Optimal Switching between Locking Down and Opening the Economy Because of an Infection

Maxim Bichuch [*]

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Abstract

We consider a two-regime switching model with the goal of minimizing the expected discounted cumulative combination of the utility of the number of infections together with the utility of an economical indicator. We assume the two regimes choices are between opening and and locking down the economy, and the choice affects the infection rate. We also assume that the economy level also has a small influence on both the infection rate and on the cumulative function being minimized. We then asymptotically find the value function and the boundaries of the switching regions, and perform a numerical calibration to draw conclusions about optimal lockdown in a pandemic.

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1 Introduction

Optimal switching is the problem of finding an optimal sequence of stopping times for switching between different regimes in the underlying stochastic processes. Generally it can be characterized as optimal timing under uncertainty. The problem have many classical applications including finance and economics, such as pricing real options. For example the works of Brennan and Schwartz [1985] on optimal extraction, Dixit [1989] on optimal production and Trigeorgis [1993] on real options, along with many other types of applications, see e.g. Benaroch et al. [2012], Parpas and Webster [2014], Sødal et al. [2008] for applications respectively in manufacturing; network and energy systems; shipping.

The general switching problem, however, is often intractable. Even the case with only three regimes and one-dimensional underlying diffusion, as in Pham et al. [2009] may not admit a fully explicit solution. Therefore it is not surprising that most explicit solutions have been provided in a two-regime switching models, see e.g. Tang and Yong [1993], Brekke and Oksendal [1994], Duckworth and Zervos [2001], Zervos [2003]. A typical approach to the problem is using viscosity solutions technique as in Ly Vath and Pham [2007]. This approach identifies the solution to the appropriate Hamilton-Jacobi-Bellman (HJB) equation and the value function. This in turn allows to prove that the value function is smooth inside each of the regions and calculate it through the smooth fit principle across the regions. Other popular approaches include Bayraktar and Egami [2010] who used optimal stopping times technique, and Hamadène and Jeanblanc [2007] who used techniques from the theory of Backward Stochastic Differential Equations.

[*]Department of Applied Mathematics and Statistics, Johns Hopkins University 3400 North Charles Street, Baltimore, MD 21218. mbichuch@jhu.edu. Work is partially supported by NSF grant DMS-1736414. Research is partially supported by the Acheson J. Duncan Fund for the Advancement of Research in Statistics.

A common alternative, for when an explicit solution is not available is to use asymptotic expansion. The literature is vast, and we refer to classical works of Fouque et al. [2001] and Fouque et al. [2017] and books of Fouque et al. [2000] and Fouque et al. [2011] and examples therein. The key idea is to find a case which can be easily solved and then asymptotically expand around it, to find an asymptotic correction, and thereby obtain an asymptotic expansion in the desired problem. This is then often combined with viscosity solution theory as in the works of Janecek and Shreve [2004], Bichuch and Sircar [2019]. This also allows to construct sub- and supermartingales and sandwich the value function in between then to identify the first couple of terms in the asymptotic expansion as in Bichuch [2012] and Bichuch and Fouque [2019].

In this paper we apply this asymptotic expansion method to switching problems. While this method has been used before, e..g in Tsekrekos and Yannacopoulos [2016], the expansion proofs so far seem to be heuristic. In this paper we provide a rigorous proof for such an asymptotic expansion in a special case. The asymptotic expansion method proposed below is broad, and can be applied to a wide range of optimal switching problems in both the number of states and the type of diffusions, however in general the proof is difficult, and requires special conditions. Therefore, we simplify it, by solving a two-regime optimal switching problem with two-dimensional coupled but simple diffusions, asymptotically in the sensitivity to the second diffusion. The asymptotic expansion is performed around the explicitly solvable case of two-regime switching problem with a one-dimensional diffusion. The main contribution of this paper is in adapting the sub- and super-martingale proof method of Bichuch [2012, 2014] to optimal switching problems, and thereby obtaining a rigorously constructed viscosity sub- and super-solutions, and "nearly-optimal" switching policies.

Another motivation for the underlying problem is to understand when is it optimal for the economies of countries to lock down and open up. That is to provide an optimization problem whose solution is to close when the infection rate is high, and open back up when it is lower, as is known to be historically optimal in past pandemic (e.g. Correia et al. [2020]). The goal is not simply to minimize the number of infections/fatalities, such as in Gonzalez-Eiras and Niepelt [2020], Acemoglu et al. [2020], or to also take into account constraints that keep the economy from completely crashing as in Alvarez et al. [2020], or to simply concentrate purely on the economical effect of the lockdown as in Moser and Yared [2020]. But to minimize the cumulative combination of the infections and the economical state. In this paper we consider a two-regime switching mode, where the goal is to minimize a cumulative discounted number of infections together with an inverse of an economic indicator. We perform an asymptotic analysis when the sensitivity to the economical factor is small, and find the asymptotic expansions of the value function and of the boundaries of the stopping regions. We then calibrate the model parameters to data and discuss the obtained results.

The structure of the paper is as follows: in Section 2 we formulate the general type of switching problem we want to solve. The problem is then specialized to a two-regime case and some preliminary results are presented in Section 3. The main proofs are in Section 4. The numerical study is done in Section 5. We conclude in Section 6.

2 Model Formulation

In this section we formulate a general N-regime switching model, and set up the optimization problem. We start with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, on which two correlated Brownian Motions B, W are given with $d\langle B, W\rangle_t = \rho dt$ with $|\rho| < 1$.

Denote X_t to the be the number of sick people at time t, and an economy statistic, (e.g. the GDP or the total stock market) by $\tilde{Y}_t$. Assume that there are N different society states, where for example $i = 1$ means stay-at-home order, $i = 2$ partial opening, low risk activities allowed, with some social distance required, ..., $i = N$ business as usual. Assume that the rate of transmission depends on the state i, and that it is possible to switch between the states. Assuming that this happens at stopping times $0 = \tau_0 < \tau_1 < \tau_2 ...$, satysfying $\lim_{n\to\infty} \tau_n = \infty$ a.s., by switching to regime $\iota_n \in \mathcal{F}_{\tau_n},\ n \geq 0$, this defines a control $\alpha = (\tau_n, \iota_n)_{n\geq 0}$, and the states process

$$I_t^i = \sum_{n\geq 0} \iota_n \mathbb{I}_{[\tau_n,\tau_{n+1})}(t),\ I_{0-} = i.$$

We denote the set of all such switching controls α by $\mathcal{A}$. We assume that there is some economical cost g_{ij} for switching states from i to j, caused by the necessity to adapt the economy to the new state. So we will assume that $g_{ij} > 0$, if $j < i$, i.e there is a cost for closing the economy, and $g_{ij} < 0$ if $j > i$, i.e. there is a benefit to the economy opening up. It is customary to assume that $g_{ii} = 0$, and we also require the triangular inequality type condition to hold, namely:

$$g_{ik} < g_{ij} + g_{jk},\ j \neq i, k,$$

stating that it is better to switch from i to k directly, rather than switching from i to j and then immediately from j to k.

The evolution of the number of infections X_t is then given by:

$$dX_t = \tilde{\mu}(X_t, \tilde{Y}_t, I_t^i)X_t dt + \sigma(I_t^i)X_t dW_t,\ X_0 = x.$$

Similarly, we also define the evolution of the economy:

$$d\tilde{Y}_t = \tilde{\theta}(I_t^i)\tilde{Y}_t dt + \tilde{\nu}(I_t^i)\tilde{Y}_t dB_t,\ \tilde{Y}_0 = \tilde{y}. \tag{2.1}$$

Our goal is to investigate the optimal cumulative discounted utility of the number of infections and the economy, when optimally switching between the different states is allowed. That is our goal is to find the value functions

$$\tilde{v}_i(x, \tilde{y}) = \inf_{\alpha\in\mathcal{A}} J_i(x, \tilde{y}, i, \alpha),\ i = 1, ..., N,$$

where

$$J(x, \tilde{y}, i, \alpha) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(X_t) + \frac{\varepsilon}{\mathcal{U}_q(\tilde{Y}_t)}\right)dt + \sum_{n\geq 1} \mathrm{e}^{-\beta\tau_n} g_{\iota_{n-1},\iota_n}\right].$$

Here we assumed that $\mathcal{U}_p, \mathcal{U}_q$ are utility functions. In what follows, we will concentrate on a power utility $\mathcal{U}_p(x) = x^p,\ \mathcal{U}_q(y) = y^q,\ 0 < p, q < \infty$. Additionally, we assume that there is a strong emphasis on public health over the economy, and this is expressed in the small coefficient $\varepsilon > 0$. Note that typically, when the utility of the economy is being maximized, it is assumed to be increasing and concave. A classical example is $-\frac{1}{\mathcal{U}_q(\cdot)},\ 0 < q < \infty$, see e.g. Karatzas and Shreve [1998][Ch. 3]. In our case a change in sign is needed, since the goal is to find a minimum, rather than a maximum. The utility for the number of infections is also chosen to be power, for tractability reasons.

