ALGEBRAIC GEOMETRY AND REPRESENTATION THEORY IN THE STUDY OF MATRIX MULTIPLICATION COMPLEXITY AND OTHER PROBLEMS IN THEORETICAL COMPUTER SCIENCE

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Abstract. Many fundamental questions in theoretical computer science are naturally expressed as special cases of the following problem: Let G be a complex reductive group, let V be a G-module, and let v, w be elements of V. Determine if w is in the G-orbit closure of v. I explain the computer science problems, the questions in representation theory and algebraic geometry that they give rise to, and the new perspectives on old areas such as invariant theory that have arisen in light of these questions. I focus primarily on the complexity of matrix multiplication.

1. Introduction

1.1. **Goals of this article.** To give an overview of uses of algebraic geometry and representation theory in algebraic complexity theory, with an emphasis on areas that are ripe for further contributions from geometers.

To give a history of, and discussion of recent breakthroughs in, the use of geometry in the study of the complexity of matrix multiplication, which is one of the most important problems in algebraic complexity theory.

To convince the reader of the utility of a geometric perspective by explaining how the fundamental theorem of linear algebra is a pathology via secant varieties.

1.2. The complexity of matrix multiplication.

1.2.1. *Strassen's magnificent failure.* In 1968, while attempting to prove the standard row-column way of multiplying matrices is optimal (at least for 2×2 matrices over $\mathbb{F}_2$), Strassen instead discovered that $\mathbf{n} \times \mathbf{n}$ matrices over any field could be multiplied using $\mathcal{O}(\mathbf{n}^{2.81})$ arithmetic operations instead of the usual $\mathcal{O}(\mathbf{n}^3)$ in the standard algorithm [102], see §11 for his algorithm. Ever since then it has been a fundamental question to determine just how efficiently matrices can be multiplied. There is an astounding conjecture that as the size of the matrices grows, it becomes almost as easy to multiply matrices as it is to add them.

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1.2.2. *The complexity of matrix multiplication as a problem in geometry.* The above-mentioned astounding conjecture may be made precise as follows: let

$$\omega := \inf_\tau \{n \times n \text{ matrices may be multiplied using } \mathcal{O}(n^\tau) \text{ arithmetic operations}\}$$

Classically one has $\omega \leq 3$ and Strassen showed $\omega \leq 2.81$. The astounding conjecture is that $\omega = 2$. The quantity ω is called the *exponent of matrix multiplication* and it is a fundamental constant of nature. As I explain below, Bini [15], building on work of Strassen, showed ω can be expressed in terms of familiar notions in algebraic geometry.

Matrix multiplication is a bilinear map $M_{\langle \mathbf{n} \rangle} : \mathbb{C}^{n^2} \times \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$, taking a pair of $n \times n$ matrices to their product: $(X, Y) \mapsto XY$. In general, one may view a bilinear map $\beta : A^* \times B^* \to C$ as a trilinear form, or *tensor*, $T_\beta \in A \otimes B \otimes C$. In the case of matrix multiplication, the trilinear form is $(X, Y, Z) \mapsto \text{trace}(XYZ)$. A basic measure of the complexity of a tensor $T \in A \otimes B \otimes C$ is its *rank*, the smallest r such that $T = \sum_{j=1}^r e_j \otimes f_j \otimes g_j$ for some $e_j \in A$, $f_j \in B$, $g_j \in C$. This is because rank one tensors correspond to bilinear maps that can be computed by performing one scalar multiplication. Let $\mathbf{R}(T)$ denote the rank of the tensor T. Strassen showed $\mathbf{R}(M_{\langle \mathbf{n} \rangle}) = \mathcal{O}(\mathbf{n}^\omega)$, so one could determine ω by determining the growth of the rank of the matrix multiplication tensor.

1.2.3. *Bini's sleepless nights.* Shortly after Strassen's discovery about multiplying 2×2 matrices with 7 multiplications instead of 8, Bini wondered if 2×2 matrices, where the first matrix had one entry zero, could be multiplied with five multiplications instead of the usual six. (His motivation was that such an algorithm would lead to efficient usual matrix multiplication algorithms for larger matrices.) With his collaborators Lotti and Romani [16] they performed a computer search for a better algorithm, using numerical methods. There seemed to be a problem with their code, as each time the program appeared to start to converge to a rank five decomposition, the coefficients in the terms would blow up.

I had the priviledge of meeting Bini, who told me the story of how he could not sleep at night because no matter how hard he checked, he could not find an error in the code and would lie awake trying to figure out what was going wrong. Then one evening, he finally realized *there was no problem with the code*! I explain the geometry of his discovery in §2.

1.3. **Things computer scientists think about.** Computer scientists are not usually interested in a single polynomial or fixed size matrix, but rather *sequences* $\{P_n\}$ of polynomials (or matrices) where the number of variables and the degree grow with n. In what follows I sometimes suppress reference to the sequence.

1.3.1. *Efficient algorithms.* This is probably the first thing that comes to mind to a person on the street, however even here there is a twist: while computer scientists are interested in, and very good at, constructing efficient algorithms, they are also often content just to prove the *existence* of efficient algorithms. As we will see in §10, this is very much the case for the complexity of matrix multiplication.

1.3.2. *Hay in a haystack.* [1] Much of theoretical computer science deals with the problem of finding explicit examples of objects. A generic (or random) sequence of polynomials will be difficult to evaluate, but it is a fundamental problem to find an explicit sequence of polynomials that is difficult to evaluate. (This uses the computer scientist's definition of "explicit", which

[1] Phrase due to H. Karloff

may not be the first thing that comes to a mathematician's mind.) Several instances of this problem are discussed in this article: §3.1, §3.2, and §3.4.

1.3.3. *"Lower bounds: Complexity Theory's Waterloo".* [2] Computer scientists are perhaps most interested in proving there are no efficient algorithms for certain tasks such as the traveling salesperson problem. More precisely, they are interested in distinguishing tasks admitting efficient algorithms from those that do not. They have not been so successful in this, but at least they can often show that if task X is difficult, then so is task Y, e.g., the abundance of **NP**-complete problems.

1.4. **Overview.** In §2 I explain how the fundamental theorem of linear algebra is a pathology via the geometry of secant varieties, which also explains the geometry underlying Bini's discovery. In §3 I discuss several problems in complexity theory that are strongly tied to algebraic geometry and representation theory. Sections §4, §5, §7, §8, §9 and §10 all discuss aspects of the complexity of matrix multiplication, with the first five discussing lower bounds and the last discussing geometric aspects of upper bounds. I also include a section, §6, which discuses other open questions about tensors, many of which are also related to complexity theory. For readers not familiar with Strassen's algorithm, I include it in an appendix §11. The background required for the sections varies considerably, with almost nothing assumed in §2, where for some of the problems in §3, I assume basic terminology from invariant theory.

1.5. **Notation and Conventions.** Let A, B, C, V denote complex vector spaces of dimensions $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{v}$. Let $a_1, \cdots, a_{\mathbf{a}}$ be a basis of A, and $\alpha^1, \cdots, \alpha^{\mathbf{a}}$ its dual basis of A^*. Similarly $b_1, \cdots, b_{\mathbf{b}}$ and $c_1, \cdots, c_{\mathbf{c}}$ are bases of B and C respectively, with dual bases $\beta^1, \cdots, \beta^{\mathbf{b}}$ and $\gamma^1, \cdots, \gamma^{\mathbf{c}}$.

The tensor product $A \otimes B \otimes C$ denotes the space of trilinear maps $A^* \times B^* \times C^* \to \mathbb{C}$, which may also be thought of as the space of linear maps $A^* \to B \otimes C$ etc... A tensor $T \in A \otimes B \otimes C$ is A-*concise*, if the linear map $T_A : A^* \to B \otimes C$ is injective and it is *concise* if it is A, B, and C-concise. Informally this means that T may not be put in a smaller space.

The group of invertible linear maps $A \to A$ is denoted $GL(A)$ and the set of all linear maps is denoted $\mathrm{End}(A) = A^* \otimes A$.

Informally, the symmetry group of a tensor $T \in A \otimes B \otimes C$ is its stabilizer under the natural action of $GL(A) \times GL(B) \times GL(C)$. For a tensor $T \in A \otimes B \otimes C$, let G_T denote its symmetry group. One says T' is *isomorphic* to T if they are in the same $GL(A) \times GL(B) \times GL(C)$-orbit. I identify isomorphic tensors.

The transpose of a matrix M is denoted $M^{\mathbf{t}}$.

For a set $S \subset V$, define the *ideal* of S, $I_S := \{\text{polys } P \mid P(s) = 0 \forall s \in S\}$ and define the *Zariski closure* of S, $\overline{S} := \{v \in V \mid P(v) = 0 \forall P \in I_S\}$. In our situation the Zariski closure will coincide with the Euclidean closure. (This happens whenever $\overline{S}$ is irreducible and S contains a Zariski-open subset of $\overline{S}$, see, e.g., [91, Thm 2.33] or [71, §3.1.6]).

Projective space is $\mathbb{P}V = V \backslash \{0\} / \mathbb{C}^*$. For the purposes of this article, a *projective variety* is the common zero set of a collection of homogeneous polynomials on V considered as a subset of $\mathbb{P}V$.

Let $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A \otimes B \otimes C)$ denote the variety of rank one tensors, called the *Segre variety.*

[2] Title of Chapter 14 of [11]

For a subset $Y \subset \mathbb{C}^N$, let $\langle Y \rangle \subset \mathbb{C}^N$ denote its linear span and I use the same notation for $Y \subset \mathbb{P}V$.

