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Peak Estimation for Uncertain and Switched Systems

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***Abstract*— Peak estimation bounds extreme values of a function of state along trajectories of a dynamical system. This paper focuses on extending peak estimation to continuous and discrete settings with time-independent and time-dependent uncertainty. Techniques from optimal control are used to incorporate uncertainty into an existing occupation measure-based peak estimation framework, which includes special consideration for handling switching-type (polytopic) uncertainties. The resulting infinite-dimensional linear programs can be solved approximately with Linear Matrix Inequalities arising from the moment-SOS hierarchy.**

I. Introduction

Peak estimation under uncertainty aims to bound extreme values of a state function subject to an adversarial noise process. Examples include finding the maximum height of an aircraft subject to wind, the maximum voltage in a transmission line subject to thermal noise, and the maximum speed of a motor subject to impedance within a tolerance. A system with finite-dimensional state $x \in \mathbb{R}^{N_x}$ evolves under Ordinary Differential Equation (ODE) dynamics defined by a locally Lipschitz vector field f perturbed by uncertainty over the time-range $t \in [0, T]$. The time-independent uncertainty $\theta \in \Theta \subset \mathbb{R}^{N_\theta}$ is fixed (such as the unknown mass of a system component within tolerance), while the time-dependent uncertainty $w(t)$ may change arbitrarily in time within the region $W \subset \mathbb{R}^{N_w}$. Let $x(t \mid x_0, \theta, w(t))$ denote a trajectory in time starting from an initial point x_0 subject to uncertainties $(\theta, w(t))$. The uncertain peak estimation problem maximizing a state cost $p(x)$ along trajectories with variables $(t, x_0, \theta, w(t))$ may be posed as,

$$P^* = \max_{t\in[0,T],\, x_0\in X_0,\, \theta\in\Theta,\, w(t)} p(x(t \mid x_0, \theta, w(t))) \tag{1}$$
$$\dot{x}(t) = f(t, x(t), \theta, w(t)), \quad w(t) \in W \quad \forall t \in [0, T].$$

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This paper proposes an infinite-dimensional linear program (LP) in occupation measures to upper bound the quantity P^* from (1). Measure-based peak estimation was introduced in [1] and [2] for a stochastic setting, and was numerically approximated by a discretizing set of finite LPs. The work in [3] forms a sum-of-squares program from an LP dual to the measure LP in [2]. Each of these are variations on the optimal control framework in [4], [5], with an optimal stopping cost rather than an integral (running) cost.

Occupation measure-based bounds for uncertain peak estimation may be developed by adapting methods from optimal control. Time-dependent uncertainty is an instance of an adversarial optimal control which aims to maximize the state function. Time-independent parameter uncertainty may be incorporated by adding states, and switched systems can be analyzed by splitting the occupation measure [6]. The true peak cost P^* is upper bounded with an infinite dimensional LP in occupation measures. The infinite LP is then truncated into a sequence of Linear Matrix Inequalities (LMIs) by the moment-SOS hierarchy [7].

This paper has the following structure: Section II reviews preliminaries such as occupation measures and peak estimation. Section III presents uncertainty models, and a unified uncertain peak estimation model is presented in Section IV. Section V extends uncertain peak estimation to discrete systems. The paper is concluded in Section VI.

II. Preliminaries

A. Notation

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}^n$ be an n-dimensional real Euclidean space, and $\mathbb{R}[x]$ be the set of polynomials in x with real-valued coefficients. For a set $X \subseteq \mathbb{R}^n$, the sets $C(X)$ and $C_+(X)$ are respectively the set of continuous functions on X and its nonnegative subcone. The subcone $C^1(X) \subset C(X)$ is composed of continuous functions on X with continuous first derivatives. $\mathcal{M}_+(X)$ is the set of nonnegative Borel measures over X, and a duality pairing exists $\langle f, \mu \rangle = \int_X f(x) d\mu(x)$ for all $f \in C(X)$, $\mu \in \mathcal{M}_+(X)$. For every linear operator $\mathcal{L}$, there exists a unique linear adjoint $\mathcal{L}^\dagger$ such that $\langle \mathcal{L}f, \mu \rangle = \langle f, \mathcal{L}^\dagger \mu \rangle$, $\forall f, \mu$ is satisfied. An indicator function is $I_A(x) = 1$ for a subset $A \subseteq X$ if $x \in A$ and $I_A(x) = 0$ otherwise. The measure of a set $A \subseteq X$ with respect to μ is $\mu(A) = \int_A d\mu = \int_X I_A(x) d\mu$. The quantity $\mu(X) = \langle 1, \mu \rangle$ is known as the 'mass', and μ is a probability measure if $\mu(X) = 1$. The Dirac delta $\delta_{x'} \in \mathcal{M}_+(X)$ is a probability measure supported only on $x = x'$. For measures $\mu \in \mathcal{M}_+(X)$, $\nu \in \mathcal{M}_+(Y)$, the product measure satisfies $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$ for all $A \in X$, $B \in Y$.

The projection map $\pi^x : X \times Y \to X$ returns only the x coordinate $(x, y) \to x$. The pushforward by a function f is the linear operator $f_\#$ satisfying $\langle v(x), f_\# \mu \rangle = \langle v(f(x)), \mu \rangle$ for any test function $v \in C(X)$ and measure $\mu \in \mathcal{M}_+(X)$. The x-marginal of a measure $\mu \in \mathcal{M}_+(X \times Y)$ may be expressed as the pushforward of a projection $\pi^x_\# \mu$ with duality pairing $\langle v(x), \pi^x_\# \mu \rangle = \int_{X \times Y} v(x) d\mu(x, y)$ holding for all test functions $v(x) \in C(X)$.

