Systemic Risk: the Effect of Market Confidence

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Abstract

In a crisis, when faced with insolvency, banks can sell stock in a dilutive offering in the stock market and borrow money in order to raise funds. We propose a simple model to find the maximum amount of new funds the banks can raise in these ways. To do this, we incorporate market confidence of the bank together with market confidence of all the other banks in the system into the overnight borrowing rate. Additionally, for a given cash shortfall, we find the optimal mix of borrowing and stock selling strategy. We show the existence and uniqueness of Nash equilibrium point for all these problems. Finally, using this model we investigate if banks have become safer since the crisis. We calibrate this model with market data and conduct an empirical study to assess safety of the financial system before, during after the last financial crisis.

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1 Introduction

Traditionally risk management has considered how the risks affect a particular institution, while ignoring how these risks might affect the financial system as a whole. In contrast, systemic risk considers how this risk spreads throughout the financial system through the interactions of the banks in the system. Such a spread of defaults is also known as contagion. It can occur through both local and global connections, e.g., contractual obligations and impacts on borrowing rates and liquidity respectively. Such a systemic event caused the last financial crisis, during which the entire financial network was threatened with insolvency. It became apparent that good models and understanding of systemic risk are vital.

Eisenberg & Noe (2001) introduced a network framework which models default contagion through contractual obligations and found the resulting equilibrium payments. The model has been widely used by both regulators and academics e.g. Anand, Craig, & Von Peter (2015), Halaj & Kok (2015), Boss, Elsinger, Summer, & Thurner (2004), Elsinger, Lehar, & Summer (2013), Upper (2011), Gai, Haldane, & Kapadia (2011), Bardoscia, Barucca, Brinley Codd, & Hill (2017).

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Many other extensions have been considered, a survey of which can be found in e.g. Weber & Weske (2017), Staum (2013), Hüser (2015). Liquidity, capital and the connectivity of the financial network are highly related to systemic risk. Freixas, Parigi, & Rochet (2000) investigated the ability of the banking system to withstand financial contagion and examined the too-big-to-fail policy. Diamond & Rajan (2005) showed that financial contagion shrinks the common pool of liquidity, creating or exacerbating aggregate liquidity shortage, which in turn leads to additional contagion and even a total meltdown of the system. Cifuentes, Ferrucci, & Shin (2005) stated that capital requirements can cause abnormal effects between portfolios valuations and systemic resilience. Gaspar, Pérez-Quirós, & Rodríguez Mendizábal (2004) studied systemic risk from the perspective of monetary policy. Their model showed how operational framework of monetary policy can affect the elasticity of supply of funds by banks throughout the reserve maintenance period. Nier, Yang, Yorulmazer, & Alentorn (2007) found that low level of equity increases number of contagious defaults and that contagion is non-monotonic on degree of connectivity. The financial crisis also pointed out the need of finding a signal that can help the banks to monitor the financial situation and survive during the crisis. Gray, Merton, & Bodie (2007) used contingent claims analysis to measure systemic risk in a structural approach. Acharya, Pedersen, Philippon, & Richardson (2017) used systemic expected shortfall as a measure of systemic risk. Additional empirical research including Huang, Zhou, & Zhu (2009) studied expected credit losses by data of CDS and stock return correlations. De Jonghe (2010) estimated tail betas of European financial firms to measure systemic risk. Billio, Getmansky, Lo, & Pelizzon (2012) measured systemic risk through Granger causality test. Mencia (2009) stated that rating is a good factor to predict liquidity shock, with lower rating banks are more likely to receive liquidity shocks. Akram & Christophersen (2010) argued that there is evidence that credit ratings, liquidity demand and supply have stronger effects on interest rate since the start of the current financial crisis, and there is a relatively large variation in actual overnight interest rates over time and across banks. Their analysis has suggested that such variation, can be partly ascribed to bank's characteristics. In particular, domestic banks, which are considered to be 'too big to fail' and 'too connected to fail' are able to borrow at relatively lower rates than other banks.

The size, connectivity degree and other such factors of the banks are all realizations of the market confidence. Hence, many studies focus on how to incorporate confidence. Arinaminpathy, Kapadia, & May (2012) emphasized the importance of market confidence. They pointed out that the importance of relatively large, well-connected banks in system stability scales is proportionately larger than their size: the impact of their collapse arise not only from their connectivity, but also from their effect on confidence in the system. Confidence and liquidity have also been noted in Glasserman & Young (2015) as examples of systemic contagion that may generate big losses. Confidence is a natural way to measure this risk, and credit rating and overnight rates were found to be important predictors of bank's heath. Bräuning & Fecht (2016) investigated the effect that lending relationship has on the availability and pricing of interbank liquidity. Their result implied relationship lenders are more likely to provide cheaper liquidity to their closest borrowers, and particularly opaque borrowers obtain liquidity at lower rates when borrowing from their relationship lenders.

In this paper, we use equity price-to-book (P/B) ratio as a proxy of market confidence. Chan, Hamao, & Lakonishok (1991) , Fama & French (1992) and Vassalou & Xing (2004) hypothesized that P/B ratio can be used as a proxy for default risk and showed that it is an indicator for distress. Similarly, Switzer & Wang (2013) showed that higher P/B ratio is associated with lower default probability, and Pae, Thornton, & Welker (2005) hypothesized that it also affects liquidity. Higher risk of default is associated with higher borrowing costs. Therefore, it is natural to use the P/B ratio as confidence/liquidity proxy and to generalize it from individual firm level to a systemic level, which is what we propose.

It has been the practice to ensure that banks are well capitalized. However, arguably a major reason for the downfall of Bear Stearns and Lehman Brother was the drying-up of liquidity to support their day-to-day operations. While after the crisis the banks reduced their exposure to short-term liquidity risks, Onaran (2018), they are still significantly exposed to drying up of the repo market, and therefore confidence and liquidity need to be taken into account when considering the capitalization of a bank. When banks need to raise funds, they can do it by borrowing. Popular alternatives, include fire sales and dilutive offering, i.e. selling additional equity of the company. While in reality these happen on a different time scale, because our model is static we will assume that both borrowing and stock sale are instantaneous and happen at the same time. This assumption is typical, e.g. Bichuch & Feinstein (2019) discussed fire sale and borrowing. In this paper, we assume the means to raise funds are through borrowing and stock sales, by issuing new shares or through a dilutive offering. We find the optimal strategy for the banks to recapitalize using these two methods. Such transactions affect the entire financial sector, thus decreasing the confidence in all the banks. It is assumed that both actions lower the market's confidence in the bank and increase the overnight rate, as shown e.g. by Rochet & Tirole (1996) and Elyasiani, Mester, & Pagano (2014), and therefore leads to an increase in borrowing costs, and a decrease in banks' stock price. The overnight interest rate has been believed to be a good proxy for liquidity issues as evidence provided by Furfine (2000), Furfine (2001) and Iori, Jafarey, & Padilla (2006). Hence, we incorporate market confidence into the ability to cover shortfall and show that the cash reserve on the balance sheet offer an incomplete picture of actual reserves of the bank. We explain the lack of confidence spreads in the bank system when facing liquidity problem through the increasing overnight borrowing rates of the banks in the system.

The rest of the paper is organized in the following way. In Section 2, we introduce the model and study the maximal amount the banks can raise by only selling/issuing stocks, with no borrowing. In Section 3, we add borrowing into consideration, and find the optimal strategy for the banks to raise cash by borrowing and selling stocks. In both sections above, we model the overnight interest rate of each bank as a function of: 1) the number of shares of stock this bank and other banks sell, 2) the amount of money borrowed by this bank and the amounts borrowed by all other banks. This sets up the optimization problems as a non-cooperate concave "n"-person game. We show the existence and uniqueness of Nash equilibrium for these optimization problems, and establish a slightly different sufficient condition for the uniqueness of equilibrium than Rosen (1965) [Theorem 6]. In Section 4, we study the optimal strategy for the banks to recover their cash shortfall while minimizing their financing cost. In Section 5, we calibrate this model with real data to find the maximum amount Citi and JP Morgan Chase banks could raise over the last decades and compare how these amounts changed before, during and after the financial crisis. Finally, we conclude and summarize the main results in Section 6.

2 Optimal Strategy Raising Funds by Selling Stock

Consider a system with $m \geq 2$ banks which suffer a shortfall or alternatively face a stress test so that they need to raise money. Our goal is to see how (the lack of) confidence spreads throughout the system, or alternatively, how the liquidity contagion evolves. We first aim to find the maximum amount of funds these m banks can raise in this scenario. In this paper, we assume that the banks can raise funds in the two ways – 1) they can sell stocks; 2) they can borrow money. They can also do both at the same time. In this section, we consider the case that the banks can only sell stocks and borrowing is not allowed. We then relax this assumption in Section 3. The proposed model is static, so even though raising funds in the stock market takes time, we assume that stock sale is done instantaneously and simultaneously with borrowing (when allowed).

Denote $s_j, j \in \{1,...,m\}$ to be the number of stock shares that bank j sells, and let $\mathbf{s} = (s_1, s_2, ..., s_m)$, and $\mathbf{s}_{-j} = (s_1, ..., s_{j-1}, s_{j+1}, ...s_m)$. Such a sale affects all banks, therefore the stock price of bank $j \in \{1,...,m\}$ drops from $p_j(0,\mathbf{0})$ to $p_j(s_j, \mathbf{s}_{-j})$, after such sale. This transaction also changes the book value of the bank by $B_j(s_j, \mathbf{s}_{-j}) - B_j(0,\mathbf{0}) = p_j(s_j, \mathbf{s}_{-j})s_j$. Finally, denote C_j which is also a function of $(s_j, \mathbf{s}_{-j})$ to be bank's j market capitalization. Therefore, after the sale transaction, the market capitalization becomes $C_j(s_j, \mathbf{s}_{-j}) = p_j(s_j, \mathbf{s}_{-j})(N_j + s_j)$, where N_j is the number of outstanding shares before this transaction.

We will use bank j's price-to-book ratio $\frac{C_j}{B_j}$ as a proxy for the market's confidence in the bank. In turn, we assume that the bank's overnight borrowing rate r_j depends on the confidence in the bank, and therefore it is a function of the price-to-book ratio. Additionally, we will assume that r_j depends also on the confidence in all other banks in the system. Since both $B_j, C_j,\ j \in \{1,...,m\}$ are functions of $\mathbf{s}$, for convenience and consistency with p_j, we write $r_j(s_j, \mathbf{s}_{-j})$ as well.

Notice that high price-to-book ratio, suggests high confidence in a bank, resulting in low overnight interest rate, and vice-versa. It follows that when the bank sells its shares, its book-to-price ratio, (the reverse ratio) will increase. This implies that its overnight rate will increase after this transaction as well. Moreover, since such a transaction will decrease the supply of funds available for borrowing in the market and indicate a potential for an increase in systemic risk for the whole banking sector, the overnight rates of all other banks will increase as well.

Together these are the two mechanism that are assumed in this paper that liquidity crisis, or the lack of confidence spreads between banks. As bank j sells stocks, the stock price for all other banks decrease, dragging down their proceed from stock sales. Moreover, the transaction of bank j also changes its price-to-book ratio, which also raises the short-term borrowing rates for the other banks, which in turn may cause a liquidity crisis.

Banks utilize the overnight lending market to cover its short-term liabilities. These accounted for 44% of all banks debt in 2007, and still account for nearly a quarter of all debt in 2018 Onaran (2018). Denote L_j to be the size of bank j's overnight loan. Therefore, the funds, bank j can raise from a stock sale, are given by

$$u_j(s_j, \mathbf{s}_{-j}) = p_j(s_j, \mathbf{s}_{-j})s_j - L_j(r_j(s_j, \mathbf{s}_{-j}) - r_j(0,\mathbf{0})), \tag{1}$$

where the first term accounts for the cash raised by stock sale, and the second term is the loss on the short-term loan due to the rate increase. We assume that $u_j\colon [0, S_j] \times \prod_{i=1, i\neq j}^{m} [0, S_i] \to \mathbb{R}$, where $S_i, i = 1, 2, .., m$ is maximal number of shares bank i can sell/issue, and refer to u_j as the objective function of bank j. Denote the mapping $s_j^*\colon \prod_{i=1, i\neq j}^{m} [0, S_i] \to [0, S_j]$ to be the solution of the optimization problem for bank j given $\mathbf{s}_{-j}$, i.e.

$$s_j^*(\mathbf{s}_{-j}) = \arg\max_{s \in [0, S_j]} u_j(s, \mathbf{s}_{-j}). \tag{2}$$

Additionally, for any given $\mathbf{s} \in \prod_{i=1}^{m} [0, S_i]$, define a mapping $\mathbf{S}^*\colon \prod_{i=1}^{m} [0, S_i] \to \prod_{i=1}^{m} [0, S_i]$ as

$$\mathbf{S}^*(\mathbf{s}) = \left(s_1^*(\mathbf{s}_{-1}), ..., s_m^*(\mathbf{s}_{-m})\right)^T. \tag{3}$$

Note that the funds function u_j in (1) depends not only on the strategy s_j of the bank j, but also on the strategies of all other banks $\mathbf{s}_{-j}$. We assume that the banks can observe what other banks are doing. Therefore, the problem in (2) is not an individual optimization problem but a "m"-person non-cooperation game, and we aim to find a Nash equilibrium solution to it. We establish the existence and uniqueness of Nash equilibrium to this game under the following assumptions:

Assumption 2.1. 1. The stock price function of bank j given by

$$p_j\colon [0, S_j] \times \prod_{i=1, i\neq j}^{m} [0, S_i] \to \mathbb{R}_+ \tag{4}$$

is a twice differentiable, convex function, for any $j \in 1, 2, .., m$. Moreover, for any given $\mathbf{s}_{-j} \in \prod_{i=1,i\neq j}^{m}[0, S_i]$, the function $s \mapsto p_j(s, \mathbf{s}_{-j})$ satisfies

$$\frac{\partial p_j(s, \mathbf{s}_{-j})}{\partial s_j} < 0, \tag{5}$$

and for all $j = 1, 2..., m$, the function $s \mapsto s p_j(s, \mathbf{s}_{-j})$ is an increasing concave function in s satisfying

$$\begin{aligned} &\frac{\partial p_j(S_j, \mathbf{s}_{-j})}{\partial s_j} S_j + p_j(S_j, \mathbf{s}_{-j}) > 0, \\ &\frac{\partial^2 p_j(s, \mathbf{s}_{-j})}{\partial s_j^2}(N_j + s) + 2\frac{\partial p_j(s, \mathbf{s}_{-j})}{\partial s_j} < 0, \;\; s \in [0, S_j]. \end{aligned} \tag{6}$$

2. The overnight rate $r_j\colon [0, S_j] \times \prod_{i=1,i\neq j}^{m}[0, S_i] \to \mathbb{R}_+$ is twice differentiable function. Additionally, for $\mathbf{s}_{-j} \in \prod_{i=1,i\neq j}^{m}[0, S_i]$, the function $s \mapsto r_j(s, \mathbf{s}_{-j})$ is a convex, and $\frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j}) > 0$.

