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Robustness of Model Predictive Control to (Large) Discrete Disturbances ⋆

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Abstract: In recent years, theoretical results for model predictive control (MPC) have been expanded to address discrete actuators (decisions) and high-level planning and scheduling problems. The application of MPC-style methods to scheduling problems has been driven, in part, by the robustness afforded by feedback. The ability of MPC, and feedback methods in general, to reject small persistent disturbances is well-recognized. In many planning and scheduling applications, however, we must also consider an additional class of *discrete and infrequent* disturbances, such as breakdowns and unplanned maintenance. In this paper, we establish that nominal MPC is robust, in a stochastic context, to this class of discrete and infrequent disturbances. We illustrate these results with a nonlinear blending example.

1. INTRODUCTION

For successful implementation, model predictive control (MPC) must be robust to disturbances. While robust or stochastic MPC algorithms can be used to directly consider disturbances in the optimal control problem, the *inherent robustness* of nominal MPC (i.e., the robustness obtained solely through feedback) is often sufficient in practice. Thus, the inherent robustness of nominal MPC is a subject that has received significant attention. Inherent robustness is typically characterized by robust asymptotic (or exponential) stability of the closed-loop system subject to sufficiently small and persistent disturbances (Allan et al., 2017; Grimm et al., 2004). For many process control applications, this characterization of disturbances is reasonable. Nonlinear MPC is inherently robust if the optimal cost function satisfies certain continuity assumptions (Pannocchia et al., 2011). These results extend to MPC implementations with compact input constraints (including integrality constraints) and discontinuous optimal cost functions with specific restrictions on the terminal region and cost (Allan et al., 2017; Yu et al., 2014).

By extending theoretical results for MPC to include discrete actuators (inputs), higher-level planning and scheduling problems with numerous discrete-valued decision variables are within the purview of MPC (Rawlings and Risbeck, 2017). In these scheduling problems, the most pertinent class of disturbances to consider is not small and persistent, but discrete and infrequent (e.g., equipment breakdowns, maintenance, or delays). Typically, these discrete disturbances are large and constructing a bound for the worst deterministic performance possible, e.g., the entire facility is broken or under repair, leads to an excessively conservative bound that offers little insight. Instead, we intend to exploit the infrequent nature of these disturbances and characterize their robustness with a stochastic form of robust exponential stability.

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The notion of stochastic stability for nonlinear systems has been developed and refined over several decades (Florchinger, 1995; Kushner, 1967). Teel and co-workers provide a modern treatment of these topics and establish that stochastic Lyapunov functions ensure uniform asymptotic convergence (Teel, 2013; Teel et al., 2012). Analogous to input-to-state stability (ISS) for deterministic systems, stochastic input-to-state stability (SISS) was also defined (Tsinias, 1998; Krstic and Deng, 1998; Tang and Basar, 2001; Liu et al., 2008). However, unlike these works, we do not assume that the effect of the stochastic disturbance vanishes as the state of the system approaches the origin.

In this work, we present a stochastic definition of robustness for nonlinear systems, robust exponential stability in expectation (RESiE), subject to discrete and infrequent disturbances. We define an exponential SISS-Lyapunov function and establish that any system that admits an exponential SISS-Lyapunov function is RESiE. We then establish that *nominal* MPC under the typical assumptions required for nominal stability and an additional assumption of robust recursive feasibility is RESiE with respect to discrete and infrequent disturbances. We conclude with a nonlinear blending example to demonstrate the implications of this analysis.

Notation. Let $\mathbb{I}$ denote integers, $\mathbb{R}$ denote reals, and subscripts on these sets denote restrictions (e.g. $\mathbb{I}_{\geq 0}$ for nonnegative integers). The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. We use $|\cdot|$ to denote Euclidean norm. Sequences are denoted in bold face and we use subscripts to indicate the length of the sequence if the length is ambiguous from context (e.g. $\mathbf{w}_k$ indicates the sequence of w's from w_0 to w_k). We use $\Pr(A)$ to denote the probability of event A.

2. MODEL PREDICTIVE CONTROL

We consider a discrete-time system of the following form

$$x^+ = f(x, u, w) \tag{1}$$

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defined for the state $x \in \mathbb{X} \subseteq \mathbb{R}^n$, input $u \in \mathbb{U} \subseteq \mathbb{R}^m$, and disturbance $w \in \mathbb{W} \subseteq \mathbb{R}^p$. The successor state is denoted by x^+. The system is subject to state and input constraints of the form $(x, u) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$ and a terminal constraint $\mathbb{X}_f \subseteq \mathbb{X}$. The nominal system is described by

$$x^+ = f(x, u, 0) \tag{2}$$

For the current state x and input sequence $\mathbf{u}$, the function $\hat{\phi}(k; x, \mathbf{u})$ denotes the open-loop state solution to the nominal system (2) after $k \in \mathbb{I}_{\geq 0}$ steps.

