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The global moduli theory of symplectic varieties

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Abstract. We develop the global moduli theory of symplectic varieties in the sense of Beauville. We prove a number of analogs of classical results from the smooth case, including a global Torelli theorem. In particular, this yields a new proof of Verbitsky's global Torelli theorem in the smooth case (assuming $b_2 \geq 5$) which does not use the existence of a hyperkähler metric or twistor deformations.

1. Introduction

A *symplectic variety* X (in the sense of Beauville [7]) is a normal variety admitting a nondegenerate closed holomorphic 2-form $\sigma \in H^0(X^{\mathrm{reg}}, \Omega^2_{X^{\mathrm{reg}}})$ on its regular part which extends holomorphically on some resolution of singularities $\pi : Y \to X$. If X is compact, $H^1(X, \mathcal{O}_X) = 0$, and σ is unique up to scaling, we say X is a *primitive* symplectic variety. We consider these varieties a singular analog of (compact) irreducible symplectic manifolds which is as general as possible such that a reasonable global moduli theory can still be established.

Irreducible symplectic manifolds are one of the three main building blocks of compact Kähler manifolds with vanishing first Chern class by a theorem of Beauville–Bogomolov [6, Théorème 1], and their geometry is very rich. In particular, Verbitsky's global Torelli theorem [88, Theorem 1.17] gives a precise description of the global deformations of a symplectic manifold in terms of the Hodge structure on its second cohomology.

Recent work of Druel, Greb, Guenancia, Höring, Kebekus and Peternell [24, 25, 34, 35, 39, 42] has shown a version of the above Beauville–Bogomolov decomposition theorem for singular projective varieties with trivial canonical class, see [42, Theorem 1.5], and the "holomorphic-symplectic" factors[1)] that show up are a special case of the primitive symplectic varieties we consider. This level of generality is important because singularities are often unavoidable in higher-dimensional geometry, for instance in the minimal model program. Our results show that the geometry of singular holomorphic-symplectic varieties enjoys

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[1)] They were called *irreducible symplectic* by Greb, Kebekus and Peternell [36, Definition 8.16], where *irreducible* refers to the decomposition theorem.

the same richness as that of smooth ones, and deformation theory – especially deformations to *non-projective* varieties – is as essential a part of the picture as in the smooth case. Interestingly, whereas it has proven difficult to produce new deformation types of *smooth* irreducible symplectic varieties, in the singular case a number of "new" deformation types – that is, deformation types which do not seemingly arise from holomorphic symplectic manifolds – can be constructed, see Example 3.2 (2).

Our main result is a global Torelli theorem for primitive symplectic varieties in general with surjectivity of the period map in the $\mathbb{Q}$-factorial[2)] terminal case. Before stating the theorem, let us fix some notation. The torsion-free part $H^2(X,\mathbb{Z})_{\mathrm{tf}} := H^2(X,\mathbb{Z})/\mathrm{torsion}$ of the second cohomology of a primitive symplectic variety X carries a pure weight two Hodge structure (see Lemma 2.1) which is further endowed with an integral locally trivial deformation-invariant quadratic form q_X called the Beauville–Bogomolov–Fujiki (BBF) form (see Section 5.1). Fixing a lattice Λ and denoting its quadratic form by q, a Λ-marking of X is an isomorphism $\mu : (H^2(X,\mathbb{Z})_{\mathrm{tf}}, q_X) \xrightarrow{\cong} (\Lambda, q)$. The set of isomorphism classes of Λ-marked primitive symplectic varieties (X,μ) is given the structure of an analytic space $\mathfrak{M}_\Lambda$ by gluing the bases of *locally trivial* Kuranishi families (see Definition 6.13). In fact, $\mathfrak{M}_\Lambda$ is a not-necessarily-Hausdorff complex manifold by the unobstructedness of locally trivial deformations (see Theorem 4.7).

We obtain a period map $P : \mathfrak{M}_\Lambda \to \Omega_\Lambda$ to the period domain $\Omega_\Lambda \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ by sending (X,μ) to $\mu(H^{2,0}(X))$ and it is a local isomorphism (see Proposition 5.5). There is a Hausdorff reduction $H : \mathfrak{M}_\Lambda \to \overline{\mathfrak{M}}_\Lambda$, where $\overline{\mathfrak{M}}_\Lambda$ is a Hausdorff complex manifold and H identifies inseparable points (see Section 8), and we moreover have a factorization

$$\begin{array}{ccccc} & & \overline{\mathfrak{M}}_\Lambda & & \\ & \overset{H}{\nearrow} & & \overset{\overline{P}}{\searrow} & \\ \mathfrak{M}_\Lambda & & \xrightarrow[P]{\qquad\qquad} & & \Omega_\Lambda. \end{array}$$

We now state our main result:

Theorem 1.1. *Assume that* $\mathrm{rk}(\Lambda) \geq 5$. *Then for each connected component* $\mathfrak{M}$ *of the* Λ*-marked moduli space* $\mathfrak{M}_\Lambda$ *we have:*

(1) *The monodromy group* $\mathrm{Mon}(\mathfrak{M}) \subset \mathrm{O}(\Lambda)$ *is of finite index.*

(2) $P : \mathfrak{M} \to \Omega_\Lambda$ *is bijective over Mumford–Tate general points and in general the fibers consist of pairwise bimeromorphic varieties.*

(3) $\overline{P} : \overline{\mathfrak{M}} \to \Omega_\Lambda$ *is an isomorphism onto the complement of countably many maximal Picard rank periods.*

(4) *If moreover one point of* $\mathfrak{M}$ *corresponds to a primitive symplectic variety with* $\mathbb{Q}$*-factorial terminal singularities, then the same is true of every point and* $\overline{P} : \overline{\mathfrak{M}} \to \Omega_\Lambda$ *is an isomorphism.*

Theorem 1.1 of course also applies to the smooth case, and yields a new proof of Verbitsky's global Torelli theorem. Note that $\mathbb{Q}$-factorial terminal singularities form a natu-

2) There is a subtlety with the definition of $\mathbb{Q}$-factoriality in the analytic category: requiring every divisor to be $\mathbb{Q}$-Cartier is potentially different from requiring every rank one torsion-free sheaf to have an invertible reflexive power (see Section 2.12).

ral class of singularities for symplectic varieties – see Example 3.2 for some examples. First, such singularities are well suited to MMP techniques (see e.g. Section 7 and Example 3.2 (5)). Second, symplectic varieties always have canonical singularities, and any projective primitive symplectic variety X admits a crepant partial resolution with $\mathbb{Q}$-factorial terminal singularities X' (a so-called $\mathbb{Q}$-factorial terminalization) whose deformation theory controls that of X (see Section 5.19). Note that a $\mathbb{Q}$-factorial terminal K-trivial variety does not admit a further crepant resolution. A version of Theorem 1.1 has been proven by Menet [67] for a symplectic varieties with quotient singularities; see Example 3.2 (4) for an explicit example of a $\mathbb{Q}$-factorial terminal symplectic variety which does not have quotient singularities.

In [5, Theorem 1.3] the authors prove Theorem 1.1 (*with* surjectivity in part (3)) in the case where $\mathfrak{M}$ parametrizes primitive symplectic varieties admitting a crepant resolution. The proof crucially uses that simultaneous crepant resolutions exist in locally trivial families of such varieties, as then Verbitsky's global Torelli theorem can be applied to the crepant resolution. Note that by definition, $\mathfrak{M}$ consists of varieties of a fixed locally trivial deformation type which allows one to prove that either all varieties it parametrizes admit a crepant resolution or none.

The main difficulty in the general setting is that while one could try to reduce to the $\mathbb{Q}$-factorial terminal case by passing to a simultaneous $\mathbb{Q}$-factorial terminalization, even in this case a new strategy is needed as Verbitsky's proof (as well as Huybrechts' proof of the surjectivity of the period map [44, Theorem 8.1]) fundamentally uses the existence of hyperkähler metrics and twistor deformations. We instead prove Theorem 1.1 directly using global results on the geometry of the period domain via Ratner theory (as first investigated by Verbitsky [89, 90]) together with finiteness results coming from algebraic geometry. The surjectivity in Theorem 1.1 then follows from a generalization to the $\mathbb{Q}$-factorial terminal case of work of Kollár, Laza, Saccà and Voisin [57] on projective degenerations using MMP techniques.

In fact, there is another problem with the naive generalization of the argument of [5]: $\mathbb{Q}$-factorial terminalizations are not guaranteed to exist in the analytic setting. In the projective case the existence of a $\mathbb{Q}$-factorial terminalization is a consequence of deep results of Birkar, Cascini, Hacon and M^c^Kernan [9] on the termination of an appropriate version of the MMP, but it is not even clear *a priori* that a symplectic variety can be deformed to a projective one (although Namikawa [72] has results in this direction). For this reason, we need a projectivity criterion for symplectic varieties, analogous to Huybrechts' criterion [44, Theorem 3.11] for hyperkähler manifolds:

Theorem 1.2. *Let X be a primitive symplectic variety, and assume $\alpha \in H^2(X, \mathbb{Q})$ is a $(1,1)$-class with $q_X(\alpha) > 0$. Then X is projective.*

Corollary 1.3. *Every primitive symplectic variety is locally trivially deformation equivalent to a projective primitive symplectic variety.*

The proof uses a (weak) singular analog of the Demailly–Păun theorem on the numerical characterization of the Kähler cone.

As an application of Theorem 1.1, we can in fact conclude that terminalizations of symplectic varieties exist in the *non-projective* case, up to a bimeromorphism:

Theorem 1.4. *Let X be a primitive symplectic variety with $b_2(X) \geq 5$. Then there is a primitive symplectic variety X' that is bimeromorphic and locally trivially deformation-*

equivalent to X that admits a $\mathbb{Q}$-factorial terminalization: that is, there exists a (compact) $\mathbb{Q}$-factorial terminal Kähler variety Y and a crepant map $\pi : Y \to X'$.

We view Theorem 1.4 as an indication that the deformation theoretic tools we develop might be used to generalize the MMP for projective symplectic varieties [23, 63] to the Kähler setting, and this will be pursued in a subsequent paper.

In addition to the global arguments, the proofs of Theorems 1.1, 1.2, and 1.4 require a careful analysis of the infinitesimal locally trivial deformation theory of not-necessarily-projective symplectic varieties. There are a number of new complications all critically stemming from the fact that one can no longer bootstrap classical results on the geometry of hyperkähler manifolds via passing to a crepant resolution. In particular, we must provide:

(i) An analysis of the Hodge theory of rational and symplectic singularities in the non-projective setting, using recent results of Kebekus and Schnell [54] on extending holomorphic forms.

(ii) An adaptation of the results of Kollár, Laza, Saccà and Voisin [57] on limits of projective families in the singular setting. This requires a singular analog of a theorem of Verbitsky saying that for a primitive symplectic variety X, the cup product map $\mathrm{Sym}^k H^2(X, \mathbb{Q}) \to H^{2k}(X, \mathbb{Q})$ is injective for $2k \leq \dim X$.

(iii) A description of the deformation theory of terminalizations. In particular, this requires a careful treatment of $\mathbb{Q}$-factoriality in the analytic category, as there are several non-equivalent generalizations of the corresponding notion in the algebraic category.

Previous work. In [5] the authors extended many of the classical results about compact irreducible symplectic manifolds to primitive symplectic varieties admitting a crepant resolution through the study of their locally trivial deformations. Menet [67] has proven a version of the global Torelli theorem for certain primitive symplectic varieties with orbifold singularities using twistor deformations. There are many interesting ideas in his work that have influenced parts of the present paper, especially concerning the projectivity criterion. The local deformation theory (and in particular the local Torelli theorem) of primitive symplectic varieties has been treated by many authors, notably by Namikawa [70, 71, 73] and Kirschner [55].

Outline. In Section 2 we review basic notions and results about the Hodge theory of rational singularities, Kähler spaces, big and nef classes, and $\mathbb{Q}$-factoriality in the analytic category. Section 3 is devoted to primitive symplectic varieties and their Hodge theory. In Section 4 we show locally trivial deformations of symplectic varieties are unobstructed. In Section 5 we recall the BBF form and deduce the local Torelli theorem. We also analyze the deformation theory of $\mathbb{Q}$-factorial terminalizations and prove some topological results, including the existence of Fujiki relations and the analog of a theorem of Verbitsky discussed in (ii) above. In Section 6 we prove a (weak) singular analog of the Demailly–Păun theorem and apply it to deduce the projectivity criterion, Theorem 1.2 (see Theorem 6.9). We also prove analogs of results of Huybrechts [44] and [5] on the inseparability of bimeromorphic symplectic varieties in moduli, including part (2) of Theorem 1.1 (see Theorem 6.14 and Corollary 6.17). In Section 7 we indicate the necessary changes to [57] to show the existence of limits of projective families for which the period does not degenerate in the $\mathbb{Q}$-factorial terminal setting. In Section 8 we prove parts (1), (3), and (4) of Theorem 1.1 (see Theorem 8.2). In Section 9 we apply the deformation theory of terminalizations and the global Torelli theorem to prove Theorem 1.4 (see Theorem 9.1).

For those interested in the proof of the global Torelli theorem in the smooth case, Section 8 can be read independently, as the results used from previous sections are standard in the smooth case[3)].

Notation and conventions. A resolution of singularities of a variety X is a proper surjective bimeromorphic morphism $\pi : Y \to X$ from a nonsingular variety Y. The term variety will denote an integral separated scheme of finite type over $\mathbb{C}$ in the algebraic setting or an irreducible and reduced separated complex space in the complex analytic setting.

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2. Preliminaries

A complex variety X is said to have rational singularities if it is normal and for any resolution of singularities $\pi : Y \to X$ and any $i > 0$ one has $R^i\pi_*\mathcal{O}_Y = 0$. Recall that the Fujiki class $\mathscr{C}$ consists of all those compact complex varieties which are meromorphically dominated by a compact Kähler manifold, see [28, Definition 1.1]. This is equivalent to saying that there is a resolution of singularities by a compact Kähler manifold by [28, Lemma 1.1].

The following lemma is well known; we refer to [5, Lemma 2.1] for a proof.

Lemma 2.1. *Let $\pi : Y \to X$ be a proper bimeromorphic morphism where X is a complex variety with rational singularities. Then, $\pi^* : H^1(X,\mathbb{Z}) \to H^1(Y,\mathbb{Z})$ is an isomorphism and the sequence*

$$0 \longrightarrow H^2(X,\mathbb{Z}) \xrightarrow{\pi^*} H^2(Y,\mathbb{Z}) \longrightarrow H^0(X, R^2\pi_*\mathbb{Z})$$

is exact. In particular, if X is compact and Y is a compact manifold of Fujiki class $\mathscr{C}$, then $H^i(X,\mathbb{Z})$ carries a pure Hodge structure for $i = 1, 2$. Moreover, $\pi^*H^{1,1}(X,\mathbb{Z})$ is the subspace of $H^{1,1}(Y,\mathbb{Z})$ of all classes that vanish on the classes of π-exceptional curves.

For a complex space X, recall that $\Omega_X^{[p]}$ denotes the sheaf of reflexive p-forms:

Definition 2.2. Let X be a complex space. The module of reflexive p-forms on X is defined as

$$\Omega_X^{[p]} := (\Omega_X^p)^{\vee\vee},$$

where $F^\vee = \mathrm{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X)$ is the dual of a sheaf of $\mathcal{O}_X$-modules.

3) Except for the required results from [57], which can be quoted without modification.

If X is a reduced normal complex space and $j : U \hookrightarrow X$ denotes the inclusion of the regular locus, then $\Omega_X^{[p]} = j_*\Omega_U^p$. For a resolution of singularities $\pi : Y \to X$ we moreover have $\pi_*\Omega_Y^p = \Omega_X^{[p]}$ by [54, Corollary 1.7] if in addition X has rational singularities. If finally X is also of Fujiki class $\mathscr{C}$, then for $p + q \leq 2$ the graded pieces of the Hodge filtration can be identified with $H^q(X, \Omega_X^{[p]})$, see e.g. [5, Corollary 2.3].

2.3. Kähler spaces. The notion of a Kähler complex space, which we now recall, is due to Grauert [32, Section 13, 3., p. 346]. Recall that a *smooth function* on a complex space Z is by definition just a function $f : Z \to \mathbb{R}$ such that under a local holomorphic embedding of Z into an open set $U \subset \mathbb{C}^n$, there is a smooth (i.e., C^∞) function on U (in the usual sense) that restricts to f on Z.

Definition 2.4. Let Z be a complex space. A *Kähler form* for Z is given by an open covering $Z = \bigcup_{i\in I} U_i$ and smooth strictly plurisubharmonic functions $\varphi_i : U_i \to \mathbb{R}$ such that on $U_{ij} := U_i \cap U_j$ the function $\varphi_i|_{U_{ij}} - \varphi_j|_{U_{ij}}$ is pluriharmonic, i.e., locally the real part of a holomorphic function.

There are two important sheaves related to Kähler forms. We denote by PH_Z the sheaf of pluriharmonic functions on Z and by C_Z^∞ the sheaf of smooth real-valued functions on Z. Then we have the sequences

$$0 \to \mathrm{PH}_Z \to C_Z^\infty \to C_Z^\infty/\mathrm{PH}_Z \to 0 \tag{2.1}$$

and

$$0 \to \mathbb{R}_Z \xrightarrow{i} \mathcal{O}_Z \xrightarrow{R} \mathrm{PH}_Z \to 0, \tag{2.2}$$

where i stands for multiplication by $\sqrt{-1}$ and R is given by taking the real part. Thus, a Kähler form on Z gives rise to an element $\omega \in H^0(Z, C_Z^\infty/\mathrm{PH}_Z)$. Successively applying the connecting homomorphisms of (2.1) and (2.2), we obtain two classes $[\omega] \in H^1(Z, \mathrm{PH}_Z)$ and $[\omega] \in H^2(Z, \mathbb{R})$. The latter is called *the Kähler class of* ω.

Definition 2.5. Let Z be a reduced complex space. A Kähler class on Z is a class $\kappa \in H^2(Z, \mathbb{R})$ which is the Kähler class of some Kähler form on Z. The Kähler cone is the set

$$\mathcal{K}_Z := \{\alpha \in H^2(Z, \mathbb{R}) \mid \alpha = [\omega] \text{ for some Kähler form } \omega\}.$$

Remark 2.6. There are several things we wish to observe.