Next, to make things more tractable we will assume the following model parameters:

$$\tilde{\theta}(i) = \tilde{\theta}_i,\ \tilde{\nu}(i) = \tilde{\nu}_i,\ \sigma(i) = \sigma_i,$$

all to be constant, only depending on state i. This is as opposed to the drift in X, which we assume to be

$$\tilde{\mu}(x, \tilde{y}, i) = \mu_i + \varepsilon\tilde{y}^{-q_1} - \varepsilon^2\tilde{y}^{-2q_1}, \tag{2.2}$$

where $0 < q_1 < \infty$.

We motivate the evolution of X, the number of sick people as follows. We assume that X_t is significantly smaller than the susceptible population, and thus (on average) additional $\bar{\mu}(x, \tilde{y}, i)$ people will get sick over time dt from every (currently) sick person, whereas proportion r of the currently sick will recover, for a total rate of $\tilde{\mu}(x, \tilde{y}, i) = \bar{\mu}(x, \tilde{y}, i) - r$ of additional infections. Alternatively, in the SIS model, which assumes that a recovery does not confer a (long lasting) immunity, the (deterministic term) in the change in the number of sick people is $dX_t = \left(\tilde{\mu}(x, \tilde{y}, i) - \bar{\mu}(x, \tilde{y}, i)\frac{X_t}{\text{(Total Population)}}\right)X_t dt$. If the model is simplified one step further, by assuming that the Total Population far exceeds X_t and neglecting that term, we recover the drift in our model.

The goal of this work is to perform an asymptotic analysis in $\varepsilon > 0$ small. It is also interesting to incorporate the entire SIS model, i.e. to allow the drift of X_t to be $\left(\tilde{\mu}(x, \tilde{y}, i) - \varepsilon\bar{\mu}(x, \tilde{y}, i)X_t + \varepsilon\tilde{Y}_t^{-q_1} - \varepsilon^2\tilde{Y}_t^{-2q_1}\right)X_t$ and perform an asymptotic expansion. We leave this work for future research.

To motivate the geometrical Brownian Motion model assumption for the evolution of the economy $\tilde{Y}$, we think of the market (e.g. the S&P index) as a proxy to the economy and refer to the classical assumptions in Mathematical Finance, e.g. Karatzas and Shreve [1998][Ch. 3].

The rational for the $O(\varepsilon)$ term in (2.2) being that a good economical state, with high $\tilde{y}$ lowers the average rate of new infections, (e.g stimulus payments, have both benefited the economy and allowed people to stay home and decreased the number of new infections). The $O(\varepsilon^2)$ term was added for technical reasons, to keep the drift of X bounded in $\tilde{Y}$.

It is more convenient to work with $Y_t = \frac{1}{\tilde{Y}_t}$. For convenience, we may refer to Y as the inverse of the economy. Therefore, we have that

$$dY_t = \theta(I_t^i)Y_t dt + \nu(I_t^i)Y_t dB_t,\ Y_0 = y, \tag{2.3}$$

where $\theta(i) = -\tilde{\theta}_i + \tilde{\nu}_i^2,\ \nu(i) = -\tilde{\nu}_i,\ y = \frac{1}{\tilde{y}}$. Let also

$$v_i(x, y) = \tilde{v}_i(x, \tilde{y})$$

Then the drift of X_t becomes $\mu(x, \tilde{y}, i) = \mu_i + \varepsilon y^{q_1} - \varepsilon^2 y^{2q_1}$ and the SDE changes to

$$dX_t = \mu(X_t, Y_t, I_t^i)X_t dt + \sigma(I_t^i)X_t dW_t,\ X_0 = x. \tag{2.4}$$

We will also assume for convenience that $\rho = 0$, though the calculations can easily be extended to other values $\rho \in (-1, 1)$.

It is well known (e.g. Pham [2009][Chapter 5.3]) that the value functions v_i are the viscosity solutions of the HJB equation:

$$\max\{\beta v_i - \mathcal{L}_i^{\varepsilon} v_i - \mathcal{U}_p(x) - \varepsilon\mathcal{U}_q(y), v_i - \min_{j \neq i}(v_j + g_{ij})\} = 0\ , i = 1, ..., N, \tag{2.5}$$

where

$$\mathcal{L}_i^{\varepsilon} = (\mu_i + \varepsilon y^{q_1})x\partial_x + \frac{\sigma_i^2 x^2}{2}\partial_{xx}^2 + \theta_i y\partial_y + \frac{\nu_i^2 y^2}{2}\partial_{yy}^2.$$

Observe that

$$\mathcal{L}_i^{\varepsilon} = \mathcal{L}_i^0 + \varepsilon\mathcal{L}^1 - \varepsilon^2\mathcal{L}^2,$$

where

$$\begin{aligned}
\mathcal{L}_i^0 &= \mu_i x\partial_x + \frac{(\sigma_i x)^2}{2}\partial_{xx}^2 + \theta_i y\partial_y + \frac{\nu_i^2 y^2}{2}\partial_{yy}^2, \\
\mathcal{L}^j &= y^{jq_1} x\partial_x, \quad j = 1, 2.
\end{aligned}$$

Since the solution to the original problem (2.5) is not known to exists in closed form, we perform an asymptotic expansion in $\varepsilon > 0$. Our goal is to find the asymptotic expansion of the value functions:

$$v_i(x,y) = v_i^0(x,y) + v_i^1(x,y)\varepsilon + ..., \tag{2.6}$$

asymptotically in $\varepsilon > 0$ small and the asymptotic expansions of the optimal stopping regions.

3 Main Example

In this section we heuristically find the asymptotic expansion of the value functions (2.6) and of the optimal switching regions. The rigorous proof of which are then presented in the next section. For tractability we will also assume that $N = 2$. While these asymptotic expansion calculations can be performed without having explicit solutions, their availability significantly simplifies the exposition. Therefore, we limit the presentation to two-regime switching problem and constant coefficients dynamics (2.4), (2.1), (2.3), but stress that it can be expanded to more general dynamics and other N, such as the case $N = 3$ in Pham et al. [2009].

We proceed to solve the HJB equation by powers of ε. We continue to assume that $g_{21} > 0$, and $g_{12} < 0$, with the asymptotic expansion

$$g_{ij} = g_{ij}^0 + \varepsilon g_{ij}^1(y^q + y^{q_1}).$$

Therefore, we will assume that

$$g_{12}^0 + g_{21}^0 > 0,\ g_{12}^1 + g_{21}^1 \geq 0,\ g_{21}^k > 0,\ g_{12}^k < 0,\ k = 0, 1, \tag{3.1}$$

We will start with heuristic asymptotic of the HJB equation:
Order $O(1)$:
At the order $O(1)$, the HJB equation is particularly simple, and is (heuristically) obtained from (2.5) by expanding in powers of ε and then considering the $O(1)$ term. We have that

$$\begin{aligned}
&\max\{\beta v_1^0 - \mathcal{L}_1^0 v_1^0 - \mathcal{U}_p(x), v_1^0 - v_2^0 - g_{12}^0\} = 0,\\
&\max\{\beta v_2^0 - \mathcal{L}_2^0 v_2^0 - \mathcal{U}_p(x), v_2^0 - v_1^0 - g_{21}^0\} = 0.
\end{aligned} \tag{3.2}$$

Note that the entire probabilistic formulation of the problem at $O(1)$ is independent of y, and so we will look for solutions v_i^0 to be functions of x and independent of y. By uniqueness of the solution to the HJB equation (3.2) this guess will ultimately turn to out be correct. We will utilize this ansatz in this section, and in the next section we will give a probabilistic proof, that will not rely on this uniqueness fact. Our first assumption, is about the infection minimization problem with no switching:

Assumption 3.1. *We will assume that $\beta > 0$ is big enough so that*

$$K_i^0 = \frac{1}{\beta - \mu_i p + \frac{\sigma_i^2}{2} p(1-p)} > 0,\ i = 1, 2,$$

and that $K_1^0 < K_2^0$.

The later makes sense as $\hat{V}_1^0(x) < \hat{V}_2^0(x)$, where

$$\hat{V}_i^0(x) = K_i^0 x^p,\ i = 1, 2,$$

are the solutions to the infection minimization problem with no switching allowed (i.e. the second order PDE parts of (3.2)) and therefore with constant state i.

