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# Nonlinear Data-Driven Control via State-Dependent Representations

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*Abstract*— **Recently, there has been renewed interest in data-driven control, that is, the design of controllers directly from observed data. In the case of linear time-invariant (LTI) systems, several approaches have been proposed that lead to tractable optimization problems. On the other hand, the case of nonlinear dynamics is considerably less developed, with existing approaches limited to at most rational dynamics and requiring the solution to a computationally expensive Sum of Squares (SoS) optimization. Since SoS problems typically scale combinatorially with the size of the problem, these approaches are limited to relatively low order systems. In this paper, we propose an alternative, based on the use of state-dependent representations. This idea allows for synthesizing data-driven controllers by solving at each time step an on-line optimization problem whose complexity is comparable to the LTI case. Further, the proposed approach is not limited to rational dynamics. The main result of the paper shows that the feasibility of this on-line optimization problem guarantees that the proposed controller renders the origin a globally asymptotically stable equilibrium point of the closed-loop system. These results are illustrated with some simple examples. The paper concludes by briefly discussing the prospects for adding performance criteria.**

## I. Introduction

Data-driven control (DDC), as a rapidly developing field, has attracted increasing attention in recent years. Compared with model-based control (MBC), it has the advantage that full knowledge of the model is not required. Rather, the controller is obtained directly from the data, avoiding issues such as inaccurate modeling or the high cost of SysId. Some early work towards DDC includes [1], [2], [3]. These methods assume a reference transfer function for the closed-loop system. Then the parameter is tuned to minimize the error between the reference and true signal. Later research focuses on state-space based design. Work along these lines includes, but is not limited to, [4], [5], [6], [7], [8] for linear systems, and [9], [10], [11], [12] for nonlinear systems. Of particular interest is [4], which establishes the equivalence between DDC and MBC for noise-free LTI systems. On the other hand, handling noisy measurements requires considering a data-driven robust design.

The design of guaranteed robust data-driven controllers, even for linear systems, is a relatively new research area. In classical robust control, uncertainty descriptions are either available or assumed and validated via model (in)validation. On the other hand, these descriptions are not available in the data-driven case, necessitating the use of a set-membership approach. In simple words, the noisy data-driven case requires designing a controller that stabilizes all possible systems compatible with the noisy measurements and some priors (the consistency set). To the best of our knowledge, the earliest work pursuing this approach is [13] where the designed controller can stabilize all systems with noise constrained by an $\ell_\infty$ bound. Further work in the $\ell_\infty$ norm bounded scenario includes [14], where exploiting duality led to the data-driven quadratic stabilization of a continuous system. Recently, [15] proposed a non-conservative method for the data-driven robust design problem with finite horizon $\ell_2$ bounded noise, based on a novel matrix S-lemma. Robust stability is achieved by using this lemma to establish a connection between two quadratic matrix inequalities (QMIs): one defined by the noisy measurements, and the second defined by the stability requirement. If the former set is a subset of the latter set, the controller is guaranteed to stabilize all plants in the consistency set.

Data-driven control of nonlinear systems is considerably less developed. An approach guaranteed to generate robust nonlinear controllers was proposed in [12]. While successful, at the present time this approach is limited to rational dynamics. Further, since it requires the solution to a computationally expensive SoS optimization, it is limited to relatively low order systems. An alternative approach was proposed in [11], but since it is also based on SoS optimization, it suffers from the same scaling problems.

To address these difficulties, in this paper we propose a new approach to address the nonlinear data-driven control (NLDDC) problem. This approach, based on combining the S-lemma based framework developed in [15] with state-dependent representations [16], allows for synthesizing data-driven controllers by solving at each time step an on-line optimization problem whose complexity is comparable to the LTI case. Our main result shows that the resulting controllers are guaranteed to render the origin a globally asymptotically stable equilibrium point, as long as a given linear matrix inequality (LMI) remains feasible along the trajectory. The advantages of the proposed approach over existing techniques are two-fold: (i) its computational complexity and scaling are comparable to those of *linear* data-driven control, and (ii) it is not limited to rational nonlinearities.

The remainder of the paper is organized as follows: Section II covers the notation and background knowledge required to make the paper self–contained. Section III presents the main result of the paper: an on-line optimization-based algorithm for finding robust data-driven controllers for nonlinear systems, along with the supporting theoretical results.

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Section IV illustrates the efficiency of the proposed method with three different systems. Finally, section V concludes the paper and provides directions for further research.

## II. Preliminaries

### A. Notation

We use the standard linear algebra notations. $\mathbb{R}$ and $\mathbb{R}^n$ denote the real numbers and the real $n$-dimensional vector space, respectively. $\mathbf{x} \in \mathbb{R}^n$ is a vector and $\mathbf{X} \in \mathbb{R}^{m\times n}$ is a matrix. $\mathbf{X} \succeq 0$ indicates a positive semi-definite (PSD) matrix. $\textbf{\textit{Tr}}(\mathbf{X})$ is the trace of the matrix. $\textbf{\textit{ker}}(\mathbf{X})$ is the kernel of the matrix. $\textbf{\textit{Kron}}(\mathbf{X},\mathbf{Y})$ denotes the Kronecker product of two matrices. $\mathbf{I}_n$ is the $n \times n$ identity matrix ($n$ may be omitted when clear from the context) and $\mathbf{0}$ is the zero matrix of suitable size.