(1) It follows from the definition that for a compact complex space Z the Kähler cone $\mathcal{K}_Z$ is open in the image of $H^1(Z, \mathrm{PH}_Z) \to H^2(Z, \mathbb{R})$. Indeed, being strictly plurisubharmonic is stable under small perturbations and $H^0(Z, C_Z^\infty/\mathrm{PH}_Z) \to H^1(Z, \mathrm{PH}_Z)$ is surjective as C_Z^∞ is a fine sheaf.

(2) We can describe the Kähler forms alternatively as follows: these are Kähler forms ω on Z^{reg} in the usual sense such that for every $p \in Z$ there is an open neighborhood $p \in U \subset Z$ and a closed embedding $U \hookrightarrow V$ into a smooth Kähler manifold, where the restriction of the Kähler form of V to U^{reg} equals $\omega|_{U^{\mathrm{reg}}}$.

(3) Let us observe that by applying the (real) operator $i\partial\bar\partial$ a Kähler form also gives rise to a global section of $\mathcal{A}_Z^{1,1}$, where $\mathcal{A}_Z^{p,q}$ denotes the sheaf of smooth (p,q)-forms with

$\mathbb{C}$-coefficients on Z – which is defined in the same manner as the sheaf of C^∞-functions. This is because $\partial\bar{\partial}\varphi_i = \partial\bar{\partial}\varphi_j$ on U_{ij} as $\partial\bar{\partial}$ annihilates pluriharmonic functions. The cohomology class of ω in $H^2(Z, \mathcal{A}_Z^\bullet)$ is the image of the Kähler class under the natural map induced by the morphism $\mathbb{R}_Z \to \mathcal{A}_Z^\bullet$.

Let us recall the following properties of Kähler spaces. We will use throughout the text, sometimes without explicit mention.

Proposition 2.7. *The following statements hold:*

(1) *Every subspace of a Kähler space is Kähler.*

(2) *A smooth complex space is Kähler if and only if it is a Kähler manifold in the usual sense.*

(3) *Every reduced Kähler space has a resolution of singularities by a Kähler manifold.*

Proof. This is a consequence of [86, Section II, 1.3.1 Proposition]. □

The proposition in particular implies that compact Kähler spaces are of Fujiki class $\mathscr{C}$ so that their singular cohomology groups carry a mixed Hodge structure. For $X \in \mathscr{C}$, we may thus define

$$\begin{aligned} H^{k,k}(X, \mathbb{R}) &:= \operatorname{Hom}(\mathbb{R}(-k), H^{2k}(X, \mathbb{R})) \\ &= F^k H^{2k}(X, \mathbb{C}) \cap H^{2k}(X, \mathbb{R}). \end{aligned} \tag{2.3}$$

Note that the weights that show up in the mixed Hodge structure on $H^k(X, \mathbb{Z})$ are $\leq k$ – the argument for class $\mathscr{C}$ varieties is the same as in the algebraic case, cf. [78, Theorem 5.39].

Proposition 2.8. *Let X be a reduced compact Kähler space. Then $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$.*

Proof. The claim is easily verified using a construction of Ancona and Gaveau [2] some properties of which we briefly recall. In this proof, all references are to [2] if not mentioned otherwise. For a reduced complex space X, in [2, Section II.2] they construct a complex $\Lambda_X^\bullet$ which is a fine resolution of the constant sheaf $\mathbb{C}_X$. In fact, $\Lambda_X^\bullet$ is not unique but we may fix one such complex once and for all. A section of $\Lambda_X^\bullet$ by [2, Section II.2, Definition 2.1] is a collection of differential forms (of shifted degrees) on an associated hypercovering $\{X_\ell \to X\}_{\ell \in L}$, where the X_ℓ are smooth. In [2, Section II.3] they use this complex to construct Deligne's mixed Hodge structure on $H^k(X, \mathbb{Z})$ if X is Kähler (or more generally of Fujiki class $\mathscr{C}$). As discussed in [2, Section II.2.8], the complex $\mathcal{A}_X^\bullet$ of smooth differential forms on X (introduced in Remark 2.6 above) is a subcomplex of $\Lambda_X^\bullet$ and this inclusion clearly sends the filtration

$$F^p \mathcal{A}_X^k := \bigoplus_{r \geq p} \mathcal{A}_X^{r,k-r}$$

to the Hodge filtration. For a Kähler form $\omega = \{\varphi_i\}_{i \in I} \in H^0(X, C_X^\infty / \mathrm{PH}_X)$, the claim now follows, because $\sqrt{-1}\partial\bar{\partial}\{\varphi_i\}_{i \in I} \in F^1 \mathcal{A}_X^1(X)$. □

Observe that if in addition X has rational singularities, the claim of the proposition simply follows from Lemma 2.1 and strictness of the pullback for the Hodge filtration.

2.9. Big and nef cohomology classes. We briefly recall the definition of $\partial\bar\partial$-cohomology for a complex manifold X. As before, we denote $\mathcal{A}_X^k$ respectively $\mathcal{A}_X^{p,q}$ the sheaf of differential k-forms respectively (p,q)-forms with values in $\mathbb{C}$. Then $\partial\bar\partial$-cohomology is defined as

$$H^{p,q}_{\partial\bar\partial}(X) := \frac{\ker\bigl(d : \mathcal{A}_X^{p,q}(X) \to \mathcal{A}_X^{p+q+1}(X)\bigr)}{\operatorname{im}\bigl(i\partial\bar\partial : \mathcal{A}_X^{p-1,q-1}(X) \to \mathcal{A}_X^{p,q}(X)\bigr)}.$$

Similarly, we write $H^{p,p}_{\partial\bar\partial}(X,\mathbb{R})$ if we take cohomology of $\mathbb{R}$-valued differential forms (which is different from zero only for $p = q$). Note that $i\partial\bar\partial$ in the above formula defines a real operator.

In algebraic geometry, bigness and nefness are important notions for line bundles. In the complex analytic world, these notions can also be defined for real cohomology classes as we now recall.

Definition 2.10. Let X be a compact complex manifold. We call a cohomology class $\alpha \in H^{1,1}_{\partial\bar\partial}(X,\mathbb{R})$ *nef* if for some hermitian form ω on X and for every $\varepsilon > 0$ it can be represented by a smooth $(1,1)$-form η_ε such that $\eta_\varepsilon \geq -\varepsilon\omega$. A *Kähler current* is a closed positive $(1,1)$-current T such that $T \geq \omega$ in the sense of currents. A class $\alpha \in H^{1,1}_{\partial\bar\partial}(X)$ is called *big* if it can be represented by a Kähler current.

We refer to [37, Chapter 3, 1.] or [20, Chapter 1] for a general reference on currents and notions of positivity.

Remark 2.11. On compact manifolds of Fujiki class $\mathscr{C}$ (in particular on compact Kähler manifolds) the natural map from $\partial\bar\partial$-cohomology to de Rham cohomology is injective and gives an identification of $H^{p,q}_{\partial\bar\partial}(X)$ with $H^{p,q}(X)$. This follows directly from the $\partial\bar\partial$-lemma, see e.g. [18, (5.21) and (5.22) Theorem] for manifolds of class $\mathscr{C}$.

2.12. $\mathbb{Q}$-factoriality. Let us spend a moment to discuss the notion of $\mathbb{Q}$-factoriality. A normal algebraic variety Z is called $\mathbb{Q}$-*factorial* if for every Weil divisor D on Z there is $m \in \mathbb{N}$ such that mD is Cartier. In the algebraic category, $\mathbb{Q}$-factoriality is local for the Zariski topology. Recall from [41, Proposition 2.7] that Weil divisor classes are in bijective correspondence with isomorphism classes of reflexive sheaves of rank one: to a Weil divisor D on Z one associates the sheaf $\mathcal{O}_Z(D)$ defined by

$$U \mapsto \mathcal{O}_Z(D)(U) := \{f \in \mathbb{C}(Z) \mid D|_U + \operatorname{div}(f|_U) \geq 0\},$$

which is easily seen to be reflexive. So $\mathbb{Q}$-factoriality can be equivalently characterized using reflexive sheaves.

Finally, assume that Z is compact, let $\pi : Z' \to Z$ be a resolution of singularities, and let $E_1, \dots, E_m$ be the prime divisors contained in the exceptional locus $\operatorname{Exc}(\pi)$. By [58, (12.1.6) Proposition], the variety Z is $\mathbb{Q}$-factorial if and only if

$$\operatorname{im}\bigl(H^2(Z',\mathbb{Q}) \to H^0(Z, R^2\pi_*\mathbb{Q}_{Z'})\bigr) = \operatorname{im}\left(\bigoplus_{i=1}^m \mathbb{Q}[E_i] \to H^0(Z, R^2\pi_*\mathbb{Q}_{Z'})\right). \tag{2.4}$$

See also [73, Section 12 (i)] for an argument for the *only if*-direction. We summarize this discussion as follows.

Lemma 2.13. *Let Z be a normal algebraic variety over $\mathbb{C}$. Then the following are equivalent:*

(1) *Z is $\mathbb{Q}$-factorial.*

(2) *Every Zariski open subset $U \subset Z$ is $\mathbb{Q}$-factorial.*

(3) *For every reflexive sheaf L on Z of rank 1, there is $n \in \mathbb{N}$ such that $(L^{\otimes n})^{\vee\vee}$ is invertible.*

If in addition Z is compact and has rational singularities, the above statements are equivalent to:

(4) *Equality* (2.4) *holds for some resolution $\pi : Z' \to Z$.*

Proof. For the equivalence of (1) and (3) one only needs that for a Weil divisor D on Z we have

$$\mathcal{O}_Z(nD) = (\underbrace{\mathcal{O}_Z(D) \otimes \ldots \otimes \mathcal{O}_Z(D)}_{n\text{-times}})^{\vee\vee}$$

which can be obtained by pushforward and the fact that it holds on the regular part. □

In the analytic category, the situation is a little more subtle. We have several different notions which turn out to be non-equivalent, see Proposition 2.15 and Example 9.3.

Definition 2.14. A normal complex analytic variety Z is called *divisorially $\mathbb{Q}$-factorial* if for every Weil divisor D on Z there is $m \in \mathbb{N}$ such that mD is Cartier and it is called *$\mathbb{Q}$-factorial* if for every reflexive sheaf L on Z of rank 1, there is $n \in \mathbb{N}$ such that $(L^{\otimes n})^{\vee\vee}$ is invertible. We say that Z is *locally analytically (divisorially) $\mathbb{Q}$-factorial* if every open set $U \subset X$ in the Euclidean topology is (divisorially) $\mathbb{Q}$-factorial.

Clearly, local analytic (divisorial) $\mathbb{Q}$-factoriality implies (divisorial) $\mathbb{Q}$-factoriality. The converse however is not true. The reason is that there are usually many more local divisors than global divisors, e.g. one cannot obtain a global divisor by taking the closure of a divisor on a small open subset. There might be no global divisors at all, see e.g. Example 9.3, which is also the reason why divisorial $\mathbb{Q}$-factoriality is not the right property to ask for and one should rather work with $\mathbb{Q}$-factoriality (defined in terms of rank one reflexive sheaves).

Proposition 2.15. *Let Z be a normal complex analytic variety and consider the following statements:*

(1) *Z is locally analytically $\mathbb{Q}$-factorial.*

(2) *Z is locally analytically divisorially $\mathbb{Q}$-factorial.*

(3) *Z is $\mathbb{Q}$-factorial.*

(4) *Z is divisorially $\mathbb{Q}$-factorial.*

Then we have the following implications:

$$\begin{array}{ccc} (1) & \Longrightarrow & (3) \\ \Downarrow & & \Downarrow \\ (2) & \Longrightarrow & (4). \end{array}$$

Moreover, suppose Z is also compact of class $\mathscr{C}$ with rational singularities. Then Z is $\mathbb{Q}$-factorial if and only if for some resolution $\pi : Z' \to Z$ we have

$$\operatorname{im}\big(\operatorname{Pic}(Z')_{\mathbb{Q}} \to H^0(Z, R^2\pi_*\mathbb{Q}_{Z'})\big) = \operatorname{im}\left(\bigoplus_{i=1}^{m} \mathbb{Q}[E_i] \to H^0(Z, R^2\pi_*\mathbb{Q}_{Z'})\right). \tag{2.5}$$

Proof. The implications (1) ⇒ (3) ⇒ (4) and (1) ⇒ (2) ⇒ (4) are immediate.

The last part is a slight adaption of Kollár and Mori [58, (12.1.6) Proposition], replacing (2.4) with (2.5) which is what is actually used there. Briefly, if Z is $\mathbb{Q}$-factorial, then for any line bundle M on Z', the sheaf $L := (\pi_*M)^{\vee\vee}$ is reflexive and therefore $\pi^*((L^k)^{\vee\vee}) \cong M^k(E)$ for some divisor E whose support is contained in the exceptional locus. Hence,

$$\operatorname{Pic}(Z')_{\mathbb{Q}} = \pi^* \operatorname{Pic}(Z)_{\mathbb{Q}} + \sum_i \mathbb{Q}[E_i],$$

which implies (2.5). Conversely, if (2.5) is satisfied, then for any rank one reflexive sheaf L on Z we can find a divisor E whose support is contained in the exceptional locus and for which $M := (\pi^*L)^{\vee\vee}(E)$ is numerically trivial on fibers. But then by [58, (12.1.4) Proposition], $\pi_*(M^k)$ is a line bundle for some k, and therefore by normality $(L^k)^{\vee\vee}$ is invertible. □

3. Symplectic varieties

For the remainder of this paper, we will use the term (primitive) symplectic variety in the following sense.

Definition 3.1. Following Beauville [7], a *symplectic variety* is a pair (X, σ) consisting of a normal variety X and a closed holomorphic symplectic form $\sigma \in H^0(X^{\mathrm{reg}}, \Omega_X^2)$ on X^{reg} such that there is a resolution of singularities $\pi : Y \to X$ for which $\pi^*\sigma$ extends to a holomorphic form on Y. A *primitive symplectic variety* is a normal compact Kähler variety X such that $H^1(X, \mathcal{O}_X) = 0$ and $H^0(X^{\mathrm{reg}}, \Omega_X^2) = \mathbb{C}\sigma$ such that (X, σ) is a symplectic variety.

Greb, Kebekus and Peternell introduced a notion of irreducible holomorphic-symplectic variety (more restrictive than ours) in [36, Definition 8.16] which serves as one of the three building blocks in a decomposition theorem (due to Druel, Greb, Guenancia, Höring, Kebekus and Peternell, see introduction for references). Matsushita [66, Definition 1.6] introduced the related notion of cohomologically irreducible symplectic varieties. The definition we use here appeared before in Schwald [84, Definition 1] for projective varieties under the name irreducible symplectic. We chose to work with the above definition because it seems to be the most general framework that allows for a general moduli and deformation theory similar to the smooth case. We prefer however the name *primitive* over irreducible symplectic for the lack of a decomposition theorem. This fits together with Menet's usage [67, Definition 3.1].

Example 3.2. (1) If X is a primitive symplectic variety, then so is:

- any contraction, that is, X' for any proper bimeromorphic $f : X \to X'$ onto a normal Kähler space,
- any quotient of X by a finite group of symplectic automorphisms [7, Proposition 2.4],
- any small locally trivial deformation (see Corollary 4.11 below).

(2) By Nikulin [74] any symplectic involution ι of a K3 surface S has eight fixed points. The quotient X of the Hilbert scheme $S^{[n]}$ of $n \geq 3$ points by ι has $\mathbb{Q}$-factorial terminal singularities by [59, Proposition 5.15] and Theorem 3.4 (3) below.

For $n = 2$, X has $\binom{8}{2} = 28$ isolated singularities and a K3 surface of transverse A_1 singularities, corresponding to the 28 fixed reduced subschemes and the closure of the locus of reduced orbits, respectively (see for example [13, Section 16]). It is therefore not terminal. The $\mathbb{Q}$-factorial terminalization Y is obtained by blowing up the K3 surface. The second Betti number of X is 15, and so the locally trivial deformation space of X is 13-dimensional while Y deforms in one dimension higher (see Theorem 4.7 below). A complete projective family of this deformation type is produced in [14]; see [27] for some other "new" deformation types.

(3) There is a cubic fourfold $Z \subset \mathbb{P}^5$ with an order 11 automorphism (see for example [68]). Its Fano variety of lines F has a symplectic automorphism σ with isolated fixed points, and the quotient $X = F/\sigma$ is a $\mathbb{Q}$-factorial terminal primitive symplectic variety with $b_2 = 3$. It follows from [67, Theorem 3.17 and Theorem 5.4] that the only deformation of X is the twistor deformation.

(4) Let S be a projective K3 surface, and $v \in H^*(S, \mathbb{Z})$ an algebraic Mukai vector with $v^2 > 0$. Then for $k \geq 1$, the moduli space $X = M(kv)$ of stable sheaves of Mukai vector kv with respect to a generic polarization is a primitive symplectic variety. Moreover, X is always locally factorial and terminal [51, Theorem A] unless $k = 2$ and $v^2 = 2$ (in which case X admits a resolution by an irreducible symplectic manifold – the O'Grady tenfold [76]). The singularities of $M(kv)$ can be non-quotient singularities, as the completions of the local rings are often not (even analytically) $\mathbb{Q}$-factorial – see [51, Remark 6.3]. This is because analytically locally or étale locally, these examples admit small crepant resolutions (but not globally).

(5) Forthcoming work of Saccà [80] shows using MMP techniques that a (projective) Lagrangian fibration which extends in codimension 2 admits a compactification as a $\mathbb{Q}$-factorial terminal symplectic variety. This for example applies to show that if $f : X \to B$ is a Lagrangian fibration of a smooth (projective) irreducible symplectic variety which is smooth over $U \subset B$, then any fibration isogenous to $f^{-1}(U) \to U$ admits such a compactification.

(6) For a possibly singular cubic fourfold $Y \subset \mathbb{P}^5$ not containing a plane, it was shown in [62, Theorem 3.3] that the variety $M_1(Y)$ of lines on Y is a symplectic variety birational to the second punctual Hilbert scheme of an associated K3 surface. Hence, $M_1(Y)$ admits a crepant resolution by an irreducible symplectic manifold, see [62, Corollary 5.6]. A similar statement is deduced for the target space $Z(Y)$ of the MRC-fibration of the Hilbert scheme compactification of the space of twisted cubics on Y, see [62, Theorem 1.1, Corollaries 5.5 and 6.2].

Note that even for smooth X the notion of a primitive symplectic variety is a priori more general than that of an irreducible symplectic manifold. However, we do not know if there are smooth primitive symplectic varieties which are not irreducible symplectic manifolds. By Lemma 3.3 below such a variety must have dimension ≥ 6.