For a group G, a G-module V, and $v \in V$, $G \cdot v$ denotes the orbit of v, so $\overline{G \cdot v}$ is its orbit closure.

The space of d-way symmetric tensors is denoted $S^d V$, which may be identified with the homogeneous degree d polynomials on V^*. The variety of rank one symmetric tensors is denoted $v_d(\mathbb{P}V) \subset \mathbb{P}S^d V$ and is called the *Veronese variety.*

The space of d-way skew-symmetric tensors is denoted $\bigwedge^d V$, and the *Grassmannian* is the variety $G(d,V) := \mathbb{P}\{X \in \bigwedge^d V \mid X = v_1 \wedge \cdots \wedge v_d \mid \text{some } v_1, \cdots, v_d \in V\}$. It admits the geometric interpretation of the set of d-planes through the origin in V.

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2. The fundamental theorem of linear algebra is an extreme pathology

When researchers first encounter tensors they are often surprised how their intuition from linear algebra fails and they view tensors as strange objects. The goal of this section is to convince the reader that it is not tensors, but matrices that are strange.

2.1. The fundamental theorem of linear algebra.

Theorem 2.1 (Fundamental theorem of linear algebra)**.** *Fix bases $\{a_i\}$, $\{b_j\}$ of A, B and for $r \leq \min\{\mathbf{a}, \mathbf{b}\}$, set $I_r = \sum_{k=1}^r a_k \otimes b_k$. The following quantities all equal the* **rank** *of $T \in A \otimes B$:*

- **(R)** *The smallest r such that T is a sum of r rank one elements. i.e., such that $T \in \mathrm{End}(A) \times \mathrm{End}(B) \cdot I_r$.*
- **($\underline{\mathbf{R}}$)** *The smallest r such that T is a limit of a sum of r rank one elements, i.e., such that $T \in \overline{GL(A) \times GL(B) \cdot I_r}$.*
- **($\mathbf{ml}_A$)** $\dim A - \dim \ker(T_A : A^* \to B)$.
- **($\mathbf{ml}_B$)** $\dim B - \dim \ker(T_B : B^* \to A)$.
- **($\underline{\mathbf{Q}}$)** *The largest r such that $I_r \in \overline{GL(A) \times GL(B) \cdot T}$.*
- **(Q)** *The largest r such that $I_r \in \mathrm{End}(A) \times \mathrm{End}(B) \cdot T$.*

2.2. The fundamental theorem fails miserably for tensors.

Now consider a tensor $T \in A \otimes B \otimes C$. Recall that $T \in A \otimes B \otimes C$ has *rank one* if there exists $a \in A$, $b \in B$, and $c \in C$ such that $T = a \otimes b \otimes c$. For $r \leq \min\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, write $I_r = \sum_{\ell=1}^r a_\ell \otimes b_\ell \otimes c_\ell$.

Definition 2.2.

- **($\mathbf{R}(T)$)** The *rank* of T is the smallest r such that T is a sum of r rank one tensors i.e., such that $T \in \mathrm{End}(\mathbb{C}^r) \times \mathrm{End}(\mathbb{C}^r) \times \mathrm{End}(\mathbb{C}^r) \cdot I_r$, allowing re-embeddings of T to $\mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$.
- **($\underline{\mathbf{R}}(T)$)** The *border rank* of T is the smallest r such that T is a limit of rank r tensors, i.e. such that $T \in \overline{GL(\mathbb{C}^r) \times GL(\mathbb{C}^r) \times GL(\mathbb{C}^r) \cdot I_r}$, allowing re-embeddings.
- **(ml)** The *multi-linear ranks* of T are $(\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)) := (\mathrm{rank} T_A, \mathrm{rank} T_B, \mathrm{rank} T_C)$.
- **($\underline{\mathbf{Q}}(T)$)** The *border subrank* of T is the largest r such that $I_r \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$.

($\mathbf{Q}(T)$) The *subrank* of T is the largest r such that $I_r \in \mathrm{End}(A)\times\mathrm{End}(B)\times\mathrm{End}(C)\cdot T$.

Proposition 2.3. *One has*

$$\mathbf{Q}(T) \le \underline{\mathbf{Q}}(T) \le \min\{\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)\}$$
$$\le \max\{\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)\} \le \underline{\mathbf{R}}(T) \le \mathbf{R}(T)$$

and all inequalities may be strict, even when $\mathbf{a} = \mathbf{b} = \mathbf{c}$.

Say $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$, then it has been known for some time that if T is generic then $\underline{\mathbf{R}}(T) = \mathbf{R}(T) \simeq \frac{m^2}{3}$, and this is largest possible $\underline{\mathbf{R}}$. However the precise generic values were not determined until Lickteig determined them in 1985 [82]. The symmetric case was studied by Terracini in 1916, who mostly solved it but it was not finished (for polynomials of arbitrary degree) until 1995 when it was solved by Alexander and Hirschowitz [3].

2.3. Geometry of Bini's insight. Consider the following three pictures:

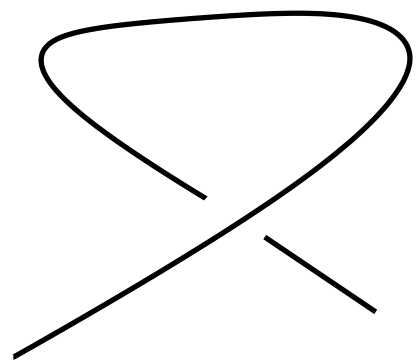

FIGURE 1. Imagine this curve represents the $3m-2$ dimensional set of tensors of rank one sitting in the m^3 dimensional space of tensors.

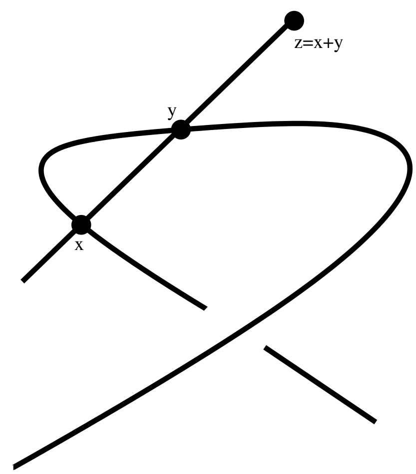

FIGURE 2. Tensors of rank two correspond to points on a secant line to the set of tensors of rank one

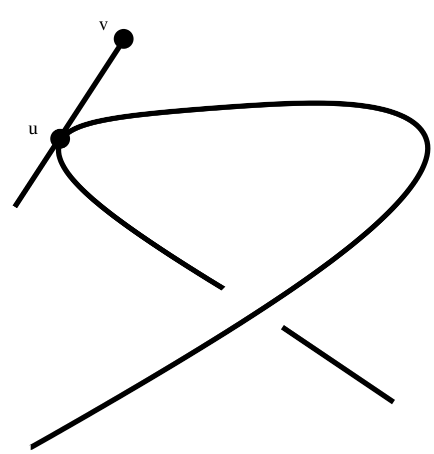

FIGURE 3. The limit of secant lines is a tangent line!

Given a curve or other variety with large codimension, most points that lie on a secant line lie on just one secant line and points that lie on a secant line do not in general lie on a tangent line. Contrast this with the case of a plane curve, where all points lie on a family of secant lines.

Bini's (re)discovery was that tensor rank is not semi-continuous precisely because secant lines may limit to tangent lines, and he coined the term border rank to include the limits. (The classical Italian algebraic geometers knew of the lack of semi-contiuity 100 years ago.) Bini then went on to prove that border rank is also a legitimate measure of the complexity of matrix multiplication, namely $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) = \mathcal{O}(n^\omega)$ [15].

Now we see that Strassen's result $\mathbf{R}(M_{\langle \mathbf{n} \rangle}) = \mathcal{O}(n^\omega)$ is not immediately useful for algebraic geometry: if P is a polynomial, $P(T_t) = 0$ for $t > 0$ implies $P(T_0) = 0$, but the limit of tensors of rank at most r need not have rank at most r. I.e., one cannot describe rank via zero sets of polynomials. In contrast, for matrices matrix rank equals matrix border rank and is given by polynomials. For this reason, we will be primarily concerned with border rank of tensors, which by definition is closed under taking limits.

As I explain below, essentially all known examples of smooth geometric objects with large codimension with the property that any time a point is on a secant line to the object, it is on a family of secant lines, comes from "minimal rank" matrices in some (restricted) space of matrices.

2.4. **Geometric context: secant varieties.** Let $X \subset \mathbb{P}V$ be a projective variety. Define

$$\sigma_r(X) := \overline{\{z \in \mathbb{P}V \mid \exists x_1, \cdots, x_r \in X \mid z \in \langle x_1, \cdots, x_r \rangle\}}$$

the variety of secant $\mathbb{P}^{r-1}$'s to X. A naïve dimension count shows that we expect $\dim(\sigma_r(X))$ to be $\min\{r \dim X + r - 1, \dim \mathbb{P}V\}$, because we get to pick r points on X and a point in their span. If this count fails one says the variety is *defective*. Defectivity is a pathology.

The fact that the fundamental lemma of linear algebra is a pathology may be rephrased as saying the Segre variety $Seg(\mathbb{P}A \times \mathbb{P}B) \subset \mathbb{P}(A \otimes B)$ of rank one matrices has defective secant varieties. The secant varieties of three and higher order tensors are generally not defective, the (m, m, m) case mentioned above proved by Lickteig has only one exception, $m = 3$ and $r = 4$. See [1] for the state of the art in the general tensor case.

