Observation of Wigner-Dyson level statistics in a classically integrable system

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Resonances in particle transmission through a 1D finite lattice are studied in the presence of a finite number of impurities. Although this is a one-dimensional system that is classically integrable and has no chaos, studying the statistical properties of the spectrum such as the level spacing distribution and the spectral rigidity shows the same statistics as the one obtained for chaotic systems. Using a dimensionless parameter that reflects the degree of state localization, we demonstrate how the transition from Poisson-level statistics to the Wigner-Dyson is affected by state localization. The resonance positions are calculated using both the Wigner-Smith time delay and a Siegert state method, which are in good agreement. Our results show the dependence of the level statistics on the localization length as it evolves from a Poisson distribution to Wigner-Dyson.

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I. INTRODUCTION

In 1984, Bohigas, Giannoni, and Schmit (BGS) stated the celebrated conjecture [1] that describes the statistical properties of chaotic spectra. This conjecture states that a quantum system whose classical analog is chaotic will have an energy-level spectrum that obeys Wigner-Dyson statistics characteristic of the Gaussian orthogonal ensemble (GOE) of random matrix theory (RMT). While GOE-level statistics have been confirmed for many classically chaotic systems, GOE statistics has also been found in the bound-level spectrum of systems that display no classical chaos [2–4]. The present study extends the analysis to the resonance spectrum in a simple 1D system by documenting that an Anderson localization-type model Wigner-Dyson GOE energy-level statistics in some parameter ranges. Such result shows that the level repulsion, as implied by GOE-level statistics, is not exclusively limited to chaotic systems. In other words, the *converse* of the BGS conjecture *(i.e., the statement that observation of GOE-level statistics would imply that the corresponding Newtonian system exhibits chaos)* is not true. The study in this paper serves as a counterexample. Classical chaos is a consequence of the nonlinearity of the Newtonian equations of motion, while Schrödinger's equation is linear and, strictly speaking, has no chaos. Nevertheless, quantum signatures of chaos can arise and are exhibited by the statistical properties of the quantum energy-level spectra, such as the level spacing distribution and the spectral rigidity (SR) [1,5].

The spectrum of random matrices was studied by Wigner in 1951 [6–8], who demonstrated the existence of a few universal classes based on the symmetry imposed on such matrices. In the present study, the two classes considered are the GOE and the Poisson distribution.

The claimed connection between chaos and RMT is the essence of the BGS conjecture, which relates the spectral properties of quantum systems whose classical Hamiltonians are irregular (chaotic) to the GOE class (for systems that satisfy time-reversal invariance). The BGS conjecture is formally stated as follows: Spectra of time-reversal-invariant systems whose classical analogs are K systems [9] show the same fluctuation properties as predicted by GOE [1]. In short, the BGS paper asserts *classical chaos implies quantum GOE*. On the other hand, the quantum spectrum of a classically regular system in two or more dimensions follows Poissonian behavior [1,10,11], except for systems with oscillatorlike or otherwise very simple spectra.

The goal of the present paper is to explore a one-dimensional quantum system whose classical analog is regular. Our analysis demonstrates that such systems have Wigner-Dyson level statistics in some regimes. Moreover, two limiting cases are discussed: the regular case that give the known Poisson distribution for classically integrable systems and a second case whose quantum spectrum matches the GOE statistics for both the nearest-neighbor distribution and the SR. The nature of the distributions is shown to hinge on a dimensionless parameter that represents the degree of state localization similarly to the results in Refs. [2,4], thereby extending the connection between the statistical properties of the quantum spectrum in such systems and the phenomenon of Anderson localization [12,13] to include both bound and resonance states. While RMT formally deals with discrete

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spectra only, our study discusses it in the context of very narrow resonances, which is justifiable because of their generally extreme narrowness.