Lemma 3.3. *Let X be a smooth primitive symplectic variety of dimension ≤ 4. Then X is an irreducible symplectic manifold (in the classical sense).*

Proof. For $\dim X = 2$ this is well known, so let us assume $\dim X = 4$.

If X is a smooth primitive symplectic variety in our sense, the Beauville–Bogomolov decomposition theorem yields that a finite topological cover $\widetilde{X} \to X$ of X splits as a product

$\widetilde{X} \cong H \times C \times T$, where H is a product of irreducible symplectic manifolds, C a product of strict Calabi–Yau varieties, and T a complex torus. From the existence of a symplectic form on $\widetilde{X}$ (by pullback from X) we deduce that the factor C is trivial.

By assumption, $H^1(\mathcal{O}_X) = 0$ and thus $H^3(\mathcal{O}_X) = 0$ by Serre duality. Moreover, by the unicity of the symplectic form we in fact have $\chi(\mathcal{O}_X) = 3$. If there is a torrus factor, then $\chi(\mathcal{O}_{\widetilde{X}}) = 0$ contradicting $\chi(\mathcal{O}_{\widetilde{X}}) = d\chi(\mathcal{O}_X)$, where d is the degree of the cover, so the factor T is trivial. If $\widetilde{X}$ is a product of K3 surfaces, then $\chi(\mathcal{O}_{\widetilde{X}}) = 4$, which is impossible. Thus, $\widetilde{X}$ is irreducible symplectic, so that $d = 1$, and thus X is irreducible symplectic as well. □

It is unclear whether the statement of Lemma 3.3 holds in higher dimensions. It is worthwhile noting that there is a singular example of a primitive symplectic variety due to Matsushita [65], see also [83, Lemma 15] and [84, Example 29], which has the right cohomological invariants but is a torus quotient. Schwald's account nicely illustrates how the geometry of primitive symplectic varieties may deviate from the one of irreducible symplectic manifolds.

We collect the following basic results about symplectic varieties which are due to work of Beauville, Kaledin, and Namikawa; we give precise references in the proof.

Theorem 3.4 (Beauville, Kaledin, and Namikawa). *The following statements hold:*

(1) *A normal variety is symplectic if and only if it has only rational Gorenstein singularities and its smooth part admits a holomorphic symplectic form. In particular, a symplectic variety has rational singularities.*

(2) *Let X be a symplectic variety and consider the stratification $X = X_0 \supset X_1 \supset \dots$, where X_{i+1} is the singular part of X_i endowed with the reduced structure. Then the normalization of every irreducible component of X_i is a symplectic variety. In particular, the singular locus of a symplectic variety has even codimension.*

(3) *A symplectic variety X has terminal singularities if and only if* $\operatorname{codim}_X X^{\text{sing}} \geq 4$.

Proof. At least for algebraic varieties, this result is well known. We give a sketch of the argument and comment on why the arguments hold in the analytic context as well.

(1) The *only if* direction is proven in [7, Proposition 1.3] and is valid in the analytic context as well. The converse follows from [54, Corollary 1.7].

(2) The existence of the stratification is [50, Theorem 2.3]. It is not claimed there that $X_{i+1} = (X_i^{\text{sing}})_{\text{red}}$, however, that is how the stratification is constructed, see [50, Proposition 3.1]. The decomposition a priori only holds on the formal level by Kaledin's result, however by [4, Corollary (1.6)] a formal isomorphism implies the existence of an isomorphism of analytic germs. We refer to Remark 3.6 for why Kaledin's results also apply in the analytic situation.

(3) For algebraic varieties, this statement is [69, Corollary 1]. The proof is a bit involved so we take the opportunity to use Kebekus–Schnell's functorial pullback of reflexive differential forms and Kaledin's decomposition theorem to write down a simple proof that also works in the analytic setting. We do not claim originality, the argument expands on an observation by Namikawa (see [69, footnote on page 1 and Section 11]).

By [50, Theorem 2.3], the codimension of the singular locus is even, and if $x \in X^{\text{sing}}$ is a general point of an irreducible component of X^{sing} of codimension 2, the germ (X, x) is isomorphic to the product of a smooth germ and the germ of rational double point. Such a product however does not have terminal singularities. If codim $X^{\mathrm{sing}} \geq 4$, we take a resolution $\pi : Y \to X$ and assume that $E \subset Y$ is a divisor with vanishing discrepancy. Then Y is symplectic at the generic point of E and $\pi(E) \subset X^{\mathrm{sing}}$. Let us consider a diagram

$$\begin{array}{ccccc} E' & \longrightarrow & E & \hookrightarrow & Y \\ & & \psi & & \\ \pi' \downarrow & & & & \downarrow \pi \\ \Sigma & & \xrightarrow{\quad \varphi \quad} & & X, \end{array}$$

where Σ is a resolution of $\pi(E)$ and $E' \to E$ is a resolution. Then by [54, Theorem 14.1] one can pullback the symplectic form along φ such that $\pi'^*\varphi^*\sigma = \psi^*\pi^*\sigma$. The pullback $\psi^*\pi^*\sigma$ has one-dimensional radical at the general point of E' and $\varphi^*\sigma$ is generically symplectic by Kaledin's result. This is a contradiction to $\dim \Sigma \leq \dim X - 4$. □

As a direct consequence of Theorem 3.4 and Lemma 2.1 we infer:

Corollary 3.5. *Let X be a compact symplectic variety. Then the Hodge structure on $H^2(X, \mathbb{Z})_{\mathrm{tf}}$ is pure.* □

Remark 3.6. Kaledin's article [50] is formulated for complex algebraic varieties, but his results are used in Theorem 3.4 for arbitrary symplectic varieties. Let us comment on why they carry over to the analytic setting. The crucial ingredient from *algebraic* geometry in Kaledin's proofs is the use of functorial mixed Hodge structures on cohomology groups of complex projective algebraic varieties and there is no such structure on the cohomology of arbitrary complex varieties. However, Kaledin only uses it for fibers of resolutions of singularities which, also in the analytic category, can be chosen projective. Actually, these fibers are always compact complex varieties of Fujiki class $\mathscr{C}$, which is sufficient.

With this in mind, Kaledin's proofs work almost literally for analytic varieties. More precisely, one first shows using mixed Hodge structures that Kaledin's proofs yield analogs of [50, Lemma 2.7] and [50, Lemma 2.9] in the analytic setting. These are the key technical ingredients to prove the stratification and formal product decomposition [50, Theorem 2.3] as well as [50, Theorem 2.5] which relates the symplectic and Poisson structure. Other than mixed Hodge theory, Kaledin mainly uses Poisson structures, commutative algebra, or direct geometric arguments which all make sense also in our setting. Finally, also semi-smallness (see [50, Lemma 2.11]) is a consequence of geometric properties of the symplectic form and [50, Lemma 2.9].

4. Deformation theory

Definition 4.1. A deformation of a compact complex space Z is a flat and proper morphism $\mathscr{X} \to S$ of complex spaces together with a distinguished point $0 \in S$ and an isomorphism of the fiber of $\mathscr{X} \to S$ over 0 with Z. A deformation $\pi : \mathscr{X} \to S$ is called *locally trivial* at $0 \in S$ if for every $p \in Z = \pi^{-1}(0)$ there exist open neighborhoods $\mathscr{U} \subset \mathscr{X}$ of p and $S_0 \subset S$ of 0 such that $\mathscr{U} \cong U \times S_0$ over S_0, where $U = \mathscr{U} \cap Z$. The deformation is called locally trivial if it is locally trivial at each point of S. We speak of a *locally trivial family* or *locally trivial morphism* $\pi : \mathscr{X} \to S$ if we do not specify $0 \in S$ and the fiber over it.

For most properties and statements we should rather speak about the morphism of space germs $(\mathscr{X}, Z) \to (S, 0)$. All deformation theoretic statements have to be interpreted as statements about germs. Considering deformations and locally trivial deformations gives rise to two deformation functors; in fact, the functor $\mathcal{D}_Z^{\mathrm{lt}}$ of locally trivial deformations of Z is a subfunctor in the functor $\mathcal{D}_Z$ of all deformations of Z. They have tangent spaces $T_{\mathcal{D}_Z^{\mathrm{lt}}} = H^1(Z, T_Z)$ and if Z is reduced $T_{\mathcal{D}_Z} = \mathrm{Ext}^1(\Omega_Z, \mathcal{O}_Z)$, respectively. Note that $H^1(Z, T_Z)$ is a subset of $\mathrm{Ext}^1(\Omega_Z, \mathcal{O}_Z)$ by the local-to-global spectral sequence for Ext. We refer to [85, Proposition 1.2.9] (which actually works for arbitrary schemes) respectively [85, Theorem 2.4.1 (iv)]. Even though Sernesi's book treats deformations of algebraic schemes, the arguments apply literally for deformations of complex spaces, mainly because zero-dimensional complex spaces are nothing else but zero-dimensional $\mathbb{C}$-schemes of finite type.

4.2. Versality and universality. Recall that a deformation $(\mathscr{X}, Z) \to (S, 0)$ is called *versal* if for every deformation $(\mathscr{X}', Z) \to (S', 0)$ of Z there is a map $\varphi : (S', 0) \to (S, 0)$ of (germs of) complex spaces such that $\mathscr{X} \times_S S' \cong \mathscr{X}'$. It is called *miniversal* if moreover the differential $T_{\varphi,0} : T_{S',0} \to T_{S,0}$ is uniquely determined. The deformation is called *universal* if furthermore the map φ is unique. Clearly, every universal deformation is miniversal and every miniversal deformation is versal. The different notions of versality are defined analogously for other deformation problems such as locally trivial deformations.

4.3. Existence of a miniversal deformation. Recall that miniversal[4)] deformations exist by [33, Hauptsatz, p. 140], see also [22, Théorème principal, p. 598]. More precisely, it is shown in [33] that there exist miniversal deformations $\mathscr{X} \to S$ of a given compact complex space Z which are versal in every point of S. We will frequently write $S = \mathrm{Def}(Z)$. The family $\mathscr{X} \to \mathrm{Def}(Z)$ is called the *Kuranishi family* and $\mathrm{Def}(Z)$ is called *Kuranishi space*.

If Z is a complex space satisfying $H^0(Z, T_Z) = 0$, then every miniversal deformation is universal.

4.4. Locally trivial miniversal deformations. Recall from [26, (0.3) Corollary] that for a miniversal deformation $\mathscr{X} \to \mathrm{Def}(Z)$ of a compact complex space Z there exists a closed complex subspace $\mathrm{Def}^{\mathrm{lt}}(Z) \subset \mathrm{Def}(Z)$ of the Kuranishi space parametrizing locally trivial deformations of Z. More precisely, the restriction of the miniversal family to this subspace, which by abuse of notation we denote also by $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(Z)$, is a locally trivial deformation of Z and is miniversal for locally trivial deformations of Z. When speaking about locally trivial deformations we will usually use the terms versal, miniversal, universal with respect to the functor of locally trivial deformations.

Lemma 4.5. *Let S be a complex space and let $f : \mathscr{X} \to S$ be a locally trivial family whose fiber X above a point $0 \in S$ is a primitive symplectic variety. Denote by $j : \mathscr{U} \to \mathscr{X}$ the inclusion of the regular locus. Then in a neighborhood of $0 \in S$ we have:*

(1) *$L := (f \circ j)_* \Omega^2_{\mathscr{U}/S}$ is an invertible sheaf and compatible with arbitrary base change.*

(2) *The following natural map is an isomorphism:*

$$T_{\mathscr{X}/S} \otimes f^* L \to j_* \Omega_{\mathscr{U}/S}. \tag{4.1}$$

[4)] Note that Grauert uses the term complete (resp. versal) for what we call versal (resp. miniversal). Nowadays, our terminology seems to be more common; some authors use semi-universal instead of miniversal.

Proof. By local triviality, the sheaves $j_*\Omega^p_{\mathscr{U}/S}$, $j_*T_{\mathscr{U}/S}$, $\Omega_{\mathscr{X}/S}$, $T_{\mathscr{X}/S}$ are all flat over S and compatible with arbitrary base change. As push forward is compatible with flat base change, invertibility of L can be tested on the completion. By the theorem on formal functions we may reduce (1) to the case where S is the spectrum of an artinian local $\mathbb{C}$-algebra of finite type. Then by the primitivity assumption on X and [5, Lemma 2.4], the sheaf L is invertible and compatible with arbitrary base change in a neighborhood of 0. As every section of L determines a morphism $T_{\mathscr{U}/S} \to \Omega_{\mathscr{U}/S}$, we obtain a canonical morphism $j_*T_{\mathscr{U}/S} \otimes f^*L \to j_*\Omega_{\mathscr{U}/S}$ and (4.1) is just the composition with $T_{\mathscr{X}/S} \to j_*T_{\mathscr{U}/S}$ tensored with the pullback of L. It then follows that (4.1) is an isomorphism in a neighborhood of 0 because it is over the special fiber. □

Lemma 4.6. *Let X be a primitive symplectic variety. Then $H^0(X, T_X) = 0$ and every miniversal deformation of X is universal.*

Proof. Let $\pi : Y \to X$ be a resolution of singularities by a Kähler manifold and denote by $j : U \hookrightarrow X$ the inclusion of the regular part. Then we have $T_X \cong \pi_*\Omega_Y$ by Lemma 4.5 and [54, Corollary 1.8]. Consequently,

$$H^0(X, T_X) = H^0(Y, \Omega_Y) \cong H^{1,0}(Y)$$

by the Dolbeault isomorphism and the complex conjugate of the latter is

$$H^{0,1}(Y) \cong H^1(Y, \mathcal{O}_Y) = H^1(X, \mathcal{O}_X)$$

again by Dolbeault and by rationality of singularities. We conclude the proof with the observation that $H^1(X, \mathcal{O}_X) = 0$ by definition of a primitive symplectic variety. □

The proof of the following result is similar to the proof of [5, Theorem 4.1]. For lack of a crepant resolution, some minor changes are necessary which is why we include a proof.

Theorem 4.7. *Let X be a primitive symplectic variety. Then the space $\mathrm{Def}^{\mathrm{lt}}(X)$ of locally trivial deformations of X is smooth of dimension $h^{1,1}(X)$.*

Proof. Smoothness is deduced using Kawamata–Ran's T^1-lifting principle [52, 53, 79], see also [38, Section 114], [61], [60, VI.3.6] for more details. We have to show the following. Let $\mathscr{X} \to S$ be a locally trivial deformation of X, where $S = \operatorname{Spec} R$ for some Artinian local $\mathbb{C}$-algebra R with residue field $\mathbb{C}$, let $S' \subset S$ be a closed subscheme, and let

$$\mathscr{X}' := \mathscr{X} \times_S S' \to S'$$

be the induced deformation. Then we need to prove that the canonical morphism

$$H^1(T_{\mathscr{X}/S}) \to H^1(T_{\mathscr{X}'/S'})$$

is surjective.

Let $j : \mathscr{U} \hookrightarrow \mathscr{X}$ the inclusion of the regular part. By Lemma 4.5, it suffices to show that $H^1(j_*\Omega_{\mathscr{U}/S}) \to H^1(j'_*\Omega_{\mathscr{U}'/S'})$ is surjective where $j : \mathscr{U}' = \mathscr{U} \times_S S' \to \mathscr{X}'$ is the regular part of $\mathscr{X}' \to S'$. However, by [5, Lemma 2.4] the R-module $H^1(j_*\Omega_{\mathscr{U}/S})$ is locally free and compatible with arbitrary base change. In other words,

$$H^1(j_*\Omega_{\mathscr{U}'/S'}) = H^1(j_*\Omega_{\mathscr{U}/S}) \otimes_R R',$$

where $S' = \operatorname{Spec} R'$, and the map is clearly surjective. Thus, it follows from the T^1-lifting criterion that the space $\mathrm{Def}^{\mathrm{lt}}(X)$ is smooth.

Recall that the tangent space to $\mathrm{Def}^{\mathrm{lt}}(X)$ at the origin is $H^1(T_X) \cong H^1(j_*\Omega_U)$, which by [5, Corollary 2.3] has dimension $h^{1,1}(X)$. By the smoothness assertion we proved before the dimension of the tangent space is the dimension of $\mathrm{Def}^{\mathrm{lt}}(X)$. □

As an application, we deduce the existence of a simultaneous resolution.

Definition 4.8. Let $\mathscr{X} \to S$ be a flat morphism between complex spaces with reduced and connected fibers. A *simultaneous resolution* of $\mathscr{X} \to S$ is a proper bimeromorphic S-morphism $\pi : \mathscr{Y} \to \mathscr{X}$ such that $\mathscr{Y} \to S$ is smooth. A simultaneous resolution is called *strong* if moreover π is an isomorphism over the complement of the singular locus of $\mathscr{X} \to S$.

It follows from the definition that for every $s \in S$ the fiber $\mathscr{Y}_s \to \mathscr{X}_s$ is a resolution of singularities. It is well known that simultaneous resolutions do not always exist. For example, let $f : \mathscr{X} \to S$ be a family of elliptic curves, where $\mathscr{X}$ is smooth and S is a smooth curve. Suppose that there is a point $0 \in S$ such that f is smooth over $S \setminus \{0\}$ and $\mathscr{X}_0 = f^{-1}(0)$ is a reduced nodal rational curve. If there were a simultaneous resolution $\pi : \mathscr{Y} \to \mathscr{X}$, the exceptional set of π would be a divisor $E \subset \mathscr{Y}$. Then $\pi(E) \subset \mathscr{X}$ would be a finite set which contradicts smoothness of $\mathscr{Y} \to S$ because this map would have some reducible fibers.

Lemma 4.9. *Let $\mathscr{X} \to S$ be a locally trivial deformation of a reduced compact complex space X over a reduced complex space S and let $\mathscr{U} \to S$ be the regular part of $\mathscr{X} \to S$. Then there exists a simultaneous resolution $\pi : \mathscr{Y} \to \mathscr{X}$ of $\mathscr{X}$ which is obtained by successive blowing ups along centers which are smooth over S. Moreover, π can be chosen to be an isomorphism over $\mathscr{U}$.*

Proof. By [8], resolution of singularities works algorithmically, see also [92]. Given a global embedding $X \subset M$ into a smooth space M, Bierstone and Milman define an invariant $\iota := \mathrm{inv}^e_X : M \to \Gamma$ with values in an ordered set in [8, Theorem 1.14 and Remark 1.16] such that the locus where ι is maximal is smooth and Zariski closed. As explained in [8, proof of Theorem 1.6, p. 285], successively blowing up the maximal locus of ι gives an algorithmic resolution. The invariant ι a priori depends on the embedding $X \subset M$. However, it is explained in [8, 13.] that it is in fact independent of the local embedding. It only depends on the local ring at the point and on the history of the blow up (which is how they obtain resolution results without the hypothesis of X being embedded).