Remark 2.2. Under Assumption 2.1, for any given $j \in 1, 2, .., m$ and $\mathbf{s}_{-j} \in \prod_{i=1,i\neq j}^{m}[0, S_i]$ the function $s \mapsto u_j(s, \mathbf{s}_{-j})$ is a twice differentiable concave function in s.

Example 2.3. Examples of inverse demand price functions p_j , $j = 1, ..., m$ satisfying Assumption 2.1.1 include:

a. Linear inverse demand:

$$p_j(s_j, \mathbf{s}_{-j}) = p_j(0, \mathbf{0})(1 - a_j s_j - \epsilon \sum_{i=1, i\neq j}^{m} s_i), 0 < a_j < \frac{1}{2S_j}, 0 < \epsilon \ll a_j. \tag{7}$$

b. Exponential inverse demand:

$$p_j(s_j, \mathbf{s}_{-j}) = p_j(0, \mathbf{0})e^{-a_j s_j - \epsilon \sum_{i=1,i\neq j}^{m} s_i}, 0 < a_j < \frac{1}{S_j}, 0 < \epsilon \ll a_j. \tag{8}$$

Example 2.4. Examples of interest rate functions r_j, $j = 1, ..., m$ that satisfy Assumption 2.1.2 include:

a. Linear interest rate:

$$\begin{aligned} r_j(s_j, \mathbf{s}_{-j}) = \beta_{0,j} + \beta_{1,j} s_j + \beta_{2,j} \sum_{i=1, i\neq j}^{m} s_i, \\ \beta_{k,j} > 0, \;\; j = 1, ..., m, k = 0, 1, 2. \end{aligned} \tag{9}$$

b. Quadratic interest rate:

$$\begin{aligned} r_j(s_j, \mathbf{s}_{-j}) = \beta_{0,j} + \beta_{1,j} s_j + \beta_{2,j} s_j^2 + \beta_{3,j} \sum_{i=1, i\neq j}^{m} s_i + \beta_{4,j} \sum_{i=1, i\neq j}^{m} s_i^2, \\ \beta_{k,j} > 0, \;\; j = 1, ..., m, k = 0, 1, ..., 4, \end{aligned} \tag{10}$$

in addition to satisfying Assumption 2.1.2, is strictly convex on $[0, S_j] \times \prod_{i=1,i\neq j}^{m}[0, S_i]$.

c. Interest rate as a function of book-to-price ratio:

$$r_j(s_j, \mathbf{s}_{-j}) = \beta_{2,j} \sum_{i \neq j} \frac{B_i(s_i, \mathbf{s}_{-i})}{C_i(s_i, \mathbf{s}_{-i})} + \beta_{1,j} \frac{B_j(s_j, \mathbf{s}_{-j})}{C_j(s_j, \mathbf{s}_{-j})} + \beta_{0,j}$$
$$\beta_{k,j} > 0, \; j = 1, ..., m, k = 0, 1, 2, \tag{11}$$

together with either linear inverse demand price function Example 2.3.a or exponential inverse demand price function Example 2.3.b (used to define $B_j(s_j, \mathbf{s}_{-j}), C_j(s_j, \mathbf{s}_{-j})$).
Let $j = 1, ..., m$. Since r_j is a linear combination of $\frac{B_i}{C_i}$ with positive coefficients, to verify that r_j satisfies Assumption 2.1.2 it is sufficient to verify that

$$\frac{\partial \frac{B_j}{C_j}}{\partial s_j}(0, \mathbf{s}_{-j}) > 0, \tag{12}$$

$$\frac{\partial^2 \frac{B_j}{C_j}}{\partial s_j^2}(s_j, \mathbf{s}_{-j}) \geq 0, \tag{13}$$

$$\frac{\partial^2 \frac{B_i}{C_i}}{\partial s_j^2}(s_i, \mathbf{s}_{-i}) \geq 0. \tag{14}$$

In case of linear inverse demand price function further assume

$$\frac{C_j(0, \mathbf{0})}{B_j(0, \mathbf{0})} < 2, \; 0 < \epsilon < \frac{1}{20} \frac{1}{\sum_{i \neq j} S_j}, \text{ and } \frac{3}{N_j} \leq a_j \leq \frac{9}{20} \frac{1}{S_j}. \tag{15}$$

Recall that

$$B_j(s_j, \mathbf{s}_{-j}) = B_j(0, \mathbf{0}) + s_j p_j (1 - a_j s_j - \epsilon \sum_{i \neq j} s_i), \tag{16}$$

$$C_j(s_j, \mathbf{s}_{-j}) = (N_j + s_j) p_j (1 - a_j s_j - \epsilon \sum_{i \neq j} s_i), \tag{17}$$

$$= C_j(0, \mathbf{0})(1 - a_j s_j - \epsilon \sum_{i \neq j} s_i) + s_j p_j (1 - a_j s_j - \epsilon \sum_{i \neq j} s_i). \tag{18}$$

We calculate that

$$\frac{\partial \frac{B_j}{C_j}}{\partial s_j}(s_j, \mathbf{s}_{-j}) = \frac{\frac{\partial B_j}{\partial s_j} C_j - B_j \frac{\partial C_j}{\partial s_j}}{C_j^2}(s_j, \mathbf{s}_{-j}), \tag{19}$$

$$\frac{\partial B_j}{\partial s_j}(s_j, \mathbf{s}_{-j}) = p_j (1 - 2 a_j s_j - \epsilon \sum_{i \neq j} s_i), \tag{20}$$

$$\frac{\partial C_j}{\partial s_j}(s_j, \mathbf{s}_{-j}) = -a_j p_j N_j + p_j (1 - 2 a_j s_j - \epsilon \sum_{i \neq j} s_i). \tag{21}$$

Conditions (15) ensure that $\frac{\partial C_j}{\partial s_j} < 0$, which implies for $(s_j, \mathbf{s}_{-j}) \in [0, S_j] \times \prod_{i \neq j} [0, S_i]$

$$\frac{\partial \frac{B_j}{C_j}}{\partial s_j}(s_j, \mathbf{s}_{-j}) > 0. \tag{22}$$

Choosing $s_j = 0$ leads to (12).

Using the fact that

$$\frac{\partial^2 B_j}{\partial s_j^2}(s_j, \mathbf{s}_{-j}) = \frac{\partial^2 C_j}{\partial s_j^2}(s_j, \mathbf{s}_{-j}) = -2a_j p_j, \tag{23}$$

we calculate that

$$\frac{\partial^2 \frac{B_j}{C_j}}{\partial s_j^2} = \frac{(\frac{\partial^2 B_j}{\partial s_j^2} C_j - B_j \frac{\partial^2 C_j}{\partial s_j^2}) C_j^2 - 2C_j \frac{\partial C_j}{\partial s_j}(\frac{\partial B_j}{\partial s_j} C_j - B_j \frac{\partial C_j}{\partial s_j})}{C_j^4} \tag{24}$$

$$\geq \frac{2a_j p_j B_j(0, \mathbf{0}) \left(1 + \frac{(N_j a_j - (1 - a_j s_j - \epsilon \sum_{i \neq j} s_i))^2}{N_j a_j}\right)}{C_j^2} \tag{25}$$

$$- \frac{a_j p_j C_j(0, \mathbf{0}) \left((1 - \epsilon \sum_{i \neq j} s_i) + \frac{(1 - 2a_j s_j - \epsilon \sum_{i \neq j} s_i)^2}{a_j N_j}\right)}{C_j^2} > 0, \tag{26}$$

where the positivity follows again from (15). This shows (13).

Finally, to show (14), we have that

$$\frac{\partial^2 \frac{B_i}{C_i}}{\partial s_j^2} = \frac{-2C_i \frac{\partial C_i}{\partial s_j} \frac{\partial \frac{B_i}{C_i}}{\partial s_j}}{C_i^4}. \tag{27}$$

From (17),

$$\frac{\partial C_i}{\partial s_j} = -(N_i + s_i) p_i \epsilon < 0, \tag{28}$$

and we have that

$$\frac{\partial \frac{B_i}{C_i}}{\partial s_j} = \frac{\frac{\partial B_i}{\partial s_j} C_i - B_i \frac{\partial C_i}{\partial s_j}}{C_i^2} = \frac{\epsilon p_i B_i(0, \mathbf{0})(N_i + s_i)}{C_i^2} > 0. \tag{29}$$

Thus (14) holds.

For exponential inverse demand function case, we slightly modify the conditions to be

$$\frac{C_j(0, \mathbf{0})}{B_j(0, \mathbf{0})} < \frac{\sqrt{2} - 1}{2}, \text{ and } \frac{3}{N_j} < a_j < \frac{1}{S_j}. \tag{30}$$

Equation (19) is still valid, and conditions (30), again ensure that $\frac{\partial C_j}{\partial s_j} < 0$, which again leads to (22) and (12) as before.

Next, similar to (24) we have that

$$\frac{\partial^2 \frac{B_j}{C_j}}{\partial s_j^2} \geq \frac{a_j p_j \, \mathrm{e}^{-a_j s_j - \epsilon \sum_{i \neq j} s_i}}{C_j^2} \tag{31}$$

$$\times \left(\left(a_j(N_j + s_j) - 2 + \frac{2}{a_j(N_j + s_j)} \right) B_j(0, \mathbf{0}) - \left(1 + a_j s_j + \frac{4(1 - a_j s_j)}{a_j N_j} \right) C_j(0, \mathbf{0}) \right).$$

Conditions (30) again ensure that (13) holds.

Finally, we calculate that

$$\frac{\partial \frac{B_i}{C_i}}{\partial s_j}(s_i, \mathbf{s}_{-i}) = \frac{\epsilon(N_i + s_i) B_i(0, \mathbf{0}) e^{-a_i s_i - \epsilon \sum_{l \neq i} s_l}}{C_i^2} > 0. \tag{32}$$

Equation (27) still holds. Using (28) together with (32) we conclude that (14) holds.

The conditions (15) and (30) are only sufficient conditions and are most likely not necessary, and hence not unique. They are used to ensure that interest rate as a function of book-to-price ratio satisfies Assumption 2.1.2 when using linear inverse demand function and exponential inverse demand function respectively. The financial intuition for the first assumption in (15) is that (intuitively) we need the interest rate to be convex function of the stock sale amounts. Because of the form of the interest rate function this requires an upper bound on the price-to-book ratio. In other words, the market confidence needs to be bounded from above. This is not a surprising assumption to hold, especially during a financial crisis. The other assumptions on ϵ and a_j in (15) provide bounds on the impact of additional stock sales on the of stock price. Similar intuition holds for the assumptions in (30).

Theorem 2.5. *(Existence of Nash Equilibrium) Under Assumptions 2.1, there exists Nash equilibrium to the optimization problem (2) denoted by* $\mathbf{s}^{**} = (s_1^*(\mathbf{s}_{-1}^{**}), ..., s_m^*(\mathbf{s}_{-m}^{**})) \in \prod_{j=1}^m [0, S_j]$.

Proof. For any $j \in 1, 2, ..., m$ and $\mathbf{s}_{-j} \in \prod_{i=1, i\neq j}^m [0, S_i]$, the first order optimality condition for an interior maximizer of the objective function $u_j(s, \mathbf{s}_{-j})$ is

$$p_j(s, \mathbf{s}_{-j}) + s\frac{\partial p_j(s, \mathbf{s}_{-j})}{\partial s_j} - L_j \frac{\partial r_j}{\partial s_j} = 0, \tag{33}$$

which is equivalent to

$$\left(p_j(s, \mathbf{s}_{-j}) + s\frac{\partial p_j(s, \mathbf{s}_{-j})}{\partial s_j}\right)\left(\frac{\partial r_j}{\partial s_j}(s, \mathbf{s}_{-j})\right)^{-1} = L_j, \tag{34}$$

due to the fact that $\frac{\partial r_j}{\partial s_j}(s, \mathbf{s}_{-j}) > 0$, for $s \in [0, S_j]$, in turn which follows from Assumptions 2.1. Moreover, according to the same assumption, the derivative of the left hand side of (34),

$$\frac{\left(s\frac{\partial^2 p_j(s, \mathbf{s}_{-j})}{\partial s_j^2} + 2\frac{\partial p_j(s, \mathbf{s}_{-j})}{\partial s_j}\right)\frac{\partial r_j}{\partial s_j} - \frac{\partial^2 r_j}{\partial s_j^2}\left(p_j(s, \mathbf{s}_{-j}) + s\frac{\partial p_j(s, \mathbf{s}_{-j})}{\partial s_j}\right)}{\left(\frac{\partial r_j}{\partial s_j}\right)^2}, \tag{35}$$

is always negative, which implies the left hand side of (34) is strictly decreasing with respect to $s \in [0, S_j]$.

Then, for this $\mathbf{s}_{-j} \in \prod_{i=1, i\neq j}^m [0, S_i]$, we have the following:

1. If

$$\frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j}) \leq \frac{p_j(0, \mathbf{s}_{-j})}{L_j}, \tag{36}$$

   that is

$$p(0, \mathbf{s}_{-j})\left(\frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j})\right)^{-1} \geq L_j, \tag{37}$$

then as the left hand side of (34) is strictly decreasing in s, there is at most one $s \in [0, S_j]$ that solves (34). Moreover, if such solution to (34) exists, it is unique. Denote it $s_j^0(\mathbf{s}_{-j})$. Otherwise, if there is no solution on $[0, S_j]$, set $s_j^0(\mathbf{s}_{-j}) = S_j$. The intuition behind the condition (36) is that an (infinitesimal) initial stock sale, raises more money than the cost increase incurred on the short term obligation L_j due to an increase in the interest rate r_j. Therefore, it is optimal to sell stocks and $s_j^0(\mathbf{s}_{-j}) \geq 0$, as we have obtained. If no solution of (34) exist on $[0, S_j]$, then it is optimal to sell all the bank owns, and $s_j^0(\mathbf{s}_{-j}) = S_j$.

2. Otherwise, if

$$\frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j}) > \frac{p_j(0, \mathbf{s}_{-j})}{L_j}, \tag{38}$$

that is

$$p(0, \mathbf{s}_{-j}) \left(\frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j}) \right)^{-1} < L_j, \tag{39}$$

then there is no solution to (34) on $s \in [0, S_j]$, because the left hand side of (34) is strictly decreasing in s. The intuition in this case is that under the condition (38), an (infinitesimal) stock sale, is insufficient to cover the additional interest rate expense, therefore selling stock is not optimal. Thus set $s_j^*(\mathbf{s}_{-j}) = 0$.

