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Extension and trace results for doubling metric measure spaces and their hyperbolic fillings

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Abstract In this paper we study connections between Besov spaces of functions on a compact metric space Z, equipped with a doubling measure, and the Newton–Sobolev space of functions on a uniform domain X_ε. This uniform domain is obtained as a uniformization of a (Gromov) hyperbolic filling of Z. To do so, we construct a family of hyperbolic fillings in the style of Bonk–Kleiner [9] and Bourdon–Pajot [13]. Then for each parameter $\beta > 0$ we construct a lift μ_β of the doubling measure ν on Z to X_ε, and show that μ_β is doubling and supports a 1-Poincaré inequality. We then show that for each θ with $0 < \theta < 1$ and $p \geq 1$ there is a choice of $\beta = p(1-\theta)\log\alpha$ such that the Besov space $B^\theta_{p,p}(Z)$ is the trace space of the Newton–Sobolev space $N^{1,p}(X_\varepsilon, \mu_\beta)$ when $\varepsilon = \log\alpha$. Finally, we exploit the tools of potential theory on X_ε to obtain fine properties of functions in $B^\theta_{p,p}(Z)$, such as their quasicontinuity and quasieverywhere existence of L^q-Lebesgue points with $q = s_\nu p/(s_\nu - p\theta)$, where s_ν is a doubling dimension associated with the measure ν on Z. Applying this to compact subsets of Euclidean spaces improves upon a result of Netrusov [42] in $\mathbf{R}^n$.

Key words and phrases: Gromov hyperbolic space, hyperbolic filling, Poincaré inequality, doubling measure, uniform space, uniformization, trace and extension, Besov space, Besov capacity, Sobolev capacity, quasicontinuity, Lebesgue point.

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Contents

1 Introduction 2
2 Gromov hyperbolic spaces 6
3 Construction of hyperbolic fillings 9
4 The uniformized boundary $\partial_\varepsilon X$ 12
5 Uniformizing a hyperbolic filling with parameter $\varepsilon \leq \log\alpha$ 17
6 Roughly similar equivalence 19
7 Trees 22
8 Geodesics in the hyperbolic filling 24
9 Measures, function spaces and capacities 29

10 Lifting doubling measures from Z to its hyperbolic filling X **31**
11 Traces to Z from the hyperbolic filling X **37**
12 Extension from Z to its hyperbolic filling X **44**
13 Properties of Besov functions on Z **47**
References **51**

1. Introduction

Much of the current trend in first-order analysis (such as Sobolev type spaces and potential theory) on metric measure spaces assumes that the underlying space is at least locally compact, doubling and supports a Poincaré type inequality, see for example [2] and [27]. On doubling spaces that do not support any Poincaré inequality, there are other possible choices of function spaces that are, however, nonlocal. This means that the energy of functions in these spaces depends on their global behavior and can be nonzero on subsets where the functions vanish or are constant. Examples of such spaces include Besov, Triebel–Lizorkin and (fractional) Hajłasz–Sobolev spaces. As locality is a highly useful tool in potential theory and variational problems, it is desirable to seek an alternative approach to studying function spaces on nonsmooth metric spaces without Poincaré inequalities.

Using a construction termed *hyperbolic filling*, Bonk–Kleiner [9, Theorem 11.1] and Bourdon–Pajot [13] connected compact doubling metric spaces to Gromov hyperbolic spaces. *Gromov hyperbolicity* is a notion of negative curvature in the nonsmooth metric setting and, unlike Alexandrov curvature which covers all scales, it captures negative curvature at large scales by requiring that every point in a geodesic triangle is within a bounded distance from the other two sides. Gromov hyperbolicity has proven to be a highly useful tool in studying the conformal geometry of hyperbolic groups (Bridson–Haefliger [14] and Gromov [22]) and in understanding uniform domains (Bonk–Heinonen–Koskela [8], Bonk–Schramm [12] and Herron–Shanmugalingam–Xie [28]). We refer the interested reader to [14] and Buyalo–Schroeder [16] for more on synthetic notions of curvature in the metric setting and the hyperbolic filling technique, respectively.

In the current paper we contribute to the study of nonlocal analysis on compact doubling metric measure spaces by introducing measures to the above construction of hyperbolic fillings and by subsequently linking the nonlocal Besov spaces on compact doubling metric measure spaces to the Newtonian (Sobolev) spaces on uniformizations of their hyperbolic fillings, see Theorem 1.1.

One of the main results of the paper is that *every* Besov space $B^\theta_{p,p}(Z)$ with $0<\theta<1$ and $p \ge 1$ arises as a trace of a Newtonian space on a uniform domain, equipped with a doubling measure supporting a 1-Poincaré inequality. Specifically, we prove the following theorem.

Theorem 1.1. *Let Z be a compact metric space equipped with a doubling measure ν, X be a hyperbolic filling of Z with parameters $\alpha, \tau > 1$, and X_ε be the uniformization of X with parameter $\varepsilon = \log \alpha$.*

Then for each parameter $\beta > 0$ we can equip X_ε with a measure μ_β induced by ν so that μ_β is doubling and supports a 1-Poincaré inequality both on X_ε and its completion $\overline{X}_\varepsilon$. Moreover, for each $0<\theta<1$ and $1 \le p < \infty$, the Besov space $B^\theta_{p,p}(Z)$ is the trace space of the Newtonian space $N^{1,p}(X_\varepsilon, \mu_\beta)$ with $\beta = \varepsilon p(1-\theta)$.

More precisely, $\operatorname{Tr} \circ E$ is the identity map on $B^\theta_{p,p}(Z)$, where

$$\operatorname{Tr} : N^{1,p}(X_\varepsilon, \mu_\beta) \to B^\theta_{p,p}(Z) \quad \text{and} \quad E : B^\theta_{p,p}(Z) \to N^{1,p}(X_\varepsilon, \mu_\beta)$$

are bounded linear trace and extension operators, respectively.

In fact, we construct Tr for each $\theta \le 1 - \beta/\varepsilon p$ (see Theorem 11.3), and E for each $\theta \ge 1 - \beta/\varepsilon p$ (see Theorem 12.1). Roughly speaking, μ_β is constructed so that its β/ε-codimensional Hausdorff measure is equivalent to the measure ν on Z. Thus Theorem 1.1 follows immediately from combining Theorems 11.3 and 12.1.

The smoothness exponent $\theta = 1 - \beta/\varepsilon p > 0$ exactly corresponds to, and generalizes, the case of d-sets in unweighted $\mathbf{R}^n$, considered by Jonsson–Wallin [29], [30]. We also show that for ν-a.e. $z \in Z$ the trace $\operatorname{Tr} u(z)$ is achieved in three different ways, namely, as averaged pointwise limits (11.4) and (11.5), by Lebesgue point integral averages (11.17) and as a pointwise restriction from the Newtonian space $N^{1,p}(\overline{X}_\varepsilon)$.

Our study includes $p = 1$ and shows that the Besov space $B^\theta_{1,1}(Z)$ is the trace space of the Newtonian space $N^{1,1}(X_\varepsilon, \mu_\beta)$ when $\beta = (1-\theta)\varepsilon = (1-\theta)\log\alpha$. This is in contrast to the result of Gagliardo [19] that the trace space of the Sobolev space $W^{1,1}(\Omega)$ is $L^1(\partial\Omega)$ when Ω is a Lipschitz domain in $\mathbf{R}^n$. This trace operator is nonlinear, which is necessary according to a result due to Peetre [44]. See Malý [39, Section 7] and Malý–Shanmugalingam–Snipes [40, Theorem 1.2] for metric space analogs of this. The key difference between the setting of [40] and the current paper is that in [40] the measure on $\partial\Omega$ has codimension 1 relative to the measure on Ω, while we have codimension $\beta/\varepsilon < p$ which precludes us from having $\beta/\varepsilon = 1$ when $p = 1$. See also [39, Section 7] for the importance of this difference.

To prove the above theorem, for each choice of $\alpha, \tau > 1$ and $0 < \varepsilon \le \log\alpha$, we construct a hyperbolic filling X of the metric space Z. Roughly speaking, is an infinite graph whose vertices correspond to maximal α^{-n}-separated subsets of Z for each positive integer n. The role of τ is to define nearness between two points in each of these sets as in (3.2). We then equip X with the uniformized metric

$$d_\varepsilon(x,y) = \inf_\gamma \int_\gamma e^{-\varepsilon d(\cdot, v_0)}\, ds,$$

where $d(\,\cdot\,, v_0)$ denotes the graph distance to the root v_0 of the hyperbolic filling and the infimum is taken over all curves in X joining x to y.

Along the way, we explore how the choice of parameters affects the structure of the hyperbolic fillings X of Z and their uniformizations X_ε:

- Hyperbolic fillings are Gromov hyperbolic for all $\alpha, \tau > 1$ (Theorem 3.4) but not when $\tau = 1$ (Example 8.8).
- The uniformization X_ε is a uniform space for all $\varepsilon \le \log\alpha$ (Theorem 5.1).
- The boundary of the uniformization X_ε, with $\varepsilon = \log\alpha$, is biLipschitz equivalent to Z when Z is compact (Proposition 4.4).

Subsequently, for a doubling measure ν on Z, we construct a lift of ν to a measure μ on X which is uniformly locally doubling and supports a uniformly local 1-Poincaré inequality. We then show that for every $\beta > 0$, the corresponding weighted measure

$$d\mu_\beta(x) = e^{-\beta d(x,x_0)}\, d\mu(x) \simeq \operatorname{dist}_\varepsilon(x, \partial_\varepsilon X)^{\beta/\varepsilon}\, d\mu(x) \tag{1.1}$$

is globally doubling and supports a global 1-Poincaré inequality on X_ε and its closure $\overline{X}_\varepsilon$, see Theorem 10.3. This gives us the flexibility to choose β for each $0 < \theta < 1$ and $p \ge 1$ so that $\theta = 1 - \beta/p\varepsilon$ and thus see the nonlocal Besov space $B^\theta_{p,p}(Z)$ as the trace space of the Newtonian space $N^{1,p}(X_\varepsilon, \mu_\beta)$, with the advantage that the Newtonian energy is local, and that the theory for Newtonian spaces is more developed than the theory for Besov spaces on metric spaces.

Invoking the regularity properties of Newtonian spaces, we then easily obtain several regularity results for Besov functions on Z:

Corollary 1.2. *Let Z be a compact metric space equipped with a doubling measure ν. Then for every $0 < \theta < 1$, Lipschitz functions are dense in $B^{\theta}_{p,p}(Z)$ and every function in $B^{\theta}_{p,p}(Z)$ has a representative that is quasicontinuous with respect to the Besov capacity.*

If $p > s_\nu/\theta$, where s_ν is the growth exponent of ν from (10.13), then functions in $B^{\theta}_{p,p}(Z)$ have Hölder continuous representatives.

If ν is in addition reverse-doubling, then functions in $B^{\theta}_{p,p}(Z)$ belong to $L^q(Z)$ for $q = s_\nu p/(s_\nu - p\theta)$ and have L^q-Lebesgue points outside a set of zero $B^{\theta}_{p,p}(Z)$-capacity.

Our results apply also to compact subsets of $\mathbf{R}^n$. On $\mathbf{R}^n$, the corresponding Sobolev-type higher integrability result is due to Peetre [43, Théorème 8.1], while Netrusov [42, Proposition 1.4] obtained the Lebesgue point result for $q < np/(n - p\theta)$. Even though $\mathbf{R}^n$ is not compact, the above corollary allows us to improve upon Netrusov's result in the Euclidean setting by including $q = np/(n - p\theta)$, see Proposition 13.6. These results show that our exponent q is optimal.

The above density and quasicontinuity results are known to hold when the Besov space is defined in terms of atomic decompositions or sequences of fractional Hajłasz gradients (see for example Han–Müller–Yang [23, Definition 5.29], Koskela–Yang–Zhou [34, Definitions 1.2 and 4.4] and Heikkinen–Koskela–Tuominen [24, Definition 2.5, Theorems 1.1 and 1.2]). Such spaces coincide with our Definition 9.7 when Z is unbounded and "reverse-doubling", by [34, Theorem 4.1]. In particular, the definitions are equivalent if Z is uniformly perfect with ν doubling, for example when $Z = \mathbf{R}^n$. With our assumptions on Z, it is not clear whether those definitions agree with the nonlocal integral definition considered here. Note that the integral in Definition 9.7 coincides with the ones defining the classical fractional Sobolev spaces in Euclidean spaces and is naturally related to nonlocal minimization problems for the fractional p-Laplacian, as in Caffarelli–Silvestre [17] and Ferrari–Franchi [18].

Higher integrability and Hölder continuity of Besov functions for large p appears also in Malý [39, Corollary 3.18], where it is obtained as a consequence of embeddings into Hajłasz–Sobolev spaces, provided by Lemma 6.1 in Gogatishvili–Koskela–Shanmugalingam [21]. Our approach based on hyperbolic fillings is different. Traces of Newtonian functions on uniform domains in metric spaces are also studied in [39, Theorem 1.1] by means of Lebesgue point averages and Poincaré inequalities. Our proof in the setting of hyperbolic fillings is more direct and rather elementary (albeit a bit lengthy) and is based only on the basic properties of upper gradients. In particular, the Poincaré inequality is not used. Moreover, we show that for *any* compact doubling space Z, the Besov space $B^{\theta}_{p,p}(Z)$ with $0 < \theta < 1$ is the trace of some Newtonian space, and that the trace can be obtained as a pointwise restriction from the Newtonian space $N^{1,p}(\overline{X}_\varepsilon)$, see Theorem 1.1.

Let us compare our definition of Besov spaces with some other function spaces on boundaries of hyperbolic fillings. Assuming Z to be uniformly perfect and Ahlfors Q-regular, the Besov space considered in Bourdon–Pajot [13] corresponds to our $B^{\theta}_{p,p}(Z)$ with $\theta = Q/p$ and is shown to be isomorphically equivalent to the first cohomology group $\ell_p H^1(X)$. For Ahlfors Q-regular spaces Z, the papers [10], [11] and [48] by Bonk, Saksman and Soto define certain function spaces on Z by means of Poisson-type extensions as in our Theorem 12.1, and using the counting measure on the collection of all edges in the hyperbolic filling. In [10] they show that if Z supports a Q-Poincaré inequality then their space $A^p(Z)$ coincides with the Hajłasz–Sobolev space $M^{1,Q}(Z)$. The function spaces considered in [11] are of Triebel–Lizorkin type, while the ones in [48] are identified with the Hajłasz–Besov spaces $\dot{N}^s_{p,q}(Z)$, defined by atomic decompositions in Koskela–Yang–Zhou [34, Definition 1.2].

While these results are interesting, from our point of view it is somewhat unsatisfactory that the energy of functions considered in [10], [11] and [48] does not take into full account the measure ν on Z and that ν is not related to a measure on the hyperbolic filling.

Unlike in [48], our definition of Besov spaces is based on integrals directly on the metric space Z, rather than on sequence spaces, see Definition 9.7. Moreover, the smoothness of the corresponding Poisson extension on the hyperbolic filling X, is controlled by a measure on X that is compatible with the measure ν on Z.

The structure of this paper is as follows. The necessary background related to metric notions and Gromov hyperbolic spaces is given in Section 2, while notions related to Newtonian and Besov spaces are given in Section 9.

In Section 3 we describe the construction of the hyperbolic filling X of a general bounded metric space Z, associated with the parameters $\alpha, \tau > 1$, and show that it indeed forms a Gromov hyperbolic space. Subsequently, in Section 4 we describe the uniformization X_ε of X, with parameter $\varepsilon > 0$, in the style of Bonk–Heinonen–Koskela [8], adapted to the setting of hyperbolic fillings. In this section we also explore links between the boundary $\partial_\varepsilon X$ of the uniformized space and the original metric space Z. In particular, the results in Section 4 show why the bound $\varepsilon \le \log \alpha$ is natural.

The primary goal of Section 5 is to prove that the uniformization X_ε of X yields a uniform space when $\varepsilon \le \log \alpha$. The general results of [8] imply that X_ε is a uniform space for sufficiently small ε, but our direct proof for hyperbolic fillings covers all $\varepsilon \le \log \alpha$, which is vital for our further results. Observe that general Gromov hyperbolic spaces do not always yield a uniform space when uniformized, see for example Lindquist–Shanmugalingam [35, Section 4].

Given that all $\alpha, \tau > 1$ generate a hyperbolic filling of Z, it is worth exploring how the choice of these parameters affects the structure of the hyperbolic filling. The rough similarity between an arbitrary locally compact roughly starlike Gromov hyperbolic space X and the hyperbolic filling $\widehat{X}$ of its uniformized boundary $\partial_\varepsilon X$ is for small ε proved in Section 6, without any limitations on α in terms of ε. Trees and hyperbolic fillings of their uniformized boundaries, as well as some counterexamples, are considered in Section 7. In Section 8 we show that if $\tau \ge (\alpha+1)/(\alpha-1)$, then we have good control over geodesics in the hyperbolic filling. As the discussion in these three sections is provided to explore the hyperbolic fillings and uniformization further, those who are interested only in the theory of Besov spaces may skip these three sections without confusion.

In the rest of the paper we consider a compact metric space Z equipped with a doubling measure ν, and its hyperbolic filling X as well as the uniformization X_ε for $0 < \varepsilon \le \log \alpha$. Following the description of notions related to Newtonian and Besov spaces given in Section 9, we describe in Section 10 our method of lifting up the measure ν on Z to a measure μ on X. In that section we also show that the uniformization μ_β of the measure μ, given for $\beta > 0$ by (1.1), is globally doubling and globally supports a 1-Poincaré inequality on the uniformized space X_ε and its completion $\overline{X}_\varepsilon$.

The trace and extension results from Theorem 1.1 are proved in their specific forms as Theorems 11.1, 11.3 and 12.1, respectively. Finally, the results stated in Corollary 1.2 are obtained in Section 13 by exploiting the perspective of the Besov spaces as traces of Newtonian spaces.

The third author communicated the results of this paper with Butler, who made use of some of the techniques of this paper together with the tools of the Buseman function to independently derive some of the results we obtain in Sections 3–5, with a focus on unbounded doubling metric spaces, see [15]. We do not address the issue of unbounded doubling metric spaces here, but the interested readers may consult [15]. However, his construction of the hyperbolic filling differs slightly from ours in that he requires (3.4) instead of (3.3). The results in [15] also require that

$$\tau \geq \min\biggl\{3, \frac{1}{1-1/\alpha}\biggr\}$$

(the parameter a in [15] corresponds to our $1/\alpha$), but we do not require any such constraint except in Section 8.

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2. Gromov hyperbolic spaces

In this section we will introduce Gromov hyperbolic spaces and uniform spaces and discuss relevant background results. In the later part of the paper we will need background results on upper gradients, Newtonian (Sobolev) spaces, Besov spaces, Poincaré inequalities etc. This background discussion will be provided in Section 9.

A *curve* is a continuous mapping from an interval. Unless stated otherwise, we will only consider curves which are defined on compact intervals. We denote the length of a curve γ by $\ell(\gamma)$, and a curve is *rectifiable* if it has finite length. Rectifiable curves can be parametrized by arc length ds.

A metric space $X=(X,d)$ is *geodesic* if for each $x,y \in X$ there is a curve γ with end points x and y and length $\ell(\gamma)=d(x,y)$. X is a *length space* if

$$d(x,y) = \inf_{\gamma} \ell(\gamma) \quad \text{for all } x,y \in X,$$

where the infimum is taken over all curves γ from x to y.

A metric space is *proper* if all closed bounded sets are compact. We denote balls in X by $B(x,r)=\{y \in X : d(y,x)<r\}$ and the scaled concentric ball by $\lambda B(x,r)=B(x,\lambda r)$. In metric spaces it can happen that balls with different centers and/or radii denote the same set. We will however adopt the convention that a ball comes with a predetermined center and radius. Sometimes (especially when dealing with several different spaces simultaneously) we will write B_X and d_X to indicate that these notions are taken with respect to the metric space X. When we say that $x \in \gamma$ we mean that $x=\gamma(t)$ for some t. If γ is noninjective, then this t may not be unique, but we are always implicitly referring to a specific such t. If $x_1,x_2 \in \gamma$, then γ_{x_1,x_2} denotes the subcurve between x_1 and x_2.

Definition 2.1. A complete unbounded geodesic metric space X is *Gromov hyperbolic* if there is a *hyperbolicity constant* $\delta \geq 0$ such that whenever $[x,y]$, $[y,z]$ and $[z,x]$ are geodesics in X, every point $w \in [x,y]$ lies within a distance δ of $[y,z]\cup[z,x]$.

The ideal Gromov hyperbolic space is a metric tree, which is Gromov hyperbolic with $\delta=0$. A *metric tree* is a tree where each edge is considered to be a geodesic of unit length.

Definition 2.2. An unbounded metric space X is *roughly starlike* if there are some $x_0 \in X$ and $M>0$ such that whenever $x \in X$ there is a geodesic ray γ in X, starting from x_0, such that $\operatorname{dist}(x,\gamma) \leq M$. A *geodesic ray* is a curve $\gamma:[0,\infty) \to X$ with infinite length such that $\gamma|_{[0,t]}$ is a geodesic for each $t>0$.

If X is a roughly starlike Gromov hyperbolic space, then the roughly starlike condition holds for every choice of x_0, although M may change.

Definition 2.3. A nonempty open set $\Omega \subsetneq X$ in a metric space X is an A-*uniform domain*, with $A \geq 1$, if for every pair $x, y \in \Omega$ there is a rectifiable arc length parametrized curve $\gamma : [0, \ell(\gamma)] \to \Omega$ with $\gamma(0) = x$ and $\gamma(\ell(\gamma)) = y$ such that $\ell(\gamma) \leq Ad(x, y)$ and

$$d_\Omega(\gamma(t)) \geq \frac{1}{A} \min\{t, \ell(\gamma) - t\} \quad \text{for } 0 \leq t \leq \ell(\gamma), \tag{2.1}$$

where

$$d_\Omega(z) = \operatorname{dist}(z, X \setminus \Omega), \quad z \in \Omega.$$

The curve γ is said to be an A-*uniform curve*. A noncomplete metric space (Ω, d) is A-*uniform* if it is an A-uniform domain in its completion.

The completion of a locally compact uniform space is always proper, by Proposition 2.20 in Bonk–Heinonen–Koskela [8]. Unlike the definition used in [8], we do not require uniform spaces to be locally compact.

We assume for the rest of this section that X is a roughly starlike Gromov δ-hyperbolic space. We also fix a point $x_0 \in X$ and let M be the constant in the roughly starlike condition with respect to x_0.

The point x_0 will serve as a center for the uniformization X_ε of X. Following Bonk–Heinonen–Koskela [8], we define, for a fixed $\varepsilon > 0$, the *uniformized metric* d_ε on X as

$$d_\varepsilon(x, y) = \inf_\gamma \int_\gamma \rho_\varepsilon \, ds, \quad \text{where } \rho_\varepsilon(x) = e^{-\varepsilon d(x, x_0)}$$

and the infimum is taken over all rectifiable curves γ in X joining x to y. Note that if γ is a compact curve in X, then ρ_ε is bounded from above and away from 0 on γ, and in particular γ is rectifiable with respect to d_ε if and only if it is rectifiable with respect to d.

Definition 2.4. The set X, equipped with the metric d_ε, is denoted by X_ε and called the *uniformization* of X, even when we do not know whether it is a uniform space. We let $\overline{X}_\varepsilon$ be the completion of X_ε, and let $\partial_\varepsilon X = \overline{X}_\varepsilon \setminus X_\varepsilon$ be the *uniformized boundary* of X_ε (or X).

The uniformization X_ε need not be a uniform space, as shown in Lindquist–Shanmugalingam [35, Section 4]. If X is locally compact and ε is sufficiently small, then $\partial_\varepsilon X$ as a set is independent of ε and depends only on the Gromov hyperbolic structure of X, see e.g. [8, Section 3]. The notation adopted in [8] is $\partial_G X$. On the other hand, if ε is large, then it is possible for $\partial_\varepsilon X$ to change, see for example Proposition 4.1 below.

When writing e.g. B_ε, $\operatorname{diam}_\varepsilon$ and $\operatorname{dist}_\varepsilon$, the subscript ε indicates that these notions are taken with respect to $(\overline{X}_\varepsilon, d_\varepsilon)$. We also define

$$d_\varepsilon(x) = \operatorname{dist}_\varepsilon(x, \partial_\varepsilon X).$$

The length of the curve γ with respect to d_ε is denoted by $\ell_\varepsilon(\gamma)$. The arc length ds_ε with respect to d_ε satisfies

$$ds_\varepsilon = \rho_\varepsilon \, ds.$$

It follows that X_ε is a length space, and thus also $\overline{X}_\varepsilon$ is a length space. By a direct calculation (or [8, (4.3)]), $\operatorname{diam}_\varepsilon \overline{X}_\varepsilon = \operatorname{diam}_\varepsilon X_\varepsilon \leq 2/\varepsilon$.

The following important theorem is due to Bonk–Heinonen–Koskela [8]; see [5, Theorem 2.6] for this version. By the Hopf–Rinow theorem (see Gromov [22, p. 9] for a suitable version), X is proper if and only if X is locally compact.

Theorem 2.5. *Assume that X is locally compact. There is a constant $\varepsilon_0(\delta) > 0$, determined by δ alone, such that if $0 < \varepsilon \le \varepsilon_0(\delta)$, then X_ε is an A-uniform space for some A depending only on δ. Moreover, $\overline{X}_\varepsilon$ is a compact geodesic space.*

If $\delta = 0$, then $\varepsilon_0(0)$ can be chosen arbitrarily large.

There is also a converse, again due to Bonk–Heinonen–Koskela [8]. Namely, if (Y, d) is a locally compact uniform space, then equipping Y with the *quasihyperbolic metric* k_Y gives a Gromov hyperbolic space, where

$$k_Y(x, y) := \inf_\gamma \int_\gamma \frac{ds(t)}{\operatorname{dist}_d(\gamma(t), \partial Y)} \quad \text{for } x, y \in Y,$$

with the infimum taken over all rectifiable curves γ in Y with end points x and y.

We recall, for further reference, the following key estimates from [8].

Lemma 2.6. ([8, Lemma 4.16]) *Assume that X is locally compact. Let $\varepsilon > 0$. If $x \in X$, then*

$$\frac{e^{-\varepsilon d(x,x_0)}}{e\varepsilon} \le \operatorname{dist}_\varepsilon(x, \partial_\varepsilon X) =: d_\varepsilon(x) \le C_0 \frac{e^{-\varepsilon d(x,x_0)}}{\varepsilon}, \tag{2.2}$$

where $C_0 = 2e^{\varepsilon M} - 1$. In particular, $\varepsilon d_\varepsilon(x) \simeq \rho_\varepsilon(x)$, and $x \to \partial_\varepsilon X$ with respect to d_ε if and only if $d(x, x_0) \to \infty$.