Therefore, we may apply the same argument in the relative setting for locally trivial deformations. Given a point $p \in \mathscr{X}$ mapping to $s \in S$, we choose neighborhoods $\mathcal{V}$ of p in $\mathscr{X}$ and S_0 of s in S and a trivialization $\varphi : \mathcal{V} \xrightarrow{\cong} V \times S_0$, where $V = \mathcal{V} \cap \mathscr{X}_s$. The maximal locus of the Bierstone–Milman invariant ι defines a smooth closed subset $C \subset V^{\mathrm{sing}}$ of the singular locus $V^{\mathrm{sing}} \subset V$. By local triviality, the singular locus $\mathcal{V}^{\mathrm{sing}}$ of $\mathcal{V} \to S_0$ is identified under φ with $V^{\mathrm{sing}} \times S_0$. Thanks to the above mentioned independence of ι, the closed subsets $C \times S_0$ glue to give a center $\mathscr{C} \subset \mathscr{X}$ for a blow up and $\mathscr{C}$ is smooth over S. Moreover, the blow up of $\mathscr{X}$ in $\mathcal{C}$ is by construction again locally trivial over S, hence we can repeat the process and obtain the sought-for resolution $\pi : \mathscr{Y} \to \mathscr{X}$. □

Remark 4.10. As the morphism $\pi : \mathscr{Y} \to \mathscr{X}$ from the preceding lemma is obtained by successive blow ups in centers which are smooth over S, every such blow up family is locally trivial over S and moreover, also the morphism π is itself locally trivial. More precisely, for every open sets $\mathcal{V} \subset \mathscr{X}$ and $S_0 \subset S$ admitting a trivialization $\varphi : \mathcal{V} \xrightarrow{\cong} V \times S_0$ where V is the intersection of $\mathcal{V}$ with some fiber over a point of S_0, there is a trivialization $\phi : \pi^{-1}(\mathcal{V}) \to \pi^{-1}(V) \times S_0$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(\mathcal{V}) & \longrightarrow & \pi^{-1}(V) \times S_0 \\ \downarrow & & \downarrow \\ \mathcal{V} & \longrightarrow & V \times S_0 \end{array}$$

commutes (and similarly for any intermediate step of the resolution procedure).

Corollary 4.11. *Every small locally trivial deformation of a primitive symplectic variety X is a primitive symplectic variety. In particular, the locally trivial Kuranishi family of a primitive symplectic variety is universal (for locally trivial deformations) for all of its fibers.*

Proof. Let $f : \mathscr{X} \to S$ be a small locally trivial deformation of $X = f^{-1}(0)$, $0 \in S$. First note that X has canonical, hence rational singularities by Theorem 3.4, so by [71, Proposition 5], nearby fibers remain Kähler. We choose a simultaneous resolution $\pi : \mathscr{Y} \to \mathscr{X}$ over S, denote by $j : \mathscr{U} \to \mathscr{X}$ the inclusion of the regular locus, and consider the canonical morphism $f_*\pi_*\Omega^2_{\mathscr{Y}/S} \to (f \circ j)_*\Omega^2_{\mathscr{U}/S}$. Both sheaves are locally free and compatible with arbitrary base change, the former by the argument of [17, Théorème 5.5] – see e.g. [5, Lemma 2.4] for the necessary changes in the analytic category – the latter by Lemma 4.5. As X is a primitive symplectic variety, both sheaves are invertible and the above morphism is an isomorphism at the point corresponding to X, hence in a small neighborhood. We thus find a relative holomorphic 2-form ω on $\mathscr{U}$ whose pullback extends to a holomorphic 2-form on $\mathscr{Y}$. As the restriction ω_0 to the fiber $X = \mathscr{X}_0$ is nondegenerate, the same is true for the restriction ω_s to $\mathscr{X}_s$ for $s \in S$ close to 0. Hence, the nearby fibers $\mathscr{X}_s$ are symplectic varieties whose symplectic form is unique up to scalars. By semi-continuity, $H^1(\mathscr{X}_s, \mathcal{O}_{\mathscr{X}_s}) = 0$ for all s in a neighborhood of $0 \in S$, and so the first claim follows. The last claim follows directly from Lemma 4.6 and openness of versality, see [33, Hauptsatz, p 140]. □

4.12. Deformations of line bundles. Let X be a primitive symplectic variety and L a line bundle on it. We will frequently consider deformations of the pair (X, L). For this purpose one considers the morphism $d\log : \mathcal{O}_X^\times \to \Omega_X$, $f \mapsto \frac{df}{f}$ and the induced first Chern class morphism

$$\mathrm{c}_1 : H^1(X, \mathcal{O}_X^\times) \to H^1(X, \Omega_X) \to H^1(X, \Omega_X^{[1]})$$

which takes values in the cohomology of reflexive differentials. Recall that

$$H^1(X, \Omega_X^{[1]}) \cong H^{1,1}(X)$$

by [5, Corollary 2.3].

Lemma 4.13. *Let L be a nontrivial line bundle on X. Then the canonical projection* $\mathrm{Def}^{\mathrm{lt}}(X,L) \to \mathrm{Def}^{\mathrm{lt}}(X)$ *is a closed immersion and identifies* $\mathrm{Def}^{\mathrm{lt}}(X,L)$ *with a smooth hypersurface whose tangent space is equal to*

$$\ker\Big(H^1(X,T_X) \xrightarrow{\cup\, c_1(L)} H^2(X,\mathcal{O}_X)\Big),$$

where the map is given by contraction and cup product.

Proof. We have a canonical map

$$H^1(X,\Omega_X^{[1]}) = \mathrm{Ext}^1_X(\mathcal{O}_X,\Omega_X^{[1]}) \to \mathrm{Ext}^1_X(T_X,\mathcal{O}_X)$$

given by sending an extension to its dual (observe that we have $\mathcal{E}xt^1_X(\mathcal{O}_X,\mathcal{O}_X)=0$). Therefore, $c_1(L) \in H^1(X,\Omega_X^{[1]})$ gives rise to an extension

$$0 \to \mathcal{O}_X \to E_L \to T_X \to 0$$

and the sheaf E_L is shown to control the deformation theory of the pair (X,L) in the sense that $H^1(X,E_L)$ is the tangent space to the functor $\mathcal{D}_{(X,L)}$ of deformations of the pair (X,L) and $H^2(X,E_L)$ is an obstruction space, see e.g. [85, Theorem 3.3.11]. The proof there is written for nonsingular projective varieties only, however, the argument is the same for *locally trivial* deformations of compact complex spaces. The rest of the proof is exactly as in [44, 1.14]. □

5. The Beauville–Bogomolov–Fujiki form and local Torelli

In this section, we develop the theory of the Beauville–Bogomolov–Fujiki (BBF) form for primitive symplectic varieties. Thanks to previous works by several authors (see Section 5.1) such a form exists and was known to share many properties with its counterpart in the smooth case. After a brief summary of these results with no claim for originality, the first fundamentally new result is the local Torelli theorem for locally trivial deformations, see Proposition 5.5, which was established for $\mathbb{Q}$-factorial terminal varieties by Namikawa [71, Theorem 8]. With this at hand, we prove many advanced features of the BBF form that are known in the smooth case: the higher degree Fujiki relations in Proposition 5.15, a Riemann–Roch-type formula in Corollary 5.16, and the non-existence of subvarieties of odd dimension on a general deformation in Corollary 5.18.

The material developed in this section is essential in the proof of the projectivity criterion in Section 6.

5.1. The Beauville–Bogomolov–Fujiki form. Let X be a primitive symplectic variety. Due to the work of Namikawa [71], Kirschner [55], Matsushita [66], and Schwald [84] there is a nondegenerate quadratic form $q_X : H^2(X,\mathbb{R}) \to \mathbb{R}$ whose associated bilinear form has signature $(3, b_2(X)-3)$. As for irreducible symplectic manifolds, we will refer to q_X as the *Beauville–Bogomolov–Fujiki (BBF) form*, see Definition 5.4. We will use it to establish a local Torelli theorem in Proposition 5.5 and we will see in Proposition 5.15 that it satisfies analogous Fujiki relations as it does for irreducible symplectic manifolds.

We will first recall the following definition, see [55, Definition 3.2.7] and also [84, Definition 20].

Definition 5.2. Let X be a compact complex variety of Fujiki class $\mathscr{C}$ and dimension $2n$ with rational singularities let $\sigma \in H^{2,0}(X)$ be the cohomology class of a holomorphic 2-form on X^{reg} (recall from Lemma 2.1 that the Hodge structure on $H^2(X,\mathbb{Z})$ is pure). We denote by $\int_X : H^{4n}(X,\mathbb{Z}) \to \mathbb{Z}$ the cap product with the fundamental class. Then one defines a quadratic form $q_{X,\sigma} : H^2(X,\mathbb{C}) \to \mathbb{C}$ via

$$q_{X,\sigma}(\alpha) := \frac{n}{2}\int_X (\sigma\bar{\sigma})^{n-1}\alpha^2 + (1-n)\int_X \sigma^n\bar{\sigma}^{n-1}\alpha \int_X \sigma^{n-1}\bar{\sigma}^n\alpha. \tag{5.1}$$

If X is a primitive symplectic variety, one can also define a form $q_{Y,\sigma}$ on a resolution of singularities $\pi : Y \to X$ by the analog of formula (5.1), where σ is replaced by the extension of the symplectic form to Y and $q_{X,\sigma}$ is the restriction to $H^2(X,\mathbb{Q}) \subset H^2(Y,\mathbb{Q})$. This is Namikawa's approach, see [71], and both are equivalent by [84, Corollary 22]. Note that Schwald assumes X to be projective but this is in fact not used in the argument.

The following result is already contained in the work of Namikawa [71], Matsushita [65], Kirschner [55], Schwald [84]. Let us emphasize that the projectivity hypothesis which is sometimes made is in fact not necessary. Denote by $b_i(X) := \dim_{\mathbb{Q}} H^i(X,\mathbb{Q})$, $i \in \mathbb{N}_0$ the i-th Betti number.

Lemma 5.3. *Let X be as in Definition* 5.2. *Then the quadratic form*

$$q_{X,\sigma} : H^2(X,\mathbb{R}) \otimes H^2(X,\mathbb{R}) \to \mathbb{R}(-2)$$

is a morphism of $\mathbb{R}$-Hodge structures. If X is a primitive symplectic variety, then $q_{X,\sigma}$ is nondegenerate and has signature $(3, b_2(X)-3)$. *Furthermore, if σ is chosen such that $\int_X(\sigma\bar{\sigma})^n = 1$, then $q_{X,\sigma}$ does not depend on σ.*

Proof. It is immediate from (5.1) that $q_{X,\sigma}$ is defined over $\mathbb{R}$ so that the statements of the lemma make sense. The first statement is easily verified. The statement about the signature (and hence also nondegeneracy) is [84, Theorem 2]. The statement about independence of $q_{X,\sigma}$ for normalized σ is [84, Lemma 24]. □

Definition 5.4. Let X be a primitive symplectic variety of dimension $2n$ and let $\sigma \in H^{2,0}(X)$ be the cohomology class of a holomorphic symplectic 2-form on X^{reg} satisfying $\int_X(\sigma\bar{\sigma})^n = 1$. Then the *Beauville–Bogomolov–Fujiki (BBF) form* is the quadratic form $q_X := q_{X,\sigma}$, up to scaling.

It is not hard now to deduce a local Torelli theorem for locally trivial deformations. Preliminary versions have been established by Namikawa [70], Kirschner [55, Theorem 3.4.12], Matsushita [66], and the authors [5].

Proposition 5.5 (Local Torelli theorem). *Let X be a primitive symplectic variety, let q_X be its BBF form, and let*

$$\Omega(X) := \{[\sigma] \in \mathbb{P}(H^2(X,\mathbb{C})) \mid q_X(\sigma) = 0,\ q_X(\sigma,\bar{\sigma}) > 0\}$$

be the period domain for X inside $\mathbb{P}(H^2(X,\mathbb{C}))$. If $f : \mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ denotes the universal *locally trivial deformation of X and $X_t := f^{-1}(t)$, then the local period map*

$$\wp : \mathrm{Def}^{\mathrm{lt}}(X) \to \Omega(X), \quad t \mapsto H^{2,0}(X_t). \tag{5.2}$$

is a local isomorphism.

Proof. Let us denote by $j : \mathscr{U} \to \mathscr{X}$ the inclusion of the regular locus. By Lemma 4.5, the sheaf $L := (f \circ j)_* \Omega^2_{\mathscr{U}/S}$ is invertible and compatible with arbitrary base change. From this and [5, Corollary 2.3] we deduce that the subbundle $L \subset H^2(X, \mathbb{C}) \otimes \mathcal{O}_{\mathrm{Def}^{\mathrm{lt}}(X)}$ defines the period map $\mathrm{Def}^{\mathrm{lt}}(X) \to \mathbb{P}(H^2(X, \mathbb{C}))$ which therefore is holomorphic. We will argue as in [6, Théorème 5] to prove that it takes values in $\Omega(X)$. The statement is local, so it suffices to show that $q_X(\sigma_t) = 0$, where σ_t is a section of $f_* \Omega^2_{\mathscr{X}/S}$ evaluated at $t \in S$ for t sufficiently close to the origin. This is done in the same way as in the first paragraph of the proof of [6, Théorème 5 (b)]. Let $j : U \hookrightarrow X$ denote the inclusion of the regular part. It is well known that the differential of $\wp$ at zero can be described as the map

$$H^1(X, T_X) \to \mathrm{Hom}(H^0(X, j_* \Omega^2_U), H^1(X, j_* \Omega^1_U))$$

given by cup product and contraction. This is clearly an isomorphism as $H^0(X, j_* \Omega^2_U)$ is spanned by the symplectic form. Therefore, the map (5.2) is an isomorphism in a neighborhood of zero. □

Remark 5.6. Namikawa assumes $\mathbb{Q}$-factorial terminal singularities for his local Torelli theorem [71, Theorem 8], and in this case all deformations are locally trivial. Proposition 5.5 shows that in fact local triviality (and not the kind of singularities) is the essential ingredient.

The local Torelli theorem can be exploited just as for irreducible symplectic manifolds. We start with the integrality of the quadratic form.

Lemma 5.7. *The BBF form q_X is up to a multiple a nondegenerate quadratic form $H^2(X, \mathbb{Z}) \to \mathbb{Z}$. Moreover, it is invariant under locally trivial deformations.*

Proof. The second statement is a consequence of the first, so we are left to prove integrality. This is done as in [6, Théorème 5 (a)]: we deduce from the local Torelli Theorem 5.5 the following formula. For every $\lambda \in H^2(X, \mathbb{C})$ we denote $v(\lambda) := \int_X \lambda^{2n}$, where $2n = \dim X$. Note that for a locally trivial deformation $f : \mathscr{X} \to S$ of X, if λ is a section of $R^2 f_* \mathbb{C}$, then $v(\lambda)$ is locally constant as it can be computed on a simultaneous resolution. For every $\alpha \in H^2(X, \mathbb{C})$ we have

$$v(\lambda)^2 q_X(\alpha) = q_X(\lambda)\left((2n-1)v(\lambda)\int_X \lambda^{2n-2}\alpha^2 - (2n-2)\left(\int_X \lambda^{2n-1}\alpha\right)^2\right). \tag{5.3}$$

This formula immediately shows that some real multiple of q_X is defined over $\mathbb{Z}$. □

Remark 5.8. As a consequence of Lemma 5.7, we will always normalize the BBF form q_X so it is a (usually primitive) integral form.

For the sake of completeness, let us summarize a statement that is well known in the smooth case.

Corollary 5.9. *Let X be a primitive symplectic variety and let L be a line bundle on it. Under the local isomorphism* $\mathrm{Def}^{\mathrm{lt}}(X) \to \Omega(X)$ *by the period map, the subspace* $\mathrm{Def}^{\mathrm{lt}}(X, L)$ *of deformations of the pair (X, L) is identified with* $\mathbb{P}(\mathrm{c}_1(L)^{\perp}) \cap \Omega(X)$. □

We will frequently simply write $\alpha^{\perp}$ instead of $\mathbb{P}(\alpha^{\perp}) \cap \Omega(X)$ for a class $\alpha \in H^2(X, \mathbb{C})$.

5.10. A theorem of Verbitsky. Let X be a primitive symplectic variety of dimension $2n = \dim X$. In Section 7 we will need the following analog of a theorem of Verbitsky [87, Theorem 1.5] (see also [10] and [38, Proposition 24.1]):

Proposition 5.11. *Let $S^* H^2(X, \mathbb{C})$ be the image of the cup product map*

$$\mathrm{Sym}^* H^2(X, \mathbb{C}) \to H^*(X, \mathbb{C}).$$

Then

$$S^* H^2(X, \mathbb{C}) \cong \mathrm{Sym}^* H^2(X, \mathbb{C}) / \langle x^{n+1} \mid q_X(x) = 0 \rangle.$$

Proof. The proof in [10] carries through with very mild modifications, and we summarize the main points. We have the following purely algebraic fact:

Lemma 5.12. *Let (H, q) be a complex vector space with a nondegenerate quadratic form q, and let A^* be a graded quotient of* $\mathrm{Sym}^* H$ *by a graded ideal I^* such that:*

(1) $A^{2n} \neq 0$,

(2) $I^* \supset \langle x^{n+1} \mid q(x) = 0 \rangle$.

Then $I^* = \langle x^{n+1} \mid q(x) = 0 \rangle$.

Take $(H, q) = (H^2(X, \mathbb{C}), q_X)$ and $A^* = S^* H^2(X, \mathbb{C})$. Observe that the first condition in the lemma is met. Indeed, let w be a generator of the $H^{2,0}$-part of $H^2(X, \mathbb{C})$. Since for any resolution $\pi : Y \to X$ we have an injection $\pi^* : H^2(X, \mathbb{C}) \to H^2(Y, \mathbb{C})$, it follows that $\pi^* w$ is the class of an extension of a symplectic form. As $(\pi^* w)^n (\overline{\pi^* w})^n \neq 0$, we then have $w^n \overline{w}^n \neq 0$.

