CANTOR DYNAMICS OF RENORMALIZABLE GROUPS

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Abstract. A group Γ is said to be finitely non-co-Hopfian, or renormalizable, if there exists a self-embedding $\varphi\colon \Gamma \to \Gamma$ whose image is a proper subgroup of finite index. Such a proper self-embedding is called a renormalization for Γ. In this work, we associate a dynamical system to a renormalization φ of Γ. The discriminant invariant $\mathcal{D}_\varphi$ of the associated Cantor dynamical system is a profinite group which is a measure of the asymmetries of the dynamical system. If $\mathcal{D}_\varphi$ is a finite group for some renormalization, we show that Γ/C_φ is virtually nilpotent, where C_φ is the kernel of the action map. We introduce the notion of a (virtually) renormalizable Cantor action, and show that the action associated to a renormalizable group is virtually renormalizable. We study the properties of virtually renormalizable Cantor actions, and show that virtual renormalizability is an invariant of continuous orbit equivalence. Moreover, the discriminant invariant of a renormalizable Cantor action is an invariant of continuous orbit equivalence.

1. Introduction

A countable group Γ is *co-Hopfian* if every monomorphism $\varphi\colon \Gamma \to \Gamma$ is an isomorphism [4], and is said to be *non-co-Hopfian* otherwise. If there exists a self-embedding φ whose image is a proper subgroup of finite index, then Γ is said to be *finitely non-co-Hopfian* [55]. A proper self-embedding $\varphi\colon \Gamma \to \Gamma$ with finite index is called a *renormalization* of Γ, in analogy with the case for $\Gamma = \mathbb{Z}^n$. If Γ admits a renormalization, then it is said to be *renormalizable.*

The free abelian group $\mathbb{Z}^n$ is renormalizable, as are many finitely generated nilpotent groups. There are also many examples of renormalizable groups which are not nilpotent, as described for example in [18, 24, 28, 29, 30, 46, 48, 55]. On the other hand, the free group $\mathbb{Z}^{\star n} = \mathbb{Z} \star \cdots \star \mathbb{Z}$ for $n \geq 2$ is non-co-Hopfian, but is not renormalizable. The classification of non-co-Hopfian groups in general appears to be a difficult problem.

There is a related concept of a *scale-invariant* group, introduced by Benjamini (see [52, Section 9.2]). A *scale* for Γ is a descending chain of finite index subgroups $\mathcal{S} = \{\Gamma_\ell \mid \ell \geq 1\}$ whose intersection is a finite group, and such that for each ℓ, there exists an isomorphism $\phi_\ell\colon \Gamma \to \Gamma_\ell$. Benjamini asked if a scale-invariant group must be virtually nilpotent? Nekrashevych and Pete [46, Theorem 1.1] gave examples of scale-invariant groups which are not virtually nilpotent. In the same work, the authors defined the notion of a *strongly scale-invariant* group, as a renormalizable group Γ such that the collection of subgroups $\{\Gamma_\ell = \varphi_\ell(\Gamma) \mid \ell \geq 0\}$ is a scale for Γ. Then [46, Question 1.1] asks if a strongly scale-invariant group must be virtually nilpotent? The results of our work give a partial answer to this question. We introduce a profinite group $\widehat{\Gamma}_\varphi$ naturally associated to a renormalization φ, and show in Proposition 5.2 that φ induces an open embedding $\widehat{\varphi}_0\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$. Corollary 1.5 states that if both the intersections $\cap_{\ell>0}\, \varphi^\ell(\Gamma)$ and $\cap_{\ell>0}\, \widehat{\varphi}_0^\ell(\widehat{\Gamma}_\varphi)$ are finite groups, then Γ is virtually nilpotent. In other words, we answer in the affirmative the question of Nekrashevych and Pete above under a stronger assumption, that both Γ and the profinite group $\widehat{G}_\varphi$ admit a scale.

Version date: February 4, 2020.

2010 *Mathematics Subject Classification.* Primary: 20F18, 37B50, 37C85.

OL is supported by FWF Project P31950-N35.

WVL is supported by NSF Grant DMS-1855371 and the Max Planck Institute for Mathematics.

Keywords: non-co-Hopfian groups, minimal Cantor actions, odometer actions, renormalization.

Our approach to the study of renormalizable groups is based on the study of the Cantor dynamical systems naturally associated to their renormalizations. An action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is said to be a *Cantor action* if Γ is a finitely generated group, $\mathfrak{X}$ is a Cantor metric space, and the action is minimal. The basic properties of Cantor actions are discussed in Section 2. In Section 3, we associate an equicontinuous Cantor dynamical system $(X_\varphi, \Gamma, \Phi_\varphi)$ to a renormalization φ of Γ. The renormalization map φ for Γ induces a *renormalization of the action* of Γ, as explained in Definitions 7.1 and 7.2. A key idea of this work is to study the regularity properties of the action, as discussed in Section 4, which leads to a proof of the fundamental technical result Proposition 5.2.

The *discriminant invariant* $\mathcal{D}$ of an equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is the profinite group defined in Definition 2.5. The isomorphism class of $\mathcal{D}$ depends only on the conjugacy class of the action, and has other invariance properties [20, 21, 35, 37]. For the Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$ associated to a renormalization φ, its discriminant invariant is denoted by $\mathcal{D}_\varphi$, and is given by formula (12) which provides an effective way to calculate it. If Γ is abelian, the discriminant $\mathcal{D}_\varphi$ is the trivial group for any renormalization.

Let $C(\mathcal{G}_\varphi)$ be the largest normal subgroup of the intersection $K(\mathcal{G}_\varphi) = \bigcap_{\ell > 0} \varphi^\ell(\Gamma)$.

THEOREM 1.1. *Let Γ be a finitely generated group, and let $\varphi\colon \Gamma \to \Gamma$ be a renormalization of Γ.*

(1) If $\mathcal{D}_\varphi$ is the trivial group, then $\Gamma/C(\mathcal{G}_\varphi)$ is nilpotent.
(2) If $\mathcal{D}_\varphi$ is a finite group, then $\Gamma/C(\mathcal{G}_\varphi)$ is virtually nilpotent.

The proof of Theorem 1.1 given in Section 6, and uses Theorem 5.3 which is based on the results in Reid [49], quoted as Theorem 5.1 below, and Theorem 4.4 and Proposition 5.2 in this work.

We give an example in Section 8.2 of a renormalization of the Heisenberg group for which $\mathcal{D}_\varphi$ is a Cantor group. Thus, while the assumption that $\mathcal{D}_\varphi$ is finite is sufficient to conclude that $\Gamma/C(\mathcal{G}_\varphi)$ is nilpotent, it is not a necessary condition. On the other hand, there are renormalizations of the Heisenberg group for which $\mathcal{D}_\varphi$ is the trivial group. The known examples of renormalizations suggest that it is an interesting problem to study the collection of all renormalizations for a given group Γ, even for the simplest non-abelian nilpotent groups.

Next, we introduce some properties of the Cantor actions associated to renormalizations.

A Cantor action $(\mathfrak{X}, G, \Phi)$ is *free* if for any $g \in \Gamma$ which is not the identity, the action $\Phi(g)$ has no fixed points. The action is *topologically free*, as in Definition 2.1, if the set of points fixed by at least one element of the group is a meager set. The notion of a *quasi-analytic* Cantor action, as in Definition 4.1, was introduced in the works [21, 35] as a generalization of the notion of a topologically free action. The quasi-analytic property of a Cantor action is a fundamental property of renormalizable groups and actions.

THEOREM 1.2. *The Cantor action $\Phi_\varphi\colon \Gamma \times X_\varphi \to X_\varphi$ associated to a renormalization φ is quasi-analytic. Hence, if the action Φ_φ is also effective, then it is topologically free.*

Given a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, let $\Phi(\Gamma) \subset Homeo(\mathfrak{X})$ denote the image subgroup. If the action is equicontinuous, then the closure $\mathfrak{G}(\Phi) \equiv \overline{\Phi(\Gamma)} \subset Homeo(\mathfrak{X})$ in the *uniform topology of maps* is a separable profinite group. This is discussed further in Section 2.2. For the Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$ associated to a renormalization φ, we denote this closure by $\widehat{\Gamma}_\varphi$. Then Theorem 4.4 implies that the profinite action $\widehat{\Phi}_\varphi\colon \widehat{\Gamma}_\varphi \times X_\varphi \to X_\varphi$ is quasi-analytic, which implies Theorem 1.2. The quasi-analytic property is used in the proof of the following, which is a restatement of Proposition 5.2:

THEOREM 1.3. *Let φ be a renormalization of the finitely generated group Γ. Then φ induces an injective homomorphism $\widehat{\varphi}_0\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ whose image is a clopen subgroup of $\widehat{\Gamma}_\varphi$.*

This is proved in Section 5, where we use this to show Theorem 5.3, which yields:

THEOREM 1.4. *Let φ be a renormalization of the finitely generated group Γ, and $\widehat{\varphi}_0\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ the induced contraction mapping on $\widehat{\Gamma}_\varphi$. Then $\mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}_0^n(\widehat{\Gamma}_\varphi)$.*

Theorems 1.1 and 1.4 yield an answer to the profinite version of the Nekrashevych-Pete question:

COROLLARY 1.5. *Let φ be a renormalization of the finitely generated group Γ. Suppose that*

$$K(\mathcal{G}_\varphi) = \bigcap_{\ell > 0} \varphi^\ell(\Gamma) \subset \Gamma \quad , \quad \mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}_0^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi \tag{1}$$

are both finite groups, then Γ is virtually nilpotent, and if both are trivial, then Γ is nilpotent.

The notion of *isomorphism* of Cantor actions, given in Definition 2.6, is a generalization of the usual notion of conjugacy of topological actions. For $\Gamma = \mathbb{Z}$, isomorphism corresponds to the notion of "flip conjugacy" introduced in the work of Boyle and Tomiyama [10]. *Return equivalence* as given in Definition 2.7, is a form of "virtual isomorphism" for equicontinuous Cantor actions.

The standard notion of *continuous orbit equivalence* for Cantor systems, as given in Definition 2.8, requires only that the orbits of two actions agree in a continuous manner. For a Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$, the isomorphism class of the reduced group C^*-algebra $C^*_r(X_\varphi, \Gamma, \Phi_\varphi)$ and its K-theory groups $K_*(C^*_r(X_\varphi, \Gamma, \Phi_\varphi))$, are invariants of its continuous orbit equivalence class. In particular, they provide invariants of the conjugacy class of the renormalization φ. For example, the limit group invariants defined in [26] for $\Gamma = \mathbb{Z}^n$ are of this form, and for Γ nilpotent there are analogous K-theoretic invariants of its renormalizations.

As mentioned above, our study of renormalizable groups naturally suggests a related notion, that of a *renormalizable* equicontinuous Cantor action, as introduced in Definition 7.1. It is modeled on the concept of a renormalizable dynamical system, and also that of self-similar groups [45] and percolation theory [52, Section 9.2]. We also introduce a variant of this notion in Definition 7.2, that of *virtually renormalizable* actions. The study of renormalizable Cantor actions is motivated, in part, by the examples of Cantor actions defined recursively, in terms of the action of a finite set of generators on a d-adic tree for $d \geq 3$, where there is an embedding $\varphi \colon \Gamma \to \Gamma$ whose image is a subgroup of the stabilizer group of a branch of a tree. The image $\varphi(\Gamma) \subset \Gamma$ need not be of finite index in Γ, even though the stabilizer group of a branch always has finite index in Γ. The following result shows that if an equicontinuous Cantor action is quasi-analytic, then it arises from a renormalization of the acting group Γ.

THEOREM 1.6. *Let φ be a renormalization of Γ, then the Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$ is virtually renormalizable. Conversely, suppose that the equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable and locally quasi-analytic, then Γ is renormalizable, and there is a renormalization φ such that $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to $(X_\varphi, \Gamma, \Phi_\varphi)$.*

This is proved in Section 7. An equicontinuous Cantor action which is not locally quasi-analytic must be *wild*, a notion introduced in the works [35, 36, 37]. We thus obtain the following dichotomy:

COROLLARY 1.7. *A renormalizable Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is either quasi-analytic, and hence Γ is renormalizable, or the action is wild.*

This result motivates the study of the invariants of renormalizable Cantor actions, both to understand the invariants of the renormalization map, and to discover invariants of these actions which distinguish between the quasi-analytic and wild cases of Corollary 1.7. Our final results in this work considers their invariant properties under continuous orbit equivalence.

THEOREM 1.8. *Let $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ be equicontinuous Cantor actions which are continuously orbit equivalent. If $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable and locally quasi-analytic, then $(\mathfrak{X}', \Gamma', \Phi')$ is virtually renormalizable.*

Theorems 1.6 and 1.8 combine to yield an important consequence, that the isomorphism class of the discriminant group $\mathcal{D}_\varphi$ associated to φ is an invariant of continuous orbit equivalence.

THEOREM 1.9. *Let $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ be equicontinuous Cantor actions associated to renormalizations $\varphi \colon \Gamma \to \Gamma$ and $\varphi' \colon \Gamma' \to \Gamma'$, respectively. If the actions are continuously orbit equivalent, then the isomorphism class of their discriminants $\mathcal{D}_\varphi$ and $\mathcal{D}'_{\varphi'}$ are isomorphic.*

Examples and applications of our results are discussed in Section 8.