II. INTRODUCING THE MODEL

The model considered in this paper is a particle moving through a one-dimensional lattice with the lattice potential energy modeled by a sum of delta functions, one per lattice site. Thus the Hamiltonian is given by

$$H = \frac{P^2}{2m} + \sum_{n=-N/2}^{N/2} \alpha_n \delta(x - na). \tag{1}$$

Here $N + 1$ is the total number of lattice sites and α_n is the strength of the nth delta function. In the case of a perfectly clean periodic but finite lattice that we consider here, all α_n are the same and all are negative, but we will keep the notation general for now because part of our analysis will be an exploration of the effect of impurities. The reason behind choosing the delta function potential is the fact that an attractive delta function potential admits one bound state, so one can imagine the system as having one atomic state around each atomic site, which allows us to treat both the bound states and scattering dynamics. Modeling the lattice by considering one atomic state at each site and treating the effect of site-to-site tunneling as an effective hopping parameter has been studied for years and is referred to as the tight-binding approximation [12,13]. One of our goals is to compare the exact solution with the results of the tight-binding model. Our results show limitations of this approximation that become relevant in the context of transmission through a finite lattice.

In the tight-binding approximation, two things are assumed: First, there exists one atomic state around each lattice site and, second, hopping occurs only between nearest neighbors [14]. The Hamiltonian in this case is written as

$$H = \sum_i \epsilon_i a_i^\dagger a_i + \sum_{<i,j>} t_{i,j} a_i^\dagger a_j, \tag{2}$$

where ϵ_i is the energy of the atomic state site i and $t_{i,j}$ is the tunneling amplitude from site j to site i, and the sum in the second term is taken for nearest neighbors where $j = i \pm 1$. Both can be calculated from the potential introduced in Eq. (1). If the lattice is periodic and all atomic sites are identical, then this model can be solved analytically [15]. However, there is much interesting physics to study when impurities are placed in the lattice, such as the transport across the lattice. In that case, the periodicity is broken and there is no general analytical solution. However, the spectrum can be obtained by writing the Hamiltonian in a matrix form and diagonalizing it [13].

The tight-binding approximation is useful in many cases, especially for the $N \to \infty$ limiting case. Most of the physics of bound states can be studied within this simple approximation. However, tight-binding has limitations, such as the fact that a set of N negative delta functions do not necessarily support N bound states. Even two delta functions in one dimension do not necessarily have two bound states, which can cause difficulties whenever finite lattices are considered. Second, within tight binding, one can only study bound states, which does not allow exploration of the physics of scattering processes, such as transmission resonances.

The approach used in this paper is to consider a finite lattice as a finite range potential. Then one column of the the scattering (S) matrix is obtained for a particle incident from $-\infty$, i.e., from the left, and the second column corresponds to a particle incident from $+\infty$, i.e., from the right, and all desired observables calculated. The resonance positions and widths can be calculated from either the Wigner-Smith time-delay maxima [16–18] or by imposing outgoing-wave Siegert state boundary conditions [19,20] and determining eigenvalues of a complex symmetric Hamiltonian. A numerical solution is obtained for lattices with different values of the lattice size and the lattice constant. A main goal here is to study the real part of the resonance energy-level distribution in the first energy band. In all calculations, atomic units are used, i.e., with $\hbar = a_0 = m_e = E_h = 1$.

III. DESCRIPTION OF THE CALCULATIONS

The solution to the time-independent Schrödinger equation for the Hamiltonian introduced in Eq. (1) has the following form for particles incident from the left:

$$\psi(x) = \begin{cases} e^{iqx} + r(q)e^{-iqx} & \text{if } x < \frac{-Na}{2} \\ t(q)e^{iqx} & \text{if } x > \frac{Na}{2}, \end{cases} \tag{3}$$

where r and t are the reflection and transmission amplitudes, respectively, and q is the momentum of the incident particle. To obtain both r and t, the solution inside the lattice has to be obtained.