For a horizon of length N, we define the set of feasible $(x, \mathbf{u})$ pairs $\mathcal{Z}_N$, the set of feasible $\mathbf{u}$ for a given state $\mathcal{U}_N(x)$, and the set of admissible initial states $\mathcal{X}_N$ as

$$\begin{aligned}\mathcal{Z}_N &= \{(x, \mathbf{u}) : (\hat{\phi}(k; x, \mathbf{u}), u(k)) \in \mathbb{Z}\ \forall k \in \mathbb{I}_{0:N-1} \\ &\qquad \hat{\phi}(N; x, \mathbf{u}) \in \mathbb{X}_f\} \\ \mathcal{U}_N(x) &= \{\mathbf{u} : (x, \mathbf{u}) \in \mathcal{Z}_N\} \\ \mathcal{X}_N &= \{x : \mathcal{U}_N(x) \neq \emptyset\}\end{aligned}$$

We define the stage cost $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, terminal cost $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and the controller's objective function

$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(\hat{\phi}(k; x, \mathbf{u}), u(k)) + V_f(\hat{\phi}(N; x, \mathbf{u}))$$

The optimal control problem for $x \in \mathcal{X}_N$ is defined as

$$\mathbb{P}_N(x) : V_N^0(x) = \min_{\mathbf{u} \in \mathcal{U}_N(x)} V_N(x, \mathbf{u})$$

and we denote the optimal solution(s) to $\mathbb{P}_N(x)$ as $\mathbf{u}^0(x)$. The MPC control law $\kappa_N(x) := u^0(0; x)$ is defined as the first input in $\mathbf{u}^0(x)$.[1] Note that this control law is based solely on the *nominal* system model, i.e., $f(x, u, 0)$, and does not consider uncertainty within the optimization problem.

For the controlled system, the state evolves according to

$$x^+ = f_{cl}(x, w) = f(x, \kappa_N(x), w) \tag{3}$$

We define the closed-loop state solution to (3) at time $k \in \mathbb{I}_{\geq 0}$ as $\phi(k; x, \mathbf{w}_k)$ given the initial condition x at $k = 0$ and the disturbance sequence $\mathbf{w}_k = (w_0, \ldots, w_{k-1})$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space for the random sequence $\mathbf{w} : \Omega \to \mathbb{W}^\infty$ of independent, identically distributed random variables, i.e, $\mathbf{w} = \{w_i\}_{i=0}^\infty$ for $w_i : \Omega \to \mathbb{W}$. In particular, we have the probability measure $\Pr(w_i \in F) = P(\{\omega \in \Omega : w_i(\omega) \in F\})$, i.e., the probability that w_i is in the Borel measurable set F. From the i.i.d. property, each random variable has the same probability measure $\mu : 2^{\mathbb{W}} \to [0, 1]$ defined as $\mu(F) = P(\{\omega \in \Omega : w_i(\omega) \in F\})$. Since we are interested in discrete disturbances, we assume that $\mathbb{W}$ is a countable set and therefore Ω is a countable set as well. Thus, we may define expected value as

$$\mathbb{E}\left[g(\phi(k; x, \mathbf{w}_k))\right] = \sum_{\omega \in \Omega} g(\phi(k; x, \mathbf{w}_k(\omega)))P(\omega)$$

in which $g : \mathbb{X} \to \mathbb{R}$ is a lower-bounded function.[2] We also define conditional expected value given $x(k)$ as

$$\begin{aligned}\mathbb{E}_{|x(k)}\left[g(f_{cl}(x(k), w_k))\right] &= \sum_{\omega \in \Omega} g(f(x(k), w_k(\omega)))P(\omega) \\ &= \sum_{w \in \mathbb{W}} g(f(x(k), w_k))\mu(w_k)\end{aligned}$$

in which $g : \mathbb{X} \to \mathbb{R}$ is a lower-bounded function and $x(k) = \phi(k; x, \mathbf{w}_k)$.

We consider the following typical assumptions for nominal MPC (Rawlings et al., 2020, sec. 2.2, 2.4).

Assumption 1. (Continuity of system and cost). The model $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$, stage cost $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and terminal cost $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous. The function $\ell(x, u)$ is lower bounded for all $(x, u) \in \mathbb{Z}$. Furthermore, we have that $f(0, 0, 0) = 0$, $\ell(0, 0) = 0$, and $V_f(0) = 0$.

Assumption 2. (Properties of constraint set). The sets $\mathbb{Z}$ and $\mathbb{X}_f \subseteq \mathbb{X}$ are closed and contain the origin. The set $\mathbb{U}$ is compact and contains the origin.