Section 9 discusses open problems. In particular, the works [35, 37] study the relations between the discriminant invariant for a Cantor action and the wild property for the action. It is an interesting problem to further explore this relation for renormalizable actions, as these include many class of branch groups and related constructions, as in [5, 6, 45, 46, 47, 32].

2. Cantor actions

In this section, we recall some of the properties of Cantor actions. A basic reference is [3].

2.1. **Basic concepts.**

For an action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ on a topological space $\mathfrak{X}$, let $g \cdot x = \Phi(g)(x)$.

Let $(\mathfrak{X}, \Gamma, \Phi)$ denote an action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$. The orbit of $x \in \mathfrak{X}$ is the subset $\mathcal{O}(x) = \{g \cdot x \mid g \in \Gamma\}$. The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x)$ is dense in $\mathfrak{X}$.

An action $(\mathfrak{X}, \Gamma, \Phi)$ is *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $g \in \Gamma$ we have $d_{\mathfrak{X}}(x, y) < \delta$ implies that $d_{\mathfrak{X}}(g \cdot x, g \cdot y) < \varepsilon$. The property of being equicontinuous is independent of the choice of the metric on $\mathfrak{X}$.

An action $(\mathfrak{X}, \Gamma, \Phi)$ is *effective*, or *faithful*, if the action homomorphism $\Phi\colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ has trivial kernel. The action is *free* if for all $x \in \mathfrak{X}$ and $g \in G$, $g \cdot x = x$ implies that $g = e$, the identity of the group. The *isotropy group* of $x \in \mathfrak{X}$ is the subgroup

$$\Gamma_x = \{g \in \Gamma \mid g \cdot x = x\} \ . \tag{2}$$

Let $\mathrm{Fix}(g) = \{x \in \mathfrak{X} \mid g \cdot x = x\}$, and introduce the *isotropy set*

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists\ g \in \Gamma\ ,\ g \neq id\ ,\ g \cdot x = x\} = \bigcup_{e \neq g \in \Gamma} \mathrm{Fix}(g)\ . \tag{3}$$

DEFINITION 2.1. [10, 41, 50] $(\mathfrak{X}, \Gamma, \Phi)$ *is said to be* topologically free *if* $\mathrm{Iso}(\Phi)$ *is meager in* $\mathfrak{X}$.

Note that if $\mathrm{Iso}(\Phi)$ is meager, then $\mathrm{Iso}(\Phi)$ has empty interior.

The notion of topologically free Cantor actions was introduced by Boyle in his thesis [9], and later used in the works by Boyle and Tomiyama [10] for the study of classification of Cantor actions, by Renault [50] for the study of the C^*-algebras associated to Cantor actions, and by Li [41] for proving rigidity properties of equicontinuous Cantor actions.

Now assume that $\mathfrak{X}$ is a Cantor space. Let $\mathrm{CO}(\mathfrak{X})$ denote the collection of all clopen (closed and open) subsets of $\mathfrak{X}$, which forms a basis for the topology of $\mathfrak{X}$. For $\phi \in \mathrm{Homeo}(\mathfrak{X})$ and $U \in \mathrm{CO}(\mathfrak{X})$, the image $\phi(U) \in \mathrm{CO}(\mathfrak{X})$. The following result is folklore, and a proof is given in [36, Proposition 3.1].

PROPOSITION 2.2. *A Cantor action* $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ *is equicontinuous if and only if the orbit of every* $U \in \mathrm{CO}(\mathfrak{X})$ *is finite for the induced action* $\Phi_*\colon \Gamma \times \mathrm{CO}(\mathfrak{X}) \to \mathrm{CO}(\mathfrak{X})$.

Let $(\mathfrak{X}, \Gamma, \Phi)$ be an equicontinuous Cantor action. We say that $U \subset \mathfrak{X}$ is *adapted* to the action if U is a non-empty clopen subset, and for any $g \in \Gamma$, if $\Phi(g)(U) \cap U \neq \emptyset$ then $\Phi(g)(U) = U$. The proof of Proposition 3.1 in [36] shows that given $x \in \mathfrak{X}$ and a clopen set W with $x \in W$, there is an adapted clopen set U with $x \in U \subset W$.

The key property of adapted sets, is that for U adapted, the set of "return times" to U,

$$\Gamma_U = \{g \in \Gamma \mid g \cdot U \cap U \neq \emptyset\} \tag{4}$$

is a subgroup of Γ, called the *stabilizer* of U. Then for $g, g' \in \Gamma$ with $g \cdot U \cap g' \cdot U \neq \emptyset$ we have $g^{-1}\, g' \cdot U = U$, hence $g^{-1}\, g' \in \Gamma_U$. Thus, the translates $\{g \cdot U \mid g \in \Gamma\}$ form a finite clopen partition of $\mathfrak{X}$, and are in 1-1 correspondence with the quotient space $X_U = \Gamma/\Gamma_U$. Then Γ acts by permutations of the finite set X_U and so the stabilizer group $\Gamma_U \subset G$ has finite index. The action of $g \in \Gamma$ on X_U is trivial precisely when g is a stabilizer of each coset $h \cdot \Gamma_U$, so $g \in C_U$ where $C_U = \bigcap_{h \in \Gamma} h\ \Gamma_U\ h^{-1} \subset \Gamma_U$ is the largest normal subgroup of Γ contained in Γ_U. The action of the finite group $Q_U \equiv \Gamma/C_U$ on X_U by permutations is a finite approximation of the action of Γ on $\mathfrak{X}$, and the isotropy group of the identity coset $e \cdot \Gamma_U$ is $D_U \equiv \Gamma_U/C_U \subset Q_U$.

DEFINITION 2.3. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an equicontinuous Cantor action. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ is an* adapted neighborhood basis *at $x \in \mathfrak{X}$ for the action Φ, if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 0$, each U_ℓ is adapted to the action Φ, and the intersection $\cap_{\ell>0} U_\ell = \{x\}$.*

Given $x \in \mathfrak{X}$ and $\varepsilon > 0$, Proposition 2.2 implies there exists an adapted clopen set $U \in \mathrm{CO}(\mathfrak{X})$ with $x \in U$ and $\mathrm{diam}(U) < \varepsilon$. Thus, one can choose a descending chain $\mathcal{U}$ of adapted sets in $\mathrm{CO}(\mathfrak{X})$ whose intersection is x, which shows the following:

PROPOSITION 2.4. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an equicontinuous Cantor action. Given $x \in \mathfrak{X}$, there exists an adapted neighborhood basis $\mathcal{U}$ at x for the action Φ.*

2.2. **The profinite model.** Given an equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, let $\Phi(\Gamma) \subset \mathrm{Homeo}(\mathfrak{X})$ denote the image subgroup. Then the closure $\mathfrak{G}(\Phi) \equiv \overline{\Phi(\Gamma)} \subset \mathrm{Homeo}(\mathfrak{X})$ in the *uniform topology of maps* is a separable profinite group. This group is identified with the *Ellis group* for the action, as defined in [3, 22, 23]; see also [20, Section 2]. Each element $\widehat{g} \in \mathfrak{G}(\Phi)$ is the uniform limit of a sequence of maps $\{\Phi(g_i) \mid i \geq 1\} \subset \Phi(\Gamma)$. We sometimes denote the limit $\widehat{g}$ by (g_i).

For example, if G is an abelian group, then $\mathfrak{G}(\Phi)$ is a compact totally disconnected abelian group, which can be thought of as the group of asymptotic motions of the system. When G is non-abelian, the action closure $\mathfrak{G}(\Phi)$ can have much more subtle algebraic properties.

Let $\widehat{\Phi} \colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$ denote the induced action of $\mathfrak{G}(\Phi)$ on $\mathfrak{X}$. For $\widehat{g} \in \mathfrak{G}(\Phi)$ we write its action on $\mathfrak{X}$ by $\widehat{g} \cdot x = \widehat{\Phi}(\widehat{g})(x)$. If the action $\Phi \colon G \times \mathfrak{X} \to \mathfrak{X}$ is minimal, then the group $\mathfrak{G}(\Phi)$ acts transitively on $\mathfrak{X}$. Given $x \in \mathfrak{X}$, introduce the isotropy group at x,

$$\mathfrak{G}(\Phi)_x = \{\widehat{g} \in \overline{\Phi(G)} \mid \widehat{g} \cdot x = x\} \subset \mathrm{Homeo}(\mathfrak{X}) \ , \tag{5}$$

which is a closed subgroup of $\mathfrak{G}(\Phi)$, and thus is either finite, or is a Cantor group.

DEFINITION 2.5. *The discriminant of an equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is $\mathcal{D} = \mathfrak{G}(\Phi)_x$.*

There is a natural identification $\mathfrak{X} \cong \mathfrak{G}(\Phi)/\mathfrak{G}(\Phi)_x$ of left G-spaces, and thus the conjugacy class of $\mathfrak{G}(\Phi)_x$ in $\mathfrak{G}(\Phi)$ is independent of the choice of x. Thus, to be precise, $\mathcal{D}$ denotes the conjugacy class of $\mathfrak{G}(\Phi)_x$. If $\mathcal{D}$ is the trivial group, then $\mathfrak{X}$ is identified with a profinite group, and the action is free. However, there exists examples of free equicontinuous Cantor actions for which the group $\mathcal{D}$ is non-trivial. The first such examples were constructed by Fokkink and Oversteegen in [25, Section 8], and further examples are constructed in [21, Section 10].

2.3. **Equivalence of Cantor actions.** We recall three notions of equivalence of Cantor actions which we use in this work. The first and strongest notion is the following, as used in [14, 36, 41]:

DEFINITION 2.6. *Cantor actions $(\mathfrak{X}_i, \Gamma_i, \Phi_i)$, for $i = 1, 2$, are said to be* isomorphic *if there is a homeomorphism $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ and a group isomorphism $\Theta \colon \Gamma_1 \to \Gamma_2$ so that*

$$\Phi_1(g) = h^{-1} \circ \Phi_1(\Theta(g)) \circ h \in \mathrm{Homeo}(\mathfrak{X}_1) \text{ for all } g \in \Gamma_1 \ . \tag{6}$$

The notion of *return equivalence* for equicontinuous Cantor actions is weaker than the notion of isomorphism, and is natural when considering the Cantor systems in the works [35, 36, 37].

Throughout this work, by a small abuse of notion, for an equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, we use Φ_U to denote both the restricted action $\Phi_U \colon \Gamma_U \times U \to U$ and the induced quotient action $\Phi_U \colon H_U \times U \to U$ for $H_U = \Phi(G_U) \subset \mathrm{Homeo}(U)$.

DEFINITION 2.7. *Equicontinuous Cantor actions $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are* return equivalent *if there exists an adapted set $U \subset \mathfrak{X}$ for the action Φ and an adapted set $V \subset \mathfrak{X}'$ for the action Φ', such that the restricted actions (U, H_U, Φ_U) and (V, H'_V, Φ'_V) are isomorphic.*

The notion of *continuous orbit equivalence* for Cantor actions was introduced in [9, 10], and plays a fundamental role in various approaches to the classification of these actions [50]. Consider the equivalence relation on $\mathfrak{X}$ defined by an action $(\mathfrak{X}, G, \Phi)$,

$$\mathcal{R}(\mathfrak{X}, G, \Phi) \equiv \{(x, g \cdot x)) \mid x \in \mathfrak{X}, g \in G\} \subset \mathfrak{X} \times \mathfrak{X} \ . \tag{7}$$

Given actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', H, \Psi)$, we say they are *orbit equivalent* if there exist a bijection $h \colon \mathfrak{X} \to \mathfrak{X}'$ which maps $\mathcal{R}(\mathfrak{X}, G, \Phi)$ to $\mathcal{R}(\mathfrak{X}', H, \Psi)$, and similarly for the inverse map h^{-1}.

DEFINITION 2.8. *Let $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', H, \Psi)$ be Cantor actions. A* continuous orbit equivalence *between the actions is a homeomorphism $h \colon \mathfrak{X} \to \mathfrak{X}'$ which is an orbit equivalence, and satisfies the locally constant properties:*

1. *for each $x \in \mathfrak{X}$ and $g \in G$, there exists $\alpha(g, x) \in H$ and an open set $x \in U_x \subset \mathfrak{X}$ such that $\Psi(\alpha(g, x)) \circ h|U_x = h \circ \Phi(g)|U_x$;*
2. *for each $y \in \mathfrak{X}'$ and $k \in H$, there exists $\beta(k, y) \in G$ and an open set $y \in V_y \subset \mathfrak{X}'$ such that $\Phi(\beta(k, y)) \circ h|V_y = h \circ \Psi(k)|V_y$.*

Note in particular that these conditions imply that the functions $\alpha \colon G \times \mathfrak{X} \to H$ and $\beta \colon H \times \mathfrak{X}' \to G$, defined by (1) and (2) in Definition 2.8, are continuous, as the groups G and H have the discrete topology. However, additional hypotheses are required to conclude that the maps α and β are cocycles over the actions. The works [14, 36, 41] discuss these notions of equivalences as they apply to equicontinuous Cantor actions.

