f_N^m$ is discrete in $\mathbb{S}^2$.

We now consider real and imaginary parts,

$$\Re f_N^m = a_N^m(z), \quad \Im f_N^m = b_N^m(z).$$

Then,

$$f_N^m(z)e^{-im\theta} = \left(a_N^m(z) + ib_N^m(z)\right)(\cos m\theta - i\sin m\theta),$$

so that with $\psi_N^m = u_N^m + iv_N^m$,

$$\begin{cases} u_N^m = a_N^m \cos m\theta + b_N^m \sin m\theta, \\ v_N^m = b_N^m \cos m\theta - a_N^m \sin m\theta. \end{cases} \tag{2.6}$$

We denote the nodal sets of the real, resp. imaginary, parts of the lift by

$$\mathcal{N}_{\Re\psi_N^m}\{p \in P_h : \Re\psi_N^m(p) = 0\}, \text{ resp. } \mathcal{N}_{\Im\psi_N^m} = \{p \in P_h : \Im\psi_N^m(p) = 0\}.$$

The analysis is the same for real and imaginary parts.

Since $\Delta_{\mathbb{S}^3}$ is a real operator, the real and imaginary parts (1.1) satisfied the modified eigenvalue system,

$$\begin{cases} \Delta_G u_N^m = -\lambda_N^m u_N^m, \\ \Delta_G v_N^m = -\lambda_N^m v_N^m, \\ \frac{\partial}{\partial\theta} u_j = m v_j, \quad \frac{\partial}{\partial\theta} v_j = -m u_j. \end{cases}$$

### 2.5 Review of the results of [8]

The main result [8, Theorem 1.5] pertains to nodal sets of real/imaginary parts of equivariant eigenfunctions of the Kaluza–Klein Laplacian on $\mathbb{S}^1$ bundles $\pi : M \to X$ over Riemannian surfaces $X$. A Kaluza–Klein metric $G$ is specified by a metric $h$ on $X$ and a connection $\nabla$ on $\pi : M \to X$. It is proved in [8] that, for a generic Kaluza–Klein metric, every equivariant eigenfunction has 0 as a regular value, and the eigenspaces are of real dimension 2 (corresponding to an equivariant eigenfunction and its complex conjugate).

**Theorem 2.3.** Suppose that the data $(g, h, \nabla)$ of the Kaluza–Klein metric satisfy the condition that every equivariant eigenfunction has 0 as a regular value and that the eigenspaces are of real dimension 2. Then,

(1) The eigenspace of $\Delta_G$ corresponding to $\lambda = \lambda_{m,j} = \lambda_{-m,j}$ is spanned by $\varphi_{m,j}$ and $\varphi_{-m,j} = \overline{\varphi_{m,j}}$. In particular, any real eigenfunction with the eigenvalue $\lambda_{m,j}$ is a constant multiple of $T_\theta(\Re\varphi_{m,j})$, where $T_\theta$ is the $\mathbb{S}^1$ action on $P$ parameterized by $\theta$.

(2) For $m \neq 0$, the nodal sets of $\Re\varphi_{m,j}$ are connected.

(3) For $m \neq 0$, the number of nodal domains of $\Re\varphi_{m,j}$ is 2.

For the sake of completeness, we briefly review here the proof of (2) and (3) in the setting of the present paper in order to bridge (i) of Theorem 1.1 and the regularity of 0 of $\psi_N^m : \mathbb{S}^3 \to \mathbb{C}$. We temporarily assume that 0 is a regular value of almost every equivariant spherical harmonic $\psi_N^m$ for every $N, m$ (proved in Section 5.5).

As discussed above, $\mathbb{S}^3$ is Kaluza–Klein with respect to the standard metric on $\mathbb{S}^2$ and Riemannian connection.

Fix a sufficiently small open ball $U \subset \mathbb{S}^2$ and a local frame giving a trivialization $\pi^{-1}U \cong \mathbb{S}^1 \times U$ where $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map. In this trivialization, one may write $\psi_N^m = e^{im\theta} f(p)$ for some function $f : U \to \mathbb{C}$. Taking the real part of $\psi_N^m$, we obtain

$$u_N^m = \Re(\psi_N^m) = f_1(p)\cos m\theta + f_2(p)\sin m\theta,$$

where $f = f_1 - if_2$. Observe that if $f(p) \neq 0$, then there are $2m$ distinct $\theta$ that make this expression vanish. This implies that the nodal set of $u_N^m$ in $\pi^{-1}U$ minus

$$\bigcup_{\{p:f(p)=0\}} \pi^{-1}p$$

is $2m$-covering of $U$ minus the zero set of $f$. In particular, if there is at least one $p \in U$ such that $f(p) = 0$, then the nodal set of $u_N^m$ in $\pi^{-1}U$ is connected. By taking any collection of $U$ that covers $\mathbb{S}^2$, one sees that the existence of at least one zero of $\psi_N^m$ implies that the nodal set of $u_N^m$ being connected. So the 1st part of Theorem 1.1 (i) follows by observing that $\psi_N^m$ corresponds to a section of $L^m \to \mathbb{S}^2$ (§2.2), which must vanish at least at a point because $L^m \to \mathbb{S}^2$ is nontrivial.

For the 2nd assertion of Theorem 1.1(i), we need the topological argument from [8, Section 8], where we proved (1) discreteness of zeros and (2) existence of at least one zero of $\psi_N^m$ imply the number of nodal domains being 2. Therefore, we prove in Proposition 5.5 that with probability 1, all zeros of $\psi_N^m$ are regular, which will complete the proof.

## 3 Genus of the Nodal Set: Proof of Lemma 1.2

In this section, we relate the genus of the nodal set of $\Re\psi_N^m$ to the number of zeros of $f_N^m$, under the assumption of all zeros of $f_N^m$ being regular (Lemma 1.2). We first recall a few lemmata (see for instance [12, Chapter 4]):

**Lemma 3.1.** Let $X$ be a topological space and let $A, B$ be topological subspaces whose interior cover $X$. Then

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B),$$

where $\chi(\cdot)$ is the Euler characteristic of $\cdot$.

**Lemma 3.2.** Let $X$ be a $n$-covering of $M$. Then we have

$$\chi(X) = n\chi(M).$$

Now we are ready to prove Lemma 1.2, again modulo Bertini's theorem (Section 5.5).

**Proof.** To simplify the notation, let $\psi = \psi_N^m$ and $f = f_N^m$. Let $\{z_j\}_{j=1,2,\ldots,k}$ be the complete set of zeros of $f$. We first note that

$$Z_{\Re\psi} - \cup_{j=1}^k \{(z_j, \theta) \ : \ \theta \in [0, 2\pi]\}$$

is $m$-covering of $k$-punctured sphere. So the Euler characteristic of

$$Z_{\Re\psi} - \cup_{j=1}^k \{(z_j, \theta) \ : \ \theta \in [0, 2\pi]\}$$

is $m(2-k)$ by Lemma 3.2.