Assumption 3. (Terminal control law). There exists a terminal control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that for all $x \in \mathbb{X}_f$, $(x, \kappa_f(x)) \in \mathbb{Z}$, $f(x, \kappa_f(x), 0) \in \mathbb{X}_f$, and

$$V_f(f(x, \kappa_f(x), 0)) \leq V_f(x) - \ell(x, \kappa_f(x))$$

Assumption 4. (Exponential cost bounds). There exist constants $a \geq 1$, $c_1, c_2 > 0$ such that

$$\ell(x, u) \geq c_1 |x|^a \tag{4}$$

$$V_N^0(x) \leq c_2 |x|^a \tag{5}$$

for all $(x, u) \in \mathbb{Z}$ and $x \in \mathcal{X}_N$.

Assumptions 1 and 2 are sufficient to establish that the optimization problem $\mathbb{P}_N(x)$ is well-defined (Rawlings et al., 2020, Proposition 2.4). We note compact $\mathbb{U}$ admits integrality constraints, e.g., $u \in \mathbb{U} := \{0, 1\}$ is allowed. Thus, discrete-valued inputs can be considered without additional modifications. The addition of Assumption 3 ensures that the nominal closed-loop system (i.e., $w = 0$) satisfies the typical optimal cost decrease condition (Rawlings et al., 2020, pp. 116-117).

Lemma 5. If Assumptions 1-3 hold, then

$$V_N^0(f_{cl}(x, 0)) \leq V_N^0(x) - \ell(x, \kappa_N(x)) \tag{6}$$

and $f_{cl}(x, 0) \in \mathcal{X}_N$ for all $x \in \mathcal{X}_N$.

For economic MPC, in which economic performance (determined by the stage cost $\ell(\cdot)$) is paramount relative to stability, Assumption 3 is sufficient to achieve an asymptotic average performance guarantee (Amrit et al., 2011, Theorem 18). The addition of Assumption 4 ensures that the origin is exponentially stable for the nominal closed-loop system (Rawlings et al., 2020, pp. 114-119).

The exponential upper bound for the optimal cost function can be difficult to directly verify for nonlinear systems. For compact $\mathcal{X}_N$, we can verify this upper bound using properties of the terminal set and terminal cost (Rawlings et al., 2020, p. 141). The exponential upper bound for $V_N^0(x)$, however, may also hold for unbounded $\mathcal{X}_N$ (e.g., stable linear systems) so we include the more general version in Assumption 4.

3. ROBUSTNESS TO DISCRETE DISTURBANCES

Before we derive properties for the closed-loop trajectory, we must first ensure that the MPC optimization problem

[1] If there are multiple solutions to $\mathbb{P}_N(x)$, we assume some selection rule is applied such that $\kappa_N(x)$ is a single-valued mapping. We note that subsequent results hold for any such selection rule.

[2] By restricting our attention to countable Ω, we ensure that expected value is well-defined without verifying that $\hat{\phi}(k; x, \mathbf{w})$ is a measurable function.

and therefore the closed-loop trajectory is well-defined for all potential realizations of the disturbance. For sufficiently small disturbances, we can establish that MPC is robustly recursively feasible, i.e, the optimization problem remains feasible along the closed-loop trajectory for all $w \in \mathbb{W}$ (Allan et al., 2017). For discrete disturbances that have a large effect on the state of the system, we cannot apply this approach. Instead, we are required to make the following assumption.

Assumption 6. (Recursive feasibility). The set $\mathcal{X}_N$ is robustly positive invariant for the system $x^+ = f_{cl}(x,w)$; $w \in \mathbb{W}$, i.e., if $x \in \mathcal{X}_N$ then $f_{cl}(x,w) \in \mathcal{X}_N$ for all $w \in \mathbb{W}$.

Clearly, Assumption 6 does not apply to all systems and disturbances. We therefore focus on a class of systems and disturbances that do not render the optimization problem infeasible, but cannot be assumed to be arbitrarily small. In general, we must ensure the MPC problem is robustly recursively feasible *by design* to consider discrete disturbances. Nonetheless, there exist many applications of MPC that admit this assumption. Many higher-level applications of MPC that may encounter discrete disturbances, such as production scheduling, are robustly recursively feasible if sufficiently long horizons and reasonable state constraints are used. We present such an example in Section 4. In addition to robust recursive feasibility, we also require a cost increase bound for the closed-loop system.

Assumption 7. (Maximum cost increase). There exist constants $b_1, b_2 \in \mathbb{R}_{\geq 0}$ such that

$$V_N^0(f_{cl}(x,w)) \leq V_N^0(x) + b_1\ell(x,\kappa_N(x)) + b_2 \tag{7}$$

for all $x \in \mathcal{X}_N$ and $w \in \mathbb{W}$.