In summary, each bank j chooses to sell $s_j^*(\mathbf{s}_{-j})$ shares provided that other banks are selling $\mathbf{s}_{-j}$ shares of their stock which is given by

$$s_j^*(\mathbf{s}_{-j}) = \begin{cases} 0 & : \frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j}) > \frac{p_j(0, \mathbf{s}_{-j})}{L_j}, \\ s_j^0(\mathbf{s}_{-j}) & : \text{otherwise.} \end{cases} \tag{40}$$

In fact, $s_j^* : \prod_{i=1, i \neq j}^m [0, S_i] \to [0, S_j]$ is continuous in each of its components. Indeed, s_j^* is continuous as a function of $\mathbf{s}_{-j}$ in the region where

$$\begin{aligned} \frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j}) &< \frac{p_j(0, \mathbf{s}_{-j})}{L_j}, \\ \frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j}) &> \frac{p_j(0, \mathbf{s}_{-j})}{L_j}. \end{aligned} \tag{41}$$

As $\frac{\partial r_j}{\partial s_j}(0, s_{-j}) \uparrow \frac{p_j(0, \mathbf{s}_{-j})}{L_j}$ implies $s_j^*(\mathbf{s}_{-j}) \downarrow 0$, we get the continuity at point $\frac{\partial r_j}{\partial s_j}(0, \mathbf{s}_{-j}) = \frac{p_j(0, \mathbf{s}_{-j})}{L_j}$. Hence, the mapping $\mathbf{S}^*$ defined in (3) is also continuous.

Therefore, by Brouwer's fixed point Theorem, there exists an equilibrium stock selling strategy given by $s^{**} = (s_1^*(\mathbf{s}_{-1}^{**}), ..., s_m^*(\mathbf{s}_{-m}^{**})) \in \prod_{j=1}^m [0, S_j]$. $\square$

We now turn to focus on the uniqueness of the Nash equilibrium strategy.

Define

$$F(\mathbf{s}) = \left(\frac{\partial u_1}{\partial s_1}(\mathbf{s}), ..., \frac{\partial u_m}{\partial s_m}(\mathbf{s}) \right)^T, \tag{42}$$

and let $J(\mathbf{s})$ be the Jacobian matrix of the mapping F, given by

$$(J(\mathbf{s}))_{k,j} = \frac{\partial^2 u_k}{\partial s_j \partial s_k}(\mathbf{s}), \; j, k = 1, .., m. \tag{43}$$

Lemma 2.6. Under Assumptions 2.1, suppose also that the Jacobian matrix $J(\mathbf{s})$ is strictly diagonally dominant for on $\prod_{i=1}^{m}[0, S_i]$, i.e.

$$\left|\frac{\partial^2 u_i}{\partial s_i^2}\right| > \sum_{k=1,k\neq i}^{m}\left|\frac{\partial^2 u_i}{\partial s_k \partial s_i}\right|,\ i = 1, ..., m, \tag{44}$$

then there exists a unique Nash equilibrium.

Proof. According to Theorem 2.5, there exists a fixed point of the mapping $\mathbf{S}^*$. This is a Nash equilibrium. We prove the uniqueness of the fixed point by contradiction. Suppose that there are two distinct fixed points $\mathbf{S}^*(\mathbf{s}) = \mathbf{s},\ \mathbf{S}^*(\tilde{\mathbf{s}}) = \tilde{\mathbf{s}},$ for $\mathbf{s}, \tilde{\mathbf{s}} \in \prod_{i=1}^{m}[0, S_i], \mathbf{s} \neq \tilde{\mathbf{s}}$. The goal is to show that in L_∞ norm

$$\|\tilde{\mathbf{s}} - \mathbf{s}\|_\infty = \left\|\mathbf{S}^*\left(\tilde{\mathbf{s}}\right) - \mathbf{S}^*\left(\mathbf{s}\right)\right\|_\infty < \|\tilde{\mathbf{s}} - \mathbf{s}\|_\infty, \tag{45}$$

thereby reaching a contradiction.

Let $j \in \{1, 2, .., m\}$. There are three possible cases for the relationships between the values of $\frac{\partial u_j}{\partial s_j}\left(s_j^*\left(\mathbf{s}_{-j}\right), \mathbf{s}_{-j}\right)$ and $\frac{\partial u_j}{\partial s_j}\left(s_j^*\left(\tilde{\mathbf{s}}_{-j}\right), \tilde{\mathbf{s}}_{-j}\right)$.
In case when

$$\frac{\partial u_j}{\partial s_j}\left(s_j^*\left(\mathbf{s}_{-j}\right), \mathbf{s}_{-j}\right) = \frac{\partial u_j}{\partial s_j}\left(s_j^*\left(\tilde{\mathbf{s}}_{-j}\right), \tilde{\mathbf{s}}_{-j}\right), \tag{46}$$

then from Gabay & Moulin (1978) [Theorem 4.1], we get that

$$\left|s_j^*\left(\mathbf{s}_{-j}\right) - s_j^*\left(\tilde{\mathbf{s}}_{-j}\right)\right| < \|\mathbf{s} - \tilde{\mathbf{s}}\|_\infty. \tag{47}$$

Next, consider the case when

$$\frac{\partial u_j}{\partial s_j}\left(s_j^*\left(\mathbf{s}_{-j}\right), \mathbf{s}_{-j}\right) < \frac{\partial u_j}{\partial s_j}\left(s_j^*\left(\tilde{\mathbf{s}}_{-j}\right), \tilde{\mathbf{s}}_{-j}\right). \tag{48}$$

In this case, either $s_j^*\left(\mathbf{s}_{-j}\right)$ or $s_j^*\left(\tilde{\mathbf{s}}_{-j}\right)$ is not an internal point of the interval $[0, S_j]$ (or both). Thus either $s_j^*\left(\mathbf{s}_{-j}\right) \in \{0, S_j\}$ or $s_j^*\left(\tilde{\mathbf{s}}_{-j}\right) \in \{0, S_j\}$. If $s_j^*\left(\mathbf{s}_{-j}\right) = s_j^*\left(\tilde{\mathbf{s}}_{-j}\right)$, then we trivially have that $0 = \left|s_j^*\left(\mathbf{s}_{-j}\right) - s_j^*\left(\tilde{\mathbf{s}}_{-j}\right)\right| < \|\mathbf{s} - \tilde{\mathbf{s}}\|_\infty$. Otherwise, $s_j^*\left(\mathbf{s}_{-j}\right) \neq s_j^*\left(\tilde{\mathbf{s}}_{-j}\right)$. Then, from the optimality of the mapping s_j^*, it follows that either

$$s_j^*\left(\mathbf{s}_{-j}\right) = 0, s_j^*\left(\tilde{\mathbf{s}}_{-j}\right) \in (0, S_j), \tag{49}$$

or

$$s_j^*\left(\mathbf{s}_{-j}\right) \in (0, S_j), s_j^*\left(\tilde{\mathbf{s}}_{-j}\right) = S_j, \tag{50}$$

or

$$s_j^*\left(\mathbf{s}_{-j}\right) = 0, s_j^*\left(\tilde{\mathbf{s}}_{-j}\right) = S_j. \tag{51}$$

Therefore in either case $s_j^*\left(\mathbf{s}_{-j}\right) < s_j^*\left(\tilde{\mathbf{s}}_{-j}\right)$. Define $\mathbf{x} = \left(s_j^*\left(\mathbf{s}_{-j}\right), \mathbf{s}_{-j}\right)$ and $\tilde{\mathbf{x}} = \left(\mathbf{s}_j^*\left(\tilde{\mathbf{s}}_{-j}\right), \tilde{\mathbf{s}}_{-j}\right)$.
Let

$$f(t) = \frac{\partial u_j\left(\mathbf{x} + t(\tilde{\mathbf{x}} - \mathbf{x})\right)}{\partial s_j}. \tag{52}$$

According to Lagrange Mean Value Theorem, there exists $t_0 \in (0, 1)$ such that

$$f'(t_0) = \sum_{i=1,i\neq j}^{m}\frac{\partial^2 u_j\left(\mathbf{y}\right)}{\partial s_j \partial s_i}\left(\tilde{s}_i - s_i\right) + \frac{\partial^2 u_j\left(\mathbf{y}\right)}{\partial s_j^2}\left(s_j^*\left(\tilde{\mathbf{s}}_{-j}\right) - s_j^*\left(\mathbf{s}_{-j}\right)\right) > 0, \tag{53}$$

where $\mathbf{y} = \mathbf{x} + t_0 (\tilde{\mathbf{x}} - \mathbf{x})$. From which we conclude that

$$-\frac{\partial^2 u_j(\mathbf{y})}{\partial s_j^2}\left(s_j^*(\tilde{\mathbf{s}}_{-j}) - s_j^*(\mathbf{s}_{-j})\right) < \sum_{i=1, i\neq j}^{m} \frac{\partial^2 u_j(\mathbf{y})}{\partial s_j \partial s_i}(\tilde{s}_i - s_i). \tag{54}$$

By concavity of function u_j, together with the strictly diagonal dominance condition (44), we get that $\frac{\partial^2 u_j(\mathbf{y})}{\partial s_j^2} < 0$ is strictly negative. Recall that $s_j^*(\mathbf{s}_{-j}) < s_j^*(\tilde{\mathbf{s}}_{-j})$, thus it follows that the left hand side of (54) is positive. Therefore

$$\left|\frac{\partial^2 u_j(\mathbf{y})}{\partial s_j^2}\right|\left(s_j^*(\tilde{\mathbf{s}}_{-j}) - s_j^*(\mathbf{s}_{-j})\right) < \left|\sum_{i=1, i\neq j}^{m} \frac{\partial^2 u_j(\mathbf{y})}{\partial s_j \partial s_i}(\tilde{s}_i - s_i)\right| \leq \sum_{i=1, i\neq j}^{m} \left|\frac{\partial^2 u_j(\mathbf{y})}{\partial s_j \partial s_i}(\tilde{s}_i - s_i)\right| \tag{55}$$

$$\leq \sum_{i=1, i\neq j}^{m} \left|\frac{\partial^2 u_j(\mathbf{y})}{\partial s_j \partial s_i}\right| \|\tilde{\mathbf{s}} - \mathbf{s}\|_\infty < \left|\frac{\partial^2 u_j(\mathbf{y})}{\partial s_j^2}\right| \|\tilde{\mathbf{s}} - \mathbf{s}\|_\infty, \tag{56}$$

where the last inequality follows from the strictly diagonal dominance property. We conclude that

$$|s_j^*(\tilde{\mathbf{s}}_{-j}) - s_j^*(\mathbf{s}_{-j})| < \|\tilde{\mathbf{s}} - \mathbf{s}\|_\infty. \tag{57}$$

The case of reverse inequality in (48) can be treated by renaming $\mathbf{s}$ and $\tilde{\mathbf{s}}$.

We have showed that

$$\left|s_j^*(\mathbf{s}_{-j}) - s_j^*(\tilde{\mathbf{s}}_{-j})\right| < \|\mathbf{s} - \tilde{\mathbf{s}}\|_\infty, \forall j \in \{1, 2, .., m\}, \tag{58}$$

and thus (45) follows, and we reach the desired contradiction. Hence, the fixed point of the mapping $\mathbf{S}^*(\cdot)$ is unique. □

The strictly diagonally dominant condition of $J(\mathbf{s})$ is not very intuitive, therefore we formulate the uniqueness theorem with slightly more financially intuitive condition

Theorem 2.7. *(Uniqueness of Nash Equilibrium) In addition to Assumptions 2.1, suppose also that for each $j = 1, 2, .., m$,*

$$-s_j \frac{\partial^2 p_j}{\partial s_j^2}(\mathbf{s}) + L_j \frac{\partial^2 r_j}{\partial s_j^2}(\mathbf{s}) \geq \sum_{i=1, i\neq j}^{m} \left| s_j \frac{\partial^2 p_j}{\partial s_i \partial s_j}(\mathbf{s}) - L_j \frac{\partial^2 r_j}{\partial s_i \partial s_j}(\mathbf{s})\right|, \tag{59}$$

then the Nash equilibrium $\mathbf{s}^{**}$ is unique.

Proof. According to Lemma 2.6, it suffices to prove the Jacobian matrix $J(\mathbf{s})$ is strictly diagonally dominant. We calculate that

$$J(\mathbf{s}) = 2\begin{bmatrix} \frac{\partial p_1}{\partial s_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\partial p_m}{\partial s_m} \end{bmatrix} + \begin{bmatrix} s_1 \frac{\partial^2 p_1}{\partial s_1^2} - L_1 \frac{\partial^2 r_1}{\partial s_1^2} & \ldots & s_1 \frac{\partial^2 p_1}{\partial s_m \partial s_1} - L_1 \frac{\partial^2 r_1}{\partial s_m \partial s_1} \\ \ldots & \ldots & \ldots \\ s_m \frac{\partial^2 p_m}{\partial s_1 \partial s_m} - L_m \frac{\partial^2 r_m}{\partial s_1 \partial s_m} & \ldots & s_m \frac{\partial^2 p_m}{\partial s_m^2} - L_m \frac{\partial^2 r_m}{\partial s_m^2} \end{bmatrix}. \tag{60}$$

Condition (59) ensures that the second matrix on the right hand side is a diagonally dominant matrix with negative diagonal elements. Since the stock price p_j is a strictly decreasing function with respect to s_j, $\frac{\partial p_j}{\partial s_j} < 0$, the sum of the diagonal elements

$$2\frac{\partial p_j}{\partial s_j} + s_j\frac{\partial^2 p_j}{\partial s_j^2} - L_j\frac{\partial^2 r_j}{\partial s_j^2} < s_j\frac{\partial^2 p_j}{\partial s_j^2} - L_j\frac{\partial^2 r_j}{\partial s_j^2} < 0. \tag{61}$$

Then, for any $j \in \{1, 2, ..., m\}$,

$$\left|2\frac{\partial p_j}{\partial s_j} + s_j\frac{\partial^2 p_j}{\partial s_j^2} - L_j\frac{\partial^2 r_j}{\partial s_j^2}\right| > -s_j\frac{\partial^2 p_j}{\partial s_j^2} + L_j\frac{\partial^2 r_j}{\partial s_j^2} \geq \sum_{i=1, i\neq j}^{m}\left|s_j\frac{\partial^2 p_j}{\partial s_i \partial s_j} - L_j\frac{\partial^2 r_j}{\partial s_i \partial s_j}\right|. \tag{62}$$

That is the sum of the two matrix on the right hand side of equation (60), which equals $J(\mathbf{s})$, is strictly diagonally dominant. Hence, by Lemma 2.6, the equilibrium point is unique. □

Remark 2.8. The intuition behind the condition (59) is that it ensures that the effect of bank's own transactions on its marginal cost is greater than the combined effects of all other banks.