Now we apply Lemma 3.1 with $X = Z_{\Re\psi}$, $A = Z_{\Re\psi} - \cup_{j=1}^k \{(z_j, \theta) \ : \ \theta \in [0, 2\pi]\}$, and $B$ equal to a sufficiently small open neighborhood of $\bigcup_{j=1}^k \{(z_j, \theta) \ : \ \theta \in [0, 2\pi]\}$ in $Z_{\Re\psi}$. Then $B$ is homotopic to $\cup_{j=1}^k \{(z_j, \theta) \ : \ \theta \in [0, 2\pi]\}$, which has Euler characteristic equal to 0, and $A \cap B$ is $m$-covering of a disjoint union of punctured disks, which also has Euler characteristic equal to 0. This implies that $X = Z_{\Re\psi}$ has Euler characteristic $m(2-k)$, and therefore the conclusion follows. ■

## 4 Gaussian Random Equivariant Spherical Harmonics

The space $\mathcal{H}_N^m$ has thus been identified as the complex vector space,

$$\mathcal{H}_N^{p,q} = \left\{ \sum_{|\alpha|=p, |\beta|=q}^N c_{\alpha,\beta}^{p,q} z^\alpha \bar{z}^\beta, \ \ c_{\alpha,\beta}^{p,q} \in \mathbb{C} \right\}, \ \ 2p - N = m.$$

The basis elements $z^\alpha \bar{z}^\beta$ are orthogonal on $\mathbb{S}^3$ but are not of norm 1. To compute the norms, it is advantageous to relate integrals over $\mathbb{S}^3$ with Gaussian integrals over $\mathbb{C}^2$, i.e., to use the measure $e^{-|Z|^2} dL(Z)$ where $dL$ is Lebesgue measure. The calculations are done in [6, p. 98] and one finds that (using multi-index notation $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$),

$$\int_{\mathbb{S}^3} |z^\alpha|^2 \mathrm{d}V = \frac{2\pi \alpha!}{(|\alpha| + 2)!}.$$

Henceforth, we denote the orthonormalized monomials by $\hat{s}_{N,\alpha,\bar{\beta}} := \frac{z^\alpha \bar{z}^\beta}{||z^\alpha \bar{z}^\beta||}$.

**Definition 4.1.** The Gaussian random equivariant spherical harmonic $\psi_N^m \in \mathcal{H}_N^m = \mathcal{H}_N^{p,q}, p - q = m$ is defined by the series

$$\psi_N^m(Z) = \sum_{|\alpha|=p, |\beta|=q}^N a_{\alpha,\beta} \hat{s}_{N,\alpha,\bar{\beta}},$$

where the coefficients $a_{\alpha,\beta}$ are independent standard complex normal Gaussians.

When $p = m = N$, these polynomials are known as the $SU(2)$ Gaussian holomorphic polynomials; for general $(p,q)$, we call them $SU(2)^{p,q}$ polynomials. They are sometimes called "poly-analytic" functions. The subspace $\mathcal{H}_N^0$ consists of invariant eigenfunction pulled back from $\mathbb{S}^2$. They are real valued and so the equivariant nodal sets are not intersections of two real nodal sets.

### 4.1 Zero set measures

If $f : \mathbb{S}^3 \to \mathbb{R}$ is a real-valued function, thenwe denote its zero set by $\mathcal{Z}_f$. We also denote by $[\mathcal{Z}_f]$ the positive measure defined by the linear functional

$$\langle [\mathcal{Z}_f], g\rangle := \int_{\mathcal{Z}_f} g \mathrm{d}\mathcal{H}^2, \quad g \in C(\mathbb{S}^3),$$

where $d\mathcal{H}^2$ is the induced surface measure on $Z_f$ (i.e., 2D Hausdorff measure). If $f$ is a random function, then $[\mathcal{Z}_f]$ is a random measure and the integral $\langle [\mathcal{Z}_f], g\rangle$ is a real random variable. When $f = \Re \psi_N^m$ with $N = p+q, m = p-q$, we denote by $\mathbb{E}_N^{p,q}[\mathcal{Z}_{\psi_N^m}]$ the measure whose integral against the test function $g \in C(\mathbb{S}^3)$ is $\mathbb{E}\langle [\mathcal{Z}_{\psi_N^m}], g\rangle$.

### 4.2 Covariance kernel

The covariance kernel of the Gaussian random equivariant spherical harmonics is the integral kernel of the orthogonal projection, $\Pi_N^m : L^2(\mathbb{S}^3) \to \mathcal{H}_N^m$, given by

$$\Pi_N^m(x,y) := \frac{1}{4\pi^2} \iint \exp(-im\theta_1 + im\theta_2) \Pi_N(r_{\theta_1}x, r_{\theta_2}y) \mathrm{d}\theta_1 \mathrm{d}\theta_2,$$

where $\Pi_N : L^2(\mathbb{S}^3) \to \mathcal{H}_N$ is the covariance kernel of the Gaussian random spherical harmonics.

Explicit calculations to follow are based on the identity

$$\Pi_N(x,y) = U_N(x \cdot y), \tag{4.1}$$

where

$$U_N(\cos\theta) = \frac{\sin(N+1)\theta}{\sin\theta} = \frac{e^{i(N+1)\theta} - e^{-i(N+1)\theta}}{e^{i\theta} - e^{-i\theta}} = e^{iN\theta} + e^{i(N-2)\theta} + \ldots + e^{-iN\theta}.$$

is the Chebyshev polynomial of the 2nd kind. One may derive (4.1) from [13, Chapter IV.2] (or [9, Chapter 2.3]) while observing that Gegenbauer polynomial $C_N^{\frac{d-1}{2}}(t)$ with $d = 3$ is a constant multiple of $U_N(t)$. For completeness, we include our version of the proof:

**Proof of (4.1).** We first claim that there exists a homogeneous real polynomial $P_N(t)$ of degree $N$ so that $\Pi_N(x,y) = P_N(\langle x,y\rangle)$ where $\langle x,y\rangle$ is the inner product.

The oriented isometry group of $\mathbb{S}^3$ is $G := SO(4)$. Since the Laplacian $\Delta_{\mathbb{S}^3}$ commutes with $G$, we have $\Pi_N(gx,gy) = \Pi_N(x,y)$ for all $g \in G$. Hence, $\Pi_N(x,y) = \Pi_N(y^{-1}x, I)$, where $I$ denotes the identity element. Thus, $\Pi_N$ is a convolution operator on $\mathbb{S}^3$. It is well known that any such point-pair invariant kernel on a symmetric space is a function of the distance $r(x,y)$ between $x$ and $y$. For $x,y \in \mathbb{S}^3$, $r(x,y)$ is the angle $\theta$ defined by $\langle x,y\rangle = \cos\theta$. Hence, $\Pi_N(x,y) = P_N(\langle x,y\rangle$ for some function $P_N$. Since $\Pi_N(x,y)$ is a homogeneous polynomial of degree $2N$ in $(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4$, which is homogeneous of degree $N$ in both $x,y$, it follows that $P_N$ is a homogeneous polynomial of degree $N$.