B. Peak Estimation and Occupation Measures

The standard (no uncertainty) peak estimation setting involves a trajectory $x(t \mid x_0)$ starting at the initial point $x_0 \in X_0 \subset X$ evolving according to dynamics $\dot{x}(t) = f(t, x(t))$ in a space X. The program to find the maximum value of a state function $p(x)$ along trajectories is,

$$P^* = \max_{t \in [0,T],\, x_0 \in X_0} p(x(t \mid x_0)), \quad \dot{x}(t) = f(t, x(t)). \tag{2}$$

The extremum P^* may be bounded through the use of occupation measure relaxations [2]. An optimal trajectory satisfying $P^* = p(x^*) = p(x(t^* \mid x_0^*))$ is described by a triple (x_0^*, t^*, x^*) [8]. The initial probability measure $\mu_0 \in \mathcal{M}_+(X_0)$ is distributed over the set of initial conditions. The peak probability measure $\mu_p \in \mathcal{M}_+([0, T] \times X)$ is a free-time terminal measure. For an optimal stopping time t^* and subsets $A \subseteq [0, t^*],\ B \subseteq X$, the μ_0-averaged occupation measure $\mu \in \mathcal{M}_+([0, T] \times X)$ has a definition [2],

$$\mu(A \times B) = \int_{[0,t^*] \times X_0} I_{A \times B}\left((t, x(t \mid x_0)\right) dt\, d\mu_0(x_0). \tag{3}$$

The measure $\mu(A \times B)$ yields the average amount of time a trajectory with initial condition x_0 drawn from μ_0 will spend in the region $A \times B$.

The Lie derivative operator $\mathcal{L}_f$ may be defined for all test functions $v \in C^1([0, T] \times X)$,

$$\mathcal{L}_f v(t, x) = \partial_t v(t, x) + f(t, x) \cdot \nabla_x v(t, x). \tag{4}$$

The three measures $(\mu_0,\ \mu_p,\ \mu)$ are linked by Liouville's equation for all test functions,

$$\langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x), \mu \rangle. \tag{5}$$

Liouville's equation ensures that initial conditions distributed as μ_0 are connected to terminal points distributed as μ_p by trajectories following the polynomial vector field f. Two consequences of (5) are that $\langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle$ $(v(t, x) = 1)$ and that $\langle 1, \mu \rangle = \langle t, \mu_p \rangle$ $(v(t, x) = t)$. Equation (5) may be expressed in a weak sense using the adjoint relaton $\langle \mathcal{L}_f v, \mu \rangle = \langle v, \mathcal{L}_f^\dagger \mu \rangle$,

$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu. \tag{6}$$

A convex measure relaxation of problem (2) is,

$$\begin{aligned} p^* = \max \quad & \langle p(x), \mu_p \rangle && \text{(7a)} \\ & \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu && \text{(7b)} \\ & \langle 1, \mu_0 \rangle = 1 && \text{(7c)} \\ & \mu, \mu_p \in \mathcal{M}_+([0, T] \times X) && \text{(7d)} \\ & \mu_0 \in \mathcal{M}_+(X_0). && \text{(7e)} \end{aligned}$$

Constraint (7c) ensures that both μ_0 and μ_p are probability measures. The objective (7a) is the expectation of $p(x)$ with respect to the peak measure μ_p. Program (7) has a dual problem over continuous functions,

$$\begin{aligned} d^* = \min_{\gamma \in \mathbb{R}} \quad & \gamma && \text{(8a)} \\ & \gamma \geq v(0, x) & \forall x \in X_0 \quad & \text{(8b)} \\ & \mathcal{L}_f v(t, x) \leq 0 & \forall (t, x) \in [0, T] \times X \quad & \text{(8c)} \\ & v(t, x) \geq p(x) & \forall (t, x) \in [0, T] \times X \quad & \text{(8d)} \\ & v \in C^1([0, T] \times X). && \text{(8e)} \end{aligned}$$

The variable $v(t, x)$ is termed an auxiliary function in [3], and is an upper bound on the cost function $p(x)$ by (8d). The graph $(t, x(t \mid x_0))$ is contained in the sublevel set $\{(t, x) \mid v(t, x) \leq \gamma\}$ for all $x_0 \in X_0$. Programs (7) and (8) satisfy strong duality ($p^* = d^*$) when the set $[0, T] \times X$ is compact (Theorem C.20 of [7]). The measure solution produces an upper bound $p^* \geq P^*$, and this bound is tight with $p^* = P^*$ when the set $[0, T] \times X$ is compact (Sec. 2.3 of [3] and [4]).

The work in [2] approximates Problems (7) and (8) by a discretized linear program over a fine mesh. The method in [3] bounds (8) with a sum-of-squares (SOS) strengthening of polynomial nonnegativity constraints. The SOS strengthening produces a converging sequence of upper bounds to $p^* = d^*$ when $[0, T] \times X$ is compact. Optimal trajectories can be localized by sublevel sets of $v(t, x)$ and $\mathcal{L}_f v(t, x)$ following the method in [3].

C. Moment-SOS Hierarchy

The α-moment of a measure μ for a multi-index $\alpha \in \mathbb{N}^n$ is $y_\alpha = \langle x^\alpha, \mu \rangle$. The moment sequence y is the infinite collection of moments $\{y_\alpha\}_{\alpha \in \mathbb{N}^n}$ of the measure μ. There exists a linear (Riesz) functional L_y converting a polynomial $p(x) \in \mathbb{R}[x]$ into a linear combination of moments in y,

$$L_y(p) = L_y\left(\textstyle\sum_{\alpha \in \mathbb{N}} p_\alpha x^\alpha\right) = \textstyle\sum_{\alpha \in \mathbb{N}} p_\alpha y_\alpha. \tag{9}$$

The moment matrix $\mathbb{M}[y]$ is a square symmetric matrix of infinite size and is indexed by monomials (α, β) as $\mathbb{M}[y]_{\alpha,\beta} = y_{\alpha+\beta}$ [7]. If a polynomial $p = \sum_\alpha p_\alpha x^\alpha$ with coefficients p_α is treated as a vector $\mathbf{p}$, evaluation of $\langle p(x)^2, \mu \rangle$ is equivalent to $\mathbf{p}^T \mathbb{M}[y] \mathbf{p}$ by the Riesz functional L_y. Nonnegativity of $\langle p(x)^2, \mu \rangle$ for all $p(x) \in \mathbb{R}[x]$ requires that $\mathbb{M}[y]$ is Positive Semidefinite (PSD).