Note also that the condition of Lemma 2.6 is very closely related to the sufficient condition of Rosen (1965)[Theorem 6]. The latter being that $J(\mathbf{s}) + J(\mathbf{s})^T$ is positive definite. However, they are not equivalent as there exists strictly (row) diagonally dominant matrices J, not satisfying the positiveness condition of $J(\mathbf{s}) + J(\mathbf{s})^T$. Finally, the simple self-containing proof is itself noteworthy.

Example 2.9. Consider $m \geq 2$ identical banks. Suppose that the price function is given by the linear inverse demand function from Example 2.3.a, and the quadratic interest rate function is given in Example 2.4.b. Then, the sufficient and necessary condition for Nash equilibrium $(s_1^*, ..., s_m^*)$ is

$$\frac{du_j}{ds}(s_j^*, \mathbf{s}_{-j}^*)(s - s_j^*) \leq 0, \; s \in [0, S_j], j = 1, ..., m. \tag{63}$$

Indeed, fix $1 \leq j \leq m$, if $s_j^* \in (0, S_j)$ then to satisfy (63), we must have

$$\frac{du_j}{ds}(s_j^*, \mathbf{s}_{-j}^*) = 0. \tag{64}$$

Moreover, from Assumption 2.1, it follows that u_j are concave functions, and therefore $u_j(\cdot, \mathbf{s}_{-j}^*)$ is maximized at s_j^*. If s_j^* is on the boundary of $[0, S_j]$, in order to satisfy (63) we must have

$$\begin{cases} \frac{du_j}{ds}(s_j^*, \mathbf{s}_{-j}^*) \leq 0, & s_j^* = 0 \\ \frac{du_j}{ds}(s_j^*, \mathbf{s}_{-j}^*) \geq 0, & s_j^* = S_j. \end{cases} \tag{65}$$

In either case, from concavity, it again follows that u_j is maximized at $(s_1^*, ..., s_m^*)$. For convenience we will drop the index identifying the bank, and denote

$$S_1 = ... = S_m = S, \; L_1 = ... = L_m = L, \; a_1 = ... = a_m = a, \tag{66}$$
$$p_1(0, \mathbf{0}) = ... = p_m(0, \mathbf{0}) = p^0, \; \beta_{i,1} = ... = \beta_{i,m} = \beta_i, \; i = 1, 2. \tag{67}$$

As the optimal strategies for the m identical banks must be the same, denote a Nash equilibrium point for this game $\mathbf{s}^* = (s^*, s^*, ..., s^*) \in \mathbb{R}^m$, and $\mathbf{s}_-^* = (s^*, ..., s^*) \in \mathbb{R}^{m-1}$. From (63), the Nash equilibrium $\mathbf{s}^*$ is the unique solution to

$$\frac{du}{ds}(s^*, \mathbf{s}_-^*)(s - s^*) = \Big(-((2a + (m-1)\epsilon)p^0 + 2L\beta_2)s^* + p^0 - L\beta_1\Big)(s - s^*) \leq 0, \quad \text{for all } s \in [0, S]. \tag{68}$$

Moreover, assuming that $\epsilon < 2a/(m-1)$ guarantees that the condition of Theorem 2.7 holds, so that the Nash equilibrium point is unique. Therefore, either there exists $s^* \in (0, S)$ such that $\frac{du}{ds}(s^*, \mathbf{s}^*_-) = 0$, or for $s^* = 0$, we must have $\frac{du}{ds}(s^*, \mathbf{s}^*_-) \leq 0$, or for $s^* = S$, we must have $\frac{du}{ds}(s^*, \mathbf{s}^*_-) \geq 0$. A calculation then shows that the optimal choice for each bank is

$$s^* = \left(0 \vee \frac{p^0 - L\beta_1}{(2a + (m-1)\epsilon)p^0 + 2L\beta_2}\right) \wedge S. \tag{69}$$

Thus, either the banks sells nothing, $s^* = 0$, or it sells all its available stock $s^* = S$, or its stock sale is proportional to $p^0 - L\beta_1$, which is the maximum marginal profit the banks can get from a stock sale. In this case, the sale is also inversely proportional to its marginal cost $2ap^0 + 2L\beta_2$ together with the aggregate marginal cost of the other banks $(m-1)\epsilon p^0$.

3 Optimal Strategy Selling Stock and Borrowing

We now relax the original assumption and allow banks to borrow cash, in addition to selling stock. Our goal is still to find the maximum amount of funds the banks in the system can raise. This completes the contagion circle. Previously, a bank selling stock, will lower the stock price for all banks, including itself, and also raise the short-term interest rate, again for all banks. Now, if a bank takes on additional debt, we assume that it will both lower the stock price for the bank (and possibly for other banks), and raise the interest rate for all the banks in the system. This is again because of the decreasing confidence in this, and (possibly) other banks. Though note that borrowing, as a debt, does not affect the total equity directly, according to International Accounting Standards Committee and International Accounting Standards Board (2000). To accommodate this change, we change the definitions introduced in the previous section by adding the dependency on debt d, in addition to their dependencies on stock sale s. Denote

$$(\mathbf{s}, \mathbf{d}) = (s_1, ..., s_m, d_1, ..., d_m), \tag{70}$$

and let

$$(\mathbf{s}_{-i}, \mathbf{d}_{-i}) = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_m, d_1, ..., d_{i-1}, d_{i+1}, ..., d_m), \ i = 1, ..., m, \tag{71}$$

where d_j denotes the amount of money the bank $j \in \{1, ..., m\}$ borrows. Then bank j stock price function becomes $p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})$. After the joint stock sale and borrowing transaction, the banks book value and market capitalization become

$$B_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = B_j(0, 0, \mathbf{0}, \mathbf{0}) + p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})s_j \tag{72}$$

and

$$C_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})(N_j + s_j). \tag{73}$$

Similarly, the overnight interest rate now becomes

$$r_j = r_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}). \tag{74}$$

Hence, for any fixed $j \in 1, 2, ..., m$, the amount of funds the bank j can raise as a function of (s_j, d_j) given the actions of other banks $(\mathbf{s}_{-j}, \mathbf{d}_{-j})$ is

$$v_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = s_j p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) + d_j - L_j\left(r_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) - r_j(0, 0, \mathbf{0}, \mathbf{0})\right), \tag{75}$$

The effect of borrowing on stock price is complicated. When a large number of banks face shortfall, the common sense is the stock prices decrease as borrowing increases, whereas the marginal costs increases. These facts suggest that the stock price function p_j is a decreasing concave function with respect to $\mathbf{d} \in \prod_{i=1}^m [0, D_i]$, for any given $\mathbf{s} \in \prod_{i=1}^m [0, S_i]$, where D_j is the maximum amount of cash the bank $j \in \{1, ..., m\}$ can borrow.

Example 3.1. Examples of inverse demand price functions p_j , $j = 1, ..., m$ satisfying Assumption 2.1.1 include:

a. Linear inverse demand:

$$\begin{aligned} &p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = p_j(0, 0, \mathbf{0}, \mathbf{0})(1 - a_j s_j - b_j d_j - \epsilon_1 \sum_{k=1,k\neq j}^{m} s_k - \epsilon_2 \sum_{k=1,k\neq j}^{m} d_k), \\ &0 < a_j < \frac{1}{2S_j}, 0 < b_j < \frac{1}{3D_j}, 0 < \epsilon_1 \ll a_j, 0 < \epsilon_2 \ll b_j. \end{aligned} \tag{76}$$

b. 2^{nd} order inverse demand:

$$\begin{aligned} &p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = p_j(0, 0, \mathbf{0}, \mathbf{0})(1 - a_j s_j - b_j d_j^2 - \epsilon_1 \sum_{k=1,k\neq j}^{m} s_k - \epsilon_2 \sum_{k=1,k\neq j}^{m} d_k^2), \\ &0 < a_j < \frac{1}{2S_j}, 0 < b_j < \frac{1}{3D_j^2}, 0 < \epsilon_1 \ll a_j, 0 < \epsilon_2 \ll b_j. \end{aligned} \tag{77}$$

c. Exponential inverse demand:

$$\begin{aligned} &p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = p_j(0, 0, \mathbf{0}, \mathbf{0}) exp(-a_j s_j - b_j d_j^2 - \epsilon_1 \sum_{\substack{k=1\\k\neq j}}^{m} s_k - \epsilon_2 \sum_{\substack{k=1\\k\neq j}}^{m} d_k^2), \\ &0 < a_j < \frac{1}{S_j}, 0 < b_j < \frac{1}{D_j^2}, 0 < \epsilon_1 \ll a_j, 0 < \epsilon_2 \ll b_j. \end{aligned} \tag{78}$$

Example 3.2. Examples of interest rate functions $r_j,\ j = 1, ..., m$ that satisfy Assumption 2.1.2 include:

a. Linear interest rate model:

$$\begin{aligned} &r_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = \beta_{0,j} + \beta_{1,j} s_j + \beta_{2,j} \sum_{i=1,i\neq j}^{m} s_i + \beta_{3,j} d_j + \beta_{4,j} \sum_{i=1,i\neq j}^{m} d_i, \\ &\beta_{k,j} > 0, \ , k = 0, 1, ..., 4, j = 1, ..., m. \end{aligned} \tag{79}$$

b. Quadratic interest rate model:

$$\begin{aligned} &r_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = \beta_{0,j} + \beta_{1,j} s_j + \beta_{2,j} s_j^2 + \beta_{3,j} \sum_{i=1,i\neq j}^{m} s_i + \beta_{4,j} \sum_{i=1,i\neq j}^{m} s_i^2 + \beta_{5,j} d_j + \beta_{6,j} d_j^2 \\ &+ \beta_{7,j} \sum_{i=1,i\neq j}^{m} d_i + \beta_{8,j} \sum_{i=1,i\neq j}^{m} d_i^2, \\ &\beta_{k,j} > 0, \ k = 0, 1, .., 8, j = 1, ..., m. \end{aligned} \tag{80}$$

c. Interest rate as a function of book-to-price ratio:

$$r_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = \beta_{0,j} + \beta_{1,j}\frac{B_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})}{C_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})} + \beta_{2,j}\sum_{i\neq j}\frac{B_i(s_i, d_i, \mathbf{s}_{-i}, \mathbf{d}_{-i})}{C_i(s_i, d_i, \mathbf{s}_{-i}, \mathbf{d}_{-i})}, \quad \beta_{k,j} > 0,\ k = 0,1,2, j = 1,...,m. \tag{81}$$

The calculation showing that this interest rate model satisfies Assumption 2.1.2 is identical to the one in Example 2.4.c, because they have identical dependencies on $\mathbf{s}$.

Define a mapping $\bar{F}\colon \prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i] \to \mathbb{R}^{2m}$ as $\bar{F}(\mathbf{s}, \mathbf{d}) = \left(\frac{\partial v_1}{\partial s_1}, \frac{\partial v_1}{\partial d_1}, \cdots, \frac{\partial v_m}{\partial s_m}, \frac{\partial v_m}{\partial d_m}\right)^T$, and let $\bar{J}(\mathbf{s}, \mathbf{d})$ be the Jacobian matrix of $\bar{F}$.

Similar to the argument in Section 2, we assume that all the banks in the system will take actions at the same time so that this problem becomes a "m"-person non-cooperate game. Hence, our goal is to investigate the existence and uniqueness of Nash equilibrium in this system. The following assumption is then needed to give a definite answer.

Assumption 3.3. 1. For each $j = 1, ..., m$, given $(\mathbf{s}_{-j}, \mathbf{d}_{-j}) \in \prod_{i=1,i\neq j}^m [0, S_i] \times \prod_{i=1,i\neq j}^m [0, D_i]$, the function $v_j(\cdot, \cdot; \mathbf{s}_{-j}, \mathbf{d}_{-j}) : [0, S_j] \times [0, D_j] \mapsto \mathbb{R}$ is strictly concave and twice differentiable.

2. The Jacobian matrix $\bar{J}$ is strictly (row) diagonally dominant on $\prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i]$.

Equivalently to (2), we define the optimization mapping

$$(s_j^*(\mathbf{s}_{-j}, \mathbf{d}_{-j}), d_j^*(\mathbf{s}_{-j}, \mathbf{d}_{-j}))\colon \prod_{\substack{i=1\\ i\neq j}}^m [0, S_i] \times \prod_{\substack{i=1\\ i\neq j}}^m [0, D_i] \to [0, S_j] \times [0, D_j],\ j = 1, ..., m, \tag{82}$$

as

$$(s_j^*(\mathbf{s}_{-j}, \mathbf{d}_{-j}), d_j^*(\mathbf{s}_{-j}, \mathbf{d}_{-j})) = \underset{(s,d)\in[0,S_j]\times[0,D_j]}{\arg\max}\ v_j(s, d, \mathbf{s}_{-j}, \mathbf{d}_{-j}). \tag{83}$$

Theorem 3.4. *Under Assumption 3.3 the Nash equilibrium for the optimization problem (83)*

$$(\mathbf{s}^{**}, \mathbf{d}^{**}) = (s_1^*(\mathbf{s}_{-1}^{**}, \mathbf{d}_{-1}^{**}), ..., s_m^*(\mathbf{s}_{-m}^{**}, \mathbf{d}_{-m}^{**}), d_1^*(\mathbf{s}_{-1}^{**}, \mathbf{d}_{-1}^{**}), ..., d_m^*(\mathbf{s}_{-m}^{**}, \mathbf{d}_{-m}^{**})) \tag{84}$$

exists and is unique.