To complete the proof, we need to show that $P_N = U_N$. To this end, we compute the character $\chi_N$ of the representation of $SU(2) \simeq \mathbb{S}^3$ on the space $\mathcal{H}_N$ of spherical harmonics of degree $N$. A maximal abelian subgroup is given by diagonal unitary matrices $D(\theta) := \mathrm{Diag}(e^{i\theta}, e^{-i\theta})$ of determinant 1. As is wellknown (cf. [4]), the character is given by

$$\chi_N(\theta) = e^{iN\theta} + e^{i(N-2)\theta} + \ldots + e^{-iN\theta} = U_N(\cos\theta).$$

On the other hand,

$$\begin{aligned}
\chi_N(\theta) = \mathrm{Tr} D(\theta)\Pi_N &= \int_{\mathbb{S}^3} \Pi_N(x, D(\theta)x)\mathrm{d}x \\
&= \int_{\mathbb{S}^3} \Pi_N(D(\theta), I)\mathrm{d}x = \Pi_N(D(\theta), I) \\
&= P_N(\langle (e^{i\theta}, e^{-i\theta}), (1,0,0,0)\rangle) = P_N(\cos\theta).
\end{aligned}$$

where d$x$ is normalized Haar measure. It follows that $P_N = U_N$. ■

### 4.3 Real versus complex Gaussian ensembles

The purpose of this section is to show that the real parts of complex Gaussian $SU(2)^{p,q}$ are real Gaussian spherical harmonics in the standard sense employed by Nazarov–Sodin, Sarnak, etc. Taking the real part defines the map

$$\Re : \psi_N^m \in (\mathcal{H}_N^m, \gamma_N^m) \to \Re\psi_N^m \in \mathcal{H}_N^m + \mathcal{H}_N^{-m}.$$

Its image is a real subspace we denote by $\Re\mathcal{H}_N^m \subset \mathcal{H}_N$. If we push forward the complex Gaussian measure $\gamma_N^m$ on $\mathcal{H}_N^m$, then we get a Gaussian measure on $\Re\mathcal{H}_N^m$. Our claim is that this measure coincides with the real Gaussian measure on $\mathcal{H}_N$ conditioned on $\Re\mathcal{H}_N^m$. We denote by $\mathbb{E}_N^m$ the expectation with respect to $\gamma_N^m$ and $E_{N,m}$ the conditional expectation of $\gamma_N$ conditioned on $\Re\mathcal{H}_N^m$.

**Lemma 4.2.** $\mathbb{E}_N^{p,q}[\mathcal{Z}_{\Re\psi_N^m}] = \mathbb{E}_{N,m}[\mathcal{Z}_{\Re\psi_N^m}]$. In fact, for any functional $G$ of $\Re\psi_N^m$, we have

$$\mathbb{E}_N^{p,q} G(\mathcal{Z}_{\Re\psi_N^m}) = \mathbb{E}_{N,m} G(\mathcal{Z}_{\Re\psi_N^m}).$$

**Proof.** We assume $p+q=N, p-q=m$. The real part of a complex combination

$$\sum_{|\alpha|=p,|\beta|=q} a_{\alpha,\beta}\hat{s}_N^{\alpha,\beta} = \sum_{\alpha,\beta}[A_{\alpha,\beta}+iB_{\alpha,\beta}][u_N^{\alpha,\beta}+iv_N^{\alpha,\beta}]$$

equals

$$\sum_{|\alpha|=p,|\beta|=q}[A_{\alpha,\beta}u_N^{\alpha,\beta} - B_{\alpha,\beta}v_N^{\alpha,\beta}].$$

Here, $A_{p,q}, B_{p,q}$ are independent $N(0,1)$ random variables. If we condition on $B_{\alpha,\beta} = 0$, then we get the conditional real Gaussian ensemble.

We consider the measure-valued random variable $[\mathcal{Z}_{\Re\psi_N^m}]$, as well as the Euler characteristic of $\mathcal{Z}_{\Re\psi_N^m}$ (or any functional $G$ of $\Re\psi_N^m$) as a function $F(A-B)$ where $A$ and $B$ are independent $N(0,1)$ vectors. That is, the Gaussian measure on the coefficients $A$ and $B$ is the product $d\gamma(A)d\gamma(B)$. The conditional Gaussian ensemble is $d\gamma(A)\delta_0(B)$. So the Lemma boils down to proving that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(A-B)\mathrm{d}\gamma(A)\mathrm{d}\gamma(B) = \int_{\mathbb{R}^N} F(A)\mathrm{d}\gamma(A).$$

Since $\gamma * \gamma = \gamma$, the left side equals

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(C)\mathrm{d}\gamma(A)\mathrm{d}\gamma(C-A) = \int_{\mathbb{R}^N} F(C)\mathrm{d}\gamma * \gamma(C) = \int_{\mathbb{R}^N} F(C)\mathrm{d}\gamma,$$

as claimed. ■

## 5 Kac–Rice Formula

In view of Lemma 1.2, the proof of Theorem 1.1 is reduced to the calculation of $\mathbb{E}_{N,m}\#\{f_N^m = 0\}$. To this end, we use the Kac–Rice formula. The Kac–Rice formula in the setting of random smooth sections of complex line bundles over Kähler manifolds is proved in [5]. It makes use of the canonical lift of sections to equivariant functions on the associated $\mathbb{S}^1$ bundle and is therefore well adapted to our setting. We briefly review the formula and then move on to the calculation of $\mathbb{E}_{N,m}\#\{f_N^m = 0\}$.

### 5.1 Review of the Kac–Rice formula in our setting

We closely follow the exposition in [5, Section 5.4].

To state the Kac–Rice result, we need some notation and background. The pullback of the Dirac mass $\delta_0$ at $0 \in \mathbb{C}$ is given by

$$\delta_0(f) = \sum_{x:f=0} \frac{\delta_x}{|df \wedge d\bar{f}|}.$$

Here, we use that the Jacobian $J_f$ of $f : \mathbb{C} \to \mathbb{C}$ is given by $J_f = |df \wedge d\bar{f}|$. The equivariant lift of $f$ to the circle bundle is $fe^{-m\varphi/2}$ where $\varphi$ is the Kähler potential of the Fubini–Study metric. The pullback of $\delta_0$ under this complex-valued function is

$$\delta_0(e^{-m\varphi/2}f) = \sum_{x:e^{-m\varphi/2}f=0} \frac{\delta_x}{J_{e^{-m\varphi/2}f}} = e^{m\varphi} \sum_{x:f=0} \frac{\delta_x}{J_f},$$

since $De^{-m\varphi/2}f = e^{-m\varphi/2}Df$ at a zero.