Recall that the switching regions, i.e. the sets where it is optimal to immediately switch, are $\tilde{S}_i = \left\{(x, y) \in \mathbb{R}^{+2} | v_i^0 = v_j^0 + g_{ij}^0,\ j \neq i\right\}$, $i = 1, 2$. In this case, it follows from Ly Vath and Pham [2007][Theorem 4.1] that the switching regions are $\tilde{\mathcal{S}}_2 = [\underline{x}_*^0, \infty) \times \mathbb{R}^+$, $\tilde{\mathcal{S}}_1 = [0, \overline{x}_*^0] \times \mathbb{R}^+$ and

$$v_1^0(x) = \begin{cases} Ax^{m_1^-} + K_1^0 x^p & x > \overline{x}_*^0, \\ Bx^{m_2^+} + K_2^0 x^p + g_{12}^0 & x \leq \overline{x}_*^0, \end{cases}$$

$$v_2^0(x) = \begin{cases} Bx^{m_2^+} + K_2^0 x^p & x < \underline{x}_*^0, \\ Ax^{m_1^-} + K_1^0 x^p + g_{21}^0 & x \geq \underline{x}_*^0. \end{cases}$$

Here, we set

$$m_i^{\pm} = \frac{-\mu_i + \frac{\sigma_i^2}{2} \pm \sqrt{\left(\mu_i - \frac{\sigma_i^2}{2}\right)^2 + 2\beta\sigma_i^2}}{\sigma_i^2},$$

to be the roots of

$$\beta - \mu_i m - \frac{\sigma_i^2}{2} m(m-1) = 0,$$

and therefore are used to construct $w_i(x) = Ax^{m_i^+} + Bx^{m_i^-}$ the general solution to the homogeneous PDEs $\beta w_i - \left(\mu_i x \partial_x + \frac{\sigma_i^2 x^2}{2} \partial_{xx}^2\right) w_i = 0$. Additionally, we define

$$\overline{x}_*^0 = \left(\frac{g_{12}^0 m_1^-(m_2^+ - p) z^{m_2^+} - g_{12}^0 m_2^+(m_1^- - p) z^{m_1^-} + g_{21}^0 p(m_2^+ - m_1^-)}{(K_1^0 - K_2^0)(m_1^- - p)(p - m_2^+)\left(z^{m_1^-} - z^{m_2^+}\right)} \right)^{\frac{1}{p}}, \tag{3.3}$$

$$\underline{x}_*^0 = z\overline{x}_*^0, \tag{3.4}$$

$$A = \frac{(\overline{x}_*^0)^{-m_1^-}\left((K_1^0 - K_2^0)(m_2^+ - p)(\overline{x}_*^0)^p - g_{12}^0 m_2^+\right)}{m_1^- - m_2^+}, \tag{3.5}$$

$$B = \frac{(\overline{x}_*^0)^{-m_2^+}\left((K_1^0 - K_2^0)(m_1^- - p)(\overline{x}_*^0)^p - g_{12}^0 m_1^-\right)}{m_1^- - m_2^+}, \tag{3.6}$$

and where z is (the unique) solution on $\left(\left(-\frac{g_{21}^0}{g_{12}^0}\right)^{1/m_2^+}, \infty\right)$ to

$$m_1^-(p - m_2^+)(1 - z^{m_1^- - p})(g_{12}^0 z^{m_2^+} + g_{21}^0) + m_2^+(m_1^- - p)(1 - z^{m_2^+ - p})(g_{12}^0 z^{m_1^-} + g_{21}^0) = 0. \tag{3.7}$$

We will rigorously prove this in the next section. For this we need an assumption, so that the order $O(1)$ switching problem will be well defined.

Assumption 3.2. *In addition to Assumption 3.1, assume that* (3.1) *holds, and that* $\beta > 0$ *is big enough, so that* $m_i^+ > \max\{p, 1\}$, *and* $m_i^- < 0$, $i = 1, 2$.

From here together with the definition $\overline{x}_*^0$ in (3.3) it follows that $\overline{x}_*^0 > 0$. From (3.1) it follows that $\left(-\frac{g_{21}^0}{g_{12}^0}\right)^{1/m_2^+} > 1$. Therefore, $\underline{x}_*^0 > \overline{x}_*^0$. Finally from (3.5) and (3.6), using (3.3), (3.4), (3.7) it follows that $A, B > 0$ respectively.

Recall Ly Vath and Pham [2007][Theorem 4.1] that these were found by solving the following system of equations:

$$\begin{aligned}
A(\overline{x}_*^0)^{m_1^-} + K_1^0(\overline{x}_*^0)^p &= B(\overline{x}_*^0)^{m_2^+} + K_2^0(\overline{x}_*^0)^p + g_{12}^0, \qquad (3.8)\\
m_1^- A(\overline{x}_*^0)^{m_1^- - 1} + pK_1^0(\overline{x}_*^0)^{p-1} &= m_2^+ B(\overline{x}_*^0)^{m_2^+ - 1} + pK_2^0(\overline{x}_*^0)^{p-1},\\
A(\underline{x}_*^0)^{m_1^-} + K_1^0(\underline{x}_*^0)^p + g_{21}^0 &= B(\underline{x}_*^0)^{m_2^+} + K_2^0(\underline{x}_*^0)^p, \qquad (3.9)\\
m_1^- A(\underline{x}_*^0)^{m_1^- - 1} + pK_1^0(\underline{x}_*^0)^{p-1} &= m_2^+ B(\underline{x}_*^0)^{m_2^+ - 1} + pK_2^0(\underline{x}_*^0)^{p-1}.
\end{aligned}$$

Order $O(\varepsilon)$: The next step becomes more difficult. We want to find functions v_1^1, v_2^1 satisfying:

$$\begin{aligned}
\left(\beta v_1^1 - \mathcal{L}_1^0 v_1^1 - \mathcal{L}^1 v_1^0\right)(x,y) - \mathcal{U}_q(y) &= 0, \quad x > \overline{x}_*^0, y \in \mathbb{R}, \qquad (3.10)\\
v_1^1(x,y) - v_2^1(x,y) - g_{12}^1(y^q + y^{q_1}) &= 0, \quad x \le \overline{x}_*^0, y \in \mathbb{R},\\
\left(\beta v_2^1 - \mathcal{L}_1^0 v_2^1 - \mathcal{L}^1 v_2^0\right)(x,y) - \mathcal{U}_q(y) &= 0, \quad x < \underline{x}_*^0, y \in \mathbb{R}, \qquad (3.11)\\
v_1^1(x,y) - v_2^1(x,y) + g_{21}^1(y^q + y^{q_1}) &= 0, \quad x \ge \underline{x}_*^0, y \in \mathbb{R}.
\end{aligned}$$

Similarly, to the $O(1)$ case, we can find particular solutions $\hat{V}_1^1, \hat{V}_2^1$ of (3.10) and (3.11) respectively, and they are given as:

$$\begin{aligned}
\hat{V}_1^1 &= K_1^1 y^q + \bar{K}_1^1 y^{q_1} x^{m_1^-} + \hat{K}_1^1 y^{q_1} x^p,\\
\hat{V}_2^1 &= K_2^1 y^q + \bar{K}_2^1 y^{q_1} x^{m_2^+} + \hat{K}_2^1 y^{q_1} x^p,
\end{aligned}$$

where we set the constants to be:

$$\begin{aligned}
K_i^1 &= \frac{1}{\beta - \theta_i q + \frac{\nu_i^2}{2} q(1-q)}, \; i = 1,2,\\
\bar{K}_1^1 &= \frac{m_1^- A}{-\theta_1 q_1 + q_1(1-q_1)\frac{\nu_1^2}{2}}, \; \bar{K}_2^1 = \frac{m_2^+ B}{-\theta_2 q_1 + q_1(1-q_1)\frac{\nu_2^2}{2}},\\
\hat{K}_i^1 &= \frac{pK_i^0}{\beta - p\mu_i + (1-p)p\frac{\sigma_i^2}{2} - q_1\theta_i + q_1(1-q_1)\frac{\nu_i^2}{2}}, \; i = 1,2.
\end{aligned}$$

For future reference, we also define

$$\tilde{K}_i^1 = \frac{1}{\beta - \theta_i q_1 + \frac{\nu_i^2}{2} q_1(1-q_1)}, \; i = 1,2.$$

We now want to find a more general solutions of the homogeneous portion of the PDEs (3.10) and (3.11). For this, we define:

$$d_i^{\pm}(r) = \frac{-\mu_i + \frac{\sigma_i^2}{2} \pm \sqrt{\left(\mu_i - \frac{\sigma_i^2}{2}\right)^2 + \left(2\beta - 2r\theta_i + (1-r)r\nu_i^2\right)\sigma_i^2}}{\sigma_i^2}, \; r > 0, \qquad (3.12)$$

and $\beta > 0$ is assumed to be big enough so that $d_i^{\pm}(q_1), d_i^{\pm}(q)$ are well defined. Then let

$$v_1^1(x,y) = \begin{cases} \hat{V}_1^1(x,y) + A_1 y^q x^{d_1^-(q)} + A_2 y^{q_1} x^{d_1^-(q_1)} & x > \overline{x}_*^0, y \in \mathbb{R},\\ \hat{V}_2^1(x,y) + B_1 y^q x^{d_2^+(q)} + B_2 y^{q_1} x^{d_2^+(q_1)} + g_{12}^1(y^q + y^{q_1}) & x \le \overline{x}_*^0, y \in \mathbb{R}, \end{cases}$$

$$v_2^1(x,y) = \begin{cases} \hat{V}_2^1(x,y) + B_1 y^q x^{d_2^+(q)} + B_2 y^{q_1} x^{d_2^+(q_1)} & x < \underline{x}_*^0, y \in \mathbb{R},\\ \hat{V}_1^1(x,y) + A_1 y^q x^{d_1^-(q)} + A_2 y^{q_1} x^{d_1^-(q_1)} + g_{21}^1(y^q + y^{q_1}) & x \ge \underline{x}_*^0, y \in \mathbb{R}. \end{cases}$$

In the above we need to determine $A_i, B_i,\ i = 1, 2$. This can be easily done, once we recognize that because these equalities must hold for all $y \in \mathbb{R}$ then they must hold for each power of y separately. Therefore from the continuity across the boundary condition, it follows that we must have:

$$K_1^1 + A_1(\overline{x}_*^0)^{d_1^-(q)} = K_2^1 + B_1(\overline{x}_*^0)^{d_2^+(q)} + g_{12}^1, \tag{3.13}$$

$$K_1^1 + A_1(\underline{x}_*^0)^{d_1^-(q)} + g_{21}^1 = K_2^1 + B_1(\underline{x}_*^0)^{d_2^+(q)}, \tag{3.14}$$

$$\hat{K}_1^1(\overline{x}_*^0)^p + \bar{K}_1^1(\overline{x}_*^0)^{m_1^-} + A_2(\overline{x}_*^0)^{d_1^-(q_1)} = \hat{K}_2^1(\overline{x}_*^0)^p + \bar{K}_2^1(\overline{x}_*^0)^{m_2^+} + B_2(\overline{x}_*^0)^{d_2^+(q_1)} + g_{12}^1, \tag{3.15}$$

$$\hat{K}_1^1(\underline{x}_*^0)^p + \bar{K}_1^1(\underline{x}_*^0)^{m_1^-} + A_2(\underline{x}_*^0)^{d_1^-(q_1)} + g_{21}^1 = \hat{K}_2^1(\underline{x}_*^0)^p + \bar{K}_2^1(\underline{x}_*^0)^{m_2^+} + B_2(\underline{x}_*^0)^{d_2^+(q_1)}, \tag{3.16}$$

where the first two equations are the coefficients of y^q and the last two are the coefficients of y^{q_1}.