Thus, it remains to verify the second condition. We have the following:

Lemma 5.13. *We have $w^{n+1} = 0$.*

Proof. For a resolution $\pi : Y \to X$, the map $\pi^* : \mathrm{gr}_m^W H^m(X, \mathbb{C}) \to H^m(Y, \mathbb{C})$ is injective. Thus, the $(m, 0)$-part of the mixed Hodge structure on $H^m(X, \mathbb{C})$ is 0 for $m > 2n$. □

To finish, just as in [10], since the period map is an étale map of $\mathrm{Def}^{\mathrm{lt}}(X)$ onto the irreducible quadric $(q_X = 0)$ by Proposition 5.5, applying Lemma 5.13 to nearby deformations yields $(q_X(x) = 0) \subset (x^{n+1} = 0)$. □

5.14. Fujiki relations. Fujiki [30, Theorem 4.7] first established interesting relations between the self intersection of a given cohomology class and powers of the BBF form on symplectic manifolds. It seems that Matsushita [65, Theorem 1.2], [66, Proposition 4.1] was the first to prove the $(k = \dim X)$ Fujiki relation in the singular setting. He required the varieties to be projective and to have $\mathbb{Q}$-factorial, terminal singularities only and Schwald extended his statement to projective primitive symplectic varieties in [84]. We need a more general statement for the projectivity criterion in the next paragraph. Generalizing to the Kähler setup is not difficult, basically the existing proofs in the projective case work literally.

A small argument instead is needed when comparing powers of the BBF form to integration over certain very general homology classes. The first results in this direction in the singular case can be found in [65, Lemma 2.4].

Proposition 5.15 (Fujiki relations). *Let X be a primitive symplectic variety and let $\phi \in \operatorname{Sym}^k H^2(X,\mathbb{Q})^\vee$ which is of type $(-k,-k)$ for all small deformations of X. Then if k is odd, we have $\phi = 0$, while if k is even, there exists a constant $c = c(\phi) \in \mathbb{Q}$ such that $\phi = cq_X^{k/2}$, where $q_X^{k/2} \in \operatorname{Sym}^{k/2} H^2(X,\mathbb{R})^\vee$ is the symmetrization of $q_X^{\otimes k/2}$. In particular, for all $\alpha \in H^2(X,\mathbb{C})$ we have*

$$\phi(\alpha^k) = c \cdot q_X(\alpha)^{k/2}.$$

Proof. Using Proposition 5.5, we see that the Mumford–Tate group of $H^2(X',\mathbb{Z})$ for a very general locally trivial deformation X' of X is $\mathrm{SO}(H^2(X',\mathbb{Q}), q_{X'})$. The representation of $\mathrm{SO}(H^2(X,\mathbb{Q}), q_X)$ on $\operatorname{Sym}^k H^2(X,\mathbb{Q})^\vee$ has no invariants for odd k, while for even k the only invariant is $q_X^{k/2}$ up to scaling. □

Corollary 5.16. *Let X be a primitive symplectic variety. There is a (unique) polynomial $f_X(t) \in \mathbb{Q}[t]$ such that for any line bundle L on X, $\chi(L) = f_X(q_X(\mathrm{c}_1(L)))$ and $f_{X'} = f_X$ for any locally trivial deformation X' of X. Moreover,*

Proof. As X has rational singularities, for a resolution $\pi : Y \to X$ we have

$$\chi(L) = \chi(\pi^* L) = \int_Y \pi^* \mathrm{ch}(L)\mathrm{td}(Y).$$

Since $\pi^* : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ is an injection of Hodge structures, it follows that

$$\chi(L) = \sum_k \phi_k(\mathrm{c}_1(L)^k)$$

for Hodge classes $\phi_k \in \operatorname{Sym}^k H^2(X,\mathbb{Q})^\vee$. Moreover, from the existence of a simultaneous resolution $\mathscr{Y} \to \mathscr{X}$ of the universal locally trivial deformation $\mathscr{X}$ of X, it follows that the ϕ_k are locally constant and of type $(-k,-k)$ everywhere. Now apply the proposition. □

For a compact complex space W of dimension k, we denote by $[W] \in H_{2k}(W,\mathbb{Z})$ the *cycle class*, that is, the sum over the fundamental classes of the irreducible components of dimension k weighted by their multiplicities. We write $\int_W : H^{2k}(W,\mathbb{Z}) \to \mathbb{Z}$ for the cap product with the cycle class. Hodge classes as in Proposition 5.15 can be constructed via the following lemma.

Lemma 5.17. *Let X be a primitive symplectic variety and $f : \mathscr{X} \to S$ a locally trivial deformation. Let $\mathscr{W} \subset \mathscr{X}$ be a closed subvariety that is flat over S with fiberwise dimension k. Then $\int_{\mathscr{W}_s}$ defines a section of $\operatorname{Sym}^k R^2 f_* \mathbb{Q}^\vee$ which is of type $(-k,-k)$.*

Proof. It suffices to show that for any sufficiently small Euclidean open set $U \subset S$, the cycle class $[\mathscr{W}_u]$ is constant in Borel–Moore homology $H^{\mathrm{BM}}_{2k}(f^{-1}(U), \mathbb{Q})$. This is done as in [31, Lemma 19.1.3]. □

For the following corollary, the term *very general* is to be interpreted in terms of locally trivial deformations, i.e., outside a countable union of proper subvarieties in the base of the locally trivial Kuranishi family.

Corollary 5.18. *Let X be a very general primitive symplectic variety. Then X does not contain odd-dimensional closed subvarieties.*

Proof. By the lemma, for a k-dimensional subvariety W we have a Hodge class $\phi = \int_W$ in $\operatorname{Sym}^k H^2(X, \mathbb{Q})^\vee$. By taking a Kähler class $\omega \in H^2(X, \mathbb{R})$, we see that $\int_W \omega^k > 0$ and thus ϕ is nonzero, a contradiction. □

5.19. $\mathbb{Q}$-factoriality and $\mathbb{Q}$-factorial terminalizations. We first deduce the invariance of $\mathbb{Q}$-factoriality under locally trivial deformations for primitive symplectic varieties.

Lemma 5.20. *Let X be a primitive symplectic variety. Then every small locally trivial deformation of X is $\mathbb{Q}$-factorial if and only if X is $\mathbb{Q}$-factorial.*

Proof. Let $\pi : Y \to X$ be a resolution and consider $H^2(X, \mathbb{Q}) \subset H^2(Y, \mathbb{Q})$ via pullback. Using Lemma 4.9, we choose a simultaneous resolution $\mathscr{Y} \to \mathscr{X}$ of the universal locally trivial deformation $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$. Recall that by Proposition 5.5, we can think of $\mathrm{Def}^{\mathrm{lt}}(X)$ as an open subset of the local period domain $\Omega(X)$.

For an element $\lambda \in H^2(X, \mathbb{Q})$ with $q_X(\lambda) > 0$, let $T_\lambda \subset \Omega(X)$ be the locus for which $\lambda \in H^{2,0}(X) \oplus H^{0,2}(X)$. Note that T_λ is a totally real half-dimensional closed subvariety of $\Omega(X)$ (see Section 8.3). We first claim that we may choose the element λ so that T_λ meets the image of $\mathrm{Def}^{\mathrm{lt}}(X)$. Indeed, note that $q_X(\sigma) = 0$ is equivalent to $q_X(\mathrm{Re}(\sigma)) = q_X(\mathrm{Im}(\sigma))$ and $q_X(\mathrm{Re}(\sigma), \mathrm{Im}(\sigma)) = 0$. Thus, taking λ to be a rational class sufficiently close to $\mathrm{Re}(\sigma)$, then taking $R = \lambda$ and I to be the projection of $\mathrm{Im}(\sigma)$ to $R^\perp$ scaled so that $q_X(R) = q_X(I)$, we can make $\sigma' = R + iI \in T_\lambda$ arbitrarily close to σ.

Now, choosing such a λ, in the notation of Lemma 5.7 we have that $v(\lambda) \neq 0$ by Proposition 5.15. Observe that the Fujiki constant is nonzero since $q_X(\sigma + \bar{\sigma}) \neq 0 \neq v(\sigma + \bar{\sigma})$. Define a quadratic form Q_λ on $H^2(Y, \mathbb{Q})$ by the right-hand side of equation (5.3) divided by $q_{Y,\sigma}(\lambda) = q_{X,\sigma}(\lambda)$. Note that:

(1) Q_λ is rational.

(2) Q_λ restricts to (a nonzero multiple of) q_X on $H^2(X, \mathbb{Q})$.

(3) If $\lambda \in H^{2,0}(X) \oplus H^{0,2}(X)$, then

$$v(\lambda)^2 q_{Y,\sigma}(\alpha) = q_{X,\sigma}(\lambda) Q_\lambda(\alpha)$$

for all $\alpha \in H^2(Y, \mathbb{C})$, as in [6, Théorème 5 (c)].

We now claim that Q_λ is a morphism of Hodge structures. For this, we consider Q_λ as a quadratic from on the local system of weight two Hodge structures associated to the family $\mathscr{Y} \to \mathrm{Def}^{\mathrm{lt}}(X) \subset \Omega(X)$. In view of (1), it suffices to show that Q_λ is a morphism of $\mathbb{R}$-Hodge structures. By (3) and Lemma 5.3, this is the case for all periods in $T_\lambda \cap \mathrm{Def}^{\mathrm{lt}}(X)$ and this set is nonempty and open in T_λ by the above. But the Hodge locus of Q_λ is certainly an analytic subset of $\mathrm{Def}^{\mathrm{lt}}(X)$ and therefore must be all of $\mathrm{Def}^{\mathrm{lt}}(X)$ as T_λ is totally real and $\dim_{\mathbb{R}} T_\lambda = \dim_{\mathbb{C}} \mathrm{Def}^{\mathrm{lt}}(X)$.

Now, q_X is nondegenerate by Lemma 5.3, so by property (2) the Q_λ-orthogonal space $H^2(X,\mathbb{Q})^\perp \subset H^2(Y,\mathbb{Q})$ is a rational complement to $H^2(X,\mathbb{Q})$ and is Hodge–Tate. Thus, condition (2.5) is equivalent to (2.4). Using Lemma 4.9 again, we see that the validity of (2.4) is clearly invariant under locally trivial deformations. We therefore conclude by Proposition 2.15. □

The rest of this section will be devoted to relating the locally trivial deformation theory of a projective primitive symplectic variety X to that of a $\mathbb{Q}$-factorial terminalization, which will play a role in the proof of surjectivity of the period map. We start with the following slight generalization of [5, Lemma 3.5]. The proof is literally the same as in [5] so we omit it here.

Lemma 5.21. *Let $\pi : Y \to X$ be a proper bimeromorphic morphism between primitive symplectic varieties. Then $\pi^* : H^2(X,\mathbb{C}) \to H^2(Y,\mathbb{C})$ is injective and the restriction of q_Y to $H^2(X,\mathbb{C})$ is equal to q_X. We have an orthogonal decomposition*

$$H^2(Y,\mathbb{Q}) = \pi^* H^2(X,\mathbb{Q}) \oplus N_{\mathbb{Q}},$$

where $N := \tilde{q}_Y^{-1}(N_1(Y/X))$, which is negative definite. □

Let X, Y be normal compact complex varieties with rational singularities and $\pi : Y \to X$ a proper bimeromorphic morphism. It follows that $\pi_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^1\pi_*\mathcal{O}_Y = 0$ so that by [58, Proposition 11.4], there is a commutative diagram

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{P} & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathrm{Def}(Y) & \xrightarrow{p} & \mathrm{Def}(X) \end{array} \tag{5.4}$$

for the miniversal families of deformations of X and Y. Consider the case that $\pi : Y \to X$ is a $\mathbb{Q}$-factorial terminalization of a projective primitive symplectic variety. We will show below (Proposition 5.22) that the locally trivial deformations of X are identified via p with the locus of deformations of Y where the classes of contracted curves remain Hodge.

Proposition 5.22. *Let X, Y be projective primitive symplectic varieties and $\pi : Y \to X$ a proper bimeromorphic morphism. Assume Y is $\mathbb{Q}$-factorial and terminal. Let $N \subset H^2(Y,\mathbb{C})$ be the q_Y-orthogonal complement to $H^2(X,\mathbb{C}) \subset H^2(Y,\mathbb{C})$ and consider diagram* (5.4). *Denote by* $\mathrm{Def}(Y,N) \subset \mathrm{Def}(Y)$ *the subspace of deformations such that classes in N remain of type* $(1,1)$. *Then the following holds:*

(1) $p^{-1}(\mathrm{Def}^{\mathrm{lt}}(X)) = \mathrm{Def}(Y,N) \subset \mathrm{Def}(Y)$.

(2) *The restriction* $p : \mathrm{Def}(Y,N) \to \mathrm{Def}^{\mathrm{lt}}(X)$ *is an isomorphism.*

Proof. By Theorem 4.7 respectively [73, Main Theorem], the two spaces $\mathrm{Def}^{\mathrm{lt}}(X)$ and $\mathrm{Def}(Y)$ are smooth of dimension $h^{1,1}(X)$ and $h^{1,1}(Y)$, respectively. Moreover, by [73, Theorem 1], $\mathrm{Def}(X)$ is smooth while $p : \mathrm{Def}(Y) \to \mathrm{Def}(X)$ is finite and, as both are of the same dimension, surjective.

Now, $\mathrm{Def}(Y, N) \subset \mathrm{Def}(Y)$ is a smooth subvariety of codimension $m := \dim N$ whose tangent space is identified with $H^{1,1}(X)$ under the period map, see Lemma 4.13. By Corollary 4.11, the fibers of the universal deformations $\mathscr{Y} \to \mathrm{Def}(Y)$ and $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ are primitive symplectic varieties. Therefore, [5, Lemma 2.2] entails that the second cohomology of locally trivial deformations of X form a vector bundle on $\mathrm{Def}^{\mathrm{lt}}(X)$, in particular, we have $h^{1,1}(\mathscr{X}_{p(t)}) = h^{1,1}(X)$. Thus, by the decomposition $H^2(Y, \mathbb{C}) = N \oplus H^2(X, \mathbb{C})$ from Lemma 5.21 we see that the space $N_1(\mathscr{Y}_t/\mathscr{X}_{p(t)})$ of curves contracted by $P_t : \mathscr{Y}_t \to \mathscr{X}_t$ has dimension m for all $t \in p^{-1}(\mathrm{Def}^{\mathrm{lt}}(X))$. As N is the orthogonal complement of $H^2(X, \mathbb{C})$, it also varies in a local system. Using the period map this shows that $p^{-1}(\mathrm{Def}^{\mathrm{lt}}(X)) = \mathrm{Def}(Y, N)$.

One shows as in [63, Proposition 2.3 (ii)] that p is an isomorphism, see also [5, Proposition 4.5]. □

We will need the following corollary in Section 8. For a projective primitive symplectic variety X and a $\mathbb{Q}$-factorial terminalization $\pi : Y \to X$, let $g : \mathscr{Y} \to \mathrm{Def}(Y)$ and $f : \mathscr{X} \to \mathrm{Def}(X)$ be the universal deformations, and let $f' : \mathscr{X}' \to \mathrm{Def}(Y)$ be the pullback of $\mathscr{X}$ to $\mathrm{Def}(Y)$ along p as in (5.4). Then $P' : \mathscr{Y} \to \mathscr{X}'$ is a simultaneous $\mathbb{Q}$-factorial terminalization by [73, Main Theorem] and Lemma 5.20. Consider the constant second Betti number locus

$$B_X := \{t \in \mathrm{Def}(Y) \mid \mathrm{rk}(R^2 f'_* \mathbb{Q}_{\mathscr{X}'})_t = b_2(X)\}$$

which is a (reduced) closed analytic subspace of $\mathrm{Def}(Y)$.

Corollary 5.23. *In the above setup,* $B_X = \mathrm{Def}(Y, N)$.

Proof. Certainly $B_X \supset \mathrm{Def}(Y, N)$ by the proposition and [5, Lemma 2.4]. By Lemma 5.21 and proper base change we have an injection

$$0 \to (R^2 f'_* \mathbb{Q}_{\mathscr{X}'})_t \to (R^2 g_* \mathbb{Q}_{\mathscr{Y}})_t$$

for all $t \in \mathrm{Def}(Y)$. The restrictions $(R^2 g_* \mathbb{Q}_{\mathscr{X}'})|_{B_X}$ and $(R^2 f'_* \mathbb{Q}_{\mathscr{Y}})|_{B_X}$ are local systems as therefore is the orthogonal $(R^2 f'_* \mathbb{Q}_{\mathscr{X}'})|^{\perp}_{B_X}$ in $(R^2 f'_* \mathbb{Q}_{\mathscr{Y}})|_{B_X}$. We must then have the equality $(R^2 f'_* \mathbb{Q}_{\mathscr{X}'})^{\perp}_t = N$ for all $t \in B_X$, but since $(R^2 f'_* \mathbb{Q}_{\mathscr{X}'})^{\perp}_t$ is Hodge–Tate, we obtain the reverse inclusion $B_X \subset \mathrm{Def}(Y, N)$. □

6. The projectivity criterion

In this section we formulate and prove an analog of Huybrechts' projectivity criterion [44, Theorem 3.11] (see also [45]) in the singular setup. Note that for orbifold singularities, the question has been examined by Menet [67]. We use several of his as well as of Huybrechts' arguments.

6.1. A singular version of the Demailly–Păun theorem. We do not know whether the analog of Demailly–Păun's celebrated theorem [21, Main Theorem 0.1] on the numerical characterization of the Kähler cone of a compact Kähler manifold holds for singular varieties.

One may however easily deduce from it that a similar statement holds which is good enough for our purposes. For this purpose, we introduce a notion for cohomology classes that behave as if they were Kähler classes.

Recall from (2.3) that we defined $H^{1,1}(X,\mathbb{R}) = F^1H^2(X,\mathbb{C}) \cap H^2(X,\mathbb{R})$ for a reduced compact complex space of class $\mathscr{C}$.

Definition 6.2. Let X be a reduced compact complex space of class $\mathscr{C}$ and consider a class $\kappa \in H^{1,1}(X,\mathbb{R})$. We say that κ is *Demailly–Păun* if for every compact complex manifold V and for every generically finite morphism $f : V \to X$ the class $f^*\kappa$ is big and nef. We denote by $\mathrm{DP}(X) \subset H^{1,1}(X,\mathbb{R})$ the convex cone consisting of all Demailly–Păun classes. We refer to it as the *Demailly–Păun cone*.