A basic semialgebraic set $\mathbb{K} = \{x \mid g_i(x) \geq 0,\ i = 1, \ldots, N_c\}$ may be the support set for a measure $\mu \in \mathcal{M}_+(\mathbb{K})$. Because μ is supported over the region $\{x \mid g_i(x) \geq 0\}$, the evaluation $\langle p(x)^2 g_i(x), \mu \rangle$ is nonnegative for all polynomials $p(x) \in \mathbb{R}[x]$. The PSD localizing matrix associated with $g_i(x) \in \mathbb{R}[x]$ and the moment sequence y is,

$$\mathbb{M}[g_i y]_{\alpha,\beta} = \textstyle\sum_{\gamma \in \mathbb{N}^n} g_{i\gamma} y_{\alpha+\beta+\gamma}. \tag{10}$$

A necessary condition for a moment sequence y to correspond with moments of a representing measure on $\mathbb{K}$ is that $\mathbb{M}[y]$ and all $\mathbb{M}[g_i y]$ are PSD. This necessary condition is sufficient if $\mathbb{K}$ is Archimedean [9]. A degree-d finite truncation of these matrices keeps moments up to order $2d$, which are located in the upper-left corners of the infinite dimensional matrices. The quantities d_i can be defined for each constraint $g_i(x) \geq 0$ as $d_i = \lceil \deg(g_i)/2 \rceil$. The truncated moment matrix $\mathbb{M}_d[y]$ has size $\binom{n+d}{d}$ corresponding to the monomials of x with degree $\leq d$, and the localizing matrix $\mathbb{M}_{d-d_i}[g_i y]$ has size $\binom{n+d-d_i}{d-\lceil \deg(g_i)/2 \rceil}$. An infinite dimensional LP in measures may be posed with a polynomial objective $p(x)$ and m polynomial constraint functions $a_j(x) \in \mathbb{R}[x], \ \forall j = 1, \ldots, m$ with $b \in \mathbb{R}^m$ as,

$$p^* = \max_{\mu \in \mathcal{M}_+(X)} \langle p, \mu \rangle \tag{11a}$$

$$\langle a_j(x), \mu \rangle = b_j \qquad \forall j = 1, \ldots, m. \tag{11b}$$

The degree-d finite truncation of (11) is an LMI with an $\binom{n+2d}{2d}$-dimensional vector of moments y as a variable,

$$p_d^* = \max_y \textstyle\sum_\alpha p_\alpha y_\alpha \tag{12a}$$

$$\mathbb{M}_d(y) \succeq 0, \ \mathbb{M}_{d-d_i}(g_i y) \succeq 0 \quad \forall i = 1, \ldots, N_c \tag{12b}$$

$$\textstyle\sum_\alpha a_{j\alpha} y_\alpha = b_j \qquad \forall j = 1, \ldots, m. \tag{12c}$$

Increasing d results in a decreasing sequence of upper bounds $p_d^* \geq p_{d+1}^* \geq \ldots \geq p^*$, which is convergent if $\mathbb{K}$ is Archimedean. The refinement of upper bounds to (11) by LMIs of increasing complexity is the moment-SOS hierarchy [7]. The moment-SOS relaxation to the peak estimation program (7) is available in Equation (15) of [8], which is an LMI in moment sequences (y_0, y_p, y) up to degree $2d$ of the measures (μ_0, μ_p, μ). These moment relaxations are dual to the SOS programs in [3]. Near-optimal trajectories extremizing $p(x)$ may be recovered from LMI solutions if the moment matrices for μ_0, μ_p obey rank conditions [8].

III. Uncertainty Models

This section summarizes techniques for incorporating uncertainty into occupation-measure based frameworks, and briefly notes their application to peak estimation. The methods mentioned here arose from optimal control and the approximation of reachability sets. The two basic types of uncertainty are time-independent ($\theta \in \Theta$) and time-dependent ($w(t) \in W$). It is assumed that Θ and W are compact basic semialgebraic sets, just like X and X_0.

A. Time-Independent Uncertainty

Time-independent uncertainty θ_ℓ for $\ell = 1 \ldots N_\theta$ may take values in a set $\Theta \subseteq \mathbb{R}^{N_\theta}$, and typically arises in systems with parameter tolerances. The time-independent θ may start at any value in $\Theta \subset \mathbb{R}^{N_\theta}$ and is then constant along trajectories. By the methods in [5], [6], the state space may be extended into $X \times \Theta$ by adding new states θ with constant dynamics $\dot{\theta}_\ell = \mathcal{L}_f \theta_\ell = 0$ for each $\ell = 1 \ldots N_\theta$.

B. Time-Dependent Uncertainty

Systems with time-dependent uncertainty may have the noise process $w(t)$ change arbitrarily quickly in W over time t. Such bounded time-varying noise may be found in driving or piloting tasks with changing winds. The disturbance $w(t)$ is a Borel measurable function of time rather than the Itô-type stochastic process considered in [1]. For an input $w(t) \in W$ and a subset $D \subseteq W$, the disturbance-occupation measure $\mu^w(A \times B \times D)$ is,

$$\int_{[0,T] \times X_0} I_{A \times B \times D}((t, x(t), w(t)) \mid x_0) dt \, d\mu_0(x_0). \tag{13}$$

The disturbance $w(t)$ may be relaxed into a distribution $d\omega(w \mid x, t)$, which is known as a Young Measure [10], [4]. The disturbance-occupation measure μ^w can be disentangled into $d\mu^w(t, x, w) = dt \, d\xi(x \mid t) \, d\omega(w \mid x, t)$ for conditional distributions ξ, ω. Liouville's equation with a relaxed disturbance $\omega(w \mid x, t)$ influencing dynamics $f(t, x, w)$ for all $v(t, x) \in C^1([0, T] \times X)$ is,

$$\langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x), \mu^w \rangle. \tag{14a}$$

Equivalent expressions are formed by rearranging operators,

$$\langle v, \mu_p \rangle = \langle v, \delta_0 \otimes \mu_0 \rangle + \langle \mathcal{L}_f v, \mu^w \rangle \qquad \forall v, \tag{14b}$$

$$\langle v, \mu_p \rangle = \langle v, \delta_0 \otimes \mu_0 \rangle + \langle v, \mathcal{L}_f^\dagger \mu^w \rangle \qquad \forall v, \tag{14c}$$

$$\langle v, \mu_p \rangle = \langle v, \delta_0 \otimes \mu_0 + \pi_\#^{tx} \mathcal{L}_f^\dagger \mu^w \rangle \qquad \forall v. \tag{14d}$$