This bound is notably weaker than the bound required for an ISS-Lyapunov function and the optimal cost may *increase* proportional to the current stage cost. Indeed, the increase in optimal cost may even *grow* as x increases for an equivalent disturbance. While Assumption 7 is difficult to verify for an arbitrary system, we can apply the following lemma.

Lemma 8. Let Assumptions 1-6 hold. If there exists $e_1, e_2 \geq 0$ such that

$$|f(x,u,w) - f(x,u,0)| \leq e_1|x| + e_2 \tag{8}$$

for all $(x,u) \in \mathbb{Z}$ and $w \in \mathbb{W}$, then Assumption 7 holds.

Proof. Choose $x \in \mathcal{X}_N$ and $w \in \mathbb{W}$. From (5) and (8), we have

$$\begin{aligned} V_N^0(f_{cl}(x,w)) &\leq c_2|f_{cl}(x,w)|^a \\ &\leq c_2\left(|f_{cl}(x,0)| + |f_{cl}(x,w) - f_{cl}(x,0)|\right)^a \\ &\leq c_2\left(|f_{cl}(x,0)| + e_1|x| + e_2\right)^a \\ &\leq 2^a c_2|f_{cl}(x,0)|^a + 2^a c_2 e_1^2|x|^a + c_2(2e_2)^a \end{aligned} \tag{9}$$

From (4) and $V_f(\cdot) \geq 0$, we have $c_1|x|^a \leq V_N^0(x)$ for all $x \in \mathcal{X}_N$. We use this bound with (5) and (6) to give

$$\begin{aligned} |f_{cl}(x,0)|^a &\leq \frac{1}{c_1}V_N^0(f_{cl}(x,0)) \\ &\leq \frac{1}{c_1}V_N^0(x) - |x|^a \leq \frac{c_2 - c_1}{c_1}|x|^a \end{aligned} \tag{10}$$

We substitute (10) into (9) to get the bound

$$V(f_{cl}(x,w)) \leq \tilde{b}_1|x|^a + b_2 = c_1|x|^a + (\tilde{b}_1 - c_1)|x|^a + b_2$$

in which $\tilde{b}_1 = 2^a c_2((c_2 - c_1)/c_1 + e_1^2)$ and $b_2 = c_2(2e_2)^2$. We use (4) and $c_1|x|^a \leq V_N^0(x)$ to give

$$V(f_{cl}(x,w)) \leq V_N^0(x) + b_1\ell(x,\kappa_N(x)) + b_2$$

for all $x \in \mathcal{X}_N$ and $w \in \mathbb{W}$ in which $b_1 = \tilde{b}_1/c_1 - 1$. □

In many applications of MPC with discrete actuators, economic performance, defined by the stage cost, is more important than stabilizing a specific setpoint and Assumption 4 does not necessarily hold. We can establish, however, that the closed-loop trajectory satisfies the following robust performance guarantee.

Theorem 9. Let Assumptions 1-3, 6, and 7 hold. Then there exists $\delta \in (0,1]$ and $\bar{\gamma}(\cdot) \in \mathcal{K}$ such that for disturbance distributions satisfying $\Pr(|w| > 0) = \varepsilon$ the system $x^+ = f_{cl}(x,w)$; $w \in \mathbb{W}$ satisfies

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \mathbb{E}\left[\ell(x(k),u(k))\right] \leq \bar{\gamma}(\varepsilon) \tag{11}$$

in which $x(k) = \phi(k;x,\mathbf{w}_k)$ and $u(k) = \kappa_N(x(k))$, for all $\varepsilon \in [0,\delta]$ and $x \in \mathcal{X}_N$.

Proof. We consider the evolution of the system with and without the disturbance. Let $x \in \mathcal{X}_N$ and $x^+ = f_{cl}(x,w)$. If $w = 0$, then the standard cost decrease applies, i.e., the inequality in (6). We combine this bound with the bound from Assumption 7 using the indicator function $I_{>0}(|w|)$, which takes value unity when the random variable $|w| > 0$ and zero otherwise so that $\mathbb{E}\left[I_{>0}(|w|)\right] = \Pr(|w| > 0) = \varepsilon$.

$$\begin{aligned} V_N^0(x^+) \leq V_N^0(x) &- (1 - I_{>0}(|w|))\ell(x,\kappa_N(x)) \\ &+ I_{>0}(|w|)(b_1\ell(x,\kappa_N(x)) + b_2) \end{aligned}$$

Taking the expected value we have

$$\mathbb{E}_{|x}\left[V_N^0(x^+)\right] \leq V_N^0(x) - (1 - \varepsilon - b_1\varepsilon)\ell(x,\kappa_N(x)) + b_2\varepsilon$$

We choose $\delta < 1/(1+b_1)$ and for all $\varepsilon \leq \delta$, we have

$$\mathbb{E}_{|x}\left[V_N^0(x^+)\right] \leq V_N^0(x) - b_3\ell(x,\kappa_N(x)) + b_2\varepsilon$$

with $b_3 = (1 - (1+b_1)\delta) > 0$. Note that $\delta \in (0,1]$.