In the domain $x \in [(n - \frac{1}{2})a, (n + \frac{1}{2})a]$, the general solution is

$$\Psi_n(x) = A_n e^{iqx} + B_n e^{-iqx} \equiv \psi_n^+ + \psi_n^-, \tag{4}$$

where $q = \frac{\sqrt{2mE}}{\hbar}$ and m is the particle mass. After applying the wave-function continuity and derivative discontinuity conditions, the solutions in any two adjacent regions separated by a lattice constant a are related by the transfer matrix [21] as follows:

$$\begin{pmatrix} \psi_{n+1}^+ \\ \psi_{n+1}^- \end{pmatrix} = \begin{pmatrix} e^{iqa} \frac{1+m\alpha_n}{iq} & \frac{m\alpha_n}{iq} \\ \frac{-m\alpha_n}{iq} & e^{-iqa} \frac{1-m\alpha_n}{iq} \end{pmatrix} \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix}$$
$$\equiv T(\alpha_n, q) \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix}. \tag{5}$$

To obtain the full S matrix, both transmission (reflection) amplitudes should also be obtained in the case of scattering by particles incident from the right, denoting them as $t'(q)$ and $r'(q)$. In one dimension, the S matrix has the form [22,23]

$$S = \begin{pmatrix} t & r' \\ r & t' \end{pmatrix}. \tag{6}$$

The dramatic qualitative difference between the periodic and the disordered cases become clear from plots of the Wigner-Smith time delay [16], $Q = iS\frac{dS^\dagger}{dE}$, and the phase shifts [17]. The trace of Q gives the total time delay. Figure 1 shows that there are no resonances in the exactly periodic case with no impurities, and only a simple, regular oscillation