The measures of the two summands on the right hand side of (14c) reside in different spaces, as $\delta_0 \otimes \mu_0 \in \mathcal{M}_+([0, T] \times X)$ while $\mathcal{L}_f^\dagger \mu^w \in \mathcal{M}_+([0, T] \times X \times W)$. The (t, x)-marginalization $\pi_\#^{tx} \mathcal{L}_f^\dagger \mu^w \in \mathcal{M}_+([0, T] \times X)$ allows the measures to be added together inside the duality pairing in (14d). The duality pairings $\langle v(t, x), \mathcal{L}_f^\dagger \mu^w \rangle$ and $\langle v(t, x), \pi_\#^{tx} \mathcal{L}_f^\dagger \mu^w \rangle$ are equal for all $v \in C^1([0, T] \times X)$ because $v(t, x)$ is not a function of w. The weak disturbed Liouville's Equation is derived from (14d) by treating $\forall v(t, x) \in C^1([0, T] \times X)$ as implicit,

$$\mu_p = \delta_0 \otimes \mu_0 + \pi_\#^{tx} \mathcal{L}_f^\dagger \mu^w. \tag{15}$$

Time-varying disturbances may be incorporated into peak estimation by letting $\mu \in \mathcal{M}_+([0, T] \times X \times W)$ be a disturbance-occupation measure of the form in (13) obeying a disturbed Liouville equation (15). The support sets of the measures $\mu_0 \in \mathcal{M}_+(X_0), \ \mu_p \in \mathcal{M}_+([0, T] \times X)$ are unchanged when time-dependent uncertainty is added.

C. Switching Uncertainty

An approach for analyzing switched systems with occupation measures is presented in [6]. Let $\{X^k\}_{k=1}^{N_s}$ be a closed cover of X with N_s switching modes. The sets X^k are not necessarily disjoint, and together satisfy $\cup_k X^k = X$ (definition of closed cover). Each region X^k has dynamics $\dot{x} = f_k(t, x)$ for some locally Lipschitz vector field f_k. The closed cover formalism generalizes partitions of X (deterministic dynamics) and arbitrary switching where $X^k = X \ \forall k$ (polytopic uncertainty). Polytopic uncertainty is a model with dynamics $f(t, x, k) = \sum_k w_k f_k(t, x)$ where the disturbance $w_k \in \mathbb{R}_+^{N_s}$ satisfies $\sum_k w_k = 1$. Trajectories from a switching system are equipped with a function $S : [0, T] \to 1 \ldots, N_s$ yielding the resident subsystem at time t^-. Such a trajectory under switching may be written as $x(t \mid x_0, S(t))$. The switched measure program introduces an occupation measure $\mu_k \in \mathcal{M}_+([0,T] \times X^k)$ for each subsystem f_k,

$$\mu = \sum_k \mu_k \qquad \mathcal{L}^\dagger \mu = \sum_k \mathcal{L}_k^\dagger \mu_k. \tag{16}$$

A valid auxiliary function $v(t,x)$ from (8c) must decrease along all subsystems [11], [12]. Problem (8) may be modified for switching by enlarging constraint (8c) to,

$$\mathcal{L}_{f_k} v(t,x) \leq 0 \quad \forall (t,x) \in [0,T] \times X_k, \ k = 1 \dots N_s. \tag{17}$$

IV. Continuous-Time Uncertain Peak Estimation

This section combines the uncertainty formulations from section III to form a pair of primal-dual infinite-dimensional LPs. The variables $\theta \in \Theta, w \in W$ will respectively denote time-independent and time-dependent uncertainties of sizes N_θ, N_w. The dynamics f have N_s switching subsystems $f_k(t,x,\theta,w)$ which are valid in regions $X_k \subseteq X$.

A. Continuous-Time Measure Program

A combined uncertain peak estimation measure program is detailed in Program (18) with indices $k = 1, \dots, N_s$ for the switching subsystems,

$$\begin{aligned}
p^* = \max \quad & \langle p(x), \mu_p \rangle && \text{(18a)}\\
& \mu_p = \delta_0 \otimes \mu_0 + \textstyle\sum_k \pi_\#^{tx\theta} \mathcal{L}_{f_k}^\dagger \mu_k && \text{(18b)}\\
& \mu_0(X_0) = 1 && \text{(18c)}\\
& \mu_k \in \mathcal{M}_+([0,T] \times X \times \Theta \times W) \qquad \forall k && \text{(18d)}\\
& \mu_p \in \mathcal{M}_+([0,T] \times X \times \Theta) && \text{(18e)}\\
& \mu_0 \in \mathcal{M}_+(X_0 \times \Theta). && \text{(18f)}
\end{aligned}$$

Theorem 4.1: The solution p^* to program (18) will yield an upper bound to P^* in (1).

Proof: First assume $N_s = 1$ with $X^1 = X$, so there is only one switching domain. An optimal achievement of (1) reaching the peak value of P^* may be characterized by the tuple $(x_0^*, t^*, x_p^*, \theta^*, w^*(t))$. The peak value $p(x_p^*) = P^*$ is achieved by following the trajectory $x(t \mid x_0^*, \theta^*, w^*(t))$ until time $t = t^*$. Measures (μ_0, μ_p, μ) may be defined from this optimal tuple such that the measures satisfy constraints (18b)-(18f). The initial measure and peak measure may be set to $\mu_0 = \delta_{x=x_0^*}$ and $\mu_p = \delta_{t=t^*} \otimes \delta_{x=x_p^*} \otimes \delta_{\theta=\theta^*}$ based on the optimal tuple. The measure $\mu \in \mathcal{M}_+([0,T] \times X \times \Theta \times W)$ may be defined as the unique occupation measure satisfying,

$$\langle \tilde{v}, \mu \rangle = \int_{t=0}^{t^*} \tilde{v}(t, x(t \mid x_0^*, \theta^*, w^*(t)), \theta^*, w^*(t)) dt, \tag{19}$$

for all test functions $\tilde{v} \in C([0,T] \times X \times \Theta \times W)$. The measures (μ_0, μ_p, μ) satisfy constraints (18b)-(18f), so $p^* \geq P^*$ when $N_s = 1$.