Corollary 2.7. ([5, Corollary 2.9 and its proof]) *Assume that X is locally compact and that $0 < \varepsilon \le \varepsilon_0(\delta)$, where $\varepsilon_0(\delta)$ is given by Theorem 2.5. Let $x, y \in X$. Then*

$$\frac{d_\varepsilon(x, y)^2}{d_\varepsilon(x) d_\varepsilon(y)} \lesssim \exp(\varepsilon d(x, y)).$$

If $\varepsilon d(x, y) \ge 1$ then

$$\exp(\varepsilon d(x, y)) \simeq \frac{d_\varepsilon(x, y)^2}{d_\varepsilon(x) d_\varepsilon(y)},$$

where the comparison constants depend only on δ, M and $\varepsilon_0(\delta)$.

Here and later, we write $a \lesssim b$ if there is an implicit constant $C > 0$ such that $a \le Cb$, and analogously $a \gtrsim b$ if $b \lesssim a$. We also use the notation $a \simeq b$ to mean $a \lesssim b \lesssim a$.

In the later part of the paper we will equip uniformizations of Gromov hyperbolic spaces and their boundaries with doubling measures. In the first part of the paper, the following metric doubling condition will instead play a role in a few places, but for most results no doubling assumption is needed.

A metric space (Y, d) is *doubling* (or *metrically doubling*) if there is a constant $N_d \ge 1$ such that whenever $z \in Y$ and $r > 0$, the ball $B(z, r)$ can be covered by at most N_d number of balls with radius $\frac{1}{2}r$. Doubling is a uniform version of total boundedness. In particular, if Y is complete and doubling, then Y is also *proper*.

A Borel regular measure μ on Y is *doubling* if there is a constant $C > 0$ such that

$$0 < \mu(2B) \le C\mu(B) < \infty \quad \text{for all balls } B \subset Y.$$

If Y carries a doubling measure, then Y is necessarily doubling. The converse is not true in general. However, if Y is a complete doubling measure space, then Luukkainen–Saksman [37] has shown that Y carries a doubling measure. For more on doubling spaces and doubling measures, see Heinonen [25, Section 10.13].

3. Construction of hyperbolic fillings

The technique of hyperbolic fillings of doubling metric spaces was first considered in Buyalo–Schroeder [16, Chapter 6], and then used in Bourdon–Pajot [13] and Bonk–Saksman [10]. The constructions are different in these papers, see below.

We construct the hyperbolic filling as follows: We assume that a bounded metric space Z is given, and fix the parameters $\alpha, \tau > 1$ and a point $z_0 \in Z$. By scaling we can assume that $0 \le \operatorname{diam} Z < 1$. As mentioned in Section 2, we later want to equip Z with a doubling measure, but to begin with no such requirement is needed.

We set $A_0 = \{z_0\}$ and note that $Z = B_Z(z_0, 1)$. By a recursive construction using Zorn's lemma or the Hausdorff maximality principle, for each positive integer n we can choose a maximal α^{-n}-separated set $A_n \subset Z$ such that $A_n \subset A_m$ when $m \ge n \ge 0$. A set $A \subset Z$ is *α^{-n}-separated* if $d_Z(z, z') \ge \alpha^{-n}$ whenever $z, z' \in A$ are distinct. Then the balls $B_Z(z, \frac{1}{2}\alpha^{-n})$, $z \in A_n$, are pairwise disjoint. Since A_n is maximal, the balls $B_Z(z, \alpha^{-n})$, $z \in A_n$, cover Z. Here and from now on, n and m will always be nonnegative integers.

Next, we define the "vertex set"

$$V = \bigcup_{n=0}^{\infty} V_n, \quad \text{where } V_n = \{(x, n) : x \in A_n\}. \tag{3.1}$$

Note that a point $x \in A_n$ belongs to A_k for all $k \ge n$, and so shows up as the first coordinate in infinitely many points in V. Given two different vertices $(x, n), (y, m) \in V$, we say that (x, n) is a *neighbor* of (y, m) (denoted $(x, n) \sim (y, m)$) if and only if $|n - m| \le 1$ and

$$\tau B_Z(x, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m}) \ne \varnothing, \quad \text{if } m = n, \tag{3.2}$$

$$B_Z(x, \alpha^{-n}) \cap B_Z(y, \alpha^{-m}) \ne \varnothing, \quad \text{if } m = n \pm 1. \tag{3.3}$$

We let *the hyperbolic filling* X be the graph formed by the vertex set V together with the above neighbor relation (edges), and consider X to be a *metric graph* where the edges are unit intervals. As usual for graphs, we do not consider a vertex to be its own neighbor. The distance between two points in X is the length of the shortest curve between them. Since X is a metric graph, it is easy to see that these shortest curves exist, and thus X is a geodesic space.

If $(x, n) \sim (y, n + 1)$ we say that $(y, n + 1)$ is a *child* of (x, n) while (x, n) is a *parent* of $(y, n + 1)$. (We use this terminology also for rooted trees.) In general, each vertex has at least one child, and all vertices but for the root $v_0 = (z_0, 0)$ have at least one parent. An edge $(x, n) \sim (y, m)$ is *horizontal* if $m = n$ and *vertical* if $m = n \pm 1$.

We will show that the hyperbolic filling X is always a Gromov hyperbolic space, but first we compare our construction with those in Buyalo–Schroeder [16, Chapter 6], Bourdon–Pajot [13] and Bonk–Saksman [10]. In [13], Bourdon and Pajot use the same construction as we do with $\alpha = e$ and $\tau = 1$. It is pointed out in [10] that the choice $\tau = 1$ causes problems in the proof of the hyperbolicity of the "hyperbolic filling", more specifically in the proof of [13, Lemme 2.2]. Indeed, in Example 8.8 we construct a "hyperbolic filling" with $\tau = 1$ and $\alpha = 2$ which is not a Gromov hyperbolic space.

According to Bonk–Saksman [10], it is enough to enlarge the balls with a factor > 1, but they make the specific choices $\alpha = \tau = 2$. Their construction is however slightly different from ours: Instead of (3.3), they require that (with $\tau = 2$)

$$\tau B_Z(x, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m}) \ne \varnothing, \quad \text{even if } m = n \pm 1. \tag{3.4}$$

Thus, the hyperbolic fillings in [10] contain more vertical edges than those considered in this paper (with $\alpha = \tau = 2$).

Buyalo and Schroeder, in [16, Chapter 6], use a similar construction with $\alpha \ge 6$ (i.e. $r = 1/\alpha \le \frac{1}{6}$ in their notation) and $\tau = 2$, but impose a different condition when $m = n \pm 1$, namely (with $\tau = 2$)

$$\tau B_Z(y, \alpha^{-m}) \subset \tau B_Z(x, \alpha^{-n}), \quad \text{if } m = n + 1.$$

Buyalo and Schroeder show that their hyperbolic filling is Gromov hyperbolic. Bonk and Saksman [10] refer to Bourdon–Pajot [13] for a proof, but mention that the proof in [13] is problematic for $\tau = 1$, as considered in [13]. Both [13] and [10] have stronger assumptions on Z than here.

When $\tau \ge (\alpha + 1)/(\alpha - 1)$, we have more concrete information on geodesics in the hyperbolic filling, see Section 8. However, a wider choice of $\tau > 1$ yields a wider variety of hyperbolic fillings. For example, if Z is a Cantor set, obtained as the uniformized boundary of an infinite tree and equipped with the induced ultrametric, then its hyperbolic filling with the choice of $\tau < \alpha$ gives back the original tree, whereas the choice of $\tau \ge \alpha$ does not give a tree as the hyperbolic filling of Z, see Section 7. On the other hand, the estimates regarding traces and extensions of Sobolev and Besov spaces in Sections 11–12 (related to the uniformization of hyperbolic fillings) are not affected by the precise values of α and τ.

In the rest of the section, we assume that Z is a metric space with $\operatorname{diam} Z < 1$ *and that X is a hyperbolic filling, as constructed above, with parameters $\alpha, \tau > 1$.*

We consider the projection maps $\pi_1 : V \to Z$ and $\pi_2 : V \to \{0, 1, 2, ...\}$ given by $\pi_1((x, n)) = x$ and $\pi_2((x, n)) = n$. We also set $v_0 := (z_0, 0)$ and use the *Gromov product*

$$(v|w)_{v_0} = \tfrac{1}{2}[d_X(v_0, v) + d_X(v_0, w) - d_X(v, w)], \quad v, w \in V.$$

It follows easily from the construction that the hyperbolic filling is connected. The following lemma is a more precise version of this.

Lemma 3.1. *For all $v \in V$ we have $d_X(v, v_0) = \pi_2(v)$. In particular, V is connected in the graph sense, and X in the metric sense.*

Proof. The first claim is clear if $v = v_0$. So suppose that $v = (x, n)$ for some positive integer n. By the construction of A_j, there are $x_j \in A_j$, such that $x \in B_Z(x_j, \alpha^{-j})$, $j = 0, 1, ..., n$. In particular, $x = x_n$,

$$v_j := (x_j, j) \in V \quad \text{and} \quad x \in B_Z(x_{j-1}, \alpha^{-(j-1)}) \cap B_Z(x_j, \alpha^{-j})$$

for $j = 1, ..., n$. It follows that $v_0 \sim v_1 \sim ... \sim v_n$, and thus $d_X(v, v_0) \le n$. As all other paths connecting these two points have length at least n, we have the required conclusion of the first claim. The last part follows directly. □

Using that every vertex has at least one child, the following consequence of the construction in the proof of Lemma 3.1 is immediate. We will use similar properties many times in this paper without further ado.

Corollary 3.2. (a) *For every vertex v there is a geodesic ray starting at v_0 and containing v.*

(b) *Every geodesic ray starting at v_0 consists solely of vertical edges.*

(c) *Any geodesic from any $x \in X$ to the root v_0 contains at most a half of a horizontal edge.*

(d) *X is roughly starlike with $M = \frac{1}{2}$.*

Next, we provide a proof of the hyperbolicity for all parameters $\alpha, \tau > 1$. The ideas are similar to those in [13] and [16]. In particular the following lemma was inspired by [13, Lemme 2.2]. As mentioned above, when $\tau = 1$ it is possible for the "hyperbolic filling" to be nonhyperbolic, see Example 8.8 below.

Lemma 3.3. *Let $v = (z, n)$ and $w = (y, m)$ be two vertices in X. Then*

$$\alpha^{-(v|w)_{v_0}} \simeq d_Z(z, y) + \alpha^{-n} + \alpha^{-m},$$

with comparison constants depending only on α and τ.

Proof. Without loss of generality, we can assume that $n \le m$. If $z = y$ then $d_X(v, w) = m - n$ and therefore $(v|w)_{v_0} = \frac{1}{2}(n + m - (m - n)) = n$, and so the statement holds in this case. Assume therefore that $z \ne y$.

Let l be the smallest nonnegative integer such that $\alpha^{-l} \le \tau - 1$, and k be the smallest nonnegative integer such that $\alpha^{-k-1} < d_Z(z, y)$. For $j = 0, 1, \dots$, let $z_j, y_j \in A_j$ be such that

$$d_Z(z, z_j) < \alpha^{-j} \quad \text{and} \quad d_Z(y, y_j) < \alpha^{-j}.$$

Clearly, we can choose $z_j = z$ for $j \ge n$ and $y_j = y$ for $j \ge m$. We shall distinguish two cases:

If $k_0 := \min\{k - l, n\} \ge 0$, then $\alpha^{k_0 - k} \le \alpha^{-l} \le \tau - 1$, and the triangle inequality shows that

$$d_Z(z, y_{k_0}) \le d_Z(z, y) + d_Z(y, y_{k_0}) < \alpha^{-k} + \alpha^{-k_0} = (\alpha^{k_0 - k} + 1)\alpha^{-k_0} \le \tau\alpha^{-k_0}.$$

Hence $z \in \tau B_Z(z_{k_0}, \alpha^{-k_0}) \cap \tau B_Z(y_{k_0}, \alpha^{-k_0})$, from which it follows that

$$(z, n) \sim (z_{n-1}, n - 1) \sim \dots \sim (z_{k_0}, k_0) \sim (y_{k_0}, k_0) \sim \dots \sim (y_{n-1}, m - 1) \sim (y, m)$$

where the middle edge may collapse into a single vertex. Thus,

$$d_X(v, w) \le n + m + 1 - 2k_0,$$

and consequently, $(v|w)_{v_0} = \frac{1}{2}(n + m - d_X(v, w)) \ge k_0 - \frac{1}{2}$. If $k_0 < 0$ then clearly $(v|w)_{v_0} \ge 0 > k_0$. In both cases we thus have that

$$\alpha^{-(v|w)_{v_0}} \le \alpha^{-k_0 + 1/2} \le \alpha^{1/2}(\alpha^{l-k} + \alpha^{-n}) \le \alpha^{l+3/2}(d_Z(z, y) + \alpha^{-n} + \alpha^{-m}).$$

Note that l only depends on α and τ.

Conversely, let $w_0 \sim w_1 \sim \dots \sim w_N$ be a geodesic from $v = w_0$ to $w = w_N$. Note that

$$d_X(v, w) = N \ge m - n \quad \text{and} \quad (v|w)_{v_0} = \tfrac{1}{2}(n + m - N).$$

Moreover, by the construction of the hyperbolic filling, $\pi_2(w_j) \ge n - j$ and $\pi_2(w_{N-i}) \ge m - i$ for all $i, j = 0, 1, \dots, N$. Therefore

$$\alpha^{-\pi_2(w_j)} \le \alpha^{j-n} \quad \text{and} \quad \alpha^{-\pi_2(w_{N-i})} \le \alpha^{i-m}.$$

The triangle inequality then yields that for all $k_1 = 0, 1, \dots, N$,

$$\begin{aligned} d_Z(z, y) + \alpha^{-n} + \alpha^{-m} &\le \sum_{j=1}^{N} d_Z(\pi_1(w_{j-1}), \pi_1(w_j)) + \alpha^{-n} + \alpha^{-m} \\ &< \alpha^{-n} + \sum_{j=1}^{N} (\tau\alpha^{-\pi_2(w_{j-1})} + \tau\alpha^{-\pi_2(w_j)}) + \alpha^{-m} \\ &\le 2\tau \sum_{j=0}^{k_1 - 1} \alpha^{j-n} + 2\tau \sum_{i=0}^{N-k_1} \alpha^{i-m} \\ &< \frac{2\tau}{\alpha - 1}(\alpha^{k_1 - n} + \alpha^{N - k_1 - m + 1}), \end{aligned}$$

where the sum $\sum_{j=0}^{k_1-1}$ is empty when $k_1 = 0$. Choosing k_1 to be the smallest integer $\geq \frac{1}{2}(N+n-m)$ gives that

$$d_Z(z,y) + \alpha^{-n} + \alpha^{-m} < \frac{4\tau\alpha}{\alpha-1}\alpha^{(N-n-m)/2} = \frac{4\tau\alpha}{\alpha-1}\alpha^{-(v|w)_{v_0}}. \qquad \square$$

Theorem 3.4. *There is a constant $C \geq 0$, depending only on α and τ, such that if u, v and w are three vertices in V, then*

$$(v|w)_{v_0} \geq \min\{(v|u)_{v_0}, (w|u)_{v_0}\} - C. \tag{3.5}$$

In particular, X is Gromov hyperbolic.

Proof. It suffices to prove (3.5) since Gromov hyperbolicity is equivalent to it, see Bonk–Heinonen–Koskela [8] or Bridson–Haefliger [14, p. 411, Proposition 1.22].

Let $v = (z,n)$, $w = (y,m)$ and $u = (x,k)$. Then clearly,

$$d_Z(z,y) + \alpha^{-n} + \alpha^{-m} \leq (d_Z(z,x) + \alpha^{-n} + \alpha^{-k}) + (d_Z(x,y) + \alpha^{-k} + \alpha^{-m}),$$

and Lemma 3.3 implies that

$$\alpha^{-(v|w)_{v_0}} \lesssim \alpha^{-(v|u)_{v_0}} + \alpha^{-(u|w)_{v_0}} \leq 2\alpha^{-\min\{(v|u)_{v_0},(u|w)_{v_0}\}}.$$

Taking logarithms concludes the proof. $\square$

4. The uniformized boundary $\partial_\varepsilon X$

In this section, we assume that Z is a metric space with $\operatorname{diam} Z < 1$*, and let X be a hyperbolic filling of Z with parameters $\alpha, \tau > 1$.*

In this section we will look at how Z relates to the uniformized boundary $\partial_\varepsilon X$ of X, see Section 2 for the definitions. We use the root $v_0 = (z_0, 0)$ as the uniformization center x_0. We will show that if $\varepsilon \leq \log\alpha$ and Z is complete, then $\partial_\varepsilon X$ is snow-flake equivalent to Z, with exponent $\sigma = \varepsilon/\log\alpha$, see Proposition 4.4 for further details. In particular, $\partial_\varepsilon X$ and Z are biLipschitz equivalent if $\varepsilon = \log\alpha$.

Before showing the equivalence of $\partial_\varepsilon X$ and Z, we take a look at the case $\varepsilon > \log\alpha$. Towards the end of the section we will also study how the degree of the vertices in X depends on properties of Z. In this section we are not concerned with whether the uniformization X_ε is a uniform domain or not. This question will be considered in Section 5.

Note that $\rho_\varepsilon \simeq \rho_\varepsilon(v)$ on every edge $[v,w] \subset X$ and thus, using also the basic facts from Corollary 3.2, for all $x \in X$,

$$d_\varepsilon(x) \simeq \int_{d_X(x,v_0)}^\infty \rho_\varepsilon\, ds = \frac{1}{\varepsilon}\rho_\varepsilon(x), \tag{4.1}$$

with equality if x is a vertex.

The following result shows that if $\varepsilon > \log\alpha$, then $\partial_\varepsilon X$ often becomes just one point.

Proposition 4.1. *Assume that there is $L < \infty$ such that any two points in the d_Z-completion $\overline{Z}$ of Z can be connected by a curve in $\overline{Z}$ of length at most L. If $\varepsilon > \log\alpha$, then $\partial_\varepsilon X$ consists of just one point.*

This shows that when $\varepsilon > \log\alpha$ and Z has at least two points, there is no natural connection between Z and $\partial_\varepsilon X$. On the other hand, it is easy to see that if Z consists of finitely many points, then $\partial_\varepsilon X$ is also finite and there is a natural biLipschitz map between these sets. (We leave the details regarding finite sets Z to the interested reader.) In particular, some connectivity assumption is necessary in Proposition 4.1.

Proof. Let $F_n = \{x \in X : d_X(x, v_0) \le n\}$ and let $x, x' \in X \setminus F_n$ be arbitrary. Then there are d_X-geodesics from x and x' to v_0 which contain vertices $v, v' \in V_n$, respectively. By the connectivity assumption on $\overline{Z}$, there is a sequence $\{w_j\}_{j=0}^m$ of points in $\overline{Z}$ such that $v = (w_0, n)$, $v' = (w_m, n)$, $m \le 3L\alpha^n/(\tau - 1)$ and (if $m \ge 1$)

$$d_Z(w_j, w_{j-1}) \le \tfrac{1}{3}(\tau - 1)\alpha^{-n}, \quad j = 1, \dots, m.$$

For each $j = 0, \dots, m$ there are points $w'_j \in Z$ and $z_j \in A_n$ such that

$$d_Z(w'_j, w_j) \le \tfrac{1}{3}(\tau - 1)\alpha^{-n} \quad \text{and} \quad d_Z(w'_j, z_j) < \alpha^{-n}.$$

We can choose $z_0 = w'_0 = w_0$ and $z_m = w'_m = w_m$. As

$$\begin{aligned} d_Z(w'_j, z_{j-1}) &\le d_Z(w'_j, w_j) + d_Z(w_j, w_{j-1}) + d_Z(w_{j-1}, w'_{j-1}) + d_Z(w'_{j-1}, z_{j-1}) \\ &< (\tau - 1)\alpha^{-n} + \alpha^{-n} = \tau\alpha^{-n} \end{aligned}$$

when $j \ge 1$, we see that $v = (z_0, n) \sim (z_1, n) \sim \dots \sim (z_m, n) = v'$. It follows that

$$\begin{aligned} d_\varepsilon(x, x') \le d_\varepsilon(x, v) + d_\varepsilon(v, v') + d_\varepsilon(v', x') &< 2\int_n^\infty e^{-\varepsilon t}\, dt + me^{-\varepsilon n} \\ &< \frac{2e^{-\varepsilon n}}{\varepsilon} + e^{-\varepsilon n}\frac{3L\alpha^n}{\tau - 1} = \frac{2e^{-\varepsilon n}}{\varepsilon} + \frac{L}{\tau - 1}e^{n(\log\alpha - \varepsilon)}. \end{aligned}$$

Taking supremum over all $x, x' \in X \setminus F_n$ shows that

$$\operatorname{diam}_\varepsilon \partial_\varepsilon X \le \operatorname{diam}_\varepsilon (X \setminus F_n) \to 0, \quad \text{as } n \to \infty,$$

since $\varepsilon > \log\alpha$. □

The easiest example of a space Z applicable in Proposition 4.1 is $Z = [0, \frac{1}{2}]$. The following example shows that it is possible to apply this proposition to a noncompact complete space.

Example 4.2. Let Z consist of countably many copies of $[0, \frac{1}{4}]$, all glued together at 0, and equipped with the inner length metric, so that Z is geodesic. It is easy to see that Z is noncompact and complete, and that $\operatorname{diam} Z = \frac{1}{2}$. Thus Proposition 4.1 applies.

Before proceeding, we deduce the following lemma which will be used several times in this paper.

Lemma 4.3. *Assume that $z, y \in Z$ are such that $d_Z(z, y) \le \alpha^{-k}$ for some nonnegative integer k. For $j = 0, 1, \dots,$ let $z_j, y_j \in A_j$ be such that $d_Z(z, z_j) < \alpha^{-j}$ and $d_Z(y, y_j) < \alpha^{-j}$. Then*

$$v_0 \sim (z_1, 1) \sim \dots \sim (z_k, k) \sim (z_{k+1}, k+1) \sim \dots. \tag{4.2}$$

Let l be the smallest nonnegative integer such that $\alpha^{-l} \le \tau - 1$. Then, for any $m, n \ge h := \max\{k - l, 0\}$,

$$(z_n, n) \sim \dots \sim (z_h, h) \sim (y_h, h) \sim \dots \sim (y_m, m), \tag{4.3}$$

where the middle edge collapses into a single vertex if $z_h = y_h$. This path γ has lengths $\ell_X(\gamma)$ and $\ell_\varepsilon(\gamma)$ (with respect to d_X resp. d_ε) satisfying

$$\begin{aligned} d_X((z_n, n), (y_m, m)) &\le \ell_X(\gamma) \le n + m + 1 - 2h \le n + m + 1 + 2l - 2k, \\ d_\varepsilon((z_n, n), (y_m, m)) &\le \ell_\varepsilon(\gamma) \le \frac{4}{\varepsilon}e^{-\varepsilon h} \le \frac{4}{\varepsilon}e^{-\varepsilon(k-l)}. \end{aligned} \tag{4.4}$$

Note that z_j and y_j exist by the construction of A_j, and that $y_0 = z_0$ with z_0 being the unique element in A_0 as before.

Proof. Since $z \in B_Z(z_j, \alpha^{-j}) \cap B_Z(z_{j+1}, \alpha^{-(j+1)})$ for each $j = 0, 1, \dots$, we directly see that (4.2) holds. By the triangle inequality and the choice of l we see that

$$d_Z(z, y_h) \le d_Z(z, y) + d_Z(y, y_h) < \alpha^{-k} + \alpha^{l-k} \le \tau\alpha^{l-k}.$$

Moreover, $z \in \tau B_Z(z_h, \alpha^{-h})$ and so $(z_h, h) \sim (y_h, h)$ or $z_h = y_h$. Therefore (4.3) follows from (4.2) (and the corresponding path for y). The estimate (4.4) follows directly from (4.3).

To estimate $\ell_\varepsilon(\gamma)$, recall that by Lemma 3.1, $d_X(v, v_0) = \pi_2(v)$ for all $v \in V$ and that $\rho_\varepsilon(x) = e^{-\varepsilon d_X(x,v_0)}$ for all $x \in X$. Hence

$$\ell_\varepsilon(\gamma) \le \int_h^n e^{-\varepsilon t}\,dt + \int_h^m e^{-\varepsilon t}\,dt + 2\int_h^{h+1/2} e^{-\varepsilon t}\,dt \le \frac{4}{\varepsilon}e^{-\varepsilon h} \le \frac{4}{\varepsilon}e^{-\varepsilon(k-l)},$$

where the last integral estimates the d_ε-length of the (possibly collapsed) horizontal edge $(z_h, h) \sim (y_h, h)$. $\square$

We next show that $\partial_\varepsilon X$ is snowflake equivalent to the completion $\overline{Z}$ of Z. In particular, $\partial_\varepsilon X$ is *biLipschitz* equivalent to $\overline{Z}$ when $\varepsilon = \log\alpha$.

Proposition 4.4. *Fix $0 < \varepsilon \le \log\alpha$. Then for all vertices $v, w \in X$,*

$$d_Z(\pi_1(v), \pi_1(w))^\sigma \le C_1 d_\varepsilon(v, w), \quad \text{where } C_1 = (2\tau\alpha)^\sigma \text{ and } \sigma = \frac{\varepsilon}{\log\alpha} \le 1. \tag{4.5}$$

Moreover, $\partial_\varepsilon X$ is snowflake-equivalent to the completion $\overline{Z}$ of Z, that is, there is a natural homeomorphism $\Psi : \overline{Z} \to \partial_\varepsilon X$ such that for all $z, y \in \overline{Z}$,

$$\frac{d_Z(z, y)^\sigma}{C_1} \le d_\varepsilon(\Psi(z), \Psi(y)) \le C_2 d_Z(z, y)^\sigma, \tag{4.6}$$

where $C_2 = 4\alpha^{(l+1)\sigma}/\varepsilon$ and l is the smallest nonnegative integer such that $\alpha^{-l} \le \tau - 1$.

Proof. Let

$$w_0 \sim w_1 \sim \dots \sim w_N$$

be a path γ in X connecting $w_0 = v$ to $w_N = w$. We can assume without loss of generality that $\pi_2(w_N) \le \pi_2(w_0)$. Then by the construction of the hyperbolic filling,

$$d_Z(\pi_1(w_0), \pi_1(w_N)) \le \sum_{i=1}^N d_Z(\pi_1(w_{i-1}), \pi_1(w_i)) \le 2\tau \sum_{i=1}^N \alpha^{-\pi_2(w_i)}. \tag{4.7}$$

Moreover, for each i,

$$\ell_\varepsilon([w_{i-1}, w_i]) \ge \int_{\pi_2(w_i)}^{\pi_2(w_i)+1} e^{-\varepsilon t}\,dt > e^{-\varepsilon(\pi_2(w_i)+1)} = e^{-\varepsilon}(\alpha^{-\pi_2(w_i)})^\sigma.$$

Summing over all i and using the elementary inequality $(\sum_{i=1}^N a_i)^\sigma \le \sum_{i=1}^N a_i^\sigma$ for $\sigma \le 1$, together with (4.7), yields

$$\ell_\varepsilon(\gamma) = \sum_{i=1}^N \ell_\varepsilon([w_{i-1}, w_i]) \ge e^{-\varepsilon} \sum_{i=1}^N (\alpha^{-\pi_2(w_i)})^\sigma \ge e^{-\varepsilon}\left(\frac{d_Z(\pi_1(w_0), \pi(w_N))}{2\tau}\right)^\sigma.$$

Taking the infimum over all such paths γ gives (4.5).