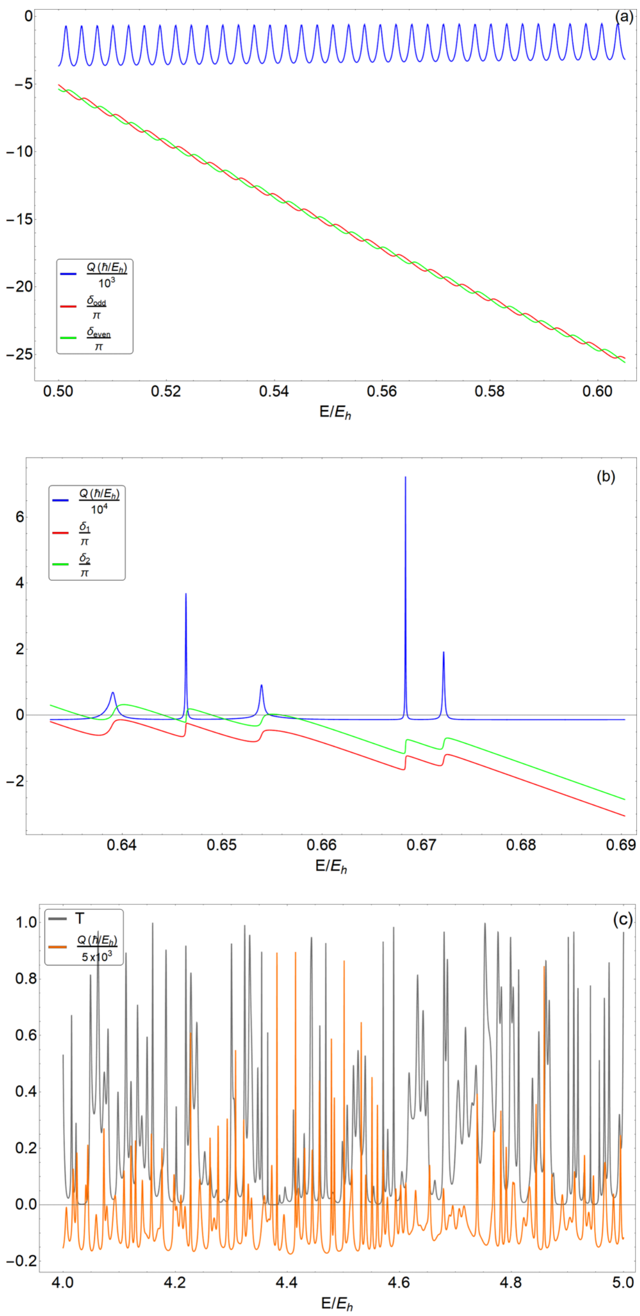

FIG. 1. (a) The blue curve is the total time delay and the green (red) curves are the even- (odd-) parity phase shifts for the periodic case with $\alpha_n = -1.5$ for all sites, $a = 0.8$ and $m = 1$. (b) The blue curve is the total time delay, the red and the green curves are the eigenphase shifts for a lattice with impurities where $\alpha_n = -1.5$ for 90% of the atoms and $\alpha_n = -1.9$ for the rest. The same mass and lattice constant as in the periodic case are used. (c) The transmission probability and the total time delay are plotted versus the collision energy where $\alpha_n = -1.5$ for 95% of the atoms and $\alpha_n = -1.9$ for the rest. The gray and orange curves are the transmission coefficient and the total time delay, respectively.

of the total time delay as a function of collision energy. On the contrary, when impurities are present, there are many narrow resonances. Associated with each resonance is a peak in the time delay and a clear rise in the sum of the eigenphase shift sum by π radians as a function of energy. The statistical properties of all the resonances in the first band show chaos signatures in the system, as is demonstrated next.

The resonances can also be calculated using a different method: The Siegert state [20] boundary conditions allows only outgoing waves, and they take the form

$$\psi(x) = \begin{cases} Ae^{-iqx} & \text{if } x < \frac{-Na}{2} \\ Be^{iqx} & \text{if } x > \frac{Na}{2}. \end{cases} \tag{7}$$

With this boundary condition, the Hamiltonian is non-Hermitian and the spectrum is complex. Each eigenvalue can be written as $E_j = E_{0j} - i\Gamma_j/2$, where E_{0j} is the position of the jth resonance and Γ_j is the width [20,24]. After applying the boundary conditions in Eq. (7), the energies are given by the roots of following equation:

$$M_{2,2} = 0, \tag{8}$$

where M is the total transfer matrix given by $M(q) = \prod_{n=1}^{N} T(\alpha_n, q)$.

Since the Hamiltonian in Eq. (1) is one-dimensional and has no classical chaos [11], the nearest-neighbor distribution of the resonances is expected to follow a Poisson distribution, $P(s) = e^{-s}$, and no level repulsion is expected as shown in most of the studied cases [25]. On the other hand, classically chaotic systems are expected to have GOE statistics, and their level spacing distribution is expected to follow the Wigner-Dyson distribution [7], $P(s) = \frac{\pi}{2} s e^{-\frac{\pi s^2}{4}}$. The one key difference between the two distributions is that in chaotic systems, there is strong level repulsion [26], leading to a vanishing of the nearest-neighbor level distribution in the limit of zero spacing; on the other hand, this feature is not mentioned in the BGS conjecture to arise in classically regular systems. The BGS conjecture only connects classically nonintegrable systems with GOE statistics in the quantum spectrum. Both distributions can be written more compactly in a convenient form as different limiting cases of the following:

$$P^{\beta}(s) = A(\beta)\left(\frac{\pi s}{2}\right)^{\beta} e^{-(\pi^2\beta/16)s^2 - [B(\beta) - (\pi\beta/4)]s}, \tag{9}$$

where $\beta = 0(1)$ corresponds to the Poisson (Wigner-Dyson) distribution. $A(\beta)$, and $B(\beta)$ are fixed by the conditions $\int_0^\infty P^\beta(s)ds = \int_0^\infty sP^\beta(s)ds = 1$. β is the Izrailev parameter [27] that reflects the degree of chaos. Equation (9) is not the only function that can be used. For example, one can use the Brody distribution [28], but it was shown by Izrailev in Refs. [27,29] that the Izrailev parameter is related to the state localization. Following Wigner's early studies of RMT [6,7], extensive efforts have generalized the statistical properties of both Hermitian and non-Hermitian systems [8,30–34].

IV. RESULTS AND DISCUSSION

In this section, we present the main results and address conclusions about the statistical distribution of the system. A discussion about the bound-state spectrum is also shown to limit the validity of the tight-binding model, notably for finite lattices.