Optimal trajectories arising from a system with $N_s > 1$ may be described in a tuple as $(x_0^*, t^*, x_p^*, \theta^*, w^*(t), S^*(t))$, where $S^*(t)$ is the sequence of switches undergone between times $t \in [0, t^*]$. The measures μ_0 and μ_p may remain the same as in the non-switched case. Switching occupation measures μ_k may be set to the unique occupation measure supported on the graph $(t, x(t \mid x_0^*, \theta^*, w^*(t)), \theta^*, w^*(t))$ between times $t \in [0, t^*]$ when $S(t) = k$. These occupation measures satisfy constraints (18b) and (18d), proving that there exists a feasible solution to (18b)-(18f) with objective P^* for the case of switching. ■

B. Continuous-Time Function Program

Dual variables $v(t,x,\theta) \in C^1([0,T] \times X \times \Theta)$ and $\gamma \in \mathbb{R}$ can be defined to find the Lagrangian of (18).

$$\begin{aligned}
\mathscr{L} = & \langle p(x), \mu_p \rangle + \langle v(t,x,\theta), \delta_0 \otimes \mu_0 + \textstyle\sum_k \pi_\#^{tx\theta} \mathcal{L}_{f_k}^\dagger \mu_k \rangle \\
& + \langle v(t,x,\theta), -\mu_p \rangle + \gamma(1 - \langle 1, \mu_0 \rangle).
\end{aligned}$$

The resulting dual program in (v, γ) is,

$$\begin{aligned}
d^* = \min_{\gamma \in \mathbb{R}} \quad & \gamma && \text{(20a)}\\
& \forall (x,\theta) \in X_0 \times \Theta : \\
& \quad \gamma \geq v(0,x,\theta) && \text{(20b)}\\
& \forall (t,x,\theta,w) \in [0,T] \times X_k \times \Theta \times W : \quad \forall k \\
& \quad \mathcal{L}_{f_k} v(t,x,\theta) \leq 0 && \text{(20c)}\\
& \forall (t,x,\theta) \in [0,T] \times X \times \Theta : \\
& \quad v(t,x,\theta) \geq p(x) && \text{(20d)}\\
& v(t,x,\theta) \in C^1([0,T] \times X \times \Theta). && \text{(20e)}
\end{aligned}$$

Theorem 4.2: There is no duality gap between (18) and (20) when the set $[0,T] \times X \times \Theta \times W$ is compact.

Proof: Necessary and sufficient conditions for there to be no duality gap between measure and function programs are if all measures are bounded and if the affine map is closed in the weak-* topology (Theorem C.20 of [7]). A measure is bounded if all of its finite-degree moments are bounded. Boundedness will hold if the mass of the measure is bounded and the support of the measure is compact. In (18) μ_0 and μ_p each have mass 1, and the mass of $\sum_k \mu_k \leq T$ by Liouville's equation. Compactness of $[0,T] \times X \times \Theta \times W$ therefore assures that all measures are bounded. The image of the affine map $(\mu_0, \mu_p, \mu_k) \to (\delta_0 \otimes \mu_0 + \sum_k \pi_\#^{tx\theta} \mathcal{L}_{fk}^\dagger \mu_k - \mu_p, \mu_0)$ induced by constraints (18b)-(18c) is closed in the weak-* topology. Strong duality therefore holds by closure and boundedness of measures. ■

The measure μ_0 has $N_x + N_\theta$ variables, and μ_p has $1 + N_x + N_\theta$ variables. The N_s occupation measures μ_k each have $1 + N_x + N_\theta + N_w$ variables. If the switching structure was not taken into account by the methods of section III-C, there would be a single occupation measure μ with $1 + N_x + N_\theta + N_w + N_s$ variables. The affine uncertainty structure breaks up the large μ (in terms of the number of variables) into N_s smaller measures (μ_k).

C. Continuous-Time LMI Relaxation

The compact (Archimedean) basic semialgebraic sets in the uncertain peak estimation setting are

$$\begin{aligned}
X &= \{x \mid g_i(x) \geq 0 && \mid \ i = 1, \dots, N_c\} && \text{(21a)}\\
X_0 &= \{x \mid g_{0i}(x) \geq 0 && \mid \ i = 1, \dots, N_c^0\} && \text{(21b)}\\
X^k &= \{x \mid g_{ki}(x) \geq 0 && \mid \ i = 1, \dots, N_c^k\} && \text{(21c)}\\
\Theta &= \{\theta \mid g_{\theta i}(\theta) \geq 0 && \mid \ i = 1, \dots, N_c^\theta\} && \text{(21d)}\\
W &= \{w \mid g_{wi}(w) \geq 0 && \mid \ i = 1, \dots, N_c^w\}. && \text{(21e)}
\end{aligned}$$

The localizing matrix degree bound d_i is $d_i = \lceil \deg(g_i)/2 \rceil$, and similar quantities d_{0i}, $d_{\theta i}$, d_{wi}, d_{ki} are defined on corresponding polynomials. Monomials forming moments may be indexed as $x^\alpha t^\beta \theta^\gamma w^\eta$ for multi-indices $\alpha \in \mathbb{N}^{N_x}$, $\beta \in \mathbb{N}$, $\gamma \in \mathbb{N}^{N_\theta}$, $\eta \in \mathbb{N}^{N_w}$. Define $y^0 = \{y^0_{\alpha\gamma}\}$, $y^p = \{y^p_{\alpha\beta\gamma}\}$ as the moment sequences for measures μ_0 and μ_p. The moment sequence for the occupation measure μ^k is $y^k = \{y^k_{\alpha\beta\gamma\eta}\}$ for each switching subsystem k. The Liouville equation (18b) with test function $v(t, x, \theta) = x^\alpha t^\beta \theta^\gamma$ has the form,

$$\begin{aligned} 0 = &\langle x^\alpha t^\beta \theta^\gamma, \delta_0 \otimes \mu_0 \rangle - \langle x^\alpha t^\beta \theta^\gamma, \mu_p \rangle \\ &+ \textstyle\sum_k \langle \mathcal{L}_{f_k(t,x,\theta,w)}(x^\alpha t^\beta \theta^\gamma), \mu_k \rangle. \end{aligned} \tag{22}$$