Let $z \in \overline{Z}$ and find a sequence $z_j \in A_j$ such that $d_Z(z_j, z) < \alpha^{-j}$. Then from Lemma 4.3 we see that $v_0 \sim (z_1, 1) \sim \dots \sim (z_j, j) \sim (z_{j+1}, j+1) \sim \dots$. Moreover, when $i > j$,

$$d_\varepsilon((z_i, i), (z_j, j)) \le \int_j^\infty e^{-\varepsilon t}\, dt = \frac{e^{-\varepsilon j}}{\varepsilon}.$$

It follows that $\{(z_j, j)\}$ is a Cauchy sequence in X_ε. Set $\Psi(z) = \lim_{j\to\infty}(z_j, j)$. From the construction of X_ε it is clear that $\Psi(z) \notin X_\varepsilon$, and so this point belongs to $\partial_\varepsilon X$. If $z_j^* \in A_j$ is such that $d_Z(z_j^*, z) < \alpha^{-j}$, then $z \in B_Z(z_j, \alpha^{-j}) \cap B_Z(z_j^*, \alpha^{-j})$ and thus $(z_j^*, j) \sim (z_j, j)$. Hence $d_\varepsilon((z_j^*, j), (z_j, j)) \lesssim e^{-\varepsilon j}$ and so $\lim_{j\to\infty}(z_j^*, j) = \lim_{j\to\infty}(z_j, j)$, which shows that $\Psi(z)$ is well-defined and gives the map $\Psi : \overline{Z} \to \partial_\varepsilon X$.

When $z, y \in \overline{Z}$ with $z \ne y$ and k is a nonnegative integer such that $\alpha^{-k-1} < d_Z(z, y) \le \alpha^{-k}$, we pick z_j and y_j as in Lemma 4.3. Then applying Lemma 4.3, we obtain (upon noting that $e^\varepsilon = \alpha^\sigma$)

$$d_\varepsilon(\Psi(z), \Psi(y)) = \lim_{n\to\infty} d_\varepsilon((z_n, n), (y_n, n)) \le \frac{4e^{l\varepsilon}}{\varepsilon} e^{-\varepsilon k} < \frac{4\alpha^{(l+1)\sigma}}{\varepsilon} d_Z(z, y)^\sigma,$$

i.e. the last inequality in (4.6) holds.

Conversely, applying (4.5) to $v = (y_n, n)$ and $w = (z_n, n)$, and letting $n \to \infty$, shows the first inequality in (4.6). In particular, Ψ is injective.

Finally, it only remains to show that Ψ is surjective. Suppose that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence of points in X_ε with limit $x \in \partial_\varepsilon X$. Then (4.1) shows that $\lim_{j\to\infty} d_X(v_0, x_j) = \infty$. By the construction of X, for each j we can find a vertex $v_j \in X$ such that $d_X(v_j, x_j) \le \frac{1}{2}$. Then, with $z_j := \pi_1(v_j) \in Z$, we know that $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence in X_ε, and so by (4.5) we also have that $\{z_j\}_{j=1}^\infty$ is a Cauchy sequence in Z. Hence, there is a point $z_\infty \in \overline{Z}$ such that $\lim_{j\to\infty} d_Z(z_j, z_\infty) = 0$. Lemma 4.3 again tells us that $\lim_{j\to\infty} d_\varepsilon(v_j, \Psi(z_\infty)) = 0$, and so as $\lim_{j\to\infty} d_\varepsilon(v_j, x_j) = 0$, we must have $x = \Psi(z_\infty)$. Thus Ψ is surjective, completing the proof. □

The construction of the hyperbolic filling as given above works for any bounded metric space, but the resulting hyperbolic filling can have vertices with infinite degree. The following proposition shows that if the metric space is doubling, then the degree is well-controlled. Recall that the *degree* of a vertex is the number of neighbors it has.

Proposition 4.5. *The hyperbolic filling X has uniformly bounded degree if and only if Z is doubling.*

The uniformity and doubling constants depend only on α, τ and each other.

Proof. Assume first that Z is doubling. Let $(x, n) \in X$ and set $A(x, n) \subset V$ to be the collection of all neighbors of (x, n). For $(y, m) \in A(x, n)$ we know that $|m - n| \le 1$ and $\tau B_Z(x, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m})$ is nonempty. Hence $d_Z(x, y) < \tau(\alpha + 1)\alpha^{-n}$ and so $y \in \tau(\alpha + 1)B_Z(x, \alpha^{-n})$. Since Z is assumed to be doubling, there is a positive integer N independent of n, such that the ball $\tau(\alpha + 1)B_Z(x, \alpha^{-n})$ can be covered by balls $B_1, \dots, B_N$ of radius $\frac{1}{2}\alpha^{-n-1}$, see Heinonen [25, p. 81].

Now, for each $m \in \{n - 1, n, n + 1\}$, we have $\frac{1}{2}\alpha^{-m} \ge \frac{1}{2}\alpha^{-n-1}$ and the balls $B_Z(y, \frac{1}{2}\alpha^{-m})$, $y \in A_m$, are pairwise disjoint. It follows that each ball B_j, $j = 1, \dots, N$, can contain at most one point from A_m. Hence, there are at most N such $y \in A_m$ satisfying $(y, m) \in A(x, n)$. Since this is true for each $m \in \{n, n \pm 1\}$, we conclude that the cardinality of $A(x, n)$ is at most $3N$, that is, X is of uniformly bounded degree.

Conversely, assume that X has a uniformly bounded degree. Let $\zeta \in Z$ and $0 < r \le 1$. Let k be the smallest nonnegative integer such that $\min\{1, 3r\} \le \alpha^{-k}$. For this choice of k, let n be the smallest integer such that $n \ge k$ and $\alpha^{-n} \le \frac{1}{2}r$. Note that $n \le k + l'$ for some l' depending only on α.

Since the balls $B_Z(z, \alpha^{-n})$ with $z \in A_n$ cover Z, for every $\xi \in B_Z(\zeta, r)$ there is $z \in A_n$ such that

$$\xi \in B_Z(z, \alpha^{-n}) \subset B_Z(z, \tfrac{1}{2}r).$$

Moreover,

$$d_Z(z, \zeta) \le d_Z(z, \xi) + d_Z(\xi, \zeta) < \alpha^{-n} + r \le \tfrac{3}{2}r.$$

It follows that the balls $B_Z(z, \alpha^{-n})$ with $z \in A := A_n \cap B_Z(\zeta, \frac{3}{2}r)$ cover $B(\zeta, r)$. To estimate the cardinality of A, note that any two points $z, z' \in A$ satisfy $d_Z(z, z') \le \min\{1, 3r\} \le \alpha^{-k}$. Lemma 4.3 and the above observation that $n \le k + l'$ then imply that (with l as in Lemma 4.3),

$$d_X((z, n), (z', n)) \le 2n + 1 + 2(l - k) \le 1 + 2(l + l'),$$

which only depends on α and τ. By assumption there is a uniform bound on the degrees in X and hence also on the number of vertices in balls with a fixed radius. Thus there is a uniform bound on the cardinality of A, i.e. Z is doubling. □

Proposition 4.6. *Let $\varepsilon > 0$ and let X_ε be the uniformization of X, as defined in Section 2. Then the following are equivalent*:

(a) *Z is totally bounded*;
(b) *each vertex layer V_n (as defined in (3.1)) is finite*;
(c) *every vertex in X has finite degree*;
(d) *X, and equivalently X_ε, is locally compact*;
(e) *$\overline{X}_\varepsilon$ is compact.*

If $\varepsilon \le \log \alpha$, then the following condition is also equivalent to those above:

(f) *$\partial_\varepsilon X$ is compact.*

Moreover, $\overline{X}_\varepsilon$ is geodesic whenever (a)–(e) hold.

Note that we do not require X_ε to be uniform. Since (a)–(c) are independent of ε, so are (d) and (e). Furthermore, (f) is also independent of ε provided that $\varepsilon \le \log \alpha$. Example 4.2 shows that (f) is not equivalent to the other statements when $\varepsilon > \log \alpha$.

Proof. (a) $\Leftrightarrow$ (b) It follows directly from the definition of total boundedness that Z is totally bounded if and only if all A_n are finite sets, or equivalently all V_n are finite.

(b) $\Rightarrow$ (c) Let $v = (x, n)$ be a vertex. Then all neighbors of v belong to the finite set $V_{n-1} \cup V_n \cup V_{n+1}$, i.e. v has finite degree.

$\neg$(b) $\Rightarrow$ $\neg$(c) Let m be the least index such that V_m is infinite (which exists as (b) fails). As $V_0 = \{v_0\}$, we must have $m \ge 1$. Each vertex in V_m has at least one parent in V_{m-1}. As V_{m-1} is finite and V_m is infinite, there must be a vertex in V_{m-1} which has infinitely many children, and hence has infinite degree.

(c) $\Leftrightarrow$ (d) This is easily seen to be true. Note that X and X_ε have the same topology, and are thus simultaneously locally compact or not.

(b) $\Rightarrow$ (e) Let $F_n = \{x \in X : d_X(x, v_0) \le n\}$. Since each V_j is finite, it follows that F_n is a union of finitely many compact intervals and so is compact.

Consider $x \in X \setminus F_n$ and let γ be a geodesic from x to v_0. As $d_X(x, v_0) > n$, there is some point $v \in \gamma$ such that $d_X(v, v_0) = n$. Since n is an integer, it follows that $v \in V_n$. Hence

$$\operatorname{dist}_\varepsilon(x, V_n) \le \int_n^\infty e^{-\varepsilon t}\, dt = \frac{e^{-\varepsilon n}}{\varepsilon}.$$

This inequality also holds for $x \in \partial_\varepsilon X$, since $\partial_\varepsilon X \subset \overline{X \setminus F_n}$ (where the closure is with respect to the d_ε metric).

Let $\eta > 0$ and choose n so that $e^{-\varepsilon n}/\varepsilon < \eta$. Then $\overline{X}_\varepsilon \setminus F_n \subset \bigcup_{y \in V_n} B_\varepsilon(y, \eta)$, and as F_n is compact and V_n is finite we see that there is a finite η-net for $\overline{X}_\varepsilon$. Since $\eta > 0$ was arbitrary, $\overline{X}_\varepsilon$ is totally bounded and thus compact (as it is complete by definition).

(e) $\Rightarrow$ (b) As $V_n \subset \overline{X}_\varepsilon$, it is also compact with respect to the metric d_ε. The metrics d_X and d_ε are biLipschitz equivalent on V_n and thus V_n is compact also with respect to d_X. Since distinct points in V_n are at least a distance 1 apart, it follows that V_n is a finite set.

Next, we assume that $\varepsilon \le \log \alpha$ and consider (f).

(e) $\Rightarrow$ (f) This is trivial.

(f) $\Rightarrow$ (a) It follows from Proposition 4.4 that $\overline{Z}$ is homeomorphic to $\partial_\varepsilon X$, and is thus also compact. Hence Z is totally bounded.

Finally, as $\overline{X}_\varepsilon$ is a length space, it follows from Ascoli's theorem that it is geodesic if it is compact. □

5. Uniformizing a hyperbolic filling with parameter $\varepsilon \le \log \alpha$

In this section, we assume that Z is a metric space with $\operatorname{diam} Z < 1$*, and let X be a hyperbolic filling of Z with parameters $\alpha, \tau > 1$.*

The aim of this section is to show that the uniformization X_ε is a uniform domain when $\varepsilon \le \log \alpha$. (Recall Definition 2.3 of uniform spaces and uniform curves.) Since X is Gromov hyperbolic (by Theorem 3.4), it follows from Bonk–Heinonen–Koskela [8, Theorem 2.6] (see Theorem 2.5) that X_ε is a uniform space for sufficiently small $\varepsilon > 0$, when X is locally compact. In the later part of this paper we are interested in uniformizing (a locally compact) X with respect to $\varepsilon = \log \alpha$, and so we cannot rely on Theorem 2.5 or [8]. Therefore we provide a direct proof here, which also avoids assuming local compactness. As we saw in Section 4, it is natural to assume that $\varepsilon \le \log \alpha$ in order for the boundary $\partial_\varepsilon X$ to be homeomorphic to Z (or its completion $\overline{Z}$ if Z is not complete).

Theorem 5.1. *For all $0 < \varepsilon \le \log \alpha$, the uniformized space X_ε is uniform with the uniformity constant depending only on α, τ and ε. Moreover, for all $x', x'' \in X$,*

$$d_\varepsilon(x', x'') \simeq e^{-\varepsilon (x'|x'')_{v_0}} \min\{d_X(x', x''), 1\}, \tag{5.1}$$

with comparison constants depending only on α, τ and ε.

Proof. First we show that curves given by Lemma 4.3, connecting vertices in X, are quasiconvex curves (i.e. having length at most a constant multiple of the distance between their endpoints). These curves will be subsequently used to construct uniform curves connecting pairs of points that are “far apart”. Throughout the proof, we let l be the smallest nonnegative integer such that $\alpha^{-l} \le \tau - 1$.

Let $v = (z, n)$ and $w = (y, m)$ be two distinct vertices in V with $z \in A_n \subset Z$ and $y \in A_m \subset Z$. We can assume that $n \le m$. Let k be the largest nonnegative integer such that $k \le n$ and $d_Z(z, y) \le \alpha^{-k}$. Consider a curve γ connecting v to w, as in (4.3) of Lemma 4.3. Then by Lemma 4.3 we have

$$\ell_\varepsilon(\gamma) \le \frac{4}{\varepsilon} e^{-\varepsilon(k-l)} = \frac{4e^{\varepsilon(l+1)}}{\varepsilon} (\alpha^{-k-1})^\sigma, \quad \text{where } \sigma = \frac{\varepsilon}{\log \alpha} \le 1.$$

We now show that $\ell_\varepsilon(\gamma)$ is comparable to $d_\varepsilon(v,w)$. If $k<n$ then the comparison follows from the choice of k and from (4.5), which together imply that

$$\ell_\varepsilon(\gamma) \lesssim (\alpha^{-k-1})^\sigma < d_Z(z,y)^\sigma \le (2\tau\alpha)^\sigma d_\varepsilon(v,w).$$

On the other hand, if $k=n$, then any injective curve γ' connecting v to w starts with an edge $v\sim v'$, and thus

$$\ell_\varepsilon(\gamma') \ge d_\varepsilon(v,v') \ge \int_k^{k+1} e^{-\varepsilon t}\,dt \ge e^{-\varepsilon(k+1)} = (\alpha^{-k-1})^\sigma \gtrsim \ell_\varepsilon(\gamma).$$

Taking infimum over all such curves γ' shows that even when $k=n$ we have

$$\ell_\varepsilon(\gamma) \lesssim d_\varepsilon(v,w). \tag{5.2}$$

Now assume that $x',x''\in X$ are two arbitrary distinct points and consider an injective curve $\widehat{\gamma}$ from x' to x'' with $\ell_\varepsilon(\widehat{\gamma}) < 2d_\varepsilon(x',x'')$. If $\widehat{\gamma}$ contains at most one vertex, then $d_X(x',x'')<2$, and by (4.1),

$$\ell_\varepsilon(\widehat{\gamma}) \simeq d_\varepsilon(x',x'') \simeq e^{-\varepsilon d_X(x',x_0)} d_X(x',x'') \lesssim d_\varepsilon(x) \quad \text{for all } x\in\widehat{\gamma}, \tag{5.3}$$

and thus $\widehat{\gamma}$ is a uniform curve. By the triangle inequality,

$$\begin{aligned}(x'|x'')_{v_0} &= \tfrac12[d_X(x',x_0)+d_X(x'',x_0)-d_X(x',x'')]\\ &\ge \tfrac12[d_X(x',x_0)+d_X(x',x_0)-2d_X(x',x'')] > d_X(x',x_0)-2,\end{aligned}$$

and by the triangle inequality again, $(x'|x'')_{v_0} \le d_X(x',x_0)$. Inserting this into (5.3) shows that (5.1) holds in this case.

If $\widehat{\gamma}$ contains at least two vertices, then let $v=(z,n)$ and $w=(y,m)$ be the first resp. last vertex in $\widehat{\gamma}$. Let k be the largest nonnegative integer such that both $k\le\min\{n,m\}$ and $d_Z(z,y)\le\alpha^{-k}$. Let γ be a curve as described at the beginning of this proof, connecting the vertices v and w. The desired uniform curve $\widetilde{\gamma}$ between x' and x'' is then obtained by appending the segments $[x',v]$ and $[w,x'']$ to γ. By (5.2),

$$\begin{aligned}\ell_\varepsilon(\widetilde{\gamma}) &\le d_\varepsilon(x',v)+\ell_\varepsilon(\gamma)+d_\varepsilon(w,x'')\\ &\lesssim d_\varepsilon(x',v)+d_\varepsilon(v,w)+d_\varepsilon(w,x'') = \ell_\varepsilon(\widehat{\gamma}) < 2d_\varepsilon(x',x'').\end{aligned}$$

We next show that $\widetilde{\gamma}$ satisfies the twisted cone condition (2.1) in Definition 2.3. Note that $\widetilde{\gamma}$ need not be injective and recall from Section 2 what $y\in\widetilde{\gamma}$ and $\widetilde{\gamma}_{y,y'}$ mean in such a case.

Let $v'=(z_h,h)\in\gamma$ and $w'=(y_h,h)\in\gamma$, where $h=\max\{k-l,0\}$, z_h and y_h are as given for v and w by Lemma 4.3. Recall that γ consists of two vertical segments, one from v to v' and the other from w to w', together with the (possibly collapsed) horizontal edge $[v',w']$. Let $x\in\widetilde{\gamma}$ be arbitrary and consider the subcurve $\widetilde{\gamma}_{x,x'}$ of $\widetilde{\gamma}$ from x to x'. We shall distinguish three basic situations and their symmetric equivalents. If $x\in\widetilde{\gamma}_{x',v}$, then

$$\ell_\varepsilon(\widetilde{\gamma}_{x,x'}) = d_\varepsilon(x,x') \le e^{-\varepsilon(n-1)} \simeq d_\varepsilon(x).$$

If $x\in\widetilde{\gamma}_{v,v'}$, then $\widetilde{\gamma}_{v,x}$ is a vertical segment, and hence, using (4.1) and that $\pi_2(v)=n\ge d_X(x,v_0)$,

$$\ell_\varepsilon(\widetilde{\gamma}_{x,x'}) = d_\varepsilon(x',v) + \int_{d_X(x,v_0)}^n e^{-\varepsilon t}\,dt \le e^{-\varepsilon(n-1)} + \int_{d_X(x,v_0)}^\infty e^{-\varepsilon t}\,dt \lesssim d_\varepsilon(x).$$

If $x \in \widetilde{\gamma}_{v',w'}$, then $\pi_2(v') = h \le n$, and thus

$$\begin{aligned} \ell_\varepsilon(\widetilde{\gamma}_{x,x'}) &\le d_\varepsilon(x',v) + \int_h^n e^{-\varepsilon t}\,dt + 2\int_h^{h+1/2} e^{-\varepsilon t}\,dt \\ &\le e^{-\varepsilon(n-1)} + 3\int_h^\infty e^{-\varepsilon t}\,dt \lesssim e^{-\varepsilon h} \simeq d_\varepsilon(x). \end{aligned}$$

The case when $x \in \widetilde{\gamma}_{w',x''}$ is treated similarly. Thus, $\widetilde{\gamma}$ is a uniform curve.

To prove (5.1) also in this case, note that by Lemma 4.3, and using that k is the largest nonnegative integer $\le \min\{n,m\}$ such that $d_Z(z,y) \le \alpha^{-k}$,

$$\begin{aligned} d_\varepsilon(x',x'') \le d_\varepsilon(x',v) + d_\varepsilon(v,w) + d_\varepsilon(w,x'') &\lesssim e^{-\varepsilon n} + e^{-\varepsilon k} + e^{-\varepsilon m} \\ &\lesssim d_Z(z,y)^\sigma + \alpha^{-\sigma n} + \alpha^{-\sigma m}. \end{aligned}$$

Conversely, (4.5) shows that

$$2d_\varepsilon(x',x'') > \ell_\varepsilon(\widehat{\gamma}) \ge d_\varepsilon(v,w) \gtrsim d_Z(z,y)^\sigma.$$

Since also $d_\varepsilon(v,w) \gtrsim e^{-\varepsilon n} + e^{-\varepsilon m} = \alpha^{-\sigma n} + \alpha^{-\sigma m}$, we get by Lemma 3.3 that

$$d_\varepsilon(x',x'') \simeq d_Z(z,y)^\sigma + \alpha^{-\sigma n} + \alpha^{-\sigma m} \simeq \alpha^{-\sigma(v|w)_{v_0}} \simeq e^{-\varepsilon(x'|x'')_{v_0}}. \qquad \square$$

6. Roughly similar equivalence

In this section, we want to show that every locally compact roughly starlike Gromov hyperbolic space is roughly similar to any hyperbolic filling of its uniformized boundary when the uniformization index ε is small enough to guarantee that the uniformized space is a uniform space.

Let X be a locally compact roughly starlike Gromov hyperbolic space. For $0 < \varepsilon \le \varepsilon_0(\delta)$, where $\varepsilon_0(\delta)$ is given by Theorem 2.5, we uniformize X, with uniformization point x_0, to obtain X_ε equipped with the metric d_ε and having boundary $\partial_\varepsilon X$. It will be convenient to use the scaled metric

$$\hat{d}_\varepsilon = \frac{\varepsilon}{e} d_\varepsilon$$

on $\overline{X}_\varepsilon$, and correspondingly $\widehat{\operatorname{dist}}_\varepsilon$ and $\widehat{\operatorname{diam}}_\varepsilon$. A consequence is that $\hat{d}_\varepsilon(x_0) = 1/e$ and

$$\frac{1}{e} \le \widehat{\operatorname{diam}}_\varepsilon X \le \frac{2}{e} < 1 \tag{6.1}$$

and thus $\widehat{\operatorname{diam}}_\varepsilon \partial_\varepsilon X \le 2/e < 1$. Note that if X is the half-line, then $\partial_\varepsilon X$ consists of just one point and thus has diameter 0.

Definition 6.1. Let (W,d_W) and (Y,d_Y) be metric spaces. A (not necessarily continuous) map $\Phi : W \to Y$ is a *rough similarity* if there are $C \ge 0$ and $L \ge 1$ such that every point in Y is at most a distance C from $\Phi(W)$, and for all $x,x' \in W$,

$$L d_W(x,x') - C \le d_Y(\Phi(x),\Phi(x')) \le L d_W(x,x') + C. \tag{6.2}$$

We refer interested readers to Bonk–Schramm [12, Section 2] for more on rough similarity between Gromov hyperbolic spaces.

Theorem 6.2. *Assume that X is a locally compact roughly starlike Gromov hyperbolic space and that $0 < \varepsilon \le \varepsilon_0(\delta)$, where $\varepsilon_0(\delta)$ is given by Theorem 2.5. Let $Z = \partial_\varepsilon X$ be the uniformized boundary of X equipped with the metric $\hat{d}_\varepsilon$. Let $\widehat{X}$ be* *any hyperbolic filling of Z, constructed with parameters $\alpha, \tau > 1$ and the maximal α^{-n}-separated sets $A_n \subset Z$.*

Consider the mapping $\Phi : X \to \widehat{X}$, defined for $x \in X$ with

$$\alpha^{-n-1} < \hat{d}_\varepsilon(x) \le \alpha^{-n}, \quad n = 0, 1, \dots,$$

by $\Phi(x) = (z, n)$, where z is chosen to be a nearest point in A_n to x, i.e. $\hat{d}_\varepsilon(x,z) = \widehat{\operatorname{dist}}_\varepsilon(x, A_n)$.

Then Φ is a rough similarity with $L = (\log \alpha)/\varepsilon = 1/\sigma$ and C depending only on α, τ, $\varepsilon_0(\delta)$, δ and M.

By combining Bonk–Schramm [12, Theorem 8.2] with Proposition 4.4 we know that X and $\widehat{X}$ are roughly similar if $\varepsilon > 0$ is small enough to guarantee that ∂X_ε is snowflake equivalent to the visual boundary of X as in Bonk–Heinonen–Koskela [8]. However, as this rough similarity is obtained from [12] via comparison with the cone $\operatorname{Con}(\partial Y_\varepsilon)$, here we give a more direct construction of the rough similarity between X and $\widehat{X}$. In so doing, we also demonstrate how the parameters α, τ and ε affect the rough similarity constants.

We chose to point out the dependence on $\varepsilon_0(\delta)$ separately in Theorem 6.2 even though $\varepsilon_0(\delta)$ is supposed to be determined solely by δ, because should a better upper bound for ε in Theorem 2.5 and Corollary 2.7 be known in the future, the result in this theorem will also be valid for the enlarged range of ε. Note that $Z = \partial_\varepsilon X$ is compact by Theorem 2.5, and thus so is A_n, which shows that the nearest points above exist. Proposition 7.3 shows that Z can be nondoubling.

To prove this theorem, we need the following lemma. Let N_0 be the smallest integer $\ge 1/\log \alpha$. For each $n \ge N_0$ we set

$$S_n = \{x \in X_\varepsilon : \hat{d}_\varepsilon(x) = \alpha^{-n}\}. \tag{6.3}$$

By the choice of N_0 we know that $S_n \ne \varnothing$.

Lemma 6.3. *Suppose that the assumptions in Theorem 6.2 hold. Fixing $n \ge N_0$, let $z \in \partial_\varepsilon X$ and let x be a nearest point in S_n to z. Then*

$$\alpha^{-n} = \hat{d}_\varepsilon(x) \le \hat{d}_\varepsilon(z,x) \le \alpha^{-n} e.$$

As noted above, $\overline{X}_\varepsilon$ is compact. Therefore S_n is also compact, and hence the nearest points referred to above do exist.