### B. State-Dependent Representations

In [16], Cloutier *et. al.* proposed to stabilize continuous time nonlinear systems through the use of State-Dependent Riccati Equations (SDRE). The main idea here was to recast the nonlinear dynamics into a linear-like form by using state-dependent representations and design a controller using the (pointwise in the state) Riccati equations associated with these representations. Existence of these representations and the properties of the associated SDRE controllers are discussed for instance in [17]. Motivated by these results, in this paper we will use discrete time state-dependent representations, e.g. given the nonlinear system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{g}(\mathbf{x}_k)\mathbf{u}_k \tag{1}$$

we will represent it by

$$\mathbf{x}_{k+1} = \mathbf{A}(\mathbf{x}_k)\mathbf{x}_k + \mathbf{B}(\mathbf{x}_k)\mathbf{u}_k \tag{2}$$

where

$$\mathbf{f}(\mathbf{x}_k) = \mathbf{A}(\mathbf{x}_k)\mathbf{x}_k \quad \mathbf{g}(\mathbf{x}_k) = \mathbf{B}(\mathbf{x}_k) \tag{3}$$

**Example.**

$$\begin{aligned}\mathbf{x}_{k+1} &= \begin{bmatrix} x_{k,1}^2 \\ x_{k,1}x_{k,2} + x_{k,2}\end{bmatrix} + \begin{bmatrix} x_{k,1} & 1 \\ 1 & 0\end{bmatrix}\mathbf{u}_k \\ &= \begin{bmatrix} x_{k,1} & 0 \\ x_{k,2} & 1\end{bmatrix}\mathbf{x}_k + \begin{bmatrix} x_{k,1} & 1 \\ 1 & 0\end{bmatrix}\mathbf{u}_k\end{aligned} \tag{4}$$

This formulation will be key in using linear DDC control tools to synthesize controllers for nonlinear systems.

### C. Matrix S-lemma

For ease of reference, we restate Theorem 13 from [15]:

**Theorem 1.** *Let* $\mathbf{M},\mathbf{N} \in \mathbb{R}^{(n+k)\times(n+k)}$ *be symmetric matrices, partitioned as*

$$\mathbf{M} = \begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{12}^T & \mathbf{M}_{22}\end{bmatrix} \quad \mathbf{N} = \begin{bmatrix}\mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{12}^T & \mathbf{N}_{22}\end{bmatrix} \tag{5}$$

*Define a set*

$$\mathcal{S}_N \doteq \left\{\mathbf{Z} \in \mathbb{R}^{n\times k} \;\middle|\; \begin{bmatrix}\mathbf{I} \\ \mathbf{Z}\end{bmatrix}^T \mathbf{N} \begin{bmatrix}\mathbf{I} \\ \mathbf{Z}\end{bmatrix} \succeq 0\right\} \tag{6}$$

*assume that*
*a. there exists some matrix* $\bar{\mathbf{Z}} \in \mathbb{R}^{n\times k}$ *satisfying*

$$\begin{bmatrix}\mathbf{I} \\ \bar{\mathbf{Z}}\end{bmatrix}^T \mathbf{N} \begin{bmatrix}\mathbf{I} \\ \bar{\mathbf{Z}}\end{bmatrix} \succ 0 \tag{7}$$

*b.* $\mathbf{M}_{22} \preceq 0$, $\mathbf{N}_{22} \preceq 0$ *and* $\textbf{\textit{ker}}(\mathbf{N}_{22}) \subseteq \textbf{\textit{ker}}(\mathbf{N}_{12})$
*Then we have*

$$\begin{bmatrix}\mathbf{I} \\ \mathbf{Z}\end{bmatrix}^T \mathbf{M} \begin{bmatrix}\mathbf{I} \\ \mathbf{Z}\end{bmatrix} \succ 0 \text{ for all } \mathbf{Z} \in \mathcal{S}_N \tag{8}$$

*if and only if there exist scalars* $\tau \geq 0$ *and* $\eta > 0$ *such that*

$$\mathbf{M} - \tau\mathbf{N} \succeq \begin{bmatrix}\eta\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{9}$$

This theorem plays a central role in establishing the connection between the noisy measurements and the stabilizing controller.

### D. Problem Statement

The goal of this paper is to solve a robust NLDDC problem. Formally, the problem of interest is

**Problem 1.** *Consider a system*

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{g}(\mathbf{x}_k)\mathbf{u}_k + \mathbf{w}_k \tag{10}$$

*with unknown dynamics* $\mathbf{f},\mathbf{g}$ *and* $\ell_2$ *bounded noise* $\mathbf{w}_k$. *Given the measurements* $\mathbf{x}_k,\mathbf{u}_k$, *a bound* $\epsilon$ *on the energy of the noise and a priori structural information on* $\mathbf{f},\mathbf{g}$, *find a state-feedback controller* $\mathbf{u}_k = \mathbf{K}_k\mathbf{x}_k$ *that renders the origin a globally asymptotically stable equilibrium point of the closed-loop system.*

## III. Main Results

In this section, we reformulate **Problem 1** as a convex optimization problem over LMIs. The key idea is to first represent the nonlinear dynamics in a state-dependent form and proceed as if the system is "linear", by solving, at each time $k$, a "frozen" quadratic regulation problem of the form:

$$\min \sum_{i=k}^{\infty} \mathbf{x}_i^T\mathbf{Q}\mathbf{x}_i + \mathbf{u}_i^T\mathbf{R}\mathbf{u}_i \tag{11}$$

$$\text{s.t. } \mathbf{x}_{i+1} = \mathbf{A}(\mathbf{x}_k)\mathbf{x}_i + \mathbf{B}(\mathbf{x}_k)\mathbf{u}_i \tag{12}$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n} \succ 0, \mathbf{R} \in \mathbb{R}^{m\times m} \succ 0$ are state and control weights. That is, at each time $k$ we obtain a state-dependent representation, and then "freeze" these dynamics and proceed by treating the system as LTI. The goal is to obtain, at each time $k$, a robust data-driven controller that solves the problem above for all "linear" systems in the consistency set. To this effect, we use eq. (9) in [8], to rewrite Problem (11)-(12) as (for simplicity, we omit $\mathbf{x}$, i.e. $\mathbf{A}(\mathbf{x}_k) = \mathbf{A}_k$)

$$\begin{aligned}\min\ & \mathbf{Tr}(\mathbf{Q}\mathbf{Y}_k) + \mathbf{Tr}(\mathbf{R}\mathbf{L}_k) \\ \text{s.t. } & \mathbf{Y}_k - (\mathbf{A}_k + \mathbf{B}_k\mathbf{K}_k)\mathbf{Y}_k(\mathbf{A}_k + \mathbf{B}_k\mathbf{K}_k)^T - \mathbf{I} \succ 0 \\ & \mathbf{L}_k - \mathbf{K}_k\mathbf{Y}_k\mathbf{K}_k^T \succeq 0\end{aligned} \tag{13}$$