Proof. Define $T^* : \prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i] \to \prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i]$ as

$$T^*(\mathbf{s}, \mathbf{d}) = (s_1^*(\mathbf{s}_{-1}, \mathbf{d}_{-1}), ..., s_m^*(\mathbf{s}_{-m}, \mathbf{d}_{-m}), d_1^*(\mathbf{s}_{-1}, \mathbf{d}_{-1}), ..., d_m^*(\mathbf{s}_{-m}, \mathbf{d}_{-m}))^T, \tag{85}$$

The first step is to show that T^* is subcontraction mapping, that is to show that for any two distinct points $(\mathbf{s}^0, \mathbf{d}^0), (\mathbf{s}^1, \mathbf{d}^1) \in \prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i]$, we have that

$$\left\|T^*(\mathbf{s}^0, \mathbf{d}^0) - T^*(\mathbf{s}^1, \mathbf{d}^1)\right\|_\infty < \left\|(\mathbf{s}^0, \mathbf{d}^0) - (\mathbf{s}^1, \mathbf{d}^1)\right\|_\infty. \tag{86}$$

Fix $j \in \{1, 2, ..., m\}$, and note that it is sufficient to show that

$$\left|s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) - s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)\right| < \left\|(\mathbf{s}^0, \mathbf{d}^0) - (\mathbf{s}^1, \mathbf{d}^1)\right\|_\infty, \tag{87}$$

$$\left|d_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) - d_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)\right| < \left\|(\mathbf{s}^0, \mathbf{d}^0) - (\mathbf{s}^1, \mathbf{d}^1)\right\|_\infty. \tag{88}$$

Consider (87) first. If $s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) = s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)$, then

$$0 = \left| s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) - s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0) \right| < \left\| (\mathbf{s}^0, \mathbf{d}^0) - (\mathbf{s}^1, \mathbf{d}^1) \right\|_\infty \tag{89}$$

and there is nothing to prove. Thus assume that $s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) \neq s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)$.

Recall that $\left(s_j^*(\mathbf{s}_{-j}^i, \mathbf{d}_{-j}^i), d_j^*(\mathbf{s}_{-j}^i, \mathbf{d}_{-j}^i)\right)$, $i = 0, 1$ are the maximizers of (83). Then consider the following cases, based on whether the partial derivatives are zero or not.

Let

$$\mathbf{x}_j^i = (s_j^*(\mathbf{s}_{-j}^i, \mathbf{d}_{-j}^i), d_j^*(\mathbf{s}_{-j}^i, \mathbf{d}_{-j}^i), \mathbf{s}_{-j}^i, \mathbf{d}_{-j}^i), \ i = 0, 1, \tag{90}$$

and consider the case when

$$\frac{\partial v_j(\mathbf{x}_j^1)}{\partial s_j} = \frac{\partial v_j(\mathbf{x}_j^0)}{\partial s_j}. \tag{91}$$

Define

$$\phi(\theta) = \frac{\partial v_j \left(\mathbf{x}_j^0 + \theta(\mathbf{x}_j^1 - \mathbf{x}_j^0)\right)}{\partial s_j}, \ \theta \in [0, 1]. \tag{92}$$

Then, $\phi(0) = \phi(1)$. By Rolle's theorem, there is a $\theta_0 \in (0, 1)$ such that $\phi'(\theta_0) = 0$. Let

$$\mathbf{z}_0 = \mathbf{x}_j^0 + \theta_0(\mathbf{x}_j^1 - \mathbf{x}_j^0), \tag{93}$$

then

$$\phi'(\theta_0) = \sum_{i=1, i\neq j}^m \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial s_i}(s_i^1 - s_i^0) + \sum_{i=1}^m \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial d_i}(d_i^1 - d_i^0) + \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j^2}(s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) - s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)) = 0. \tag{94}$$

Then, from the triangular inequality it follows that

$$\left| \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j^2} \right| \left| s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) - s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0) \right| \leq \sum_{i=1, i\neq j}^m \left| \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial s_i} \right| \left| s_i^1 - s_i^0 \right| + \sum_{i=1}^m \left| \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial d_i} \right| \left| d_i^1 - d_i^0 \right|. \tag{95}$$

From strictly diagonal dominance condition of $\bar{J}$, it follows that

$$\begin{aligned} &\left(\sum_{i=1, i\neq j}^m \left| \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial s_i} \right| + \sum_{i=1}^m \left| \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial d_i} \right| \right) \left\| (\mathbf{s}^0, \mathbf{d}^0) - (\mathbf{s}^1, \mathbf{d}^1) \right\|_\infty \\ &< \left| \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j^2} \right| \left\| (\mathbf{s}^0, \mathbf{d}^0) - (\mathbf{s}^1, \mathbf{d}^1) \right\|_\infty . \end{aligned} \tag{96}$$

Combining inequality (95) and (96), we get

$$\left| s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) - s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0) \right| < \left\| (\mathbf{s}^0, \mathbf{d}^0) - (\mathbf{s}^1, \mathbf{d}^1) \right\|_\infty . \tag{97}$$

Next, consider the case that

$$\frac{\partial v_j(\mathbf{x}_j^0)}{\partial s_j} < \frac{\partial v_j(\mathbf{x}_j^1)}{\partial s_j}. \tag{98}$$

In this case, we assert that

$$s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0) \leq s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1). \tag{99}$$

The condition (98) excludes the case $\frac{\partial v_j(\mathbf{x}_j^0)}{\partial s_j} = \frac{\partial v_j(\mathbf{x}_j^1)}{\partial s_j} = 0$, which is the case that both points $(s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0), d_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)), (s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1), d_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1))$ can be interior points of $[0, S_j] \times [0, D_j]$. Therefore we need to consider two scenarios:

First, when $\frac{\partial v_j(\tilde{\mathbf{s}}_j^0, \tilde{\mathbf{d}}_j^0)}{\partial s_j} < 0$, corresponds to the scenario when $s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0) = 0$. Therefore, we readily get that $0 = s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0) \leq s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1)$.

Otherwise, we must have that $\frac{\partial v_j(\tilde{\mathbf{s}}_j^1, \tilde{\mathbf{d}}_j^1)}{\partial s_j} > 0$. In this scenario, $s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) = S_j$, and thus $s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0) \leq s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) = S_j$.

Again, we may assume that $s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0) < s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1)$, and define $\phi(\theta)$ as in (92). According to Lagrange Mean Value theorem, there is $\theta_0 \in (0, 1)$ such that for $\mathbf{z}_0 = \mathbf{x}_j^0 + \theta_0(\mathbf{x}_j^1 - \mathbf{x}_j^0)$, we get

$$\phi'(\theta_0) = \sum_{i=1, i\neq j}^{m} \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial s_i}(s_i^1 - s_i^0) + \sum_{i=1}^{m} \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial d_i}(d_i^1 - d_i^0) + \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j^2}(s_j^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) - s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)) > 0. \tag{100}$$

We then have

$$-\frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j^2}(s^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) - s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)) < \sum_{i=1, i\neq j}^{m} \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial s_i}(s_i^1 - s_i^0) + \sum_{i=1}^{m} \frac{\partial^2 v_j(\mathbf{z}_0)}{\partial s_j \partial d_i}(d_i^1 - d_i^0). \tag{101}$$

By the concavity of v_j, the strictly diagonal dominance condition of $\bar{J}$, and the fact that $s^*(\mathbf{s}_{-j}^1, \mathbf{d}_{-j}^1) > s_j^*(\mathbf{s}_{-j}^0, \mathbf{d}_{-j}^0)$, the left hand side of (101) is positive. Then, from strictly diagonal dominant property and triangular inequality, we again get (97)

Similar argument shows (88), and therefore since (87) and (88) are true for any $j \in 1, 2, ...m$, it follows that T^* is subcontraction mapping.

In particular, T^* is a continuous function, and thus by Brouwer's fixed point Theorem there exists a fixed point for T^* in $\prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i]$. Such a fixed point (84) is therefore a Nash equilibrium, the uniqueness of which follows from the subcontraction property of T^*.

□

Example 3.5. Similar to Example 2.9, consider the case when the m banks are identical. Let the price function be the linear inverse demand function from Example 3.1.a, and the overnight interest rate model be the quadratic interest model from Example 3.2.b. For convenience we will drop the index identifying the bank, and in addition to the already defined notation in (66), denote also

$$D_1 = ... = D_m = D, \ p_1(0, 0, \mathbf{0}, \mathbf{0}) = ... = p_m(0, 0, \mathbf{0}, \mathbf{0}) = p^0, \tag{102}$$

$$\beta_{i,1} = ... = \beta_{i,m} = \beta_i, \ i = 1, 2, ..., 6. \tag{103}$$

As all the banks are identical, they must have the same optimal strategy denoted as (s^*, d^*). Also, let $\mathbf{s}_- = (s, ..., s) \in \mathbb{R}^{m-1}, \mathbf{d}_- = (d, ..., d) \in \mathbb{R}^{m-1}$. The identical banks also have the same objective function v, given by (75). Further assume that

$$\beta_2 > 0 \wedge \frac{(b + (m-1)\epsilon_1 + (m-1)\epsilon_2)p^0}{2L}, \text{ and } \beta_6 > \frac{bp^0}{2L}. \tag{104}$$

These conditions guarantee that the objective function v satisfies Assumption 3.3. The sufficient and necessary condition for (s^*, d^*) being the optimal strategy of each bank is,

$$\frac{\partial v}{\partial s}(s^*, d^*; \mathbf{s}^*_-, \mathbf{d}^*_-)(s - s^*) + \frac{\partial v}{\partial d}(s^*, d^*; \mathbf{s}^*_-, \mathbf{d}^*_-)(d - d^*) \leq 0, \text{for any } s \in [0, S], d \in [0, D], \tag{105}$$

where

$$\frac{\partial v}{\partial s}(s, d; \mathbf{s}_-, \mathbf{d}_-) = -((2a + (m-1)\epsilon_1)p^0 + 2\beta_2 L)s - (b + (m-1)\epsilon_2)p^0 d + p^0 - \beta_1 L, \tag{106}$$

$$\frac{\partial v}{\partial d}(s, d; \mathbf{s}_-, \mathbf{d}_-) = -bp^0 s - 2\beta_6 L d + 1 - \beta_5 L. \tag{107}$$

Given the concavity of v from Assumption 3.3, the intuition for this condition is similar to the intuition for the condition (63) in Example 2.9. Let the unconstrained optimal point (s^0, d^0) be the solution to the linear system $\frac{\partial v}{\partial s}(s, d; \mathbf{s}_-, \mathbf{d}_-) = 0$, $\frac{\partial v}{\partial s}(s, d; \mathbf{s}_-, \mathbf{d}_-) = 0$,

$$s^0 = \frac{2\beta_6 L(p^0 - \beta_1 L) - (b + (m-1)\epsilon_2)p^0(1 - \beta_5 L)}{2((2a + (m-1)\epsilon_1)p^0 + 2\beta_2 L)\beta_6 L - (b + (m-1)\epsilon_2)b(p^0)^2}, \tag{108}$$

$$d^0 = \frac{((2a + (m-1)\epsilon_1)p^0 + 2\beta_2 L)(1 - \beta_5 L) - bp^0(p^0 - \beta_1 L)}{2((2a + (m-1)\epsilon_1)p^0 + 2\beta_2 L)\beta_6 L - (b + (m-1)\epsilon_2)b(p^0)^2}. \tag{109}$$

We now need to consider the different cases, whenever $(s^0, d^0) \notin [0, S] \times [0, D]$. A technical, but straightforward calculation shows that the optimal strategy (s^*, d^*) is given by

$$(s^*, d^*) = \begin{cases} (s^0, d^0) & : (s^0, d^0) \in [0, S] \times [0, D], \\ (0, 0) & : s^0 < 0, d^0 < 0, \\ (S, D) & : s^0 > S, d^0 > D, \\ (0, \frac{1-\beta_5 L}{2\beta_6} \vee 0) & : s^0 < 0, d^0 \in [0, D], \\ (S, \frac{(1-\beta_5 L) - bp^0 S}{2\beta_6 L} \wedge D) & : s^0 > S, d^0 \in [0, D], \\ (0 \vee \frac{p^0 - \beta_1 L}{(2a + (m-1)\epsilon_1 p^0) + 2\beta_2 L}, 0) & : s^0 \in [0, S], d^0 < 0, \\ (\frac{p^0 - \beta_1 L - (b + (m-1)\epsilon_2)p^0 D}{(2a + (m-1)\epsilon_1 p^0) + 2\beta_2 L} \wedge S, D) & : s^0 \in [0, S], d^0 > D, \\ (0, 0 \vee \frac{1-\beta_5 L}{2\beta_6 L} \wedge D) & : s^0 < 0, d^0 > D, p^0 - \beta_1 L - (b + (m-1)\epsilon_2)D \leq 0, \\ (\frac{p^0 - \beta_1 L - (b + (m-1)\epsilon_2)p^0 D}{(2a + (m-1)\epsilon_1 p^0) + 2\beta_2 L} \wedge S, D) & : s^0 < 0, d^0 > D, p^0 - \beta_1 L - (b + (m-1)\epsilon_2)D > 0, \\ (0 \vee \frac{p^0 - \beta_1 L}{(2a + (m-1)\epsilon_1) + 2\beta_2 L} \wedge S, 0) & : s^0 > S, d^0 < 0, 1 - \beta_5 L - bp^0 S \leq 0, \\ (S, \frac{1 - \beta_5 L - bp^0 S}{2\beta_6 L} \wedge D) & : s^0 > S, d^0 < 0, 1 - \beta_5 L - bp^0 S > 0. \end{cases} \tag{110}$$

When $(s^*, d^*) = (s^0, d^0)$, the optimal number of stock to sell and the optimal amount of money to borrow are inversely proportional to the difference between two terms. The first term is the the product of $(2a + (m-1)\epsilon_1)p^0$ – the marginal cost from stock sale with respect to the amount of stock sale s, and $2\beta_6 L$ – the marginal cost of borrowing with respect to d. The second term is the product of exogenous effects: $(b + (m-1)\epsilon_2)p^0$ – the marginal cost in stock sale with respect to borrowing d, and bp^0 – the marginal cost of borrowing with respect to s. While the magnitude of optimal stock sale depends on the difference between two terms . The first term being the product of marginal profit from the stock sale with respect to s and the marginal cost of borrowing with respect to d. While the second term is the product of the marginal profit for borrowing with respect to d and the marginal cost from the stock price with respect to d.

4 Optimal Strategy to Recover Shortfall

In this section, we turn to discuss one more case that the banks in the system face a shortfall and need to raise funds to cover it. This is a more classical scenario used e.g. in Bichuch & Feinstein (2019). Without loss of generality, assume that all banks $j = 1, ..., m$ in the system face shortfall $M_j > 0$, otherwise, the system can be shrunk accordingly, and that none of the banks are in default. In other words, they can sell and borrow so as to be able to cover their debt. Otherwise, similar to Bichuch & Feinstein (2019), we may assume that these banks will not make any transaction and will be taken over by the regulator. We also continue with our previous assumptions that the bank j, $j = 1, 2, 3, ..., m$ can raise fund through stock sale and borrowing, and it can sell at most S_j shares of its stock and can borrow at most D_j dollars. Then, the cost of such a transaction as a function of $(\mathbf{s}, \mathbf{d})$ to the bank j is given by

$$w_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = s_j(p_j(0, 0; \mathbf{0}, \mathbf{0}) - p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})) + r_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})d_j. \tag{111}$$

Here, the first and the second terms represent the opportunity loss due to selling the stock at the reduced stock price, and the cost of the newly issued debt respectively. The purpose of each bank is to recover its shortfall with financing cost as small as possible. That is for the bank j minimizes

$$\min_{(s_j, d_j) \in [0, S_j] \times [0, D_j]} w_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}), \text{ subject to } d_j + s_j p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = M_j. \tag{112}$$

As we argued in previous section, debt financing does not pull down the stock price significantly, especially when the amount of borrowing is not so big. As the shortfall needed to be recovered, M_j, is not very large, for any given number of shares of its stock s_j, the total amount of raising fund increases with respect to borrowing d_j. These facts lead to the assumptions below, with which we can proof the existence and uniqueness of Nash equilibrium strategy in this scenario.