To my knowledge, essentially all known smooth varieties with degenerate $r = 2$ secant varieties come from matrices: the rank one matrices $Seg(\mathbb{P}A \times \mathbb{P}B) \subset \mathbb{P}(A \otimes B)$, the rank one symmetric matrices $v_2(A) \subset \mathbb{P}(S^2 A)$, the rank two skew-symmetric matrices (i.e., minimal rank skew-symmetric matrices) $G(2, A) \subset \mathbb{P}(\bigwedge^2 A)$, the Cayley plane discussed below, and the closed orbit in the adjoint representation of a complex simple Lie algebra $G/P_{\tilde{\alpha}} \subset \mathbb{P}\mathfrak{g}$ [63] (which consists of $v_2(\mathbb{P}A)$, the two planes isotropic for a quadratic form on A, denoted $G_Q(2, A)$, the traceless rank one matrices when $\mathfrak{g} = \mathfrak{sl}_n$, and five exceptional cases of "minimal rank" matrices in some space of matrices with extra structure). When the dimension of the secant variety differs from the expected dimension by more than one, one can take hyperplane sections to get further examples.

Secant varieties of projective varieties have been studied for a long time, dating back at least to the Italian school in the 1800's. They show up in numerous geometric situations. For example, Zak (see [106]) proved a linear approximation to Hartshorne's famous conjecture on complete intersections [54] that was also conjectured by Hartshorne, which amounts to proving that if X is smooth of dimension n, not contained in a hyperplane, and the codimension of X is less than $\frac{n}{2}$, then $\sigma_2(X) = \mathbb{P}V$. Moreover, Zak classified the exceptional cases with codimension $\frac{n}{2}$, now called Severi varieties. They turn out to be the projective planes over the composition algebras $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$, where $\mathbb{A}$ is one of $\mathbb{R}^{\mathbb{C}} = \mathbb{C}$, $\mathbb{C}^{\mathbb{C}} = \mathbb{C} \oplus \mathbb{C}$, $\mathbb{H}^{\mathbb{C}}$, or $\mathbb{O}^{\mathbb{C}}$, the last two cases are the complexified quaternions and octonions, and $\mathcal{J}_3(\mathbb{A})$ denotes the (complex) vector space of 3×3 $\mathbb{A}$-Hermitian matrices. It is an open problem to determine if a gap larger than 8 (which occurs for $\mathbb{O}^{\mathbb{C}}\mathbb{P}^2$) between the expected and actual dimension occurs for a smooth variety, see [80]. Moreover, using the geometry of secant and tangential varieties, one can obtain a proof of the Killing-Cartan classification of complex simple Lie algebras via a constructive procedure starting with $\mathbb{P}^1$, see [72].

3. Problems in complexity with interesting geometry

The bulk of this article discusses the history and recent developments using geometry in the study of matrix multiplication. Here I discuss several other questions with interesting geometry. I ignore numerous fundamental developments that have not yet been cast in geometric language. I briefly mention two such developments now: Hardness v. randomness (see, e.g., [60]), shows that if certain conjecturally hard problems are truly hard, then many other problems, for which only a randomized efficient algorithm is known, admit deterministic efficient algorithms, and conversely, if these problems cannot be derandomized, then the conjectured hard problem is not hard after all. The PCP theorem [12] says that any putative proof may be rewritten in such a way that its correctness is checkable by looking at only a few probabilistically chosen symbols, See [11] for excellent discussions of these and other omissions.

3.1. **P v. NP and variants.** The famous $\mathbf{P} \neq \mathbf{NP}$ conjecture of Cook, Karp, and Levin has origins in the 1950's work of John Nash (see [92, Chap. 1]), researchers in the Soviet Union (see, e.g, [104]), and, most poetically, Gödel, who asked if one could quantify the idea of intuition (see [99, Appendix]). One of L. Valiant's algebraic versions of the problem [105] is as follows: Valiant showed that any polynomial $p(x_1, \cdots, x_N)$ may be realized as the determinant of some $n \times n$ matrix of affine linear forms in the x_i, where the size n depends on p. He showed the size is a good measure of the complexity of a polynomial. The "permanent v. determinant" version of the $\mathbf{P} \neq \mathbf{NP}$ conjecture is that the size $n(m)$ of the matrix needed to compute the permanent $\mathrm{perm}_m \in S^m\mathbb{C}^{m^2}$ of an $m \times m$ matrix[3] grows faster than any polynomial in m. For those familiar with the traveling salesman version of $\mathbf{P} \neq \mathbf{NP}$, the "easy to verify" proposed answer is replaced by an "easy to write down" polynomial sequence, and the conjecture amounts to saying a polynomial sequence that is easy to write down in general will not be easy to compute. Here I discuss the permanent v. determinant version of the problem.

In algebraic geometry one generally prefers to work with homogeneous polynomials, so instead of asking to write the permanent as a determinant of affine linear functions, following K. Mulmuley and M. Sohoni, one adds a variable to homogenize the problem:

Conjecture 3.1. *[Rephrasing of Valiant's conjecture as in [89]] Let ℓ be a linear coordinate on $\mathbb{C}^1$ and consider any linear inclusion $\mathbb{C}^1 \oplus \mathbb{C}^{m^2} \to \mathbb{C}^{n^2}$, so in particular $\ell^{n-m}\,\mathrm{perm}_m \in S^n\mathbb{C}^{n^2}$. Let $n(m)$ be a polynomial. Then for all sufficiently large m,*

$$[\ell^{n-m}\,\mathrm{perm}_m] \notin \mathrm{End}(\mathbb{C}^{n^2}) \cdot [\det_{n(m)}].$$

Remark 3.2. The best known $n(m)$ is $n = 2^m - 1$ due to B. Grenet [49]. His expression has interesting geometry. Note that given an expression for the permanent as a determinant, one

[3] The permanent is the polynomial that is the same as the determinant, only without the minus signs.

gets a family of such by the action of $G_{\mathrm{perm}_m} \times G_{\det_n}$ on $(\mathbb{C}^1 \oplus \mathbb{C}^{m^2})^* \otimes \mathbb{C}^{n^2}$, and we can consider the stabilizer $\Gamma \subset G_{\mathrm{perm}_m} \times G_{\det_n}$ of a decomposition. One has $G_{\mathrm{perm}_m} = [(\mathbb{C}^*)^{\times 2m-2} \times \mathfrak{S}_m \times \mathfrak{S}_m] \rtimes \mathbb{Z}_2$ and this is the largest possible stabilizer. One has $\Gamma_{Grenet} = (\mathbb{C}^*)^{\times m} \times \mathfrak{S}_m$ [77] and moreover, if one insists on a stabilizer this size, N. Ressayre and I also showed Grenet's $n(m) = 2^m - 1$ is the smallest possible [77]. In particular, one could prove Valiant's conjecture by proving that for any determinantal expression for the permanent, that there exists a slightly larger one with symmetry Γ_{Grenet}. Similarly, one can also have $\Gamma = G_{\mathrm{perm}_m}$ for a similar (but larger) cost in size.

To further geometrize the problem, Mulmuley and Sohoni proposed a stronger conjecture: Let

$$\mathcal{D}et_n := \overline{GL_{n^2} \cdot [\det_n]},$$

and let

$$\mathcal{P}erm_n^m := \overline{GL_{n^2} \cdot [\ell^{n-m} \mathrm{perm}_m]}.$$

Conjecture 3.3. *[89] Let $n(m)$ be a polynomial. Then for all sufficiently large m,*

$$\mathcal{P}erm_{n(m)}^m \notin \mathcal{D}et_{n(m)}.$$

This is stronger as in general, for a polynomial $P \in S^d\mathbb{C}^N$, the inclusion $\mathrm{End}(\mathbb{C}^N) \cdot P \subset \overline{GL_N \cdot P}$ is strict.

Both $\mathcal{P}erm_n^m$ and $\mathcal{D}et_n$ are *invariant* under GL_{n^2} so their ideals are GL_{n^2}-modules. The original idea of [89,90] was to solve the problem by finding a sequence, depending on n, of GL_{n^2}-modules M_n such that $M_n \subset I[\mathcal{D}et_n]$ and $M_n \notin I[\mathcal{P}erm_n^m]$. The initial idea in [89] was to look for not any module, but an invariant to separate the orbit closures. The mathematical issues raised by this program were analyzed in [30]. Work of Ikenmeyer and Panova [59] and Bürgisser, Ikenmeyer and Panova [29] shows this is not possible but other paths using representation theory are still open, see [83–88].

This program has inspired a tremendous amount of work and breathed new life into invariant theory. I will loosely refer to the inspired work as *geometric complexity theory* (GCT) even for problems not directly related to Valiant's conjecture.

3.2. **Explicit Noether Normalization.** The classical Noether normalization lemma may be stated geometrically as follows: given an affine variety $X^n \subset \mathbb{A}^{n+a} = V$, there exists a $W = \mathbb{A}^a \subset \mathbb{A}^{n+a}$ such that the projection of X to $V/W = \mathbb{A}^n$ will be a finite to one surjection. In fact a general or "random" $\mathbb{A}^a$ will do. The problem of *explicit Noether normalization* is to find an explicit $\mathbb{A}^a$ that has this property. In [86], Mulmuley proposes this problem as a possible path to resolve Valiant's conjecture.