Second, we need to recall the joint probability density $D_N^m(x, \xi; z)$ of the random variables

$$X_z(f_N^m) := f_N^m(z), \Xi_z(f_m^N) := df_N^m(z)$$

and in particular the “conditional” density $D(0, \xi; z)$. The joint probability density is given by

$$D(x, \xi; z) = \frac{\exp\langle -\Delta^{-1}v, v\rangle}{\pi^3 \det \Delta}, \qquad v = \begin{pmatrix} x \\ \xi \end{pmatrix}, \tag{5.1}$$

where $\Delta = \Delta_N^m(z)$ is the covariance matrix of $(X_z, \Xi_z)$,

$$\begin{aligned}
\Delta_N^m(z) &= \begin{pmatrix} A_N^m & B_N^m \\ B_N^{m*} & C_N^m \end{pmatrix}, \\
\left(A_N^m\right) &= \mathbb{E}\left(X_z \bar{X}_z\right) = \frac{1}{d_N^m} \Pi_N^m(z, 0; z, 0), \\
\left(B_N^m\right) &= \mathbb{E}\left(X_z \overline{\Xi}_z\right) = \frac{1}{d_N^m} \overline{\nabla}^2 \Pi_N^m(z, 0; z, 0), \\
\left(C^N\right) &= \mathbb{E}\left(\Xi_z \overline{\Xi}_z\right) = \frac{1}{d_N^m} \nabla^1 \overline{\nabla}^2 \Pi_N^m(z, 0; z, 0).
\end{aligned} \tag{5.2}$$

Here, $\nabla_z^1$, respectively $\nabla_z^2$, denotes the differential operator on $X \times X$ given by applying $\nabla_z$ to the 1st, respectively 2nd, factor. For notational simplicity, we often drop the super- and sub-scripts $(N, m)$ in what follows.

As discussed in [5] and as is wellknown, $D(0, \xi; z)$ is then given by

$$D(0, \xi; z) = Z(z) D_\Lambda(\xi; z), \tag{5.3}$$

where

$$D_\Lambda(\xi; z) = \frac{1}{\pi^2 \det \Lambda} \exp\left(-\langle \Lambda^{-1}\xi, \xi\rangle\right) \tag{5.4}$$

is the Gaussian density with covariance matrix

$$\Lambda = C - B^* A^{-1} B \tag{5.5}$$

and where

$$Z(z) = \frac{\det \Lambda}{\pi \det \Delta} = \frac{1}{\pi \det A}. \tag{5.6}$$

The formula (5.3) for $D(0, \xi; z)$ simplifies to

$$D(0, \xi; z) = \frac{1}{\pi^3 \det A \det \Lambda} e^{-\langle \Lambda^{-1}\xi, \xi\rangle}. \tag{5.7}$$

**Proposition 5.1.** Let $s = fe$ in a local frame and let $\hat{s} = fe^{-m\varphi/2}$. Then, $\mathbb{E}\mathcal{Z}_s$ is the measure on $\mathbb{CP}^1$ given by

$$\mathbb{E}(\mathcal{Z}_s) \quad = \quad \int_{\mathbb{C}^2} \left|\xi \wedge \bar{\xi}\right| D(0, \xi : x) \mathrm{d}L(\xi),$$

where $dL$ is Lebesgue measure and where $D(x, \xi; z)$ is the joint probability density of $(f(z), df(z))$, given by (5.7).

**Proof.** By definition,

$$\begin{aligned} \mathbb{E}(\langle \mathcal{Z}_s, \psi\rangle) \quad &= \quad \mathbb{E} \int_{\mathbb{CP}^1} \psi(z) \delta_0(fe^{-m\varphi/2}) \left|\mathrm{d}(f(z, \bar{z}) e^{-m\varphi/2}) \wedge \bar{\mathrm{d}}(f(z, \bar{z}) e^{-m\varphi/2})\right| \\ &= \quad \mathbb{E} \int_{\mathbb{CP}^1} \psi(z) \delta_0(f) \left|\mathrm{d}(f(z, \bar{z})) \wedge \bar{\mathrm{d}}(f(z, \bar{z}))\right|. \end{aligned}$$

Here, $\left|\mathrm{d}(f(z, \bar{z}) e^{-n\varphi/2}) \wedge \bar{\mathrm{d}}(f(z, \bar{z}) e^{-n\varphi/2})\right|$ is a density (the absolute value of a volume form). We then replace the $\delta_0(f)$ by the Fourier integral, to get

$$\mathbb{E}(\langle \mathcal{Z}_s, \psi\rangle) = \int_{\mathbb{C}} \psi(z) \mathbb{E}\left(e^{i\Re f(z)\bar{t}} \left|\mathrm{d}f \wedge \mathrm{d}\bar{f}\right|\right) \mathrm{d}L(t).$$

In general, if $F$ is a complex Gaussian random field and $\Phi$ is a (possibly nonlinear) functional, then

$$\mathbb{E}(\Phi(F(z), dF(z))) = \int_{\mathbb{C}\times\mathbb{C}^2} \Phi(z,\xi)\mathbb{P}(F(z) = x, \mathrm{d}F(z) = \xi) = \int_{\mathbb{C}\times\mathbb{C}^2} \Phi(z,\xi)D(x,\xi;z)\mathrm{d}z\mathrm{d}\xi.$$

Therefore,

$$\mathbb{E}(e^{i\Re f(z)\bar{t}}\left|\mathrm{d}f\wedge\mathrm{d}\bar{f}\right|) = \int_{\mathbb{C}}\int_{\mathbb{C}^2} e^{i\Re x\bar{t}}\left|\xi\wedge\bar{\xi}\right| D(x,\xi;z)\mathrm{d}z\mathrm{d}\xi.$$

Using that $\int_{\mathbb{C}} e^{i\Re x\bar{t}}\mathrm{d}L(t) = \delta_0(x)$, we get

$$\int_{\mathbb{C}} \mathbb{E}(e^{i\Re f(z)\bar{t}}\left|\mathrm{d}f\wedge\mathrm{d}\bar{f}\right|)\mathrm{d}L(t) = \int_{\mathbb{C}^2}\left|\xi\wedge\bar{\xi}\right|\ D(0,\xi;z)\mathrm{d}L(z)\mathrm{d}L(\xi).$$

It remains to compute $D(x,\xi;z)$, and we outline the calculation in [5] using the real linear 1-jet map $\mathcal{J} := J^1_z$, which is locally written in terms of an orthonormal basis $\{f_j\}$ as

$$\mathcal{J}(a) = (x,\xi) := (\sum_j a_jf_j(z), \sum a_jDf_j(z)),\ \ J_z(a) = \sum a_jDf_j(z)). \tag{5.8}$$