3. Renormalizable groups

In this section, we construct the Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$ associated to a renormalization $\varphi \colon \Gamma \to \Gamma$, and give some of the basic properties of this action.

Set $\Gamma_0 = \Gamma$, and for $\ell \geq 1$, recursively define subgroups $\Gamma_\ell \subset \Gamma$, where $\Gamma_\ell = \varphi(\Gamma_{\ell-1}) \equiv \varphi^\ell(\Gamma)$.

Let $\mathcal{G}_\varphi \equiv \{\Gamma_\ell \mid \ell \geq 0\}$ denote the descending group chain, where each Γ_ℓ has finite index in Γ. Denote the intersection of the group chain by $K(\mathcal{G}_\varphi) \equiv \bigcap_{\ell > 0} \Gamma_\ell$. If $K(\mathcal{G}_\varphi)$ is a finite group, then the group Γ is said to be *strongly scale-invariant*, in the terminology of Nekrashevych and Pete [46].

Let $X_\ell = \Gamma/\Gamma_\ell$ be the finite coset space. Note that X_ℓ is not necessarily a group, as the subgroup Γ_ℓ is not assumed to be normal in Γ. Note that Γ acts transitively on the left on X_ℓ, and the inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a natural Γ-invariant quotient map $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. The inverse limit space

$$X_\varphi \equiv \varprojlim \ \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \tag{8}$$

with the Tychonoff topology is a Cantor space. The actions of Γ on the factors X_ℓ induce a minimal equicontinuous action on X_φ, denoted by $\Phi_\varphi \colon \Gamma \times X_\varphi \to X_\varphi$ or by $(X_\varphi, \Gamma, \Phi_\varphi)$.

Let $\widehat{\Gamma}_\varphi \subset \mathrm{Homeo}(X_\varphi)$ denote the closure of the image $\Phi_\varphi(\Gamma)$. That is, $\widehat{\Gamma}_\varphi = \mathfrak{G}(\Phi_\varphi)$ as defined in Section 2.2. The isomorphism class of the profinite group $\widehat{\Gamma}_\varphi$ is an invariant of the conjugacy class in $\mathrm{Homeo}(X_\varphi)$ of the action Φ_φ.

Let G denote the profinite completion of Γ. Then there is a homomorphism $\psi \colon \Gamma \to G$ with dense image, and the kernel of ψ is the group $N(\psi)$ given by the intersection of all normal subgroups of finite index in Γ. Thus, $N(\psi)$ is trivial exactly when the group Γ is residually finite. The map $\Phi_\varphi \colon \Gamma \to \widehat{\Gamma}_\varphi$ induces a surjective map $\widehat{\Phi}_\varphi \colon G \to \widehat{\Gamma}_\varphi$ of profinite groups, and an action $\widehat{\Phi}_\varphi \colon G \times X_\varphi \to X_\varphi$.

The embedding φ induces a mapping denoted by $\lambda_\varphi \colon X_\varphi \to X_\varphi$, which is defined as the shift map on sequences as follows. A point $\widehat{x} \in X_\varphi$ is defined by an equivalence class of sequences $\widehat{x} = (g_0, g_1, g_2, \ldots)$ with each $g_\ell \in \Gamma$ satisfying the relations $g_\ell = g_{\ell+1} \mod \Gamma_\ell$ for all $\ell \geq 0$. Then $\lambda_\varphi(\widehat{x}) = (e, \varphi(g_0), \varphi(g_1), \varphi(g_2), \ldots)$ is well-defined, and is a contraction on X_φ. Let $x_\varphi \in X_\varphi$ be the unique fixed point for λ_φ. Then $x_\varphi = (e, e, e, \ldots)$ where $e \in \Gamma$ is the identity.

Let $G_\varphi \equiv G_{x_\varphi} \subset G$ denote the isotropy subgroup for x_φ of the action $\widehat{\Phi}_\varphi$, and let $N(\widehat{\Phi}_\varphi) \subset G$ denote the kernel of the action map $\widehat{\Phi}_\varphi$. Then $N(\widehat{\Phi}_\varphi) \subset G_\varphi$, and we have:

LEMMA 3.1. *The* discriminant group $\mathcal{D}_\varphi$ *of* $(X_\varphi, \Gamma, \Phi_\varphi)$ *is* $\mathcal{D}_\varphi = \widehat{\Phi}_\varphi(G_\varphi) \subset \widehat{\Gamma}_\varphi$.

For $k \geq 0$, define

$$U_k = \{(g_0, g_1, g_2, \ldots) \in X_\varphi \mid g_i = e \text{ for } 0 \leq i \leq k\} = \varprojlim \, \{p_{\ell+1} \colon \Gamma_k/\Gamma_{\ell+1} \to \Gamma_k/\Gamma_\ell \mid \ell \geq k\} \ , \tag{9}$$

which is a clopen subset of X_φ adapted to the action Φ_φ, with stabilizer subgroup $\Gamma_{U_k} = \Gamma_k$.

Recall that the proper embedding $\varphi \colon \Gamma \to \Gamma$ induces a contraction mapping $\lambda_\varphi \colon X_\varphi \to X_\varphi$, and observe that $\lambda_\varphi \colon U_\ell \to U_{\ell+1}$ is a homeomorphism onto for all $\ell \geq 0$. The clopen sets $\{U_k \mid k \geq 0\}$ form an adapted neighborhood basis at $x_\varphi = \bigcap_{k>0} U_k$ which is the unique fixed point for λ_φ.

As the orbit of x_φ is dense in X_φ, for any non-empty open subset $U \subset X$ there exists $g \in \Gamma$ so that $\Phi(g)(x_\varphi) \in U$. It follows that there also exists $k > 0$ such that $\Phi(g)(U_k) \subset U$.

For each $\ell \geq 1$, let C_ℓ denote the largest normal subgroup (the *core*) of the stabilizer group Γ_ℓ, so

$$C_\ell \ = \ \bigcap_{g \in \Gamma} \ g \ \Gamma_\ell \ g^{-1} \ \subset \Gamma_\ell \ . \tag{10}$$

As Γ_ℓ has finite index in Γ, the same holds for C_ℓ. Observe that for all $\ell \geq 1$, we have $C_{\ell+1} \subset C_\ell$. Introduce the quotient group $Q_\ell = \Gamma/C_\ell$ with identity element $e_\ell \in Q_\ell$. There are natural quotient maps $q_{\ell+1} \colon Q_{\ell+1} \to Q_\ell$, and we can form the inverse limit Cantor group

$$\widehat{\Gamma}_\infty = \varprojlim \, \{q_{\ell+1} \colon Q_{\ell+1} \to Q_\ell \mid \ell \geq 0\} \ . \tag{11}$$

THEOREM 3.2. [20, Theorem 4.4] *There is a natural isomorphism* $\widehat{\tau} \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\infty$ *which identifies the discriminant group* $\mathcal{D}_\varphi$ *with the inverse limit group*

$$\mathcal{D}_\infty = \varprojlim \, \{q_{\ell+1} \colon \Gamma_{\ell+1}/C_{\ell+1} \to \Gamma_\ell/C_\ell \mid \ell \geq 0\} \subset \widehat{\Gamma}_\infty \ . \tag{12}$$

There is an interpretation of the group $\mathcal{D}_\infty$ as an *asymptotic defect* of the Γ-action on X_∞ which we elaborate on. Suppose that Γ_ℓ is a normal subgroup so that the quotient Γ/Γ_ℓ is a group. Then Γ/Γ_ℓ acts transitively on X_ℓ without fixed points. For example, if Γ is abelian then this is always true. In general, for the normal core $C_\ell \subset \Gamma_\ell$, the finite group $Q_\ell = \Gamma/C_\ell$ acts transitively on X_ℓ and the finite subgroup $D_\ell = \Gamma_\ell/C_\ell$ is the "defect" for the action of Q_ℓ on X_ℓ being a free action. Then $\mathcal{D}_\infty$ is the inverse limit of these finite defects, and provides a measure of the deviation of the action $\widehat{\Phi}_\infty$ of $\widehat{\Gamma}_\infty$ on X_∞ from being free.

Associated to the group chain $\mathcal{G}_\varphi$, there are two subgroups,

$$K(\mathcal{G}_\varphi) = \bigcap_{\ell > 0} \Gamma_\ell \quad , \quad C(\mathcal{G}_\varphi) = \bigcap_{g \in \Gamma} g \ K(\mathcal{G}_\varphi) \ g^{-1} \ . \tag{13}$$

where $C(\mathcal{G}_\varphi)$ is the largest normal subgroup of Γ contained in $K(\mathcal{G}_\varphi)$. Note that for any $g \in C(\mathcal{G}_\varphi)$, the action of $\Phi_\varphi(g)$ on X_φ is trivial.

4. Regularity of Cantor actions

A Cantor space $\mathfrak{X}$ is totally disconnected, so one cannot define the usual notions of regularity for an action based on the derivatives of the action maps. There is an alternative approach, based on the notion of *quasi-analytic* actions, which was introduced in the works [1, 2] by Álvarez López, Candel, and Moreira Galicia.

DEFINITION 4.1. *An action* $\Phi \colon H \times \mathfrak{X} \to \mathfrak{X}$*, where* H *is a topological group and* $\mathfrak{X}$ *is a Cantor space, is said to be* quasi-analytic (or QA) *if for each clopen set* $U \subset \mathfrak{X}$*, if the action of* $g \in H$ *satisfies* $\Phi(g)(U) = U$ *and the restriction* $\Phi(g)|U$ *is the identity map on* U*, then* $\Phi(g)$ *acts as the identity on all of* $\mathfrak{X}$.

If $\mathfrak{X}$ is a Cantor group for which the action Φ is defined by group multiplication, so that the action is induced by a group homomorphism $\Phi\colon H \to \mathfrak{X}$, then the action is quasi-analytic. A topologically free action, as in Definition 2.1, is quasi-analytic. Conversely, the Baire Category Theorem implies that an effective quasi-analytic action of a countable group is topologically free [50, Section 3].

A local formulation of the QA condition actions was introduced in the works [21, 35], and has proved very useful for the study of the dynamical properties of equicontinuous Cantor actions.

DEFINITION 4.2. *An action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$, where H is a topological group and $\mathfrak{X}$ a Cantor metric space with metric $d_{\mathfrak{X}}$, is* locally quasi-analytic *(or* LQA*), if there exists $\varepsilon > 0$ such that for any non-empty open set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \varepsilon$, and for any non-empty open subset $V \subset U$, if the action of $g \in H$ satisfies $\Phi(g)(V) = V$ and the restriction $\Phi(g)|V$ is the identity map on V, then $\Phi(g)$ acts as the identity on all of U.*

Examples of equicontinuous Cantor actions which are locally quasi-analytic, but not quasi-analytic, are given in [21, 35].

If $(\mathfrak{X}, H, \Phi)$ is an equicontinuous Cantor action which is not quasi-analytic, then the isotropy group defined in (5) is non-trivial. On the other hand, there are actions with non-trivial isotropy group that are quasi-analytic (see Section 8.2 below, and the examples in [21]). Finally, we define:

DEFINITION 4.3. *An equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is said to be* stable *if the associated profinite action $\widehat{\Phi}\colon \widehat{\Gamma} \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic. The action is said to be* wild *otherwise.*

Now let $\varphi\colon \Gamma \to \Gamma$ be a renormalization of the finitely generated group Γ. We assume the notation and results from Sections 1, 2, and 3. In particular, X_φ is the Cantor space defined in (8), and $(X_\varphi, \Gamma, \Phi_\varphi)$ is the associated equicontinuous Cantor action on this space. Let G denote the profinite completion of Γ. Its action on X_φ is denoted $\widehat{\Phi}_\varphi\colon G \times X_\varphi \to X_\varphi$. Then $\widehat{\Gamma}_\varphi = \widehat{\Phi}_\varphi(G) \subset \mathrm{Homeo}(X_\varphi)$, and $G_\varphi \equiv G_{x_\varphi}$ is the isotropy group at x_φ of the action $\widehat{\Phi}_\varphi$ of G on X_φ.

THEOREM 4.4. *Let Γ be a finitely generated group and $\varphi\colon \Gamma \to \Gamma$ a renormalization of Γ. Then the profinite action $\widehat{\Phi}_\varphi\colon G \times X_\varphi \to X_\varphi$ is quasi-analytic.*

Proof. Let $g \in G$ be such that $\widehat{\Phi}_\varphi(g)$ acts non-trivially on X_φ. Suppose there exists a non-empty open set $U \subset X_\varphi$ such that $\widehat{\Phi}_\varphi(g)$ acts on U as the identity.

The orbit of every point of X_φ is dense in X_φ under the action of Γ, so there exists $h \in \Gamma$ such that $\Phi_\varphi(h)(x_\varphi) \in U$. Set $g' = h^{-1}gh$ so that $\widehat{\Phi}_\varphi(g')$ fixes the open set $U' = \Phi_\varphi(h^{-1})(U)$. In particular, $\widehat{\Phi}_\varphi(g')$ fixes x_φ and hence $g' \in G_\varphi$. Thus, we can assume without loss of generality that $\widehat{\Phi}_\varphi(g)$ acts as the identity on U and $x_\varphi \in U$, so that $g \in G_\varphi$.