Proof. Find a sequence $z_k \in X$ such that $z_k \to z$ with respect to $\hat{d}_\varepsilon$. Since X is roughly starlike, there is a sequence of arc length parametrized d_X-geodesic rays $\gamma_k : [0, \infty) \to X$ starting from x_0, and a sequence of points $w_k \in \gamma_k$, such that $\operatorname{dist}_X(w_k, z_k) \le M$. For all $y \in [z_k, w_k]$ (where $[z_k, w_k]$ is any d_X-geodesic from z_k to w_k),

$$d_X(y, x_0) \ge d_X(z_k, x_0) - M \quad \text{and hence} \quad \rho_\varepsilon(y) \le e^{\varepsilon M} \rho_\varepsilon(z_k).$$

It follows that

$$\hat{d}_\varepsilon(z_k, w_k) \le \frac{\varepsilon}{e} \int_{[z_k, w_k]} \rho_\varepsilon \, ds \le \frac{M\varepsilon}{e} e^{\varepsilon M} \rho_\varepsilon(z_k) \to 0, \quad \text{as } k \to \infty,$$

showing that $w_k \to z$. The ray γ_k intersects S_n and for sufficiently large k there exists $y_k = \gamma_k(t_k) \in \gamma_k \cap S_n$ such that $w_k \in \gamma_k((t_k, \infty))$. As $y_k \in S_n$, using (2.2) we see that

$$\alpha^{-n} = \hat{d}_\varepsilon(y_k) = \frac{\varepsilon}{e} d_\varepsilon(y_k) \ge \frac{e^{-\varepsilon d_X(y_k, x_0)}}{e^2} = \frac{e^{-\varepsilon t_k}}{e^2}.$$

Consequently,

$$\hat{d}_\varepsilon(w_k, y_k) \le \frac{\varepsilon}{e}\int_{t_k}^{\infty} e^{-\varepsilon t}\, dt = \frac{e^{-\varepsilon t_k}}{e} \le \alpha^{-n} e.$$

This finally implies, since $w_k \to z$, that

$$\begin{aligned}\widehat{\operatorname{dist}}_\varepsilon(z, S_n) &\le \hat{d}_\varepsilon(z, y_k) \le \hat{d}_\varepsilon(z, w_k) + \hat{d}_\varepsilon(w_k, y_k)\\ &\le \hat{d}_\varepsilon(z, w_k) + \alpha^{-n} e \to \alpha^{-n} e, \qquad \text{as } k \to \infty,\end{aligned}$$

which proves the last inequality in the statement of the lemma. The remaining (in)equalities are clear from the definitions. □

Proof of Theorem 6.2. Let $x, x' \in X$. Let n and m be the largest integers such that $\hat{d}_\varepsilon(x) \le \alpha^{-n}$ and $\hat{d}_\varepsilon(x') \le \alpha^{-m}$, respectively. Note that $n, m \ge 0$ by (6.1). Let $v = \Phi(x) = (z, n)$ and $w = \Phi(x') = (y, m)$, with $z \in A_n$ and $y \in A_m$. Note that

$$\hat{d}_\varepsilon(x, z) < 2\alpha^{-n} \quad \text{and} \quad \hat{d}_\varepsilon(x', y) < 2\alpha^{-m}. \tag{6.4}$$

The triangle inequality and Lemma 3.3 imply that

$$\hat{d}_\varepsilon(x, x') \le \hat{d}_\varepsilon(z, y) + 2(\alpha^{-n} + \alpha^{-m}) \simeq \alpha^{-(v|w)_{v_0}}.$$

It then follows from Corollary 2.7 that either $\varepsilon d_X(x, x') < 1$ or

$$\exp(\varepsilon d_X(x, x')) \simeq \frac{d_\varepsilon(x, x')^2}{d_\varepsilon(x) d_\varepsilon(x')} \lesssim \frac{\alpha^{-2(v|w)_{v_0}}}{\hat{d}_\varepsilon(x)\hat{d}_\varepsilon(x')} \le \frac{\alpha^{-(n+m-d_{\widehat{X}}(v,w))}}{\alpha^{-n-1}\alpha^{-m-1}} \simeq \alpha^{d_{\widehat{X}}(v,w)}.$$

Taking logarithms proves the first inequality in (6.2) of Definition 6.1.

For the second inequality we distinguish two cases. Assume, without loss of generality, that $n \le m$. If $\hat{d}_\varepsilon(x, x') > (1 - 1/\alpha)\hat{d}_\varepsilon(x)$, then $\alpha^{-n} < \alpha \hat{d}_\varepsilon(x) \lesssim \hat{d}_\varepsilon(x, x')$ and hence, by Lemma 3.3 and (6.4),

$$\alpha^{-(v|w)_{v_0}} \simeq \hat{d}_\varepsilon(z, y) + \alpha^{-n} + \alpha^{-m} < \hat{d}_\varepsilon(x, x') + 3(\alpha^{-n} + \alpha^{-m}) \lesssim \hat{d}_\varepsilon(x, x').$$

This together with Corollary 2.7 implies that

$$\alpha^{d_{\widehat{X}}(v,w)} = \frac{\alpha^{-2(v|w)_{v_0}}}{\alpha^{-n}\alpha^{-m}} \lesssim \frac{\hat{d}_\varepsilon(x, x')^2}{\hat{d}_\varepsilon(x)\hat{d}_\varepsilon(x')} \lesssim \exp(\varepsilon d_X(x, x')).$$

Taking logarithms proves the second inequality in (6.2) in this case.

If $\hat{d}_\varepsilon(x, x') \le (1 - 1/\alpha)\hat{d}_\varepsilon(x)$, then

$$\alpha^{-m} \ge \hat{d}_\varepsilon(x') \ge \hat{d}_\varepsilon(x) - \hat{d}_\varepsilon(x, x') \ge \frac{\hat{d}_\varepsilon(x)}{\alpha} > \alpha^{-n-2},$$

and hence $n + 1 \ge m \ge n$. Let l and t be the smallest nonnegative integers such that $\alpha^{-l} \le \tau - 1$ and $5 \le \alpha^t$. Then, by (6.4),

$$\hat{d}_\varepsilon(z, y) \le \hat{d}_\varepsilon(z, x) + \hat{d}_\varepsilon(x, x') + \hat{d}_\varepsilon(x', y) < 2\alpha^{-n} + \hat{d}_\varepsilon(x) + 2\alpha^{-m} \le 5\alpha^{-n} \le \alpha^{t-n}.$$

If $n \ge t$, then by Lemma 4.3 (with $k = n - t \ge 0$),

$$d_{\widehat{X}}(v, w) \le n + m + 1 + 2l - 2(n - t) \le 2(l + t + 1).$$

If on the other hand $n < t$, then

$$d_{\widehat{X}}(v, w) \le n + m \le 2n + 1 < 2t + 1 \le 2(l + t + 1).$$

Thus, the second inequality in (6.2) holds also in the case $\hat{d}_\varepsilon(x,x') \le (1-1/\alpha)\hat{d}_\varepsilon(x)$ by choosing $C \ge 2(l+t+1)$.

To verify that some C'-neighborhood of $\Phi(X)$ contains $\widehat{X}$, note that every point in $\widehat{X}$ is within a distance $\frac{1}{2}$ of the set V of vertices in $\widehat{X}$. So it suffices to show that if $(y,n) \in V$, then there is some $x \in X$ such that $d_{\widehat{X}}((y,n),\Phi(x)) \le C''$.

As before, let l and t be the smallest nonnegative integers such that $\alpha^{-l} \le \tau - 1$ and $5 \le \alpha^t$. Note that $t \ge 1$. Recall the definition of S_n from (6.3). Let x be a nearest point in S_{n+l+t} to y. Then $\hat{d}_\varepsilon(y,x) \le \alpha^{-n-l-t}e$ by Lemma 6.3. By the construction of Φ, the point x has a nearest point z_{n+l+t} in A_{n+l+t} such that $(z_{n+l+t}, n+l+t) = \Phi(x)$ and $\hat{d}_\varepsilon(x, z_{n+l+t}) < 2\alpha^{-n-l-t}$. For $j = n, \dots, n+l+t-1$, find $z_j \in A_j$ such that $\hat{d}_\varepsilon(z, z_j) < \alpha^{-j}$. Hence, by the choice of l and t,

$$\hat{d}_\varepsilon(y,z_n) \le \hat{d}_\varepsilon(y,x) + \hat{d}_\varepsilon(x,z) + \hat{d}_\varepsilon(z,z_n) < \alpha^{-n-l-t}e + 2\alpha^{-n-l-t} + \alpha^{-n} < \tau\alpha^{-n}.$$

Since also $y \in \tau B_Z(y,\alpha^{-n})$ and $z \in B_Z(z_j,\alpha^{-j})$, $j = n, \dots, n+l+t$, we see that

$$(y,n) \sim (z_n,n) \sim (z_{n+1},n+1) \sim \dots \sim (z_{n+l+t}, n+l+t) = \Phi(x),$$

where the first edge collapses into a single vertex if $y = z_n$. This implies that $d_{\widehat{X}}((y,n),\Phi(x)) \le l+1$. □

7. Trees

In this section, we will obtain a sharper version of Theorem 6.2 in the case when X is a tree, namely we get an isometry rather than a mere rough similarity, provided that the parameters are chosen appropriately. Note that since X is a rooted tree, $\delta = 0$ and we can choose $\varepsilon(0)$ arbitrarily, by Theorem 2.5, so in this case there is no upper bound on ε in Theorem 6.2. Recall that an *isometry* is a 1-biLipschitz map, i.e. a rough similarity with $L = 1$ and $C = 0$.

Theorem 7.1. *Let X be a metric tree, rooted at x_0, such that every vertex $x \in X$ has at least one child. Consider the uniformized boundary $\partial_\varepsilon X$ of X, with parameter $\varepsilon > 0$, and let $1 < \tau < \alpha = e^\varepsilon$ be fixed. Let $Z = \partial_\varepsilon X$ be equipped with the metric*

$$d_Z(\zeta,\xi) := \frac{\varepsilon\tau}{2\alpha} d_\varepsilon(\zeta,\xi).$$

Then X is isometric to any hyperbolic filling of Z, constructed with the parameters α and τ.

Note that $\operatorname{diam} Z \le \tau/\alpha < 1$. In Remark 7.2 below we show that if $\tau \ge \alpha$, then the hyperbolic filling is never a tree. Thus the range $1 < \tau < \alpha$ for τ in Theorem 7.1 is optimal.

Proof. Let $\widehat{X}$ be a hyperbolic filling of Z, constructed from a maximal α^{-n}-separated set $A_n \subset Z$, with parameters τ and α. It suffices to show that the sets of vertices in X and $\widehat{X}$, respectively, are isometric, since the extension to the edges is straightforward.

To start with, note that if $\zeta, \xi \in Z$ have a common ancestor $x \in X$ at distance $n \ge 0$ from the root x_0, then

$$d_Z(\zeta,\xi) \le \frac{\varepsilon\tau}{\alpha}\int_n^\infty e^{-\varepsilon t}\,dt = \tau\alpha^{-n-1} < \alpha^{-n}, \tag{7.1}$$

with equality if and only if ζ and ξ do not have a common ancestor at distance $n+1$ from x_0.

It follows that for every $n \geq 0$, the metric space Z can be written as a finite union of open balls of radius α^{-n}, namely those consisting exactly of all descendants in Z of some vertex $x \in X$ with $d_X(x, x_0) = n$. Moreover, every two points in such a ball satisfy (7.1), and these balls are disjoint and can also be written as balls centered at any of the points in it, with radius $\tau\alpha^{-n}$. Indeed, if $d_Z(\zeta, \eta) < \tau\alpha^{-n}$, then we know by (7.1) and the comment after it that ζ and η have a common ancestor at distance n from the root x_0, and so $d_Z(\zeta, \eta) \leq \tau\alpha^{-(n+1)} < \alpha^{-n}$. Thus, A_n contains exactly one point in each of these balls and this correspondence defines a bijection F between the vertices in X at level n and the set $A_n \subset Z$. More precisely, $F(x)$ is the unique descendant of x belonging to A_n. Define the mapping $\widehat{F}$ from vertices in X to vertices in $\widehat{X}$ as

$$\widehat{F}(x) = (F(x), d_X(x, x_0)).$$

To show that $\widehat{F}$ is an isometry between the two sets of vertices, assume that $x \sim y$ in X. Without loss of generality, we may assume that x is a parent of y and that $d_X(x, x_0) = n$. Then both $F(x)$ and $F(y)$ have x as a common ancestor and hence by (7.1), $d_Z(F(x), F(y)) < \alpha^{-n}$, which yields

$$F(y) \in B_Z(F(x), \alpha^{-n}) \cap B_Z(F(y), \alpha^{-n-1}).$$

Therefore $\widehat{F}(x) = (F(x), n) \sim (F(y), n+1) = \widehat{F}(y)$.

Conversely, assume that $\widehat{F}(x) \sim \widehat{F}(y)$. Then $|n - m| \leq 1$, where $n = d_X(x, x_0)$ and $m = d_X(y, x_0)$. By the construction of $\widehat{X}$, there exists $\zeta \in Z$ such that

$$d_Z(\zeta, F(x)) < \tau\alpha^{-n} \quad \text{and} \quad d_Z(\zeta, F(y)) < \tau\alpha^{-m}.$$

As pointed out after (7.1), the first inequality implies that ζ and $F(x)$ have a common ancestor at distance n from the root x_0 and this ancestor must be x since there is only one ray in X from x_0 to $F(x)$. Similarly, ζ and $F(y)$ have y as a common ancestor at distance m from the root x_0.

Because there is only one ray in X from x_0 to ζ, this implies that $x = y$ when $m = n$ and contradicts the assumption $\widehat{F}(x) \sim \widehat{F}(y)$. Consequently, there are no horizontal edges in $\widehat{X}$. If $m \neq n$, then we can assume that $m = n + 1$ and conclude that x is the parent of y in the above ray, and so $x \sim y$. Thus, $\widehat{F} : X \to \widehat{X}$ is an isometry. □

Remark 7.2. If Z is not a singleton in Theorem 7.1 (that is, X is not the half-line), then for $\tau \geq \alpha$ and a scaling of d_Z so that $\operatorname{diam} Z < 1$, the hyperbolic filling $\widehat{X}$ of Z will always contain horizontal edges and is thus not a tree.

More precisely, let $d_Z(\zeta, \xi) := \frac{1}{2}\varepsilon\kappa d_\varepsilon(\zeta, \xi)$, where $0 < \kappa < 1$, so that $\operatorname{diam} Z \leq \kappa < 1$. Let $l \geq 0$ be the smallest integer such that $\kappa < \alpha^{-l}$. Then (7.1) becomes $d_Z(\zeta, \xi) \leq \kappa\alpha^{-n} < \alpha^{-n-l}$, and so Z can be written as a finite disjoint union of open balls of radius α^{-n-l}, each consisting exactly of all descendants in Z of some vertex $x \in X$ with $d_X(x, x_0) = n$. Thus, A_{n+l} contains exactly one point in each of these balls.

However, if $\zeta, \xi \in A_{n+l}$ are descendants of two distinct vertices $x, y \in X$ at distance n from the root, having the same parent, then because $\tau \geq \alpha$, we have

$$d_Z(\zeta, \xi) \leq \kappa\alpha^{1-n} < \alpha^{1-n-l} \leq \tau\alpha^{-n-l}.$$

We therefore see that $\zeta \in \tau B_Z(\zeta, \alpha^{-n-l}) \cap \tau B_Z(\xi, \alpha^{-n-l})$ and thus the vertices $(\zeta, n+l)$ and $(\xi, n+l)$ in $\widehat{X}$ will be neighbors connected by a horizontal edge.

We also give the following characterizations.

Proposition 7.3. *Let X be a rooted tree such that every vertex $x \in X$ has at least one child, and let $\varepsilon > 0$. Then the following are true:*

(a) *The uniformized boundary $\partial_\varepsilon X$ is compact if and only if every vertex has a finite number of children.*

(b) *The uniformized boundary $\partial_\varepsilon X$ is doubling if and only if there is a uniform bound on the number of children for each vertex.*

Proof. Let $1 < \tau < \alpha = e^\varepsilon$ and let $\widehat{X}$ be any hyperbolic filling of $\partial_\varepsilon X$, with parameters α and τ. By Theorem 7.1, $\widehat{X}$ is isometric to X. Part (a) now follows from Proposition 4.6, while part (b) follows from Proposition 4.5. □

8. Geodesics in the hyperbolic filling

In this section, except for Examples 8.7 and 8.8, we fix an arbitrary parameter $\alpha > 1$ and assume that

$$\tau \geq \frac{\alpha + 1}{\alpha - 1}. \tag{8.1}$$

We also assume that Z is a metric space with $\operatorname{diam} Z < 1$, and let X be a hyperbolic filling of Z with the parameters α and τ.

In this section we study how the geodesics in a hyperbolic filling look like under the above restriction relating τ and α. We do not use these precise properties of geodesics in the rest of the paper, and so in the other sections we do not require this limit on τ. However, in other applications the structure of the geodesics is quite useful to know. As we gain control of the geodesics in a straightforward manner under the above constraint on τ, we have included this study here as well for the convenience of the reader and for possible future applications. In Example 8.7 we show that most of the properties obtained in this section can fail when τ is close to 1. We end the section with Example 8.8 showing that when $\tau = 1$ it is possible for the "hyperbolic filling" to be nonhyperbolic.

Uniformizations will not play any role in this section. We will only study geodesics between vertices in X and all the geodesics are with respect to the d_X-metric.

Lemma 8.1. *Assume that (8.1) holds. If $(x,n) \sim (z,n+1) \sim (y,n)$ is a segment in a path with $x \neq y$, then $(x,n) \sim (y,n)$ and the path is not a geodesic.*

Proof. By the hypothesis of this lemma, we have $B_Z(x,\alpha^{-n}) \cap B_Z(z,\alpha^{-(n+1)}) \neq \varnothing$ and so, using (8.1),

$$d_Z(x,z) < \frac{\alpha+1}{\alpha}\alpha^{-n} < \tau\alpha^{-n} \quad \text{and similarly,} \quad d_Z(y,z) < \tau\alpha^{-n}.$$

It follows that $z \in \tau B_Z(x,\alpha^{-n}) \cap \tau B_Z(y,\alpha^{-n})$ and consequently $(x,n) \sim (y,n)$. So the length of the path can be reduced by one by replacing the segment $(x,n) \sim (z,n+1) \sim (y,n)$ with the edge $(x,n) \sim (y,n)$. Hence it cannot be a geodesic. □

Lemma 8.2. *Assume that (8.1) holds. If $(x_1,n) \sim (x_2,n+1) \sim (y_2,n+1) \sim (y_1,n)$ is a segment in a path with $x_1 \neq y_1$, then $(x_1,n) \sim (y_1,n)$ and the path is not a geodesic.*

Proof. Since $(x_2,n+1) \sim (y_2,n+1)$, there exists

$$b \in \tau B_Z(x_2,\alpha^{-(n+1)}) \cap \tau B_Z(y_2,\alpha^{-(n+1)}).$$

Similarly, as $(x_1, n) \sim (x_2, n+1)$, we have $B_Z(x_1, \alpha^{-n}) \cap B_Z(x_2, \alpha^{-(n+1)}) \neq \varnothing$, and so

$$d_Z(x_1, x_2) < \frac{\alpha+1}{\alpha}\alpha^{-n}.$$

Therefore,

$$d_Z(x_1, b) \leq d_Z(x_1, x_2) + d_Z(x_2, b) < \frac{\alpha+1}{\alpha}\alpha^{-n} + \tau\alpha^{-(n+1)} \leq \tau\alpha^{-n}.$$

Similarly, $d_Z(y_1, b) < \tau\alpha^{-n}$. Hence $b \in \tau B_Z(x_1, \alpha^{-n}) \cap \tau B_Z(y_1, \alpha^{-n})$, which shows that $(x_1, n) \sim (y_1, n)$.

Finally, replacing the segment $(x_1, n) \sim (x_2, n+1) \sim (y_2, n+1) \sim (y_1, n)$ with the edge $(x_1, n) \sim (y_1, n)$ reduces the length of the path by 2, and thus the original path is not a geodesic. □

Next, we show that there are no geodesics going first down (i.e. away from the root) and then back up. The first part of this lemma will also be used when proving the structure Lemma 8.6.

Lemma 8.3. *Assume that (8.1) holds. If $(x, n) \sim (y, n+1) \sim (z, n+1)$ is a segment in a geodesic, then there is some $y' \in A_n$ such that $(x, n) \sim (y', n) \sim (z, n+1)$, and replacing the segment $(x, n) \sim (y, n+1) \sim (z, n+1)$ with $(x, n) \sim (y', n) \sim (z, n+1)$ also gives a geodesic.*

Consequently, if $(x_0, n_0) \sim (x_1, n_1) \sim \dots \sim (x_m, n_m)$ is a geodesic, then there are no indices $0 \leq i < j < k \leq m$ with $n_j > \max\{n_i, n_k\}$.

Proof. By the choice of A_n there is some $y' \in A_n$ such that $d_Z(z, y') < \alpha^{-n}$. We immediately have that $(y', n) \sim (z, n+1)$. Since we started with a geodesic, we see that $x \neq y'$. Lemma 8.2 therefore implies that $(x, n) \sim (y', n)$. Since $(x, n) \sim (y, n+1) \sim (z, n+1)$ is a geodesic segment, it follows that $(x, n) \sim (y', n) \sim (z, n+1)$ is also a geodesic segment, which proves the first claim.

Next, assume that there would exist a geodesic violating the second part. Because of Lemma 8.1, after restricting to a subpath we may assume that it is of the form

$$(x_0, n) \sim (x_1, n+1) \sim \dots \sim (x_{m-1}, n+1) \sim (x_m, n). \tag{8.2}$$

Applying the first part of the lemma iteratively shows that there are $y_1, \dots, y_{m-2}$ such that

$$(x_0, n) \sim (y_1, n) \sim \dots \sim (y_{m-2}, n) \sim (x_{m-1}, n+1) \sim (x_m, n).$$

As it has the same length as (8.2), it is also a geodesic, but this contradicts Lemma 8.1. □

Lemma 8.4. *Assume that (8.1) holds. Let $v = (x, n) \in V$. Then the following are true*:

(a) *If*

$$v_0 \sim v_1 \sim \dots \sim v_n = v \quad \text{and} \quad v_0 \sim w_1 \sim \dots \sim w_n = v$$

are two geodesics, then for $j = 1, \dots, n-1$ we have $v_j \sim w_j$.

(b) *If $\hat{v} = (y, n-1) \sim v$, then there is a geodesic $v_0 \sim v_1 \sim \dots \sim v_n = v$ such that $v_{n-1} = \hat{v}$.*

In Example 8.7 below we show that (a) can fail drastically if τ is close to 1. Recall that here we assume that τ satisfies (8.1), and that v_0 is the root of X. Part (b) holds for any hyperbolic filling, also without the requirement (8.1).

Proof. To verify (a), note that by Lemma 3.1, $v_j = (x_j, j)$ and $w_j = (y_j, j)$ for some $x_j, y_j \in A_j$, $j = 0, \ldots, n$. Then

$$w_{n-1} = (y_{n-1}, n-1) \sim v = (x, n) \sim (x_{n-1}, n-1) = v_{n-1}$$

and it follows from Lemma 8.1 that $v_{n-1} \sim w_{n-1}$. Now an inductive application of Lemma 8.2 gives the desired conclusion.

The second claim follows from the fact that the concatenation of the edge $v \sim \hat{v}$ to any of the geodesics connecting $\hat{v}$ to the root vertex v_0 gives a geodesic. □

Lemma 8.5. *Assume that* (8.1) *holds. Let n_0 be the smallest positive integer such that*

$$n_0 \alpha^{1-n_0} \le \frac{1}{\alpha+1}.$$

Then there is no horizontal geodesic of length $\ge 2n_0$, i.e., if $m \ge 2n_0$ and

$$(y_0, n) \sim (y_1, n) \sim \ldots \sim (y_m, n), \tag{8.3}$$

then (8.3) *is not a geodesic.*

If we drop the assumption (8.1) then the proof below shows that the same conclusion holds provided that n_0 is the smallest positive integer such that $2n_0\alpha^{1-n_0} \le 1 - 1/\tau$.

Proof. We may assume that $m = 2n_0$. For $0 \le j \le n$, there is some $x_j \in A_j$ so that $y_0 \in B_Z(x_j, \alpha^{-j})$. Necessarily, $x_0 = z_0$ and $x_n = y_0$. Then for each $j = 0, \ldots, n-1$ we have that $y_0 \in B_Z(x_j, \alpha^{-j}) \cap B_Z(x_{j+1}, \alpha^{-(j+1)})$, and so

$$(y_0, n) \sim (x_{n-1}, n-1) \sim (x_{n-2}, n-2) \sim \ldots \sim (x_1, 1) \sim v_0. \tag{8.4}$$

Similarly, we can find $z_j \in A_j$ so that $y_m \in B_Z(z_j, \alpha^{-j})$, $1 \le j < n$, and

$$(y_m, n) \sim (z_{n-1}, n-1) \sim (z_{n-2}, n-2) \sim \ldots \sim (z_1, 1) \sim v_0. \tag{8.5}$$

If $n \le n_0 - 1$, then combining (8.4) and (8.5) gives us a path from (y_0, n) to (y_m, n) (through the root v_0) of length at most $2n \le 2(n_0 - 1) < m$, and thus (8.3) is not a geodesic.

Now assume that $n \ge n_0$. Since

$$d_Z(y_j, y_{j+1}) < 2\tau\alpha^{-n} \quad \text{for } 0 \le j \le n_0 - 1,$$

we get that $d_Z(y_0, y_{n_0}) < 2n_0\tau\alpha^{-n}$. Let $k = n + 1 - n_0$. Then $1 \le k \le n$ and

$$\begin{aligned} d_Z(x_k, y_{n_0}) &\le d_Z(x_k, y_0) + d_Z(y_0, y_{n_0}) \\ &< \alpha^{-k} + 2n_0\tau\alpha^{-n} = \alpha^{-k}(1 + 2n_0\tau\alpha^{1-n_0}) \le \tau\alpha^{-k}, \end{aligned}$$

and in particular $y_{n_0} \in \tau B_Z(x_k, \alpha^{-k})$. Similarly, $y_{n_0} \in \tau B_Z(z_k, \alpha^{-k})$, and thus

$$(x_k, k) \sim (z_k, k).$$

It follows that

$$(y_0, n) \sim (x_{n-1}, n-1) \sim \ldots \sim (x_k, k) \sim (z_k, k) \sim \ldots \sim (y_{2n_0}, n)$$

is a path of length $2(n-k)+1 = 2n_0 - 1 < 2n_0$ showing that (8.3) is not a geodesic. □

More general geodesics can be more complicated. However, we have the following lemma, which can be used to obtain potentially simpler geodesics.

Lemma 8.6. *Assume that* (8.1) *holds. If* $v = (x, k_0)$ *and* $w = (y, k_m)$ *are two distinct vertices with* $d_X(v, w) = m$*, then there is a geodesic*

$$v = (x_0, k_0) \sim (x_1, k_1) \sim \dots \sim (x_{m-1}, k_{m-1}) \sim (x_m, k_m) = w$$

consisting of at most two vertical and one horizontal segments. More precisely, there exist integers $0 \le j_0 \le j_1 \le m$*, with* $j_1 - j_0 \le 2n_0 - 1$ *where* n_0 *is as in Lemma* 8.5*, such that*

$$\begin{aligned} k_{j+1} &= k_j - 1 \quad \text{for } 0 \le j < j_0, \\ k_{j+1} &= k_j + 1 \quad \text{for } j_1 < j \le m, \\ k_j &= k_{j_0} = k_{j_1} \quad \text{for } j_0 \le j \le j_1. \end{aligned}$$

This geodesic minimizes $\sum_j k_j$ over all the geodesics between v and w, and has a similar shape to the path identified in the latter part of Lemma 4.3.

Proof. As X is connected, there is a geodesic between v and w. By the second part of Lemma 8.3 this geodesic does not contain any subpath going first down and then up. We can therefore modify this geodesic iteratively using the first part of Lemma 8.3 to obtain a geodesic of the type described above. That $j_1 - j_0 \le 2n_0 - 1$ follows from Lemma 8.5. □

We end this section with two examples. In the first one we show that, in contrast to the fact obtained in Lemma 8.4 (a), the distance between different geodesics connecting a pair of points can be large when τ is small in comparison with α.

The first example is tailored so that it can be used iteratively in the second example, producing a nonhyperbolic "hyperbolic filling" when $\tau = 1$. This illustrates the dependence of the Gromov constant δ on α and τ.