Assumption 4.1.

1. $(\mathbf{s}, \mathbf{d}) \mapsto d_j + s_j p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})$ is a concave, twice continuously differentiable function, $j = 1, 2, ..., m$.
2. $D_j > M_j, j = 1, 2, ..., m$.
3. For each $j = 1, ..., m$, given $(\mathbf{s}_{-j}, \mathbf{d}_{-j}) \in \prod_{i=1, i\neq j}^m [0, S_i] \times \prod_{i=1, i\neq j}^m [0, D_i]$,
$$w_j(\cdot, \cdot; \mathbf{s}_{-j}, \mathbf{d}_{-j}) : [0, S_j] \times [0, D_j] \mapsto \mathbb{R} \tag{113}$$
is a strictly convex twice differentiable function.
4. Let $\tilde{F}(\mathbf{s}, \mathbf{d}) = \left(\frac{\partial w_1}{\partial s_1}, \frac{\partial w_1}{\partial d_1}, \cdots, \frac{\partial w_m}{\partial s_m}, \frac{\partial w_m}{\partial d_m}\right)^T$, and let $\tilde{J}(\mathbf{s}, \mathbf{d})$ be the Jacobian matrix of $\tilde{F}$. $\tilde{J}(\mathbf{s}, \mathbf{d})$ is a strictly diagonal dominant matrix with any $(\mathbf{s}, \mathbf{d}) \in \prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i]$.

With these assumptions, we can prove the existence and uniqueness of Nash equilibrium in this game.

Theorem 4.2. *Under Assumptions 4.1, there exists an unique Nash equilibrium for the minimization problem (112), selling stock and borrowing strategy:*

$$(\mathbf{s}^{**}, \mathbf{d}^{**}) = (s_1^*(\mathbf{s}_{-1}^{**}, \mathbf{d}_{-1}^{**}), ..., s_m^*(\mathbf{s}_{-m}^{**}, \mathbf{d}_{-m}^{**}), d_1^*(\mathbf{s}_{-1}^{**}, \mathbf{d}_{-1}^{**}), ..., d_m^*(\mathbf{s}_{-m}^{**}, \mathbf{d}_{-m}^{**})) \in \prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i].$$

Proof. First, we relax the restriction and show that there is an unique Nash equilibrium for

$$\min_{(s_j,d_j)\in[0,S_j]\times[0,D_j]} w_j(s_j,d_j,\mathbf{s}_{-j},\mathbf{d}_{-j}), \text{ subject to } d_j + s_j p_j(s_j,d_j,\mathbf{s}_{-j},\mathbf{d}_{-j}) \geq M_j. \tag{114}$$

For any $j = 1,2,...,m$, let $U_j \subset \prod_{i=1}^m [0,S_i] \times \prod_{i=1}^m [0,D_i]$ be a set that satisfies constraint

$$d_j + s_j p_j(s_j,d_j,\mathbf{s}_{-j},\mathbf{d}_{-j}) \geq M_j \tag{115}$$

and $U = \bigcap_{j=1}^m U_j$. According to Assumption 4.1.2, $(\mathbf{0}, D_1, D_2, ..., D_m)$ must be in U, then we know U is not empty. By concavity and continuity of function $d_j + s_j p_j(s_j,d_j,\mathbf{s}_{-j},\mathbf{d}_{-j})$, Assumption 4.1.2, U_j is compact and convex for each $j = 1,2,...,m$, therefore so is U.

Let also $U_j^{(\mathbf{s}_{-j},\mathbf{d}_{-j})} = \{(s_j,d_j) : (\mathbf{s},\mathbf{d}) \in U_j\}$, $j = 1,...,m$. Then, any Nash equilibrium for the constrained optimization problem (114), must also be a Nash equilibrium to the optimization problem

$$\min_{(s_j,d_j)\in U_j^{(\mathbf{s}_{-j},\mathbf{d}_{-j})}} w_j(s_j,d_j,\mathbf{s}_{-j},\mathbf{d}_{-j}) \tag{116}$$

as well. And the opposite is also true. Hence, we only need to show the existence and uniqueness of Nash equilibrium of optimization problem (116) and this can be proved by similar argument of Theorem 3.4 with Assumption 4.1.3 and 4.1.4.

Next, let's prove that it is also the Nash equilibrium of constrained optimization problem (112). Assume $(\mathbf{s}^{**},\mathbf{d}^{**}) = (s_1^*(\mathbf{s}_{-1}^{**},\mathbf{d}_{-1}^{**}),...,s_m^*(\mathbf{s}_{-m}^{**},\mathbf{d}_{-m}^{**}),d_1^*(\mathbf{s}_{-1}^{**},\mathbf{d}_{-1}^{**}),...,d_m^*(\mathbf{s}_{-m}^{**},\mathbf{d}_{-m}^{**}))$ is the Nash equilibrium of optimization problem (114) with some $j \in \{1,2,...,m\}$ such that

$$d_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}) + s_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}) p_j(s_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}), d_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}), \mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}) > M_j. \tag{117}$$

Then, $(s_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}), d_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}))$ must be a interior point of $U_j^{(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**})}$. Hence, we can reduce $d_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**})$ a little to $\tilde{d}_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}))$ to make sure that $(s_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}), \tilde{d}_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}))$ is still in $U_j^{(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**})}$ so that the shortfall coverage constraint is still satisfied. With the fact that given $s_j, \mathbf{s}_{-j}$ and $\mathbf{d}_{-j}$, w_j is strictly decreasing function with respect to d_j, we have

$$w_j(s_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}), \tilde{d}_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}); \mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}) < w_j(s_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}), d_j^*(\mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}); \mathbf{s}_{-j}^{**},\mathbf{d}_{-j}^{**}), \tag{118}$$

which is a contradiction with the definition of Nash equilibrium. That is the Nash equilibrium

$$(\mathbf{s}^{**},\mathbf{d}^{**}) = (s_1^*(\mathbf{s}_{-1}^{**},\mathbf{d}_{-1}^{**}),...,s_m^*(\mathbf{s}_{-m}^{**},\mathbf{d}_{-m}^{**}),d_1^*(\mathbf{s}_{-1}^{**},\mathbf{d}_{-1}^{**}),...,d_m^*(\mathbf{s}_{-m}^{**},\mathbf{d}_{-m}^{**})) \tag{119}$$

of optimization problem (114) must satisfy the equality constraint for optimization problem (112) of any $j = 1,2,...,m$. Therefore, there exist an unique Nash equilibrium of optimization problem (112). □

Example 4.3. Consider again $m \geq 2$ identical banks, and consider the linear inverse demand function from Example 3.1.a with parameter $b = \epsilon_1 = \epsilon_2 = 0$ together with the linear interest rate function from Example 3.2.a. For convenience we will again drop the index identifying the bank, and use the same notation as in (102) and (103). Additionally, assume that

$$0 < \beta_2 < \frac{\beta_1}{2p^0}, \; 0 < \beta_4 < \frac{\beta_2}{2p^0}. \tag{120}$$

These constraints guarantee that Assumption 4.1 is satisfied. Recall the constraint in the optimization problem (112),

$$d = M - sp^0(1 - as). \tag{121}$$

We will assume that $M \geq \frac{p^0}{4a}$, which ensures that $d \geq 0$. Additionally, Assumption 2.1.1 and Assumption 4.1.2 guarantee that $d = M - p^0 s + ap^0 s^2 \leq D$ holds true. This eliminates the borrowing component in the original "m"-bank game (112), and the optimization problem becomes

$$\min_{s\in[0,S]} \tilde{w}(s, \mathbf{s}_{-j}),\ j = 1, 2, .., m, \tag{122}$$

where

$$\begin{aligned}
\tilde{w}(s; \mathbf{s}_{-j}) &= a^2(p^0)^2\beta_2 s^4 + \left(ap^0\beta_1 - 2a(p^0)^2\beta_2\right) s^3 \\
&+ \left(ap^0 + ap^0\beta_0 - p^0\beta_1 + ((p^0)^2 + 2ap^0M)\beta_2 - ap^0\beta_3 \sum_{\substack{i=1\\ i\neq j}}^{m} s_i + ap^0\beta_4 \sum_{\substack{i=1\\ i\neq j}}^{m}(M - p^0 s_i + ap^0 s_i^2)\right) s^2 \\
&+ \left(M\beta_1 - p^0\beta_0 - 2p^0M\beta_2 - p^0\beta_3 \sum_{\substack{i=1\\ i\neq j}}^{m} s_i - p^0\beta_4 \sum_{\substack{i=1\\ i\neq j}}^{m}(M - p^0 s_i + ap^0 s_i^2)\right) s \\
&+ \beta_0 M + \beta_2 M^2 + \beta_3 M \sum_{\substack{i=1\\ i\neq j}}^{m} s_i + M\beta_4 \sum_{\substack{i=1\\ i\neq j}}^{m}(M - p^0 s_i + ap^0 s_i^2).
\end{aligned} \tag{123}$$

As before, since all the banks are identical, their optimal choices must be the same. Similarly to above, denote $\mathbf{s}_- = (s, ..., s) \in \mathbb{R}^{m-1}, \mathbf{d}_- = (d, .., d) \in \mathbb{R}^{m-1}$. The sufficient and necessary condition for the optimal strategy s^* in (122) is

$$\frac{d\tilde{w}}{ds}(s^*, \mathbf{s}_-^*)(s - s^*) \geq 0,\ \ s \in [0, S], \tag{124}$$

where

$$\begin{aligned}
\frac{d\tilde{w}}{ds}(s; \mathbf{s}_-) = &\left(4a^2(p^0)^2\beta_2 + (m-1)a^2(p^0)^2\beta_4\right) s^3 \\
&+ \left(3ap^0\beta_1 - 6a(p^0)^2\beta_2 + (m-1)ap^0\beta_3 - 2(m-1)a(p^0)^2\beta_4\right) s^2 \\
&+ \left(2ap^0 + 2ap^0\beta_0 - 2p^0\beta_1 + (p^0 + 2ap^0M)(2\beta_2 + (m-1)\beta_4) - (m-1)p^0\beta_3\right) s \\
&+ \left(M\beta_1 - p^0\beta_0 - 2p^0M\beta_2 - (m-1)p^0M\beta_4\right).
\end{aligned} \tag{125}$$

The two conditions in (120) guarantee that $\frac{d\tilde{w}}{ds}(s, \mathbf{s}_-)$ is strictly increasing on $[0, \infty)$. We now have two cases:

If

$$M\beta_1 - p^0\beta_0 - 2p^0M\beta_2 - (m-1)p^0M\beta_4 > 0, \tag{126}$$

$\frac{d\tilde{w}}{ds}(s, \mathbf{s}_-) = 0$ has no solution on $[0, S]$, and for any $s \in [0, S]$, $\frac{d\tilde{w}}{ds}(s, \mathbf{s}_-) > 0$. Hence, $s^* = 0$, and $(s^*, d^*) = (0, M)$.

Otherwise, if

$$M\beta_1 - p^0\beta_0 - 2p^0M\beta_2 - (m-1)p^0M\beta_4 \leq 0, \tag{127}$$

$\frac{d\tilde{w}}{ds}(s, \mathbf{s}_-) = 0$ has an unique solution s^0 on $[0, \infty)$, and for any $s > s^0$, $\frac{d\tilde{w}}{ds}(s, \mathbf{s}_-) > 0$. Hence, $s^* = s^0 \wedge S$, and $d^* = M - s^*p^0(1 - as^*)$.

In summary,

$$(s^*, d^*) \tag{128}$$
$$= \begin{cases} (0, M) & : M\beta_1 - p^0\beta_0 - 2p^0 M\beta_2 - (m-1)p^0 M\beta_4 > 0, \\ (s^0 \wedge S, M - (s^0 \wedge S)p^0(1 - a(s^0 \wedge S))) & : M\beta_1 - p^0\beta_0 - 2p^0 M\beta_2 - (m-1)p^0 M\beta_4 \leq 0. \end{cases}$$

5 Empirical Analysis

We now investigate if the current financial system is more stable than it was before the crisis by considering the ability of banks to raise extra capital first through selling stocks alone, and then through both selling stocks and borrowing. We concentrate on two examples: JP Morgan Chase and Citi banks. Both banks were strong before the financial crisis 2008. The difference between them is that the former became even (relatively) stronger by absorbing Bear-Stearns, while the latter was on the brink of failing and survived, arguably, only due to government's help.

Our model is a one step static model, in the sense that all actions such as borrowing and selling stocks are assumed to be happening at the same time. Hence, the optimal strategies and other empirical results are static and are independent of time. They reflect what banks should do based on the conditions at each fixed time point if they were faced with liquidity issues at that time. We then perform this computation repeatedly through time, to obtain a time series.