The solution to this system yields:

$$\begin{aligned}
A_1 &= \frac{(\overline{x}_*^0)^{d_2^+(q)}(K_2^1 - K_1^1 - g_{21}^1) - (\underline{x}_*^0)^{d_2^+(q)}(K_2^1 - K_1^1 + g_{12}^1)}{(\overline{x}_*^0)^{d_2^+(q)}(\underline{x}_*^0)^{d_1^-(q)} - (\overline{x}_*^0)^{d_1^-(q)}(\underline{x}_*^0)^{d_2^+(q)}}, \\
B_1 &= \frac{(\overline{x}_*^0)^{d_1^-(q)}(K_2^1 - K_1^1 - g_{21}^1) - (\underline{x}_*^0)^{d_1^-(q)}(K_2^1 - K_1^1 + g_{12}^1)}{(\overline{x}_*^0)^{d_2^+(q)}(\underline{x}_*^0)^{d_1^-(q)} - (\overline{x}_*^0)^{d_1^-(q)}(\underline{x}_*^0)^{d_2^+(q)}}, \\
A_2 &= \frac{\left((\hat{K}_2^1 - \hat{K}_1^1)(\underline{x}_*^0)^p + \bar{K}_2^1(\underline{x}_*^0)^{m_2^+} - \bar{K}_1^1(\underline{x}_*^0)^{m_1^-} - g_{21}^1\right)(\overline{x}_*^0)^{d_2^+(q_1)}}{(\overline{x}_*^0)^{d_2^+(q_1)}(\underline{x}_*^0)^{d_1^-(q_1)} - (\overline{x}_*^0)^{d_1^-(q_1)}(\underline{x}_*^0)^{d_2^+(q_1)}} \\
&\quad - \frac{\left((\hat{K}_2^1 - \hat{K}_1^1)(\overline{x}_*^0)^p + \bar{K}_2^1(\overline{x}_*^0)^{m_2^+} - \bar{K}_1^1(\overline{x}_*^0)^{m_1^-} + g_{12}^1\right)(\underline{x}_*^0)^{d_2^+(q_1)}}{(\overline{x}_*^0)^{d_2^+(q_1)}(\underline{x}_*^0)^{d_1^-(q_1)} - (\overline{x}_*^0)^{d_1^-(q_1)}(\underline{x}_*^0)^{d_2^+(q_1)}}, \\
B_2 &= \frac{\left((\hat{K}_2^1 - \hat{K}_1^1)(\underline{x}_*^0)^p + \bar{K}_2^1(\underline{x}_*^0)^{m_2^+} - \bar{K}_1^1(\underline{x}_*^0)^{m_1^-} - g_{21}^1\right)(\overline{x}_*^0)^{d_1^-(q_1)}}{(\overline{x}_*^0)^{d_2^+(q_1)}(\underline{x}_*^0)^{d_1^-(q_1)} - (\overline{x}_*^0)^{d_1^-(q_1)}(\underline{x}_*^0)^{d_2^+(q_1)}} \\
&\quad - \frac{\left((\hat{K}_2^1 - \hat{K}_1^1)(\overline{x}_*^0)^p + \bar{K}_2^1(\overline{x}_*^0)^{m_2^+} - \bar{K}_1^1(\overline{x}_*^0)^{m_1^-} + g_{12}^1\right)(\underline{x}_*^0)^{d_1^-(q_1)}}{(\overline{x}_*^0)^{d_2^+(q_1)}(\underline{x}_*^0)^{d_1^-(q_1)} - (\overline{x}_*^0)^{d_1^-(q_1)}(\underline{x}_*^0)^{d_2^+(q_1)}}.
\end{aligned}$$

We can now state the rest of our assumptions. These last set of assumptions are needed in addition to the previous assumptions to rigorously prove the $O(\varepsilon)$ expansion in the switching problem.

Assumption 3.3. *In addition to Assumption 3.2, we will also assume that $\beta > 0$ is big enough so that $d_1^-(q_1) < 0,\ d_1^-(q) < 0,\ d_2^+(q_1) > \max\{1, p\},\ d_2^+(q) > \max\{1, p\}$, where $d^\pm$ were defined in (3.12). Also, assume that if $\sigma_i > \sigma_j,\ i, j = 1, 2,\ i \neq j$, then $\beta > 0$ is big enough such that all the other terms $\left(\sigma_i^2(m_1^- + m_2^+ - 1) + 2\mu_i\right),\ \left(\sigma_i^2(d_1^-(q) + d_2^+(q) - 1) + 2\mu_i\right),\ \left(\sigma_i^2(d_1^-(q_1) + d_2^+(q_1) - 1) + 2\mu_i\right) < 0$ are all negative, and vice versa, if $\sigma_i < \sigma_j$, then all the above quantities are positive.*

Moreover, we will also assume that

$$K_2^1 > K_1^1,\ \tilde{K}_2^1 > \tilde{K}_1^1,\ \bar{K}_2^1 > \bar{K}_1^1,\ \hat{K}_2^1 > \hat{K}_1^1, \tag{3.17}$$

and that $\beta > 0$ is big enough, to make all of the quantities in (3.17) positive.

Notably, we will assume that the long-term growth of the economy (after) lockdown is at least as large as when it is open. The true price of the lockdown is expressed in the cost g_{21}, while K_i^1 are the long-term growth of the (inverse) of the economy *given a constant state*. Therefore, intuitively after a big contraction we expect a big expansion. The numerical calibration of Section 5 in fact finds that they most likely identical, and the difference is well within the statistical error: $K_2^1 - K_1^1 \approx 3 \times 10^{-6}$.

It now follows from (3.4), (3.17) that

$$A_1, A_2, B_1, B_2 > 0.$$

Finally, the switching regions will be updated to

$$\mathcal{S}_2 = [\underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot), \infty) \times \mathbb{R}^+, \ \mathcal{S}_1 = (0, \overline{x}_*^0 + \varepsilon \overline{x}_*^1(\cdot)] \times \mathbb{R}^+, \tag{3.18}$$

where $\underline{x}_*^1, \overline{x}_*^1$ are defined to satisfy the smooth pasting condition, i.e. they solve:

$$(\partial^+_{xx} v_1^0)(\overline{x}_*^0)\overline{x}_*^1 + (\partial^+_x v_1^1)(\overline{x}_*^0, y) = (\partial^-_{xx} v_1^0)(\overline{x}_*^0)\overline{x}_*^1 + (\partial^-_x v_1^1)(\overline{x}_*^0, y),$$
$$(\partial^+_{xx} v_2^0)(\underline{x}_*^0)\underline{x}_*^1 + (\partial^+_x v_2^1)(\underline{x}_*^0, y) = (\partial^-_{xx} v_2^0)(\underline{x}_*^0)\underline{x}_*^1 + (\partial^-_x v_2^1)(\underline{x}_*^0, y).$$