We may regard $\mathcal{J}$ as a map from $a\in\mathbb{C}^N$ into $(x,\xi)\in\mathbb{C}\times\mathbb{C}^2$. The joint probability density is the push forward of the measure $e^{-|a|^2/2}da$ under $J^1_z$,

$$\mathcal{J}_* e^{-|a|^2}\mathrm{d}a = D(x,\xi;z)\mathrm{d}L(x)\mathrm{d}L(\xi)\ \text{ i.e., } D(x,\xi;z) = \int_{\mathcal{J}^{-1}(x,\xi)} e^{-|a|^2/2}\mathrm{d}\dot{a},$$

where $\mathrm{d}\dot{a}$ is the surface Lebesgue measure on the subspace $\mathcal{J}^{-1}(x,\xi)$. This follows from general principles on pushing forward complex Gaussians under complex linear maps $F:\mathbb{C}^d\to\mathbb{C}^n$, whereby

$$F_* e^{-|a|^2}da = \gamma_{FF^*},\ \ \text{i.e.,}\ \ \mathcal{J}(x,\xi) = \frac{1}{\det\mathcal{J}\mathcal{J}^*}e^{-\langle[\mathcal{J}\mathcal{J}^*]^{-1}(x,\xi),(x,\xi)\rangle}. \tag{5.9}$$

As in [5], one shows that $\mathcal{J}\mathcal{J}^* = \Delta^m_N$ above. Conditioning on $x=0$ then gives (5.3). ■

### 5.2 Symmetries and application to Lemma 1.3

Note that $\Pi_N(\Phi(x),\Phi(y)) = \Pi_N(x,y)$ for any isometry $\Phi\in SO(4)$. However, this is not true for $\Pi^m_N(x,y)$. To understand the symmetries of $\Pi^m_N(x,y)$, we first identify $(z_1,z_2)\in\mathbb{C}^2$ in (2.1) with unit quaternion $p = z_1 + z_2 j$. Then the action $r_\theta$ (2.3) is equivalent to the left multiplication by $e^{i\theta}$.

We recall that the map from $SU(2) \times SU(2)$ to $SO(4)$ given by mapping a pair of unit quaternions $(p, q)$ to the map $\Phi_{p,q} : x \mapsto \bar{p}xq$ is a surjective homomorphism with the kernel $\{(1, 1), (-1, -1)\}$. Observe that among $\Phi_{p,q}$, $\Phi_{1,q}$ for any fixed $q$ commutes with the action $r_\theta$, by the associativity of multiplication of quaternions. Hence, $\Phi_{1,q}$ leaves $\Pi_N^m(x, y)$ invariant because

$$\Pi_N(r_{\theta_1}\Phi_{1,q}(x), r_{\theta_2}\Phi_{1,q}(y)) = \Pi_N(\Phi_{1,q}(r_{\theta_1}x), \Phi_{1,q}(r_{\theta_2}y)) = \Pi_N(r_{\theta_1}x, r_{\theta_2}y).$$

We also infer that $\Phi_{1,q}$ is well defined on the fibers of the Hopf fibration $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ and therefore induces $SU(2)$-action on $\mathbb{S}^2$. This induces a surjective homomorphism $SU(2) \to SO(3)$ with the kernel $\{1, -1\}$, and it is well known that $SO(3)$ acts doubly transitively on $\mathbb{S}^2$.

Now to prove Lemma 1.3, we need to calculate $\Delta_N^m$ and $\Lambda$ in (5.4). Equivalently, for a local orthonormal frame $E = \{\partial_\theta, \mathbf{e}_1, \mathbf{e}_2\}$ near $x \in \mathbb{S}^3$, we need to calculate

$$\Pi_N^m(x, x), \ \left( \mathbf{e}_1^y \Pi_N^m(x, y)\big|_{x=y} \quad \mathbf{e}_2^y \Pi_N^m(x, y)\big|_{x=y} \right)$$

and

$$\begin{pmatrix} \mathbf{e}_1^x \mathbf{e}_1^y \Pi_N^m(x, y)\big|_{x=y} & \mathbf{e}_1^x \mathbf{e}_2^y \Pi_N^m(x, y)\big|_{x=y} \\ \mathbf{e}_2^x \mathbf{e}_1^y \Pi_N^m(x, y)\big|_{x=y} & \mathbf{e}_2^x \mathbf{e}_2^y \Pi_N^m(x, y)\big|_{x=y} \end{pmatrix}.$$

We then deduce from $\Phi_{1,q}$-invariance of $\Pi_N^m(x, y)$ and the discussion above that these quantities do not depend on the choice of $x \in \mathbb{S}^3$ and $E = \{\partial_\theta, \mathbf{e}_1, \mathbf{e}_2\}$:

**Lemma 5.2.** $\Delta_N^m$ and $\Lambda$ in (5.4) are constant matrices.

### 5.3 Chebyshev calculations

In this section, we compute $\Delta_N^m$ and $\Lambda$ in (5.4) explicitly. Firstly, from Lemma 5.2, it is sufficient to compute the matrix at $x = (\alpha, \varphi, \theta) = (\pi/4, 0, 0)$ in the coordinate system (2.2).

Note that $\Pi_N^m(x, y) = 0$, if $2 \nmid N - m$ or if $|m| > N$, by Proposition 2.2. So we assume that $2 | N - m$ and $|m| \le N$ in this section.

**Lemma 5.3.** Let $U_N(x)$ be the Chebyshev polynomial of the 2nd kind. Assume that $m \in \mathbb{Z}$ and $N \in \mathbb{N}$ satisfies $|m| \le N$.

Then we have

$$\frac{1}{2\pi}\int \cos(m\theta)U_N(\cos\theta)\mathrm{d}\theta = \begin{cases} 1 & \text{if } 2|N-m \\ 0 & \text{otherwise.} \end{cases}$$

$$\frac{1}{2\pi}\int \cos(m\theta)U_N'(\cos\theta)\cos\theta\mathrm{d}\theta = \begin{cases} \frac{N^2-m^2}{2}+N & \text{if } 2|N-m \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Recall that

$$U_N(\cos\theta) = \frac{\sin(N+1)\theta}{\sin\theta} = \frac{e^{i(N+1)\theta}-e^{-i(N+1)\theta}}{e^{i\theta}-e^{-i\theta}} = e^{iN\theta}+e^{i(N-2)\theta}+\ldots+e^{-iN\theta}.$$

Therefore, the 1st integral is 1 if $m$ is equal to one of $N, N-2, \ldots, -N$ and 0 otherwise. To compute the 2nd integral, we first differentiate the above equation to get

$$\begin{aligned} -U_N'(\cos\theta)\sin\theta &= iNe^{iN\theta}+i(N-2)e^{i(N-2)\theta}+\ldots+i(-N)e^{-iN\theta} \\ &= iN(e^{iN\theta}-e^{-iN\theta})+i(N-2)(e^{i(N-2)\theta}-e^{-i(N-2)\theta})+\ldots \end{aligned}$$

and therefore

$$\begin{aligned} U_N'(\cos\theta) &= 2N(e^{i(N-1)\theta}+e^{i(N-3)\theta}+\ldots+e^{-i(N-1)\theta}) \\ &\quad +2(N-2)(e^{i(N-3)\theta}+e^{i(N-5)\theta}+\ldots+e^{-i(N-3)\theta})+\ldots. \end{aligned}$$