We use OLS to fit a linear model between overnight interest rate and banks' book-to-price ratios. Although the adjusted R^2 for the interest rate models of both banks are higher using the entire original data—31 March 1998 to 23 June 2017—for which we obtain 0.972 and 0.969 for JP Morgan and Citi bank respectively, we decided to use the data from 31 Dec 2005 to 31 Dec 2016, corresponding to dates closer to the financial crisis and the subsequent recovery (the resulting R^2 are presented in Table 1). In Section 5.2, we use data with quarterly frequency in the entire original dataset to estimate the optimal strategy for the banks to raise money. In Section 5.2.1, we estimate the optimal strategy without borrowing. While in Section 5.2.2, the optimal choices for the banks are studied when borrowing is allowed. In Section 5.3, we verify that the objective functions v_j, $j \in \{JPM, C\}$ from (75), constructed using the 2^{nd} order inverse demand price function from Example 3.1.b, and the interest rate model from Example 3.2.c, satisfy Assumption 3.3 on domain

$$(\mathbf{s}, \mathbf{d}) \in \prod_{i=1}^{2} [\eta_i, S_i] \times [0, D_i], \ \eta_i > 0. \tag{129}$$

Finally, in Section 5.4, we present the empirical result for optimal strategy to recover a given shortfall from Section 4

5.1 Estimate Overnight Interest Rate Function

Denote

$$y_j(\mathbf{s}, \mathbf{d}) = \frac{B_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})}{C_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j})}, j \in \{1, ..., m\} \tag{130}$$

to be the book-to-price ratios of the banks. Recall that we previously assumed that the short-term rate is a function of the book-to-price ratios. Therefore, we write

$$r_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}) = \tilde{r}_j(y_j(\mathbf{s}, \mathbf{d}), y_1(\mathbf{s}, \mathbf{d}), ..., y_{j-1}(\mathbf{s}, \mathbf{d}), y_{j+1}(\mathbf{s}, \mathbf{d}), ..., y_m(\mathbf{s}, \mathbf{d})). \tag{131}$$

We then proceed to approximate the overnight interest rate functions $\tilde{r}_j$ of bank j by using Ordinary Least Square method, as follows. The general regression model for the overnight interest rate $\tilde{r}_j$ is

$$\tilde{r}_j(y_j(\mathbf{s},\mathbf{d}), y_1(\mathbf{s},\mathbf{d}), ..., y_{j-1}(\mathbf{s},\mathbf{d}), y_{j+1}(\mathbf{s},\mathbf{d}), ..., y_m(\mathbf{s},\mathbf{d})) \sim \beta_0 + \beta_1 y_j + \beta_2 \sum_{i \neq j} y_i + \beta_3 \, OIS, \quad (132)$$

where OIS is the overnight interest swap rate . The parameters of linear estimations of JP Morgan Chase and Citi bank are listed in Table 1.

Table 1: Overnight Interest Rate Model $\tilde{r}_j(y_j(\mathbf{s},\mathbf{d}), y_1(\mathbf{s},\mathbf{d}), ..., y_{j-1}(\mathbf{s},\mathbf{d}), y_{j+1}(\mathbf{s},\mathbf{d}), ..., y_m(\mathbf{s},\mathbf{d})) \sim \beta_0 + \beta_1 y_j + \beta_2 \sum_{i \neq j} y_i + \beta_3 \, OIS$, for JP Morgan Chase and Citi between 31 Dec 2005 and 31 Dec 2016.

Parameters	JP Morgan	Citi
β_0	-0.15312	-0.13966
β_1	0.17551	0.11374
β_2	0.11146	0.19447
β_3	1.1742	1.1636
R^2	0.937	0.928

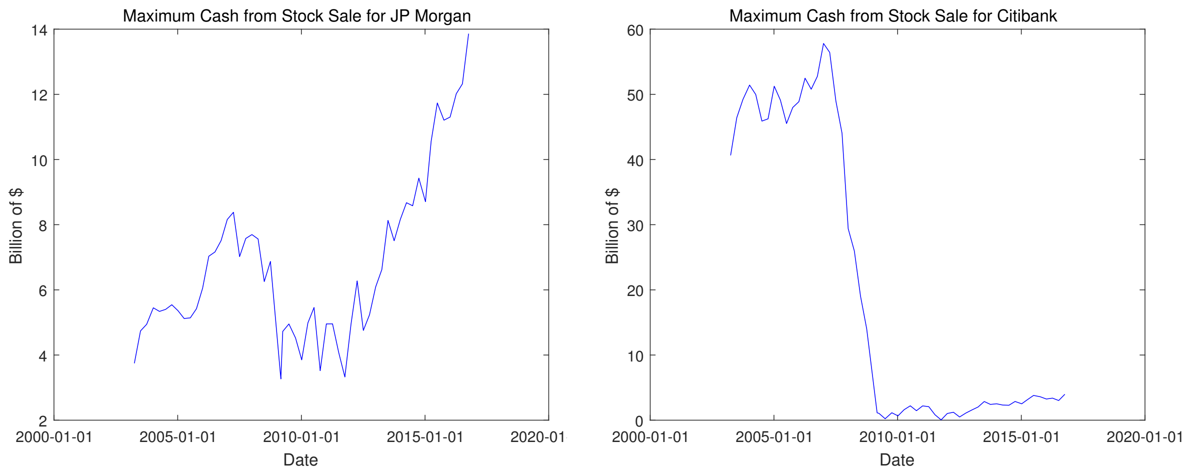
The high R^2s suggest that the linear models are very good approximations for the short-rate $\tilde{r}_j(y_j(\mathbf{s},\mathbf{d}), y_1(\mathbf{s},\mathbf{d}), ..., y_{j-1}(\mathbf{s},\mathbf{d}), y_{j+1}(\mathbf{s},\mathbf{d}), ..., y_m(\mathbf{s},\mathbf{d})), j = \{1, 2, .., m\}$.

5.2 The Optimal Strategies to Raise Cash

5.2.1 The Optimal Strategies to Raise Cash only with Stock Sale

We first estimate the optimal strategy to raise money by selling stock alone. We use the linear inverse demand price function from Example 2.3.a and choose $a_{JPM} = \frac{1}{900}$ for JP Morgan and $a_C = \frac{1}{450}$ for Citi bank. The denominators in this choice of parameters correspond to approximately a third of the number of outstanding shares in millions for each bank. We use $S_{JPM} = 400$ for JP Morgan and $S_C = 200$ for Citi. We let $\epsilon = 10^{-5}$, chosen to be much smaller than $a_{JPM} \wedge a_C$. For the interest rate, we use the model in (132), which is exactly the model in Example 2.4.c. As the number of outstanding shares of JP Morgan during the 31 March 1998 to 23 June 2017 period, was always above 3000 million, $a_{JPM} = \frac{1}{900}$ always satisfies condition (15). While for Citi, the outstanding shares was around 500 million before 30 Sep 2009 and was above 2000 million starting from 30 Sep 2009. To satisfies condition (15), we did a split adjustment to Citi's data after which the number of outstanding shares for Citi was always above 2000 million. These settings guarantee that Assumption 2.1 is satisfied. The left and right graphs of Figure 1 show the maximal amount of funds that JP Morgan Chase and Citi can raise respectively through only selling their stock as a function of time and Figure 2 shows how the market's confidence (price-to-book ratio) changes. As expected, there is a peak of the amount of cash that can be raised, before the financial crash for both banks, which then dramatically drops during the crash. Similarly, both banks' confidence hits a low during the crisis. For JP Morgan the minimum of funds that can be raised is reached some time after the crash and the acquisition of Bear-Stearns, as the markets stabilized, and has been increasing since, as both the market's confidence in JP Morgan and banking system overall recovers. Unlike JP Morgan, the recovery of Citi bank has been less significant. The main reason for these different recoveries is shown in Figure 2. Although there is a recovery of market's confidence,

Figure 1: The maximum amount of funds JP Morgan and Citi banks can raise through stock sale alone.



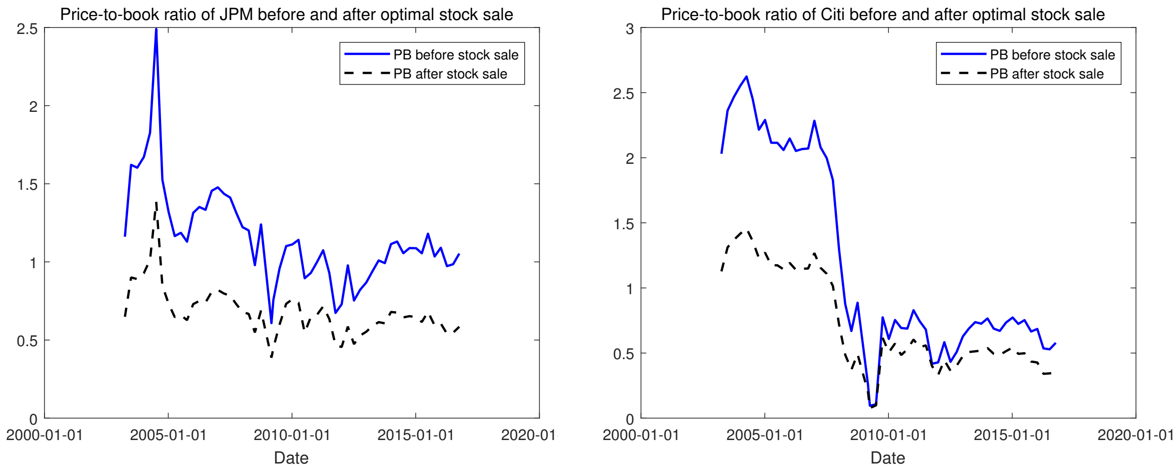
for which we use price-to-book ratio as a proxy, for Citi bank, it has not recovered to its original level, as compared with JP Morgan's price-to-book ratio. The continuous low market's confidence prevents Citi bank from raising funds through stock sale. Additionally Figure 2 illustrates the dip in confidence in case a bank decides to raise cash. We observe that for both JP Morgan and Citi, such a dip is larger during normal times, than during the crisis, as during the crisis the confidence is already low, and so an additional dip in confidence due to stock sale appears to be less significant than it would have been during non-crisis times.

5.2.2 The Optimal Strategies to Raise Cash with Stock Sale and Borrowing

Next, we repeat the optimization when borrowing is also allowed. The book-to-price ratio used here is identical to the book-to-price ratio without borrowing and the price function is the 2^{nd} order inverse demand price function from Example 3.1.b. For the interest rate, we use the model in (132), which is exactly the model in Example 3.2.c. For the empirical experiment in this part, we continue to use $a_{JPM} = \frac{1}{900}, S_{JPM} = 400$ for JP Morgan and $a_C = \frac{1}{450}, S_C = 200$ for Citi to be consistent with previous setting, which ensures that the price functions $p_j(s_j, d_j, \mathbf{s}_{-j}, \mathbf{d}_{-j}), j \in \{JPM, C\}$ satisfy Assumption 2.1, and the additional requirements of Example 3.1.b. Additionally, we choose $b_j = 1500, D_j = 15, j \in \{JPM, C\}$ for both banks, which is approximately equal to the market capitalization sizes (in billions) of the banks during 2008 financial crisis and to match the fact that a little borrowing has little effect on the stock price. Finally, we choose $\epsilon_1 = 10^{-5}, \epsilon_2 = 10^{-6}$ to ensure that the effects of other banks are not as significant as that of the bank itself. Figures 3 and 4 are plotted based on the above parameters.

Figure 3 provides the optimal function value for each bank. Similar to the scenario in Section 5.2.1, when borrowing is not an option, we observe that both banks experienced a big decline in the amount of money they can raise during the crash, and both recovered from these lows. However, Citi's recovery is only partial as opposed to a more robust recovery by JP Morgan. Figure 4 shows the maximum amount of funds that can be raised by JP Morgan and Citi through both stock sale and borrowing component-wise. Recall both stock sale and borrowing, incur costs and reduce market confidence. Therefore, the actual amounts that can be raised in Figure 3, are the sum of

Figure 2: Changes in market confidence (price-to-book ratio) of JP Morgan and Citi banks as a result of stock sale transaction that raises the maximum amount of funds for each of the banks. The blue solid curves are the market's confidence of JP Morgan and Citi banks before the stock sale transaction; the black dashed curves are the market's confidence of the banks after the stock sale respectively.



the red curve and blue curve minus the black dashed curve in Figure 4.

Additionally, comparing Figure 3 with Figure 1, we observe that both banks can raise more funds when borrowing is allowed, especially Citi. It appears that JP Morgan cannot raise substantially more funds than before by using stock sale and borrowing together. The main reason appears to be the significant cost increase that happened after the financial crisis, as is shown in the left graph of Figure 4. This forces JP Morgan to rely more on stock sale than on borrowing. Moreover, Figure 4 points out that before the crash, the optimal strategy for Citi to raise fund was to sell its stock with little borrowing, while after the crash it became optimal to rely almost exclusively on borrowing. This is not surprising, since the government ultimately converted its loan to stock, and signifies a continuation of low confidence in the bank, as debt holders are reimbursed before the shareholders, who are always the last to be reimbursed in case of a default. Whereas it is commonly assumed that Citi's too big to fail status will prevent any losses to the debt holders. The situation is almost reversed for JP Morgan, whose balance sheet now implies that it can raise significant funds by selling stock, due to high market confidence. This in addition to borrowing funds cheaply, likely partially due to its too big to fail status.

These findings are also supported in Figure 5, which similar to Figure 2 illustrates the changes to price-to-book ratio (our proxy for confidence) as a result of raising the funds. We observe that the original price-to-book ratio in Figure 5 reaches the lowest point of about 0.6 for JP Morgan (which is above 1 for most of the time), while the average price-to-book ratio for Citi after the crash is only about 0.5 comparing with values over 2, before the crash. The relatively more robust market confidence of JP Morgan makes it possible for the bank to raise funds through stock sales even in financial crisis period. Moreover, comparing with stock sale, borrowing is less sensitive to market confidence, although it still declined during the crash. The change in confidence is very similar to Figure 2, so even though the amount of funds that both banks can raise is larger, the corresponding drop in confidence is almost identical, signifying that the fund raising can now be done cheaper.

Figure 3: Maximum amount of funds that JP Morgan and Citi banks can raise through stock sale and borrowing.

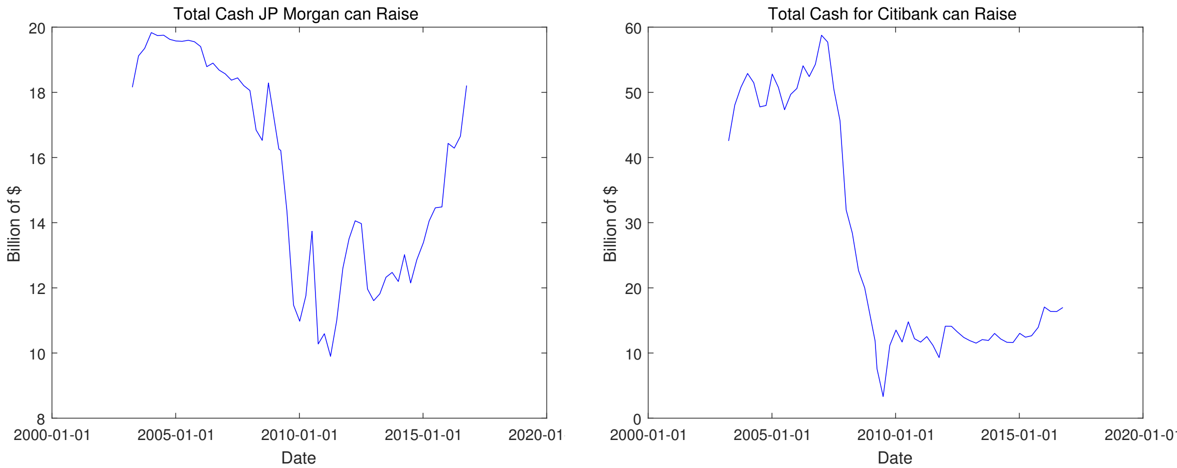


Figure 4: Changes in proceeds from stock sale and borrowing, together with the cost incurred in a transaction where a maximum amount are being raised by JP Morgan and Citi banks. The red solid curves are the funds that can be raised from stock sale, the blue solid curves are the funds that can be raised from borrowing, the black dashed curves are the amount of cost increase of the transaction.