Motivated by [15] we will solve a robust version of (13) by minimizing an upper bound for all possible "linear" systems. To this effect, we define two sets: $S_1$, the set of all systems compatible with the noisy measurements, and $S_2$, the set of all systems that can be stabilized by a given state-feedback controller. Finally, we apply **Theorem 1** to enforce the inclusion $S_1 \subseteq S_2$. To begin with, we make two assumptions on the a-priori information.

**Assumption 1.** *There exist known basis functions* $\mathbf{F} \in \mathbb{R}^{n_f \times 1}, \mathbf{G} \in \mathbb{R}^{n_g \times m}$ *that span* $\mathbf{f}, \mathbf{g}$.

With this assumption, it is not hard to see that (10) can be rewritten as

$$\mathbf{x}_{k+1} = \boldsymbol{\alpha}\mathbf{F}(\mathbf{x}_k) + \boldsymbol{\beta}\mathbf{G}(\mathbf{x}_k)\mathbf{u}_k + \mathbf{w}_k \tag{14}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^{n \times n_f}$, $\boldsymbol{\beta} \in \mathbb{R}^{n \times n_g}$ are the (unknown) coefficients of $\mathbf{f}, \mathbf{g}$ and $n_f, n_g$ the corresponding size of basis functions. For example, (4) can be parameterized as

$$\begin{aligned}\mathbf{x}_{k+1} = &\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{k,2} \\ x_{k,1}^2 \\ x_{k,1}x_{k,2} \end{bmatrix} + \\ &\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{Kron}(\mathbf{I}_2, \begin{bmatrix} 1 \\ x_{k,1} \end{bmatrix})\mathbf{u}_k\end{aligned} \tag{15}$$

Next we excite the system for $T$ time steps and collect samples of the states and inputs in matrix form

$$\begin{aligned}\mathbf{X} &\doteq [\mathbf{x}_0 \dots \mathbf{x}_T] \\ \mathbf{U} &\doteq [\mathbf{u}_0 \dots \mathbf{u}_{T-1}]\end{aligned} \tag{16}$$

and define the matrices

$$\begin{aligned}\mathbf{X}_+ &\doteq [\mathbf{x}_1 \dots \mathbf{x}_T] \\ \mathbf{X}_- &\doteq [\mathbf{F}(\mathbf{x}_0) \dots \mathbf{F}(\mathbf{x}_{T-1})] \\ \mathbf{U}_- &\doteq [\mathbf{G}(\mathbf{x}_0)\mathbf{u}_0 \dots \mathbf{G}(\mathbf{x}_{T-1})\mathbf{u}_{T-1}] \\ \mathbf{W}_- &\doteq [\mathbf{w}_0 \dots \mathbf{w}_{T-1}]\end{aligned} \tag{17}$$

Clearly (14) satisfies:

$$\mathbf{X}_+ = \boldsymbol{\alpha}\mathbf{X}_- + \boldsymbol{\beta}\mathbf{U}_- + \mathbf{W}_- \tag{18}$$

Finally, we introduce an assumption on the noise

**Assumption 2.**

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{W}_-^T \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{12}^T & \boldsymbol{\Phi}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{W}_-^T \end{bmatrix} \succeq 0 \tag{19}$$

*for known matrices* $\boldsymbol{\Phi}_{11} = \boldsymbol{\Phi}_{11}^T$, $\boldsymbol{\Phi}_{12}$ *and* $\boldsymbol{\Phi}_{22} = \boldsymbol{\Phi}_{22}^T \prec 0$.

In particular, if $\boldsymbol{\Phi}_{12} = 0$ and $\boldsymbol{\Phi}_{22} = -\mathbf{I}$, (19) reduces to $\mathbf{W}_-\mathbf{W}_-^T \preceq \boldsymbol{\Phi}_{11}$, that is simply a bound on the covariance of the noise.
By substitution of $\mathbf{W}_-$ in (18), we have

$$\begin{bmatrix} \mathbf{I} \\ \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \end{bmatrix}^T \begin{bmatrix} \mathbf{I} & \mathbf{X}_+ \\ \mathbf{0} & -\mathbf{X}_- \\ \mathbf{0} & -\mathbf{U}_- \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{12}^T & \boldsymbol{\Phi}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{X}_+ \\ \mathbf{0} & -\mathbf{X}_- \\ \mathbf{0} & -\mathbf{U}_- \end{bmatrix}^T \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \end{bmatrix} \succeq 0 \tag{20}$$

Define $S_1$ as the set of possible system parameters compatible with the measurements

$$S_1 \doteq \{\boldsymbol{\alpha}, \boldsymbol{\beta} \mid (20) \text{ holds}\} \tag{21}$$