For $x \in \mathcal{X}_N$, we denote the closed-loop state trajectory as $x(k) = \phi(k;x,\mathbf{w}_k)$ and the input trajectory as $u(k) = \kappa_N(x(k))$. By the independence of w and the law of iterated expectation, we have

$$\mathbb{E}\left[V_N^0(x(k+1))\right] - \mathbb{E}\left[V_N^0(x(k))\right] \leq -b_3\mathbb{E}\left[\ell(x(k),u(k))\right] + b_2\varepsilon$$

for all $k \in \mathbb{I}_{\geq 0}$. We take the sum from $k = 0$ to $T - 1$ with $T \geq 1$, divide by T, and rearrange to give

$$\frac{b_3}{T}\sum_{k=0}^{T-1}\mathbb{E}\left[\ell(x(k),u(k))\right] \leq \frac{V_N^0(x) - \mathbb{E}\left[V_N^0(x(T))\right]}{T} + b_2\varepsilon$$

By Assumption 1, there exists $M \in \mathbb{R}$ such that $V_N^0(x) \geq M$ for all $x \in \mathcal{X}_N$ and therefore

$$\frac{1}{T}\sum_{k=0}^{T-1}\mathbb{E}\left[\ell(x(k),u(k))\right] \leq \frac{V_N^0(x) - M}{b_3 T} + \bar{\gamma}(\varepsilon)$$

in which $\bar{\gamma}(\varepsilon) = (b_2/b_3)\varepsilon \in \mathcal{K}$. Thus, as $T \to \infty$, the initial and final costs vanish and we have

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1}\mathbb{E}\left[\ell(x(k),u(k))\right] \leq \bar{\gamma}(\varepsilon)$$

which completes the proof. □

The inequality in (11) ensures that the asymptotic average of the expected value of the stage cost is bounded for fixed

ε. Furthermore, as $\varepsilon \to 0$ we asymptotically approach the nominal performance guarantee (in expected value).

For tracking MPC, robustness is characterized through input-to-state stability in which the "input" to the closed-loop system is the disturbance. With small and persistent disturbances, the bound constructed for the closed-loop state is based on size of the disturbance. For discrete and infrequent disturbances, we instead construct a bound for the expected value of the closed-loop state that is based on the probability of the disturbance occurring.

Definition 10. (RESiE to discrete, infrequent disturbances). The origin is RESiE to discrete, infrequent disturbances on the robustly positive invariant set $\mathcal{X}_N$ for the system $x^+ = f_{cl}(x,w);\ w \in \mathbb{W}$ if there exists $\delta \in (0,1]$, $\lambda \in (0,1)$, $\rho > 0$, and $\gamma(\cdot) \in \mathcal{K}$ such that for disturbance distributions satisfying $\Pr(|w| > 0) = \varepsilon$, the closed-loop trajectory satisfies

$$\mathbb{E}\left[|\phi(k;x,\mathbf{w}_k)|\right] \leq \lambda^k \rho |x| + \gamma(\varepsilon) \tag{12}$$

for all $\varepsilon \in [0,\delta]$, $x \in \mathcal{X}_N$, and $k \in \mathbb{I}_{\geq 0}$.

Note that we can convert (12) to a similar bound for confidence intervals of $|\phi(k;x,\mathbf{w}_k)|$ using Markov's inequality. Analogous to an ISS-Lyapunov function, we define an exponential Stochastic ISS-Lyapunov function as follows.

Definition 11. (Exponential SISS-Lyapunov function). The function $V : \mathcal{X}_N \to \mathbb{R}_{\geq 0}$ is an exponential Stochastic ISS-Lyapunov function on the robust positive invariant set $\mathcal{X}_N$ for the system $x^+ = f_{cl}(x,w);\ w \in \mathbb{W}$ if there exists $\delta \in (0,1]$, $a \geq 1$, $c_1, c_2, c_3 > 0$, and $\sigma(\cdot) \in \mathcal{K}$ such that for disturbances satisfying $\Pr(|w| > 0) = \varepsilon$, we have

$$c_1|x|^a \leq V(x) \leq c_2|x|^a \tag{13}$$

$$\mathbb{E}_{|x}\left[V(f_{cl}(x,w))\right] \leq V(x) - c_3|x|^a + \sigma(\varepsilon) \tag{14}$$

for all $\varepsilon \in [0,\delta]$ and $x \in \mathcal{X}_N$.

We also establish that for any system that admits an exponential SISS-Lyapunov function, the origin is RESiE to discrete, infrequent disturbance.

Proposition 12. If a system $x^+ = f_{cl}(x,w);\ w \in \mathbb{W}$ admits an exponential SISS-Lyapunov function on the robustly positive invariant set $\mathcal{X}_N$, then the origin is RESiE to discrete, infrequent disturbances.