TABLE I. The transition, as measured by β in Eq. (9), from the Poisson to Wigner distribution is presented as a function of the localization parameter $Z = \frac{<\sigma_x>}{L}$. The values in this table are taken for different numbers of impurities, namely, from 2% to 20%.

Z	0.0419	0.0483	0.0987	0.1105	0.1239	0.1397	0.1429	0.1446
β	0.0002	0.3016	0.6049	1.0654	1.2972	1.6773	1.9281	2.7226
χ_r^2	0.4506	0.2930	0.1811	0.0942	0.0324	0.6854	0.1861	0.6066

A. The level statistics

The results obtained from both the time-delay analysis and Eq. (8) agree, and they show that the character of the nearest-neighbor spacing (NNS) distribution depends strongly on the extent to which the energy eigenstates are localized. For different lattice parameters, different values of β are found to fit Eq. (9) for different values of localization length. To see the dependence quantitatively, it is convenient to define the dimensionless parameter $Z = \frac{<\sigma_x>}{L}$, where $< \sigma_x >$ is the average uncertainty in the position, taken over the domain $x \in [-\frac{L}{2}, +\frac{L}{2}]$, averaged over all resonance states in the first band, and $L = Na$ is the length of the lattice. Evidently, Z represents a statistical measure of how localized the resonance energy eigenstates are. Table I shows that the transition in the level statistics between the Poisson behavior and the GOE occurs in a way that is consistent with the claim that the level statistics depend on the state localization. Moreover, some intermediate values of β from fitting the NNS distribution are also obtained for different lattices with different values of Z. Values of $\beta > 1$ are also obtained as the localization length increases, which shows even more level repulsion. The value of $\beta = 2$ corresponds to the Gaussian unitary ensemble which describes the class of n x n Hermitian matrices and is also found as described in both Fig. 2 and Table I. The level repulsion obtained in such systems is a result of the state localization and found to fit the Izrailev distribution for all values of β, including the ensembles of RMT. Here, we draw attention to the Poisson and the GOE distributions as two limiting cases. In addition to the NNS distribution, the calculation of the SR [5,35] shows similarly good agreement for the two limiting cases, and each one corresponds to the same statistics consistent with the nearest-neighbor distribution as shown in Fig. 2. The results, shown in both Fig. 2 and Table I, are calculated only for resonance states while bound states are not considered. Hence, our claim of chaotic behavior in the spectrum of this classically nonchaotic system applies only to resonance states because only positive energy solutions have a nontrivial classical analog. A study published in 1991 by Heiss and Kotzé [3] has shown a connection between the distribution of exceptional points and particular properties of level spacing distribution. A similar transition between the Poissonian and GOE statistics was shown to depend on the density of the exceptional points [36] of the underlying Hamiltonian. Here, we consider the dependence on the localization length in the lattice. The fractional values of β represent the so-called *intermediate quantum chaos* [27] and they do not correspond to any of the RMT classes [8], however, the values help to visualize how the unusual level repulsion emerges in the system.

Classical chaos is absent in one dimension because of the integrability of Newton's equations of motion, which implies that any small change in the initial condition cannot produce a drastic change in the classical trajectory of the particle. In other words, the Lyapunov exponent [10,11,37–39] always vanishes in any systems whose classical Hamiltonian is given by Eq. (1) with the replacement of each delta function by a very narrow Gaussian or any other attractive well. The meaning of the results in both Table I and Fig. 2 is that there exists a one-dimensional system that is classically regular but which displays similar signatures to chaotic systems in its quantum spectrum. This surprising result serves as a powerful example of an integrable system whose quantum spectrum admits similar level statistics to the universal behavior of chaotic systems in this remarkably simple one-dimensional system. This result shows that the BGS conjecture only gives predictions for nonintegrable systems, but the distinction between chaos and integrability cannot be fully explored just by studying the level spacing of the quantum spectra. Examples of so-called quantum chaos have been studied in detail in more complex systems such as a Rydberg atom in a magnetic field, three-dimensional lattices, and chaotic systems exhibiting closed orbit signatures [40–43]. But those systems mentioned are either higher-dimensional or else many-body systems whose classical analogs are irregular.