Because the 2nd integral does not depend on the sign of $m$, we assume without loss of generality that $m\geq 0$. Since $\cos(m\theta)\cos\theta = \frac{1}{2}(\cos((m+1)\theta)+\cos((m-1)\theta))$, we have

$$D_N^m = [N+(N-2)+\ldots+(m+2)]+[N+(N-2)+\ldots+m] = \frac{1}{2}(N^2-m^2)+N$$

when $2|N-m$ and 0 otherwise. ∎

**Theorem 5.4.** We have

$$\Delta_N^m = \frac{1}{N+1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{N^2-m^2}{2}+N & \frac{im}{2} \\ 0 & -\frac{im}{2} & \frac{N^2-m^2}{2}+N \end{pmatrix}$$

and

$$\Lambda = \frac{1}{N+1}\begin{pmatrix} \frac{N^2-m^2}{2}+N & \frac{im}{2} \\ -\frac{im}{2} & \frac{N^2-m^2}{2}+N \end{pmatrix}.$$

**Proof.** Firstly, we have

$$\begin{aligned}\Pi_N^m(x,x) &= \frac{1}{4\pi^2}\iint \exp(-im\theta_1 + im\theta_2)\Pi_N(r_{\theta_1}x, r_{\theta_2}x)\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ &= \frac{1}{4\pi^2}\iint \exp(-im\theta_1 + im\theta_2)U_N(\cos(\theta_1-\theta_2))\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ &= \frac{1}{2\pi}\int \exp(-im\theta)U_N(\cos\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int \cos(m\theta)U_N(\cos\theta)\mathrm{d}\theta = 1,\end{aligned}$$

by Lemma 5.3. For $\nu = \alpha$ or $\varphi$, we have

$$\begin{aligned}\partial_{\nu(y)}\Pi_N^m(x,y)\big|_{x=y} &= \frac{1}{4\pi^2}\partial_{\nu(y)}\iint \exp(-im\theta_1 + im\theta_2)\Pi_N(r_{\theta_1}x, r_{\theta_2}y)\mathrm{d}\theta_1\mathrm{d}\theta_2\big|_{x=y} \\ &= \frac{1}{4\pi^2}\iint \exp(-im\theta_1 + im\theta_2)U_N'(\cos(\theta_1-\theta_2))\partial_{\nu(y)}(r_{\theta_1}x\cdot r_{\theta_2}y)\big|_{x=y}\mathrm{d}\theta_1\mathrm{d}\theta_2.\end{aligned}$$

If we write

$$\begin{aligned}r_{\theta_1}x = (&\sin\alpha(x)\cos(\varphi(x)+\theta_1), \sin\alpha(x)\sin(\varphi(x)+\theta_1), \cos\alpha(x)\cos(\theta_1-\varphi(x)), \\ &\cos\alpha(x)\sin(\theta_1-\varphi(x)))\end{aligned}$$

and

$$\begin{aligned}r_{\theta_2}y = (&\sin\alpha(y)\cos(\varphi(y)+\theta_2), \sin\alpha(y)\sin(\varphi(y)+\theta_2), \cos\alpha(y)\cos(\theta_2-\varphi(y)), \\ &\cos\alpha(y)\sin(\theta_2-\varphi(y))),\end{aligned}$$

then

$$\begin{aligned}\partial_{\alpha(y)}(r_{\theta_1}x\cdot r_{\theta_2}y)\big|_{x=y=(\alpha,\varphi,0)} &= \sin\alpha\cos\alpha\cos(\varphi+\theta_1)\cos(\varphi+\theta_2) + \sin\alpha\cos\alpha\sin(\varphi+\theta_1)\sin(\varphi+\theta_2) \\ &\quad - \sin\alpha\cos\alpha\cos(\theta_1-\varphi)\cos(\theta_2-\varphi) - \sin\alpha\cos\alpha\sin(\theta_1-\varphi)\sin(\theta_2-\varphi),\end{aligned}$$

which is 0, and

$$\partial_{\varphi(y)}(r_{\theta_1}x\cdot r_{\theta_2}y)\big|_{x=y=(\alpha,\varphi,0)} = -\sin^2\alpha\cos(\varphi+\theta_1)\sin(\varphi+\theta_2)+\sin^2\alpha\sin(\varphi+\theta_1)\cos(\varphi+\theta_2)$$
$$+\cos^2\alpha\cos(\theta_1-\varphi)\sin(\theta_2-\varphi)-\cos^2\alpha\sin(\theta_1-\varphi)\cos(\theta_2-\varphi),$$

which simplifies to

$$\cos(2\alpha)\sin(\theta_2-\theta_1).$$

Therefore,

$$\partial_{\alpha(y)}\Pi_N^m(x,y)\big|_{x=y} = 0$$
$$\begin{aligned}\partial_{\varphi(y)}\Pi_N^m(x,y)\big|_{x=y} &= -\frac{\cos(2\alpha)}{4\pi^2}\iint \exp(-im\theta_1+im\theta_2)U_N'(\cos(\theta_1-\theta_2))\sin(\theta_1-\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &= -\frac{\cos(2\alpha)}{2\pi}\int\exp(-im\theta)U_N'(\cos\theta)\sin\theta\mathrm{d}\theta\\ &= -\frac{im\cos(2\alpha)}{2\pi}\int\exp(-im\theta)U_N(\cos\theta)\mathrm{d}\theta\\ &= -im\cos(2\alpha),\end{aligned}$$

by Lemma 5.3. Likewise,

$$\partial_{\nu_1}\partial_{\nu_2}\Pi_N^m(x,y)\big|_{x=y} = \frac{1}{4\pi^2}\iint\exp(-im(\theta_1-\theta_2))$$
$$\times(U_N''(\cos(\theta_1-\theta_2))\partial_{\nu_1}(r_{\theta_1}x\cdot r_{\theta_2}y)\partial_{\nu_2}(r_{\theta_1}x\cdot r_{\theta_2}y)|_{x=y}+U_N'(\cos(\theta_1-\theta_2))\partial_{\nu_1}\partial_{\nu_2}(r_{\theta_1}x\cdot r_{\theta_2}y)|_{x=y})\mathrm{d}\theta_1\mathrm{d}\theta_2,$$

and we have

$$\begin{aligned}\partial_{\alpha(x)}\partial_{\alpha(y)}(r_{\theta_1}x\cdot r_{\theta_2}y)\big|_{x=y=(\alpha,\varphi,0)} &= \cos(\theta_1-\theta_2)\\ \partial_{\alpha(x)}\partial_{\varphi(y)}(r_{\theta_1}x\cdot r_{\theta_2}y)\big|_{x=y=(\alpha,\varphi,0)} &= \cos\alpha\sin\alpha\sin(\theta_1-\theta_2)\\ \partial_{\varphi(x)}\partial_{\alpha(y)}(r_{\theta_1}x\cdot r_{\theta_2}y)\big|_{x=y=(\alpha,\varphi,0)} &= -\cos\alpha\sin\alpha\sin(\theta_1-\theta_2)\\ \partial_{\varphi(x)}\partial_{\varphi(y)}(r_{\theta_1}x\cdot r_{\theta_2}y)\big|_{x=y=(\alpha,\varphi,0)} &= \cos(\theta_1-\theta_2).\end{aligned}$$