Algebraically, given a ring R, one is interested in finding an explicitly generated (in the sense of computer science, in particular, efficiently describable, and I remind the reader that one is really dealing with sequences of rings) subring S such that R is integral over S. Of particular interest is the case R is the ring of invariants in the coordinate ring of some G-variety Z (i.e., the X in question is the GIT quotient of Z). M. Forbes and A. Shpilka [44] give an effective solution to this in the case R is the ring of invariants of the set of r-tuples of $n \times n$ matrices under the action of GL_n by simultaneous conjugation. This ring is generated by traces of words in the matrices [93, 95]. A key ingredient here is the use of *pseudo-randomness*: deterministic processes that are not too expensive yet are able to imitate randomness sufficiently to resolve the problem.

There is work still to be done here. One wants to find a smallest subring S that works. One way to measure smallness is the number of separating invariants that are used to generate it. Ideally one would get a set of generators whose size matches the dimension of the variety in question, and generically this is possible. Here the dimension is polynomial in (n,r), one writes "$poly(n,r)$". The articles [93,95] give an exponential in n,r size S and [44] gives an explicit set of size $poly(n,r)^{\mathcal{O}(\log(nr))}$ and the open problem is to reduce it to polynomial size.

3.3. **Algorithms in invariant theory.** Thanks to the GCT program, computer scientists have recognized that the orbit closure containment problem, and the related orbit closure intersection and orbit equality problems provide a natural framework for many questions in computer science. One insight of [28], by P. Burgisser, C. Franks, A. Garg, R. Oliveira, M. Walter, and A. Wigderson, is that many problems in complexity and other applications may be phrased as follows: Let G be a complex reductive algebraic group, let $K \subset G$ be a maximal compact subgroup with Lie algebra $\mathfrak{k}$, so $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{k}$ is a real Lie algebra. Then $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, and G-modules V come equipped with a K-invariant inner product, in particular a norm. Then given $v \in V$, the problem is to compute its *capacity*

$$\mathrm{cap}(v) := \min_{w \in \overline{G \cdot v}} ||w||.$$

Readers familiar with the Kempf-Ness theorem [64] will recognize w as a vector satisfying $\mu(w) = 0$, where $\mu : \mathbb{P}V \to (\mathfrak{g}/\mathfrak{k})^*$ is the *moment map*, the normalized derivative of the norm map $G \to \mathbb{R}$, $g \mapsto ||g \cdot v||^2$ at the identity.

They point out that already when G is abelian, this encompasses all linear programming problems. Incidences of the general problem have been known for some time in the community of geometers, e.g., the famous Horn problem, see, e.g., [46]. A very special case is null cone membership: determine if v has capacity zero.

The problem with just checking the boundary of the moment polytope is that in general, the moment polytope is defined by too many inequalities to make such a check efficient.

Their new ingredient to this well-studied problem is the introduction of algorithms to solve, or approximately solve the problem. They use gradient descent methods. In essence, given $\epsilon > 0$, the algorithms, after some specified amount of computation, either take one within ϵ of a w achieving the minimum, or stay a specified distance from it. A variant that they also study is an algorithm to compute an element of G that takes one close to w.

3.4. **Elusive functions.** R. Raz defines the following "hay in a haystack" approach to Valiant's conjecture. Consider a linear projection of a Veronese variety $v_r(\mathbb{P}^{s-1}) \subset \mathbb{P}S^r\mathbb{C}^s$ via $proj : \mathbb{P}S^r\mathbb{C}^s \dashrightarrow \mathbb{P}^m$, and let $\Gamma_{r,s} := proj \circ v_r : \mathbb{P}^{s-1} \dashrightarrow \mathbb{P}^m$ be the composition of the projection with the Veronese map. A map $f : \mathbb{P}^n \to \mathbb{P}^m$ is said to be (r,s)-*elusive* if $f(\mathbb{P}^n)$ is not contained in the image of any such $\Gamma_{r,s}$.

Theorem 3.4. *[94] Let m be super-polynomial in n, and $s \geq m^{\frac{9}{10}}$. If there exists an explicit $(s,2)$-elusive $f : \mathbb{P}^n \to \mathbb{P}^m$, then Valiant's conjecture is true.*

Theorem 3.5. *[94] Let $r(n) = \log(\log(n))$, $s(n) = n^{\log(\log(\log(n)))}$, $m = n^r$, and let C be a constant. If there exists an explicit (s,r)-elusive $f : \mathbb{P}^n \to \mathbb{P}^m$, then Valiant's conjecture is true.*

By a dimension count, a general polynomial in either range will be elusive. The problem of finding explicit elusive functions seems to be worth further study with the tools of algebraic geometry. See [50] for recent work in this direction.

3.5. **Shallow circuits.** The usual model of computation in algebraic complexity theory is the *arithmetic circuit*, which is a directed graph used to encode a polynomial allowing additions and multiplications and the size of the circuit (essentially the number of additions and multiplications) measures its complexity. There have been results regarding restricted circuits, where, e.g., one is just allowed a round of sums, then a round of products then a third round of sums, called "$\Sigma\Pi\Sigma$ circuits". Then if one can prove a strong lower bound for computing the permanent perm_m in such a restricted model (roughly exponential in $\sqrt{m}\log^{\frac{3}{2}} m$), one can prove Valiant's conjecture. See, e.g., [2, 21, 51, 65, 103].

The essential point here for geometry is that the Zariski closure of the set of polynomials that $\Sigma\Pi\Sigma$ circuits of fixed size and "fainin" can produce corresponds to a well-studied, albeit little understood, object in representation theory and algebraic geometry: secant varieties of the (simplest) *Chow variety.*

The Chow variety $Ch_n(W) \subset \mathbb{P}S^nW$ is the set of polynomials that are products of linear forms. When $\mathbf{w} = \dim W \geq n$, it is the orbit closure $\overline{GL(W)\cdot[x_1\cdots x_n]}$. It is a longstanding open problem to understand its ideal dating back to Brill, Gordan, Hermite and especially Hadamard [52]. See [22, 23, 57, 68] for its connections to interesting questions in algebraic geometry and representation theory. See [71, Chap. 9] for an exposition of the state of the art.

Other models of shallow circuits give rise to other interesting secant varieties.

3.6. **Identifiability.** While the following problem in applications initially comes from engineering, it is similar enough in spirit that I include it here. A basic geometric fact is that under a Veronese re-embedding points become more independent. A collection of r points is in *general linear position* if for all k, no subset of k of them lies on a $\mathbb{P}^{k-2}$. More generally, a collection of r points is in d*-general position* if for all k, no subset of k them lies on a higher dimensional space of hypersurfaces of degree d than holds for a generic set of k points. Under the d-th Veronese re-embedding $v_d : \mathbb{P}V \to \mathbb{P}S^dV$, given by $[x] \mapsto [x^d]$, collections of points become in $(d-1)$-general position. P. Comon [36] exploited this in signal processing. See [70, Chap. 12] for a geometric discussion of how this was exploited.

The general question that arises in applications is as follows: given a tensor of rank r with a rank r decomposition, determine if the decomposition is unique (up to trivialities). The first, and still most important, result in this regard is Kruskal's theorem [66], which assures uniqueness in a certain range if the points giving rise to the decomposition are in general linear position. In [42] it was shown that Kruskal's theorem is sharp. On the other hand, tensors are known to be generally identifiable well beyond the Kruskal range [32, 33]. In a series of papers, generalizations of Kruskal's theorem that exploit more subtle geometric information have extended identifiability. For example, a first step beyond Kruskal's bound is obtained in [10] by exploiting Castenuovo's theorem that if a set of $2n+3$ points lies on a $\binom{n}{2}$-dimensional space of quadrics, then the points all lie on a rational normal curve. The current state of the art is [9], where further advanced tools (minimal free resolution, liason, Hilbert functions) are used.

4. Representation theory and border rank

When I was first introduced to the problem of matrix multiplication, $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle})$ was not known in any case except the trivial $\mathbf{n}=1$, and it was an often stated open problem just to determine $\underline{\mathbf{R}}(M_{\langle 2\rangle})$. I now explain why I thought this would be easy to resolve using representation theory.

Definition 4.1. Given a variety $X\subset \mathbb{P}V$, one says X is a *G-variety* if it is invariant under the action of some group $G\subset GL(V)$, i.e., $\forall g\in G$, $\forall x\in X$, $g\cdot x\in X$.

For example, $\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)) := \{[T]\mid \underline{\mathbf{R}}(T)\leq r\}$, is a $G=GL(A)\times GL(B)\times GL(C)\subset GL(A{\otimes}B{\otimes}C)$ variety.

As mentioned above, a fundamental observation about G-varieties is that their ideals are G-modules.

In particular, if G is reductive, one could in principle determine the ideal $I(X)$ in any given degree d by decomposing S^dV^* as a G-module and then testing highest weight vectors on random points of X. If the polynomial vanishes at a general point, then the module is in the ideal and otherwise it is not.

In the special case where $X=G/P\subset \mathbb{P}V_\lambda$ is homogeneous (as with the Segre variety), Kostant showed that the ideal is generated in degree two by $V_{2\lambda}^{\perp}\subset S^2V^*$, where λ is the highest weight of the irreducible G-module V (Kostant's proof appeared in an appendix to the unpublished [47]. See [70, §16.2] for a proof or [67] for a proof with an extension to the infinite dimensional case).