The Wigner-Dyson distribution has been observed in other one-dimensional systems [4,44] whose Hamiltonian has the same form in Eq. (2) with disorder present. We emphasize that the main differences between the models presented in Refs. [2,4,44] and our study are the following: First, in the present model, the randomness introduced in the system has identical strength but is placed randomly, while in Refs. [2,4,44] and other studies the randomness is usually introduced with a random strength at each lattice point. Second, while most of the studies of the transport phenomenon are done for $E < 0$, the analysis in this paper is performed for resonant states at $E > 0$ and relies on a different method of computing the quantum-level spectrum. Both cases confirm the assertion that the GOE-level statistics can occur in classically integrable systems; this demonstrates that the spectrum of both bound and resonance states can exhibit level statistics that is not always simply Poissonian, even for classically regular systems.

Another study by Ujfalusi *et al.* [45] explores whether there are any one-dimensional systems whose statistical spectrum exhibits chaotic signatures. In that treatment, a Wigner-Dyson distribution for the level spacing of the energies is assumed and then inverted to predict the corresponding local potential energy of the system, which yielded a potential energy function with many sharp peaks. Their result is

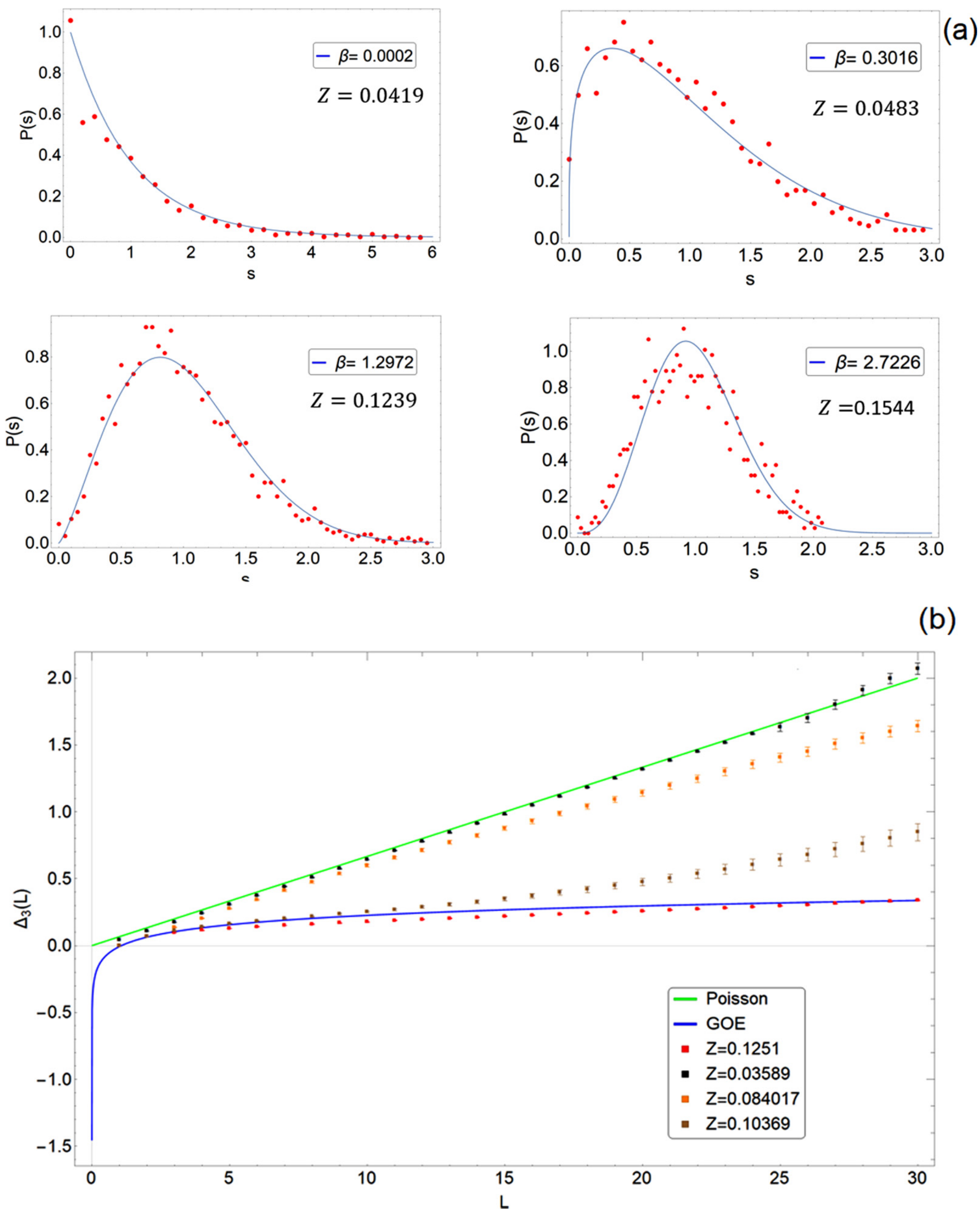

FIG. 2. (a) The nearest-neighbor spacing distribution of the resonances is shown for different values of Z. The blue curve is the Izrailev distribution for the corresponding fitting parameter β. The red dots show the calculated nearest-neighbor level distribution obtained from the solution of Eq. (8). (b) The spectral rigidity is shown for different values of β. The red dots are the values of the SR calculated for $Z = 0.1251$ matching the GOE curve, while the black dots are those calculated for $Z = 0.0358$ matching the Poisson curve.

consistent with the results in the present paper, in particular, for Hamiltonians with sharp irregular shapes like the delta function. One can view the treatment in Ref. [45] as essentially solving the inverse problem of quantum chaos in one dimension.

B. Bound states

In a one-dimensional infinite lattice, all states are localized in the presence of *any* percentage of impurities as shown by Anderson [12,13]. Anderson localization has since been studied in many one-dimensional systems [13,46] and, in most of the cases studied in the literature, the tight-binding approximation is implemented with either periodic or vanishing boundary conditions. Many results in the literature document the lack of diffusion in one dimension. However, in Fig. 1, the transmission probability $T = \cos^2(\delta_1 - \delta_2)$ obtained from the eigenphase