Therefore,

$$\partial_{\alpha(x)}\partial_{\alpha(y)}\Pi_N^m(x,y)\big|_{x=y=(\alpha,\varphi,0)} = \frac{1}{2\pi}\int \cos(m\theta)U_N'(\cos\theta)\cos\theta\,\mathrm{d}\theta = \frac{N^2-m^2}{2}+N$$

$$\partial_{\alpha(x)}\partial_{\varphi(y)}\Pi_N^m(x,y)\big|_{x=y=(\alpha,\varphi,0)} = im\cos\alpha\sin\alpha$$

$$\partial_{\varphi(x)}\partial_{\alpha(y)}\Pi_N^m(x,y)\big|_{x=y=(\alpha,\varphi,0)} = -im\cos\alpha\sin\alpha.$$

For the last case,

$$\begin{aligned}
&\partial_{\varphi(x)}\partial_{\varphi(y)}\Pi_N^m(x,y)\big|_{x=y=(\alpha,\varphi,0)}\\
=&\frac{1}{2\pi}\int \cos(m\theta)\cos^2(2\alpha)U_N''(\cos\theta)\sin^2\theta\,\mathrm{d}\theta + \frac{1}{2\pi}\int \cos(m\theta)U_N'(\cos\theta)\cos\theta\,\mathrm{d}\theta\\
=&\frac{\cos^2(2\alpha)}{2\pi}\int (\cos(m\theta)\sin\theta)'\,U_N'(\cos\theta)\mathrm{d}\theta + \frac{1}{2\pi}\int \cos(m\theta)U_N'(\cos\theta)\cos\theta\,\mathrm{d}\theta\\
=&-\frac{m\cos^2(2\alpha)}{2\pi}\int U_N'(\cos\theta)\sin(m\theta)\sin\theta\,\mathrm{d}\theta + \frac{1+\cos^2 2\alpha}{2\pi}\int \cos(m\theta)U_N'(\cos\theta)\cos\theta\,\mathrm{d}\theta\\
=&\frac{m^2\cos^2(2\alpha)}{2\pi}\int U_N(\cos\theta)\cos(m\theta)\mathrm{d}\theta + \frac{1+\cos^2 2\alpha}{2\pi}\int \cos(m\theta)U_N'(\cos\theta)\cos\theta\,\mathrm{d}\theta\\
=&m^2\cos^2(2\alpha)+(1+\cos^2(2\alpha))\left(\frac{N^2-m^2}{2}+N\right).
\end{aligned}$$

Finally, recall that

$$\Delta_N^m = \frac{1}{N+1}\begin{pmatrix} A & B\\ B^* & C\end{pmatrix},$$

where

$$A = \Pi_N^m(x,x), B = \begin{pmatrix} \mathbf{e}_1^y\Pi_N^m(x,y)\big|_{x=y} & \mathbf{e}_2^y\Pi_N^m(x,y)\big|_{x=y}\end{pmatrix}$$

and

$$C = \begin{pmatrix} \mathbf{e}_1^x\mathbf{e}_1^y\Pi_N^m(x,y)\big|_{x=y} & \mathbf{e}_1^x\mathbf{e}_2^y\Pi_N^m(x,y)\big|_{x=y}\\ \mathbf{e}_2^x\mathbf{e}_1^y\Pi_N^m(x,y)\big|_{x=y} & \mathbf{e}_2^x\mathbf{e}_2^y\Pi_N^m(x,y)\big|_{x=y}\end{pmatrix}$$

where $\{\partial_\theta, \mathbf{e}_1, \mathbf{e}_2\}$ is a local orthonormal frame. (So $A$, $B$, and $C$ are $1\times 1$, $1\times 2$, and $2\times 2$ complex matrices.) Above computation with $(\mathbf{e}_1, \mathbf{e}_2) = (\partial_\theta, \partial_\varphi)\big|_{(\alpha,\varphi)=(\pi/4,0)}$ imply that

$$\Delta_N^m = \frac{1}{N+1}\begin{pmatrix} 1 & 0 & -im\cos(2\alpha) \\ 0 & \frac{N^2-m^2}{2}+N & im\cos\alpha\sin\alpha \\ im\cos(2\alpha) & -im\cos\alpha\sin\alpha & m^2\cos^2(2\alpha)+(1+\cos^2(2\alpha))\left(\frac{N^2-m^2}{2}+N\right)\end{pmatrix}\Bigg|_{(\pi/4,0)}$$

$$= \frac{1}{N+1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{N^2-m^2}{2}+N & \frac{im}{2} \\ 0 & -\frac{im}{2} & \frac{N^2-m^2}{2}+N\end{pmatrix}.$$

∎

**Proof of Lemma 1.3.** From Proposition 5.1 and Theorem 5.4, we see that

$$\mathbb{E}(\#\{f_N^m = 0\}) = \int_{\mathbb{C}^2} |\xi\wedge\bar{\xi}|\frac{N+1}{\det\Lambda}\exp(-\langle\Lambda^{-1}\xi,\xi\rangle)dL(\xi)$$

$$= \int_{\mathbb{C}^2} |\xi\wedge\bar{\xi}|\frac{N+1}{\det\Lambda}\exp(-\langle\Lambda^{-1/2}\xi,\Lambda^{-1/2}\xi\rangle)\mathrm{d}L(\xi),$$

where

$$\Lambda = \frac{1}{N+1}\begin{pmatrix}\frac{N^2-m^2}{2}+N & \frac{im}{2} \\ -\frac{im}{2} & \frac{N^2-m^2}{2}+N\end{pmatrix}.$$

By change of variables $\Lambda^{-1/2}\xi = \zeta$, we have

$$\mathbb{E}(\#\{f_N^m = 0\}) = \frac{N+1}{\pi^3}\int_{\mathbb{C}^2} |\Lambda^{1/2}\zeta\wedge\overline{\Lambda^{1/2}\zeta}|\exp(-|\zeta|^2)\mathrm{d}L(\zeta).$$