The nested clopen sets $\mathcal{U} = \{U_\ell \mid \ell \geq 0\}$ form a neighborhood basis at x_φ so there exists some $k_0 > 0$ such that $U_k \subset U$ for all $k \geq k_0$. Thus, for all $k \geq k_0$ we have $\lambda_\varphi^k(X_\varphi) \subset U$.

The embedding φ induces a continuous embedding $\widehat{\varphi}\colon G \to G$, whose image is a clopen subgroup denoted by $\widehat{V}_1$. More generally, we have the clopen subset $\widehat{V}_\ell = \widehat{\varphi}^\ell(G) \subset G$ for $\ell \geq 0$.

The isotropy group is $G_\varphi = \cap_{\ell \geq 0} \widehat{\varphi}^\ell(G)$, so for $g \in G_\varphi$ and all $\ell \geq 0$, the element $g_{-\ell} = \widehat{\varphi}^{-\ell}(g) \in G$ is defined. Thus $g_\ell = \widehat{\varphi}^\ell(g)$ is defined for all $\ell \in \mathbb{Z}$. Here is the first key observation:

LEMMA 4.5. *Let $g \in G$, and suppose g acts trivially on U_{k_0}, for some $k_0 \geq 0$. Then for all $\ell \geq k_0$, we have that the action of $\widehat{\varphi}^{-\ell}(g) \in G$ on X_φ is trivial. In particular, for $\ell \geq k_0$, $g_{-\ell} \in G_\varphi$.*

Proof. For $x \in X_\varphi$ and $\ell \geq 0$, set $x_\ell = \lambda_\varphi^\ell(x)$. Choose $g_x \in G$ such that $x = g_x \cdot G_\varphi$ via the identification $X_\varphi \cong G/G_\varphi$; that is, x is represented in X_φ by the coset $g_x G_\varphi$. Recall that under this identification, for $h \in G$ the action of $\widehat{\Phi}_\varphi(h)$ on X_φ becomes multiplication by h. That is,

$\widehat{\Phi}_\varphi(h)(x) = h \cdot g_x G_\varphi = hg_x \cdot G_\varphi$. As $\widehat{\varphi}\colon G \to G$ is a homomorphism, for $\ell \geq k_0$ we calculate

$$\begin{aligned} \widehat{\Phi}_\varphi(g_{-\ell})(x) &= \widehat{\Phi}_\varphi(\widehat{\varphi}^{-\ell}(g))(x) \\ &= \widehat{\varphi}^{-\ell}(g)g_x \cdot G_\varphi \\ &= \widehat{\varphi}^{-\ell}(g)(\widehat{\varphi}^{-\ell} \circ \widehat{\varphi}^{\ell}(g_x)) \cdot G_\varphi \\ &= \widehat{\varphi}^{-\ell}(g\widehat{\varphi}^{\ell}(g_x) \cdot G_\varphi) \end{aligned} \tag{14}$$

Recall that that $\lambda_\varphi^\ell(x) \in U_\ell$ and that $\widehat{\Phi}_\varphi(g)$ acts trivially on U_ℓ for $\ell \geq k_0$, so we have

$$\widehat{\varphi}^{-\ell}(g\widehat{\varphi}^{\ell}(g_x) \cdot G_\varphi) = \widehat{\varphi}^{-\ell}(\widehat{\varphi}^{\ell}(g_x) \cdot G_\varphi) = g_x G_\varphi = x \tag{15}$$

That is, $\widehat{\Phi}_\varphi(g_{-\ell})(x) = x$ for all $x \in X_\varphi$ and $\ell \geq k_0$, as was to be shown. □

The second key observation required for the proof of Theorem 4.4 is that for $g \in G_\varphi$ the equicontinuous action of $\widehat{\Phi}_\varphi(g)$ on X_φ is approximated by the action on the finite quotient spaces X_ℓ for $\ell > 0$. That is, the assumption that $\widehat{\Phi}_\varphi(g)$ acts non-trivially on X_φ implies there exists some $m_0 > 0$ such that the induced action of $\widehat{\Phi}_\varphi(g)$ on $X_{m_0} = \Gamma/G_{m_0}$ is non-trivial for some $m_0 > 0$. Without loss of generality, we may assume that $m_0 > k_0$. Denote this action by $\widehat{\Phi}_{m_0}(g) \in \mathrm{Perm}(X_{m_0})$, where $\mathrm{Perm}(X_{m_0})$ is the finite group of permutations of the finite set X_{m_0}.

LEMMA 4.6. *There exists an increasing sequence of integers*

$$\Lambda_{m_0} = \{0 < \ell_1 < \ell_2 < \ell_3 < \cdots\}$$

such that $\widehat{\Phi}_{m_0}(g_{-\ell_i}) = \widehat{\Phi}_{m_0}(g)$ *for all* $i \geq 0$.

Proof. Recall that the elements $g_{-\ell} = \widehat{\varphi}^{-\ell}(g)$ were introduced above, where each $g_{-\ell} \in G_\varphi$ by Lemma 4.5. Thus, we can define a mapping $\sigma\colon \mathbb{N} \to \mathrm{Perm}(X_{m_0})$ by $\sigma(\ell) = \widehat{\Phi}_{m_0}(g_{-\ell})$ for $\ell \geq 0$. Note that $\mathrm{Perm}(X_{m_0})$ is a finite set, so by the pigeonhole principle, there exists some $A \in \mathrm{Perm}(X_{m_0})$ such that the set $\Lambda_A = \{\lambda \geq 0 \mid \widehat{\Phi}_{m_0}(g_{-\lambda}) = A\}$ is infinite. Let $\lambda_0 \in \Lambda_A$ be the least value for the set, then set

$$\Lambda_{m_0} = \{\ell_i = \lambda_i - \lambda_0 \mid \lambda_i \in \Lambda_A\}.$$

Then note that $\widehat{\Phi}_{m_0}(g_{-\ell_i}) = \widehat{\Phi}_{m_0}(g_{-\ell_0}) = \widehat{\Phi}_{m_0}(g_0) = \widehat{\Phi}_{m_0}(g)$, as was to be shown. □

We can now complete the proof that the action $\widehat{\Phi}_\varphi\colon G \times X_\varphi \to X_\varphi$ is quasi-analytic. If not, then there exists $g \in G$ such that $\widehat{\Phi}_\varphi(g)$ acts non-trivially on X_φ, and a non-empty open set $U \subset X_\varphi$ such that $\widehat{\Phi}_\varphi(g)$ acts on U as the identity. Then by Lemma 4.5, there exists $x \in X_\varphi$ such that $\widehat{\Phi}_\varphi(g)(x) \neq x$, while $\widehat{\Phi}_{m_0}(g_{-\ell_i})(x) = x$ for $\ell_i > k_0$, which contradicts the conclusion of Lemma 4.6. Thus, the action of $\widehat{\Phi}_\varphi$ must be quasi-analytic. □

Finally, note that Theorem 4.4 shows that the profinite action $(X_\varphi, \widehat{\Gamma}_\varphi, \widehat{\Phi}_\varphi)$ is quasi-analytic, so the same holds for the action $(X_\varphi, \Gamma, \Phi_\varphi)$ restricted to the image of Γ in G. Then the Baire Category Theorem implies (see [36, Proposition 2.2] for example) that if a Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ of a countable group Γ is quasi-analytic and effective, then it is topologically free. It follows that if the action Φ_φ is effective, then the action $(X_\varphi, \Gamma, \Phi_\varphi)$ is topologically free, as asserted in Theorem 1.2.

5. Open embeddings

In this section, given a renormalization φ, we consider properties of the open embeddings of the profinite groups G and $\widehat{\Gamma}_\varphi$ associated to the Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$. This results in a structure theory for the profinite group $\widehat{\Gamma}_\varphi$ that is the key to the proof of Theorem 1.1.

The universal property of the profinite completion of a group implies that φ induces a homomorphism denoted by $\widehat{\varphi}\colon G \to G$ of the profinite completion G of Γ, whose image is a clopen subgroup of G. We then have the following result of Reid [49], as formulated in [55, Theorem 3.10].

THEOREM 5.1 (Reid [49]). *There exist closed subgroups $C_\varphi \subset G$ and $Q_\varphi \subset G$ so that:*

(1) $G \cong C_\varphi \rtimes Q_\varphi$, where C_φ is a pro-nilpotent normal subgroup of G;
(2) C_φ is $\widehat{\varphi}$-invariant, and $\widehat{\varphi}$ restricts to an open contracting embedding on C_φ;
(3) Q_φ is $\widehat{\varphi}$-invariant, and $\widehat{\varphi}$ restricts to an automorphism of Q_φ.

Moreover, let $\widehat{e} \in G$ be the identity element, then we have

$$C_\varphi = \{g \in G \mid \lim_{n\to\infty} \widehat{\varphi}^n(g) = \widehat{e}\} \quad , \quad Q_\varphi = \bigcap_{n>0} \widehat{\varphi}^n(G) \ . \tag{16}$$

Theorem 1.2 implies that for $\widehat{g} \in G$, the action of $\widehat{\varphi}(\widehat{g})$ on X_φ is locally determined. In particular, the action $\widehat{\varphi}(\widehat{g})$ is determined by its restriction to the clopen subset $\lambda_\varphi(X_\varphi)$. We use this fact to show that φ descends to a homomorphism of $\widehat{\Gamma}_\varphi$.

PROPOSITION 5.2. *Let φ be a renormalization of the finitely-generated group Γ. Then φ induces an injective homomorphism $\widehat{\varphi}_0 \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ whose image is a clopen subgroup of $\widehat{\Gamma}_\varphi$.*

Proof. Recall that $N(\widehat{\Phi}_\varphi) \subset G$ is the kernel of the homomorphism $\widehat{\Phi}_\varphi \colon G \to \widehat{\Gamma}_\varphi \subset \mathrm{Homeo}(X_\varphi)$. By the universal property of profinite completions, the embedding $\varphi\colon \Gamma \to \Gamma$ induces a homomorphism $\widehat{\varphi}\colon G \to G$. We claim that $\widehat{\varphi}$ descends to a homomorphism

$$\widehat{\varphi}_0 \colon \widehat{\Gamma}_\varphi \cong G/N(\widehat{\Phi}_\varphi) \longrightarrow \widehat{\Gamma}_\varphi \cong G/N(\widehat{\Phi}_\varphi) \ . \tag{17}$$

For $g \in N(\widehat{\Phi}_\varphi)$, observe that $\widehat{\varphi}(g)$ acts as the identity on the clopen subset $U_1 = \lambda_\varphi(X_\varphi)$. As the action $\widehat{\Phi}_\varphi$ is quasi-analytic, this implies that $\widehat{\varphi}(g)$ acts as the identity on X_φ, and thus $\widehat{\varphi}(g) \in N(\widehat{\Phi}_\varphi)$. This shows that $\widehat{\varphi}(N(\widehat{\Phi}_\varphi)) \subset N(\widehat{\Phi}_\varphi) \subset G$, and thus we have the composition of homomorphisms

$$\widehat{\varphi}_0 \colon \widehat{\Gamma}_\varphi = G/N(\widehat{\Phi}_\varphi) \to \widehat{\varphi}(G)/\widehat{\varphi}(N(\widehat{\Phi}_\varphi)) \to G/\widehat{\varphi}(N(\widehat{\Phi}_\varphi)) \to G/N(\widehat{\Phi}_\varphi) = \widehat{\Gamma}_\varphi \tag{18}$$

which defines the map (17). We claim that $\widehat{\varphi}_0$ is injective. If not, let $\gamma \in \widehat{\Gamma}_\varphi$ such that $\widehat{\varphi}_0(\gamma) = \mathrm{id}$. That is, $\widehat{\varphi}_0(\gamma) \in \widehat{\Gamma}_\varphi$ acts as the identity on X_φ. In particular, $\widehat{\varphi}_0(\gamma)$ acts as the identity on $\lambda_\varphi(X_\varphi)$, so for $x \in X_\varphi$,

$$\lambda_\varphi(x) = \widehat{\varphi}_0(\gamma) \cdot \lambda_\varphi(x) = \lambda_\varphi(\gamma \cdot x) \ .$$

As λ_φ is an injection, we have $\gamma \cdot x = x$ for all $x \in X_\varphi$, as was to be shown. □

We use the conclusions of Theorem 5.1 and Proposition 5.2 to obtain:

THEOREM 5.3. *Let $\varphi\colon \Gamma \to \Gamma$ be a renormalization for the finitely generated group Γ, with associated Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$. Let $\widehat{\Phi}_\varphi : G \times X_\varphi \to X_\varphi$ be the action of the profinite completion G of Γ, and set $\widehat{\Gamma}_\varphi = \widehat{\Phi}_\varphi(G) \subset \mathrm{Homeo}(X_\varphi)$. Let $\widehat{\varphi}_0 \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ be the embedding induced from φ. Then there exists a closed* pro-nilpotent normal *subgroup $\widehat{N}_\varphi \subset \widehat{\Gamma}_\varphi$ so that:*

(1) $\widehat{\Gamma}_\varphi \cong \widehat{N}_\varphi \rtimes \mathcal{D}_\varphi$ is a semi-direct product;
(2) $\widehat{N}_\varphi$ is $\widehat{\varphi}_0$-invariant, and $\widehat{\varphi}_0$ restricts to an open contracting embedding on $\widehat{N}_\varphi$;
(3) $\mathcal{D}_\varphi$ is $\widehat{\varphi}_0$-invariant, and $\widehat{\varphi}_0$ restricts to an automorphism of $\mathcal{D}_\varphi$.