shifts and determined by our choice of the channel functions is enhanced and approaches unity for the narrow resonances. The study of how disorder affects the transport has been explored in detail, in terms of quantities like the Wigner-Smith time delay and the Thouless conductivity [46–49], deriving there a relation between the localization length and the disorder. However, it has usually been assumed that the disorder in the system is taken from a random distribution and all the quantities of interest such as the conductivity or the localization length are derived based on that assumption. As mentioned previously, the strength of the impurities is not taken randomly in our paper; only two different kinds of atoms have been assumed to be present and the strength of all the impurities has been taken to be the same, while those impurities are placed randomly throughout the lattice. This is why the dimensionless quantity Z introduced in Table I is—more conveniently—chosen as our measure of the localization.

The scattering solutions obtained from the S matrix analysis can also be used to calculate bound states by searching for the poles of $S(E)$ in the complex energy plane, and comparing them with the spectrum obtained from the tight-binding approximation. The Hamiltonian in Eq. (2) admits a number of bound states that is always equal to the number of lattice points. Stated differently, the Hilbert space of the particle on the lattice is given by the direct sum of every single-particle Hilbert space around each lattice point [15]. Consequently, if each attractive delta function admits one bound state, then the prediction of the tight binding gives a number of bound states that is equal to the number of lattice points. Figure 3 shows the difference between the density of states of the bound states between the solution obtained from the S matrix and the tight-binding model. Both are calculated for a periodic lattice with $N = 1000$. The main difference is in the number of bound states. As argued above, the tight binding gives 1000 bound states. However, there are only 463 bound states obtained from the S-matrix treatment, while the rest of eigenstates with bound character resonances are only quasibound. Moreover, as is shown in Fig. 3, the number of bound states can be determined by the value of the eigenphase shifts at zero energy, as predicted by Levinson's theorem in one dimension [50]. After setting the values of both phase shifts at infinite energy to zero, we have

$$\frac{\delta_o(0) + \delta_e(0)}{\pi} = \left(N_b + \frac{1}{2}\right) = \left(463 + \frac{1}{2}\right), \qquad (10)$$

where N_b is the total number of bound states and that gives exactly the same number of bound states; this Levinson's theorem result is confirmed in our study. This provides evidence of the internal consistency of the results shown in this paper.