We now have a QMI related to the measurements. The next step is to get an expression similar to (20) where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are bounded by the QMI representing all the systems stabilized by the state-feedback controller $\mathbf{u}_k = \mathbf{K}_k\mathbf{x}_k$. We consider now the state-dependent representations $\mathbf{F}(\mathbf{x}_k) = \mathbf{A}(\mathbf{x}_k)\mathbf{x}_k, \mathbf{G}(\mathbf{x}_k) = \mathbf{B}(\mathbf{x}_k)$ where $\mathbf{A} \in \mathbb{R}^{n_f \times n}$. In terms of these matrices (14) can be restated as

$$\mathbf{x}_{k+1} = \boldsymbol{\alpha}\mathbf{A}(\mathbf{x}_k)\mathbf{x}_k + \boldsymbol{\beta}\mathbf{B}(\mathbf{x}_k)\mathbf{u}_k + \mathbf{w}_k \tag{22}$$

To stabilize this system, we solve (13) with the additional constraint that the control Lyapunov function $V(\mathbf{x}_k) = \mathbf{x}_k^T\mathbf{Y}(\mathbf{x}_k)^{-1}\mathbf{x}_k$ must be decreasing along the trajectories, leading to (for simplicity, we omit x):

$$\begin{aligned}\min \ & \mathbf{Tr}(\mathbf{Q}\mathbf{Y}_k) + \mathbf{Tr}(\mathbf{R}\mathbf{L}_k) \\ \text{s.t. } & \mathbf{Y}_k - (\boldsymbol{\alpha}\mathbf{A}_k + \boldsymbol{\beta}\mathbf{B}_k\mathbf{K}_k)\mathbf{Y}_k(\boldsymbol{\alpha}\mathbf{A}_k + \boldsymbol{\beta}\mathbf{B}_k\mathbf{K}_k)^T - \mathbf{I} \succ 0 \\ & \mathbf{L}_k - \mathbf{K}_k\mathbf{Y}_k\mathbf{K}_k^T \succeq 0 \\ & V_{k-1} - V_k > 0\end{aligned} \tag{23}$$

It is not hard to see that the first inequality in (23) is equivalent to

$$\begin{bmatrix} \mathbf{I} \\ \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \end{bmatrix}^T \begin{bmatrix} \mathbf{Y}_k - \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_k\mathbf{Y}_k\mathbf{A}_k^T & -\mathbf{A}_k\mathbf{Y}_k\mathbf{K}_k^T\mathbf{B}_k^T \\ \mathbf{0} & -\mathbf{B}_k\mathbf{K}_k\mathbf{Y}_k\mathbf{A}_k^T & -\mathbf{B}_k\mathbf{K}_k\mathbf{Y}_k\mathbf{K}_k^T\mathbf{B}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \end{bmatrix} \succ 0 \tag{24}$$

Defining $\mathbf{H}_k = \mathbf{K}_k\mathbf{Y}_k$ to eliminate the bilinear term, we have the equivalent expression

$$\begin{bmatrix} \mathbf{I} \\ \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \end{bmatrix}^T \begin{bmatrix} \mathbf{Y}_k - \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_k\mathbf{Y}_k\mathbf{A}_k^T & -\mathbf{A}_k\mathbf{H}_k^T\mathbf{B}_k^T \\ \mathbf{0} & -\mathbf{B}_k\mathbf{H}_k\mathbf{A}_k^T & -\mathbf{B}_k\mathbf{H}_k\mathbf{Y}_k^{-1}\mathbf{H}_k^T\mathbf{B}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \end{bmatrix} \succ 0 \tag{25}$$

The set $S_2$ that contains all the systems stabilized by the state-feedback controller $\mathbf{u}_k = \mathbf{H}_k\mathbf{Y}_k^{-1}\mathbf{x}_k$ is given by:

$$S_2 \doteq \{\boldsymbol{\alpha}, \boldsymbol{\beta} \mid (25) \text{ holds}\} \tag{26}$$

The inclusion $S_1 \subseteq S_2$ can be enforced now using **Theorem 1**. However, before proceeding, we need to verify that the hypotheses of the theorem hold. Assumption a is trivially satisfied. To check assumption b, we first define

$$\begin{aligned}\mathbf{M} &= \left[\begin{array}{c|c} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \hline \mathbf{M}_{21} & \mathbf{M}_{22} \end{array}\right] \\ &\doteq \left[\begin{array}{c|cc} \mathbf{Y}_k - \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{A}_k\mathbf{Y}_k\mathbf{A}_k^T & -\mathbf{A}_k\mathbf{Y}_k\mathbf{K}_k^T\mathbf{B}_k^T \\ \mathbf{0} & -\mathbf{B}_k\mathbf{K}_k\mathbf{Y}_k\mathbf{A}_k^T & -\mathbf{B}_k\mathbf{K}_k\mathbf{Y}_k\mathbf{K}_k^T\mathbf{B}_k^T \end{array}\right]\end{aligned} \tag{27}$$

and

$$\begin{aligned}\mathbf{N} &= \left[\begin{array}{c|c} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \hline \mathbf{N}_{21} & \mathbf{N}_{22} \end{array}\right] \\ &\doteq \left[\begin{array}{cc} \mathbf{I} & \mathbf{X}_+ \\ \hline \mathbf{0} & -\mathbf{X}_- \\ \mathbf{0} & -\mathbf{U}_- \end{array}\right] \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{12}^T & \boldsymbol{\Phi}_{22} \end{bmatrix} \left[\begin{array}{cc} \mathbf{I} & \mathbf{X}_+ \\ \hline \mathbf{0} & -\mathbf{X}_- \\ \mathbf{0} & -\mathbf{U}_- \end{array}\right]^T\end{aligned} \tag{28}$$