For example, $S^2(A^*{\otimes}B^*)=S^2A^*{\otimes}S^2B^*\oplus\bigwedge^2A^*{\otimes}\bigwedge^2B^*$, and $I(Seg(\mathbb{P}A\times\mathbb{P}B))$ is generated in degree two by $\bigwedge^2A^*{\otimes}\bigwedge^2B^*$ which spans the two by two minors. More generally $I(\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B)))$ is generated in degree $r+1$ by the size $r+1$ minors $\bigwedge^{r+1}A^*{\otimes}\bigwedge^{r+1}B^*$.

Tensors of border rank at most two are the zero set of degree three polynomials [73], so I was optimistic. After all, to decide the border rank of $M_{\langle 2\rangle}$, one just needs to determine polynomials in the ideal of $\sigma_6(Seg(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3))$ and to test them on $M_{\langle 2\rangle}$. (Strassen had previously proved the border rank was at least six and $\sigma_7(Seg(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3))=\mathbb{P}(\mathbb{C}^4{\otimes}\mathbb{C}^4{\otimes}\mathbb{C}^4)$.) With L. Manivel, we carried out a systematic search. Unfortunately we found:

Theorem 4.2. *[73] The ideal of $\sigma_6(Seg(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3))$ is empty in degrees less than* 12.

It was probably this proposition that ended Manivel's activity in the area. Later, J. Hauenstein and C. Ikenmeyer extended the result:

Theorem 4.3. *[55] The ideal of $\sigma_6(Seg(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3))$ is empty in degrees less than* 19.

Fortunately, at the same time we showed:

Theorem 4.4. *[55] A copy of the degree* 19 *module $S_{5554}A^*{\otimes}S_{5554}B^*{\otimes}S_{5554}C^*$ is in the ideal of $\sigma_6(Seg(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3))$.*

Here I use partition notation to describe modules for the general linear group. Unfortunately, the polynomials were too complicated to test symbolically on matrix multiplication, so it was only useful to get a numerical proof. However there is an invariant in degree 20 that was easier to work with which provided the first algebraic proof of

Theorem 4.5. *[55, 69]* $\underline{\mathbf{R}}(M_{\langle 2\rangle})=7$.

The original proof in [69] was obtained using differential-geometric methods.

Clearly to deal with larger $M_{\langle \mathbf{n} \rangle}$, different methods were needed.

5. Retreat to linear algebra

A concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ must have border rank at least m, and if equality holds, one says T has *minimal border rank*. As explained in §10 below, minimal border rank tensors are important for Strassen's laser method.

Strassen's proof that $\underline{\mathbf{R}}(M_{\langle 2 \rangle}) \geq 6$ was obtained by taking advantage of the following correspondence: given a concise $T \in A \otimes B \otimes C$, one obtains an $\mathbf{a}$-dimensional space of matrices $T(A^*) \subset B \otimes C$, and T may be recovered up to isomorphism from this subspace. I.e., we have a correspondence

A-concise tensors $T \in A \otimes B \otimes C$ up to $GL(A) \times GL(B) \times GL(C)$-isomorphism $\leftrightarrow$ $\mathbf{a}$-dimensional subspaces $U \subset B \otimes C$ up to $GL(B) \times GL(C)$-isomorphism.

While spaces of linear maps are nice, even better are spaces of endomorphsims: let $\mathbf{b} = \mathbf{c} = m$ and assume there exists $\alpha \in A^*$ with $T(\alpha) : B^* \to C$ of full rank m. Then $T(A^*)T(\alpha)^{-1} \subset \operatorname{End}(C)$, and one can recover T up to isomorphism from this space as well.

Now if $\mathbf{a} = m$ and $\mathbf{R}(T) = m$, then $T(A^*)T(\alpha)^{-1}$ is an m-dimensional space of simultaneously diagonalizable endomorphisms, which, in a good basis is $\langle \gamma^1 \otimes c_1, , \cdots, \gamma^m \otimes c_m \rangle$. So if T is of minimal border rank, then $T(A^*)T(\alpha)^{-1}$ is a limit (in the Grassmannian $G(m, \operatorname{End}(C))$) of spaces of simultaneously diagonalizable endomorphisms. So the problem to determine if T has minimal border rank is reduced to determining if $T(A^*)T(\alpha)^{-1}$ is such a limit. Good news: this problem was studied classically in the linear algebra literature (e.g., [48]). Bad news: it is still open!

Nonetheless, it is easy to obtain necessary conditions: simultaneously diagonalizable matrices commute, and commutivaty is a Zariski closed condition. Call the vanishing of the commutators *Strassen's equations for minimal border rank*. Moreover, Strassen showed that the failure of commutativity (i.e., the rank of the commutator) lower bounds the border rank, which enabled the first lower bound on $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle})$:

Theorem 5.1. *[101] $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) \geq \frac{3}{2} n^2$.*

Lickteig [81] was able to improve the error term to obtain $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) \geq \frac{3}{2} n^2 + \frac{n}{2} - 1$. Then from 1985 to 2012 there was no further progress on the general case.

Taking a more abstract view of Strassen's theorem, he found equations by embedding $A \otimes B \otimes C$ into a space of matrices and then took minors. This is a longstanding trick in algebraic geometry: obtain determinantal equations of varieties by looking at rank loci of maps between vector bundles on projective space. In the situation of G-varieties, one can give an elementary description of the most useful embeddings:

Observation 5.2. *[76] Given a G-variety $X \subset \mathbb{P} V_\lambda$, say V_λ occurs as a submodule of $V_\mu \otimes V_\nu$. Then if a general point of X maps to a rank t element (as a matrix) of $V_\mu \otimes V_\nu$, then the size $rt+1$ minors restricted to V_λ are in the ideal of $\sigma_r(X)$.*

Call such an inclusion $V_\lambda \to V_\mu \otimes V_\nu$ a *Young flattening*. Several such were useful for the case X is a Veronese variety [76]. For three way tensors, the most useful inclusions have been, for values of $p < \mathbf{a}/2$, $A \otimes B \otimes C \to (\bigwedge^p A^* \otimes B) \otimes (\bigwedge^{p+1} A \otimes C)$, which we call *Koszul flattenings*. To implement them one generally restricts T to a $2p+1$ dimensional subspace of A, and it is often an art to find an explicit useful such subspace. Using a judiciously chosen $SL_2 \subset SL_A$ to define a good restriction, G. Ottaviani and I were able to show:

Theorem 5.3. *[79]* $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) \geq 2n^2 - n$.

6. Detour: Additional open questions regarding tensors

Unlike the case of linear maps, we are rather ignorant of tensor rank:

Question 6.1. For $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, what is the largest possible $\mathbf{R}(T)$?

For the state of art, see [26] and [20].

Problem 6.2. [27, Problem 15.2] Classify concise tensors of minimal border rank.

We are rather ignorant here as well: the state of the art is $m = 4$ (Friedland [45]).

As innocent as Problem 6.2 may sound, even a special case of it amounts to characterizing when a zero dimensional scheme is smoothable, a notoriously difficult problem that is known only for very small values of m: when $m \leq 7$ all are, and when $m = 8$ the problem is solved in [31]. For larger m little is known, see [61] for the state of the art as of this writing. I now explain this connection.

Let $A = B = C = \mathbb{C}^m$. Call a tensor 1_A-generic if $T(A^*) \subset B \otimes C$ contains an element of full rank m. Call a tensor 1_*-generic if it is at least one of 1_A, 1_B or 1_C-generic, and *binding* if the property holds in at least two directions, and 1-generic if it holds in all three.

M. Bläser and V. Lysikov [18] showed that if a tensor is binding, then it is the structure tensor of some (not necessarily commutative) algebra with unit. Strassen's commutivity equations for minimal border rank described above are a necessary condition for minimal border rank and satisfying them implies that the algebra is commutative, which implies the algebra is of the form $\mathbb{C}[x_1, \cdots, x_n]/\mathcal{I}$, where $\mathcal{I}$ is some ideal and $\mathbb{C}[x_1, \cdots, x_n]/\mathcal{I}$ is an m-dimensional vector space over $\mathbb{C}$. This leads to the Hilbert scheme for zero dimensional schemes of length m, which parametrizes such objects, see §9.1 for more details. Then the question becomes whether the given algebra lies in the same component as the algebra $(\mathbb{C}[x]/(x^2))^{\oplus m}$ corresponding to a concise rank m tensor.

If $\mathbf{b} = \mathbf{c} = m$ and the tensor is only 1_A-generic, there is still a geometric object one can utilize, the *Quot scheme* parametrizing modules: the Atiyah-Hitchin-Drinfeld-Manin (ADHM) correspondence associates to the tuple $(x_1, \cdots, x_{\mathbf{a}})$ of commuting $m \times m$ matrices one gets from Strassen's commutivity equations, a $\mathbb{C}[y_1, \cdots, y_{\mathbf{a}}]$-module structure on $V = \mathbb{C}^m$ defined by $y_i(v) := x_i(v)$. The *Quot scheme of points* is a moduli space for such modules. (The special case where the module has one generator is the Hilbert scheme of points.) Jelisiejew and Sivic [62] use this correspondence to prove new results about each. In particular, they classify all components when $m \leq 7$. In recent work with A. Pal and J. Jelisiejew, we use this to solve problem 6.2 under the assumption of 1_*-genericity up to $m \leq 6$.

If a tensor fails to be 1_*-generic, one is led to the problem of characterizing spaces of matrices of bounded rank - a classical but difficult topic that has only be solved for ranks up to 3 [13]. Here one is in a slightly better situation, as there are three such spaces to be considered.