Moreover, let $\widehat{e} \in \widehat{\Gamma}_\varphi$ be the identity element, then we have

$$\widehat{N}_\varphi = \{g \in \widehat{\Gamma}_\varphi \mid \lim_{n\to\infty} \widehat{\varphi}_0^n(g) = \widehat{e}\} \quad , \quad \mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}_0^n(\widehat{\Gamma}_\varphi) \ . \tag{19}$$

Proof. First, we show:

LEMMA 5.4. $\mathcal{D}_\varphi = \widehat{\Phi}_\varphi(Q_\varphi) \subset \widehat{\Gamma}_\varphi$.

Proof. Recall that the clopen neighborhoods U_ℓ of x_φ are defined by (9), and for each $\ell \geq 0$ we have $U_\ell = \lambda_\varphi^\ell(X_\varphi)$. For each $\ell \geq 0$, define the clopen subset $\widehat{U}_\ell = \{\gamma \in \widehat{\Gamma}_\varphi \mid \widehat{\Phi}_\varphi(\gamma)(U_\ell) = U_\ell\} \subset \widehat{\Gamma}_\varphi$.

Also, recall that $\mathcal{D}_\varphi = \{\gamma \in \widehat{\Gamma}_\varphi \mid \gamma \cdot x_\varphi = x_\varphi\}$. As $x_\varphi = \bigcap_{\ell \geq 0} U_\ell$, we then have $\mathcal{D}_\varphi = \bigcap_{\ell \geq 0} \widehat{U}_\ell$, and so $\widehat{U}_\ell = \widehat{\varphi}_0^\ell(\widehat{\Gamma}_\varphi)$ where $\widehat{\varphi}_0 \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ was defined in Proposition 5.2.

Recall that $\widehat{V}_\ell = \widehat{\varphi}^\ell(G) \subset G$, and thus $\widehat{U}_\ell = \widehat{\Phi}_\varphi(\widehat{V}_\ell)$. Then we have

$$\widehat{\Phi}_\varphi(Q_\varphi) = \widehat{\Phi}_\varphi \left\{ \bigcap_{\ell \geq 0} \widehat{\varphi}^\ell(G) \right\} = \bigcap_{\ell \geq 0} \widehat{\Phi}_\varphi(\widehat{\varphi}^\ell(G)) = \bigcap_{\ell \geq 0} \widehat{\Phi}_\varphi(\widehat{V}_\ell) = \bigcap_{\ell \geq 0} \widehat{U}_\ell = \mathcal{D}_\varphi \ , \tag{20}$$

as was to be shown. □

Next, set $\widehat{N}_\varphi = \widehat{\Phi}_\varphi(C_\varphi) \subset \widehat{\Gamma}_\varphi$ which is a pro-nilpotent closed subgroup. Then by an argument exactly analogous to the proof of Lemma 5.4, we have

$$\widehat{N}_\varphi = \{\gamma \in \widehat{\Gamma}_\varphi \mid \lim_{\ell \to \infty} \widehat{\varphi}_0^\ell(\gamma) = \widehat{e} \in \widehat{\Gamma}_\varphi\} \ . \tag{21}$$

This completes the proof of Theorem 5.3. □

Note that the identities (19) in Theorem 5.3 identify the images of the groups C_φ and Q_φ in $\mathrm{Homeo}(X_\varphi)$ in terms of the dynamical properties of the action $\widehat{\varphi}_0$ on $\widehat{\Gamma}_\varphi$.

The conclusions of Theorem 5.3 are illustrated in various examples of renormalizable groups and self-embeddings in Section 8, and also by the examples in the works [46, 54, 55]. Moreover, the conclusion that φ induces an automorphism of the discriminant group $\mathcal{D}_\varphi$ has applications to the constructions of examples of Cantor actions using the *Lenstra method* as given in [35, Section 8.2].

6. Finite discriminant

We next consider the consequences of Theorem 5.3, for $\mathcal{D}_\varphi$ a finite group. We first assume that the discriminant group $\mathcal{D}_\varphi$ is trivial, and show that the quotient group $\Gamma/C(\mathcal{G}_\varphi)$ is nilpotent, where $C(\mathcal{G}_\varphi)$ is the normal core of the intersection $K(\mathcal{G}_\varphi) \subset \Gamma$ associated to $\mathcal{G}_\varphi$, as defined in (13). Recall that $C(\mathcal{G}_\varphi) \subset \Gamma$ is identified with the kernel of the homomorphism $\Phi_\varphi \colon \Gamma \to \widehat{\Gamma}_\varphi \subset \mathrm{Homeo}(X_\varphi)$, and that $\widehat{\varphi}_0 \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ was defined in Proposition 5.2.

Note that φ restricts to an isomorphism of $K(\mathcal{G}_\varphi)$ by its definition, and so φ also maps $C(\mathcal{G}_\varphi)$ isomorphically to itself, and thus induces an embedding $\varphi' \colon \Gamma/C(\mathcal{G}_\varphi) \to \Gamma/C(\mathcal{G}_\varphi)$. Then without loss of generality, we can replace Γ with $\Gamma/C(\mathcal{G}_\varphi)$, so can assume that $\Phi_\varphi \colon \Gamma \to \widehat{\Gamma}_\varphi$ is an embedding, and identify Γ with its image $\Phi_\varphi(\Gamma)$. As we assume that $\mathcal{D}_\varphi$ is trivial, by Theorem 5.3 we have $\widehat{\Gamma}_\varphi \cong \widehat{N}_\varphi$ where $\widehat{N}_\varphi$ is a closed pro-nilpotent normal group.

Section 3 of the work [55] gives an overview of some of the structure theory of pro-nilpotent groups, and we recall those aspects as required for the proof of Theorem 1.1. First, $\widehat{N}_\varphi$ admits a splitting by [27, Theorem B] as $\widehat{N}_\varphi \cong \widehat{N}_\infty \times \widehat{N}_{tor}$ where $\widehat{N}_\infty$ is a torsion-free nilpotent group and $\widehat{N}_{tor}$ is a torsion group with bounded exponent, by results of [38]. We now claim:

LEMMA 6.1. *If $\mathcal{D}_\varphi$ is trivial, then $\widehat{N}_{tor}$ is the trivial group.*

Proof. Let $\pi_{tor} \colon \widehat{N}_\varphi \to \widehat{N}_{tor}$ be the projection, then the image $\pi_{tor}(\Gamma) \subset \widehat{N}_{tor}$ is dense.

The abelianization $\widehat{A}_{tor}$ of $\widehat{N}_{tor}$ is an abelian group of bounded exponent, which is trivial if and only if $\widehat{N}_{tor}$ is trivial. By Prüfer's First Theorem (see § 24 of [39]), $\widehat{A}_{tor}$ is a possibly infinite direct sum of cyclic groups. As Γ is finitely generated, the image of Γ in $\widehat{A}_{tor}$ is finite rank and dense, and therefore the abelianization $\widehat{A}_{tor}$ has finite rank. Thus, $\widehat{A}_{tor}$ is a direct sum of finitely many cyclic groups, hence is a finite group.

Note that the contraction mapping $\widehat{\varphi}\colon \widehat{N}_\varphi \to \widehat{N}_\varphi$ induces a contraction mapping $\widehat{\varphi}_{tor}\colon \widehat{N}_{tor} \to \widehat{N}_{tor}$.

The second part of Theorem B in Glöckner and Willis [27] proves the existence of a Jordan-Hölder series for bounded exponent contraction groups with each composition factor a *simple* contraction group. Here we say a contraction group with contraction α is *simple* if it has no nontrivial, proper closed normal α-invariant subgroup. Further, the simple contraction groups are classified as shifts on $F^{\mathbb{N}}$ where F is a finite simple group. By considering the first composition factor, we see that $\widehat{N}_{tor}$ has a quotient of the form $F^{\mathbb{N}}$ where F is a finite simple group. Since $\widehat{N}_{tor}$ is solvable of bounded exponent [49], we conclude that F is abelian. In particular $\widehat{N}_{tor}$ has an infinite abelian quotient, which contradicts the fact that $\widehat{A}_{tor}$ is a finite group, as shown previously. □

Now observe that by Lemma 6.1, the group $\Gamma \subset \widehat{N}_\infty$ and $\widehat{N}_\infty$ is a torsion-free nilpotent group, thus Γ is nilpotent. This concludes the proof of Theorem 1.1 in the case where $\mathcal{D}_\varphi$ is trivial.

Next, assume that $\mathcal{D}_\varphi$ is a finite group. By Theorem 5.3, we have $\mathcal{D}_\varphi = \widehat{\Phi}_\varphi(Q_\varphi) \subset \widehat{\Gamma}_\varphi$ and its intersection with $\widehat{N}_\varphi = \widehat{\Phi}_\varphi(C_\varphi)$ is the trivial subgroup. It follows that $\widehat{N}_\varphi$ is a clopen subset of $\widehat{\Gamma}_\varphi$, and so $\Lambda_\varphi \equiv \Gamma \cap \widehat{N}_\varphi$ is a dense subgroup of $\widehat{N}_\varphi$

The restriction of $\widehat{\varphi}_0$ defines a contraction mapping $\widehat{\varphi}_0\colon \widehat{N}_\varphi \to \widehat{N}_\varphi$. We can thus apply the above arguments for the trivial discriminant case to the action of Λ_φ on $\widehat{N}_\varphi$ to conclude that the group Λ_φ is nilpotent with finite index in Γ. This completes the proof of Theorem 1.1.

Finally, we give the proof of Corollary 1.5. Assume that both $\mathcal{D}_\varphi$ and the subgroup $K(\mathcal{G}_\varphi)$ in (13) are finite groups. Thus its core $C(\mathcal{G}_\varphi) \subset K(\mathcal{G}_\varphi)$ is also finite. Recall that in the above proof of Theorem 1.1, we replaced Γ with the quotient $\Gamma/C(\mathcal{G}_\varphi)$, and concluded that $\Gamma/C(\mathcal{G}_\varphi)$ contains a nilpotent subgroup of finite index. In the case where both groups $\mathcal{D}_\varphi$ and $K(\mathcal{G}_\varphi)$ are trivial, then the claim of the corollary follows directly from Theorem 1.1 and the identification in 19 of $\mathcal{D}_\varphi$ with the second intersection in (13). In the case where both groups are finite, we have that $C(\mathcal{G}_\varphi)$ is a finite normal subgroup of Γ and $\Gamma/C(\mathcal{G}_\varphi)$ contains a nilpotent subgroup of finite index, which implies that Γ contains a nilpotent subgroup of finite index. This completes the proof of Corollary 1.5.

7. Renormalizable Cantor actions

In this section, we introduce the notions of *(virtually) renormalizable* Cantor actions, and study their regularity properties and invariants, yielding proofs of Theorems 1.6, 1.8 and 1.9.

For a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ and an adapted set $U \subset \mathfrak{X}$, note that $H_U = \Phi(\Gamma_U) \subset \mathrm{Homeo}(U)$ acts faithfully on U, so (U, H_U, Φ_U) is always an effective action.

DEFINITION 7.1. *A Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is* renormalizable *if it is equicontinuous, and there exists an adapted clopen set $U \subset \mathfrak{X}$ such that the actions $(\mathfrak{X}, \Gamma, \Phi)$ and (U, H_U, Φ_U) are isomorphic (as in Definition 2.6) by a homeomorphism $\lambda\colon \mathfrak{X} \to U$ and group isomorphism $\Theta\colon \Gamma \to H_U$, and the intersection $\cap_{\ell \geq 0} \lambda^\ell(\mathfrak{X})$ is a point.*

For example, let $(X_\varphi, \Gamma, \Phi_\varphi)$ be the Cantor action associated to a renormalization φ of Γ. Suppose the action is topologically free, then it is renormalizable, where $\lambda = \lambda_\varphi$ and $\Theta = \Phi_U \circ \varphi\colon \Gamma \to H_U$ is an isomorphism. In general, for a renormalizable action, there is no requirement that the map $\Phi_U : G_U \to H_U$ is injective, and so H_U need not be identified with a subgroup of Γ.