This definition deserves a couple of comments.

Remark 6.3. (1) Every Kähler class is Demailly–Păun, in particular, $\mathrm{DP}(X) \neq \emptyset$ if X is Kähler. Indeed, every Kähler class is a $(1,1)$-class by Proposition 2.8. Then the claim follows as the pullback of a Kähler class under a generically finite morphism from a smooth variety is big and nef.

(2) We do not know of an example of a class that is Demailly–Păun but not Kähler. It seems likely that Demailly–Păun classes are the same as Kähler classes. Apart from the Demailly–Păun theorem [21, Main Theorem 0.1], evidence for this presumption is given in (3).

(3) Every rational Demailly–Păun class is Kähler. Indeed, a multiple of such a class is the first Chern class of a big line bundle L. Therefore, X is Moishezon and L is ample by the Nakai–Moishezon criterion. Note that the Nakai–Moishezon criterion holds for big line bundles on Moishezon varieties, see e.g. [56, 3.11 Theorem].

(4) A closed subvariety of a class $\mathscr{C}$ variety is again dominated by a compact Kähler manifold, see Proposition 2.7, and so it is itself class $\mathscr{C}$. Then it is immediate that the restriction of a Demailly–Păun class to a subvariety is again Demailly–Păun.

(5) The assumption that X be of class $\mathscr{C}$ is somewhat redundant but simplifies the exposition. If for some $\kappa \in H^{1,1}_{\partial\bar\partial}(X,\mathbb{R})$ the pullback $\pi^*\kappa$ along a resolution $\pi : Y \to X$ is big, then Y (and hence also X) are of class $\mathscr{C}$ by [21, Theorem 0.7].

Lemma 6.4. *Let X be a compact variety of class $\mathscr{C}$ and let $\kappa \in H^{1,1}(X,\mathbb{R})$. Then κ is Demailly–Păun if and only if for every compact complex manifold W and for every holomorphic map $\pi : W \to X$ which is bimeromorphic onto its image the class $\pi^*\kappa$ is big and nef. Moreover, the pullback of a Demailly–Păun class to an arbitrary compact complex manifold is nef.*

Proof. To prove the non-trivial direction of the first claim, let $\pi : V \to X$ be a holomorphic map from a compact complex manifold which is generically finite onto its image. We denote $\bar V := \pi(V)$ and factor π as $V \xrightarrow{\pi_1} \bar V \xrightarrow{\pi_2} X$. We then chose a diagram

$$\begin{array}{ccccc} W_2 & \xrightarrow{\phi} & V & & \\ {\scriptstyle f}\downarrow & & {\scriptstyle \pi_1}\downarrow & \overset{\pi}{\searrow} & \\ W_1 & \xrightarrow{\psi} & \bar V & \xrightarrow{\pi_2} & X, \end{array}$$

where W_1, W_2 are compact Kähler manifolds and $W_1 \to \bar{V}$, $W_2 \to V$ are bimeromorphic. By assumption, $\alpha := \psi^*\pi_2^*\kappa$ is big and nef. By a result of Păun [77, Théorème 1], nefness of α is equivalent to $f^*\alpha$ being nef. Bigness is preserved under generically finite pullbacks so that $f^*\alpha$ is big and nef. Since $W_2 \to V$ is bimeromorphic between compact complex manifolds, $\pi^*\kappa$ is big and nef as $\phi^*\pi^*\kappa = f^*\alpha$ is.

For the second statement, let $\pi : V \to X$ be a morphism from a compact complex manifold. We change the above diagram accordingly and deduce the claim by invoking Păun's result once more. □

The main result of this section is deduced from the smooth Demailly–Păun theorem and Păun's results in [77] via an inductive argument. Note that while our result is not essentially new compared to the Demailly–Păun theorem, it should be mentioned that Collins and Tosatti proved in [15, Theorem 1.1] a true generalization of the Demailly–Păun theorem for possibly singular compact subvarieties of Kähler manifolds.

Theorem 6.5. *Let X be a reduced compact complex space of class $\mathscr{C}$ and consider the cone $P \subset H^{1,1}(X, \mathbb{R})$ of all classes α on X such that for all closed analytic subsets $V \subset X$ we have*

$$\int_V \alpha^{\dim V} > 0.$$

Then the Demailly–Păun cone $\mathrm{DP}(X)$ *is empty or a connected component of P. If X is Kähler,* $\mathrm{DP}(X)$ *is the connected component of P containing the Kähler cone.*

Proof. Clearly, $\mathrm{DP}(X) \subset P$ and as the Demailly–Păun cone is convex, it is contained in a connected component of P. Moreover, if X is Kähler, then the Kähler cone is contained in $\mathrm{DP}(X)$.

For the converse, we may assume that $\mathrm{DP}(X)$ is non-empty, otherwise there is nothing to prove. Let $\alpha \in P$ be a class in the same connected component as $\mathrm{DP}(X)$. We will prove that the restriction of α to any subvariety of X is Demailly–Păun by induction on the dimension of the subvariety.

For $d = 0$ the statement is trivial. Let $V \subset X$ be a subvariety of dimension d and assume that α is Demailly–Păun on every subvariety of X of dimension strictly smaller than d. We denote by $\pi : W \to X$ the composition of a resolution of singularities of V with the inclusion $V \subset X$ where W is a compact Kähler manifold of dimension d. Such a resolution exists thanks to Proposition 2.7. By Lemma 6.4 it suffices to prove that $\pi^*\alpha$ is big and nef. Clearly, $\alpha|_V$ fulfills the hypotheses of the theorem if α does. We show first that $\pi^*\alpha$ is nef on W using the Demailly–Păun theorem on W. Let us take a Kähler class κ on W. For $0 < \varepsilon \ll 1$ the class $\alpha_W := \pi^*\alpha + \varepsilon\kappa$ satisfies $\alpha_W^d > 0$. If $Z \subset W$ is a proper analytic subvariety of dimension e, then $\pi(Z) \subset V$ is also a proper subvariety and thus $\alpha|_{\pi(Z)}$ is Demailly–Păun by the inductive hypothesis. We will show that $\int_Z \alpha_W^e > 0$. But this can be computed on a resolution of singularities, so we may without loss of generality assume that Z is nonsingular. Then $\pi^*\alpha|_Z$ is nef by Lemma 6.4 and therefore $\alpha_W|_Z$ has positive top self intersection.

As α is in the same connected component of $P = P(V)$ as the Demailly–Păun classes on V, also α_W is in the same connected component $P(W)$ as the Demailly–Păun classes on W. But by [21, Main Theorem 0.1], we have $\mathrm{DP}(W) = \mathcal{K}(W)$, where $\mathcal{K}(W)$ denotes the Kähler cone. Hence, the Demailly–Păun theorem applies and α_W is Kähler. Moreover, $\pi^*\alpha$ is nef on

W because ε was arbitrarily small. But certainly $\int_W (\pi^*\alpha)^d > 0$ and therefore $\pi^*\alpha$ is also big on W by [21, 0.4 Theorem]. This concludes the proof. □

6.6. Projectivity criterion. In this subsection, the term *very general* is to be interpreted in terms of locally trivial deformations, i.e., outside a countable union of proper subvarieties in the base of the locally trivial Kuranishi family.

Definition 6.7. Let X be a primitive symplectic variety and q_X its BBF form. We define the positive cone

$$\mathcal{C}_X := \{\alpha \in H^{1,1}(X,\mathbb{R}) \mid q_X(\alpha) > 0\}^\kappa,$$

where κ denotes the connected component containing the Kähler cone.

Theorem 6.8. *For a very general primitive symplectic variety X, the positive cone equals the Demailly–Păun cone:*

$$\mathrm{DP}(X) = \mathcal{C}_X.$$

Proof. The Demailly–Păun cone is always contained in the positive cone by Theorem 6.5. Let us prove the other inclusion. By Corollary 5.18, X does not contain any odd-dimensional subvarieties. Let $Z \subset X$ be a subvariety and denote by $2d$ its dimension. Choose a Kähler class κ on X. Then by the Fujiki relations, Proposition 5.15, there is a constant $c_Z \in \mathbb{R}$ such that for every $\alpha \in H^2(X,\mathbb{C})$ the equality

$$c_Z \cdot \int_Z (\alpha S + \kappa T)^{2d} = q_X(\alpha S + \kappa T)^d = \big(q_X(\alpha)S^2 + 2q_X(\alpha,\kappa)ST + q_X(\kappa)T^2\big)^d$$

of polynomials in the indeterminates S and T holds. Choosing $\alpha = \kappa$, we see that c_Z has to be strictly positive. From now on let $\alpha \in \mathcal{C}_X$. As also $\kappa \in \mathcal{C}_X$, Lemma 5.3 implies that $q_X(\alpha,\kappa) > 0$. The coefficients of the polynomial on the right-hand side are manifestly all positive. We conclude from looking at the left-hand side that for every $0 \le \lambda \le 1$ we have that $\lambda\alpha + (1-\lambda)\kappa$ lies in the cone P from Theorem 6.5. In particular, α is in the connected component of P containing the Kähler cone $\mathcal{K}(X)$. We conclude from Theorem 6.5 that $\alpha \in \mathrm{DP}(X)$. □

The following is the singular version of [46, Theorem 3.11] and the proof relies on important ideas of his and of Menet [67], see section 4 of Menet's article. The presentation follows [38, Proposition 26.13].

Theorem 6.9. *Let X be a primitive symplectic variety and $\alpha \in H^2(X,\mathbb{Z})$ a $(1,1)$-class. If $q(\alpha) > 0$, then X is projective.*

Note that the existence of such a class can be read off only from the period.

Proof. By the Lefschetz $(1,1)$-theorem, there is a line bundle L on X with first Chern class $\mathrm{c}_1(L) = \alpha$. We show that L is big. It suffices to do this on a resolution, say $\pi : Y \to X$, as bigness of a line bundle is a birationally invariant notion. Bigness of the line bundle π^*L is implied by bigness of $\pi^*\alpha$, see [49, Theorem 4.6]. The strategy is to infer bigness of α by approximating α on a resolution with Kähler currents on nearby varieties.

Consider the locally trivial Kuranishi family $\mathscr{X} \to S := \mathrm{Def}^{\mathrm{lt}}(X)$ and take a simultaneous resolution $\mathscr{Y} \to \mathscr{X}$ which is possible by Lemma 4.9. From now on we choose $\pi : Y \to X$ to be the special fiber of $\mathscr{Y} \to \mathscr{X}$. For a very general $t \in S$ the corresponding primitive symplectic varieties $\mathscr{X}_t$ satisfy $\mathrm{DP}(\mathscr{X}_t) = \mathscr{C}_{\mathscr{X}_t}$ thanks to Theorem 6.8. Therefore, α can be approximated by Demailly–Păun classes α_{t_i} on $\mathscr{X}_{t_i}$ where $t_i \to 0 \in S$ for $i \to \infty$, where X is the fiber of $\mathscr{X} \to S$ over 0. Consequently, $\pi^*\alpha$ can be approximated by big classes on nearby fibers $\mathscr{Y}_{t_i}$ and as in [19, Proposition 6.1], see also the proof of [38, Proposition 26.13], we deduce that $\pi^*\alpha$ is big. The key point here is to see that in the above approximation procedure, the limit of a sequence of closed positive currents are again closed and positive. This is explained in detail in the appendix by Diverio to [3]. As explained before, bigness of $\pi^*\alpha$ implies that π^*L and hence L is big. Thus, X is Moishezon. Being Kähler and having rational singularities, it must be projective by [72, Theorem 1.6]. □

The following result is the singular analog of [29, Theorem 4.8 2)], see also [44, Theorem 3.5] and [38, Proposition 26.6]. We have to change the proof slightly in the singular setting.

Corollary 6.10. *Let X be a primitive symplectic variety, $f : \mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ the universal locally trivial deformation of $X = f^{-1}(0)$, and $S \subset \mathrm{Def}^{\mathrm{lt}}(X)$ a positive-dimensional subvariety through $0 \in \mathrm{Def}^{\mathrm{lt}}(X)$. Then in every open neighborhood $U \subset S$ of 0 there is a point $s \in U$ such that the fiber $\mathscr{X}_s$ is projective.*

Proof. The proof is almost the same as in [29, Theorem 4.8 2)] respectively [44, Theorem 3.5]. We refer to these references for details and content ourselves with a sketch of proof. One restricts to a one-dimensional disk $S \subset \mathrm{Def}^{\mathrm{lt}}(X)$ and chooses a Kähler form ω on X such that the locus $S_{[\omega]} \subset \mathrm{Def}^{\mathrm{lt}}(X)$ where the class $[\omega]$ remains of type $(1,1)$ intersects S transversally. Next one chooses classes $\alpha_i \in H^2(X,\mathbb{Q})$ converging to $[\omega]$ such that the α_i are not of type $(1,1)$ on X. Then the $(1,1)$-locus $S_{\alpha_i} \subset \mathrm{Def}^{\mathrm{lt}}(X)$ intersects S in points $t_i \neq 0$ converging to 0. Now the idea is that the $(1,1)$-class α_i is Kähler on $\mathscr{X}_{t_i}$ for t_i sufficiently close to 0. In [44, Theorem 3.5] this is seen via harmonic representatives. As X is singular, we cannot argue literally the same. However, due to Lemma 4.9 we may take a simultaneous resolution $\pi : \mathscr{Y} \to \mathscr{X}$ obtained by successive blow ups. In particular, there is an $\mathbb{R}$-linear combination E of exceptional divisors such that for $e := \mathrm{c}_1(\mathscr{O}(E))$ we have that $\alpha_i - e$ is Kähler on $Y := \pi^{-1}(X)$. Now we apply the argument involving harmonic representatives to $\alpha_i - e$ and deduce that for t_i sufficiently close to 0 the variety $\mathscr{Y}_{t_i}$ is projective. Hence, also the corresponding $\mathscr{X}_{t_i}$ is projective by [72, Theorem 1.6]. □

We immediately deduce:

Corollary 6.11. *Let X be a primitive symplectic variety and let $f : \mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ be the universal locally trivial deformation of $X = f^{-1}(0)$. Then for every positive-dimensional subvariety $S \subset \mathrm{Def}^{\mathrm{lt}}(X)$ the set of points $\Sigma \subset S$ with projective fiber is dense.* □

6.12. Inseparability and moduli. Given a primitive symplectic variety X and a lattice Λ with quadratic form q, a Λ-*marking* of X is an isomorphism

$$\mu : (H^2(X,\mathbb{Z})_{\mathrm{tf}}, q_X) \xrightarrow{\cong} (\Lambda, q).$$

A Λ-*marked primitive symplectic variety* is a pair (X, μ), where X is a primitive symplectic variety and μ is a Λ-marking of X. Two Λ-marked primitive symplectic varieties (X, μ) and (X', μ') are *isomorphic* if there is an isomorphism $\varphi : X \to X'$ such that $\mu' = \mu \circ \varphi^*$.

Definition 6.13. Given a lattice Λ as above, we denote by $\mathfrak{M}_\Lambda$ the analytic coarse moduli space of Λ-*marked* primitive symplectic varieties. As a set, $\mathfrak{M}_\Lambda$ consists of isomorphism classes of Λ-marked primitive symplectic varieties (X, μ), and it is given the structure of a not-necessarily-Hausdorff complex manifold using Theorem 4.7 by identifying points in the bases of locally trivial Kuranishi families over which the fibers are isomorphic as Λ-marked varieties.

Note that this definition coincides with the usual one [44, 1.18] for irreducible symplectic manifolds due to the fact that all deformations of smooth varieties are locally trivial. The following statement of Huybrechts' carries over together with its proof.

Theorem 6.14. *Let X, X' be primitive symplectic varieties such that for some choice of marking $\mu : H^2(X, \mathbb{Z})_{\mathrm{tf}} \to \Lambda$, $\mu' : H^2(X', \mathbb{Z})_{\mathrm{tf}} \to \Lambda$ the pairs (X, μ), (X', μ') define non-separated points in the Λ-marked moduli space. Then there is a bimeromorphic map*

$$\phi : X \dashrightarrow X'.$$

Proof. Identical to [44, Theorem 4.3] using a simultaneous resolution. □

Corollary 6.15. *If (X, μ) and (X', μ') are inseparable in moduli with Mumford–Tate general periods, then $(X, \mu) = (X', \mu')$.*

Proof. By the theorem, there is a bimeromorphic $\phi : X \dashrightarrow X'$. Mumford–Tate generality implies that neither X nor X' contain compact curves. Indeed, such a curve would define a non-zero Hodge class, e.g., in $H_2(X, \mathbb{Q})$, so by the BBF form we also had a non-zero Hodge class in $H^2(X, \mathbb{Q})$. By a standard argument, bimeromorphic maps between normal varieties without curves are necessarily isomorphisms.[5)] We therefore obtain an isomorphism of Hodge structures $H^2(X, \mathbb{Z})_{\mathrm{tf}} \to H^2(X', \mathbb{Z})_{\mathrm{tf}}$ which maps a Kähler class to a Kähler class. The claim follows since the automorphism group of a Mumford–Tate general period $\mathrm{Aut}_{\mathrm{Hdg}}(H^2(X, \mathbb{Z})_{\mathrm{tf}}) = \{\pm 1\}$, since $\mathrm{End}(H^2(X, \mathbb{Q}), q_X)^{\mathrm{SO}(H^2(X,\mathbb{Q}), q_X)} = \mathbb{Q}\,\mathrm{id}$. □

We denote by $\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$ the complex unit disk and by $\Delta^* := \Delta \setminus \{0\}$ the complement of the origin. Recall that if two not necessarily $\mathbb{Q}$-factorial complex varieties are bimeromorphic, it is not in general true that we can push forward (or pull back) line bundles along the bimeromorphic map.

Theorem 6.16. *Let X, X' be projective primitive symplectic varieties and $\phi : X \dashrightarrow X'$ a birational map which is an isomorphism in codimension* 1 *such that*

$$\phi_* : \mathrm{Pic}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X')_{\mathbb{Q}}$$

5) This can be seen exactly as for projective algebraic varieties of Picard rank one by applying, e.g., [16, Lemma 1.15 (b)] to a resolution of indeterminacies and its inverse.

is well defined and an isomorphism. Then there are one parameter locally trivial deformations $f : \mathscr{X} \to \Delta$, $f' : \mathscr{X}' \to \Delta$ such that $\mathscr{X}$ and $\mathscr{X}'$ are birational over Δ and such that $\mathscr{X}^* = f^{-1}(\Delta^*) \cong (f')^{-1}(\Delta^*) = (\mathscr{X}')^*$.