Example 8.7. Let $n \ge 3$, $\alpha = 2$, $1 \le \tau < \frac{5}{4}$ and $\frac{1}{4}(\tau - 1) \le \rho < 2^{-n-1}$ with $\rho > 0$ as well when $\tau = 1$. Set

$$z_0 = 0, \quad z_1 = \tfrac{1}{2}, \quad z_\pm = \tfrac{1}{4} \pm \rho \quad \text{and} \quad Z = [z_0, z_-] \cup [z_+, z_1] =: Z_- \cup Z_+.$$

Next we choose, $A_0, A_1, \dots$, as follows:

$$\begin{aligned} &A_0 = \{z_0\}, \quad A_1 = A_2 = \{z_0, z_1\}, \\ &A'_j = \{2^{-j}k : k = 0, 1, \dots, 2^{j-1}\} \cap Z, && j = 3, 4 \dots, \\ &A_j = A'_j, && j = 3, \dots, n-1, \\ &A_j = (A'_j \setminus \{z^j_\pm\}) \cup \{z_\pm\}, && j = n, n+1, \dots, \end{aligned}$$

where $z^j_\pm$ is the point in A'_j closest to $z_\pm$. We then construct a "hyperbolic filling" based on this. In the first three levels we have

$$\begin{aligned} &(z_0, 0) \sim (z_0, 1) \sim (z_1, 1) \sim (z_0, 0), \\ &(z_0, 2) \sim (z_0, 1) \sim (z_1, 2) \quad \text{and} \quad (z_0, 2) \sim (z_1, 1) \sim (z_1, 2). \end{aligned}$$

Next, for $j = 2, \dots, n-1$, the distance between $A_j \cap Z_-$ and z_+, as well as between $A_j \cap Z_+$ and z_-, is

$$|z^j_- - z_+| = |z^j_+ - z_-| = 2^{-j} + \rho \ge \tau 2^{-j}.$$

Hence, there are no horizontal edges between the vertices (z, j) and (y, i) with

$$z \in A_j \cap Z_-, \quad y \in A_i \cap Z_+ \quad \text{for } 2 \le i, j \le n-1. \tag{8.6}$$

On the other hand, since $z_+ \in B_Z(z_-, 2^{-n}) \cap B_Z(z_+^{n-1}, 2^{1-n})$, we see that

$$(z_-, n) \sim (z_+^{n-1}, n-1) \quad \text{and similarly,} \quad (z_+, n) \sim (z_-^{n-1}, n-1). \tag{8.7}$$

Hence, there are (at least) two upward-directed geodesics $\gamma_\pm$ between (z_-, n) and $(z_0, 1)$, with γ_- passing only through vertices with the first coordinate in Z_-, while the vertices in γ_+ have the first coordinate in Z_+, except for the starting and ending vertices. It follows that the midpoints in γ_+ and γ_- have distance $\frac{1}{2}(n-1)$ to γ_- and γ_+, respectively, and so the Gromov constant $\delta \ge \frac{1}{2}(n-1)$. (If n is even the midpoints of $\gamma_\pm$ are not vertices.)

With $n = 3$, this also shows that (at least) for $\tau < \frac{5}{4}$, it can happen that

$$(z_0, 2) \sim (z_-, 3) \sim (z_1, 2), \quad \text{while} \quad (z_0, 2) \not\sim (z_1, 2),$$

i.e. both conclusions in Lemma 8.1, the last conclusion in Lemma 8.3, as well as Lemma 8.4 (a) all fail in this case. Similarly, since

$$(z_0, 2) \sim (z_+, 3) \sim (z_1, 3) \sim (z_1, 2)$$

and there is no y such that $(z_0, 2) \sim (y, 2) \sim (z_1, 3)$, the first conclusions in both Lemma 8.2 and 8.3 fail.

When $n = 4$, the geodesics $(\frac{1}{8}, 3) \sim (z_\pm, 4) \sim (\frac{3}{8}, 3)$ are the only geodesics between $(\frac{1}{8}, 3)$ and $(\frac{3}{8}, 3)$, and thus Lemma 8.6 fails. Moreover, the geodesic

$$(z_0, 3) \sim (\tfrac{1}{8}, 4) \sim (z_-, 4) \sim (\tfrac{3}{8}, 3)$$

violates both conclusions of Lemma 8.2 and the last conclusion of Lemma 8.3.

Example 8.8. Let $\alpha = 2$ and $\tau = 1$. Let $\{n_j\}_{j=0}^\infty$ be an increasing sequence of positive integers $n_j \ge 3$ and let $N_k = \sum_{j=0}^k n_j$. Choose $\rho_j < 2^{-n_j-1}$ and repeat the construction in Example 8.7 with $n = n_j$ and $\rho = \rho_j$, and call the resulting space Z_j, $j = 0, 1, \dots$.

Now, replace the interval $[0, 2^{-N_0-1}] \subset Z_0$ by a 2^{-N_0}-scaled copy of Z_1 to form the new space

$$Z_1' = Z_0 \setminus (z_-', z_+'), \quad \text{where } z_\pm' := 2^{-N_0}(\tfrac{1}{4} \pm \rho_1).$$

The first two points z_0 and 2^{-N_0-1} in $A_{N_0+1} \subset Z_1'$ are still next to each other and the corresponding vertices form the horizontal edge

$$(z_0, N_0+1) \sim (2^{-N_0-1}, N_0+1)$$

similarly to $(z_0, 1) \sim (z_1, 1)$ in Z_0.

On the other hand, in the following levels $j = N_0+2, \dots, N_1-1$, similarly to (8.6), there are no horizontal edges between the left-most interval $[0, z_-']$ and the rest of Z_1'. At the same time, similarly to (8.7), the points $z_\pm' \in A_{N_1}$ have upward-directed edges both to the interval $[0, z_-']$ and the second interval in Z_1'. Consequently, the vertex (z_-', N_1) has two upward-directed geodesics $\gamma_\pm'$ to (z_0, N_0+1), such that the midpoints of γ_+' and γ_-' have distance $\frac{1}{2}(n_1 - 1)$ to γ_-' and γ_+', respectively.

Next, the interval $[0, 2^{-N_1-1}] \subset Z_1'$ can be replaced by a 2^{-N_1}-scaled copy of Z_2, i.e. we get the new space $Z_2' = Z_1' \setminus (2^{-N_1}(\frac{1}{4} - \rho_2), 2^{-N_1}(\frac{1}{4} + \rho_2))$. Continuing in this way, we obtain a compact doubling space

$$Z' = \bigcap_{j=1}^\infty Z_j'.$$

Moreover, if $z \in Z'$ and $0 < r < \frac{1}{2}$, then $m(B_Z(z, r) \cap Z') \simeq m(B_Z(z, r))$, where m denotes the Lebesgue measure.

Since $\lim_{j\to\infty} n_j = \infty$, we can for each k find two vertices having two upward-directed geodesics $\widetilde{\gamma}_\pm$ between them such that the midpoint of $\widetilde{\gamma}_+$ has distance $\ge k$ to $\widetilde{\gamma}_-$, i.e. the hyperbolic filling of Z' does not satisfy the Gromov δ-condition, and is thus not Gromov hyperbolic.

9. Measures, function spaces and capacities

In this section, we assume that $1 \le p < \infty$ and that (Y, d) is a metric space equipped with a complete Borel measure ν such that $0 < \nu(B) < \infty$ for all balls $B \subset Y$. We call $Y = (Y, d, \nu)$ a metric measure space.

In the rest of the paper we are interested in studying the metric space Z, considered in the previous sections, together with a doubling measure on Z and Besov spaces on Z with respect to this measure. In particular, as mentioned in the introduction, we wish to associate Besov functions on Z with upper gradient-based Sobolev functions on the uniformization X_ε of the hyperbolic filling X of Z. In this section we will explain the notions related to measures and function spaces.

We follow Heinonen–Koskela [26] in introducing upper gradients as follows (they are referred to as very weak gradients in [26]). For proofs of the facts on upper gradients and Newtonian functions discussed in this section, we refer the reader to Björn–Björn [2] and Heinonen–Koskela–Shanmugalingam–Tyson [27].

Definition 9.1. A Borel function $g : Y \to [0, \infty]$ is an *upper gradient* of a function $u : Y \to [-\infty, \infty]$ if for each nonconstant compact rectifiable curve γ in Y, we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds. \tag{9.1}$$

Here x and y denote the two endpoints of γ. The above inequality should be interpreted as also requiring that $\int_\gamma g\, ds = \infty$ if at least one of $u(x)$ and $u(y)$ is not finite. If g is a nonnegative measurable function on Y and if (9.1) holds for p-almost every curve (see below), then g is a *p-weak upper gradient* of u.

We say that a property holds for *p-almost every curve* if the family Γ of all nonconstant compact rectifiable curves for which the property fails has *zero p-modulus*, i.e. there is a Borel function $0 \le \rho \in L^p(Y)$ such that $\int_\gamma \rho\, ds = \infty$ for every curve $\gamma \in \Gamma$. The p-weak upper gradients were introduced in Koskela–MacManus [33]. It was also shown therein that if $g \in L^p(Y)$ is a p-weak upper gradient of u, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of u such that $\|g_j - g\|_{L^p(Y)} \to 0$.

If u has an upper gradient in $L^p(Y)$, then it has a *minimal p-weak upper gradient* $g_u \in L^p(Y)$ in the sense that $g_u \le g$ a.e. for every p-weak upper gradient $g \in L^p(Y)$ of u, see Shanmugalingam [46]. The minimal p-weak upper gradient is well defined up to a set of measure zero.

Following Shanmugalingam [45], we define a version of Sobolev spaces on Y.

Definition 9.2. A function $u : Y \to [-\infty, \infty]$ is in the Newtonian space $\widetilde{N}^{1,p}(Y)$ if $\int_Y |u|^p\, d\mu < \infty$ and u has a p-weak upper gradient $g \in L^p(Y)$. This space is a vector space and a lattice, equipped with the seminorm $\|u\|_{N^{1,p}(Y)}$ given by

$$\|u\|_{N^{1,p}(Y)} := \left(\int_Y |u|^p\, d\nu + \inf_g \int_Y g^p\, d\nu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of u, or equivalently all p-weak upper gradients g of u (see the comments above).

The *Newtonian space* $N^{1,p}(Y) = \widetilde{N}^{1,p}(Y)/\sim$, where $\sim$ is the equivalence relation on $\widetilde{N}^{1,p}(Y)$ given by $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(Y)} = 0$. To specify the measure with respect to which the Newtonian space is taken, we will also write $\widetilde{N}^{1,p}(Y, \nu)$ and $N^{1,p}(Y, \nu)$.

Definition 9.3. The C_p^Y-*capacity* of a set $E \subset Y$ is defined as

$$C_p^Y(E) = \inf_u \|u\|_{N^{1,p}(Y)}^p,$$

where the infimum is taken over all $u \in \widetilde{N}^{1,p}(Y)$ satisfying $u \geq 1$ on E.

Note that since functions in $\widetilde{N}^{1,p}(Y)$ are defined pointwise everywhere, the requirement $u \geq 1$ on E in the definition of $C_p^Y(E)$ makes sense for an arbitrary set $E \subset Y$.

A property is said to hold *quasieverywhere* (q.e. or C_p^Y-q.e.) if the set of all points at which the property fails has C_p^Y-capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in \widetilde{N}^{1,p}(Y)$, then $v \sim u$ if and only if $v = u$ q.e. Moreover, if $u, v \in \widetilde{N}^{1,p}(Y)$ and $u = v$ a.e., then $u = v$ q.e. That means that the equivalence classes in $N^{1,p}(Y)$ are precisely made up of functions which are equal q.e., and not a.e. as in the usual Sobolev spaces. By an abuse of notation, just as for L^p-spaces, we will often not distinguish between a function in $\widetilde{N}^{1,p}(Y)$ and the corresponding equivalence class in $N^{1,p}(Y)$.

Definition 9.4. We say that Y (or the measure ν) supports a *p-Poincaré inequality* if there exist $C, \lambda > 0$ such that for each ball $B = B(x, r)$ and for all integrable functions u and upper gradients g of u on λB,

$$\fint_B |u - u_B|\, d\nu \leq Cr \biggl(\fint_{\lambda B} g^p \, d\nu \biggr)^{1/p},$$

where $u_B := \fint_B u \, d\nu = \nu(B)^{-1} \int_B u \, d\nu$.

See Björn–Björn [2] and Heinonen–Koskela–Shanmugalingam–Tyson [27] for equivalent formulations of the C_p^Y capacity and the p-Poincaré inequality.

Remark 9.5. We will primarily be interested in Newtonian spaces on the uniformization X_ε of a hyperbolic filling of Z, and on its closure $\overline{X}_\varepsilon$, in both cases equipped with the measure μ_β, $\beta > 0$, defined by (10.7) below. In particular, each edge in X is measured by a multiple of the Lebesgue measure. It is then quite easy to see that the only family of nonconstant compact rectifiable curves in X_ε which has zero p-modulus (with respect to μ_β) is the empty family. Functions in Newtonian spaces are absolutely continuous on p-almost every line, see Shanmugalingam [45]. Thus all functions $u \in \widetilde{N}^{1,p}(X_\varepsilon, \mu_\beta)$ are continuous on X_ε and absolutely continuous on each edge. Moreover, $g_u = |du/ds_\varepsilon|$ a.e. on each edge, where ds_ε denotes the arc length with respect to d_ε. In particular, each equivalence class in $N^{1,p}(X_\varepsilon, \mu_\beta)$ contains just one function, and that function is continuous. Moreover, points in X_ε have positive capacity.

For functions on $\overline{X}_\varepsilon$, the situation is not quite as simple, but the following result will be useful. A function u on Y is C_p^Y-*quasicontinuous* if for each $\eta > 0$ there is an open set $G \subset Y$ with $C_p^Y(G) < \eta$ such that $u|_{Y \setminus G}$ is continuous. Note that any $E \subset \overline{X}_\varepsilon$ with $C_p^{\overline{X}_\varepsilon}(E) = 0$ must satisfy $E \subset \partial_\varepsilon X$.

Theorem 9.6. (Björn–Björn–Shanmugalingam [4]) *Assume that Y is complete and that ν is doubling and supports a p-Poincaré inequality. Then every $u \in \widetilde{N}^{1,p}(Y)$ is C_p^Y-quasicontinuous. Moreover, C_p^Y is an outer capacity, i.e.*

$$C_p^Y(E) = \inf_{\substack{G \supset E \\ G \text{ open}}} C_p^Y(G).$$

We will use these facts together with our trace and extension results to show that Besov functions on Z have $\mathrm{Cap}_{B^\theta_{p,p}(Z)}$-quasicontinuous representatives (which is defined just as C_p^Y-quasicontinuous), see Proposition 13.3. Similarly, we will obtain density of Lipschitz functions and existence of Hölder continuous representatives in Besov spaces using our trace and extension results, and corresponding theorems for Newtonian functions.

We now give the definition of Besov spaces on metric measure spaces.

Definition 9.7. Let $\theta > 0$. We say that $u \in L^p(Y)$ is in the Besov space $B^\theta_{p,p}(Y)$ if

$$\|u\|^p_{\theta,p} := \int_Y \int_Y \frac{|u(\zeta) - u(\xi)|^p}{d(\zeta,\xi)^{p\theta}} \frac{d\nu(\xi)\, d\nu(\zeta)}{\nu(B(\zeta, d(\zeta,\xi)))} < \infty.$$

Remark 9.8. Note that $B^\theta_{p,p}(Y)$ is a Banach space with the norm given by

$$\|u\|_{B^\theta_{p,p}(Y)} = \|u\|_{\theta,p} + \|u\|_{L^p(Y)}.$$

Indeed, it is clear that this function space is a normed vector space. To see that it is complete, we argue as follows. Let $\{u_k\}_{k=1}^\infty$ be a Cauchy sequence in $B^\theta_{p,p}(Y)$. Then it is a Cauchy sequence in $L^p(Y)$, and hence it is convergent to some function $u \in L^p(Y)$. By passing to a subsequence if necessary, we also ensure that $u_k \to u$ ν-a.e. in Y. Setting the measure ν_0 on $Y \times Y$ by

$$\nu_0(E) = \int_E \frac{d(\nu \times \nu)(\xi,\zeta)}{d(\zeta,\xi)^{p\theta} \nu(B(\zeta, d(\zeta,\xi)))},$$

and defining $v_k : Y \times Y \to \mathbf{R}$ as $v_k(\xi,\zeta) = u_k(\xi) - u_k(\zeta)$, we note that

$$\|u_k\|_{\theta,p} = \|v_k\|_{L^p(Y\times Y,\nu_0)}.$$

Thus, $\{v_k\}_{k=1}^\infty$ is also a Cauchy sequence in the complete space $L^p(Y \times Y, \nu_0)$, and so converges therein to a function $v : Y \times Y \to \mathbf{R}$. Again, by passing to yet another subsequence if necessary, we may also assume that $v_k \to v$ ν_0-a.e. in $Y \times Y$. Setting $w : Y \times Y \to \mathbf{R}$ by $w(\xi,\zeta) = u(\xi) - u(\zeta)$, we see that necessarily $v = w$ ν_0-a.e. in $Y \times Y$. Therefore $v_k \to w$ in $L^p(Y \times Y, \nu_0)$, that is, $u_k \to u$ in $B^\theta_{p,p}(Y)$.

We recall the following lemma. For a proof see Gogatishvili–Koskela–Shanmugalingam [21, Theorem 5.2 and (5.1)] (where the factor of 2 should be replaced with $\alpha > 1$).

Lemma 9.9. *Assume that ν is doubling and $\theta > 0$. If $u \in B^\theta_{p,p}(Y)$, then*

$$\|u\|^p_{\theta,p} \simeq \sum_{n=0}^\infty \int_Y \fint_{B(\zeta,\alpha^{-n})} \frac{|u(\zeta) - u(\eta)|^p}{\alpha^{-n\theta p}}\, d\nu(\zeta)\, d\nu(\eta).$$

Definition 9.10. We set the *Besov capacity* of $E \subset Y$ to be the number

$$\mathrm{Cap}_{B^\theta_{p,p}(Y)}(E) := \inf_u (\|u\|^p_{\theta,p} + \|u\|^p_{L^p(Y)}),$$

where the infimum is taken over all $u \in B^\theta_{p,p}(Y)$ satisfying $u \geq 1$ a.e. on a neighborhood of E.

10. Lifting doubling measures from Z to its hyperbolic filling X

From now on, we let X be a hyperbolic filling, constructed with parameters $\alpha, \tau > 1$, of a compact metric space Z with $0 < \operatorname{diam} Z < 1$ and equipped with a doubling *measure* ν*. In this section, we also let* X_ε *be the uniformization of* X *with parameter* $0 < \varepsilon \le \log \alpha$*.*

We now focus on lifting up ν from Z as follows: Recall that the vertices in X are denoted $v = (z, n)$, where z belongs to the maximal α^{-n}-separated set $A_n \subset Z$. Note that n is the graph distance from the root $v_0 := (z_0, 0)$ to (z, n). For $(z, n) \in V$ we set

$$\hat{\mu}(\{(z,n)\}) = \nu(B_Z(z, \alpha^{-n})). \tag{10.1}$$

The measure μ on X is then given by "smearing out" $\hat{\mu}$ to X: for a Borel set $A \subset X$,

$$\mu(A) = \sum_{v \in V} \sum_{w \sim v} \bigl(\hat{\mu}(\{v\}) + \hat{\mu}(\{w\})\bigr) \mathcal{L}(A \cap [v, w]), \tag{10.2}$$

where $[v, w]$ denotes the unit interval that connects the two vertices v and w, and $\mathcal{L}$ denotes the Lebesgue measure.

Strictly speaking, it is $\hat{\mu}(v) \deg v$ that is smeared out, but because X has uniformly bounded degree (by Proposition 4.5) this is comparable to $\hat{\mu}(v)$.

Note that the vertex set V_n of points in X that are at level n from the root is composed of a maximal α^{-n}-separated set of points from Z. So by the work of Gill and Lopez [20], [36], we know that V_n, equipped with the neighborhood relationship inherited from V and with the measure $\hat{\mu}|_{V_n}$, is doubling and that a subsequence converges in the pointed measured Gromov–Hausdorff sense to a measure $\hat{\mu}_\infty$ on Z such that $\hat{\mu}_\infty \simeq \nu$.

Lemma 10.1. *Let* $(z, n), (y, m) \in V$ *with* $(y, m) \sim (z, n)$*. Then*

$$\frac{1}{C_d^N} \hat{\mu}(\{(z,n)\}) \le \hat{\mu}(\{(y,m)\}) \le C_d^N \hat{\mu}(\{(z,n)\}),$$

where N *is the smallest integer such that* $2^N \ge \alpha(1+\tau) + \tau$ *and* C_d *is the doubling constant associated with* ν*.*

Proof. Since the two points are neighbors, we have that $|n - m| \le 1$ and thus $\alpha^{-n} \le \alpha^{1-m}$. Since $\tau B_Z(z, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m}) \ne \varnothing$ by the construction of the hyperbolic filling, every $\zeta \in B_Z(z, \alpha^{-n})$ satisfies

$$d_Z(\zeta, y) \le d_Z(\zeta, z) + d_Z(z, y) < \alpha^{-n} + \tau(\alpha^{-n} + \alpha^{-m}) \le 2^N \alpha^{-m},$$

and so $B_Z(z, \alpha^{-n}) \subset 2^N B_Z(y, \alpha^{-m})$. The doubling property of ν then implies that

$$\hat{\mu}(\{(z,n)\}) = \nu(B_Z(z, \alpha^{-n})) \le C_d^N \nu(B_Z(y, \alpha^{-m})) = C_d^N \hat{\mu}(\{(y,m)\}).$$

Reversing the roles of z and y in the above argument gives the desired double inequality. □

Theorem 10.2. *Let* Y *be a metric graph equipped with the length metric* d *such that all edges have length* 1 *and assume that* Y *has uniformly bounded degree, i.e. every vertex has at most* K *neighbors. Let* $\hat{\mu}$ *be a discrete measure defined on the vertices of* Y *and such that* $\hat{\mu}(v) \simeq \hat{\mu}(w)$ *whenever* $v \sim w$*, with comparison constants independent of* v *and* w*. Consider the smeared out measure* μ *on* Y *given by* (10.2)*.*

Then for each $R_0 > 0$ *there is a constant* $C_0 \ge 1$ *such that for all balls* $B = B(x, r)$ *with* $r \le R_0$*, and every integrable function* u *and upper gradient* g *of* u *on* B*,*

$$\mu(2B) \le C_0 \mu(B) \quad \text{and} \quad \fint_B |u - u_B| \, d\mu \le C_0 r \fint_B g \, d\mu. \tag{10.3}$$

The constant C_0 *depends only on* R_0*,* K *and the comparison constants in* $\hat{\mu}(v) \simeq \hat{\mu}(w)$*.*

Proof. Since Y is a length space, it follows from Lemma 3.5 and Theorem 5.3 in Björn–Björn–Shanmugalingam [5] that it suffices to consider only the case $R_0 = \frac{1}{4}$. Let v be a nearest vertex to the center x of B, i.e. $d(x,v) = \operatorname{dist}(x,V)$. As $r \le \frac{1}{4}$, the ball $2B$ contains at most one vertex, namely v. Hence,

$$B \subset 2B \subset \bigcup_{w\sim v} [v,w] \quad \text{and} \quad d\mu = (\hat{\mu}(\{v\}) + \hat{\mu}(\{w\}))\, ds \simeq \hat{\mu}(\{v\})\, ds \text{ on each } [v,w], \tag{10.4}$$

by Lemma 10.1. Thus

$$\mu(2B) \lesssim Kr\hat{\mu}(\{v\}) \lesssim \mu(B).$$

To prove the 1-Poincaré inequality in (10.3), observe that if $v \notin B$, then B is an interval, and so the 1-Poincaré inequality for B follows from the 1-Poincaré inequality on $\mathbf{R}$ and the fact that $d\mu = C_B d\mathcal{L}$ on B. On the other hand, if $v \in B$, then

$$B = \bigcup_{w\sim v} I_w, \quad \text{where } I_w = B \cap [v,w].$$

We therefore obtain from (10.4) and the definition of upper gradients that for each $w \sim v$,

$$\begin{aligned}\int_{I_w} |u - u(v)|\, d\mu \le \int_{I_w}\int_{I_w} g(s)\, ds\, d\mu = \mu(I_w)\int_{I_w} g\, ds \lesssim r\hat{\mu}(\{v\})\int_{I_w} g\, ds \\ \simeq r\int_{I_w} g\, d\mu.\end{aligned}$$

Summing over all $w \sim v$ yields

$$\int_B |u - u(v)|\, d\mu = \sum_{w\sim v}\int_{I_w} |u - u(v)|\, d\mu \lesssim \sum_{w\sim v} r\int_{I_w} g\, d\mu = r\int_B g\, d\mu.$$

A standard argument based on the triangle inequality allows us to replace $u(v)$ on the left-hand side by u_B at the cost of an extra factor 2 on the right-hand side. □

To obtain a doubling measure on X_ε with respect to the uniformized metric d_ε, we can equip X_ε with the uniformized measure

$$d\hat{\mu}_\beta(x) = \rho_\beta(x)\, d\mu(x), \quad \text{where } \rho_\beta(x) = e^{-\beta d_X(x,v_0)} \simeq d_\varepsilon(x)^{\beta/\varepsilon}. \tag{10.5}$$

Theorems 4.9 and 6.2 in Björn–Björn–Shanmugalingam [5] (which hold for general Gromov hyperbolic spaces) then guarantee that for sufficiently large β, the obtained measure $\hat{\mu}_\beta$ is doubling and supports a 1-Poincaré inequality on X_ε as well as on $\overline{X}_\varepsilon$. More precisely, this holds for $\beta > \beta_0$, where β_0 is determined by the limitations given in [5, Theorems 4.9 and 6.2] based on the doubling constant from Theorem 10.2.

In considering the specific case of hyperbolic fillings, we are able to show that the requirement $\beta > \beta_0$ can be omitted. Just as we showed in Theorem 5.1 that the full range $0 < \varepsilon \le \log\alpha$ is possible, we can allow for the full range $\beta > 0$ in our setting. This will be important in Sections 11–13 for our trace and extension results, and their applications.

Since we consider measures (on hyperbolic fillings) that are constructed from doubling measures on Z, it is possible to start directly with the weighted discrete measure

$$\rho_\beta(v)\hat{\mu}(\{v\}) = e^{-\beta n}\nu(B_Z(z,\alpha^{-n})) \quad \text{for } v = (z,n) \in V, \tag{10.6}$$

defined on the vertices in X, and smear it out as in (10.2):

$$\mu_\beta(A) = \sum_{v\in V}\sum_{w\sim v}\big(\rho_\beta(v)\hat{\mu}(\{v\}) + \rho_\beta(w)\hat{\mu}(\{w\})\big)\mathcal{L}(A \cap [v,w]). \tag{10.7}$$

Since $\rho_\beta(v) \simeq \rho_\beta(w)$ whenever $v \sim w$, the measures μ_β and $\hat{\mu}_\beta$ are clearly comparable, and the following result also holds for $\hat{\mu}_\beta$.