Proof. We use the upper bound in (13) and (14) to give

$$\begin{aligned}\mathbb{E}_{|x}\left[V(f_{cl}(x,w))\right] &\leq V(x) - \frac{c_3}{c_2}V(x) + \sigma(\varepsilon)\\ &= \tilde{\lambda}V(x) + \sigma(\varepsilon)\end{aligned}$$

for all $x \in \mathcal{X}_N$, in which $\tilde{\lambda} := (1 - c_3/c_2) \in (0,1)$. Note that $x(k) := \phi(k;x,\mathbf{w}_k) \in \mathcal{X}_N$ for all $x \in \mathcal{X}_N$ because $\mathcal{X}_N$ is robustly positive invariant. Therefore, we have

$$\mathbb{E}_{|x(k)}\left[V(x(k+1))\right] \leq \tilde{\lambda}V(x(k)) + \sigma(\varepsilon)$$

By independence of w, the law of iterated expectation, and linearity of expected value we have

$$\mathbb{E}\left[V(x(k+1))\right] \leq \tilde{\lambda}\mathbb{E}\left[V(x(k))\right] + \sigma(\varepsilon)$$

By iteration of this bound from $\mathbb{E}\left[V(x)\right] = V(x)$, we have

$$\begin{aligned}\mathbb{E}\left[V(x(k))\right] &\leq \tilde{\lambda}^k V(x) + \sigma(\varepsilon)\sum_{i=0}^{k-1}\tilde{\lambda}^i\\ &\leq \tilde{\lambda}^k V(x) + \sigma(\varepsilon)/(1-\tilde{\lambda})\\ &\leq \tilde{\lambda}^k c_2|x|^a + \sigma(\varepsilon)/(1-\tilde{\lambda})\end{aligned} \tag{15}$$

Since we have restricted $a \geq 1$, we can apply the lower bound on $V(x)$ and Jensen's inequality to give

$$c_1\mathbb{E}\left[|x(k)|\right]^a \leq \mathbb{E}\left[c_1|x(k)|^a\right] \leq \mathbb{E}\left[V(x(k))\right] \tag{16}$$

Because $a \geq 1$, $(x+y)^{1/a}$ is sub-additive and $(x+y)^{1/a} \leq x^{1/a} + y^{1/a}$ for all $x, y \geq 0$. Combining (15) with (16) and rearranging gives

$$\mathbb{E}\left[|x(k)|\right] \leq \lambda^k \rho |x| + \gamma(\varepsilon)$$

in which $\lambda = \tilde{\lambda}^{1/a}$, $\rho = (c_2/c_1)^{1/a}$, and

$$\gamma(s) = \left(\frac{\sigma(s)}{c_1(1-\tilde{\lambda})}\right)^{1/a}$$

Note that $\alpha(s) = (s/c_1/(1-\tilde{\lambda}))^p$ for constant c and $p > 0$ is a $\mathcal{K}$-function. So $\gamma(s) = \alpha(\sigma(s))$ is the composition of two $\mathcal{K}$-functions and is therefore a $\mathcal{K}$-function. Since $\tilde{\lambda} \in (0,1)$, $\lambda \in (0,1)$ as well and the proof is complete. □

We now establish the main result of this paper: nominal MPC produces a closed-loop system that is RESiE to discrete, infrequent disturbances.

Theorem 13. Let Assumptions 1-7 hold. Then the origin is RESiE to discrete, infrequent disturbances on $\mathcal{X}_N$ for the system $x^+ = f_{cl}(x,w);\ w \in \mathbb{W}$.

Proof. From Assumption 1 and Assumption 4, we have that $c_1|x|^a \leq \ell(x,\kappa_N(x)) \leq V_N^0(x)$. From Assumption 4, we immediately have $V_N^0(x) \leq c_2|x|^a$. Following the same approach as the proof of Theorem 9, we choose $\delta < 1/(1+b_1) \in (0,1]$ such that for all $\varepsilon \in [0,\delta]$,

$$\mathbb{E}_{|x}\left[V_N^0(f_{cl}(x,w))\right] \leq V_N^0(x) - b_3\ell(x,\kappa_N(x)) + b_2\varepsilon$$

in which $b_3 = (1-(1+b_1)\delta) > 0$. We substitute in the lower bound for $\ell(\cdot)$ from Assumption 4 to give

$$\mathbb{E}_{|x}\left[V_N^0(f_{cl}(x,w))\right] \leq V_N^0(x) - c_3|x|^a + \sigma(\varepsilon)$$

in which $c_3 = c_1 b_3$ and $\sigma(\varepsilon) = b_2\varepsilon$. Thus, $V_N^0 : \mathcal{X}_N \to \mathbb{R}_{\geq 0}$ is an exponential SISS-Lyapunov function of the robustly positive invariant set $\mathcal{X}_N$. We apply Proposition 12 to complete the proof. □

In addition to robust exponential stability in 1st moment of $|\phi(k;x,\mathbf{w}_k)|$, we can also establish robust exponential stability in the 2nd moment of $|\phi(k;x,\mathbf{w}_k)|$ if we apply the usual quadratic stage and terminal costs used in tracking MPC.