As opposed to the $O(1)$ these are linear equations in $\overline{x}_*^1, \underline{x}_*^1$. It turns out they must be in fact functions of y, and are given by:

$$\begin{aligned}
\overline{x}_*^1(y) &= \frac{(\partial^-_x v_1^1)(\overline{x}_*^0, y) - (\partial^+_x v_1^1)(\overline{x}_*^0, y)}{(\partial^+_{xx} v_1^0)(\overline{x}_*^0) - (\partial^-_{xx} v_1^0)(\overline{x}_*^0)} \\
&= y^{q_1} \frac{B_2 d_1^+(q)(\overline{x}_*^0)^{d_1^+(q)-1} + \bar{K}_2^1 m_2^+(\overline{x}_*^0)^{m_2^+-1} + \hat{K}_2^1 p(\overline{x}_*^0)^{p-1} - A_2 d_1^-(q)(\overline{x}_*^0)^{d_1^-(q)-1} - \bar{K}_1^1 m_1^-(\overline{x}_*^0)^{m_1^--1} - \hat{K}_1^1 p(\overline{x}_*^0)^{p-1}}{A m_1^-(m_1^- - 1)(\overline{x}_*^0)^{m_1^--2} - B m_2^+(m_2^+ - 1)(\overline{x}_*^0)^{m_2^+-2} + p(p-1)(K_1^0 - K_2^0(\overline{x}_*^0)^{p-2}} \\
&\quad + y^q \frac{B_1 d_1^+(q)(\overline{x}_*^0)^{d_1^+(q)-1} - A_1 d_1^-(q)(\overline{x}_*^0)^{d_1^-(q)-1}}{A m_1^-(m_1^- - 1)(\overline{x}_*^0)^{m_1^--2} - B m_2^+(m_2^+ - 1)(\overline{x}_*^0)^{m_2^+-2} + p(p-1)(K_1^0 - K_2^0(\overline{x}_*^0)^{p-2}}, \\
\underline{x}_*^1(y) &= \frac{(\partial^-_x v_2^1)(\underline{x}_*^0, y) - (\partial^+_x v_2^1)(\underline{x}_*^0, y)}{(\partial^+_{xx} v_2^0)(\underline{x}_*^0) - (\partial^-_{xx} v_2^0)(\underline{x}_*^0)} \\
&= y^{q_1} \frac{B_2 d_1^+(q)(\underline{x}_*^0)^{d_1^+(q)-1} + \bar{K}_2^1 m_2^+(\underline{x}_*^0)^{m_2^+-1} + \hat{K}_2^1 p(\underline{x}_*^0)^{p-1} - A_2 d_1^-(q)(\underline{x}_*^0)^{d_1^-(q)-1} - \bar{K}_1^1 m_1^-(\underline{x}_*^0)^{m_1^--1} - \hat{K}_1^1 p(\underline{x}_*^0)^{p-1}}{A m_1^-(m_1^- - 1)(\underline{x}_*^0)^{m_1^--2} - B m_2^+(m_2^+ - 1)(\underline{x}_*^0)^{m_2^+-2} + p(p-1)(K_1^0 - K_2^0(\underline{x}_*^0)^{p-2}} \\
&\quad + y^q \frac{B_1 d_1^+(q)(\underline{x}_*^0)^{d_1^+(q)-1} - A_1 d_1^-(q)(\underline{x}_*^0)^{d_1^-(q)-1}}{A m_1^-(m_1^- - 1)(\underline{x}_*^0)^{m_1^--2} - B m_2^+(m_2^+ - 1)(\underline{x}_*^0)^{m_2^+-2} + p(p-1)(K_1^0 - K_2^0(\underline{x}_*^0)^{p-2}}.
\end{aligned}$$

Order $O(\varepsilon^2)$: We would like to stop now, but since we want to build solutions of the HJB equation (2.5), if we attempt to plugin the expansion that we have found $v_i^0(x) + \varepsilon v_i^1(x, y)$, we will have terms of $O(\varepsilon^2)$. Therefore to build sub- and super-solutions, we have to perturb them at the order of $O(\varepsilon^2)$ as follows:

$$v^2(x, y) = (1 + x^p)\left(y^{q_1} + y^{2q_1}\right) M\left(1 + y^q + y^{q_1}\right) \tag{3.19}$$

where the constant $M > 0$ big enough, to be determined later.

4 Main Results

We are now ready to state the main result. Not surprisingly first we will prove that our expansion can be used to build sub- and super-solutions. This is done in Theorem 4.1. We then could appeal to comparison (uniqueness) argument, as in Ly Vath and Pham [2007], but this requires (at most) linear growth for the comparison argument to hold. Instead, we will appeal to a standard stochastic calculus technique and using the sub- and super-slutions build sub- and supermartigales. We will then sandwich the value function between the sub- and supermartingales, and since their expansion matches at the order $O(\varepsilon)$ we will get the expansion of the value function. More importantly, in the process we will get that the stopping regions defined in (3.18) are "nearly-optimal". This is done in Theorem 4.2.

Theorem 4.1. *Let $N = 2$ and $\rho = 0$. Assume that $\sigma_1 \neq \sigma_2$, and also that g^1_{21}, g^1_{12}, satisfying (3.1), and $\left|g^1_{21}\right| >, \left|g^1_{12}\right| > 0$ small enough. Then, under Assumption 3.3 there exists $M > 0$, such that for $\varepsilon > 0$ small enough*

$$v^{\pm}_i(x,y) = v^0_i(x) + \varepsilon v^1_i(x,y) \pm \varepsilon^2 v^2(x,y),\ i = 1,2,$$

are super- and -solutions of the HJB equation (2.5).

Proof. We will show that $v^{\pm}_i,\ i = 1,2$ are viscosity super- and sub-solutions to the HJB equation (2.5). By symmetry it is sufficient to verify this for $i = 2$. To verify that v^+_2 is a viscosity super-solution of (2.5) we will show that $v^+_2 = v^+_1 + g_{21}$ on $[\underline{x}^0_*, \infty) \times \mathbb{R}^+$ and that $\beta v^+_2 - \mathcal{L}^{\varepsilon}_2 v^+_2 - \mathcal{U}_p(x) - \varepsilon \mathcal{U}_q(y) \geq 0$ on $(0, \underline{x}^0_*) \times \mathbb{R}^+$, and conclude that (2.5) holds for v^+_2. The viscosity sub-solution property is not symmetrical, as we need to show that $v^-_2 - v^-_1 - g_{21} \leq 0$ everywhere, and that $\beta v^-_2 - \mathcal{L}^{\varepsilon}_2 v^-_2 - \mathcal{U}_p(x) - \varepsilon \mathcal{U}_q(y) \leq 0$ where v^-_2 is smooth, and comment what happens at the point when it's not. There the proof is divided into a few steps, as follows.

Step 1: We need to show that

$$v^{\pm}_2 \leq v^{\pm}_1 + g_{21}. \tag{4.1}$$

Step 1.a: We first work on the $O(1)$ term, and show that

$$v^0_2 \leq v^0_1 + g^0_{21}. \tag{4.2}$$

This is trivial on $[\underline{x}^0_*, \infty) \times \mathbb{R}^+$, since $v^0_2 = v^0_1 + g^0_{21}$ by definition. Moreover, on $(0, \overline{x}^0_*) \times \mathbb{R}^+$, we have that $v^0_1 = v^0_2 + g^0_{12}$ by construction. Using the fact that $g^0_{12} > -g^0_{21}$, (4.2) now follows.

Step 1.b: Still considering the $O(1)$ term, we now show that (4.2) is true on the set $(\overline{x}^0_*, \underline{x}^0_*) \times \mathbb{R}^+$. Consider

$$G_1(x) = v^0_2 - v^0_1 - g^0_{21} = Bx^{m^+_2} - Ax^{m^-_1} + (K^0_2 - K^0_1)x^p - g^0_{21}.$$

The proof is similar to Ly Vath and Pham [2007][Theorem 4.1]. From (3.8), (3.9) we have that $G_1(\overline{x}^0_*) = -g^0_{12} - g^0_{21} < 0$, and $G_1(\underline{x}^0_*) = 0$. We claim that $G_1 \leq 0$. Indeed, G'_1 can only have at most two zeroes on $(0\,\infty)$, because G''_1 changes signs once on $(0, \infty)$. Therefore, if there would have been another root x_0 of G_1, then there would have been another point $x_1 \in (x_0, \underline{x}^0_*)$ such that $G'_1(x_1) = G'_1(\underline{x}^0_*) = G'_1(\overline{x}^0_*) = 0$. This contradicts the fact that G'_1 has (at most) two roots .

Step 1.c : We now consider the $O(\varepsilon)$ term of (4.1). It is sufficient that we show that

$$v^1_2 \leq v^1_1 + g^1_{21}(y^q + y^{q_1}). \tag{4.3}$$

Similar to *Step 1.a* we get that (4.3) holds on $[\underline{x}^0_* + \varepsilon \underline{x}^1_*(\cdot), \infty) \times \mathbb{R}^+ \cup (0, \overline{x}^0_* + \varepsilon \overline{x}^1_*(\cdot)) \times \mathbb{R}^+$.

Step 1.d : Lastly on $(\overline{x}^0_* + \varepsilon \overline{x}^1_*(\cdot), \underline{x}^0_* + \varepsilon \underline{x}^1_*(\cdot)) \times \mathbb{R}^+$, consider again

$$\begin{aligned}
v^1_2 - v^1_1 - g^1_{21}(y^q + y^{q_1}) &= y^q G_2(x) + y^{q_1} G_3(x) \\
&= y^q \left(K^1_2 - K^1_1 + B_1 x^{d^+_2(q)} - A_1 x^{d^-_1(q)} - g^1_{21}\right) \\
&\quad + y^{q_1} \left(\bar{K}^1_2 x^{m^+_2} - \bar{K}^1_1 x^{m^-_1} + (\hat{K}^1_2 - \hat{K}^1_1)x^p + B_2 x^{d^+_2(q_1)} - A_2 x^{d^-_1(q_1)} - g^1_{21}\right).
\end{aligned}$$

Similar to *Step 1.b*, from (3.13), (3.15) we have that $G_2(\overline{x}^0_* + \varepsilon \overline{x}^1_*(y)) = -g^1_{12} - g^1_{21} + O(\varepsilon^2) < 0$, and $G_2(\underline{x}^0_* + \varepsilon \underline{x}^1_*(y)) = O(\varepsilon^2)$. Similar to *Step 1.b*, we again obtain that $G_2 \leq O(\varepsilon^2)$ on the set. G_3 is treated similar.