Thus

$$\mathbf{M}_{22} = -\begin{bmatrix}\mathbf{A}_k \\ \mathbf{B}_k\mathbf{K}_k\end{bmatrix}\mathbf{Y}_k\begin{bmatrix}\mathbf{A}_k \\ \mathbf{B}_k\mathbf{K}_k\end{bmatrix}^T \preceq 0 \tag{29}$$

since $\mathbf{Y}_k \succ 0$. Similarly:

$$\mathbf{N}_{22} = -\begin{bmatrix}\mathbf{X}_{\text{-}} \\ \mathbf{U}_{\text{-}}\end{bmatrix}\mathbf{\Phi_{22}}\begin{bmatrix}\mathbf{X}_{\text{-}} \\ \mathbf{U}_{\text{-}}\end{bmatrix}^T \preceq 0 \tag{30}$$

holds since $\mathbf{\Phi}_{22} \prec 0$. Finally, We note that

$$\begin{aligned}\boldsymbol{ker}(\mathbf{N}_{22}) &= \boldsymbol{ker}\left(\begin{bmatrix}\mathbf{X}_{\text{-}} \\ \mathbf{U}_{\text{-}}\end{bmatrix}^T\right) \\ \boldsymbol{ker}(\mathbf{N}_{12}) &= \boldsymbol{ker}\left((\mathbf{\Phi_{12}} + \mathbf{X}_+\mathbf{\Phi_{22}})\begin{bmatrix}\mathbf{X}_{\text{-}} \\ \mathbf{U}_{\text{-}}\end{bmatrix}^T\right)\end{aligned} \tag{31}$$

Hence $\boldsymbol{ker}(\mathbf{N}_{22}) \subseteq \boldsymbol{ker}(\mathbf{N}_{12})$. Thus all the hypotheses of **Theorem 1** are satisfied. Directly application of the theorem to (20) and (25) yields that $S_1 \subseteq S_2$ if and only if

$$\begin{bmatrix}\mathbf{Y}_k - \mathbf{I} - \eta_k\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_k\mathbf{Y}_k\mathbf{A}_k^T & -\mathbf{A}_k\mathbf{H}_k^T\mathbf{B}_k^T \\ \mathbf{0} & -\mathbf{B}_k\mathbf{H}_k\mathbf{A}_k^T & -\mathbf{B}_k\mathbf{H}_k\mathbf{Y}_k^{-1}\mathbf{H}_k^T\mathbf{B}_k^T\end{bmatrix} - \tau_k\begin{bmatrix}\mathbf{I} & \mathbf{X}_+ \\ \mathbf{0} & -\mathbf{X}_{\text{-}} \\ \mathbf{0} & -\mathbf{U}_{\text{-}}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{11} & \mathbf{\Phi}_{12} \\ \mathbf{\Phi}_{12}^T & \mathbf{\Phi}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{X}_+ \\ \mathbf{0} & -\mathbf{X}_{\text{-}} \\ \mathbf{0} & -\mathbf{U}_{\text{-}}\end{bmatrix}^T \succeq 0 \tag{32}$$

Now we are ready to state the main theorem of the paper.

**Theorem 2.** ***Problem 1*** *is solvable if there exist matrices* $\mathbf{Y}_k \succ 0, \mathbf{H}_k, \mathbf{L}_k$ *and scalars* $\tau_k \geq 0, \eta_k > 0$ *satisfying* (33)-(35) *below. Then the state-dependent feedback gain* $\mathbf{K}_k = \mathbf{H}_k\mathbf{Y}_k^{-1}$ *renders the origin an asymptotically stable equilibrium point of* (10).

$$\begin{bmatrix}\mathbf{Y}_k - \mathbf{I} - \eta_k\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_k\mathbf{Y}_k\mathbf{A}_k^T & -\mathbf{A}_k\mathbf{H}_k^T\mathbf{B}_k^T & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_k\mathbf{H}_k\mathbf{A}_k^T & \mathbf{0} & \mathbf{B}_k\mathbf{H}_k \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_k^T\mathbf{B}_k^T & \mathbf{Y}_k\end{bmatrix} - \tau_k\begin{bmatrix}\mathbf{I} & \mathbf{X}_+ \\ \mathbf{0} & -\mathbf{X}_{\text{-}} \\ \mathbf{0} & -\mathbf{U}_{\text{-}} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{11} & \mathbf{\Phi}_{12} \\ \mathbf{\Phi}_{12}^T & \mathbf{\Phi}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{X}_+ \\ \mathbf{0} & -\mathbf{X}_{\text{-}} \\ \mathbf{0} & -\mathbf{U}_{\text{-}} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}^T \succeq 0 \tag{33}$$

$$\begin{bmatrix}\mathbf{L}_k & \mathbf{H}_k \\ \mathbf{H}_k^T & \mathbf{Y}_k\end{bmatrix} \succeq 0 \tag{34}$$

$$\begin{bmatrix}V_{k-1} & \mathbf{x}_k^T \\ \mathbf{x}_k & \mathbf{Y}_k\end{bmatrix} \succ 0 \tag{35}$$

*Proof.* The equivalence between (33) and (32) follows from applying a Schur complement to (33). Similarly, (34)-(35) follows from applying Schur complements to the last two inequalities in (23). Finally, (35) guarantees that $V(\mathbf{x}_k) = \mathbf{x}_k^T\mathbf{Y}_k^{-1}\mathbf{x}_k$ is a Lyapunov function for the closed-loop system. □

Motivated by the result above we propose the following on-line optimization based robust data–driven control law:

**Algorithm 1** Online NLDDC

1: Excite the system for $k$ time steps and collect initial data matrices $\mathbf{X}, \mathbf{U}$
2: Decide basis functions $\mathbf{F}, \mathbf{G}$, performance matrices $\mathbf{Q}, \mathbf{R}$, Initial value of the Lyapunov function $V_0$ and the noise bound $\epsilon$
3: **repeat**
4: Form data matrices $\mathbf{X}_+(\mathrm{k}), \mathbf{X}_{\text{-}}(\mathrm{k}), \mathbf{U}_{\text{-}}(\mathrm{k})$
5: Solve $\min\ \boldsymbol{Tr}(\mathbf{Q}\mathbf{Y}_k) + \boldsymbol{Tr}(\mathbf{R}\mathbf{L}_k)$ s.t. (33)-(35)
6: Apply the controller $\mathbf{u}_k = \mathbf{K}_k\mathbf{x}_k$
7: Compute $V_k = \mathbf{x}_k^T\mathbf{Y}_k^{-1}\mathbf{x}_k$
8: k = k+1
9: **until**
10: The system trajectories converge