7. Bad News for matrix multiplication lower bounds

Theorem 7.1. *[14,43] It is essentially game over for rank methods. More precisely, one cannot prove bounds stronger than* $\underline{\mathbf{R}}(T) \geq 6m$ *for* $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ *using rank methods.*

Remarkably this result was discovered essentially simultaneously by computer scientists and algebraic geometers with two completely different proofs.

I briefly explain the algebraic geometry proof as articulated by J. Buczyński (personal communication, see [71, §10.2] for more detail): a collection of r distinct points on a variety X defines a zero dimensional smooth scheme of length r, and a point on $\sigma_r(X)$ defines a zero dimensional smoothable scheme of length r (more precisely a point in the span of such). Determinantal methods detect zero dimensional schemes of length r. Moreover, zero dimensional schemes of length r, even those supported at just one point, quickly fill the ambient projective space. To rephrase (in what follows, R is a zero dimensional scheme): The secant variety is

$$\sigma_r(X) := \overline{\bigcup \{\langle R \rangle \mid \mathrm{length}(R) = r,\ \mathrm{support}(R) \subset X,\ R : \mathrm{smoothable}\}}.$$

Define the *cactus variety* [25]:

$$\kappa_r(X) := \overline{\bigcup \{\langle R \rangle \mid \mathrm{length}(R) = r,\ \mathrm{support}(R) \subset X\}}$$

Determinantal equations are equations for the cactus variety and the cactus variety fills the ambient space when r is small ($6m$ for tensors).

8. How to continue? Use more symmetry!

So far, lower bounds were obtained by exploiting symmetry of the variety $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$. But the point $M_{\langle \mathbf{n} \rangle}$ also has symmetry. Write $A = U^* \otimes V$, $B = V^* \otimes W$, $C = W^* \otimes U$. Then $M_{\langle \mathbf{n} \rangle}$ is $\mathrm{Id}_U \otimes \mathrm{Id}_V \otimes \mathrm{Id}_W$ re-ordered. Here $\mathrm{Id}_U : U \to U$ is the identity map. Recall that the $GL(U)$-module $U^* \otimes U$ decomposes as $\mathfrak{sl}(U) \oplus \langle \mathrm{Id}_U \rangle$. Thus

$$G_{M_{\langle \mathbf{n} \rangle}} \supset GL(U) \times GL(V) \times GL(W) = GL_n^{\times 3} \subset GL_{n^2}^{\times 3}.$$

How to exploit this symmetry?

Given $T \in A \otimes B \otimes C$, one has $\underline{\mathbf{R}}(T) \leq r$ if and only if there exists a curve $E_t \subset G(r, A \otimes B \otimes C)$ such that

i) For $t \neq 0$, E_t is spanned by r rank one elements, and

ii) $T \in E_0$.

Notice that if E_t is such a curve, for all $g \in G_T$, gE_t also works. This led to the following observation with M. Michałek:

Proposition 8.1. *[75] One can insist that* E_0 *be fixed by a Borel subgroup of* G_T*. In particular, for* $M_{\langle \mathbf{n} \rangle}$*, one may insist that* E_0 *is fixed by the action of triples of upper triangular* $\mathbf{n} \times \mathbf{n}$ *matrices on* $\Lambda^r((\mathbb{C}^{\mathbf{n}^2})^{\otimes 3})$.

This, combined with a border rank version of the classical substitution method (see, e.g., [4]), led to what at the time I viewed as a Phyrric victory:

Theorem 8.2. *[78]* $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n}^2 - \log_2 \mathbf{n} - 1$.

I write Phyrric because it was clear this was the limit of the method. Little did I realize that soon after, W. Buczyńska and J. Buczyński would generalize Proposition 8.1 in a way that not only allowed further progress but also gave a potential path to overcoming the cactus barrier.

9. Border Apolarity

Buczyńska-Buczyński had the following idea to use more information [24]: Instead of considering limits of r planes $\langle T_1(t), \cdots, T_r(t)\rangle$, where $T = \lim_{t\to 0} \sum T_j(t)$, consider limits of *ideals* I^t, where I^t is the ideal of $[T_1(t)] \sqcup \cdots \sqcup [T_r(t)]$.

This leads to the problem: How to take limits? A natural first idea is in the the *Hilbert scheme*.

9.1. The Hilbert Scheme of points.

One insists on saturated (with respect to the maximal ideal) ideals $I \subset Sym(V^*)$. Then, in a sufficiently high degree D, $I_D \subset S^D V^*$ determines I in all degrees, and "sufficiently high" can be made precise. Thus one is reduced to to taking limits in one fixed Grassmannian. The Hilbert scheme parametrizes saturated ideals with same *Hilbert polynomial*.

Let $I \subset Sym(V^*)$ be any ideal. Let $r_d = \dim(S^d V^*/I_d)$, so the Hilbert function is $h_I(d) := r_d$. Castelnuovo-Mumford regularity implies that if one fixes the Hilbert function, there exists an explicit $D = D(h_I)$ such that I_D determines $I_{D'}$ for all $D' > D$. Moreover $h_I(x)$ is a polynomial when $x > D$, called the Hilbert polynomial.

Bad news: The Hilbert scheme doesn't work. Consider a toy case of 3 points in $\mathbb{P}^2$: $[1,0,0], [0,1,0], [1,-1,t]$ $t \neq 0$, $(I^t)_1 = 0$ and $(I^t)_2 = \langle x_3^2 + t^2 x_1 x_2,\ x_3^2 - t x_1 x_3,\ x_1 x_3 + x_2 x_3\rangle$ But $(I^0)_1 = \langle x_3\rangle$, $(I^0)_2 = \langle x_3^2, x_1x_3 + x_2x_3\rangle$. The problem is that the ideal of the limiting scheme in a fixed degree is not the limit of spans and one loses information important for border rank decomposition.

9.2. The multigraded Hilbert scheme.

The solution is to use the *Haiman-Sturmfels multigraded Hilbert scheme* [53]: Consider the product of Grasmannians

$$G(r_1, V^*) \times G(r_2, S^2V^*) \times \cdots \times G(r_D, S^D V^*)$$

and map $I \mapsto ([I_1] \times [I_2] \times \cdots \times [I_D])$. For each $\mathbb{Z}_{\geq 0}$-valued function h, get a (possibly empty) subscheme parametrizing all ideals I with *Hilbert function* $h_I = h$. This is rigged such that limit I of ideals has same Hilbert function as ideals I^t.

Buczyńska-Buczyński show that in border rank decompositions, for $t > 0$ one may assume the points are in general position which leads to a constant Hilbert function as soon as is possible.

In the tensor case, one has more information because one has curves of points on $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. One obtains ideals in $Sym(A \oplus B \oplus C)^* = \bigoplus_{s,t,u} S^s A^* \otimes S^t B^* \otimes S^u C^*$, which is $\mathbb{Z}^{\oplus 3}$-graded. This leads to a Hilbert function that depends on three arguments: $h_I(s,t,u) := \dim(S^s A^* \otimes S^t B^* \otimes S^u C^* / I_{s,t,u})$. By the general position assumption, $h_I(s,t,u) = \min\{r, \dim S^s A^* \otimes S^t B^* \otimes S^u C^*\}$.

Instead of single curve $E_t \subset G(r, A\otimes B\otimes C)$ limiting to a Borel fixed point, for each (i,j,k) one gets a curve in $G(r, S^i A^* \otimes S^j B^* \otimes S^k C^*)$, and Buczyńska-Buczyński show that one may assume that each curve limits to a Borel fixed point.

9.3. **Consequences.** The upshot is an algorithm that either produces all normalized candidate I^0's or proves border rank $> r$ as follows:

If $\underline{\mathbf{R}}(T) \leq r$, there exists a multi-graded ideal $I = I^0$ satisfying:

(1) I is contained in the annihilator of T. This condition says $I_{110} \subset T(C^*)^{\perp}$, $I_{101} \subset T(B^*)^{\perp}$, $I_{011} \subset T(A^*)^{\perp}$ and $I_{111} \subset T^{\perp} \subset A^* \otimes B^* \otimes C^*$.

(2) For all (ijk) with $i + j + k > 1$, $\mathrm{codim} I_{ijk} = r$.

(3) each I_{ijk} is Borel-fixed.

(4) I is an ideal, so the multiplication maps $I_{i-1,j,k} \otimes A^* \oplus I_{i,j-1,k} \otimes B^* \oplus I_{i,j,k-1} \otimes C^* \to S^i A^* \otimes S^j B^* \otimes S^k C^*$ have image contained in I_{ijk}. (These are rank conditions.)

9.4. **Results.** Recall that Strassen proved $\underline{\mathbf{R}}(M_{\langle 3 \rangle}) \geq 14$, Ottaviani and I showed $\underline{\mathbf{R}}(M_{\langle 3 \rangle}) \geq 15$, and Michałek and I showed $\underline{\mathbf{R}}(M_{\langle 3 \rangle}) \geq 16$. Using border apolarity, with A. Conner and A. Harper, we showed:

Theorem 9.1. *[39]* $\underline{\mathbf{R}}(M_{\langle 3 \rangle}) \geq 17$.

The known upper bound is 20 [100]. Interestingly, we are able to construct candidate ideals for border rank 17 and we are currently attempting to determine if these ideals actually come from border rank decompositions using deformation theory.