Theorem 10.3. *For every $\beta > 0$, μ_β is doubling and supports a 1-Poincaré inequality on X_ε as well as on $\overline{X}_\varepsilon$.*

Furthermore, for all $\zeta \in \partial_\varepsilon X$ and $0 < r \le 2 \operatorname{diam}_\varepsilon X_\varepsilon \le 4/\varepsilon$,

$$\mu_\beta(B_\varepsilon(\zeta,r)) \simeq (\varepsilon r)^{\beta/\varepsilon} \nu(B_Z(\zeta,(\varepsilon r)^{1/\sigma})), \quad \text{where } \sigma = \frac{\varepsilon}{\log\alpha} \le 1 \tag{10.8}$$

and the comparison constants depend only on ε, β, α, τ and the doubling constant associated with ν.

In particular, ν is comparable to the β/ε-codimensional measure on $\partial_\varepsilon X$ generated by μ_β.

Lemma 10.4. *Let $0 < \varepsilon \le \log\alpha$. Then*

$$B_X\biggl(x, \frac{C_1 r}{\varepsilon d_\varepsilon(x)}\biggr) \subset B_\varepsilon(x,r) \subset B_X\biggl(x, \frac{C_2 r}{\varepsilon d_\varepsilon(x)}\biggr), \quad \text{if } x \in X \text{ and } 0 < r \le \tfrac12 d_\varepsilon(x),$$

where $C_1, C_2 > 0$ are independent of ε.

Proof. This was proved when $0 < \varepsilon \le \varepsilon_0(\delta)$ for general Gromov hyperbolic spaces (where δ is the Gromov hyperbolicity constant) in Theorem 2.1 in Björn–Björn–Shanmugalingam [5], with Remark 2.11 in [5] showing that $C_1, C_2 > 0$ are independent of ε. That proof only relies on the following facts which hold for *hyperbolic fillings* for all $0 < \varepsilon \le \log\alpha$:

- That X is geodesic, which follows from the definition.
- That X is locally compact and $\overline{X}_\varepsilon$ is geodesic, which follows from Proposition 4.6, as Z is equipped with a doubling measure.
- Lemma 2.6 (i.e. [8, Lemma 4.16]) which holds for arbitrary $\varepsilon > 0$. □

Proof of Theorem 10.3. We first concentrate on the doubling property. We need to consider three types of balls, namely, subWhitney balls, balls centered in (or near) $\partial_\varepsilon X$ and intermediate balls. Recall that the graph X has uniformly bounded degree, by Proposition 4.5.

1. For *subWhitney balls*, that is, balls $B_\varepsilon(x,r)$ with $r \le \frac14 d_\varepsilon(x)$, we note that $2C_2 r/\varepsilon d_\varepsilon(x) \le C_2/2\varepsilon$. Hence, Lemmas 10.1, 10.4 and Theorem 10.2 imply that

$$\begin{aligned}\mu_\beta(B_\varepsilon(x,2r)) &\le \mu_\beta\biggl(B_X\biggl(x,\frac{2C_2 r}{\varepsilon d_\varepsilon(x)}\biggr)\biggr) \simeq \rho_\beta(x)\mu\biggl(B_X\biggl(x,\frac{2C_2 r}{\varepsilon d_\varepsilon(x)}\biggr)\biggr)\\ &\simeq \mu_\beta\biggl(B_X\biggl(x,\frac{C_1 r}{\varepsilon d_\varepsilon(x)}\biggr)\biggr) \le \mu_\beta(B_\varepsilon(x,r)).\end{aligned}$$

2. If $x \in \overline{X}_\varepsilon$ and $r \ge 2d_\varepsilon(x)$, then for some $\zeta \in \partial_\varepsilon X$,

$$B_\varepsilon(x,2r) \subset B_\varepsilon(\zeta,\tfrac52 r) \quad \text{and} \quad B_\varepsilon(\zeta,\tfrac12 r) \subset B_\varepsilon(x,r).$$

It therefore suffices to estimate $\mu_\beta(B_\varepsilon(\zeta,r))$ for all $\zeta \in \partial_\varepsilon X$ and $0 < r \le 2\operatorname{diam}_\varepsilon X_\varepsilon$. From the construction of μ_β, and using also the uniformly bounded degree of X, it is clear that

$$\mu_\beta(B_\varepsilon(\zeta,r)) \simeq \sum_{v\in V\cap B_\varepsilon(\zeta,r)} e^{-\beta\pi_2(v)}\hat\mu(v).$$

Let $v = (z,n) \in V \cap B_\varepsilon(\zeta,r)$. Then $e^{-\varepsilon n} = \varepsilon d_\varepsilon(v) < \varepsilon r$ by (4.1), and hence $n \ge N$, where N is the smallest nonnegative integer such that

$$N \ge \frac{1}{\varepsilon}\log\frac{1}{\varepsilon r}. \tag{10.9}$$

Let $z_j \in A_j \subset Z$ be such that $d_Z(z_j,\zeta) < \alpha^{-j}$, $j = 0,1,\dots$. Proposition 4.4 shows that

$$d_Z(z,z_j)^\sigma \le (2\tau\alpha)^\sigma d_\varepsilon(v,(z_j,j)) \le (2\tau\alpha)^\sigma \big(d_\varepsilon(v,\zeta)+d_\varepsilon(\zeta,(z_j,j))\big).$$

Since the path $(z_0,0) \sim \dots \sim (z_j,j) \sim \dots$ is a geodesic ray in X ending at ζ (see Lemma 3.1 and Proposition 4.4), we have

$$d_\varepsilon(\zeta,(z_j,j)) = e^{-\varepsilon j}/\varepsilon. \tag{10.10}$$

Letting $j \to \infty$ then shows that

$$d_Z(z,\zeta) \le d_Z(z,z_j)+d_Z(z_j,\zeta) \le 2\tau\alpha\bigg(r+\frac{e^{-\varepsilon j}}{\varepsilon}\bigg)^{1/\sigma} + \alpha^{-j} \to 2\tau\alpha r^{1/\sigma}.$$

We therefore obtain that

$$\mu_\beta(B_\varepsilon(\zeta,r)) \lesssim \sum_{n\ge N} e^{-\beta n} \sum_{z\in A_n\cap B_Z(\zeta,3\tau\alpha r^{1/\sigma})} \nu(B_Z(z,\alpha^{-n})).$$

Since $\alpha^{-n} = (e^{-\varepsilon n})^{1/\sigma} < (\varepsilon r)^{1/\sigma}$, the bounded overlap of the balls $B_Z(z,\alpha^{-n})$ in $A_n \subset Z$ and the doubling property of ν now imply that

$$\mu_\beta(B_\varepsilon(\zeta,r)) \lesssim \sum_{n\ge N} e^{-\beta n}\nu(B_Z(\zeta,(\varepsilon r)^{1/\sigma})) \simeq (\varepsilon r)^{\beta/\varepsilon}\nu(B_Z(\zeta,(\varepsilon r)^{1/\sigma})). \tag{10.11}$$

To verify the reverse comparison, observe that by (10.9) and (10.10) we have $d_\varepsilon(\zeta,(z_N,N)) = e^{-\varepsilon N}/\varepsilon < r$. It follows that the edge $(z_N,N) \sim (z_{N+1},N+1)$ is contained in $B_\varepsilon(\zeta,r)$. Consequently, using the doubling property of ν and the fact that $d_Z(\zeta,z_N) < \alpha^{-N} = (e^{-\varepsilon N})^{1/\sigma} \simeq (\varepsilon r)^{1/\sigma}$, we conclude from (10.7) that

$$\begin{aligned}\mu_\beta(B_\varepsilon(\zeta,r)) &\gtrsim e^{-\beta N}\hat\mu(\{(z_N,N)\})\\ &\simeq (\varepsilon r)^{\beta/\varepsilon}\nu(B_Z(z_N,\alpha^{-N})) \simeq (\varepsilon r)^{\beta/\varepsilon}\nu(B_Z(\zeta,(\varepsilon r)^{1/\sigma})),\end{aligned}$$

which, together with (10.11), proves (10.8).

3. If $x \in X_\varepsilon$ and $\frac14 d_\varepsilon(x) \le r \le 2d_\varepsilon(x)$, then clearly

$$B_\varepsilon(x,\tfrac14 d_\varepsilon(x)) \subset B_\varepsilon(x,r) \subset B_\varepsilon(x,2r) \subset B_\varepsilon(x,4d_\varepsilon(x)).$$

From Proposition 4.6 we know that $\overline{X}_\varepsilon$ is compact, and thus there is $\zeta \in \partial_\varepsilon X$ such that $d_\varepsilon(\zeta,x) = d_\varepsilon(x)$. Let γ be a d_ε-geodesic from x to ζ and let $v=(z,k)$ be the vertex in γ nearest to x. As in case 1, using (10.7), Lemma 10.1, and the uniform boundedness of the degrees, we see that

$$\mu_\beta(B_\varepsilon(x,\tfrac14 d_\varepsilon(x))) \simeq \rho_\beta(x)\hat\mu(\{v\}) = \rho_\beta(x)\nu(B_Z(z,\alpha^{-k})). \tag{10.12}$$

On the other hand, by (10.8) (proved when considering case 2 above),

$$\mu_\beta(B_\varepsilon(x,4d_\varepsilon(x))) \le \mu_\beta(B_\varepsilon(\zeta,5d_\varepsilon(x))) \simeq (\varepsilon d_\varepsilon(x))^{\beta/\varepsilon}\nu(B_Z(\zeta,(5\varepsilon d_\varepsilon(x))^{1/\sigma})).$$

Note that the right-hand side of the above is comparable to the right-hand side of (10.12) since $d_\varepsilon(z,\zeta) \le d_\varepsilon(z,v)+d_\varepsilon(v,\zeta) \le 2d_\varepsilon(x)$, ν is doubling, and

$$(\varepsilon d_\varepsilon(x))^{\beta/\varepsilon} \simeq \rho_\varepsilon(x)^{\beta/\varepsilon} = \rho_\beta(x) \quad \text{and} \quad \alpha^{-k} = e^{-\varepsilon k/\sigma} \simeq \rho_\varepsilon(x)^{1/\sigma} \simeq (\varepsilon d_\varepsilon(x))^{1/\sigma}.$$

The doubling property of μ_β now follows directly in all three cases from the above estimates.

To show that μ_β supports a 1-Poincaré inequality on $\overline{X}_\varepsilon$, we proceed as in the proof of Lemma 6.1 in Björn–Björn–Shanmugalingam [5]. This is possible for all $\beta > 0$, not only $\beta > \beta_0$ as in [5], since we already know that μ_β is doubling on X_ε. Together with Theorem 10.2, it shows that there exists $c_0 > 0$ such that (10.3) holds for all subWhitney balls $B = B_\varepsilon(x, r)$ with $x \in X$, the measure μ_β, and $0 < r \le c_0 d_\varepsilon(x)$. Since X_ε is a uniform length space and μ_β is doubling on X_ε, we conclude from [5, Proposition 6.3] that μ_β supports a 1-Poincaré inequality on X_ε as well as on $\overline{X}_\varepsilon$. □

Corollary 10.5. *With the assumptions as in Theorem 10.3, we have for all $x \in \overline{X}_\varepsilon$ and $0 < r \le 2 \operatorname{diam}_\varepsilon X_\varepsilon$,*

$$\mu_\beta(B_\varepsilon(x,r)) \simeq \begin{cases} (\varepsilon r)^{\beta/\varepsilon} \nu(B_Z(\zeta, (\varepsilon r)^{1/\sigma})), & \text{if } r \ge d_\varepsilon(x) \text{ and } \zeta \in Z \text{ is a nearest point to } x, \\ r(\varepsilon d_\varepsilon(x))^{\beta/\varepsilon - 1} \hat{\mu}(\{v\}), & \text{if } r \le d_\varepsilon(x) \text{ and } v \in X \text{ is a nearest vertex to } x. \end{cases}$$

In both cases, the nearness is with respect to the metric d_ε.

Recall from (4.1) that $\alpha^{-n} = (e^{-\varepsilon n})^{1/\sigma} = (\varepsilon d_\varepsilon(v))^{1/\sigma}$ if $v = (z,n) \in V$. Moreover, if $\zeta \in Z$ and $v = (z,n) \in V$ are nearest points to x in Z and V, respectively, then by Proposition 4.4,

$$d_Z(\zeta, z)^\sigma \simeq d_\varepsilon(\zeta, z) \lesssim d_\varepsilon(x) \simeq d_\varepsilon(v).$$

It therefore follows from (10.1) and the doubling property of ν that

$$\hat{\mu}(\{v\}) = \nu(B_Z(z, (\varepsilon d_\varepsilon(v))^{1/\sigma}) \simeq \nu(B_Z(\zeta, (\varepsilon d_\varepsilon(x))^{1/\sigma})),$$

which further simplifies the formula in Corollary 10.5. Also note that since ν is doubling, $B_Z(\zeta, (\varepsilon d_\varepsilon(x))^{1/\sigma}))$ can be replaced by any ball $B_Z(\xi, (\varepsilon d_\varepsilon(x))^{1/\sigma}))$ with $\xi \in Z$ such that $d_\varepsilon(\xi, x) \lesssim d_\varepsilon(x)$, and that $\nu(B_Z(\zeta, (\varepsilon d_\varepsilon(x))^{1/\sigma})) \simeq \nu(B_\varepsilon(\zeta, \varepsilon d_\varepsilon(x)))$.

Proof of Corollary 10.5. The first case follows directly from (10.8) together with the inclusions

$$B_\varepsilon(\zeta, r) \subset B_\varepsilon(x, 2r) \subset B_\varepsilon(\zeta, 3r)$$

and the doubling property of μ_β and ν.

In the second case, Lemma 10.4 implies that

$$B_\varepsilon(x, \tfrac{1}{2}r) \subset B_X\biggl(x, \frac{C_2 r}{2\varepsilon d_\varepsilon(x)}\biggr) \subset B_X\biggl(x, \frac{C_2}{2\varepsilon}\biggr)$$

Recall from Proposition 4.5 that the graph X has uniformly bounded degree. Therefore we have by (10.7) and Lemma 10.1 that

$$\begin{aligned} \mu_\beta(B_\varepsilon(x, \tfrac{1}{2}r)) &\simeq \frac{r}{\varepsilon d_\varepsilon(x)} \biggl(\rho_\beta(v)\hat{\mu}(\{v\}) + \sum_{w \in V \cap B_\varepsilon(x, r/2)} \rho_\beta(w)\hat{\mu}(\{w\}) \biggr) \\ &\simeq \frac{r}{\varepsilon d_\varepsilon(x)} \rho_\beta(v)\hat{\mu}(\{v\}). \end{aligned}$$

Since $\rho_\beta(v) \simeq (\varepsilon d_\varepsilon(x))^{\beta/\varepsilon}$, the doubling property of μ_β concludes the proof. □

Lemma 10.6. *Assume that the measure ν on Z satisfies for all $\zeta \in Z$ and $0 < r' \le r \le 2 \operatorname{diam} Z$,*

$$\frac{\nu(B_Z(\zeta, r'))}{\nu(B_Z(\zeta, r))} \gtrsim \biggl(\frac{r'}{r}\biggr)^{s_\nu}. \tag{10.13}$$

Let $\beta > 0$ *and* $\varepsilon = \log \alpha$. *Then the measure* μ_β*, defined by* (10.6) *and* (10.7)*, satisfies for all* $x \in X_\varepsilon$ *and* $0 < r' \le r \le 2 \operatorname{diam}_\varepsilon X_\varepsilon$*,*

$$\frac{\mu_\beta(B_\varepsilon(x,r'))}{\mu_\beta(B_\varepsilon(x,r))} \gtrsim \left(\frac{r'}{r}\right)^{s_\beta}, \quad \textit{where } s_\beta = \max\{1, \beta/\varepsilon + s_\nu\}. \tag{10.14}$$

It is well known that every doubling measure ν satisfies (10.13) for some $s_\nu > 0$, see for example [25, (4.16)].

Proof. Note that $\sigma = 1$. We shall distinguish three cases:

1. If $r \le d_\varepsilon(x)$ then the second case in Corollary 10.5 applies both to r and r' and hence

$$\frac{\mu_\beta(B_\varepsilon(x,r'))}{\mu_\beta(B_\varepsilon(x,r))} \simeq \frac{r'}{r}.$$

2. If $r' \ge d_\varepsilon(x)$ then the first case in Corollary 10.5 applies both to r and r' and hence

$$\frac{\mu_\beta(B_\varepsilon(x,r'))}{\mu_\beta(B_\varepsilon(x,r))} \simeq \frac{(\varepsilon r')^{\beta/\varepsilon}\nu(B_Z(\zeta,\varepsilon r'))}{(\varepsilon r)^{\beta/\varepsilon}\nu(B_Z(\zeta,\varepsilon r))} \gtrsim \left(\frac{r'}{r}\right)^{\beta/\varepsilon+s_\nu}.$$

3. If $r' \le d_\varepsilon(x) \le r$ then by the already proved cases 1 and 2,

$$\frac{\mu_\beta(B_\varepsilon(x,r'))}{\mu_\beta(B_\varepsilon(x,r))} = \frac{\mu_\beta(B_\varepsilon(x,r'))}{\mu_\beta(B_\varepsilon(x,d_\varepsilon(x)))} \frac{\mu_\beta(B_\varepsilon(x,d_\varepsilon(x)))}{\mu_\beta(B_\varepsilon(x,r))} \gtrsim \frac{r'}{d_\varepsilon(x)} \left(\frac{d_\varepsilon(x)}{r}\right)^{\beta/\varepsilon+s_\nu}.$$

If $\beta/\varepsilon + s_\nu \ge 1$ then

$$\frac{r'}{d_\varepsilon(x)} \left(\frac{d_\varepsilon(x)}{r}\right)^{\beta/\varepsilon+s_\nu} \ge \left(\frac{r'}{d_\varepsilon(x)}\right)^{\beta/\varepsilon+s_\nu} \left(\frac{d_\varepsilon(x)}{r}\right)^{\beta/\varepsilon+s_\nu} = \left(\frac{r'}{r}\right)^{\beta/\varepsilon+s_\nu},$$

and if $\beta/\varepsilon + s_\nu \le 1$, then

$$\frac{r'}{d_\varepsilon(x)} \left(\frac{d_\varepsilon(x)}{r}\right)^{\beta/\varepsilon+s_\nu} \ge \frac{r'}{d_\varepsilon(x)} \frac{d_\varepsilon(x)}{r} = \frac{r'}{r}.$$

From the above three cases we conclude that (10.14) holds. □

11. Traces to Z from the hyperbolic filling X

Recall the standing assumptions from Section 10. *Here and in the rest of the paper, we also let* $1 \le p < \infty$ *and consider the uniformized space* $\overline{X}_\varepsilon$ *equipped with the measure* μ_β *where* $\varepsilon = \log \alpha$ *and* $\beta > 0$.

Theorem 5.1 shows that X_ε is a uniform space. From Proposition 4.4 with $\varepsilon = \log \alpha$ we know that $\partial_\varepsilon X$ is biLipschitz equivalent to Z. Hence we can replace $\partial_\varepsilon X$ by Z as well, since the Besov spaces are biLipschitz invariant. Of course, the measure on Z is pushed forward to $\partial_\varepsilon X$ via the biLipschitz identification $\Psi : Z \to \partial_\varepsilon X$. In the following, we shall therefore not distinguish between $(\partial_\varepsilon X, d_\varepsilon)$ and (Z, d_Z).

We equip the uniformized space X_ε with the doubling measure μ_β, obtained in (10.7). Equivalently, the uniformized measure $\hat{\mu}_\beta$ from (10.5), based on the smeared out measure μ from (10.2), can be used.

For the vertices in X, consider the projections $\pi_1((z,n)) = z$ and $\pi_2((z,n)) = n$. Whenever a nonvertex $x \in X$ belongs to the edge $[v,w] \subset X$, let

$$\pi_2(x) := \min\{\pi_2(v), \pi_2(w)\}. \tag{11.1}$$

Theorem 11.1. *Let $u \in \widetilde{N}^{1,p}(X_\varepsilon, \mu_\beta)$ and $0 < \theta \le 1 - \beta/\varepsilon p$. Then u has a trace $\tilde{u} \in B^\theta_{p,p}(Z)$ given by (11.4) and (11.5) below, with the (semi)norm estimates*

$$\|\tilde{u}\|_{\theta,p} \lesssim \|g_u\|_{L^p(X_\varepsilon,\mu_\beta)} \tag{11.2}$$

and

$$\|\tilde{u}\|_{L^p(Z)} \lesssim |u(v_0)| + \|g_u\|_{L^p(X_\varepsilon,\mu_\beta)} \lesssim \|u\|_{N^{1,p}(X_\varepsilon,\mu_\beta)}. \tag{11.3}$$

If $u \in \operatorname{Lip}(X_\varepsilon)$ then $\tilde{u} = \hat{u}|_Z$, where $\hat{u}$ is the unique Lipschitz extension of u to $\overline{X}_\varepsilon$.

Note that the equivalence classes in $N^{1,p}(X_\varepsilon, \mu_\beta)$ consist of one function each, see Remark 9.5.

Proof. Let $u \in \widetilde{N}^{1,p}(X_\varepsilon, \mu_\beta)$ with an upper gradient $g \in L^p(X_\varepsilon, \mu_\beta)$. For $\zeta \in Z$ and $n = 0, 1, \dots$, let $A_n(\zeta) = A_n \cap B_Z(\zeta, \alpha^{-n})$. We define

$$u_n(\zeta) = \frac{1}{\#A_n(\zeta)} \sum_{z \in A_n(\zeta)} u((z,n)), \tag{11.4}$$

where $\#A_n(\zeta)$ is the cardinality of $A_n(\zeta)$. Note that the construction of A_n, together with the doubling property of Z, shows that $1 \le \#A_n(\zeta) \le K$ for some K independent of n and ζ.

For each fixed $z \in A_n$, the function $\chi_{B_Z(z,\alpha^{-n})}$ is lower semicontinuous and thus ν-measurable. Hence also the linear combinations

$$\sum_{z \in A_n(\zeta)} u((z,n)) = \sum_{z \in A_n} u((z,n))\chi_{B_Z(z,\alpha^{-n})}(\zeta) \quad \text{and} \quad \#A_n(\zeta) = \sum_{z \in A_n} \chi_{B_Z(z,\alpha^{-n})}(\zeta)$$

are ν-measurable, and hence so is u_n. We shall show that for ν-a.e. $\zeta \in Z$, the limit

$$\tilde{u}(\zeta) = \lim_{n\to\infty} u_n(\zeta) \tag{11.5}$$

exists and defines the trace $\tilde{u} : Z \to \mathbf{R}$. To this end, note that $(z,j) \sim (y, j+1)$ whenever $z \in A_j(\zeta)$ and $y \in A_{j+1}(\zeta)$, $j = 0, 1, \dots$, since $\zeta \in B(z, \alpha^{-j}) \cap B(y, \alpha^{-j-1})$. Also,

$$\frac{1}{\#A_j(\zeta)\#A_{j+1}(\zeta)} \le 1.$$

We then have for each j,

$$\begin{aligned} |u_j(\zeta) - u_{j+1}(\zeta)| &\le \sum_{z \in A_j(\zeta)} \sum_{y \in A_{j+1}(\zeta)} |u((z,j)) - u((y,j+1))| \\ &\le \sum_{z \in A_j(\zeta)} \sum_{y \in A_{j+1}(\zeta)} \int_{[(z,j),(y,j+1)]} g\, ds_\varepsilon. \end{aligned} \tag{11.6}$$

On the edge $E = [(z,j),(y,j+1)]$, we have by (10.7) that

$$ds_\varepsilon \simeq e^{-\varepsilon j}\, d\mathcal{L} = \frac{\alpha^{-j}\, d\mu_\beta}{\mu_\beta(E)}. \tag{11.7}$$

If $p > 1$, then (11.7) and Hölder's inequality applied to (11.6) give

$$\begin{aligned} |u_j(\zeta) - u_{j+1}(\zeta)| &\le \alpha^{-j} \sum_{z \in A_j(\zeta)} \sum_{y \in A_{j+1}(\zeta)} \fint_{[(z,j),(y,j+1)]} g\, d\mu_\beta \\ &\le \alpha^{-j} \sum_{z \in A_j(\zeta)} \sum_{E \in \mathcal{E}(z,j)} \biggl(\fint_E g^p\, d\mu_\beta \biggr)^{1/p}, \end{aligned} \tag{11.8}$$

where $\mathcal{E}(z,j)$ consists of all downward-directed edges emanating from the vertex (z,j). Choose $0<\kappa<\theta p$ and insert $\alpha^{-j\kappa/p}\alpha^{j\kappa/p}$ into (11.8). Summing over j, together with another use of Hölder's inequality, this time on the sum, shows that for all $m>n\ge 0$,

$$\begin{aligned} |u_n(\zeta)-u_m(\zeta)| &\le \sum_{j=n}^{m-1} |u_j(\zeta)-u_{j+1}(\zeta)| \qquad (11.9)\\ &\le \sum_{j=n}^{\infty} \alpha^{-j\kappa/p}\alpha^{-j(1-\kappa/p)} \sum_{z\in A_j(\zeta)} \sum_{E\in\mathcal{E}(z,j)} \biggl(\fint_E g^p\, d\mu_\beta \biggr)^{1/p} \\ &\lesssim \alpha^{-n\kappa/p} \biggl(\sum_{j=n}^{\infty} \alpha^{-j(p-\kappa)} \sum_{z\in A_j(\zeta)} \sum_{E\in\mathcal{E}(z,j)} \fint_E g^p\, d\mu_\beta \biggr)^{1/p}, \end{aligned}$$

where we have also used the fact that

$$\biggl(\sum_{j=n}^{\infty} \alpha^{-j\kappa/(p-1)} \biggr)^{1-1/p} \simeq \alpha^{-n\kappa/p},$$

together with $\#A_j(\zeta)\le K$ and $\#\mathcal{E}(z,j)\le K$. For $p=1$ the estimate is simpler and Hölder's inequality is not needed, and the above estimate holds as well. We shall now see that (11.9) tends to zero as $m>n\to\infty$ for ν-a.e. $\zeta\in Z$. Thus, the sequence $\{u_n(\zeta)\}_{n=0}^{\infty}$ is a Cauchy sequence, and has a limit as $n\to\infty$, for ν-a.e. $\zeta\in Z$.