Corollary 14. Let Assumptions 1-7 hold with $a = 2$. Then the origin is robustly exponentially *mean-squared* stable for the system $x^+ = f_{cl}(x,w);\ w \in \mathbb{W}$, i.e., there exists $\delta \in (0,1]$, $\lambda \in (0,1)$, $\rho > 0$, and $\gamma \in \mathcal{K}$ such that for disturbance distributions satisfying $\Pr(|w| > 0) = \varepsilon$, the closed-loop trajectory satisfies

$$\mathbb{E}\left[|\phi(k;x,\mathbf{w}_k)|^2\right] \leq \lambda^k \rho |x| + \gamma(\varepsilon) \tag{17}$$

for all $\varepsilon \in [0,\delta]$, $x \in \mathcal{X}_N$, and $k \in \mathbb{I}_{\geq 0}$.

Proof. Starting with (15), we substitute in $c_1|x(k)|^2 \leq V(x(k))$ and rearrange to give

$$\mathbb{E}\left[|x(k)|^2\right] \leq \lambda^k \rho |x| + \gamma(\varepsilon)$$

in which $\lambda = \tilde{\lambda}$, $\rho = (c_2/c_1)$, and $\gamma(s) = \sigma(s)/(c_1(1-\tilde{\lambda}))$. Thus, $\lambda \in (0,1)$, $\gamma(\cdot) \in \mathcal{K}$, and the proof is complete. □

Since $\mathbb{E}\left[|\cdot|\right] \geq 0$, we know that the variance of $|x(k)| = |\phi(k;x,\mathbf{w}_k)|$ satisfies

$$\mathrm{Var}(|x(k)|) = \mathbb{E}\left[|x(k)|^2\right] - \mathbb{E}\left[|x(k)|\right]^2 \leq \mathbb{E}\left[|x(k)|^2\right]$$

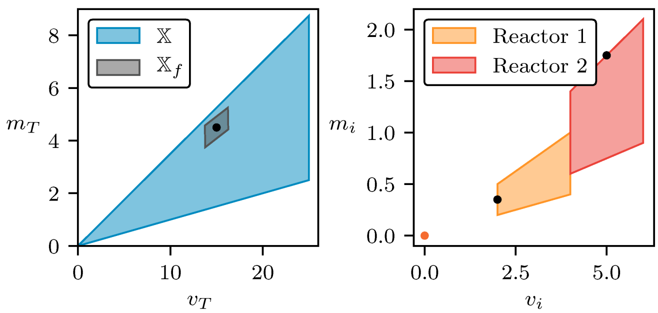

Fig. 1. Feasible state and input space for the blending example. Round black markers show steady-state values.

and therefore Corollary 14 implies robust exponential stability *in variance* as well.

4. EXAMPLE

We consider a blending example similar to the one in Rawlings and Risbeck (2017). Two batch reactors deliver product to a single holding tank from which product may be withdrawn at the start of each hour. The goal of the controller is to maintain both the total volume and concentration of product species within the tank while withdrawing 7 m^3 every hour. We consider a discrete time model of the facility as follows:

$$v_T^+ = v_T + v_1(1-d_1) + v_2(1-d_2) - v_d$$
$$m_T^+ = m_T + m_1(1-d_1) + m_2(1-d_2) - \frac{m_T}{v_T} v_d$$

in which v_T is the volume of fluid in the tank, m_T is the mass of product in the tank, v_1, m_1 and v_2, m_2 are the volumes and masses, respectively, of product produced by reactors 1 and 2. We also include the binary disturbance variables d_1 and d_2 that represent breakdowns or unplanned maintenance of reactors 1 and 2. To enforce minimum capacity requirements of each reactor we use the binary variables z_1 and z_2 to indicate if the reactor is 'on' or 'off'. We also allow the controller to select the outlet flow rate v_d. Thus, the system has two states $x = (v_T, m_T)$, seven inputs $u = (z_1, v_1, m_1, z_2, v_2, m_2, v_d)$, and two binary disturbances $w = (d_1, d_2)$.

We require that the maximum volume of the tank is not exceeded, $0 \leq v_T \leq 25$, and the concentration of product remains within acceptable bounds, $0.1v_T \leq m_T \leq 0.35v_T$.[3] We also require that $0 \leq v_d \leq 10$ and $v_d \leq v_T$ to ensure that we do not withdraw more material than is available at the start of each hour. The input constraints associated with the reactors are shown in Figure 1.