DEFINITION 7.2. *A Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is* virtually renormalizable *if is equicontinuous, and there exists an adapted set $V \subset \mathfrak{X}$ such that the restricted action (V, H_V, Φ_V) is renormalizable.*

The class of virtually renormalizable actions is much more general than the renormalizable actions, as it allows for the case when the action map $\Phi\colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ has a non-trivial kernel. In the following, we show some properties of these actions. We first show:

PROPOSITION 7.3. *Suppose that the Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable and locally quasi-analytic, then it is quasi-analytic.*

Proof. We assume there is given a homeomorphism $\lambda\colon \mathfrak{X} \to U$ and group isomorphism $\Theta\colon \Gamma \to H_U$ implementing an isomorphism of $(\mathfrak{X}, \Gamma, \Phi)$ with (U, H_U, Φ_U) as in (6).

Suppose that the map $\Phi_U\colon \Gamma_U \to H_U$ is injective, and hence is an isomorphism as it is onto by the definition of H_U. Then the composition $\varphi \equiv \Phi_U^{-1} \circ \Theta\colon \Gamma \to \Gamma$ is a proper inclusion with image $\Gamma_U \subset \Gamma$. As U is adapted, Γ_U has finite index in Γ, and thus φ is a renormalization of Γ.

It thus suffices to show that if $\Phi_U\colon \Gamma_U \to H_U$ has a non-trivial kernel $K_U \subset \Gamma$, then the action Φ is not locally quasi-analytic, which yields a contradiction. We show this using a recursive argument.

Set $U_0 = \mathfrak{X}$, then $U = \lambda(\mathfrak{X})$ is a clopen set by assumption. Then recursively define clopen sets $U_\ell = \lambda^\ell(U_0) \subset U_{\ell-1}$ for $\ell > 0$. The assumption in Definition 7.1 that the intersection $\cap_{\ell \geq 0} \lambda^\ell(\mathfrak{X})$ is a point, labeled $x_\lambda \in \mathfrak{X}$, implies that $\{U_\ell \mid \ell \geq 0\}$ is an adapted neighborhood basis at x_λ.

Now set $\Gamma_\ell = \Gamma_{U_\ell}$ for $\ell \geq 0$, and let $H_\ell = \Phi_{U_\ell}(\Gamma_\ell) \subset \mathrm{Homeo}(U_\ell)$ for $\ell \geq 0$. The associated group chain $\mathcal{G}_\varphi = \{\Gamma_\ell = \varphi^\ell(\Gamma) \mid \ell \geq 0\}$. Then $U_1 = U$, and $H_1 = H_U$. Recall that as the action of H_1 on U_1 is effective, and the actions $(\mathfrak{X}, \Gamma, \Phi)$ and (U, H_U, Φ_U) are isomorphic, so the action of Γ on $\mathfrak{X}$ is effective. That is, the kernel $K_0 \subset \Gamma$ of Φ is trivial, and $\Phi\colon \Gamma \to H_0$ is an isomorphism onto. To avoid cumbersome notation, we will identify $\Gamma = H_0$ and write the action as $g \cdot x = \Phi(g)(x)$.

Now observe that

$$\Gamma_{\ell+1} = \{g \in \Gamma \mid g \cdot U_{\ell+1} = U_{\ell+1}\} = \{g \in \Gamma_\ell \mid g \cdot U_{\ell+1} = U_{\ell+1}\} = (\Gamma_{U_\ell})_{U_{\ell+1}}$$

since $g \cdot U_{\ell+1} = U_{\ell+1}$ implies $g \cdot U_\ell = U_\ell$, as U_ℓ is an adapted clopen set and $U_{\ell+1} \subset U_\ell$.

We give the first step of the recursive argument. Define

$$K_1 \equiv \ker \{\Phi_{U_1}\colon \Gamma_1 \to H_1 \subset \mathrm{Homeo}(U_1)\} \subset \Gamma_1 \subset \Gamma . \tag{22}$$

By assumption, the subgroup K_1 is non-trivial.

Let $\Phi^1_{U_1}\colon H_1 \times U_1 \to U_1$ denote the action of H_1, and let $(H_1)_{U_2} \subset H_1$ denote the elements of H_1 which map U_2 to itself. Then introduce the subgroup $K_2' \subset (H_1)_{U_2}$ of elements which restrict to the identity on U_2. Then we have:

$$\begin{aligned} K_2' = \ker\left\{\Phi^1_{U_2}\colon (H_1)_{U_2} \to \mathrm{Homeo}(U_2)\right\} &= \ker\left\{\Phi^1_{U_2}\colon (H_1)_{\lambda(U_1)} \to \mathrm{Homeo}(\lambda(U_1))\right\} \\ &= \ker\left\{\Phi^1_{U_2}\colon \Theta(\Gamma)_{\lambda(U_1)} \to \mathrm{Homeo}(\lambda(U_1))\right\} \\ &= \Theta\left(\ker\{\Phi_{U_1}\colon \Gamma_{U_1} \to \mathrm{Homeo}(U_1)\}\right) = \Theta(K_1) , \end{aligned} \tag{23}$$

where the last equality follows using the isomorphism of $(\mathfrak{X}, \Gamma, \Phi)$ with (U_1, H_1, Φ^1).

By assumption K_1 is a non-trivial subgroup, so by (23) we have $K_2' = \Theta(K_1)$ is also non-trivial. That is, if $g \in K_1 \subset \Gamma_1$ is not the identity, then g acts non-trivially on $U_0 = \mathfrak{X}$ and restricts to the identity on U_1 by the definition (22) of K_1. Thus, $h = \Theta(g) \in H_1$ acts non-trivially on U_1 and restricts to the identity on U_2. Since $H_1 = \Phi_{U_1}(\Gamma_1)$, there exists $g' \in \Gamma_1$ such that $\Phi_{U_1}(g') = h$. We have found $g' \in \Gamma_1$, such that $g' \notin K_1$ and $g' \in K_2$. Therefore, K_1 is a non-trivial proper subgroup of K_2.

Set $K_\ell = \ker\{\Phi_{U_\ell}\colon \Gamma_\ell \to \mathrm{Homeo}(U_\ell)\}$ for $\ell \geq 2$, then by repeating the above arguments in (23), we have $K_\ell \subset K_{\ell+1} \subset \Gamma$ is a proper inclusion for all $\ell \geq 1$. As the diameter of the sets U_ℓ tends to 0 as ℓ increases, given any adapted set $V \subset \mathfrak{X}$ for the action Φ, there exist $\ell > 0$ and $\gamma \in \Gamma$ such that $O = \gamma \cdot U_\ell \subset V$. This implies that the dynamics of Γ_ℓ acting on U_ℓ is conjugate to the restricted action of Γ_V on the adapted clopen set O. Thus, there exists some element $\gamma' \in \Gamma$ such that $\gamma' \cdot O = O$ and the action of $\Phi(\gamma')$ restricted to O is non-trivial, but restricts to the identity on some open set that is a translate of $U_{\ell+1}$ in O. Thus, the action Φ is not locally quasi-analytic. □

We have the following consequence of the above proof of Proposition 7.3, which yields a proof of the second conclusion of Theorem 1.6.

PROPOSITION 7.4. *Suppose that the Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable and locally quasi-analytic, then the action is isomorphic to an action $(X_\varphi, \Gamma, \Phi_\varphi)$ associated to a renormalization $\varphi\colon \Gamma \to \Gamma$, and in particular Γ is renormalizable.*

Proof. From the proof of Proposition 7.3, $\mathcal{G}_\lambda = \{G_\ell \mid \ell \geq 0\}$ is the group chain associated to the adapted neighborhood basis $\mathcal{U}_\lambda \equiv \{U_\ell \mid \ell \geq 0\}$ at x_λ for the action Φ. Let $(X_\lambda, \Gamma, \Phi_\lambda)$ be the Cantor action associated to this group chain, as in Section 3 (see also [13, 20, 21].) It follows from the results of these papers that the Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to the action $(X_\varphi, \Gamma, \Phi_\varphi)$.

The action $(\mathfrak{X}, \Gamma, \Phi)$ is quasi-analytic by Proposition 7.3, so we have isomorphisms $\Gamma_\ell \cong H_\ell$, and in particular the composition $\varphi \equiv \Phi_{U_1}^{-1} \circ \Theta \colon \Gamma \to \Gamma$ is a proper inclusion with image $\Gamma_1 \subset \Gamma$ a subgroup of finite index. Thus, φ is a renormalization of Γ whose associated Cantor action is isomorphic to $(\mathfrak{X}, \Gamma, \Phi)$, as was needed to show. □

We next consider the invariance under continuous orbit equivalence for the renormalization property of a Cantor action. We first give the proof of Theorem 1.8, which relies on results in the works [36, 37].

7.1. **Proof of Theorem 1.8.** Assume that $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable Cantor action which is locally quasi-analytic, hence is quasi-analytic by Proposition 7.3. Let $(\mathfrak{X}', \Gamma', \Phi')$ be a Cantor action which is continuously orbit equivalent to $(\mathfrak{X}, \Gamma, \Phi)$. By Proposition 7.4, there exists a proper self-embedding $\varphi \colon \Gamma \to \Gamma$ such that the action $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to the action $(X_\varphi, \Gamma, \Phi_\varphi)$. We may thus assume that the Cantor actions $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are continuously orbit equivalent, where $(X_\varphi, \Gamma, \Phi_\varphi)$ is quasi-analytic. Then Theorem 6.9 of [37] implies that $(\mathfrak{X}', \Gamma', \Phi')$ is locally quasi-analytic.

The hypotheses of Theorem 1.5 in [36] are then satisfied, so that $(X_\varphi, \Gamma, \Phi_\varphi)$ is return equivalent to $(\mathfrak{X}', \Gamma', \Phi')$. Thus, there exists adapted sets $V \subset X_\varphi$ for the action $(X_\varphi, \Gamma, \Phi_\varphi)$ and $V' \subset \mathfrak{X}'$ for the action $(\mathfrak{X}', \Gamma', \Phi')$ so that the restricted actions (V, H_V, Φ_V) and $(V', H'_{V'}, \Phi'_{V'})$ are isomorphic, where $H_V = \Phi_V(\Gamma_V) \subset \mathrm{Homeo}(V)$ and $H'_{V'} = \Phi'_{V'}(\Gamma'_{V'}) \subset \mathrm{Homeo}(V')$.

Let $x_\varphi \in X_\varphi$ denote the fixed-point for the contraction $\lambda_\varphi \colon X_\varphi \to X_\varphi$. The action $(X_\varphi, \Gamma, \Phi_\varphi)$ is minimal, so by conjugating by an element of Γ, we can assume that $x_\varphi \in V$.

Let $h \colon V \to V'$ be a homeomorphism, and $\Theta \colon H_V \to H'_{V'}$ a group isomorphism, which realizes the isomorphism between (V, H_V, Φ_V) and $(V', H'_{V'}, \Phi'_{V'})$ as in Definition 2.6.

For the action $(X_\varphi, \Gamma, \Phi_\varphi)$, we have an adapted neighborhood basis $\{U_\ell = \lambda^\ell(X_\varphi) \mid \ell \geq 0\}$ and a group chain $\mathcal{G}_\varphi = \{\Gamma_\ell = \varphi^\ell(\Gamma) \mid \ell \geq 0\}$ as before.

Choose $\ell_0 > 0$ sufficiently large so that $U_{\ell_0} \subset V$ and $h(U_{\ell_0}) \subset V'$. Then set $W = U_{\ell_0}$. Note that $\lambda_\varphi(U_\ell) = U_{\ell+1}$ for all $\ell \geq 0$, so $W_1 = \lambda_\varphi(W) \subset W$. Set $W' = h(W) \subset V'$ and $W'_1 = h(W_1) \subset W'$. Then the restriction of φ to $\Gamma_W = \Gamma_{\ell_0}$ yields a proper self-embedding $\varphi_W \colon \Gamma_W \to \Gamma_W$.

Since the action $(X_\varphi, \Gamma, \Phi_\varphi)$ is quasi-analytic, the map $\Phi_W \colon \Gamma_W \to H_W$ is an isomorphism. Thus, φ_W induces a proper self-embedding $\widehat{\varphi}_W \colon H_W \to H_W$. Then set $H_\ell = \widehat{\varphi}_W^\ell(H_V)$ for all $\ell \geq 0$. It then follows from the constructions that the Cantor action (W, H_W, Φ_W) is isomorphic with the Cantor action associated to $\widehat{\varphi}_W \colon H_W \to H_W$.

Finally, the isomorphism between (V, H_V, Φ_V) and $(V', H'_{V'}, \Phi'_{V'})$ restricts to an isomorphism between (W, H_W, Φ_W) and $(W', H'_{W'}, \Phi'_{W'})$ which then defines a self-embedding of $H'_{W'}$. Thus, the Cantor action $(\mathfrak{X}', \Gamma', \Phi')$ is virtually renormalizable.

This completes the proof of Theorem 1.8.

7.2. **Proof of Theorem 1.9.** Let $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ be Cantor actions associated to renormalizations $\varphi \colon \Gamma \to \Gamma$ and $\varphi' \colon \Gamma' \to \Gamma'$, respectively. Assume that $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ are continuously orbit equivalent. We must show that the discriminant groups $\mathcal{D}_\varphi$ and $\mathcal{D}'_{\varphi'}$ for these actions are isomorphic.