## IV. Illustrative Examples

In this section, we illustrate the potential of the proposed approach with several examples with sinusoidal, rational and exponential dynamics. In all cases, we set $\mathbf{Q} = \mathbf{I}, \mathbf{R} = \mathbf{I}$ and $V_0 = 1000$. The noise bound $\epsilon$ is computed from the largest eigenvalue of $\mathbf{W}_{\text{-}}\mathbf{W}_{\text{-}}^T$ and we set $\mathbf{\Phi}_{11} = \epsilon\mathbf{I}, \mathbf{\Phi}_{12} = 0, \mathbf{\Phi}_{22} = -\mathbf{I}$, that is we bound the noise covariance, i.e. $\sum_k \|\mathbf{w}_k\|_2 \leq \epsilon$. The initial state and input obey the normal distribution. The noise obey Gaussian distribution with standard deviation $\sigma = 0.01$. All discrete time models are obtained from a discretization of continuous time systems using the forward Euler method $\mathbf{x}_{k+1} = \mathbf{x}_k + T_c\dot{\mathbf{x}} + \mathbf{w}_k$ with $T_c = 0.1$. We run all the simulations in MATLAB [18] and the optimization problem is solved using YALMIP [19] with the MOSEK [20] SDP solver.

**Example 1.** *Inverted pendulum with friction*

$$\mathbf{x}_{k+1} = \begin{bmatrix}x_{k,1} + T_c x_{k,2} \\ x_{k,2} + T_c\sin x_{k,1} - T_c x_{k,2}\end{bmatrix} + \begin{bmatrix}0 \\ T_c\end{bmatrix}u_k + \mathbf{w}_k$$

The goal here is to find a control action that stabilizes the pendulum in the upright position $[0, 0]$. Note that since the dynamics are not rational, this case cannot be addressed by the approaches in [12], [21]. We excited the system for 20 time steps, i.e. $k = 20$ and chose the basis $\mathbf{F} = [x_1, x_2, \sin(x_1)]^T$, $\mathbf{G} = 1$. Applying **Algorithm 1**, leads to the trajectories shown in Figures 1-2

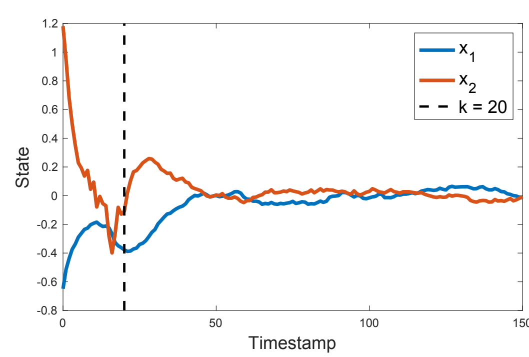


Fig. 1. System Trajectories

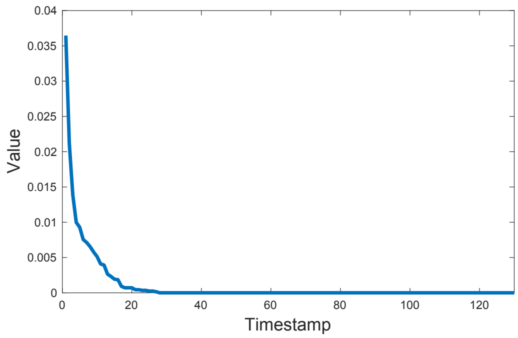


Fig. 2. Lyapunov Function $V_k$

**Example 2.** *Simple rational dynamics*

$$\mathbf{x}_{k+1} = \begin{bmatrix} x_{k,1} + T_c \frac{x_{k,1}}{1+x_{k,1}^2} \\ x_{k,2} + T_c x_{k,1} \end{bmatrix} + \begin{bmatrix} T_c \frac{1}{1+x_{k,1}^2} \\ 0 \end{bmatrix} u_k + \mathbf{w}_k$$

This system is open-loop unstable. Choosing $k = 20$ and the basis $\mathbf{F} = [x_1, x_2, \frac{x_1}{1+x_1^2}]^T$, $\mathbf{G} = \frac{1}{1+x_1^2}$ leads to

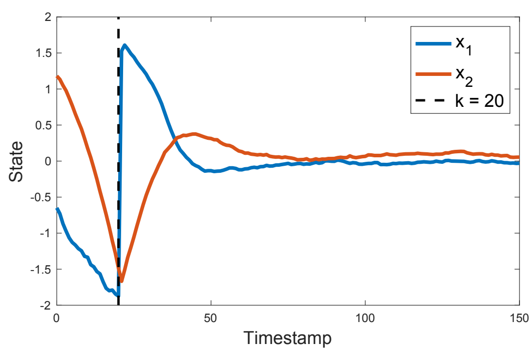


Fig. 3. System Trajectories

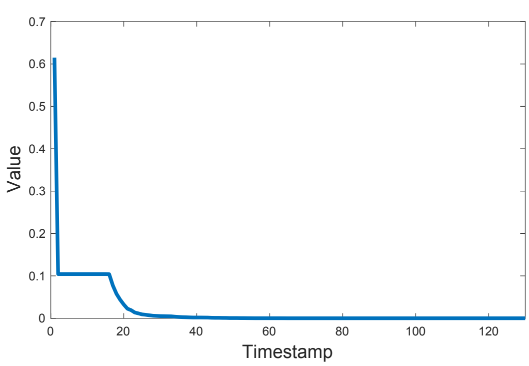


Fig. 4. Lyapunov Function $V_k$

**Example 3.** *Cart and spring model [22]*

$$\mathbf{x}_{k+1} = \begin{bmatrix} x_{k,1} + T_c x_{k,2} \\ x_{k,2} - T_c \frac{k_0}{M} e^{-x_{k,1}} x_{k,1} + T_c \frac{h}{M} x_{k,2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{T_c}{M} \end{bmatrix} u_k + \mathbf{w}_k$$