Define the operator $\text{Liou}_{\alpha\beta\gamma}(y^0, y^p, y^k)$ as the linear relation between the moment sequences induced by (22) assuming that each f_k is a polynomial vector field. Given a degree d, define the degrees d'_k as $d + \lceil \deg(f_k)/2 \rceil - 1$ for each k. The degree-d LMI relaxation of the uncertain peak estimation problem in (18) resulting in an upper bound $p^*_d \geq P^*$ is,

$$\begin{aligned}
p^*_d = &\max \quad \textstyle\sum_\alpha p_\alpha y^p_{\alpha 00} && \text{(23a)}\\
&\forall |\alpha| + |\beta| + |\gamma| \leq 2d : && \\
&\quad \text{Liou}_{\alpha\beta\gamma}(y^0, y^p, y^k) = 0 \quad \text{by (22)} && \text{(23b)}\\
&y^0_0 = 1 && \text{(23c)}\\
&\mathbb{M}_d(y^0), \mathbb{M}_d(y^p), \forall k : \mathbb{M}_{d'_k}(y^k) \succeq 0 && \text{(23d)}\\
&\mathbb{M}_{d-1}(t(T-t)y^p) \succeq 0 && \text{(23e)}\\
&\forall k : \mathbb{M}_{d'_k - 2}(t(T-t)y^k) \succeq 0 && \text{(23f)}\\
&\forall i = 1, \ldots, N^0_c : && \\
&\quad \mathbb{M}_{d-d_{0i}}(g_{0i} y^0) \succeq 0 && \text{(23g)}\\
&\forall i = 1, \ldots, N^\theta_c : && \\
&\quad \mathbb{M}_{d-d_{\theta i}}(g_{\theta i} y^0),\ \mathbb{M}_{d-d_{\theta i}}(g_{\theta i} y^p) \succeq 0 && \text{(23h)}\\
&\quad \forall k : \mathbb{M}_{d'_k - d_{\theta i}}(g_{\theta i} y^k) \succeq 0 && \text{(23i)}\\
&\forall i = 1, \ldots, N_c : && \\
&\quad \mathbb{M}_{d-d_i}(g_i y^p) \succeq 0 && \text{(23j)}\\
&\forall k,\ \forall i = 1, \ldots, N^k_c : && \\
&\quad \mathbb{M}_{d'_k - d_{ki}}(g_{ki} y^k) \succeq 0 && \text{(23k)}\\
&\forall i = 1, \ldots, N^w_c : && \\
&\quad \forall k : \mathbb{M}_{d-d_{wi}}(g_{wi} y^k) \succeq 0. && \text{(23l)}
\end{aligned}$$

Constraints (23d)- (23l) are moment and localizing matrix PSD constraints ensuring that there exist representing measures to the moment sequences (y^0, y^p, y^k) supported on the appropriate spaces. The sequence $\{p^*_d\}$ will converge to p^* monotonically from above as $d \to \infty$ if all sets in (21) are Archimedean [7].

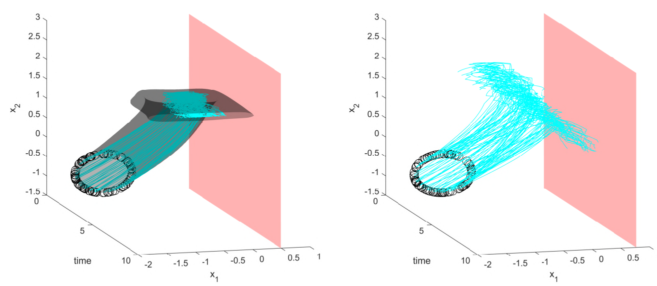
D. Continuous-Time Uncertain Examples

Code is available at `github.com/jarmill/peak`, and is written in Matlab R2020a using Gloptipoly3 [13], YALMIP [14], and Mosek 9.2 [15] to formulate and solve LMIs. Demonstrations are available in the folder `peak/experiments_uncertain` and are run here on an Intel i9 CPU at 2.30 GHz with 64.0 GB of RAM.

Dynamics based on Example 1 of [16] (adding w) are,

$$\dot{x}(t) = \begin{bmatrix} -0.5x_1 - (0.5 + w(t))x_2 + 0.5 \\ -0.5x_2 + 1 + \theta \end{bmatrix}. \tag{24}$$

Figure 1 illustrates maximization of $p(x) = x_1$ starting in $X_0 = \{x \mid (x_1+1)^2 + (x_2+1)^2 \leq 0.25\}$ for time $t \in [0, 10]$. The admissible disturbances $w(t)$ are in $w = [-0.2, 0.2]$. Fig. 1a has $\Theta = 0$ while Fig. 1b has $\Theta = [-0.5, 0.5]$ for the time-independent uncertainty $\theta \in \Theta$. In each figure, the black circles are initial conditions from the boundary of X_0, the blue curves are sampled trajectories, and the red plane are level sets for upper bounds of x_1 along trajectories. At the order $r = 4$ LMI relaxation, Fig. 1a yields a bound of $P^* \leq 0.4925$ while Fig. 1b with θ results in $P^* \leq 0.7680$. The black surface containing all trajectories in Fig. 1a is the level set $\{(t, x) \mid v(t, x) = 0.4925\}$.



(a) $\theta = 0$ (b) $\theta \in [-0.5, 0.5]$

Fig. 1: Maximize x_1 at order 4 with $w(t) \in w$

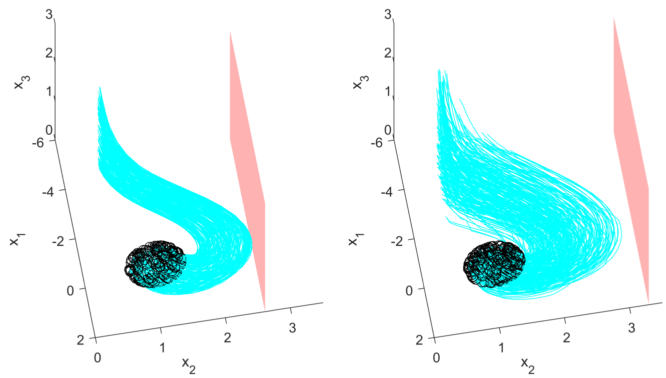
The reduced three-wave model is a nonlinear model for the interaction of three quasisynchronous waves in a plasma [17]. These dynamics with parameters (A, B, G) are,

$$\begin{aligned} \dot{x_1} &= Ax_1 + Bx_2 + x_3 - 2x_2^2 \\ \dot{x_2} &= -Bx_1 + Ax_2 + 2x_1x_2 \\ \dot{x_3} &= -Gx_3 - 2x_1x_2. \end{aligned} \tag{25}$$