Theorem 4.4 implies the profinite actions $\widehat{\Phi}_\varphi \colon G \times X_\varphi \to X_\varphi$ and $\widehat{\Phi}'_{\varphi'} \colon G' \times X'_{\varphi'} \to X'_{\varphi'}$ are quasi-analytic, and so also the actions $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ are quasi-analytic. Theorem 1.5 in [36] then implies that the actions $(\mathfrak{X}, \Gamma, \Phi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ are return equivalent.

Thus, there exist adapted sets $V \subset \mathfrak{X}$ for the action $(\mathfrak{X}, \Gamma, \Phi)$ and $V' \subset \mathfrak{X}'$ for the action $(\mathfrak{X}', \Gamma', \Phi')$ so that the restricted actions (V, H_V, Φ_V) and $(V', H'_{V'}, \Phi'_{V'})$ are isomorphic, where recall that $H_V = \Phi_V(\Gamma_V) \subset \mathrm{Homeo}(V)$ and $H'_{V'} = \Phi'_{V'}(\Gamma'_{V'}) \subset \mathrm{Homeo}(V')$. As the actions are quasi-analytic, the maps Φ_V and $\Phi'_{V'}$ are monomorphisms, hence are isomorphisms. Thus, the actions (V, Γ_V, Φ_V) and $(V', \Gamma'_{V'}, \Phi'_{V'})$ are isomorphic, induced by a homeomorphism $h\colon V \to V'$.

Let $\mathcal{D}_V$ denote the discriminant group for the restricted action (V, Γ_V, Φ_V). Then by the arguments in [35, Section 4], there is a surjective map $\rho_{\mathfrak{X},V}\colon \mathcal{D}_\varphi \to \mathcal{D}_V$ which is an isomorphism when the profinite action $\widehat{\Phi}_\varphi\colon G \times X_\varphi \to X_\varphi$ is quasi-analytic. Likewise, for the discriminant $\mathcal{D}'_{V'}$ of the action $(V', \Gamma'_{V'}, \Phi'_{V'})$, there is an isomorphism $\rho_{\mathfrak{X}',V'}\colon \mathcal{D}_{\varphi'} \to \mathcal{D}'_{V'}$.

The isomorphism class of the discriminant group is an invariant for isomorphism of Cantor actions, so we conclude $\mathcal{D}_\varphi \cong \mathcal{D}_V \cong \mathcal{D}'_{V'} \cong \mathcal{D}'_{\varphi'}$ as claimed.

This completes the proof of Theorem 1.9.

8. Applications and Examples

The classification of renormalizations has applications in a variety of contexts.

For a compact manifold M without boundary, an expansive diffeomorphism $\phi\colon M \to M$ gives rise to a renormalization $\varphi\colon \Gamma \to \Gamma$ of the fundamental group $\Gamma = \pi_1(M, x)$. In this case, Shub showed in [53] that the universal covering of M has polynomial growth type, and hence by Gromov [33] the group Γ has a finite-index nilpotent subgroup. There are a variety of constructions of expansive diffeomorphisms on nilmanifolds, and the invariants associated to the renormalization φ of Γ are then invariants of the expansive map ϕ.

The construction of generalized Hirsch foliations in [8, 34] is based on choosing a renormalization $\varphi\colon \Gamma \to \Gamma$ of the fundamental group of a compact manifold M. Thus, invariants of the renormalization yield invariants for this genre of foliated manifolds.

The classification of M-like laminations, where M is a fixed compact manifold, is reduced to the classification of renormalizations in the work [11].

These applications are all based on the constructions of renormalizations for groups with the non-co-Hopfian property. Many finitely generated nilpotent groups are renormalizable, as shown for example in [7, 12, 15, 16, 17, 40]. There is also a variety of examples of renormalizable groups which are not nilpotent, as described for example in [18, 24, 28, 29, 30, 46, 48, 55]. While these works show the existence of a proper self-embedding for a particular class of groups, they do not calculate the groups $\mathcal{D}_\varphi$ and $\widehat{N}_\varphi$ which are associated to an embedding φ by Theorem 5.3. In the following, we make these calculations for a selected set of examples of renormalizable groups.

Example 8.2 calculates the discriminant Cantor group $\mathcal{D}_\varphi$ and the induced map $\widehat{\varphi}\colon \mathcal{D}_\varphi \to \mathcal{D}_\varphi$ for an "untwisted" embedding $\varphi\colon \mathcal{H} \to \mathcal{H}$ of the Heisenberg group $\mathcal{H}$. The work [42] has a discussion of twisted and untwisted subgroups of the Heisenberg group.

Section 8.3 gives an example of a renormalizable group that arises in the study of arboreal representations of absolute Galois groups of number fields.

8.1. **Multihedral groups.** This is the simplest example of a group Γ with self-embedding and non-trivial finite discriminant group $\mathcal{D}_\varphi \subset \Gamma$.

Let $\Lambda = \mathbb{Z}^k$ be the free abelian group on k generators. Let $H \subset \mathrm{Perm}(k)$ be a non-trivial subgroup of the finite symmetric group $\mathrm{Perm}(k)$ on k symbols, and let $\mathrm{Perm}(k) \subset \mathrm{GL}(k, \mathbb{Z})$ be the standard embedding permuting the coordinates.

Let $\Gamma = \mathbb{Z}^k \rtimes H$ be the semi-direct product of these groups. For $m > 1$, define $\varphi\colon \Gamma \to \Gamma$ to be multiplication by m on the $\mathbb{Z}^k$ factor. That is, for $(\vec{v}, g) \in \Gamma$ set $\varphi(\vec{v}, g) = (m \cdot \vec{v}, g)$. Then

$$\Gamma_\ell = \{(m^\ell \cdot \vec{v}, g) \mid \vec{v} \in \mathbb{Z}^k\ ,\ g \in H\} = m^\ell \mathbb{Z}^k \rtimes H \tag{24}$$

$$K(\mathcal{G}_\varphi) = \{(0, 0, g) \mid g \in H\} \cong H\ . \tag{25}$$

where $\mathcal{G}_\varphi = \{\Gamma_\ell \mid \ell \geq 0\}$. Then we have $X_\varphi \cong \widehat{\mathbb{Z}}^k_m$. The subgroup H acts on X_φ by permutations of the coordinates, so the adjoint action on X_φ of a non-identity element $g \in H$ is a non-trivial permutation of the coordinate axes, hence is non-trivial. Thus, the normal core $C(\mathcal{G}_\varphi) \subset K(\mathcal{G}_\varphi)$ is trivial, and we have $K(\mathcal{G}_\varphi) \subset \mathcal{D}_\varphi$. Thus, a calculation shows that the normal core $C_\ell \subset \Gamma_\ell$ is the subgroup of (24) where $g = e \in H$ is the identity, so $\Gamma_\ell / C_\ell \cong H$ for all $\ell > 0$. Thus, $\mathcal{D}_\varphi \cong H$. Also, the subgroup $\widehat{N}_\varphi$ is the product of k copies of $\widehat{\mathbb{Z}}_m$, or the m-adic k-torus.

Observe that the map φ restricts to the identity on the subgroup H, while φ acts as multiplication by m on the normal subgroup $\mathbb{Z}^k$. Thus, $\widehat{\varphi}\colon \mathcal{D}_\varphi \to \mathcal{D}_\varphi$ in Theorem 5.3.3 is the identity map, and $\widehat{\varphi}\colon \widehat{N}_\varphi \to \widehat{N}_\varphi$ in Theorem 5.3.2 is induced by coordinate-wise multiplication by m on $\mathbb{Z}^k$.

8.2. **Nilpotent endomorphisms.** The 3-dimensional Heisenberg group $\mathcal{H}$ is the simplest non-abelian nilpotent group, and we give a self-embedding for which $\mathcal{D}_\varphi$ is non-trivial, and in fact is a Cantor group. This example is a special case of the general construction for self-embeddings of 2-step nilpotent groups given in Lee and Lee [40]. More generally, group chains in $\mathcal{H}$ were studied in detail by Lightwood, Şahin and Ugarcovici in [42], where they give a complete description for the subgroups of $\mathcal{H}$ and a characterization of which subgroups are normal. Group chains in $\mathcal{H}$ whose discriminant invariant is a Cantor group were constructed by Dyer in her thesis [19], and also described in [20, Example 8.1]. In the following, we construct such a group chain realized via a self-embedding of $\mathcal{H}$.

Let $\mathcal{H}$ be represented as $(\mathbb{Z}^3, *)$ with the group operation $*$, so for $x, u, y, v, z, w \in \mathbb{Z}$ we have,

$$(x, y, z) * (u, v, w) = (x + u, y + v, z + w + xv) \quad , \quad (x, y, z)^{-1} = (-x, -y, -z + xy)\ . \tag{26}$$

This is equivalent to the upper triangular representation in $\mathrm{GL}(\mathbb{Z}^3)$. In particular, we have

$$(x, y, z) * (u, v, w) * (x, y, z)^{-1} = (u, v, w + xv - yu)\ . \tag{27}$$

For integers $p, q > 0$ define $\varphi\colon \mathcal{H} \to \mathcal{H}$ by a self-embedding by $\varphi(x, y, z) = (px, qy, pqz)$. Then

$$\mathcal{H}_\ell = \varphi^\ell(\mathcal{H}) = \{(p^\ell x, q^\ell y, (pq)^\ell z) \mid x, y, z \in \mathbb{Z}\}\ .$$

Observe that the intersection $\cap_{\ell > 0} \mathcal{H}_\ell = \{e\}$. Now assume that $p, q > 1$ are distinct prime numbers. Formula (27) implies that the normal core for $\mathcal{H}_\ell$ is given by

$$C_\ell = \mathrm{core}(\mathcal{H}_\ell) = \{((pq)^\ell x, (pq)^\ell y, (pq)^\ell z) \mid x, y, z \in \mathbb{Z}\}\ .$$

Thus, the finite group

$$Q_\ell = \mathcal{H}/C_\ell = \{(x, y, z) \mid x, y, z \in \mathbb{Z}/(pq)^\ell \mathbb{Z}\}\ . \tag{28}$$

The profinite group $\widehat{\mathcal{H}}_\infty$ is the inverse limit of the quotient groups Q_ℓ so we have

$$\widehat{\mathcal{H}}_\infty = \{(x, y, z) \mid x, y, z \in \widehat{\mathbb{Z}}_{pq}\}$$

with multiplication on each finite quotient induced given by the formula (27). To identify the discriminant subgroup $\mathcal{D}_\infty$ first note

$$\mathcal{H}_\ell / C_\ell = \{(p^\ell x, q^\ell y, 0) \mid x \in \mathbb{Z}/q^\ell \mathbb{Z},\ y \in \mathbb{Z}/p^\ell \mathbb{Z}\} \subset Q_\ell\ , \tag{29}$$

$$\mathcal{H}_{\ell+1} / C_{\ell+1} = \{(p^{\ell+1} x, q^{\ell+1} y, 0) \mid x \in \mathbb{Z}/q^{\ell+1} \mathbb{Z},\ y \in \mathbb{Z}/p^{\ell+1} \mathbb{Z}\}\ . \tag{30}$$

The bonding map $q_{\ell+1}\colon \mathcal{H}_{\ell+1}/C_{\ell+1} \to \mathcal{H}_\ell / C_\ell$ from the definition (12) for $\mathcal{D}_\infty$ is induced from the inclusion $\mathcal{H}_{\ell+1} \subset \mathcal{H}_\ell$ modulo quotient by

$$\mathcal{H}_{\ell+1} \cap C_\ell = \{(p^{\ell+1} q^\ell x, p^\ell q^{\ell+1} y, (pq)^{\ell+1} z) \mid x, y, z \in \mathbb{Z}\}\ .$$

Thus, in terms of the coordinates x, y in (30) the bonding map is given by

$$q_{\ell+1}(x, y, 0) = (x \bmod q^\ell\mathbb{Z}, y \bmod p^\ell\mathbb{Z}, 0) \ .$$

It then follows by formula (12) that

$$\mathcal{D}_\varphi \cong \mathcal{D}_\infty = \varprojlim \{q_{\ell+1}\colon \mathcal{H}_{\ell+1}/C_{\ell+1} \to \mathcal{H}_\ell/C_\ell \mid \ell \geq 0\} \cong \widehat{\mathbb{Z}}_q \times \widehat{\mathbb{Z}}_p \ . \tag{31}$$

The induced map $\widehat{\varphi}\colon \mathcal{D}_\varphi \to \mathcal{D}_\varphi$ is given by multiplication by p on $\widehat{\mathbb{Z}}_q$ in the first x-coordinate, and multiplication by q on $\widehat{\mathbb{Z}}_p$ in the second y-coordinate, so that $\widehat{\varphi}$ acts as an isomorphism on $\mathcal{D}_\varphi$, as asserted in Theorem 5.3.