Step 1.e : From the definition (3.19), by continuity, and by accounting for the $O(\varepsilon^2)$ term, from *Step 1.a,c* it now follows that $v^+ \le v_1^+ + g_{21}$ on $[\underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot), \infty) \times \mathbb{R}^+$. Additionally, we have shown in *Step 1.a,b* that $v_2^0 - v_1^0 - g_{21}^0 \le 0$ everywhere on the union of $\left((\underline{x}_*^0, \infty) \cup (0, \overline{x}_*^0) \cup (\overline{x}_*^0, \underline{x}_*^0)\right) \times \mathbb{R}^+$. By continuity of v_i^0, $i = 1, 2$ we have it on the entire $\mathbb{R}^{+2}$. Therefore, for $\varepsilon > 0$ small enough, it follows that $v_2^1 - v_1^1 - g_{21}^1(y^q + y^{q_1}) \le (y^q + y^{q_1})O(\varepsilon^2)$ on $\left((\underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot), \infty) \cup (0, \overline{x}_*^0 + \varepsilon \overline{x}_*^1(\cdot)) \cup (\overline{x}_*^0 + \varepsilon \overline{x}_*^1(\cdot), \underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot))\right) \times \mathbb{R}^+$. By continuity, again it follows that $v_2^1 - v_1^1 - g_{21}^1(y^q + y^{q_1}) \le (y^q + y^{q_1})O(\varepsilon^2)$ on the entire $\mathbb{R}^{+2}$, and thus $v_2^- - v_1^- - g_{21} \le 0$.

Step 2: We now deal with the second order PDE :

$$\pm\left(\beta v_2^\pm - \mathcal{L}_2^\varepsilon v_2^\pm - \mathcal{U}_p(x) - \varepsilon \mathcal{U}_q(y)\right) \ge 0.$$

Step 2.a: A technical, but simple calculation shows that on the set $(0, \underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot)) \times \mathbb{R}^+$

$$\begin{aligned}
&\beta v_2^\pm - \mathcal{L}_2^\varepsilon v_2^\pm - \mathcal{U}_p(x) - \varepsilon \mathcal{U}_q(y) \qquad (4.4)\\
&= \varepsilon^2 \left(y^q \left(\pm M(\beta - \mu_2 q + \frac{\sigma_2^2}{2} q^2(1-q)) + O(1) \right) + y^{q_1} \left(\pm M(\beta - \mu_2 q_1 + \frac{\sigma_2^2}{2} {q_1}^2 (1 - q_1)) + O(1) \right) \right) y^{2q_1},
\end{aligned}$$

where $O(1)$ is uniform in x. Therefore, for $\varepsilon > 0$ small enough, the sign of (4.4) is the same as that of $\pm M$, for $M > 0$ big enough. Note that $M > 0$ is chosen independently of x and y.

Step 2.b: We now want to show that consider the points in the set $(\underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot), \infty) \times \mathbb{R}^+$. We first evaluate the $O(1)$ term, for which it is the same as considering the set $(\underline{x}_*^0, \infty) \times \mathbb{R}^+$. Recall that v^0 is continuously differential at $\underline{x}_*^0$, whereas the differences in the second derivatives evaluates to:

$$\lim_{x \to (\underline{x}_*^0)^+} \partial_{xx} v_2^0(x) - \lim_{x \to (\underline{x}_*^0)^-} \partial_{xx} v_2^0(x) = \frac{g_{12}^0 m_1^- m_2^+ + (K_1^0 - K_2^0)(m_1^- - p)(m_2^+ - p)(\underline{x}_*^0)^p}{(\underline{x}_*^0)^2} > 0.$$

Therefore at the order $O(1)$, using the results of the previous step, we have that

$$\begin{aligned}
&\lim_{x \to (\underline{x}_*^0)^+} \beta v_2^0 - \mathcal{L}_2^0 v_2^0 - \mathcal{U}_p(x) \qquad (4.5)\\
&= \lim_{x \to (\underline{x}_*^0)^-} \beta v_2^0 - \mathcal{L}_2^0 v_2^0 - \mathcal{U}_p(x) - \left(g_{12}^0 m_1^- m_2^+ + (K_1^0 - K_2^0)(m_1^- - p)(m_2^+ - p)(\underline{x}_*^0)^p\right) \frac{\sigma_2^2}{2} \le 0.
\end{aligned}$$

Furthermore, on the set $(\underline{x}_*^0, \infty) \times \mathbb{R}^+$, with $L = (2\mu_1 - 2\mu_2 + (1 - m_1^-)(\sigma_2^2 - \sigma_1^2))$, we find that

$$\beta v_2^0 - \mathcal{L}_2^0 v_2^0 - \mathcal{U}_p(x) = \frac{(K_1^0 - K_2^0)}{K_2^0} x^p + m_1^- L A x^{m_1^-} + \beta g_{21}^0.$$

To conclude that,

$$\beta v_2^0 - \mathcal{L}_2^0 v_2^0 - \mathcal{U}_p(x) \le 0, \ \ x > \underline{x}_*^0, \qquad (4.6)$$

consider two cases:

If $L < 0$ the derivative $\partial_x \left(\beta v_2^- - \mathcal{L}_2^0 v_2^- - \mathcal{U}_p(x)\right) < 0$ is negative, and together with (4.5), (4.6) now follows.

If $L \ge 0$, then it is sufficient to show that $\frac{(K_1^0 - K_2^0)}{K_2^0}(\underline{x}_*^0)^p + \beta g_{21}^0 \le 0$. This is true because $0 < A$ and it can be expressed as $A = \frac{(\underline{x}_*^0)^{-m_1^-}\left((K_1^0 - K_2^0)(m_2^+ - p)(\underline{x}_*^0)^p + g_{21}^0 m_2^+\right)}{m_1^- - m_2^+}$. Thus $(K_1^0 - K_2^0)(\underline{x}_*^0)^p + g_{21}^0 \frac{m_2^+}{m_2^+ - p} < 0$. We have that $\frac{1}{2}\sigma_2 m_2^+(p-1) < \frac{1}{2}\sigma_2^2 m_2^+(m_2^+ - 1) = \beta - \mu_2 p$. Therefore, $\frac{m_2^+}{m_2^+ - p} \ge \beta K_2^0 = \frac{\beta}{\beta - \mu_i p + \frac{\sigma_i^2}{2} p(1-p)}$. Thus the desired result finally follows from the fact that $(K_1^0 - K_2^0)(\underline{x}_*^0)^p + g_{21}^0 \beta K_2^0 \le (K_1^0 - K_2^0)(\underline{x}_*^0)^p + g_{21}^0 \frac{m_2^+}{m_2^+ - p} < 0$.

Step 2.c: Finally, on set $(\underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot), \infty) \times \mathbb{R}^+$ a technical calculation shows that the $O(\varepsilon)$ term evaluates to

$$\begin{aligned}
&\left(\beta v_2^1 - \mathcal{L}_2^0 v_2^1 - \mathcal{L}^1 v_2^0\right)(x,y) - \mathcal{U}_q(y) \\
&= y^q \left(\frac{1}{K_2^1}\left(g_{21}^1 + K_1^1 - K_2^1\right) + A_1 x^{d_1^-(q)} (d_1^+(q) - d_1^-(q)) \left(\frac{\sigma_2^2}{2}(d_1^-(q) + d_1^+(q) - 1) + \mu_2 \right)\right) \\
&\quad + \frac{y^{q_1}}{2} \Bigg(g_{21}^1 \frac{2}{\tilde{K}_2^1} + x^p \left(-\frac{2}{\hat{K}_1^1} \left(\frac{1}{K_1^0} - \frac{1}{K_2^0} + \frac{1}{\tilde{K}_1^1} - \frac{1}{\tilde{K}_2^1} \right)\right) \\
&\qquad + x^{d_1^-(q_1)} A_2 (d_2^+(q_1) - d_2^-(q_1)) \left(\sigma_2^2 (d_2^-(q_1) + d_2^+(q_1) - 1) + 2\mu_2 \right) \\
&\qquad + x^{m_1^-} \left(q_1 \left(\frac{1}{\tilde{K}_2^1} - \frac{1}{\tilde{K}_1^1} \right) + (m_2^+ - m_1^-) \left(\sigma_2^2 (m_1^- + m_2^+ - 1) + 2\mu_2 \right)\right)\Bigg).
\end{aligned} \tag{4.7}$$