This example aims to maximize x_2 on the three-wave system starting in $X_0 = \{x \mid (x_1+1)^2 + (x_2+1)^2 + (x_3+1)^2 \leq 0.16\}$. Order 3 LMI relaxations are used to upper bound x_2 over the region of interest $X = [-4, 3] \times [0.5, 3.6] \times [0, 4]$ and times $t \in [0, 5]$. The bound $P^* \leq 2.6108$ is produced with parameter values $A = 1$, $B = 0.5$, $G = 2$ (no uncertainty), as illustrated in Fig. 2a. Fig. 2b adds uncertainty by letting $A \in [-0.5, 1.5]$ and $B \in [0.25, 0.75]$ vary arbitrarily with time, and G now possesses parametric uncertainty in $[1.9, 2.1]$. Uncertainty in A, B are realized by switching between 4 subsystems of (25) with $(A, B) \in \{0.5, 1.5\} \times \{0.25, 0.75\}$ Uncertainty in G is implemented as $G = 2 + \theta$ where $\theta \in [-0.1, 0.1]$. The order-3 bound under uncertainty in Fig. 2b is $P^* \leq 3.296$.

V. Discrete-Time Uncertain Peak Estimation

Uncertain peak estimation can be extended to discrete-time systems, including switched discrete-time systems. A discrete-time system from times $t = 0, 1, \ldots, T$ is considered for dynamics $x_+ = f(x)$ where x_+ is the next state.



(a) no uncertainty (b) with uncertainty

Fig. 2: Maximize x_2 on three-wave system (25)

A trajectory starting at the initial condition $x_0 \in X_0$ is $x_t(x_0)$ The uncertain peak estimation problem for discrete-time systems with uncertainties (θ, w_t) and N_s subsystems with switching sequence S_t is,

$$\begin{aligned} P^* = &\max_{t,\, x_0 \in X_0,\, \theta \in \Theta,\, w_t,\ S_t} p(x_t(x_0, \theta, w_t, S_t)) && (26) \\ &x_+ = f_k(x_t, \theta, w_t) \text{ if } S_t = k \\ &w_t \in W,\ S_t \in 1, \ldots, N_s \qquad \forall t \in 0, \ldots, T. \end{aligned}$$

A. Discrete-Time Measure Background

Just as the Lie derivative $\mathcal{L}_f v$ yields the infinitesimal change in v along continuous trajectories, the quantity $v(f(x)) - v(x)$ is the change in v along a single discrete-time step. The discrete-time occupation measure for sets $A \subseteq X$ with initial conditions distributed as $\mu_0 \in \mathcal{M}_+(X_0)$ is,

$$\mu(A) = \int_{X_0} \sum_{t=0}^{T} I_A(f^t(x_0)) d\mu_0. \tag{27}$$

The quantity $\mu(A)$ is the averaged number of time steps that trajectories distributed as μ_0 spend in A. For measures $\mu_0 \in \mathcal{M}_+(X_0)$, $\mu_p \in \mathcal{M}_+(X)$, $\mu \in \mathcal{M}_+(X)$, the strong and weak discrete-time Liouville equations for all v are:

$$\langle v(x), \mu_p \rangle = \langle v(x), \mu_0 \rangle + \langle v(f(x)), \mu \rangle - \langle v(x), \mu \rangle, \tag{28}$$
$$\mu_p = \mu_0 + f_\# \mu - \mu. \tag{29}$$

Time may be optionally included in system dynamics by setting a state $t_+ = t+1$ and incorporating t into dynamics. The pushforward term in (29) would then be $v(t+1, f(t,x)) - v(t,x)$. Discrete-time systems with uncertainties (θ, w) have dynamics and Liouville equations according to,

$$x_+ = f(x_t, \theta, w_t), \qquad \mu_p = \mu_0 + \pi^{x\theta}_\# (f_\# \mu - \mu). \tag{30}$$

The uncertainty $\theta \in \Theta$ is fixed, and the time-dependent uncertainty has $w_t \in W$ for every time step $t = 0, \ldots, T$. Switching uncertainty from Section III-C with subsystems f_k valid over X_k may be realized by defining occupation measures $\mu_k \in \mathcal{M}_+(X_k \times \Theta \times W)$ such that $\mu = \sum_k \mu_k$.

B. Discrete-Time Measure Program

A measure program may be formulated to upper bound the peak-estimation task on discrete-time systems with uncertainties (θ, w) and switching between dynamics f_k over X_k. The uncertain discrete-time peak estimation measure problem with variables (μ_0, μ_k, μ_p) is,

$$\begin{aligned} p^* = \max\ & \langle p(x), \mu_p \rangle && \text{(31a)} \\ & \mu_p = \mu_0 + \pi^{x\theta}_\# \left(\textstyle\sum_k (f_{k\#}\mu_k - \mu_k)\right) && \text{(31b)} \\ & \mu_0(X_0) = 1 && \text{(31c)} \\ & T \geq \textstyle\sum_k \langle 1, \mu_k \rangle && \text{(31d)} \\ & \mu_k \in \mathcal{M}_+(X_k \times \Theta \times W) \qquad \forall k && \text{(31e)} \\ & \mu_p \in \mathcal{M}_+(X \times \Theta) && \text{(31f)} \\ & \mu_0 \in \mathcal{M}_+(X_0 \times \Theta). && \text{(31g)} \end{aligned}$$

Remark 1: The composition of pushforwards in (31b) acts as $\langle v(x,\theta), \pi^{x\theta}_\# f_{k\#}\mu_k \rangle = \langle v(f_k(x,\theta,w),\theta), \mu_k \rangle$ for all test functions $v(x,\theta) \in C(X \times \Theta)$.

Theorem 5.1: The optimum p^* of (31) is an upper bound for P^* from discrete-time program (26).