To this end, note that for $E\in\mathcal{E}(z,j)$,

$$\fint_E g^p\, d\mu_\beta \simeq \frac{\alpha^{j\beta/\varepsilon}}{\nu(B_Z(z,\alpha^{-j}))} \int_{X_V} g(x)^p \chi_E(x)\, d\mu_\beta(x), \qquad (11.10)$$

where X_V denotes the union of all vertical edges in X. Also, $z\in A_j(\zeta)$ if and only if $z\in A_j$ and $\zeta\in B_Z(z,\alpha^{-j})$. Integrating over all $\zeta\in Z$ we then obtain from (11.9) by means of Tonelli's theorem that

$$\begin{aligned} &\int_Z |u_m(\zeta)-u_n(\zeta)|^p\, d\nu(\zeta) \\ &\quad\lesssim \alpha^{-n\kappa} \int_Z \sum_{j=n}^{\infty} \sum_{z\in A_j} \frac{\alpha^{-j(p-\beta/\varepsilon-\kappa)}}{\nu(B_Z(z,\alpha^{-j}))} \chi_{B_Z(z,\alpha^{-j})}(\zeta) \\ &\qquad\times \sum_{E\in\mathcal{E}(z,j)} \int_{X_V} g(x)^p \chi_E(x)\, d\mu_\beta(x)\, d\nu(\zeta) \\ &\quad= \alpha^{-n\kappa} \int_{X_V} g(x)^p \sum_{j=n}^{\infty} \alpha^{-j(p-\beta/\varepsilon-\kappa)} \\ &\qquad\times \sum_{z\in A_j} \sum_{E\in\mathcal{E}(z,j)} \int_Z \frac{\chi_{B_Z(z,\alpha^{-j})}(\zeta)}{\nu(B_Z(z,\alpha^{-j}))}\, d\nu(\zeta) \chi_E(x)\, d\mu_\beta(x). \qquad (11.11) \end{aligned}$$

The integral over Z is clearly equal to 1. Moreover, for a.e. $x\in X$,

$$\chi_E(x)\ne 0 \quad \text{only if} \quad x\in E\in\mathcal{E}(z,j) \text{ with } j=\pi_2(x),$$

and so for a.e. $x\in X$ we have

$$\begin{aligned} \sum_{z\in A_j} \sum_{E\in\mathcal{E}(z,j)} \int_Z \frac{\chi_{B_Z(z,\alpha^{-j})}(\zeta)}{\nu(B_Z(z,\alpha^{-j}))}\, d\nu(\zeta) \chi_E(x) &= \sum_{z\in A_j} \sum_{E\in\mathcal{E}(z,j)} \chi_E(x) \\ &= \chi_{\{y\in X_V:\pi_2(y)=j\}}(x). \qquad (11.12) \end{aligned}$$

We therefore conclude that

$$\int_Z |u_m(\zeta) - u_n(\zeta)|^p \, d\nu(\zeta) \lesssim \alpha^{-n\kappa} \int_{\{y \in X_V : \pi_2(y) \ge n\}} g(x)^p \alpha^{-\pi_2(x)(p-\beta/\varepsilon-\kappa)} \, d\mu_\beta(x).$$

Since $p - \beta/\varepsilon - \kappa \ge \theta p - \kappa > 0$, we obtain that

$$\int_Z |u_m(\zeta) - u_n(\zeta)|^p \, d\nu(\zeta) \lesssim \alpha^{-n(p-\beta/\varepsilon)} \int_{X_V} g^p \, d\mu_\beta \to 0, \quad \text{as } m > n \to \infty.$$

Hence, the sequence $\{u_n\}_{n=0}^\infty$ is a Cauchy sequence both in $L^p(Z)$ and ν-a.e. in Z (recall that we have $m > n$ in the above computations). The limit (11.5) therefore exists for ν-a.e. $\zeta \in Z$, and $\tilde{u} \in L^p(Z)$.

This also shows, by letting $n = 0$ and $u_m \to \tilde{u}$, that

$$\biggl(\int_Z |\tilde{u} - u(v_0)|^p \, d\nu \biggr)^{1/p} \lesssim \int_{X_V} g^p \, d\mu_\beta,$$

where $v_0 = (z_0, 0)$. Thus the first inequality

$$\|\tilde{u}\|_{L^p(Z)} \lesssim |u(v_0)| + \|g\|_{L^p(X_\varepsilon, \mu_\beta)}$$

holds in (11.3). For the second inequality, recall the notion of capacity from Definition 9.3. Since $|u(v_0)|^p C_p^{X_\varepsilon}(\{v_0\}) \le \|u\|^p_{N^{1,p}(X_\varepsilon,\mu_\beta)}$ by the definition of $C_p^{X_\varepsilon}(\{v_0\})$, and $C_p^{X_\varepsilon}(\{v_0\}) > 0$, by Remark 9.5, we conclude that the second inequality in (11.3) holds as well.

To estimate $\|\tilde{u}\|_{\theta,p}$, we let $m \to \infty$ in (11.9) to obtain for ν-a.e. $\zeta \in Z$ and any $n \ge 0$,

$$|\tilde{u}(\zeta) - u_n(\zeta)|^p \lesssim \alpha^{-n\kappa} \sum_{j=n}^\infty \alpha^{-j(p-\kappa)} \sum_{z \in A_j(\zeta)} \sum_{E \in \mathcal{E}(z,j)} \fint_E g^p \, d\mu_\beta. \tag{11.13}$$

A similar estimate holds for ν-a.e. $\xi \in Z$. As in Lemma 4.3, we let $l \ge 0$ be the smallest integer such that $\alpha^{-l} \le \tau - 1$. Also let

$$Z_n(\zeta) = \{\xi \in Z : \alpha^{-n-l-1} < d_Z(\xi,\zeta) \le \alpha^{-n-l}\}, \quad n = 1, 2, \dots,$$

and $Z_0(\zeta) = Z \setminus \overline{B_Z(\zeta, \alpha^{-l-1})}$. Note that $\xi \in Z_n(\zeta)$ if and only if $\zeta \in Z_n(\xi)$, in which case also $\zeta \in \tau B_Z(z, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-n})$ and thus $(z,n) \sim (y,n)$ for all $z \in A_n(\zeta)$ and $y \in A_n(\xi)$ with $y \ne z$. Hence for all $\xi \in Z_n(\zeta)$, $n = 1, \dots,$

$$|u_n(\zeta) - u_n(\xi)|^p \lesssim \sum_{z \in A_n(\zeta)} \sum_{\substack{y \in A_n(\xi) \\ y \ne z}} |u((z,n)) - u((y,n))|^p, \tag{11.14}$$

while $u_0(\zeta) = u(v_0) = u_0(\xi)$ for all $\zeta, \xi \in Z$. Hölder's inequality and (11.7) with $E = [(z,n),(y,n)]$ give

$$|u((z,n)) - u((y,n))|^p \le \biggl(\int_E g \, ds_\varepsilon \biggr)^p \lesssim \alpha^{-np} \fint_E g^p \, d\mu_\beta. \tag{11.15}$$

Next, note that

$$|\tilde{u}(\zeta) - \tilde{u}(\xi)|^p \lesssim |\tilde{u}(\zeta) - u_n(\zeta)|^p + |u_n(\zeta) - u_n(\xi)|^p + |u_n(\xi) - \tilde{u}(\xi)|^p, \tag{11.16}$$

and that each of the three terms can be estimated with the aid of (11.13)–(11.15). We shall insert (11.16) into the Besov norm

$$\|\tilde{u}\|^p_{\theta,p} = \int_Z \int_{Z \setminus \{\zeta\}} \frac{|\tilde{u}(\xi) - \tilde{u}(\zeta)|^p}{d_Z(\xi,\zeta)^{\theta p}} \frac{d\nu(\xi)\, d\nu(\zeta)}{\nu(B_Z(\zeta, d_Z(\xi,\zeta)))},$$

and obtain three terms corresponding to the three terms on the right-hand side of (11.16). We next use the comparisons $d_Z(\xi,\zeta)^{\theta p} \simeq \alpha^{-n\theta p}$ and

$$\nu(B_Z(\zeta, d_Z(\xi,\zeta))) \simeq \nu(B_Z(\xi, d_Z(\xi,\zeta))) \simeq \nu(B_Z(\zeta,\alpha^{-n})) \simeq \nu(B_Z(\xi,\alpha^{-n}))$$

whenever $\xi \in Z_n(\zeta)$ (or equivalently, $\zeta \in Z_n(\xi)$). We then get $\|\tilde{u}\|^p_{\theta,p} \lesssim I_0 + II_0 + III_0$, where

$$\begin{aligned} I_0 &:= \int_Z \sum_{n=0}^\infty \int_{Z_n(\zeta)} \frac{|\tilde{u}(\zeta)-u_n(\zeta)|^p}{\alpha^{-n\theta p}} \frac{d\nu(\xi)\,d\nu(\zeta)}{\nu(B_Z(\zeta,\alpha^{-n}))}, \\ II_0 &:= \int_Z \sum_{n=0}^\infty \int_{Z_n(\zeta)} \frac{|u_n(\zeta)-u_n(\xi)|^p}{\alpha^{-n\theta p}} \frac{d\nu(\xi)\,d\nu(\zeta)}{\nu(B_Z(\zeta,\alpha^{-n}))}, \\ III_0 &:= \int_Z \sum_{n=0}^\infty \int_{Z_n(\xi)} \frac{|\tilde{u}(\xi)-u_n(\xi)|^p}{\alpha^{-n\theta p}} \frac{d\nu(\zeta)\,d\nu(\xi)}{\nu(B_Z(\xi,\alpha^{-n}))}. \end{aligned}$$

Observe that III_0 is the same as I_0 once the roles of ζ and ξ are switched, and so it suffices to find estimates for I_0 and II_0. Using (11.13)–(11.15), we find that

$$\begin{aligned} I_0 &\lesssim \int_Z \sum_{n=0}^\infty \frac{\alpha^{-n(\kappa-\theta p)}}{\nu(B_Z(\zeta,\alpha^{-n}))} \int_{Z_n(\zeta)} \sum_{j=n}^\infty \alpha^{-j(p-\kappa)} \\ &\qquad \times \sum_{z\in A_j(\zeta)} \sum_{E\in\mathcal{E}(z,j)} \fint_E g(x)^p\,d\mu_\beta(x)\,d\nu(\xi)\,d\nu(\zeta) =: I, \\ II_0 &\lesssim \int_Z \sum_{n=1}^\infty \frac{\alpha^{-n(p-\theta p)}}{\nu(B_Z(\zeta,\alpha^{-n}))} \\ &\qquad \times \int_{Z_n(\zeta)} \sum_{z\in A_n(\zeta)} \sum_{\substack{y\in A_n(\xi)\\ y\neq z}} \fint_{[(z,n),(y,n)]} g(x)^p\,d\mu_\beta(x)\,d\nu(\xi)\,d\nu(\zeta) =: II. \end{aligned}$$

To estimate I, we use (11.10) and that $z\in A_j(\zeta)$ if and only if $z \in A_j$ and $\zeta \in B_Z(z,\alpha^{-j})$. Now an argument using Tonelli's theorem as in the verification of (11.11) yields that

$$\begin{aligned} I &\simeq \sum_{n=0}^\infty \alpha^{-n(\kappa-\theta p)} \sum_{j=n}^\infty \alpha^{-j(p-\beta/\varepsilon-\kappa)} \sum_{z\in A_j} \sum_{E\in\mathcal{E}(z,j)} \\ &\quad \times \int_Z \frac{\chi_{B_Z(z,\alpha^{-j})}(\zeta)}{\nu(B_Z(z,\alpha^{-j}))} \int_{Z_n(\zeta)} \frac{d\nu(\xi)}{\nu(B_Z(\zeta,\alpha^{-n}))}\,d\nu(\zeta) \int_{X_V} g(x)^p \chi_E(x)\,d\mu_\beta(x). \end{aligned}$$

Since $Z_n(\zeta) \subset B_Z(\zeta,\alpha^{-n})$, the integral over $Z_n(\zeta)$ followed by the integral over Z is clearly at most 1. Another use of Tonelli's theorem therefore implies that

$$I \lesssim \int_{X_V} g(x)^p \sum_{n=0}^\infty \alpha^{-n(\kappa-\theta p)} \sum_{j=n}^\infty \alpha^{-j(p-\beta/\varepsilon-\kappa)} \sum_{z\in A_j} \sum_{E\in\mathcal{E}(z,j)} \chi_E(x)\,d\mu_\beta(x).$$

The last three sums are simplified using the last identity in (11.12) and we obtain

$$\begin{aligned} I &\lesssim \int_{X_V} g(x)^p \alpha^{-\pi_2(x)(p-\beta/\varepsilon-\kappa)} \sum_{n=0}^{\pi_2(x)} \alpha^{-n(\kappa-\theta p)}\,d\mu_\beta(x) \\ &\simeq \int_{X_V} g(x)^p \alpha^{-\pi_2(x)(p-\beta/\varepsilon-\theta p)}\,d\mu_\beta(x), \end{aligned}$$

because of the choices $\kappa < \theta p$ and $\alpha > 1$. Since $p - \beta/\varepsilon - \theta p \ge 0$, this yields

$$I \lesssim \int_{X_V} g^p \, d\mu_\beta.$$

To estimate II, we proceed similarly. As in (11.10), we have that when $(z,n) \sim (y,n)$,

$$\fint_{[(z,n),(y,n)]} g^p \, d\mu_\beta \simeq \frac{\alpha^{n\beta/\varepsilon}}{\nu(B_Z(z,\alpha^{-n}))} \int_{X_H} g(x)^p \chi_{[(z,n),(y,n)]}(x) \, d\mu_\beta(x),$$

where X_H denotes the union of all horizontal edges in X.

Moreover, $z \in A_n(\zeta)$ and $y \in A_n(\xi)$ if and only if $z, y \in A_n$, $\zeta \in B_Z(z,\alpha^{-n})$ and $\xi \in B_Z(y,\alpha^{-n})$. Tonelli's theorem then yields that

$$\begin{aligned} II \simeq \sum_{n=1}^{\infty} \alpha^{-n(p-\beta/\varepsilon-\theta p)} \sum_{\substack{z,y\in A_n \\ (z,n)\sim(y,n)}} \int_Z \frac{\chi_{B_Z(z,\alpha^{-n})}(\zeta)}{\nu(B_Z(z,\alpha^{-n}))} \int_{Z_n(\zeta)} \frac{\chi_{B_Z(y,\alpha^{-n})}(\xi)}{\nu(B_Z(\zeta,\alpha^{-n}))} \, d\nu(\xi) \, d\nu(\zeta) \\ \times \int_{X_H} g(x)^p \chi_{[(z,n),(y,n)]}(x) \, d\mu_\beta(x). \end{aligned}$$

Since $Z_n(\zeta) \subset B_Z(\zeta,\alpha^{-n})$, the integral over $Z_n(\zeta)$ followed by the integral over Z is at most 1, and another use of Tonelli's theorem shows that

$$II \lesssim \int_{X_H} g(x)^p \sum_{n=1}^{\infty} \alpha^{-n(p-\beta/\varepsilon-\theta p)} \sum_{\substack{z,y\in A_n \\ (z,n)\sim(y,n)}} \chi_{[(z,n),(y,n)]}(x) \, d\mu_\beta(x).$$

Moreover, $\chi_{[(z,n),(y,n)]}(x) \neq 0$ if only if $x \in [(z,n),(y,n)]$, in which case also $n = \pi_2(x)$. We therefore conclude that

$$II \lesssim \int_{X_H} g(x)^p \alpha^{-\pi_2(x)(p-\beta/\varepsilon-\theta p)} \, d\mu_\beta(x) \le \int_{X_H} g^p \, d\mu_\beta,$$

because $p - \beta/\varepsilon - \theta p \ge 0$. Combining the estimates for I and II gives the desired bound (11.2).

The fact that $\tilde{u} = \hat{u}|_Z$ when u is Lipschitz continuous on X_ε follows from the definition of $\tilde{u}$ and the fact that u has a unique Lipschitz extension to $\overline{X}_\varepsilon$. □

Recall the notion of capacity from Definition 9.3. The following proposition shows that the boundary measure ν on $Z = \partial_\varepsilon X$ is absolutely continuous with respect to the $C_p^{\overline{X}_\varepsilon}$-capacity. Note that points in X have positive $C_p^{\overline{X}_\varepsilon}$-capacity, but that it is possible to have nonempty subsets of $\partial_\varepsilon X = Z$ with zero capacity.

Proposition 11.2. *Let $E \subset \partial_\varepsilon X$. If $p > \beta/\varepsilon$ and $C_p^{\overline{X}_\varepsilon}(E) = 0$, then $\nu(E) = 0$.*

Proof. Since $C_p^{\overline{X}_\varepsilon}$ is an outer capacity by Theorem 9.6, there are open sets $G_j \supset E$ such that $C_p^{\overline{X}_\varepsilon}(G_j) < 1/j$. Then

$$E' := \bigcap_{j=1}^{\infty} G_j \supset E$$

is a Borel set with zero capacity. Let $K \subset E'$ be compact. Because μ_β is doubling and supports a p-Poincaré inequality on $\overline{X}_\varepsilon$, it follows from Kallunki–Shanmugalingam [31, Theorem 1.1] (or [2, Theorem 6.7 (xi)]) that there are $u_k \in \operatorname{Lip}(\overline{X}_\varepsilon)$

such that $u_k = 1$ on K and $\|u_k\|_{N^{1,p}(\overline{X}_\varepsilon,\mu_\beta)} < 1/k$, $k = 1, 2, \dots$. By the last part of Theorem 11.1 with $\theta = 1 - \beta/\varepsilon p > 0$,

$$\nu(K)^{1/p} \le \lim_{k\to\infty} \|u_k\|_{L^p(Z)} \lesssim \lim_{k\to\infty} \|u_k\|_{N^{1,p}(X_\varepsilon,\mu_\beta)} = 0,$$

i.e., $\nu(K) = 0$. Since E' is a Borel set and ν is a Borel regular measure, we conclude that

$$\nu(E) \le \nu(E') = \sup_{K\subset E' \text{ compact}} \nu(K) = 0. \qquad \square$$

The following result is a refinement of Theorem 11.1. In the case of regular trees, it provides a more precise trace result than Proposition 6.1 in Björn–Björn–Gill–Shanmugalingam [3]. Recall that by Theorem 7.1, every rooted tree X can be seen as a hyperbolic filling of its uniformized boundary $\partial_\varepsilon X$.

Theorem 11.3. *Let $u \in \widetilde{N}^{1,p}(X_\varepsilon, \mu_\beta)$ and $0 < \theta \le 1 - \beta/\varepsilon p$. Then u has an extension $\hat{u} \in \widetilde{N}^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$. Furthermore, the restriction $\tilde{u} := \hat{u}|_Z$ agrees with the trace of u defined earlier ν-a.e. in Z, and belongs to $B^\theta_{p,p}(Z)$ with the (semi)norm estimates*

$$\|\tilde{u}\|_{\theta,p} \lesssim \|g_u\|_{L^p(X_\varepsilon,\mu_\beta)}$$

and

$$\|\tilde{u}\|_{L^p(Z)} \lesssim |u(x_0)| + \|g_u\|_{L^p(X_\varepsilon,\mu_\beta)} \lesssim \|u\|_{N^{1,p}(X_\varepsilon,\mu_\beta)}.$$

Moreover, for $C_p^{\overline{X}_\varepsilon}$-q.e. (and thus ν-a.e.) $\zeta \in Z$ we have that

$$\lim_{r\to 0+} \fint_{X_\varepsilon \cap B_\varepsilon(\zeta,r)} |u - \tilde{u}(\zeta)|^p \, d\mu_\beta = 0. \tag{11.17}$$

Note that the extension $\hat{u}$ is not unique, but it is unique up to sets of capacity zero and thus ν-a.e. (by Proposition 11.2), since if $\hat{u}_1$ and $\hat{u}_2$ are two extensions, then they are equal μ_β-a.e., and thus $C_p^{\overline{X}_\varepsilon}$-q.e. We may therefore take the restriction of any such extension. The key observation that makes the above statement true is that the representatives in Newtonian spaces are equal q.e., not just a.e. as for standard Sobolev spaces.

The last claim of the above theorem tells us that the trace of a function in $N^{1,p}(X_\varepsilon, \mu_\beta)$, as constructed in Theorem 11.1, agrees with other notions of traces in the current literature, see e.g. Malý [39].

Proof. By Theorems 5.1 and 10.3, X_ε is a uniform domain in $\overline{X}_\varepsilon$ and μ_β is doubling and supports a p-Poincaré inequality on $\overline{X}_\varepsilon$. Thus by Proposition 5.9 in Björn–Shanmugalingam [7], X_ε is an extension domain, and thus u has an extension to $\overline{X}_\varepsilon$, denoted $\hat{u} \in N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$.

By Shanmugalingam [45, Theorem 4.1 and Corollary 3.9], there is a sequence $u_j \in \operatorname{Lip}(\overline{X}_\varepsilon)$ such that $\|u_j - \hat{u}\|_{N^{1,p}(\overline{X}_\varepsilon,\mu_\beta)} \to 0$ and $u_j(x) \to \hat{u}(x)$ for $C_p^{\overline{X}_\varepsilon}$-q.e. $x \in \overline{X}_\varepsilon$, as $j \to \infty$. Let $\tilde{u} := \hat{u}|_Z$ and $\tilde{u}_j := u_j|_Z$. By Proposition 11.2, $\tilde{u}_j(\zeta) \to \tilde{u}(\zeta)$ for ν-a.e. $\zeta \in Z$. Moreover, by Theorem 11.1, $\{\tilde{u}_j\}_{j=1}^\infty$ is a Cauchy sequence in the norm $\|\cdot\|_{B^\theta_{p,p}(Z)}$. By Remark 9.8, $B^\theta_{p,p}(Z)$ is complete with respect to $\|\cdot\|_{B^\theta_{p,p}(Z)}$, and hence we see that $\|\tilde{u}_j - \tilde{u}\|_{B^\theta_{p,p}(Z)} \to 0$ as $j \to \infty$, with the (semi)norm estimates from Theorem 11.1 preserved for $\tilde{u}$.

By [2, Theorem 5.62] or [27, Theorem 9.2.8] (for $p > 1$) and Kinnunen–Korte–Shanmugalingam–Tuominen [32, Theorem 4.1 and Remark 4.7] (for $p = 1$, see below) we know that $C_p^{\overline{X}_\varepsilon}$-q.e. point in $\overline{X}_\varepsilon$ is an $L^p(\mu_\beta)$-Lebesgue point of $\hat{u}$; hence (11.17) holds because $\mu_\beta(\partial_\varepsilon X) = 0$.

In [32, p. 404] it is assumed that $\mu(X) = \infty$, which is used in their proof of the boxing inequality. In Mäkäläinen [38], the boxing inequality is proved also when $\mu(X) < \infty$, and thus the Lebesgue point result in [32] holds also here where $\mu_\beta(\overline{X}_\varepsilon) < \infty$. $\square$

12. Extension from Z to its hyperbolic filling X

Recall the standing assumptions from Sections 10 and 11.

Theorem 11.3 related the Newtonian space $N^{1,p}(X_\varepsilon, \mu_\beta)$ to a certain range of Besov spaces of functions on $Z \equiv \partial_\varepsilon X$. The principal goal of this section is to find a counterpart of this theorem in the opposite direction. This is the purpose of the theorem below.

Theorem 12.1. *For $f \in B^\theta_{p,p}(Z)$, consider the extension*

$$Ef((z,n)) := \fint_{B_Z(z,\alpha^{-n})} f\,d\nu, \quad \text{if } (z,n) \in V \subset X,$$

extended piecewise linearly (with respect to d_ε) to each edge in X_ε, and then to the boundary $\partial_\varepsilon X$ by letting

$$Ef(\zeta) = \limsup_{r\to 0^+} \fint_{B_\varepsilon(\zeta,r)} Ef\,d\mu_\beta, \quad \zeta \in \partial_\varepsilon X, \tag{12.1}$$

so that $Ef : \overline{X}_\varepsilon \to [-\infty,\infty]$.

If $\theta \ge 1 - \beta/p\varepsilon$, then $Ef \in N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ with

$$\int_{\overline{X}_\varepsilon} g^p_{Ef}\,d\mu_\beta \lesssim \|f\|^p_{\theta,p} \quad \text{and} \quad \int_{\overline{X}_\varepsilon} |Ef|^p\,d\mu_\beta \lesssim \int_Z |f|^p\,d\nu. \tag{12.2}$$

Moreover, if $\zeta \in Z$ is an $L^q(\nu)$-Lebesgue point of f for some $q \ge 1$ then ζ is an $L^q(\mu_\beta)$-Lebesgue point of Ef, and

$$Ef(z) = f(z).$$

Let ζ be an $L^1(\nu)$-Lebesgue point of f. Then for any choice of $z_n \in A_n$ with $d_Z(z_n,\zeta) < \alpha^{-n}$ for $n = 1, 2, \dots$, we have,

$$\lim_{n\to\infty} Ef((z_n,n)) = f(\zeta).$$

Note that E is a linear operator.

Proof. If $v = (z,n) \sim (y,m) = w$, then $|m-n| \le 1$ and so, by the choice of $\varepsilon = \log\alpha$,

$$d_\varepsilon(v,w) \simeq e^{-\varepsilon n} = \alpha^{-n}.$$

The function given for $x \in [v,w]$ by

$$g_{[v,w]}(x) := \frac{|Ef(v) - Ef(w)|}{d_\varepsilon(v,w)}$$

is an upper gradient of Ef on $[v,w]$ with respect to the uniformized metric d_ε. Note that $g_{[v,w]}$ is a constant function. Because of $|m-n| \le 1$ and $(z,n) \sim (y,m)$, we have for all $\eta \in B_Z(z,\alpha^{-n})$ that

$$B_Z(y,\alpha^{-m}) \subset 4\tau B_Z(\eta,\alpha^{1-n}).$$

Hence,

$$\begin{aligned} g_{[v,w]} &\simeq \alpha^n \biggl| \fint_{B_Z(z,\alpha^{-n})} f(\zeta)\,d\nu(\zeta) - \fint_{B_Z(y,\alpha^{-m})} f(\eta)\,d\nu(\eta) \biggr| \\ &\lesssim \alpha^n \fint_{B_Z(z,\alpha^{-n})} \fint_{B_Z(y,\alpha^{-m})} |f(\zeta) - f(\eta)|\,d\nu(\eta)\,d\nu(\zeta) \\ &\lesssim \alpha^n \fint_{B_Z(z,4\tau\alpha^{1-n})} \fint_{B_Z(\eta,4\tau\alpha^{1-n})} |f(\zeta) - f(\eta)|\,d\nu(\eta)\,d\nu(\zeta). \end{aligned}$$

Now by Hölder's inequality, we see that

$$g_{[v,w]}^p \lesssim \alpha^{np(1-\theta)} \fint_{B_Z(z,4\tau\alpha^{1-n})} \fint_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta)-f(\eta)|^p}{\alpha^{-np\theta}}\, d\nu(\eta)\, d\nu(\zeta).$$

Therefore, letting $g : X \to \mathbf{R}$ be given by $g = g_{[v,w]}$ on each edge $[v, w]$, and noting from (10.6)–(10.7) that

$$d\mu_\beta \simeq \alpha^{-\beta n/\varepsilon} \nu(B_Z(z,\alpha^{-n}))\, d\mathcal{L} \quad \text{on } [v,w] \text{ with } n = \pi_2(v),$$

we obtain

$$\int_{[v,w]} g^p\, d\mu_\beta \lesssim \alpha^{n(p(1-\theta)-\beta/\varepsilon)} \int_{B_Z(z,4\tau\alpha^{1-n})} \fint_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta)-f(\eta)|^p}{\alpha^{-np\theta}}\, d\nu(\eta)\, d\nu(\zeta),$$

where $z = \pi_1(v) \in A_n \subset Z$. For each nonnegative integer n set

$$X(n) := \{x \in X : n \le \pi_2(x) < n+1\},$$

where $\pi_2(x)$ is as in (11.1). By Proposition 4.5, each vertex in X has degree at most K and thus we get integrating over $X(n)$ that

$$\begin{aligned}
\int_{X(n)} g^p\, d\mu_\beta &\le \sum_{z\in A_n} \sum_{V\ni w\sim(z,n)} \int_{[(z,n),w]} g^p\, d\mu_\beta \\
&\lesssim \alpha^{n(p(1-\theta)-\beta/\varepsilon)} \sum_{z\in A_n} \int_{B_Z(z,4\tau\alpha^{1-n})} \fint_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta)-f(\eta)|^p}{\alpha^{-np\theta}}\, d\nu(\eta)\, d\nu(\zeta) \\
&\lesssim \alpha^{n(p(1-\theta)-\beta/\varepsilon)} \int_Z \fint_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta)-f(\eta)|^p}{\alpha^{-np\theta}}\, d\nu(\eta)\, d\nu(\zeta).
\end{aligned}$$

In the last line we used the fact that the balls $B_Z(z, 4\tau\alpha^{1-n})$, $z \in A_n$, have a bounded overlap in Z because of the doubling property of Z. As $X = \bigcup_{n=0}^\infty X(n)$ with $X(n) \cap X(m) = \varnothing$ if $m \ne n$, it follows that

$$\int_{X_\varepsilon} g^p\, d\mu_\beta \lesssim \sum_{n=0}^\infty \alpha^{n(p(1-\theta)-\beta/\varepsilon)} \int_Z \fint_{B_Z(\eta,4\tau\alpha^{1-n})} \frac{|f(\zeta)-f(\eta)|^p}{\alpha^{-np\theta}}\, d\nu(\eta)\, d\nu(\zeta).$$

If $\theta \ge 1 - \beta/p\varepsilon$, it then follows from Lemma 9.9 that

$$\int_{X_\varepsilon} g^p\, d\mu_\beta \lesssim \|f\|_{\theta,p}^p < \infty. \tag{12.3}$$

As μ_β supports a 1-Poincaré inequality on X_ε, by Theorem 10.3, and X_ε is bounded, it follows that $Ef \in N^{1,p}(X_\varepsilon, \mu_\beta)$.