We modified the original system in order to make it open-loop unstable. The parameters are $M = 1, k_0 = 0.33, h = 1.1$. Again, we set $k = 20$ and choose the basis $\mathbf{F} = [\mathbf{x}_1, \mathbf{x}_2, e^{-\mathbf{x}_1}\mathbf{x}_1]^T$, $\mathbf{G} = 1$. Running **Algorithm 1**, we have

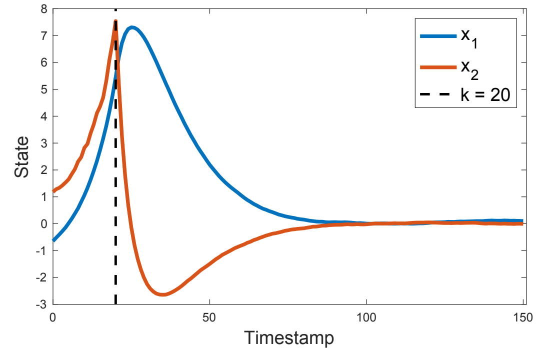


Fig. 5. System Trajectories

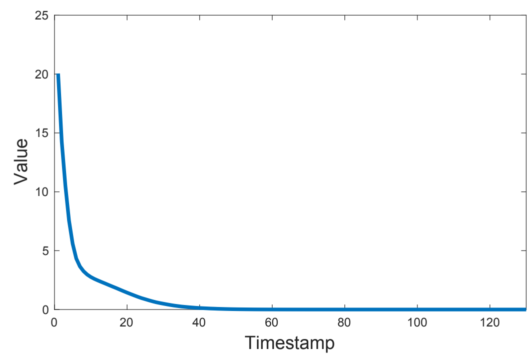


Fig. 6. Lyapunov Function $V_k$

In the remainder of this section, we comment on three issues: equivalent state-dependent realizations, the effect of noise, and the effect of the choice of basis.

**Remark 1.** *It is not hard to see that state-dependent representations are not unique. Indeed, given* $\mathbf{A}_k$*, any* $\hat{\mathbf{A}}_k = \mathbf{A}_k + \mathbf{E}_k$*, such that* $\mathbf{E}_k \mathbf{x}_k = 0$ *is also a valid representation. For instance,* (4) *can also be parameterized as*

$$\mathbf{x}_{k+1} = \begin{bmatrix} x_{k,1} & 0 \\ 0 & 1 + x_{k,1} \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} x_{k,1} & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}_k \tag{36}$$

*For the given examples, we ran our algorithm with different representations and empirically observed no noticeable effect on the stabilization of the system.*

**Remark 2.** *The amount of noise the system can tolerate depends on the quality of the data. In general, the more samples we have, the larger noise we can tolerate since the uncertainty decreases (equivalently, the consistency set is smaller). For* **Example 1***, if we increase the noise to* $\sigma = 0.05$*, then we need 80 samples to stabilize the system, i.e.* $k = 80$*. On the other hand,* $\sigma$ *= 0.1, requires* $k = 400$ *samples. However, in the case of an open-loop unstable system, the trajectories rapidly explode, preventing the collection of enough samples. For example in* **Example 2, 3***, with* $T_c = 0.1$*, we cannot collect enough samples before* *the trajectory explodes. In principle, this can be mitigated by faster sampling. However, if the model originates from the forward Euler discretization $\mathbf{x}_{k+1} = \mathbf{x}_k + T_c\dot{\mathbf{x}} + \mathbf{w}_k$, a smaller $T_c$ results in a larger relative noise level, which may require, once again, using a long horizon.*

**Remark 3.** *Effect of the dictionary choice. In all the simulations, we used a dictionary containing only the required basis functions. Using an overcomplete dictionary may lead to parameter matrices $\boldsymbol{\alpha}, \boldsymbol{\beta}$ with additional elements, which in turn, may require collecting more data to obtain comparable sized consistency sets. Let us consider using a large basis $\mathbf{F} = [x_1, x_2, x_1^2, x_1x_2, sin(x_1), x_1sin(x_1)]^T$, $\mathbf{G} = 1$ for* **Example 1**. *In this case, finding a stabilizing controller requires setting $k = 70$.*