Recall that we have assumed that $\sigma_1 \neq \sigma_2$. Therefore, by Assumption 3.3 the coefficients of $x^{d_1^-(q)}$, $x^{d_1^-(q_1)}$, x^{m_1-} have the same signs as the sign of the expression $\sigma_2 - \sigma_1$. Moreover, by Assumption 3.3 $K_1^1 - K_2^1 < 0$, and we may assume that $g_{21}^1 > 0$ is small enough, so that $g_{21}^1 + K_1^1 - K_2^1 \le 0$ too. Additionally, the sign of the coefficient of x^p is $\left(-\frac{2}{\hat{K}_1^1} \left(\frac{1}{K_1^0} - \frac{1}{K_2^0} + \frac{1}{\tilde{K}_1^1} - \frac{1}{\tilde{K}_2^1} \right)\right) < 0$, and we assume that $g_{21}^1 > 0$ is small enough so that $g_{21}^1 \frac{2}{\tilde{K}_2^1} + (\underline{x}_*^0)^p \left(-\frac{2}{\hat{K}_1^1} \left(\frac{1}{K_1^0} - \frac{1}{K_2^0} + \frac{1}{\tilde{K}_1^1} - \frac{1}{\tilde{K}_2^1} \right)\right) \le 0$. Therefore, if $\sigma_2 > \sigma_1$, then all the coefficients of $x^{d_1^-(q)}$, $x^{d_1^-(q_1)}$, x^{m_1-} are negative, and so is the entire (4.7). Otherwise, if $\sigma_1 > \sigma_2$, then the coefficients of $x^{d_1^-(q)}$, $x^{d_1^-(q_1)}$, x^{m_1-} are positive, and these terms are decreasing in x. The coefficient of x^p is still negative, and so the entire (4.7) decreases in x. Therefore, to show the negativity of (4.7), it is sufficient to verify that $\lim_{(x,y)\to((\underline{x}_*^0+\varepsilon\underline{x}_*^1(y))^+,y)} \left(\beta v_2^1 - \mathcal{L}_2^0 v_2^1 - \mathcal{L}^1 v_2^0\right)(x,y) - \mathcal{U}_q(y) < 0$.

Using the , and the fact that across $(\underline{x}_*^0 + \varepsilon \underline{x}_*^1(y), y)$ both v_2^1 and v_2^0 are continuously differentiable in x and smooth in y, but not twice continuously differentiable in x, we have that

$$\begin{aligned}
&\lim_{(x,y)\to((\underline{x}_*^0+\varepsilon\underline{x}_*^1(y))^+,y)} \left(\beta v_2^1 - \mathcal{L}_2^0 v_2^1 - \mathcal{L}^1 v_2^0\right)(x,y) - \mathcal{U}_q(y) \\
&= \lim_{(x,y)\to((\underline{x}_*^0+\varepsilon\underline{x}_*^1(y))^-,y)} \left(\beta v_2^1 - \mathcal{L}_2^0 v_2^1 - \mathcal{L}^1 v_2^0\right)(x,y) - \mathcal{U}_q(y) \\
&- \frac{\sigma_2^2}{2} \Bigg(\bar{K}_1^1 y^{q_1} m_1^-(m_1^- - 1)(\underline{x}_*^0)^{m_1^-} + p(p-1)\hat{K}_1^1 y^{q_1} x^p + A_1 d_1^-(q)(d_1^-(q)-1) y^q x^{d_1^-(q)} \\
&\qquad + A_2 d_1^-(q_1)(d_1^-(q_1)-1) x^{d_1^-(q_1)} y^{q_1} \Bigg) \\
&+ \frac{\sigma_2^2}{2} \Bigg(\bar{K}_2^1 y^{q_1} m_2^+(m_2^+ - 1)(\underline{x}_*^0)^{m_2^+} + p(p-1)\hat{K}_2^1 y^{q_1} x^p + B_1 d_2^+(q)(d_2^+(q)-1) y^q x^{d_2^+(q)} \\
&\qquad + B_2 d_2^+(q_1)(d_2^+(q_1)-1) x^{d_2^+(q_1)} y^{q_1} \Bigg).
\end{aligned}$$

This right hand is indeed negative because the fact that fact that $\lim_{(x,y)\to((\underline{x}_*^0+\varepsilon\underline{x}_*^1(y))^-,y)} \left(\beta v_2^1 - \mathcal{L}_2^0 v_2^1 - \mathcal{L}^1 v_2^0\right)(x,y) - \mathcal{U}_q(y) = 0$, together with using (3.14), (3.16), and the fact $\sigma_1 > \sigma_2$ and therefore $m_1^-(m_1^- - 1) > m_2^+(m_2^+ - 1)$, $d_1^-(q)(d_1^-(q) - 1) > d_2^+(q_1)(d_2^+(q_1) - 1)$, and $d_1^-(q_1)(d_1^-(q_1) - 1) > d_2^+(q_1)(d_2^+(q_1) - 1)$.

Step 2.d: The $O(\varepsilon^2)$ terms on $(\underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot), \infty) \times \mathbb{R}^+$ are the same as in (the right hand side of) (4.4). Therefore, the sign of the $O(\varepsilon^2)$ will the same as the sign of $\pm M$ once $M > 0$ is big enough.

Step 3: Summary. The proof is now complete. Indeed, we have shown in *Step 1.a* that $v_2^+ = v_1^+ + g_{21}$ on $[\underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot), \infty) \times \mathbb{R}^+$ and in *Step 2.a* that $\beta v_2^+ - \mathcal{L}_2^\varepsilon v_2^+ - \mathcal{U}_p(x) - \varepsilon \mathcal{U}_q(y) \geq 0$ on $(0, \underline{x}_*^0 + \varepsilon \underline{x}_*^1(\cdot)) \times \mathbb{R}^+$. Therefore, v_2^+ is a super-solution of (2.5).

Similarly, we have concluded in *Step 1.e* that $v_2^- - v_1^- - g_{21} \leq 0$ on the entire $\mathbb{R}^{+2}$. Similarly, we have shown in *Step 2.a,b,c* that $\beta v_2^- - \mathcal{L}_2^\varepsilon v_2^- - \mathcal{U}_p(x) - \varepsilon \mathcal{U}_q(y) \leq 0$ on the union $\mathbb{R}^{+2}$, except on $\{\underline{x}_*^0\} \times \mathbb{R}^+$ and on the curve $y \mapsto (\underline{x}_*^0 + \underline{x}_*^1(y), y)$. However, v_2^- is continuously differentiable even at those points, because of the construction, and it may not be twice continuously differentiable in x only. However, the second derivative in x jumps down, as shown in *Step 2.b,c*. Therefore, it must satisfy $\beta v_2^- - \mathcal{L}_2^\varepsilon v_2^- - \mathcal{U}_p(x) - \varepsilon \mathcal{U}_q(y) \leq 0$ in the viscosity sense there. This completes the proof.

□

We are now ready to bind the value functions in between the sub- and super-solutions. As mentioned earlier, we will utilize Theorem 4.1 to show that the processes $\mathrm{e}^{-\beta t} v^\pm(X_t, Y_t) + \int_0^t \mathrm{e}^{-\beta s} \left(\mathcal{U}_p(X_s) + \varepsilon \mathcal{U}_q(Y_s)\right) ds$ are sub- and super-martingales. More specifically:

Theorem 4.2. *Under the same assumptions as in Theorem 4.1, for $\varepsilon > 0$ small enough,*

$$v_i^-(x, y) \leq v_i(x, y) \leq v_i^+(x, y), \tag{4.8}$$

and therefore

$$\left|v_i(x, y) - v_i^\pm(x, y)\right| \leq (1 + x^p)\left(y^{q_1} + y^{2q_1}\right)(1 + y^q + y^{q_1})\, O(\varepsilon^2). \tag{4.9}$$

Moreover, let $(\tilde{I}, \tilde{X}, \tilde{Y})$ be the process associated with the stopping regions $\mathcal{S}_1$, $\mathcal{S}_2$ in (3.18) and starting from (i, x, y). Then the strategy $\tilde{I}_t$ is "nearly optimal", in that if followed, the difference between it and the value function is:

$$\left|\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(\tilde{X}_t) + \varepsilon \mathcal{U}_q(\tilde{Y}_t)\right) dt + \sum_{n \geq 1} \mathrm{e}^{-\beta \tau_n} g_{\iota_{n-1}, \iota_n}\right] - v_i(x, y)\right| \leq (1 + x^p)\left(y^{q_1} + y^{2q_1}\right)(1 + y^q + y^{q_1})\, O(\varepsilon^2). \tag{4.10}$$

Proof. First, we have that $v^\pm$ are continuously differentiable, and with one-sided limits for the second derivative. Let (I, X, Y) be any switching process starting from (i, x, y). Fix $T > 0$, then by Itô's lemma:

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-\beta T} v_i^-(X_T, Y_T) - v_i^-(x, y)\right] \\
&= \mathbb{E}\left[\sum_{j=0}^\infty \int_{T \wedge \tau_j}^{T \wedge \tau_{j+1}} \left(\partial_t + \mathcal{L}_2^\varepsilon\right)\left(\mathrm{e}^{-\beta t} v^-(X_t, Y_t)\right) + \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(X_t) + \varepsilon \mathcal{U}_q(Y_t)\right) dt\right] \\
&\quad - \mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(X_t) + \varepsilon \mathcal{U}_q(Y_t)\right) dt\right] \\
&\geq -E\left[\int_0^T \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(X_t) + \varepsilon \mathcal{U}_q(Y_t)\right) dt\right].
\end{aligned}$$

where in the first equality we have used the fact that the expectation of the stochastic term is zero, and Theorem 4.1 to deduce the non-negativity of the first expectation. Notice, that we have shown that $\mathrm{e}^{-\beta t} v^-(X_t, Y_t) + \int_0^t \mathrm{e}^{-\beta s}\left(\mathcal{U}_p(X_s) + \varepsilon \mathcal{U}_q(Y_s)\right) ds$ is a submartingale. Finally, we have that $\lim_{T \to \infty} \mathbb{E}\left[\mathrm{e}^{-\beta T} v^-(X_T, Y_T)\right] = 0$, because so are $\lim_{T \to \infty} \mathbb{E}[\mathrm{e}^{-\beta T} X_T^p] = \lim_{T \to \infty} \mathbb{E}[\mathrm{e}^{-\beta T} Y_T^{q+2q_1}] = \lim_{T \to \infty} \mathbb{E}[\mathrm{e}^{-\beta T} Y_T^{q_1}] = 0$. Therefore, taking an infimum over all possible strategies, starting from (i, x, y) we get:

$$v_i^-(x, y) \leq v_i(x, y). \tag{4.11}$$

Similarly, by considering $(\tilde{I}, \tilde{X}, \tilde{Y})$ to be the switching process associated with the stopping regions $\mathcal{S}_1,\ \mathcal{S}_2$ in (3.18) and starting from (i, x, y), we get that

$$
\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-\beta T}\, v_i^+(\tilde{X}_T, \tilde{Y}_T) - v_i^+(x, y)\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{\infty} \int_{T\wedge\tau_j}^{T\wedge\tau_{j+1}} \left(\partial_t + \mathcal{L}_2^{\varepsilon}\right)\left(\mathrm{e}^{-\beta t}\, v^+(\tilde{X}_t, \tilde{Y}_t)\right) + \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(\tilde{X}_t) + \varepsilon\,\mathcal{U}_q(\tilde{Y}_t)\right) dt\right] \\
&\quad - \mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(\tilde{X}_t) + \varepsilon\,\mathcal{U}_q(\tilde{Y}_t)\right) dt\right] \\
&\leq -\mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(\tilde{X}_t) + \varepsilon\,\mathcal{U}_q(\tilde{Y}_t)\right) dt\right].
\end{aligned}
$$

In other words $\mathrm{e}^{-\beta t}\, v^+(\tilde{X}_t, \tilde{Y}_t) + \int_0^t \mathrm{e}^{-\beta s}\left(\mathcal{U}_p(\tilde{X}_s) + \varepsilon\,\mathcal{U}_q(\tilde{Y}_s)\right) ds$ is a superartingale. Again, taking the limit as $T \to \infty$, we get

$$
v_i^+(x, y) \geq \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-\beta t}\left(\mathcal{U}_p(\tilde{X}_t) + \varepsilon\,\mathcal{U}_q(\tilde{Y}_t)\right) dt\right] \geq v_i(x, y), \tag{4.12}
$$

where the last inequality is be definition of the value function. Combining (4.11), (4.12) and remembering the definition of $v_i^{\pm}$ we get (4.8), (4.9), (4.10).

□

5 Numerical Case Study

We now calibrate the model parameters to the data and study the opening and closing boundaries and their approximations. We perform the calibration using he data from NY state. [1] We calibrated the parameters assuming the stay-at-home state 1 lasted between 03/14/2020 – 06/25/2020, and the subsequent open state 2 was between 06/26/2020–09/26/2020. We found $\mu_1 = -0.00056$, $\mu_2 = 0.669$, $\sigma_1 = 13.17$, $\sigma_2 = 3.468$, $\theta_1 = -0.00037$, $\theta_2 = 0.00068$, $\nu_1 = \nu_2 = 0.3206$. $q_1 = 0.0034$. Additionally, we have set $\beta = 20$, $\varepsilon = 0.3105$, $g_{12}^0 = -1$, $g_{21}^0 = 1.1$, $g_{21}^1 = 10^{-3} g_{21}^0$, $g_{12}^1 = 10^{-3} g_{12}^0$, $p = 0.5$, $q = 0.1$.

We assume the population of NY state is $20m$[2]. X is assumed to be the 7-day average number of infections in the population. First, we find that $\overline{x}_*^0 = 211.93$ and $\underline{x}_*^0 = 1245.9$. In other words, if closed the state should reopen with approximately 1 infections per 100k, whereas it should close up if more than 6 infections per 100k occur. We find this in the ballpark of the recommendations from Brown School of Public Health[3].

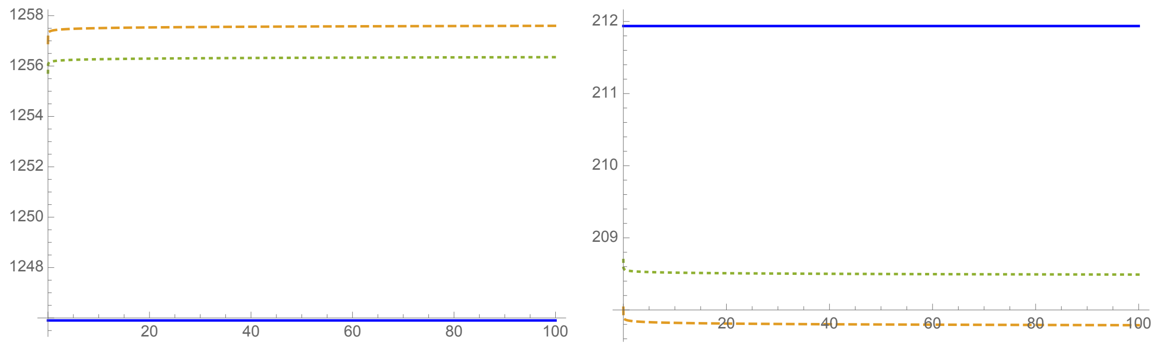
From Figure 1 we see that once the economy is added into the consideration the closing boundary (slightly) increases. The inverse happens to the opening boundaries. It is not surprising that the closing boundary increased when ε is no longer zero, as the cost of lockdown increased. The decrease of the reopening boundary may follow from the fact with $\varepsilon > 0$ the infection rate has also increased, and therefore, the lockdown needs to last longer when compared to the case $\varepsilon = 0$.

From Figure 2 we also see that the first order adjustment is a function of the economy. When the economy is doing well (and its inverse y is small) the adjustment to the opening boundary is a little higher, than it becomes when the economy underperforms. This is consistent with our assumption that in well performing economy the infection rate is a little lower, as people can afford to stay at home more, and thus

[1] NY state infection data from https://www1.nyc.gov/site/doh/covid/covid-19-data.page. NY economical data from https://fred.stlouisfed.org/series/NYNQGSP and https://fred.stlouisfed.org/series/NYINSUREDUR

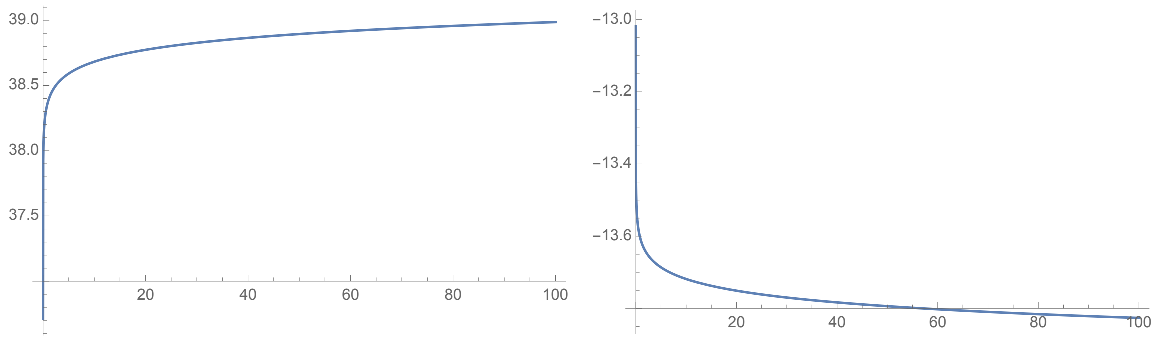
[2] https://en.wikipedia.org/wiki/New_York_(state)

[3] https://globalepidemics.org/



(a) Closing Boundary (b) Opening Boundary

Figure 1: Opening (left) and closing (right) boundaries. $O(1)$ approximation in thick blue, $O(\varepsilon)$ approximation in dashed orange, numerical solution in dotted green.



(a) $O(\varepsilon)$ adjustment to the closing boundary (b) $O(\varepsilon)$ adjustment to the opening boundary

Figure 2: Opening (left) and closing (right) boundaries with with $\varepsilon = 0.1$. $O(1)$ approximation in blue, $O(\varepsilon)$ approximation in orange, numerical solution in green.

the opening can happen a little sooner, as a function of the economy. Similarly, we observe the correction to the closing boundary is lower, when the economy is doing very well, because the minimization is being dragged up by the infection rate, therefore it is optimal to close sooner, and the economy can tolerate the closure better, and the infection rate will drop faster so that the economy can be re-opended sooner.

6 Conclusion

In conclusion we have asymptotically solved a two-regime switching problem with a two-dimensional underlying coupled diffusion. We have formulated and solved an infection optimization problem, whose solution approximately yield the popular, but heuristic strategy of opening and closing based on the number of infections per 100k population. Moreover, we have showed how the sub- and supersolutions can be used to rigorously construct bounds on an asymptotic expansion of the value function.

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