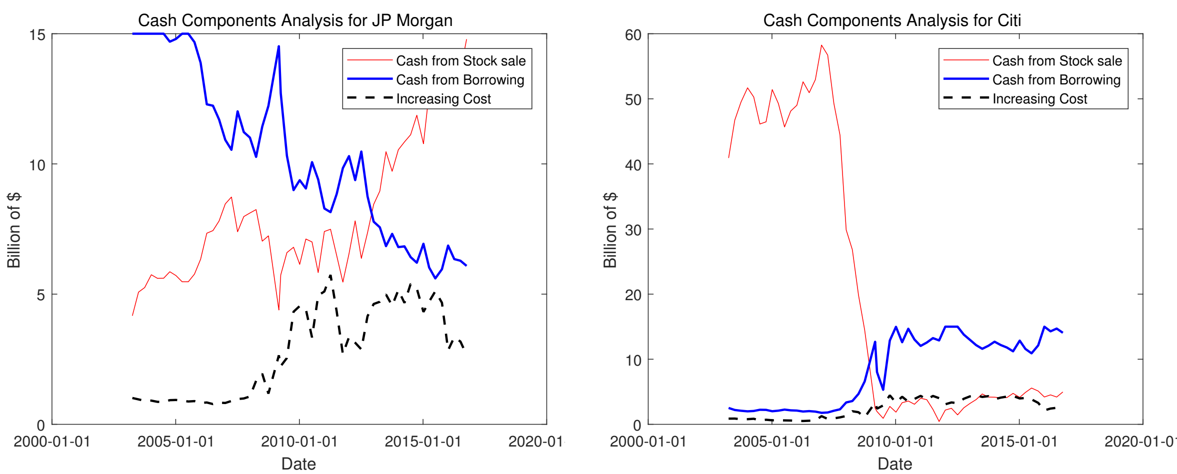
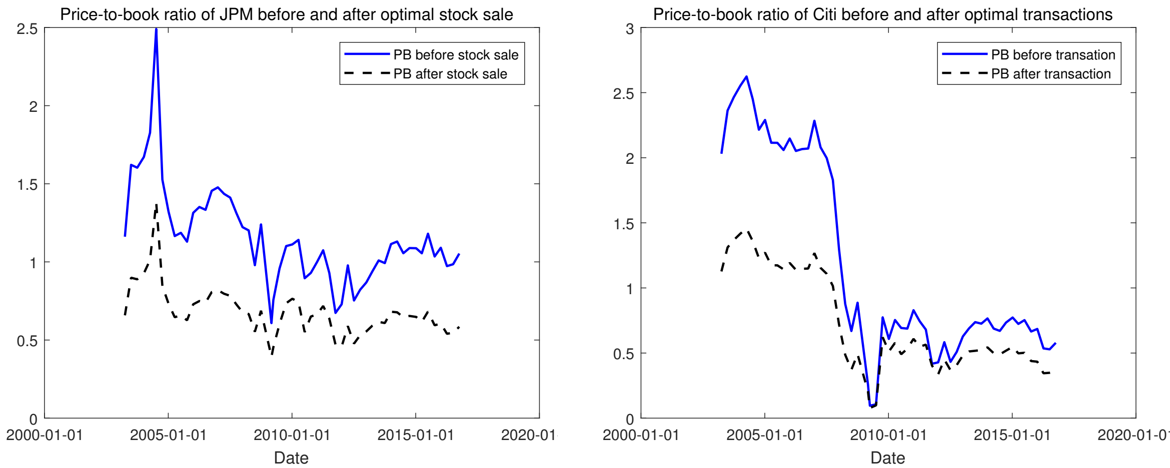


Figure 5: Changes in market confidence (price-to-book ratio) of JP Morgan and Citi banks from a transaction to raise the maximum amount of funds, when both stock sale and borrowing are utilized. The blue solid curves are the market's confidence of JP Morgan and Citi banks before the transaction; the black dashed curves are the market's confidence of the banks after the stock sale and borrowing transaction respectively.



5.3 Assumption Verification

We now verify that Assumption 3.3 holds for the two banks case, where $v_j,\ j \in \{JPM, C\}$ is constructed as in Section 5.2.2 using the 2^{nd} order inverse demand price function from Example 3.1.b, and the interest rate, we use the model in (132), which is exactly the model in Example 3.2.c. It is sufficient to check that

$$-\frac{\partial^2 v_j}{\partial s_j^2} > \left|\frac{\partial^2 v_j}{\partial d_i \partial s_j}\right| + \left|\frac{\partial^2 v_j}{\partial d_j \partial s_j}\right| + \left|\frac{\partial^2 v_j}{\partial s_i \partial s_j}\right|, \tag{133}$$

$$-\frac{\partial^2 v_j}{\partial d_j^2} > \left|\frac{\partial^2 v_j}{\partial s_i \partial d_j}\right| + \left|\frac{\partial^2 v_j}{\partial s_j \partial d_j}\right| + \left|\frac{\partial^2 v_j}{\partial d_i \partial d_j}\right|. \tag{134}$$

In practice in order to raise funds a bank will not sell a handful of stocks, but will have to sell a substantial amount. Therefore, we assume that for $j \in \{JPM, C\}$ there exists $\eta_j > 0$ such that bank j will not sell less then η_j shares, and we restrict the domain of s_j to $[\eta_j, S_j]$. It is not hard to verify that the results on the existence and uniqueness of the Nash equilibrium in Theorem 3.4 still hold with this additional restriction. Using our findings in Table 1 and the fact that $\frac{s_j}{N_j+s_j} < y_j$, for any $(\mathbf{s}, \mathbf{d}) \in \prod_{i=1}^2 [\eta_i, S_i] \times [0, D_i]$, it can be checked that all the second derivatives $\frac{\partial^2 v_j}{\partial s_j^2}, \frac{\partial^2 v_j}{\partial d_j \partial s_j}, \frac{\partial^2 v_j}{\partial s_i \partial s_j}, \frac{\partial^2 v_j}{\partial s_i \partial d_j}, \frac{\partial^2 v_j}{\partial d_i \partial s_j}, \frac{\partial^2 v_j}{\partial d_j^2} < 0$ are strictly negative. Moreover, since the numerical value of L_C, L_{JPM} are about 10^6, when the parameters $a_j, b_j,\ j \in \{C, JPM\}, \epsilon_1, \epsilon_2$ are small, then $\eta_j \geq D_j$ and $a_j > b_j D_j$ are sufficient condition for (133) and (134) to hold. Thus, the functions $v_j,\ j \in \{C, JPM\}$ given in (75) that is being used for the empirical calculation satisfies all the assumption needed in Section 3.

5.4 Optimal Strategy for Recovering Shortfall

In this section, we investigate the optimal strategy to minimize the financing costs, while raising funds to cover a given shortfall. We use the 2^{nd} order inverse demand price function from Example 3.1.b, and the interest rate function from the model in (132), which is exactly the model in Example 3.2.c, together with the parameters, $a_{JPM} = \frac{1}{900}, b_{JPM} = \frac{1}{15000}$ for JP Morgan and $a_C = \frac{1}{450}, b_C = \frac{1}{15000}$ for Citi, together with $\epsilon_1 = \epsilon_2 = 0$. These setting guarantee Assumption 4.1 holds. The values of the parameters are slightly different from the values in Section 5.2, as b_{JPM}, b_C are much smaller and $\epsilon_1 = \epsilon_2 = 0$, because in this section the banks are in danger of a default, and their shortfall is not very big. Therefore, we assume that the stock sale and borrowing done by the other bank does not affect the bank's own stock price and new borrowing of the bank does not decrease the stock price as significantly as in Section 5.2. However, these transactions are still systemic events as they will affect the overnight interest rate of every bank through decreasing the market's confidence/price-to-book ratios.

We assume that the banks (JP Morgan and Citi) need to cover \$10 billion dollar shortfall ($M = 10$) and set $S_{JPM} = 400, S_C = 200, D_{JPM} = D_C = 10$. To satisfy Assumption 4.1, set lower bound for stock sale to be $\eta_{JPM} = \eta_C = 10$ for JP Morgan and Citi respectively. That is, the domains for s_{JPM} and s_C are $[\eta_{JPM}, S_{JPM}]$, $[\eta_C, S_C]$ respectively.

The results are presented in Figure 6. We observe that both banks would have faced high costs to cover their shortfall around 2008. However, once the Federal Reserve implemented quantitative easing (QE) policy from the late 2008, the borrowing costs for the banks have significantly decreased. This ultimately helped Citi to survive the crisis, as it relied on debt financing much more than on stock sale after the financial crisis 2008. This situation is completely inverted when comparing with the situation before the crisis, where it was optimal for Citi to rely on stock sale. We attribute this to the sharp decrease in market's confidence during and after the crisis.

Figure 7 illustrates the change in confidence for both banks as a result of raising the funds for covering a given fixed shortfall. We observe that the change in confidence for JP Morgan is greatest before the crash. So not only the bank's confidence has recovered after the crash, but also an emergency fund raising would not lower the confidence as much as it would have before the crash. In case of Citi bank, the change in confidence is not very significant both before and after the crash. We hypothesize that before the crisis the market was overconfident, whereas during and after the crisis the market believes that the bank is too big to fail, and since the majority of funds are raised by borrowing, those will be repaid in any case.

6 Conclusion

In this paper, we consider the maximum amount of funds that banks can raise either through stock sales alone, or through both stock sales and borrowing, and the problem of raising funds to cover a given shortfall. We have created a simple model, incorporating the price-to-book ratio as proxy for market's confidence in these optimization problems. We have shown the existence and uniqueness of Nash equilibrium in all the optimization problems, and performed a time series empirical analysis of two banks to show how they weathered through the last financial crisis.

Our model is a one step model, in the sense that all actions such as borrowing and selling stocks are assumed to be instantaneous and happening at the same time. However, in the empirical section we use it repeatedly at every time periods, to try to repeatedly answer the same question for each of the periods. We leave the work of an extension of the model to a truly dynamic one for future research.

Figure 6: Changes in proceeds from stock sale and borrowing, together with the cost incurred in a transaction to cover a given shortfall by JP Morgan and Citi banks. The red solid curves are the funds that can be raised from stock sale, the blue solid curves are the funds that can be raised from borrowing, the black dashed curves are the amount of cost increase of the transaction.

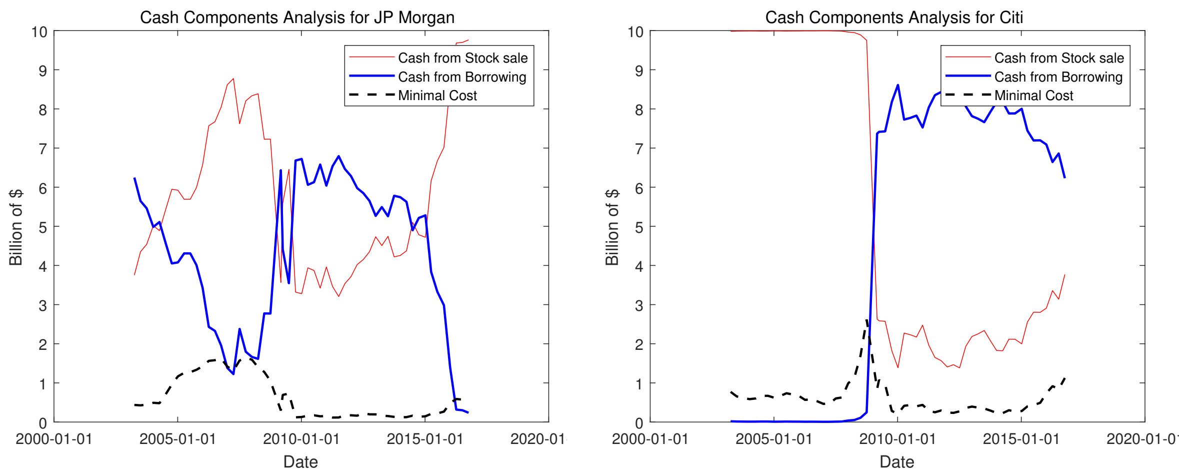
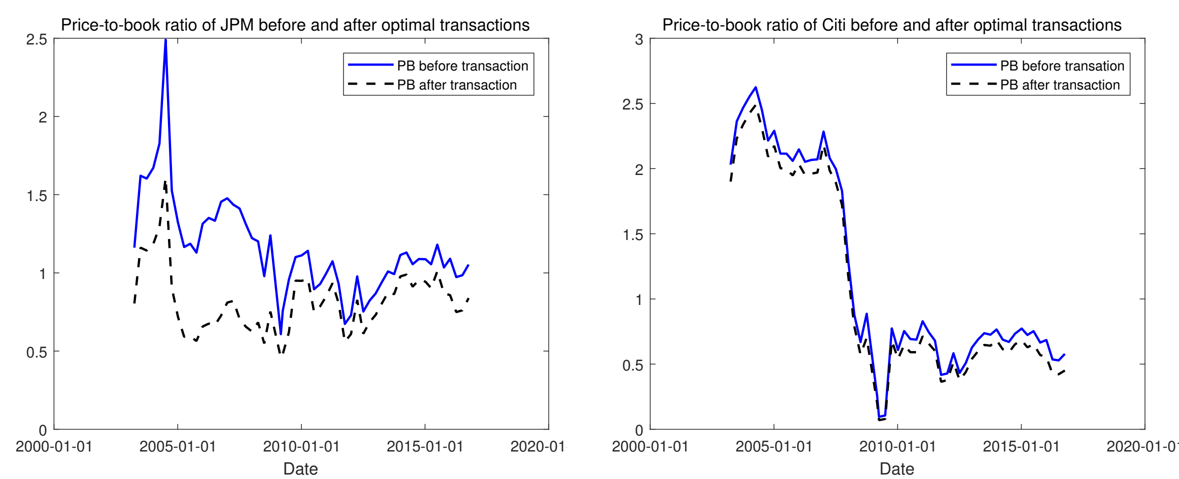


Figure 7: Changes in market confidence (price-to-book ratio) of JP Morgan and Citi banks from a transaction to cover a given shortfall, when both stock sale and borrowing are utilized. The blue solid curves are the market's confidence of JP Morgan and Citi banks before the transaction; the black dashed curves are the market's confidence of the banks after the stock sale and borrowing transaction respectively.



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