Now let

$$\zeta = \begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$$

and let $v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Observe that $\Lambda v_1 = (\mu+\nu)v_1$ and $\Lambda v_2 = (\mu-\nu)v_2$ with $\mu = \frac{1}{N+1}\left(\frac{N^2-m^2}{2}+N\right)$ and $\nu = \frac{m}{2(N+1)}$. In particular, we have

$$\begin{aligned}
|\Lambda^{1/2}\zeta \wedge \overline{\Lambda^{1/2}\zeta}| &= |(\alpha_1(\mu+\nu)^{\frac{1}{2}}v_1 + \alpha_2(\mu-\nu)^{\frac{1}{2}}v_2) \wedge \overline{(\alpha_1(\mu+\nu)^{\frac{1}{2}}v_1 + \alpha_2(\mu-\nu)^{\frac{1}{2}}v_2)}| \\
&= \left| |\alpha_1|^2(\mu+\nu)v_1 \wedge \overline{v_1} + |\alpha_2|^2(\mu-\nu)v_2 \wedge \overline{v_2} \right| \\
&= \mu \left| |\alpha_1|^2(1+\frac{\nu}{\mu}) - |\alpha_2|^2(1-\frac{\nu}{\mu}) \right| |v_1 \wedge v_2| \\
&= 2\mu \left| |\alpha_1|^2(1+\eta) - |\alpha_2|^2(1-\eta) \right|,
\end{aligned}$$

where

$$\eta = \frac{m/2}{\frac{N^2-m^2}{2}+N}.$$

Therefore,

$$\mathbb{E}(\#\{f_N^m = 0\}) = \left(\frac{N^2-m^2}{2}+N\right)\frac{1}{\pi^3}\int_{\mathbb{C}^2}\left| |\alpha_1|^2(1+\eta) - |\alpha_2|^2(1-\eta) \right| \exp(-2|\alpha|^2)\mathrm{d}L(\alpha),$$

and we evaluate the integral as follows:

$$\begin{aligned}
&\frac{1}{\pi^3}\int_{\mathbb{C}^2}\left| |\alpha_1|^2(1+\eta) - |\alpha_2|^2(1-\eta) \right| \exp(-2|\alpha|^2)\mathrm{d}L(\alpha) \\
=&\frac{4}{\pi}\int_{\mathbb{R}_+^2} r_1 r_2 \left| r_1^2(1+\eta) - r_2^2(1-\eta) \right| \exp(-2(r_1^2+r_2^2))\mathrm{d}r_1\mathrm{d}r_2 \quad \text{change of variables } \alpha_j = r_j e^{i\theta_j} \\
=&\frac{4}{\pi}\int_0^{\pi/2}\int_0^{\infty} r^3\cos\theta\sin\theta \left| r^2\cos^2\theta(1+\eta) - r^2\sin^2\theta(1-\eta) \right| \exp(-2r^2)\mathrm{d}r\mathrm{d}\theta \\
&\qquad \text{change of variables } (r_1, r_2) = (r\cos\theta, r\sin\theta) \\
=&\frac{4}{\pi}\int_0^{\infty} r^5\exp(-2r^2)\mathrm{d}r \int_0^{\pi/2}\cos\theta\sin\theta\,|\eta+\cos(2\theta)|\,\mathrm{d}\theta \\
=&\frac{1}{4\pi}\int_{-1}^{1}|\eta+t|\,\mathrm{d}t = \frac{1+\eta^2}{4\pi}.
\end{aligned}$$

In the last equality, we used the fact that $|\eta| \le \frac{1}{2}$ to evaluate the integral. ■

### 5.4 Bertini theorem

We need the following Bertini-type theorem to employ the Kac–Rice formalism and to prove Theorem 1.1:

**Proposition 5.5.** For all $m \neq 0$, 0 is almost surely a regular value of the random equivariant eigenfunction $\psi_N^m : \mathbb{S}^3 \to \mathbb{C}$.

**Proof.** We need to show that the derivative $d_x \psi_N^m : T_x\mathbb{S}^3 \to \mathbb{C}$ is surjective at each point $x$ where $\psi_N^m(x) = 0$ for almost any $\psi_N^m$. It is sufficient to prove that $\mathcal{H}_N^m$ has the "1-jet spanning property" that the 1-jet evaluation map,

$$\mathcal{J} : \mathbb{S}^3 \times \mathcal{H}_N^m \to J^1(\mathbb{S}^3, \mathbb{C}), \quad \mathcal{J}(x, \psi_N^m) = J_x^1 \psi_N^m = (\psi_N^m(x), d_x^H \psi_N^m(x)),$$

is surjective.

The 1-jet spanning property implies that at each point, $\{d_x^H \psi_N^m(x) : \psi_N^m \in \mathcal{H}_N^m\}$ spans the horizontal tangent space $H_x\mathbb{S}^3$. Since the ensemble is $SU(2)$ invariant, it suffices to prove the spanning property at a single point. Moreover, since $SU(2)$ acts transitively on the unit tangent bundle of $\mathbb{S}^2$ (or on the horizontal spaces of $\mathbb{S}^3$), failure to span is equivalent to the existence of $x$ such that $d_x^H \psi_N^m(x) = 0$ for all $\psi_N^m$ such that $\psi_N^m(x) = 0$. This is false, since Jacobi polynomials have simple zeros, as can for instance be seen from the Darboux formula.

As explained in [2, Section 4.1], the 1-jet spanning property implies that the incidence set $I := \{(x, \psi_N^m) \in \mathbb{S}^3 \times \mathcal{H}_N^m : \psi_N^m(x) = 0\}$ is a smooth submanifold, and hence by Sard's theorem applied to the projection $I \to \mathcal{H}_N^m$, the zero set

$$Z_{\psi_N^m} = \{x \in \mathbb{S}^3 : \psi_N^m(x) = 0\}$$

is a smooth 1D submanifold of $\mathbb{S}^3$ for almost all $\psi_N^m$. ■

## 6 Résumé of the Proof Theorem 1.1

Having established all of the ingredients of the proof of Theorem 1.1 outlined in the Introduction, we only review how to assemble the ingredients into a proof. We resume the discussion begun in Section 2.5. To prove (i) of Theorem 1.1, we use the proof of the same statement as Theorem 2.3 ([8, Theorem 1.5]) for general Kaluza–Klein Laplacians on circle bundles. The only point we need to establish to apply the proof is that 0 is almost surely a regular value of the equivariant eigenfunctions, and this is proved in

Theorem 5.5. The main new result is therefore (ii) of Theorem 1.1. It is based on a simple formula of Lemma 1.2 relating the genus of the nodal set with the number of zeros of $f_N^m$; this makes use of the special structure of nodal set of real parts of equivariant eigenfunctions as "helicoid covers" of $\mathbb{S}^2$. In Section 5, we use the Kac–Rice to calculate the expected number of zeros of the random $f_N^m$ and prove Lemma 1.3, concluding the proof of Theorem 1.1.

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