Proof: This proof follows the same steps as the proof to theorem 4.1. A trajectory achieving a peak value of P^* solving (26) may be expressed as a tuple $(t^*, x_0^*, x_p^*, \theta^*, w_t^*, S_t^*)$ with $P^* = p(x_p^*) = p(x_{t^*}(x_0^*, \theta^*, w_t))$. Measures may be defined from this tuple to solve problem (31). The probability distributions are $\mu_0 = \delta_{x=x_0^*}$ and $\mu_p = \delta_{x=x_p^*} \otimes \delta_{\theta=\theta^*}$. Switching measures μ_k may be chosen as the unique occupation measures satisfying,

$$\langle \tilde{v}_k, \mu_k \rangle = \sum_{t=0}^{t^*} \tilde{v}(x_t(x_0^*, \theta^*, w_t^*), \theta^*, w_t^*) I(S_t = k), \tag{32}$$

for all test functions $\tilde{v}_k \in C(X_k \times \Theta \times W)$ and for each $k = 1, \ldots, N_s$. The measures (μ_0, μ_p, μ_k) are feasible solutions to (31b)-(31g) with objective value $P^* = p(x_p^*) = \langle p(x), \mu_p \rangle$, so $p^* \geq P^*$ is a valid upper bound to (26). ∎

Remark 2: Constraint (31d) is a technique from [18] ensuring that the maximal time in optimization is T and that each μ_k has a bounded mass.

C. Discrete-Time Function Program

With dual variables $(v(x,\theta) \in C(X \times \Theta), \gamma \in \mathbb{R})$ and a new dual variable $\alpha \geq 0$, the Lagrangian of (31) is,

$$\begin{aligned} \mathscr{L} = &\langle p(x), \mu_p \rangle + \langle v(x,\theta), \mu_0 - \mu_p \rangle + \alpha(T - \langle 1, \textstyle\sum_k \mu_k \rangle) \\ &+ \langle v(x,\theta), \pi^{x\theta}_\# \textstyle\sum_k f_{k\#}\mu_k - \mu_k \rangle + \gamma(1 - \langle 1, \mu_0 \rangle). \end{aligned}$$

The corresponding dual problem is,

$$\begin{aligned} d^* = &\min_{\gamma \in \mathbb{R},\ \alpha \geq 0} \quad \gamma + T\alpha && \text{(33a)} \\ &\forall (x,\theta) \in X_0 \times \Theta: \\ &\quad \gamma \geq v(x,\theta) && \text{(33b)} \\ &\forall (x,\theta,w) \in X_k \times \Theta \times W: \quad \forall k \\ &\quad v(f_k(x,\theta,w),\theta) - v(x,\theta) \leq \alpha && \text{(33c)} \\ &\forall (x,\theta) \in X \times \Theta: \\ &\quad v(x,\theta) \geq p(x) && \text{(33d)} \\ &v(x,\theta) \in C(X \times \Theta). && \text{(33e)} \end{aligned}$$

Theorem 5.2: Strong duality $p^* = d^*$ holds between (31) and (33) if $T < \infty$ and $X \times \Theta \times W$ is compact.

Proof: This is affirmed by a similar process to Theorem 4.2. All measures have bounded finite moments given that their masses are bounded and their supports are compact. The image of the affine map in constraints (31b)-(31c) is closed in the weak-* topology, concluding the conditions for strong duality by Theorem C.20 of [7]. ■

Remark 3: The LMI for Discrete-Time Uncertain Peak Estimation is similar to LMI (23) in continuous time. If time is not included, support constraints (23e)-(23f) are unnecessary. The Liouville operator $\text{Liou}_{\alpha\gamma}(y^0, y^p, y^k)$ is now the affine relation in moments induced from Liouville's Equation (31b) for every test function $v(x, \theta) = x^\alpha \theta^\gamma$,

$$0 = \langle x^\alpha \theta^\gamma, \delta_0 \otimes \mu_0 \rangle - \langle x^\alpha \theta^\gamma, \mu_p \rangle + \textstyle\sum_k \langle (f_k(x, \theta, w)^\alpha \theta^\gamma - x^\alpha \theta^\gamma, \mu_k \rangle. \tag{34}$$

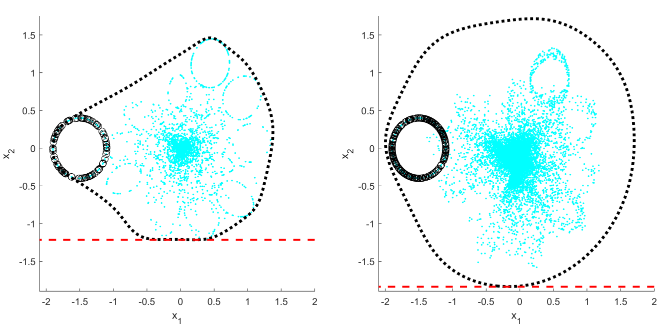
D. Discrete Example

An example to demonstrate uncertain discrete peak estimation is to minimize x_2 on the following subsystems,

$$f_1(x, w) = \begin{bmatrix} -0.3x_1 + 0.8x_2 + 0.1x_1x_2 \\ -0.75x_1 - 0.3x_2 + w \end{bmatrix} \tag{35a}$$

$$f_2(x, w) = \begin{bmatrix} 0.8x_1 + 0.5x_2 - 0.01x_1^2 \\ -0.5x_1 + 0.8x_2 - 0.01x_1x_2 + w \end{bmatrix}. \tag{35b}$$

The space under consideration is $X = [-3, 3]^2$, and the time varying uncertainty w_t satisfies $w_t \in [-0.2, 0.2] = \Delta$. The valid regions for subsystems of (35) are $X_1 = X$ and $X_2 = X \cap (x_1 \geq 0)$. When $x_1 \geq 0$ the system may switch arbitrarily between dynamics f_1 and f_2, but when $x_2 < 0$, the system only follows dynamics f_1. Figure 3 visualizes minimizing x_2 starting from the initial set $X_0 = \{x \mid (x_1 + 1.5)^2 + x_2^2 = 0.16\}$ between discrete times $t \in 0, \ldots, T$ with $T = 50$. A fourth order LMI relaxation of (31a) is solved aiming to maximize $p(x) = -x_2$. With $w = 0$ in Fig. 3a the bound is $P^* \leq 1.215$ ($\min x_2 \geq -1.215$), while the time varying w in Fig. 3b yields a bound of $P^* \leq 1.837$.



(a) $w_t = 0$ (b) $w_t \in [-0.2, 0.2]$

Fig. 3: Minimize x_2 on system (35)

VI. Conclusion

The problem of peak estimation with uncertainty may be bounded by the optimal value of an infinite-dimensional LP in occupation measures. This LP is then approximated by the moment-SOS hierarchy and Linear Matrix Inequalities. Time-independent and time-dependent uncertainties are incorporated into this measure framework for continuous-time and discrete-time systems. Future work includes uncertain peak estimation for safety verification and hybrid systems, and also exploiting specialized uncertainty structures.

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