Finally, consider the subgroup of Q_ℓ which is complementary to the subgroup $\mathcal{H}_\ell/C_\ell$,

$$N_\ell = \{(q^\ell x, p^\ell y, z) \mid x \ \in \mathbb{Z}/p^\ell\mathbb{Z}, \ y \in \mathbb{Z}/q^\ell\mathbb{Z}, z \in \mathbb{Z}/(pq)^\ell\mathbb{Z}\} \ \subset \ Q_\ell \ . \tag{32}$$

The map φ induces a map on N_ℓ given by multiplication by p in the first x-coordinate, and multiplication by q in the second y-coordinate, so the action is nilpotent on N_ℓ. The inverse limit of the groups N_ℓ is a subgroup of $\widehat{\mathcal{H}}_\infty$ identified with

$$\widehat{N}_\varphi \cong \widehat{\mathcal{H}}_\infty/\mathcal{D}_\infty \cong \{(x, y, z) \mid x \in \widehat{\mathbb{Z}}_p \ , \ y \in \widehat{\mathbb{Z}}_q \ , \ z \in \widehat{\mathbb{Z}}_{pq}\} \ ,$$

and is a pro-nilpotent group as it has the finite nilpotent groups N_ℓ as quotients. Moreover, the induced map $\widehat{\varphi}\colon \widehat{N}_\varphi \to \widehat{N}_\varphi$ is a contraction, as asserted in Theorem 5.3.

Note that if we take $p = q$ in the above calculations, so $\varphi\colon \mathcal{H} \to \mathcal{H}$ is the "diagonal expansion" by p on the abelian factor $\mathbb{Z}^2$, then $\mathcal{H}_{2\ell} \subset C_\ell$. So while each quotient $\mathcal{H}_{2\ell}/C_{2\ell}$ is non-trivial, its image under the composition of bonding maps in (12) vanishes in $\mathcal{H}_\ell/C_\ell$, hence $\mathcal{D}_\varphi$ is the trivial group in the inverse limit. Correspondingly, the inverse limit space X_φ has a well-defined group structure.

8.3. **Semi-direct product of dyadic integers with its group of units.** This example can be viewed as a more sophisticated version of Example 8.1. It arises, in particular, as the profinite arithmetic iterated monodromy group associated to a certain post-critically finite quadratic polynomial, as discussed in [44]. We give the most basic example in the following.

Let $\widehat{\Gamma} = \widehat{\mathbb{Z}}_2 \rtimes \widehat{\mathbb{Z}}_2^\times$, where $\widehat{\mathbb{Z}}_2$ is the dyadic integers, and $\widehat{\mathbb{Z}}_2^\times$ is the multiplicative group of dyadic integers. Denote by a the topological generator of the abelian group $\widehat{\mathbb{Z}}_2$, that is, a is identified with $([1]) \in \widehat{\mathbb{Z}}_2$, where $[1]$ is the equivalence class of 1 in $\mathbb{Z}/2^n\mathbb{Z}$, $n \geq 1$.

Recall that $\widehat{\mathbb{Z}}_2^\times$ is the automorphism group of $\widehat{\mathbb{Z}}_2$. The multiplicative units in the 2-adic integers can be computed by computing the units in $\mathbb{Z}/2^n\mathbb{Z}$ for any n, and taking the inverse limit (see [51, Theorem 4.4.7]) so we have $\widehat{\mathbb{Z}}_2^\times \cong \mathbb{Z}/2\mathbb{Z} \times \widehat{\mathbb{Z}}_2$. Here, $\mathbb{Z}/2\mathbb{Z}$ is generated by $([-1]) \in \widehat{\mathbb{Z}}_2^\times$, where $[-1]$ denotes the equivalence class of -1 in $\mathbb{Z}/2^n\mathbb{Z}$ for $n \geq 1$, and the the second factor is generated by $([5]) \in \widehat{\mathbb{Z}}_2^\times$, where $[5]$ is the equivalence class of 5 in $\mathbb{Z}/2^n\mathbb{Z}$ for $n \geq 1$. Denote these generators by b and c respectively. Then let

$$\Gamma \cong \langle a, b, c \mid b^2 = 1, \, bab^{-1} = a^{-1}, \, cac^{-1} = a^5, \, bcb^{-1}c^{-1} = 1\rangle, \tag{33}$$

where b and c commute since they are generators of different factors of a product space.

Define a self-embedding $\varphi\colon \Gamma \to \Gamma$ by setting $\varphi(a) = a^2$, $\varphi(b) = b$ and $\varphi(c) = c$. That is, we have

$$\Gamma_1 = \varphi(\Gamma) \cong \langle a^2, b, c \mid b^2 = 1, \, ba^2b^{-1} = a^{-2}, \, ca^2c^{-1} = (a^2)^5, \, bcb^{-1}c^{-1} = 1\rangle,$$

and so we obtain a group chain $\Gamma_\ell = \langle a^{2^\ell}, b, c\rangle$, $\ell \geq 1$. The discriminant group of the action defined by this group chain was computed in [44, Section 7]. In particular, computing the normal cores of the subgroups Γ_ℓ we obtain $C_\ell = \langle a^{2^\ell}, c^{2^\ell - 2}\rangle \subset \Gamma_\ell$, and it follows that

$$\mathcal{D}_\varphi = \varprojlim \{\Gamma_{\ell+1}/C_{\ell+1} \to \Gamma_\ell/C_\ell\} \cong \widehat{\mathbb{Z}}_2^\times \ .$$

9. Problems

The study of the properties of the dynamical systems of the form $(X_\varphi, \Gamma, \Phi_\varphi)$ suggest the following approach to the classification problem for renormalizable groups and their proper self-embeddings.

PROBLEM 9.1. *Classify the structure of renormalizable groups* Γ *which satisfy:*

(1) $\mathcal{D}_\varphi$ *is the trivial group;*
(2) $\mathcal{D}_\varphi$ *is a finite group;*
(3) $\mathcal{D}_\varphi$ *is a Cantor group.*

Case (1) is discussed further in Section 9.1 below. There are numerous and varied constructions of examples of case (2), where $\mathcal{D}_\varphi$ is a finite group. See Section 8.1 for some typical examples.

The most interesting problems arise for case (3), where $\mathcal{D}_\varphi$ is a Cantor group. Theorem 1.2 implies that all of the direct limit group invariants for Cantor actions defined in [37] are bounded for these examples. Thus, the problem is to refine the invariants constructed from the adjoint action of $\mathcal{D}_\varphi$ on the pro-nilpotent normal subgroup $\widehat{N}_\varphi \subset \widehat{\Gamma}_\varphi$ to distinguish these various examples. Note that if the group chain $\mathcal{G}_\varphi$ has trivial intersection, then the intersection $\mathcal{D}_\varphi \cap \Gamma$ is trivial, so the invariants constructed using the adjoint action of $\mathcal{D}_\varphi$ are only "seen" when considering the action of $\widehat{\Gamma}_\varphi$.

9.1. Renormalizable nilpotent groups.

Suppose that Γ admits a renormalization $\varphi\colon \Gamma \to \Gamma$, such that each of the subgroups $\Gamma_\ell = \varphi^\ell(\Gamma)$ is a *normal* subgroup of Γ. Then the third author showed in the work [54] that the quotient $\Gamma/C(\mathcal{G}_\varphi)$ must be free abelian. In particular, if the group chain $\mathcal{G}_\varphi = \{\Gamma_\ell \mid \ell \geq 0\}$ has trivial intersection, then Γ is free abelian. Theorem 1.1 is a more general form of this result, where the assumption that $\mathcal{G}_\varphi$ has finite discriminant implies that Γ is virtually nilpotent.

The remarks at the end of Section 8.2 show that $\mathcal{D}_\varphi$ is trivial when $p = q$ for the construction in Section 8.2. In fact, these remarks apply in general to the diagonal action on the nilpotent subgroup of upper triangular integer matrices, where φ is given by multiplication by a constant factor p on the super-diagonal entries; that is, those directly above the diagonal. This suggests that the non-triviality of the discriminant invariant $\mathcal{D}_\varphi$ for an endomorphism of a nilpotent group is a measure of the "asymmetry" of the embedding φ. It is an interesting problem to make this statement more precise for the general nilpotent group.

PROBLEM 9.2. *Let* Γ *be a finitely generated torsion free nilpotent group, and* φ *a renormalization such that* $\mathcal{G}_\varphi = \{\Gamma_\ell \mid \ell \geq 0\}$ *has trivial intersection. Develop the relationship between the properties of the discriminant group* $\mathcal{D}_\varphi$*, the embedding* φ*, and the nilpotent structure theory of* Γ*, as developed for example in* [12, 17]*.*

9.2. Algebraic invariants.

The reduced group C^*-algebra $C^*_r(X_\varphi, \Gamma, \Phi_\varphi)$ obtained from the group action $(X_\varphi, \Gamma, \Phi)$ is a source of invariants for the group Γ and the embedding φ. In the case when $\Gamma = \mathbb{Z}^n$ is free abelian, the work [26] shows that the ordered K-theory of this C^*-algebra is a complete invariant of the action. It is natural to ask whether similar results are possible in more generality:

PROBLEM 9.3. *Let* Γ *be a finitely generated nilpotent group, and* φ *a renormalization of* Γ*. What information about the nilpotent structure constants of* Γ *and the embedding* φ *is determined by the K-theory groups* $K_*(C^*_r(X_\varphi, \Gamma, \Phi_\varphi))$*?*

Note that by Theorem 1.9, the isomorphism class of the discriminant group $\mathcal{D}_\varphi$ is an invariant of the continuous orbit equivalence class of the Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$, and the isomorphism class of $C^*_r(X_\varphi, \Gamma, \Phi_\varphi)$ is also invariant. It seems natural that these two invariants should be closely related.

PROBLEM 9.4. *Let* Γ *be a renormalizable group. How does the algebraic structure of* $C^*_r(X_\varphi, \Gamma, \Phi_\varphi)$ *reflect the properties of the profinite group* $\mathcal{D}_\varphi$*?*

Theorem 5.3 shows that the profinite group $\widehat{\Gamma}_\varphi$ is a semi-direct product with $\mathcal{D}_\varphi$ as a factor. One approach to Problem 9.4 would be to relate the decomposition $\widehat{\Gamma}_\varphi \cong \widehat{N}_\varphi \rtimes \mathcal{D}_\varphi$ in Theorem 5.3 to the algebraic structure of $C_r^*(X_\varphi, \Gamma, \Phi_\varphi)$.

9.3. **Realization.** Given any pro-finite group $\mathcal{D}$ which is topologically countably generated, it was shown in [35, 37], using the Lenstra method, that there exists a finitely generated group Γ and Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ whose discriminant is isomorphic to $\mathcal{D}$.

PROBLEM 9.5. *Let Γ be a renormalizable group which is not virtually nilpotent, so the discriminant invariant $\mathcal{D}_\varphi$ is a Cantor group. What profinite groups can be realized as the discriminant group for a Cantor actions associated to a renormalization of Γ?*

9.4. **Renormalizable Cantor actions.** A Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ such that the group $\widehat{\Gamma}_\varphi \subset \mathrm{Homeo}(\mathfrak{X})$ is not locally quasi-analytic, and such that for every $\ell \geq 0$ the kernel $\ker \Phi_\ell$ is a finite group, are called *wild of finite type* in the work [37] of the first two authors. Examples of wild actions constructed by the same authors in [35] are of finite type. However, the examples in [35] are not renormalizable.

PROBLEM 9.6. *Do there exist renormalizable Cantor actions which are wild of finite type?*

PROBLEM 9.7. *Suppose that $(\mathfrak{X}, \Gamma, \Phi)$ is a renormalizable Cantor action which is not quasi-analytic. What can be said about the algebraic properties of Γ? For example, must Γ have exponential growth type? What can be said about the profinite group $\widehat{\Gamma}_\varphi \subset \mathrm{Homeo}(\mathfrak{X})$ for such actions?*

9.5. **Representations of Galois groups.** The works of the second author [43, 44] define the discriminant invariants associated to arboreal representations of absolute Galois groups for number fields and function fields. Such a representation is a profinite group, obtained as the inverse limit of finite Galois groups, which act on finite extensions of the ground field, obtained by adjoining the roots of the n-th iteration of the same polynomial, for $n \geq 1$.

The example given in Section 8.3 is an example of an arboreal representation of an absolute Galois group, which is isomorphic to a Cantor action associated to a renormalization. For many polynomials the associated action is known to be not locally quasi-analytic [44] and, therefore, by Theorem 1.2 it cannot be associated to a renormalization of a group. This suggests the following problem:

PROBLEM 9.8. *For which arboreal representations of absolute Galois groups does there exists a dense finitely generated group Γ and a renormalization $\varphi\colon \Gamma \to \Gamma$, such that the arboreal representation of Γ is return equivalent to a Cantor action associated to $(X_\varphi, \Gamma, \Phi_\varphi)$?*

Although, as discussed above, many arboreal representations are not associated to an finite-index embedding $\varphi\colon \Gamma \to \Gamma$, since they are associated to a structure built using iterations of the same polynomial, it is natural to look for a formalism similar to the non-co-Hopfian setting for the study of these groups. This motivated the definition of renormalizable actions in Section 7, and suggest the following interesting problem:

PROBLEM 9.9. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an equicontinuous minimal Cantor action, and suppose that $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable as in Definition 7.1. Develop a structure theory for the group obtained as the closure of the action $(\mathfrak{X}, \Gamma, \Phi)$ in $\mathrm{Homeo}(\mathfrak{X})$, analogous to Theorem 5.3.*

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