Recall that so far only $\underline{\mathbf{R}}(M_{\langle 2 \rangle})$ was known among nontrivial matrix multiplication tensors. Let $M_{\langle \mathbf{a},\mathbf{b},\mathbf{c} \rangle} \in \mathbb{C}^{\mathbf{ab}} \otimes \mathbb{C}^{\mathbf{bc}} \otimes \mathbb{C}^{\mathbf{ca}}$ denote the rectangular matrix multiplication tensor. Using border apolarity, we show

Theorem 9.2. *[39]* $\underline{\mathbf{R}}(M_{\langle 223 \rangle}) = 10$ *and* $\underline{\mathbf{R}}(M_{\langle 233 \rangle}) = 14$.

I also remark that the method gives a very short, computer free algebraic proof that $\underline{\mathbf{R}}(M_{\langle 2 \rangle}) = 7$.

All previous techniques for border rank lower bounds were useless when one of the three vector spaces has dimension much larger than the other. We also showed:

Theorem 9.3. *[39] For all* $\mathbf{n} > 25$, $\underline{\mathbf{R}}(M_{\langle 2\mathbf{nn} \rangle}) \geq \mathbf{n}^2 + 1.32\mathbf{n} + 1$ *and for all* $\mathbf{n} > 14$, $\underline{\mathbf{R}}(M_{\langle 3\mathbf{nn} \rangle}) \geq \mathbf{n}^2 + 2\mathbf{n}$.

Previously, only $\underline{\mathbf{R}}(M_{\langle 2\mathbf{nn} \rangle}) \geq \mathbf{n}^2 + 1$ and $\underline{\mathbf{R}}(M_{\langle 3\mathbf{nn} \rangle}) \geq \mathbf{n}^2 + 2$ were known. Notice that this also shows that border apolarity may be used for sequences of tensors, not just fixed small tensors.

Currently we are working to strengthen the border apolarity algorithm, to implement it more efficiently, to take into account more geometric information, and to use deformation theory to overcome the cactus barrier.

10. Strassen's laser method and geometry

In this last section I describe a program to utilize geometry to obtain upper bounds.

10.1. **Strassen's laser method and its barriers.** There was steady progress upper bounding ω from 1969 to 1988 culminating in $\omega < 2.3755$ [41]. All progress since 1984 has been obtained using methods from probability, statistical mechanics and information theory. Given a tensor $T \in A \otimes B \otimes C$, define its k-th Kronecker power $T^{\boxtimes k} := T^{\otimes k} \in (A^{\otimes k}) \otimes (B^{\otimes k}) \otimes (C^{\otimes k})$, that is one takes its k-th tensor power and considers it as a 3-way tensor instead of a $3k$-way tensor. Since we are unable to upper bound the border rank of the matrix multiplication tensor directly, the idea of Strassen's laser method is to start with a tensor where we can upper bound its border rank (e.g., a tensor of minimal border rank), take a large Kronecker power of it, and then show the Kronecker power degenerates to a large matrix multiplication tensor to get an upper bound on the border rank of the large matrix multiplication tensor (namely the bound $\underline{\mathbf{R}}(T)^k$). Here one says T degenerates to T' if $T' \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$. Thus if the original tensor is of low cost (border rank) and it produces a large matrix multiplication tensor (high value), it gives a good upper bound on ω. I emphasize that this is not done explicitly, just the existence of such a degeneration is proved using methods from information theory pioneered by Shannon [98]. All the bounds since 1988 have been obtained by using a single tensor, the "big" Coppersmith-Winograd tensor, which I'll denote $CW_q \in (\mathbb{C}^{q+2})^{\otimes 3}$ (there is one such for each q). Previous to that, the champion was the "little" Coppersmith-Winograd tensor, which I'll denote $cw_q \in (\mathbb{C}^{q+1})^{\otimes 3}$. I will not describe the method here, see e.g., [17, 27, 41] for expositions.

From 1988 until 2011 there was no progress whatsoever and starting 2011 there was incremental progress leading up to the current record in [6].

In 2014, [8] gave explanations for the halting progress, and showed there was a limit to what one could prove with CW_q (the limit is around $\omega < 2.3$). Further explorations of limits were made in [5, 7, 34].

Remark 10.1. An approach to upper bounds using the discrete Fourier transform for finite groups was proposed in [35]. This approach yields similar bounds to Strassen's laser method and faces similar barriers [19, 96].

A geometric explanation of the limits is given in [34]:

Define the *asymptotic rank* of T:

$$\underset{\sim}{\mathbf{R}}(T) := \lim_{N\to\infty} (\underline{\mathbf{R}}(T^{\boxtimes N}))^{\frac{1}{N}},$$

and the *asymptotic subrank* of T:

$$\underset{\sim}{\mathbf{Q}}(T) := \lim_{N\to\infty} (\underline{\mathbf{Q}}(T^{\boxtimes N}))^{\frac{1}{N}}.$$

For a given tensor T, a limit to its utility for the laser method is given by the ratio of these two quantities, and the tensor could potentially be used to prove ω is two only if the ratio is one, i.e., the tensor is of minimal asymptotic rank and maximal asymptotic subrank. (In [34] they take the ratio of the logs, which they call *irreversibility*.) The barriers say nothing about just how useful the tensor can be, only what one cannot do with it.

The *only* tensors we know the asymptotic rank of are those of minimal border rank.

The tensors cw_q for $q < 10$ could potentially be used to prove $\omega < 2.3$, the main obstruction to doing so is that they are not of minimal border rank $q+1$ but instead $\underline{\mathbf{R}}(cw_q) = q+2$. Moreover, the case cw_2 could potentially be used to prove ω is two. What counts is the asymptotic rank, so were $\underline{\mathbf{R}}(cw_q^{\boxtimes 2}) < \underline{\mathbf{R}}(cw_q)^2$, one could get a better upper bound on ω than the one found by Coppersmith-Winograd using the tensor.

Unfortunately for upper bounds, in [37] A. Conner, F. Gesmundo, E. Ventura and I showed $\underline{\mathbf{R}}(cw_q^{\boxtimes 2}) = \underline{\mathbf{R}}(cw_q)^2$ for $q > 2$. At the time we were unable to determine the behaviour of $cw_2^{\boxtimes 2}$

as existing techniques did not yield any meaningful bound. With the advent of border apolarity, recently in [40], A. Conner, H. Huang and I showed $\underline{\mathbf{R}}(cw_2^{\boxtimes 2}) = \underline{\mathbf{R}}(cw_2)^2$.

I bring all this up in an article on representation theory and geometry because of recent work in the search for tensors useful for the laser method. Although the recent work has yet to improve upon the exponent with geometric methods, it shows promise for the future. I emphasize that in the computer science literature, the tensors used in the laser method were found and exploited because of their combinatorial properties when expressed in a good basis. M. Michałek and I had the idea [74,75] to analyze the geometry of the tensors that have been successful in proving upper bounds on ω via the laser method, and then to find other tensors with similar geometry in the hope they might be better for the laser method. We found they had remarkable geometric properties. Among them, perhaps the most interesting property is that their symmetry groups have large dimension.

10.2. **Tensors with symmetry.** There is a slight subtlety when discussing symmetry. The map $GL(A)\times GL(B)\times GL(C)\to GL(A\otimes B\otimes C)$ has a two dimensional kernel, namely $\{(\lambda\,\mathrm{Id}_A,\mu\,\mathrm{Id}_B,\nu\,\mathrm{Id}_C)\mid \lambda\mu\nu=1\}$ and sometimes it is more convenient to express the symmetry group (resp. algebra) in $GL(A)\times GL(B)\times GL(C)$ including this kernel (resp. $\mathfrak{gl}(A)\oplus\mathfrak{gl}(B)\oplus\mathfrak{gl}(C)$). When I do this I will decorate it with a tilde.

First note that any minimal border rank tensor in $(\mathbb{C}^m)^{\otimes 3}$ has symmetry group of dimension at least $2m-2$ as that is true for $M_{\langle 1\rangle}^{\oplus m} := a_1\otimes b_1\otimes c_1+\cdots+a_m\otimes b_m\otimes c_m$ and $\sigma_m(Seg(\mathbb{P}^{m-1}\times\mathbb{P}^{m-1}\times\mathbb{P}^{m-1})) = \overline{GL_m^{\times 3}\cdot[M_{\langle 1\rangle}^{\oplus m}]}$. A rank one tensor will have the largest symmetry group (of dimension $3m^2-3m+1$) but tensors useful for the laser method have tended to be 1-generic, so one expects a much smaller symmetry group.

As I illustrate below, the tensor CW_{m-2} has a symmetry group of dimension $\frac{m^2}{2}+\frac{m}{2}$, which is quite large, so it is natural to look among 1-generic tensors with large symmetry groups to find ones useful for the laser method. This was the starting point of [38] with A. Conner, F. Gesmundo, and E. Ventura.

Let $\mathcal{B}\in\langle a_2,\cdots,a_{m-1}\rangle\otimes\langle b_2,\cdots,b_{m-1}\rangle\subset A\otimes B$ be a nondegenerate bilinear form on $\mathbb{C}^{m-2}\times\mathbb{C}^{m-2}$.