Proof. The basic strategy of [44, Theorem 4.6] remains unchanged, we will therefore only explain where we need to deviate from it. By Corollary 5.16, there are polynomials $f_X(t)$ and $f_{X'}(t)$ with rational coefficients of degree $n = \frac{\dim X}{2}$ such that for any line bundle L on X a Hirzebruch–Riemann–Roch statement of the form $\chi(X, L) = f_X(q_X(\mathrm{c}_1(L)))$ holds and similarly for X'. We may assume that $f_X \geq f_{X'}$ with respect to the lexicographic order and choose an ample line bundle L' on X' and denote by L the corresponding $\mathbb{Q}$-line bundle on X. Replacing L' by a multiple, we may assume that L is integral. Let $\pi : (\mathscr{X}, \mathscr{L}) \to S$ be a locally trivial deformation of (X, L) over a smooth one-dimensional base such that the Picard number of the general fiber of $\mathscr{X} \to S$ is one. As in [44, Theorem 4.6], using the projectivity criterion from Theorem 6.9, one shows that $h^0(\mathscr{L}_t^{\otimes m})$ for $m \gg 0$ does not depend on $t \in S$ that the associated linear system gives a meromorphic S-morphism $\mathscr{X} \dashrightarrow \mathbb{P}_S(\pi_*\mathscr{L}^\vee)$ which is bimeromorphic onto its image. We obtain $\mathscr{X}' \to S$ as the closure of this image and one verifies as in [43, Proposition 4.2] that $\mathscr{X}' \to S$ has the desired properties, in particular, that its central fiber is X'. □

This result can be reformulated as follows.

Corollary 6.17. *Let X and X' be projective primitive symplectic varieties, and let $\phi : X \dashrightarrow X'$ be a birational map which is an isomorphism in codimension* 1 *such that*

$$\phi_* : \mathrm{Pic}(X)_\mathbb{Q} \to \mathrm{Pic}(X')_\mathbb{Q}$$

is well defined and an isomorphism. Then for every choice of a marking $\mu : H^2(X, \mathbb{Z})_{\mathrm{tf}} \to \Lambda$ there exists a marking $\mu' : H^2(X', \mathbb{Z})_{\mathrm{tf}} \to \Lambda$ such that the points (X, μ) and (X', μ') are inseparable points in the moduli space $\mathfrak{M}_\Lambda$.

7. Projective degenerations

The main goal of this section is to prove the following result, which will be needed for the surjectivity of the period map in Section 8:

Theorem 7.1. *Let $f : \mathscr{X}^* \to \Delta^*$ be a projective locally trivial family of primitive symplectic varieties with $\mathbb{Q}$-factorial terminal singularities such that the monodromy of $R^2 f_* \mathbb{Q}_{\mathscr{X}^*}$ is finite. Then there is a proper locally trivial family $g : \mathscr{Y} \to \Delta$ of primitive symplectic varieties whose restriction $\mathscr{Y}|_{\Delta^*} \to \Delta^*$ is isomorphic to the restriction of the base-change of $\mathscr{X}^* \to \Delta^*$ along a finite étale cover $\Delta^* \to \Delta^*$.*

Theorem 7.1 is proven for smooth $\mathscr{X}^* \to \Delta^*$ in [57, Theorem 1.7], and the proof in our slightly more general setting involves very mild modifications of the same arguments given Proposition 5.11, albeit rearranged slightly and with some simplifications.

A crucial step is the following version of [57, Theorem 2.1] which uses the MMP to produce nice models for degenerations of K-trivial varieties.

Theorem 7.2 ([57, Theorem 2.1 and Remarks 2.3 and 2.4]). *Let $f : \mathscr{X} \to \Delta$ be a projective family whose generic fiber is a K-trivial variety with $\mathbb{Q}$-factorial terminal singularities and such that at least one component of the special fiber is not uniruled. Then there is a projective family $g : \mathscr{Y} \to \Delta$ for which:*

(1) *the restriction $\mathscr{Y}|_{\Delta^*} \to \Delta^*$ is isomorphic to the restriction of the base-change of $\mathscr{X} \to \Delta$ along a finite cover $\Delta \to \Delta$,*

(2) *the special fiber is a K-trivial variety with canonical singularities,*

(3) *the total space $\mathscr{Y}$ has terminal singularities.*

Note that the third statement follows from the proof in [57]. Theorem 7.2 reduces the proof of Theorem 7.1 to showing that the assumption on the local monodromy implies that some component of a degeneration must be non-uniruled, and this is accomplished by the following:

Proposition 7.3. *Let $f : \mathscr{X} \to \Delta$ be a flat projective family such that:*

(1) *the restriction $\mathscr{X}^* := \mathscr{X}|_{\Delta^*} \to \Delta^*$ is a locally trivial family of primitive symplectic varieties,*

(2) *the local monodromy of $R^2 f_*\mathbb{Q}_{\mathscr{X}}$ is trivial,*

(3) *the special fiber X has no multiple components,*

(4) *the total space $\mathscr{X}$ has log terminal singularities.*

Then a resolution of some component of the special fiber X has a generically nondegenerate holomorphic 2-form.

Proof. Let $2n$ be the fiber dimension of f and take $\pi : (\mathscr{Y}, Y) \to (\mathscr{X}, X)$ to be a log resolution and $g := f \circ \pi : \mathscr{Y} \to \Delta$. After possible shrinking Δ, $\mathscr{Y} \to \mathscr{X}$ is a fiberwise resolution over Δ^*. Recall that there is a specialization map $\mathrm{sp} : H^*(Y, \mathbb{Q}) \to H^*(\mathscr{Y}_\infty, \mathbb{Q})$ which is topologically constructed as follows. After possibly shrinking Δ we let $\mathscr{Y}_\infty = e^*\mathscr{Y}_{\Delta^*}$, where $e : \mathbb{H} \to \Delta^*$ is the universal cover. Then sp is the pullback along the natural map $\mathscr{Y}_\infty \to \mathscr{Y}$ composed with the isomorphism induced by the inclusion $Y \to \mathscr{Y}$ which is a homotopy equivalence. Note that sp is a ring homomorphism, and that the inclusion $\mathscr{Y}_t \to \mathscr{Y}_\infty$ of a fiber above $t \in \Delta^*$ is also a homotopy equivalence, as locally trivial families are topologically (even real analytically) trivial [1, Proposition 5.1]. We can also view $H^*(\mathscr{Y}_\infty, \mathbb{Q})$ as the nearby cycles $\psi R f_*\mathbb{Q}_{\mathscr{Y}}$ (up to a shift) and the specialization map as the natural map $i^* R f_*\mathbb{Q}_{\mathscr{Y}} \to \psi R f_*\mathbb{Q}_{\mathscr{Y}}$ by proper base-change, where $i : \{0\} \to \Delta$ is the inclusion. By Saito's theory [81, 82], this is a morphism of mixed Hodge structures, the mixed Hodge structure on $\psi R f_*\mathbb{Q}_{\mathscr{Y}}$ being the limit mixed Hodge structure.

Now for $t \in \Delta^*$, we have that the pullback $\pi_t^* : H^*(\mathscr{X}_t, \mathbb{Q}) \to H^*(\mathscr{Y}_t, \mathbb{Q})$ induces an injection $\mathrm{gr}_k^W H^k(\mathscr{X}_t, \mathbb{Q}) \to H^k(\mathscr{Y}_t, \mathbb{Q})$ for all k. By Theorem 5.11, for $k \le n$ we have an induced injection

$$\mathrm{Sym}^k H^2(\mathscr{X}_t, \mathbb{Q}) \to H^{2k}(\mathscr{Y}_t, \mathbb{Q})$$

and therefore also an injection

$$\mathrm{Sym}^k H^2(\mathscr{X}_\infty, \mathbb{Q}) \to H^{2k}(\mathscr{Y}_\infty, \mathbb{Q}).$$

Claim. *The image of the specialization* $\mathrm{sp}: H^{2k}(Y,\mathbb{Q}) \to H^{2k}(\mathscr{Y}_\infty,\mathbb{Q})$ *contains the image of* $\mathrm{Sym}^k H^2(\mathscr{X}_\infty,\mathbb{Q})$ *for* $k \le n$.

Proof. By the semisimplicity of the category of variations of polarized integral Hodge structures, $\mathrm{Sym}^k R^2 f_*\mathbb{Q}_{\mathscr{X}^*}$ is a summand of $R^{2k} g_*\mathbb{Q}_{\mathscr{Y}^*}$ for $k \le n$. By the decomposition theorem [81, Section 15.3], the intermediate extension $j_{!*}(\mathrm{Sym}^k R^2 f_*\mathbb{Q}_{\mathscr{X}^*}[1])[-2k-1]$ is a summand of $Rg_*\mathbb{Q}_{\mathscr{Y}}$, where $j:\Delta^*\to\Delta$ is the inclusion. Because the monodromy of $\mathrm{Sym}^k R^2 f_*\mathbb{Q}_{\mathscr{X}^*}$ is trivial, the specialization map

$$i^* j_{!*}(\mathrm{Sym}^k R^2 f_*\mathbb{Q}_{\mathscr{X}^*}[1]) \to \psi j_{!*}(\mathrm{Sym}^k R^2 f_*\mathbb{Q}_{\mathscr{X}^*}[1])$$

is an isomorphism, hence the claim. □

Now, $H^2(\mathscr{Y}_\infty,\mathbb{Q})$ has the same Hodge numbers as the general fiber (since the monodromy is trivial), and it follows that there is an element $w \in I^{2,0}H^2(Y,\mathbb{Q})$ mapping to a generator of $I^{2,0}H^2(\mathscr{Y}_\infty,\mathbb{Q})$. Here, $I^{2,0}$ denotes the $(2,0)$-part of the Deligne splitting, see e.g. [78, Lemma-Definition 3.4]. Moreover, $w^n \neq 0$ by the claim. The same is true on the normalization $\widetilde{Y}\to Y$, so some component of Y has a generically nondegenerate holomorphic 2-form. Finally, since $\mathscr{X}$ is log terminal, by [40, Corollary 1.5] the exceptional divisors of $\pi:\mathscr{Y}\to\mathscr{X}$ are uniruled, so the same must be true of X. □

Proof of Theorem 7.1. Obviously we may assume the monodromy of $R^2 f_*\mathbb{Q}_{\mathscr{X}^*}$ is trivial. Let $f:\mathscr{X}\to\Delta$ be a flat projective family restricting to the base change of $\mathscr{X}^*$ over Δ^*; we may assume the special fiber has no multiple component. By running the MMP as in the first part of [57, Theorem 2.1], we may assume $\mathscr{X}$ has terminal singularities, and so by Proposition 7.3 and Theorem 7.2 we may assume the special fiber X is a K-trivial variety with canonical singularities. By the proposition again and Theorem 3.4, X is symplectic. Take a $\mathbb{Q}$-factorial terminalization $\pi: Y\to X$ and consider the diagram (5.4) for π. With the notations used there, the deformation $\mathscr{Y}\to\mathrm{Def}(Y)$ is locally trivial by [73, Main Theorem]. By [73, Theorem 1] the induced map $p:\mathrm{Def}(Y)\to\mathrm{Def}(X)$ is finite and surjective. Thus, the classifying map $\Delta\to\mathrm{Def}(X)$ of the family $\mathscr{X}\to\Delta$ can be lifted to $\mathrm{Def}(Y)$ over a finite cover $\Delta'\to\Delta$. The pullback $\mathscr{Y}_{\Delta'}$ is then the claimed family; it only remains to show that it is isomorphic to the pullback $\mathscr{X}_{\Delta'}$ outside the central fiber. This is because for $t'\in\Delta'^*$ mapping to $t\in\Delta^*$ we have that $\mathscr{Y}_{t'}\to\mathscr{X}_t$ is a proper birational morphism between $\mathbb{Q}$-factorial terminal K-trivial varieties and thus is an isomorphism. □

Remark 7.4. The techniques of [57] are used to *fill in* varieties over projective period points in the interior of the period domain. We would like to point out that in the smooth case this technique of “filling holes” has been used independently by Odaka and Oshima for a different purpose, see [75, second paragraph in the first proof of Claim 8.10].

8. Monodromy and Torelli theorems

Fix a lattice Λ and denote its quadratic form by q.

Definition 8.1. A Hodge structure on Λ is *semi-polarized* (by q) if $q:\Lambda\otimes\Lambda\to\mathbb{Z}(-2)$ is a morphism of Hodge structures. We furthermore say a semi-polarized Hodge structure is *hyperkähler* if it is pure of weight two with $h^{2,0} = h^{0,2} = 1$, the signature of q is $(3, b_2 - 3)$, and q is positive-definite on the real space underlying $H^{2,0} \oplus H^{0,2}$. Hyperkähler Hodge structures on Λ are parametrized by the period domain

$$\Omega_\Lambda := \{[\sigma] \in \mathbb{P}(\Lambda_\mathbb{C}) \mid q(\sigma) = 0,\, q(\sigma, \bar{\sigma}) > 0\}.$$

Let X^+ be a primitive symplectic variety with

$$(H^2(X^+, \mathbb{Z})_{\mathrm{tf}}, q_{X^+}) \cong (\Lambda, q),$$

and let $\mathfrak{M}^+$ be the moduli space of Λ-marked locally trivial deformations of X^+. Note that $\mathfrak{M}^+$ is a union of connected components of the full moduli space $\mathfrak{M}_\Lambda$ of Λ-marked primitive symplectic varieties from Section 6.12.

Set $\Omega := \Omega_\Lambda$. We have a period map $P : \mathfrak{M}^+ \to \Omega$ which is a local isomorphism by the local Torelli theorem (Proposition 5.5). Furthermore, inseparable points of $\mathfrak{M}^+$ lie above proper Mumford–Tate subdomains of Ω by Corollary 6.15, so as in[6] [47, Corollary 4.10] we have a factorization

$$\begin{array}{ccccc} & & \overline{\mathfrak{M}}^+ & & \\ & \overset{H}{\nearrow} & & \overset{\overline{P}}{\searrow} & \\ \mathfrak{M}^+ & & \xrightarrow[P]{\qquad\qquad} & & \Omega, \end{array}$$

where H is the Hausdorff reduction of $\mathfrak{M}^+$ and $\overline{P}$ is a local homeomorphism. For each $x \in \overline{\mathfrak{M}}^+$, a local basis is provided by images $H(B)$ of open balls $x \in B \subset \mathfrak{M}^+$ over which there is a universal family for x.

Note that $\mathrm{O}(\Lambda)$ acts on each of $\mathfrak{M}^+$, $\overline{\mathfrak{M}}^+$, and Ω by changing the marking, and the three maps $H, P, \overline{P}$ respect these actions. For any connected component $\mathfrak{M}$ of $\mathfrak{M}^+$, we define $\mathrm{Mon}(\mathfrak{M}) \subset O(\Lambda)$ to be the image of the monodromy representation on second cohomology, which is defined up to conjugation.

The goal of this section is to show:

Theorem 8.2. *Assume* $\mathrm{rk}(\Lambda) \geq 5$ *and let* $\mathfrak{M}$ *be a connected component of* $\mathfrak{M}^+$.

(1) *The monodromy group* $\mathrm{Mon}(\mathfrak{M}) \subset \mathrm{O}(\Lambda)$ *is of finite index.*
(2) $\overline{P}$ *is an isomorphism of* $\overline{\mathfrak{M}}$ *onto the complement in* Ω *of countably many maximal Picard rank periods.*
(3) *If* X^+ *is* $\mathbb{Q}$*-factorial and terminal, then the same is true of every point* $(X, \mu) \in \mathfrak{M}$ *and* $\overline{P}$ *is an isomorphism of* $\overline{\mathfrak{M}}$ *onto* Ω.

Theorem 8.2 immediately yields parts (1), (3), and (4) of Theorem 1.1. Before the proof, we briefly recall the classification of orbit closures in Ω under an arithmetic lattice, which is crucial to the argument.

8.3. Reminder on orbit closures.

Definition 8.4. The *rational rank* of a hyperkähler period $p \in \Omega$ is defined as

$$\mathrm{rrk}(p) := \dim_\mathbb{Q}\big((H^{2,0} \oplus H^{0,2}) \cap \Lambda_\mathbb{Q}\big) \in \{0, 1, 2\}.$$

[6] Huybrechts uses that the inseparability only occurs above Noether–Lefschetz loci, but the same argument works for any countable union of proper complex analytic subvarieties.

We define the rational rank of a primitive symplectic variety to be the rational rank of its Hodge structure on second cohomology.

Recall that the period domain Ω can be thought of as the oriented positive Grassmannian $\mathrm{Gr}^{++}(2, \Lambda_{\mathbb{R}})$. For a rational positive-definite sublattice $\ell \subset \Lambda_{\mathbb{Q}}$ with $\mathrm{rk}(\ell) \leq 2$, we define T_ℓ to be the locus of periods for which $\ell \subset (H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$. Obviously we have that $T_\ell \supset T_{\ell'}$ if $\ell \subset \ell'$. Note that if $\mathrm{rk}(\ell) = 2$, then T_ℓ is a pair of conjugate maximal Picard rank points (and all such pairs arise this way). For $\mathrm{rk}(\ell) = 1$, the set T_ℓ is isomorphic to the space $S^+(\ell_{\mathbb{R}}^{\perp})$ positive unit-norm vectors in $\ell_{\mathbb{R}}^{\perp}$, which is a totally real submanifold of Ω of real dimension $\mathrm{rk}(\Lambda) - 2$.

The important point is that orbit closures for the action of a finite index subgroup Γ of $\mathrm{O}(\Lambda)$ on the period domain Ω are classified according to rational rank.

Proposition 8.5 ([89, Theorem 4.8] and [90, Theorem 2.5]). *Assume* $\mathrm{rk}(\Lambda) \geq 5$. *We have for* $p \in \Omega$*:*

(1) *If* $\mathrm{rrk}(p) = 0$, *then* $\overline{\Gamma \cdot p} = \Omega$.

(2) *If* $\mathrm{rrk}(p) = 1$, *then* $\overline{\Gamma \cdot p}$ *is a (countable) union of* T_ℓ *with* $\mathrm{rk}(\ell) = 1$.

(3) *If* $\mathrm{rrk}(p) = 2$, *then* $\overline{\Gamma \cdot p}$ *is a (countable) union of* T_ℓ *with* $\mathrm{rk}(\ell) = 2$.