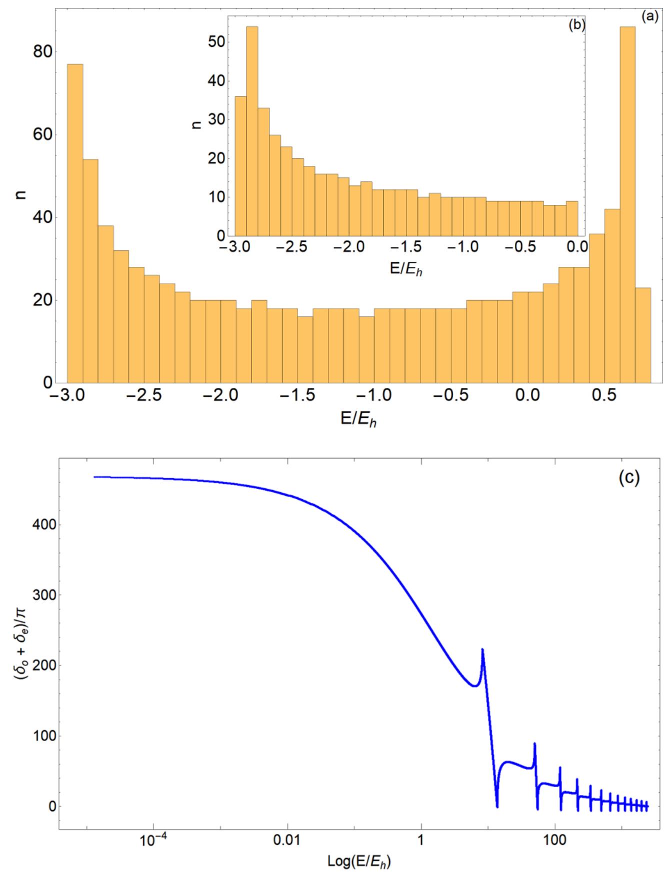

FIG. 3. (a) The density of bound states plotted versus the energy. The bars show the number of states within an energy interval, calculated from the tight-binding approximation. (b) The density of bound states plotted versus the energy. The bars show the number of states within an energy interval, obtained from the poles of the S matrix in the complex energy plane. (c) The sum of the even and odd phase shifts versus the logarithm (base 10) of the energy. The value of the zero-energy eigenphase sum divided by π fixes the number of bound states and gives the same number of bound states predicted by Levinson's theorem, namely, 463 states.

V. CONCLUSION

In conclusion, we have shown a counterexample to the converse of the BGS conjecture that states that classical chaos implies Wigner-Dyson quantum level statistics. The results of the calculations and the arguments above provide a clear observation of Wigner-Dyson level statistics in a simple, classically integrable 1D system. The Wigner-Dyson distribution is thus not exclusive to classically irregular systems. Moreover, the approach and the methods of calculations used in this paper are generally applicable to a lattice with any number of lattice sites. Although some results such as the density of states could differ from that of the known models, as shown in Fig. 3, the tight-binding theory results are recovered for the limit of infinite lattice, or large lattice constant. The work of Anderson [12] in the 1960s demonstrated that in one-dimensional disordered lattices there is no transport in the presence of even the smallest amount of disorder that breaks the periodicity of the lattice. Anderson proved his statement mathematically by considering only bound states that form a conduction band in the clean case. As shown in Table I, however, the signatures of the GOE-level statistics in this system hinge critically on the quantitative extent of state localization. These results show the value of considering aspects of such systems that go beyond the tight-binding approximation, such as studying the transmission resonances that are the focus of the present exploration.

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