Theorem 10.2. *[38] Let $m\geq 7$ and let $\dim A=\dim B=\dim C=m$. Let $T\in A\otimes B\otimes C$ be a 1-generic tensor. Then*

$$\dim G_T < \frac{m^2}{2}+\frac{m}{2}-2 \tag{1}$$

except when T is isomorphic to

$$S_{\mathcal{B}} := a_1\otimes b_1\otimes c_m + a_1\otimes b_m\otimes c_1 + a_m\otimes b_1\otimes c_1 + \textstyle\sum_{\rho=2}^{m-1} a_1\otimes b_\rho\otimes c_\rho + \sum_{\rho=2}^{m-1} a_\rho\otimes b_1\otimes c_\rho + \mathcal{B}\otimes c_1, \tag{2}$$

where $\mathcal{B}\in A\otimes B$ is one of the four following rank $m-2$ bilinear forms

$$\textstyle\sum_{\xi=2}^{p+1} a_\xi\otimes b_{\xi+p} - a_{\xi+p}\otimes b_\xi \quad m=2p \text{ even} \quad (T_{skewCW,m-2}) \tag{3}$$

$$\textstyle\sum_{\rho=2}^{m-1} a_\rho\otimes b_\rho \quad \text{all } m \quad (T_{CW,m-2}) \tag{4}$$

$$a_{m-1}\otimes b_{m-1} + \textstyle\sum_{\xi=2}^{p}\left(a_\xi\otimes b_{\xi+p-1} - a_{\xi+p-1}\otimes b_\eta\right) \quad m=2p \text{ even} \quad (T_{s+skewCW,m-2}) \tag{5}$$

$$a_{m-1}\otimes b_{m-1} + \textstyle\sum_{\xi=2}^{p}\left(a_\xi\otimes b_{\xi+p-1} - a_{\xi+p-1}\otimes b_\eta\right) \quad m=2p+1 \text{ odd} \quad (T_{s\oplus skewCW,m-2}) \tag{6}$$

All these except $T_{skewCW,m-2}$ *have* $\dim G_T = \frac{m^2}{2} + \frac{m}{2} - 2$, *and* $\dim G_{T_{skewCW,m-2}} = \frac{m^2}{2} + \frac{3m}{2} - 4$.

In particular: when m *is even, there is a unique up to isomorphism,* 1*-generic tensor* T *with maximal dimensional symmetry group, namely* $T_{skewCW,m-2}$, *and there are exactly two, up to isomorphism, additional* 1*-generic tensors* T *such that* $\dim G_T \geq \frac{m^2}{2} + \frac{m}{2} - 2$, *which are* $T_{CW,m-2}$ *and* $T_{s+skewCW,m-2}$, *where equality holds.*

When m *is odd, there are exactly two* 1*-generic tensors* T *up to isomorphism with maximal dimensional symmetry group* $\frac{m^2}{2} + \frac{m}{2} - 2$, *which are* $T_{CW,m-2}$ *and* $T_{s\oplus skewCW,m-2}$.

Remark 10.3. In [97] T. Seyannaeve decomposed $S^3\mathfrak{gl}_n$ and noticed that several of the highest weight vectors that appeared were Coppersmith-Winograd tensors. This gave rise to the idea that one might look among the highest weight vectors in $S^3\mathfrak{gl}_n$ to find ones useful for the laser method. This was carried out in [56]. This is a variant on having a large symmetry group, as highest weight vectors are preserved by a parabolic subgroup.

Call a tensor *skeletal* if it may be written in the form (2) for some nondegenerate bilinear form $\mathcal{B}$.

Proposition 10.4. *[38]*

(1) *Any* 1*-generic tensor in* $(\mathbb{C}^m)^{\otimes 3}$ *may be degenerated to a skeletal tensor.*
(2) *The only skeletal tensor of minimal border rank is the Coppersmith-Winograd tensor, which is the case of* $\mathcal{B} \in S^2\mathbb{C}^{m-2}$.
(3) *In particular any* 1*-generic minimal border rank tensor* $(\mathbb{C}^m)^{\otimes 3}$ *degenerates to* CW_{m-1}.

The result (3) originally appeared in [58], although the proof (but not the statement) was already in an early preprint version of [38].

In some sense (3) could be interpreted as saying that CW_q is the worst minimal border rank 1-generic tensor for the laser method. The question is now, whether all others are equally bad, or just that the laser method as currently practiced is not refined enough.

I exhibit the symmetry Lie algebras of the above tensors: Let $\widetilde{\mathfrak{g}}_T$ denote the Lie subalgebra of $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ annihilating T:

$$\widetilde{\mathfrak{g}}_T := \{L \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C) \mid L.T = 0\}.$$

Here $L.T$ denotes the Lie algebra action.

If $L = (U, V, W) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$, and we have bases $\{u^i_j\}, \{v^i_j\}, \{w^i_j\}$ respectively for U, V, W, the condition $L.T = 0$ is equivalent to the following system of linear equations:

$$\sum_{i'} u^i_{i'} T^{i'jk} + \sum_{j'} v^j_{j'} T^{ij'k} + \sum_{k'} w^k_{k'} T^{ijk'} = 0, \text{ for every } i, j, k. \tag{7}$$

For a skeletal 1-generic tensor:

$$\widetilde{\mathfrak{g}}_{S_{\mathcal{B}}} = \tag{8}$$
$$\left\{ \begin{pmatrix} u^1_1 & \overline{u}^{\mathbf{t}} & u^1_m \\ 0 & X - \frac{1}{2}u^1_1 \mathrm{Id} & \mathcal{B}\overline{v} - \overline{z} \\ 0 & 0 & -2u^1_1 \end{pmatrix}, \begin{pmatrix} u^1_1 & \overline{v}^{\mathbf{t}} & v^1_m \\ 0 & X - \frac{1}{2}u^1_1 \mathrm{Id} & \mathcal{B}\overline{u} - \overline{z} \\ 0 & 0 & -4u^1_1 \end{pmatrix}, \begin{pmatrix} u^1_1 & \overline{z}^{\mathbf{t}} & -u^1_m - v^1_m \\ 0 & -X^{\mathbf{t}} - \frac{1}{2}u^1_1 \mathrm{Id} & -(\overline{v} + \overline{u}) \\ 0 & 0 & -4u^1_1 \end{pmatrix} \right\}.$$

Here $\overline{u}, \overline{v}, \overline{z} \in \mathbb{C}^{m-2}$ and $X \in \mathfrak{h}_{\mathcal{B}}$.

The small Coppersmith-Winograd tensor is

$$cw_{m-2} = +\sum_{\rho=2}^{m-1} a_1 \otimes b_\rho \otimes c_\rho + \sum_{\rho=2}^{m-1} a_\rho \otimes b_1 \otimes c_\rho + \mathcal{B} \otimes c_1 \in (\mathbb{C}^{m-1})^{\otimes 3}$$

with $\mathcal{B}$ symmetric, and for other $\mathcal{B}$, write the tensors as $T_{\mathcal{B}-cw,m-2}$. Then

$$\widetilde{\mathfrak{g}}_{T_{\mathcal{B}-cw,q}} = \left\{ \left(\begin{pmatrix} -\mu - \nu & 0 \\ 0 & \lambda \operatorname{Id} + X \end{pmatrix}, \begin{pmatrix} -\lambda - \nu & 0 \\ 0 & \mu \operatorname{Id} + X \end{pmatrix}, \begin{pmatrix} -\lambda - \mu & 0 \\ 0 & \nu \operatorname{Id} + X \end{pmatrix} \right) \mid \lambda, \mu, \nu \in \mathbb{C} X \in \mathfrak{h}_{\mathcal{B}} \right\}.$$

In particular $\dim \mathfrak{g}_{T_{cw,q}} = \binom{q}{2} + 1$.

All these tensors could potentially be used to prove $\omega < 2.3$ for $q < 10$ and the case $q = 2$ could again potentially be used to prove ω is two. Notice that if we take $\mathcal{B}$ to be completely skew, we get a tensor with a larger symmetry group. This tensor $T_{skewcw,2}$ unfortunately has larger initial cost than cw_2, namely $\underline{\mathbf{R}}(T_{skewcw,2}) = 5 > 4 = \underline{\mathbf{R}}(cw_2)$, (with the same value), however in [39], using border apolarity, we proved

Theorem 10.5. $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes 2}) = 17 << 25$

which provides hope for the laser method.

Remark 10.6. The Kronecker squares of $T_{skewcw,2}, cw_2$ are familiar tensors, respectively $\det_3$ and perm_3 considered as tensors, so these results are of interest well beyond the laser method.

11. Appendix: Strassen's alogorithm

Here is Strassen's algorithm for multiplying 2×2 matrices using 7 scalar multiplications [102]: Set

$$\begin{aligned} I &= (a_1^1 + a_2^2)(b_1^1 + b_2^2), \\ II &= (a_1^2 + a_2^2)b_1^1, \\ III &= a_1^1(b_2^1 - b_2^2) \\ IV &= a_2^2(-b_1^1 + b_1^2) \\ V &= (a_1^1 + a_2^1)b_2^2 \\ VI &= (-a_1^1 + a_1^2)(b_1^1 + b_2^1), \\ VII &= (a_2^1 - a_2^2)(b_1^2 + b_2^2), \end{aligned} \tag{9}$$

Exercise 11.1: (1) Show that if $C = AB$, then

$$\begin{aligned} c_1^1 &= I + IV - V + VII, \\ c_1^2 &= II + IV, \\ c_2^1 &= III + V, \\ c_2^2 &= I + III - II + VI. \end{aligned}$$

Now notice the the entries of A, B themselves could be matrices, so this also gives, by iterating, an algorithm for multiplying $2^k \times 2^k$ matrices.

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