8.6. Proof of Theorem 8.2. We divide the proof into five steps. Parts (1), (2), and (3) are proven in Steps 4, 5 (a), and 5 (b), respectively.

Step 1. *Let* $p \in \Omega$ *be a very general period with Picard group generated by a positive vector. Then* $\overline{P}^{-1}(p)$ *is finite.*

Proof. In fact, its equivalent to show $P^{-1}(p)$ is finite by the assumption on the Picard rank. For the following lemma, we say an ample line bundle L on a primitive symplectic variety X has BBF square d if $q_X(\mathrm{c}_1(L)) = d$.

Proposition 8.7. *Pairs* (X, L) *consisting of a primitive symplectic variety* X *of a fixed locally trivial deformation type and an ample line bundle* L *with fixed BBF square form a bounded family.*

Proof. Using that the Fujiki constants are locally trivially deformation-invariant and [64, Theorem 2.4], for any such pair (X, L), the variety X can be embedded with bounded degree in $\mathbb{P}^N$ for some fixed N via the sections of some fixed power L^k. Let H be the corresponding Hilbert scheme of subschemes of $\mathbb{P}^N$ of bounded degree, and let $f : \mathscr{X} \to H$ be the universal family. Let $H' \subset H$ denote the subset over which the fibers of f are primitive symplectic. By semi-continuity and openness of symplecticity, $H' \subset H$ is open.

Lemma 8.8. *There is a stratification of* H' *by locally closed reduced subschemes over which* $\mathscr{X}$ *is locally trivial.*

Proof. There is a stratification H_i of H' by locally closed reduced subschemes along which the second Betti numbers $(R^2 f_* \mathbb{Q}_{\mathscr{X}})_t$ are constant, for instance by using étale cohomology. By Corollary 5.23, $\mathscr{X}$ is locally trivial in an analytic neighborhood of every point t in each H_i, and so $\mathscr{X}$ is locally trivial on each H_i. □

It follows from the lemma that the set of pairs (X, L) as in the statement of the proposition together with a choice of an embedding into $\mathbb{P}^N$ as above is a locally closed subscheme U of H. The $\mathbb{C}$-points of the quotient stack $[\mathrm{PGL}_{N+1}\backslash U]$ then parametrize isomorphism classes of the pairs (X, L). The PGL_{N+1} action has finite stabilizers on U by Lemma 4.6, so by general theory $[\mathrm{PGL}_{N+1}\backslash U]$ is a Deligne–Mumford stack and there is a finite-type étale atlas $S \to [\mathrm{PGL}_{N+1}\backslash U]$.

To summarize, there is (depending on the fixed locally trivial deformation type and the fixed BBF square) a finite-type scheme S and a locally trivial family $\mathscr{X} \to S$ of primitive symplectic varieties and a relatively ample $\mathscr{L}$ on $\mathscr{X}$ which has the property that every (X, L) as in the statement of the lemma appears finitely many times (and at least once) as a fiber. □

Each component S_0 of the scheme S constructed in the proof of the lemma has a period map of the form $P_v : S_0 \to \mathrm{O}(v^\perp)\backslash\Omega_{v^\perp}$ for some $v \in \Lambda$ with fixed square $q(v) = d$, where we think of $\Omega_{v^\perp} = \mathbb{P}(v^\perp) \cap \Omega$. Moreover, P_v is a local isomorphism and therefore quasifinite, as by, e.g., [11, Theorem 3.10] the fibers are algebraic.

Now, for $p \in \Omega$ as in the original claim, suppose $q(v) = d$ for a generator v of the Picard group. It follows that there are finitely many isomorphism classes of pairs (X, L), where X is a primitive symplectic variety that is locally trivially deformation-equivalent to X^+, and L is an ample bundle of BBF square d, and the primitive parts of $H^2(X, \mathbb{Z})_{\mathrm{tf}}$ and p are abstractly isomorphic as polarized Hodge structures. By the assumption on the Picard rank, there are then finitely many isomorphism classes of projective X locally-trivially deformation equivalent to X^+ and with $H^2(X, \mathbb{Z})_{\mathrm{tf}}$ abstractly isomorphic to p as semi-polarized Hodge structures. Moreover, $\mathrm{Aut}(p) = \pm 1$, so for each such X there are finitely many such isomorphisms.

To finish, by Theorem 6.9 every point in $P^{-1}(p)$ is projective and uniquely polarized by a class of BBF square d, and the claim follows. □

For the next step, let $\Omega_{\mathrm{rrk}=0} \subset \Omega$ be the rational-rank-zero locus, let $\overline{\mathfrak{M}}^+_{\mathrm{rrk}=0} \subset \overline{\mathfrak{M}}^+$ be the preimage of $\Omega_{\mathrm{rrk}=0}$ under $\overline{P}$, and let $\overline{P}_{\mathrm{rrk}=0}$ be the restriction of $\overline{P}$ to $\overline{\mathfrak{M}}^+_{\mathrm{rrk}=0}$. Note that since we are assuming $\mathrm{rk}(\Lambda) \geq 5$, every $p \in \Omega_{\mathrm{rrk}=0}$ has dense $\mathrm{O}(\Lambda)$-orbit, by Proposition 8.5.

Step 2. *The map $\overline{P}_{\mathrm{rrk}=0}$ is a covering map onto $\Omega_{\mathrm{rrk}=0}$.*

Proof. The claim follows from the following two lemmas.

Lemma 8.9. *The map $\overline{P}_{\mathrm{rrk}=0}$ has finite fibers of constant size. In particular, it is surjective onto $\Omega_{\mathrm{rrk}=0}$.*

Proof. By the previous step there is a point $p_0 \in \Omega_{\mathrm{rrk}=0}$ over which $\overline{P}^{-1}(p_0)$ is finite of size N, and therefore $\overline{P}^{-1}(p)$ is finite of size $\leq N$ for every point $p \in \Omega_{\mathrm{rrk}=0}$. Indeed, if some $p \in \Omega_{\mathrm{rrk}=0}$ had at least $N + 1$ preimages, then by Hausdorffness we can find pairwise non-intersecting open neighborhoods around any $N + 1$ points in the fiber $\overline{P}^{-1}(p)$ that map isomorphically to the same open neighborhood V of p, but p_0 has dense orbit. Interchanging p_0 and p, we see that in fact the fibers are finite of constant size. □

Lemma 8.10. *Suppose $f : X \to Y$ is a local homeomorphism between two Hausdorff*[7] *topological spaces. If f has finite fibers of constant size, it is a covering map onto its image.*

Proof. For any $y \in Y$, because $f^{-1}(y)$ is finite we may find nonintersecting open sets U_x around each point $x \in f^{-1}(y)$ on which f is a homeomorphism, and by shrinking we may further assume all the U_x have the same image U. It follows from the assumption on fiber size that $f^{-1}(U) = \bigcup_{x \in f^{-1}(y)} U_x$. □

The claim is proved. □

Step 3. *The map $\overline{P}_{\mathrm{rrk}=0}$ is an isomorphism of $\overline{\mathfrak{M}}_{\mathrm{rrk}=0}$ onto $\Omega_{\mathrm{rrk}=0}$.*

Proof. The rational-rank-zero locus is, in the notation of Section 8.3,

$$\Omega_{\mathrm{rrk}=0} := \Omega \setminus \bigcup_{\ell \neq 0} T_\ell,$$

and each T_ℓ is a closed submanifold of real codimension $\mathrm{rk}(\Lambda) - 2$. Assuming $\mathrm{rk}(\Lambda) \geq 5$, we have that $\Omega_{\mathrm{rrk}=0}$ is locally path-connected and path-connected by [88, Lemma 4.10] and moreover locally simply connected and simply connected by the following lemma, as the same is true of Ω.

Lemma 8.11. *If M is a simply connected smooth manifold and S is a countable union of closed submanifolds of (real) codimension ≥ 3, then $M \setminus S$ is simply connected.*

Proof. This argument is taken from a MathOverflow answer of Martin M. W. [93]. The result is well known when S is a single closed submanifold of codimension ≥ 3. The space of nullhomotopies $S^1 \times [0,1] \to M$ of a given path with the compact open topology is a Baire space and the set of homotopies avoiding a single closed submanifold of codimension ≥ 3 is a dense open subset. Therefore, the set of homotopies avoiding S is nonempty (and in fact dense) by definition of a Baire space. □

Thus, the claim follows from the previous step. □

Step 4. *The subgroup* $\mathrm{Mon}(\mathfrak{M})$ *has finite index in* $\mathrm{O}(\Lambda)$.

Proof. The map $\overline{P} : \overline{\mathfrak{M}}^+_{\mathrm{rrk}=0} \to \Omega_{\mathrm{rrk}=0}$ has finite degree and $\Omega_{\mathrm{rrk}=0}$ is path-connected. Therefore, $\mathfrak{M}^+$ has finitely many connected components. The group $\mathrm{Mon}(\mathfrak{M})$ is the stabilizer of the component $\mathfrak{M}$, and is therefore finite index. □

Step 5 (a). *The map $\overline{P}$ is an isomorphism of $\overline{\mathfrak{M}}$ onto the complement in Ω of countably many maximal Picard rank periods.*

Proof. By Step 3, it is enough to show that the image of $\mathfrak{M}$ under P contains the locus $\Omega_{\mathrm{rrk}\leq 1}$ of non-maximal Picard rank periods. The image is open and $\mathrm{Mon}(\mathfrak{M})$-invariant, whereas by Proposition 8.5 and the previous step a $\mathrm{Mon}(\mathfrak{M})$ orbit closure in Ω must be a union

[7] In fact, only Hausdorffness on the source is used.

of T_ℓ or all of Ω. It is therefore enough to show that for any rank one sublattice $\ell \subset \Lambda$, a very general point of T_ℓ is contained in $P(\mathfrak{M})$.

Considering a projective $(X, \mu) \in \mathfrak{M}$ with a polarization v that is orthogonal to ℓ, we obtain a period map $P_v : S_0 \to \mathrm{O}(v^\perp)\backslash\Omega_{v^\perp}$ as in Step 1 corresponding to a family of locally trivial deformations of X over S_0. The complement of $P_v(S_0)$ is a locally closed subvariety of $\mathrm{O}(v^\perp)\backslash\Omega_{v^\perp}$ and its preimage V in $\Omega_{v^\perp}$ is therefore also a locally closed analytic subvariety.

It suffices to show that $T_\ell \cap \Omega_{v^\perp}$ is not contained in V. But T_ℓ is totally real and has half the (real) dimension of Ω, so the tangent space to $T_\ell \cap \Omega_{v^\perp}$ at a point p is not contained in any proper complex subspace of $T_p\Omega_{v^\perp}$. It follows that if T_ℓ were contained in V, it must be contained in the singular locus of V, and so by induction we get a contradiction. □

Step 5 (b). *When X^+ is $\mathbb{Q}$-factorial and terminal, then the same is true of every point $(X, \mu) \in \mathfrak{M}$ and $\overline{P}$ is an isomorphism of $\overline{\mathfrak{M}}$ onto Ω.*

Proof. The first claim follows from Lemma 5.20. For the second claim, by the previous step, it remains to show $P(\mathfrak{M})$ contains all maximal Picard rank points, which are in particular projective by Theorem 6.9.

Now for any maximal Picard rank period p, let $v \in \Lambda$ be a positive vector which is Hodge with respect to p. A very general deformation of p for which v remains algebraic is in the image of P, and the period map $P_v : S_0 \to \mathrm{O}(v^\perp)\backslash\Omega_{v^\perp}$ from Step 1 is dominant, so we can find a curve $B \subset \mathrm{O}(v^\perp)\backslash\Omega_{v^\perp}$ through p such that an open set $U \subset B$ lifts to S_0, possibly after a base change. Now apply[8)] Proposition 7.1. □

This concludes the proof.

Remark 8.12. The argument given in Step 1 of the proof of the theorem together with Huybrechts' surjectivity of the period map [44, Theorem 8.1] implies that the monodromy group is of finite index for irreducible symplectic manifolds even when $b_2 = 4$ (for $b_2 = 3$ it is automatic). For an argument not using Huybrechts' theorem, see [91, Theorem 2.6].

Remark 8.13. Some ideas similar to those appearing in the proof of Theorem 8.2 have also been used recently by Huybrechts [48] to prove some finiteness results for hyperkähler manifolds, and these arguments can likely be adapted to the singular setting.

9. $\mathbb{Q}$-factorial terminalizations

If X is an algebraic variety, then by [9, Corollary 1.4.3] there exists $\mathbb{Q}$-factorial terminalization $\pi : Y \to X$. This is often crucial in the theory of singular symplectic varieties. On the other hand, even if you are mainly interested in projective symplectic varieties, it is often necessary to consider also compact Kähler varieties and certainly the methods of [9] are not yet established in the Kähler case. The main result of this section, Theorem 9.1, partially remedies this in the case of primitive symplectic varieties. If we start with a primitive symplectic variety with second Betti number ≥ 5, it establishes the existence of $\mathbb{Q}$-factorial terminalizations on a bimeromorphic model which is locally trivially deformation equivalent to the initial variety.

8) Or [57, Theorem 1.7] in the smooth case.

In fact, by Theorem 6.14 the following is slightly stronger, though we expect it to be equivalent. For a normal variety X we denote by ω_X the push forward of the canonical bundle along the inclusion of the regular locus.

Theorem 9.1. *Let X be a primitive symplectic variety satisfying $b_2(X) \geq 5$. Then there exist a primitive symplectic variety X' which is inseparable from X in (locally trivial) moduli and a $\mathbb{Q}$-factorial terminalization of X', i.e., a proper bimeromorphic morphism $\pi : Y \to X'$ such that Y has only $\mathbb{Q}$-factorial terminal singularities and $\pi^*\omega_{X'} = \omega_Y = \mathcal{O}_Y$. In particular, Y is a primitive symplectic variety.*

As a consequence of the fact that bimeromorphic varieties without compact curves are isomorphic, see e.g. the proof of Corollary 6.15, we obtain:

Corollary 9.2. *Let X be as in Theorem* 9.1*, and additionally assume it has Picard rank zero. Then X has a $\mathbb{Q}$-factorial terminalization.* □

The proof of Theorem 9.1 is obtained by combining Proposition 5.22 with Corollary 6.11, Theorems 6.14 and 8.2, and the existence of $\mathbb{Q}$-factorial terminalizations of projective varieties.

Proof of Theorem 9.1. Consider the universal locally trivial deformation $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ and choose $t \in \mathrm{Def}^{\mathrm{lt}}(X)$ nearby such that $X_0 := \mathscr{X}_t$ is projective. Take a $\mathbb{Q}$-factorial terminalization $Y_0 \to X_0$, denote N the q_{Y_0}-orthogonal complement of $H^2(X_0, \mathbb{Q})$ in $H^2(Y_0, \mathbb{Q})$, and consider the universal deformation of the pair

$$\begin{array}{ccc} \mathscr{Y}_0 & \longrightarrow & \mathscr{X}_0 \\ \downarrow & & \downarrow \\ \mathrm{Def}(Y_0, N) & \xrightarrow{\;p\;} & \mathrm{Def}^{\mathrm{lt}}(X_0) \end{array}$$

given by Proposition 5.22. By Lemma 5.20, we may assume that every fiber of the map $\mathscr{Y}_0 \to \mathrm{Def}(Y_0, N)$ is $\mathbb{Q}$-factorial and by local triviality and Theorem 3.4, every fiber has terminal singularities. In other words, for all $s \in \mathrm{Def}(Y_0, N)$ the morphism $(\mathscr{Y}_0)_s \to (\mathscr{X}_0)_{p(s)}$ is a $\mathbb{Q}$-factorial terminalization.

If $\mathrm{rrk}(X) = 0$, then by Proposition 8.5 and Theorem 8.2 there is a point $t' \in \mathrm{Def}^{\mathrm{lt}}(X_0)$ such that the fiber $X' := (\mathscr{X}_0)_{t'}$ and X are inseparable in moduli. By construction, X' has a $\mathbb{Q}$-factorial terminalization, and by Theorem 6.14, X' is bimeromorphic to X. If $\mathrm{rrk}(X) = 1$, projective periods are still dense in the orbit closure of the period of X by Theorem 6.9, so the same argument can be applied by choosing the period of X_0 to be in the orbit closure of the period of X. Finally, varieties X with $\mathrm{rrk}(X) = 2$ are projective so there the result is known anyway by [9, Corollary 1.4.3]. □

As an application, we can give examples of divisorially $\mathbb{Q}$-factorial but not $\mathbb{Q}$-factorial varieties.

Example 9.3. Consider a projective irreducible symplectic manifold Y of dimension $2n$ admitting a small contraction $\pi : Y \to X$, where X is a projective primitive symplectic variety and the exceptional locus of π is isomorphic to $\mathbb{P}^n$. As π has connected fibers, $\mathbb{P}^n$

must be contracted to a point and thus X has an isolated singularity. Such examples can be realized on the Hilbert scheme $Y = S^{[n]}$ of n points on a K3 surface S containing a smooth rational curve. As the contraction is small, the variety X is not $\mathbb{Q}$-factorial. By [5, Theorem 4.1, Propositions 4.5 and 5.8], this contraction deforms over a smooth hypersurface in $\mathrm{Def}(Y)$.

Let us denote by $\ell \subset \mathbb{P}^n \subset Y$ a line and by L the unique line bundle on Y satisfying $q_Y(\mathrm{c}_1(L), \cdot) = (\ell, \cdot)$, where the right–hand side denotes the pairing $N_1(Y)_{\mathbb{Q}} \otimes N^1(Y)_{\mathbb{Q}} \to \mathbb{Q}$. It follows that $\mathrm{c}_1(L)$ is q-orthogonal to the pullback of any ample divisor on X, hence we have $q_Y(\mathrm{c}_1(L)) = q_Y(\ell) < 0$. Replacing X by a small locally trivial deformation, we may assume:

(1) The contraction $\pi : Y \to X$ deforms and has $\mathbb{P}^n$ as its exceptional set.

(2) The varieties X and Y are Kähler and non-algebraic such that the Picard group of X is trivial and $\mathrm{Pic}(Y)$ has rank one.

(3) There are no divisors on Y, in particular, X is not $\mathbb{Q}$-factorial but divisorially $\mathbb{Q}$-factorial in the sense of Definition 2.14.

For (3), if L^n were represented by an effective divisor D, then since $q_X(D) < 0$ it is exceptional [12, Theorem 4.5] and hence uniruled [12, Proposition 4.7]. However, the only curves on Y are the ones contracted by π.

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