As in the proof of Theorem 11.3, we have an extension $u \in N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ of Ef, and $C_p^{\overline{X}_\varepsilon}$-q.e. point in $\overline{X}_\varepsilon$ is a Lebesgue point of u. As $\mu_\beta(\partial_\varepsilon X) = 0$, we see that $u(x) = Ef(x)$ for $C_p^{\overline{X}_\varepsilon}$-q.e. $x \in \partial_\varepsilon X$, where $Ef|_{\partial_\varepsilon X}$ is given by (12.1). Hence Ef is also in $N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$. Since X_ε is open in $\overline{X}_\varepsilon$, we see that the minimal p-weak upper gradients of Ef with respect to $\overline{X}_\varepsilon$ and X_ε coincide almost everywhere in X_ε, and thus the first inequality in (12.2) follows from (12.3).

To control the L^p-norm of Ef as stated in the theorem, note that for $v = (z, n) \sim w = (y, m)$,

$$\begin{aligned}
\int_{[v,w]} |Ef|^p\, d\mu_\beta &\le \mu_\beta([v,w])[|Ef(v)|^p + |Ef(w)|^p] \\
&\le \mu_\beta([v,w]) \biggl(\fint_{B_Z(z,\alpha^{-n})} |f|^p\, d\nu + \fint_{B_Z(y,\alpha^{-m})} |f|^p\, d\nu \biggr) \\
&\lesssim \mu_\beta([v,w]) \fint_{B_Z(z,4\tau\alpha^{1-n})} |f|^p\, d\nu.
\end{aligned}$$

Therefore, for each nonnegative integer n, with $X(n)$ as above, we have that

$$\begin{aligned}\int_{X(n)} |Ef|^p \, d\mu_\beta &\lesssim \alpha^{-n\beta/\varepsilon} \sum_{z\in A_n} \nu(B_Z(z,\alpha^{-n})) \fint_{B_Z(z,4\tau\alpha^{1-n})} |f|^p \, d\nu \\ &\lesssim \alpha^{-n\beta/\varepsilon} \int_Z |f|^p \, d\nu.\end{aligned}$$

It follows that

$$\int_{X_\varepsilon} |Ef|^p \, d\mu_\beta \lesssim \|f\|^p_{L^p(Z)} \sum_{n=0}^{\infty} \alpha^{-n\beta/\varepsilon} \lesssim \|f\|^p_{L^p(Z)}$$

as desired.

Assume that $q \ge 1$ and that $\zeta \in Z$ is an $L^q(\nu)$-Lebesgue point of f. Let $N \ge 0$ be a fixed but arbitrary integer and consider all $x \in X$ such that $d_\varepsilon(x,\zeta) < r := \alpha^{-N}/\varepsilon$. If x belongs to an edge $[v,w]$, then at least one of the vertices also belongs to $B_\varepsilon(\zeta,r)$, say $w=(y,m)$, and hence

$$\alpha^{-m} = e^{-\varepsilon m} = \varepsilon d_\varepsilon(w) \le \varepsilon d_\varepsilon(w,\zeta) < \varepsilon r = \alpha^{-N},$$

from which it follows that $m \ge N+1$ and thus $n \ge N$, where $v=(z,n)$. In particular, $d_\varepsilon(z,v) = d_\varepsilon(v) = e^{-\varepsilon n}/\varepsilon \le r$ and $d_\varepsilon(v,x) \le e^{-\varepsilon N} = \varepsilon r$. Proposition 4.4 then yields

$$d_Z(z,\zeta) \le 2\tau\alpha d_\varepsilon(z,\zeta) \le 2\tau\alpha(d_\varepsilon(z,v) + d_\varepsilon(v,x) + d_\varepsilon(x,\zeta)) < 2\tau\alpha(2+\varepsilon)r,$$

and similarly $d_Z(y,\zeta) < 2\tau\alpha(2+\varepsilon)r$. Since $Ef(x)$ is a convex combination of $Ef(v)$ and $Ef(w)$, we have

$$\int_{[v,w]} |Ef - f(\zeta)|^q \, d\mu_\beta \le (|Ef(v) - f(\zeta)|^q + |Ef(w) - f(\zeta)|^q)\mu_\beta([v,w]),$$

where by the definition of Ef and Hölder's inequality,

$$|Ef(v) - f(\zeta)|^q = \biggl| \fint_{B_Z(z,\alpha^{-n})} f \, d\nu - f(\zeta) \biggr|^q \le \fint_{B_Z(z,\alpha^{-n})} |f - f(\zeta)|^q \, d\nu.$$

Noting that

$$\mu_\beta([v,w]) \simeq \rho_\beta(v)\hat{\mu}(v) = e^{-\beta n}\nu(B_Z(z,\alpha^{-n}))$$

and that every vertex belongs to at most a bounded number of edges (by Proposition 4.5), we conclude that

$$\int_{B_\varepsilon(\zeta,r)} |Ef - f(\zeta)|^q \, d\mu_\beta \lesssim \sum_{n\ge N} e^{-\beta n} \sum_{z\in A_n \cap B_Z(\zeta,2\tau\alpha(2+\varepsilon)r)} \int_{B_Z(z,\alpha^{-n})} |f - f(\zeta)|^q \, d\nu.$$

Since for each $n \ge N$ we have that $\alpha^{-n} \le \alpha^{-N} = \varepsilon r$ and the balls $B_Z(z,\alpha^{-n})$, $z \in A_n$, have bounded overlap in Z, we obtain

$$\begin{aligned}\int_{B_\varepsilon(\zeta,r)} |Ef - f(\zeta)|^q \, d\mu_\beta &\lesssim \sum_{n\ge N} e^{-\beta n} \int_{B_Z(\zeta,4\tau\alpha(1+\varepsilon)r)} |f - f(\zeta)|^q \, d\nu \\ &\simeq e^{-\beta N} \int_{B_Z(\zeta,4\tau\alpha(1+\varepsilon)r)} |f - f(\zeta)|^q \, d\nu,\end{aligned}$$

where $e^{-\beta N} = (\alpha^{-N})^{\beta/\varepsilon} = (\varepsilon r)^{\beta/\varepsilon}$. Dividing by $\mu_\beta(B_\varepsilon(\zeta,r)) \simeq (\varepsilon r)^{\beta/\varepsilon}\nu(B_Z(\zeta,\varepsilon r))$ (because of Corollary 10.5) and letting $N \to \infty$ shows that ζ is an $L^q(\mu_\beta)$-Lebesgue point of Ef. That $Ef(\zeta) = f(\zeta)$ now follows directly from the definition of $Ef(\zeta)$ in (12.1) and by considering the case $q = p$ in the above discussion (recall that f is necessarily in $L^p(\nu)$ and so ν-a.e. point in Z is an $L^p(\nu)$-Lebesgue point of f).

Moreover, with z_n as in the final claim of the theorem for $n = 1, 2, \dots$, the doubling property of ν and the fact that $B_Z(z_n, \alpha^{-n}) \subset B_Z(\zeta, 2\alpha^{-n})$ yield

$$
\begin{aligned}
|Ef((z_n, n)) - f(\zeta)| &= \left| \fint_{B_Z(z_n,\alpha^{-n})} [f(y) - f(\zeta)]\, d\nu(y) \right| \\
&\lesssim \fint_{B_Z(\zeta,2\alpha^{-n})} |f(y) - f(\zeta)|\, d\nu(y) \to 0, \quad \text{as } n \to \infty. \qquad \square
\end{aligned}
$$

13. Properties of Besov functions on Z

Recall the standing assumptions from Sections 10 and 11. Since $\varepsilon = \log \alpha$, from Proposition 4.4 we know that $\partial_\varepsilon X$ can be identified with Z in a biLipschitz fashion. In this section we also fix $0 < \theta < 1$ and let $\beta = \varepsilon p(1-\theta)$.

In this section we will only consider the Besov spaces on Z that arise as traces of Newtonian functions on $\overline{X}_\varepsilon$ as in Theorem 1.1. This makes it possible to derive various regularity properties for $B^\theta_{p,p}(Z)$ from from the theory of Newtonian spaces.

Proposition 13.1. *If $0 < \theta < 1$ then Lipschitz functions are dense in $B^\theta_{p,p}(Z)$.*

Proof. Equip the uniformized hyperbolic filling X_ε with the measure μ_β, where $\beta = \varepsilon p(1-\theta)$. Theorems 11.3 and 12.1 tell us that $B^\theta_{p,p}(Z)$ is the trace space of $N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$, with comparable norms.

Since μ_β is doubling and supports a 1-Poincaré inequality on $\overline{X}_\varepsilon$, it follows from Shanmugalingam [45, Theorem 4.1 and Corollary 3.9] that Lipschitz functions are dense in $N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$. Their restrictions to Z are then dense in $B^\theta_{p,p}(Z)$. $\square$

Proposition 13.2. *Let $E \subset Z$. Then $\operatorname{Cap}_{B^\theta_{p,p}(Z)}(E) \simeq C_p^{\overline{X}_\varepsilon}(E)$.*

Proof. Let $u \in B^\theta_{p,p}(Z)$ be admissible in the definition of $\operatorname{Cap}_{B^\theta_{p,p}(Z)}(E)$, i.e. $u \ge 1$ ν-a.e. in an open neighborhood $G \subset Z$ of E. By truncation and redefinition on a set of ν-measure zero, we may assume that $u \equiv 1$ in G. Let $Eu \in N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ be its extension as guaranteed by Theorem 12.1. As all points in G are Lebesgue points for u, we see that $Eu \equiv u \equiv 1$ in G. Hence Eu is admissible in computing $C_p^{\overline{X}_\varepsilon}(E)$, and so by Theorem 12.1,

$$
C_p^{\overline{X}_\varepsilon}(E) \le \|Eu\|^p_{N^{1,p}(\overline{X}_\varepsilon,\mu_\beta)} \lesssim \|u\|^p_{B^\theta_{p,p}(Z)}.
$$

Taking infimum over all u admissible in the definition of $\operatorname{Cap}_{B^\theta_{p,p}(Z)}(E)$ proves one inequality in the statement of the lemma.

Conversely, since $C_p^{\overline{X}_\varepsilon}$ is an outer capacity by Theorems 9.6 and 10.3, for each $\eta > 0$ we can find an open set $U \subset \overline{X}_\varepsilon$ with $E \subset U$ and a function $u \in \widetilde{N}^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ such that $u \ge 1$ on U and

$$
\|u\|_{N^{1,p}(\overline{X}_\varepsilon,\mu_\beta)} < C_p^{\overline{X}_\varepsilon}(E) + \eta.
$$

By Theorem 11.3, the function $f = u|_Z \in B^\theta_{p,p}(Z)$ with

$$
\|f\|^p_{L^p(Z)} + \|f\|^p_{\theta,p} \lesssim \|u\|^p_{N^{1,p}(\overline{X}_\varepsilon,\mu_\beta)} < C_p^{\overline{X}_\varepsilon}(E) + \eta.
$$

As $f \ge 1$ on the relatively open set $G := U \cap Z$, letting $\eta \to 0$ concludes the proof. $\square$

We have now shown that for subsets of Z the two capacities are comparable. Next we turn our attention to the matter of continuity properties of Besov functions. The following result shows that functions in $B^\theta_{p,p}(Z)$ have representatives that are *quasicontinuous* with respect to the Besov capacity, i.e. such that for each $\eta > 0$ there is an open set $G \subset Z$ with $\operatorname{Cap}_{B^\theta_{p,p}(Z)}(G) < \eta$ such that $f|_{Z\setminus G}$ is continuous.

Proposition 13.3. *Let $f_0 \in B^\theta_{p,p}(Z)$. Then there is a $\operatorname{Cap}_{B^\theta_{p,p}(Z)}$-quasicontinuous function $f \in B^\theta_{p,p}(Z)$ such that $f = f_0$ ν-a.e. in Z.*

Proof. Given such a function f_0, let $Ef_0 \in N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ be its extension given by Theorem 12.1. Then $f := Ef_0|_Z = f_0$ ν-a.e. By Theorems 9.6 and 10.3, Ef_0 is $C_p^{\overline{X}_\varepsilon}$-quasicontinuous, i.e. for each $\eta > 0$ there is an open set $U \subset \overline{X}_\varepsilon$ with $C_p^{\overline{X}_\varepsilon}(U) < \eta$ such that $Ef_0|_{\overline{X}_\varepsilon \setminus U}$ is continuous. By choosing $G = U \cap Z$, we get that $f|_{Z\setminus G}$ is continuous. Moreover, by Proposition 13.2, we see that

$$\operatorname{Cap}_{B^\theta_{p,p}(Z)}(G) \simeq C_p^{\overline{X}_\varepsilon}(G) \le C_p^{\overline{X}_\varepsilon}(U) < \eta,$$

which completes the proof. □

The following result shows that Besov functions have Lebesgue points q.e., provided that the measure on Z satisfies a reverse-doubling property.

Proposition 13.4. *Assume that ν satisfies (10.13) with $s_\nu > 0$ and that there is some $\eta > 0$ such that*

$$\frac{\nu(B_Z(\zeta, r'))}{\nu(B_Z(\zeta, r))} \lesssim \left(\frac{r'}{r}\right)^\eta \tag{13.1}$$

for all $\zeta \in Z$ and all $0 < r' \le r \le 2 \operatorname{diam} Z$. Let $1 \le q \le s_\nu p/(s_\nu - p\theta)$ and $\tilde{u} \in B^\theta_{p,p}(Z)$. Then there is a function $u \in L^q(Z)$ such that $u = \tilde{u}$ ν-a.e. and

$$\lim_{r\to 0^+} \fint_{B_\varepsilon(\zeta,r)\cap Z} |u - u(\zeta)|^q \, d\nu = 0 \quad \textit{for } \operatorname{Cap}_{B^\theta_{p,p}(Z)}\textit{-q.e. } \zeta \in Z.$$

Since ν is doubling, condition (13.1) is equivalent to Z being uniformly perfect, see Martín–Ortiz [41, Lemma 7]. See [47] for a weaker Lebesgue point result when Z is not necessarily uniformly perfect. Embeddings of Besov spaces into L^q spaces were also obtained in Malý [39, Corollary 3.18 (i)] via embeddings into Hajłasz–Sobolev spaces. Proposition 13.4 will follow from our trace and extension results and the following two-weighted Poincaré type inequality, which is a special case of Björn–Kałamajska [6, Theorem 3.1].

Proposition 13.5. *Let ν and μ_β be doubling measures on $Z = \partial_\varepsilon X$ and $\overline{X}_\varepsilon$, respectively. Assume moreover that μ_β supports a p-Poincaré inequality on $\overline{X}_\varepsilon$ with dilation λ and that ν satisfies the reverse-doubling condition (13.1). Let $1 \le p < q < \infty$ and $u \in N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ be such that ν-a.e. $z \in Z$ is a μ_β-Lebesgue point of u. Then for all balls $B = B_\varepsilon(\zeta, r)$ with $\zeta \in Z$,*

$$\left(\int_{B\cap Z} |u - u_{B,\mu_\beta}|^q \, d\nu\right)^{1/q} \lesssim \Theta_q(r) \left(\int_{2\lambda B} g_u^p \, d\mu_\beta\right)^{1/p},$$

where

$$\Theta_q(r) := \sup_{0<\rho\le r} \sup_{z \in B\cap Z} \frac{\rho \nu(B_\varepsilon(z,\rho))^{1/q}}{\mu_\beta(B_\varepsilon(z,\rho))^{1/p}}.$$

Proof of Proposition 13.4. By Hölder's inequality, it suffices to consider the case of $q > p$. Using Theorem 12.1, we can find a function $u \in N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ such that $u = \tilde{u}$ ν-a.e. on Z. By [27, Lemma 9.2.4] and Theorem 11.3, we know that for $C_p^{\overline{X}_\varepsilon}$-q.e. $\zeta \in Z$,

$$\lim_{r\to 0^+} r^p \fint_{B_\varepsilon(\zeta,r)} g_u^p \, d\mu_\beta = 0 \quad \text{and} \quad \lim_{r\to 0^+} \fint_{B_\varepsilon(\zeta,r)} u \, d\mu_\beta = u(\zeta). \tag{13.2}$$

In particular, Proposition 11.2 shows that ν-a.e. $z \in Z$ is a μ_β-Lebesgue point of u.

Proposition 13.2 shows that (13.2) holds for $\operatorname{Cap}_{B^\theta_{p,p}(Z)}$-q.e. $\zeta \in Z$. For such ζ, we have by the Minkowski inequality and Proposition 13.5 that

$$\begin{aligned}\biggl(\fint_{B_\varepsilon(\zeta,r)\cap Z} |u-u(\zeta)|^q \, d\nu\biggr)^{1/q} &\lesssim \biggl|\fint_{B_\varepsilon(\zeta,r)} u\, d\mu_\beta - u(\zeta)\biggr| \\ &\quad + \frac{\Theta_q(r)\mu_\beta(B_\varepsilon(\zeta,2\lambda r))^{1/p}}{\nu(B_\varepsilon(\zeta,r))^{1/q}} \biggl(\fint_{B_\varepsilon(\zeta,2\lambda r)} g_u^p \, d\mu_\beta\biggr)^{1/p}.\end{aligned} \tag{13.3}$$

In view of (13.2) and the definition of $\Theta_q(r)$, it suffices to show that for all $0 < \rho \le r \le 2 \operatorname{diam}_\varepsilon X_\varepsilon$ and $z \in B_\varepsilon(\zeta, r)) \cap Z$,

$$\frac{\rho\nu(B_\varepsilon(z,\rho))^{1/q}}{\mu_\beta(B_\varepsilon(z,\rho))^{1/p}} \frac{\mu_\beta(B_\varepsilon(\zeta,2\lambda r))^{1/p}}{\nu(B_\varepsilon(\zeta,r))^{1/q}} \lesssim r \tag{13.4}$$

with a comparison constant independent of z, ρ and r. Theorem 10.3 and the doubling property show that

$$\frac{\rho\nu(B_\varepsilon(z,\rho))^{1/q}}{\mu_\beta(B_\varepsilon(z,\rho))^{1/p}} \simeq \rho^{1-\beta/\varepsilon p}\nu(B_\varepsilon(z,\rho))^{1/q-1/p}$$

and

$$\frac{\mu_\beta(B_\varepsilon(\zeta,2\lambda r))^{1/p}}{\nu(B_\varepsilon(\zeta,r))^{1/q}} \simeq r^{\beta/\varepsilon p}\nu(B_\varepsilon(\zeta,r))^{1/p-1/q} \simeq r^{\beta/\varepsilon p}\nu(B_\varepsilon(z,r))^{1/p-1/q}.$$

Since $1/q - 1/p < 0$ and ν satisfies (10.13), the required estimate (13.4) holds because

$$1 - \frac{\beta}{\varepsilon p} + s_\nu\biggl(\frac{1}{q} - \frac{1}{p}\biggr) \ge 0.$$

The conclusion $u \in L^q(Z)$ follows by applying (13.3) to $r = 2 \operatorname{diam} \overline{X}_\varepsilon$. □

Even though we have so far only considered compact Z, we can now apply Proposition 13.4 to obtain the following improvement of Netrusov's result [42, Proposition 1.4] in $\mathbf{R}^n$, which was obtained for $q < np/(n-p\theta)$. The lift from the compact to the unbounded case is somewhat subtle since the Besov norm is nonlocal.

Proposition 13.6. *Assume that $n \ge 1$ and that $p\theta < n$. Let $q = np/(n-p\theta)$, $\tilde{u} \in B^\theta_{p,p}(\mathbf{R}^n)$, $u(\zeta) := \limsup_{r\to 0^+} \tilde{u}_{B(\zeta,r)}$, and*

$$E = \biggl\{\zeta : \limsup_{r\to 0^+} \fint |u(x)-u(\zeta)|^q \, dx > 0\biggr\}$$

be the set of non-L^q-Lebesgue points for u. Then $\operatorname{Cap}_{B^\theta_{p,p}(\mathbf{R}^n)}(E) = 0$.

Proof. For $r > 0$, let $Z_r = \overline{B(0,r)}$. Observe that for $x \in Z_r$ and $0 < \rho \le 2r$ we have $m(B(x,\rho) \cap Z_r) \simeq m(B(x,\rho)) \simeq \rho^n$. It follows that $B^\theta_{p,p}(\mathbf{R}^n) \subset B^\theta_{p,p}(Z_r)$

for all $r > 0$. Since $\mathrm{Cap}_{B^\theta_{p,p}(\mathbf{R}^n)}$ is comparable to a countably subadditive Besov capacity on $\mathbf{R}^n$, see Adams–Hedberg [1, Propositions 2.3.6 and 4.4.3], it suffices to show that $\mathrm{Cap}_{B^\theta_{p,p}(\mathbf{R}^n)}(E \cap B(0,r)) = 0$ for all $r \geq 1$. Fix $\varepsilon > 0$ and let $R \geq 3r$. As $u \in B^\theta_{p,p}(Z_R)$, it follows from Proposition 13.4 that $\mathrm{Cap}_{B^\theta_{p,p}(Z_R)}(E \cap B(0,r)) = 0$. Proposition 13.2, together with [2, Lemma 6.15 and Theorem 4.21] and Theorem 11.3, now implies that there is a function $v : \mathbf{R}^n \to [0,1]$ with $\operatorname{supp} v \subset B(0,2r)$ such that $v \geq 1$ in an open neighborhood of $E \cap B(0,r)$ and $\|v\|^p_{B^\theta_{p,p}(Z_R)} < \varepsilon$. Now

$$\begin{aligned}\|v\|^p_{\theta,p} + \|v\|^p_{L^p(\mathbf{R}^n)} &\lesssim \|v\|^p_{B^\theta_{p,p}(Z_R)} + \int_{\mathbf{R}^n \setminus Z_R} \int_{B(0,2r)} \frac{v(\zeta)}{|\zeta - \xi|^{p\theta+n}}\, d\zeta\, d\xi \\ &\lesssim \varepsilon + \int_{\mathbf{R}^n \setminus Z_R} \frac{(2r)^n}{(|\xi| - 2r)^{p\theta+n}}\, d\xi \to \varepsilon,\end{aligned}$$

as $R \to \infty$. Hence $\mathrm{Cap}_{B^\theta_{p,p}(\mathbf{R}^n)}(E \cap B(0,r)) \lesssim \varepsilon$, and letting $\varepsilon \to 0$ concludes the proof. □

The following result extends Proposition 6.6 in Björn–Björn–Gill–Shanmugalingam [3] to general compact doubling metric measure spaces Z, and essentially recovers Corollary 3.18 (iii) in Malý [39].

Proposition 13.7. *Assume that the measure ν on Z satisfies (10.13) for all $\zeta \in Z$ and $0 < r' \leq r \leq \operatorname{diam} Z$, with exponent s_ν. Let $s_\beta = \max\{1, p(1-\theta) + s_\nu\}$. If $p > s_\beta$ then every $f \in B^\theta_{p,p}(Z)$ has a ν-a.e. representative which is $(1 - s_\beta/p)$-Hölder continuous on Z.*

Proof. By Theorem 12.1, there is a function $u \in N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ such that $u|_Z = f$ ν-a.e. By Lemma 10.6 and by $\beta = \varepsilon p(1-\theta)$, μ_β satisfies the dimension condition (10.14).

Since $p > s_\beta$, functions in $N^{1,p}(\overline{X}_\varepsilon, \mu_\beta)$ are $(1 - s_\beta/p)$-Hölder continuous with respect to d_ε, by [2, Corollary 5.49] or [27, Theorem 9.2.14]. (There is a missing local compactness assumption in [27, Theorem 9.2.14].) It follows that the trace $u|_Z$ is $(1 - s_\beta/p)$-Hölder continuous with respect to d_Z. □

Elementary calculations show that Proposition 13.7 applies in the following two cases with $0 < \theta < 1$ and $p > s_\nu/\theta$:

- If $p \geq (1 - s_\nu)/(1-\theta)$, then $s_\beta = p(1-\theta) + s_\nu \geq 1$ and $1 - s_\beta/p = \theta - s_\nu/p$. Hence every $f \in B^\theta_{p,p}(Z)$ has a ν-a.e. representative which is $(\theta - s_\nu/p)$-Hölder continuous.
- If $1 < p < (1 - s_\nu)/(1-\theta)$ (which necessarily implies that $s_\nu < \theta$), then $p(1-\theta) + s_\nu < 1 = s_\beta$ and every $f \in B^\theta_{p,p}(Z)$ has a ν-a.e. representative which is $(1 - 1/p)$-Hölder continuous.

If $\theta \geq 1$ then $B^\theta_{p,p}(Z) \subset B^{\theta'}_{p,p}(Z)$ for every $0 < \theta' < 1$ and the above two cases with θ replaced by θ' imply (upon letting $\theta' \to 1$):

- If $p > s_\nu \geq 1$, then every $f \in B^\theta_{p,p}(Z)$ has a ν-a.e. representative which is η-Hölder continuous for any $0 < \eta < 1 - s_\nu/p$.
- If $0 < s_\nu < 1 < p$, then every $f \in B^\theta_{p,p}(Z)$ has a ν-a.e. representative which is $(1 - 1/p)$-Hölder continuous.

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