## V. Conclusion

This paper proposes an on-line optimization-based method for synthesizing robust nonlinear data-driven controllers guaranteed to stabilize all plants in the consistency set. It differs from existing approaches in that is not limited to rational nonlinearities, and, more importantly, it does not require solving a computationally expensive SoS optimization. Rather, the computational complexity at each time step is comparable to that of designing a data-driven controller for a *linear* system of comparable dimensions. Our main theoretical results show that, as long as 3 LMIs feasible along the trajectory, the proposed control law is guaranteed to render the origin an asymptotically stable equilibrium point for all plants in the consistency set. These results were illustrated with several simple examples two of which involve non-rational dynamics and hence cannot be addressed with existing methods. An issue that was not addressed in this paper is the effect of the choice of state-dependent parameterizations. As noted in **Remark 1**, all such parameterizations corresponding to a given nonlinear dynamics can be described in terms of a free state-dependent matrix $\mathbf{E}_k$ such that $\mathbf{E}_k\mathbf{x}_k = 0$. In principle, this could be accommodated by replacing $\mathbf{A}_k$ with $\mathbf{A}_k + \mathbf{E}_k$ in (33) and adding the linear constraint $\mathbf{E}_k\mathbf{x}_k = 0$. However, this will lead to terms of the form $\mathbf{E}_k^T\mathbf{Y}_k\mathbf{E}_k$, so that (33) is no longer an LMI. The resulting inequality can still be solved using polynomial optimization ideas, but this will result in a poorly scaling, computationally expensive algorithm, defeating the purpose of the paper. Research is currently underway seeking parameterizations that do not substantially increase computational complexity. A second issue that has not been addressed in this paper is performance. Note that even though an equivalent LQR problem is solved at each step, in principle no claim can be made relating the Lyapunov function $V(\mathbf{x}_k) = \mathbf{x}_k^T\mathbf{Y}^{-1}(\mathbf{x}_k)\mathbf{x}_k$ to the actual closed-loop $\mathcal{H}_2$ cost for the nonlinear system. We believe that this situation can be partially alleviated by considering a robust version of the receding horizon approach pursued in [23], where the Lyapunov function obtained from the state-dependent representations is only used as a terminal cost.

## References


[1] Marco C Campi, Andrea Lecchini, and Sergio M Savaresi. Virtual reference feedback tuning: a direct method for the design of feedback controllers. *Automatica*, 38(8):1337–1346, 2002.

[2] A Karimi, L Mišković, and D Bonvin. Iterative correlation-based controller tuning. *International journal of adaptive control and signal processing*, 18(8):645–664, 2004.

[3] Hakan Hjalmarsson, Michel Gevers, Svante Gunnarsson, and Olivier Lequin. Iterative feedback tuning: theory and applications. *IEEE control systems magazine*, 18(4):26–41, 1998.

[4] Claudio De Persis and Pietro Tesi. On persistency of excitation and formulas for data-driven control. *arXiv preprint arXiv:1903.06842*, 2019.

[5] Jun Ma, Zilong Cheng, Xiaocong Li, Masayoshi Tomizuka, and Tong Heng Lee. Data-driven linear quadratic optimization for controller synthesis with structural constraints. *arXiv preprint arXiv:1912.03616*, 2019.

[6] Monica Rotulo, Claudio De Persis, and Pietro Tesi. Data-driven linear quadratic regulation via semidefinite programming. *arXiv preprint arXiv:1911.07767*, 2019.

[7] Gustavo R Gonçalves da Silva, Alexandre S Bazanella, Charles Lorenzini, and Luciola Campestrini. Data-driven lqr control design. *IEEE control systems letters*, 3(1):180–185, 2018.

[8] Claudio De Persis and Pietro Tesi. Low-complexity learning of linear quadratic regulators from noisy data. *arXiv preprint arXiv:2005.01082*, 2020.

[9] Zhongsheng Hou and Shangtai Jin. A novel data-driven control approach for a class of discrete-time nonlinear systems. *IEEE Transactions on Control Systems Technology*, 19(6):1549–1558, 2010.

[10] Marko Tanaskovic, Lorenzo Fagiano, Carlo Novara, and Manfred Morari. Data-driven control of nonlinear systems: An on-line direct approach. *Automatica*, 75:1–10, 2017.

[11] Meichen Guo, Claudio De Persis, and Pietro Tesi. Learning control for polynomial systems using sum of squares relaxations. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 2436–2441. IEEE, 2020.

[12] T Dai and M Sznaier. A semi-algebraic optimization approach to data-driven control of continuous-time nonlinear systems. *IEEE Control Systems Letters*, 5(2):487–492, 2020.

[13] Tianyu Dai and Mario Sznaier. A moments based approach to designing mimo data driven controllers for switched systems. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 5652–5657. IEEE, 2018.

[14] Tianyu Dai and Mario Sznaier. Data-driven quadratic stabilization of continuous lti systems. *IFAC-PapersOnLine*, 2019.

[15] Henk J Vanwaarde, M Kanat Camlibel, and Mehran Mesbahi. From noisy data to feedback controllers: non-conservative design via a matrix s-lemma. *IEEE Transactions on Automatic Control*, 2020.

[16] James R Cloutier, Christopher N D'Souza, and Curtis P Mracek. Nonlinear regulation and nonlinear $h_\infty$ control via the state-dependent riccati equation technique: Part 1, theory. In *Proceedings of the international conference on nonlinear problems in aviation and aerospace*, pages 117–131. Embry Riddle University, 1996.

[17] J. S. Shamma and J. R. Cloutier. Existence of sdre stabilizing feedback. *IEEE Transactions on Automatic Control*, 48(3):513–517, 2003.

[18] MATLAB. *9.6.0.1072779 (R2019a)*. The MathWorks Inc., Natick, Massachusetts, 2019.

[19] Johan Löfberg. Yalmip: A toolbox for modeling and optimization in matlab. In *Proceedings of the CACSD Conference*, volume 3. Taipei, Taiwan, 2004.

[20] MOSEK ApS. *The MOSEK optimization toolbox for MATLAB manual. Version 9.0.*, 2019.

[21] Meichen Guo, Claudio De Persis, and Pietro Tesi. Data-driven stabilization of nonlinear polynomial systems with noisy data, 2020.

[22] L Magni, G De Nicolao, Riccardo Scattolini, and F Allgöwer. Robust model predictive control for nonlinear discrete-time systems. *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, 13(3-4):229–246, 2003.

[23] Mario Sznaier, Rodolfo Suárez, and James Cloutier. Suboptimal control of constrained nonlinear systems via receding horizon constrained control lyapunov functions. *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, 13(3-4):247–259, 2003.