With a given steady state (x_{ss}, u_{ss}), we define the stage cost as $\ell(x,u) = |x - x_{ss}|_Q^2 + |u - u_{ss}|_R^2$ with diagonal Q and R. To construct the terminal cost and constraint, we linearize the system, take m_1 and v_2 as free inputs, assume $m_2 = \rho_{max} v_2$, and fix all other inputs to their steady-state values. We then determine the LQR solution and state-feedback gain for the reduced linear system with only two free inputs. This procedure produces a linear terminal

[3] Note that the reactors cannot produce concentrations that violate these bounds.

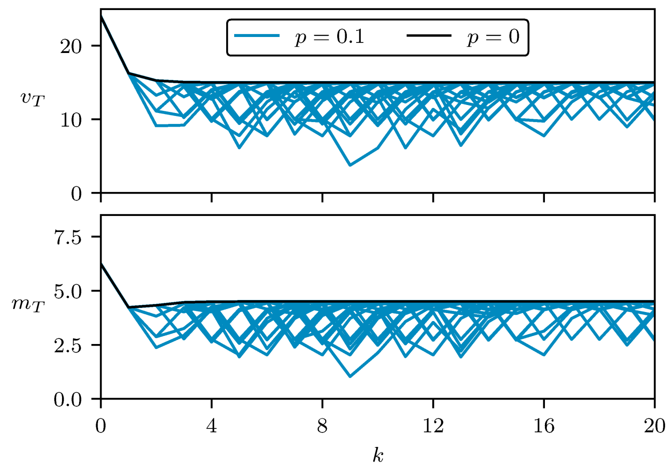

Fig. 2. Closed-loop volume and product mass for the blending example subject to breakdowns of each reactor with probability $p = 0.1$. The blue lines show closed-loop trajectories for 50 realizations of the trajectory from the initial condition $x_0 = (24, 6.24)$. The black line is the nominal trajectory, i.e, $p = 0$, for comparison.

control law $\kappa_f(x) = K(x - x_{ss}) + u_{ss}$ and terminal cost $V_f(x) = |x - x_{ss}|_P^2$. A candidate terminal set is then

$$\mathbb{X}_f = \{x \in \mathbb{X} \mid \kappa_f(x) \in \mathbb{U}\}$$

We verify that the set $\mathbb{X}_f$ is invariant under $\kappa_f(\cdot)$ and that the cost decrease condition in Assumption 3 is satisfied.

Clearly, Assumptions 1 and 2 are satisfied. From the definition of $V_f(x)$, there exists $c_f > 0$ such that $V_f(x) \leq c_f |x|^2$ for all $x \in \mathbb{X}_f$. Since $\mathcal{X}_N \subseteq \mathbb{X}$ is bounded and the origin is in the interior of $\mathbb{X}_f$, there exists $c_2 > 0$, such that $V_N^0(x) \leq c_2 |x|^2$ for all $x \in \mathcal{X}_N$. We choose positive definite Q and therefore Assumption 4 is satisfied with $a = 2$.

We note that $\mathcal{X}_N = \mathbb{X}$ for $N \geq 6$.[4] Let $x \in \mathcal{X}_N$, $\hat{x}^+ = (\hat{v}_T^+, \hat{m}_T^+) = f_{cl}(x, 0)$, and $x^+ = (v_T^+, m_T^+) = f_{cl}(x, w)$. For any value of $w \in \mathbb{W} := \{0,1\}^2$, we have that $v_T^+ \leq \hat{v}_T^+ \leq 25$. Furthermore, we restricted $v_d \leq v_T$ and therefore $0 \leq v_T^+$. We also know that the reactors cannot produce concentrations that violate the product quality state constraints. So if $0.1v_T \leq m_T \leq 0.35v_T$, then $0.1v_T^+ \leq m_T^+ \leq 0.35v_T^+$ as well. Thus, $x^+ \in \mathbb{X} = \mathcal{X}_N$ and Assumption 6 holds. The system model $f(\cdot)$ also satisfies the condition in Lemma 8 and therefore the closed-loop system is RESiE and robustly exponentially mean-squared stable to discrete, infrequent disturbances.

We simulate the closed-loop trajectory subject to breakdowns that occur with some probability p for each reactor. For each value of p we simulate the closed-loop system over 20 hours of operation and for 50 realizations of the disturbance with the initial condition $x_0 = (24, 6.24)$. The resulting state trajectories for each of the 50 realizations with $p = 0.1$ are plotted in Figure 2.

We evaluate the distance of each trajectory from the steady state for all k and calculate the sample mean and sample mean-squared for all k. These trajectories are

[4] Three steps to drain the tank from full and three steps to fill to 15 m^3 with the correct concentration.

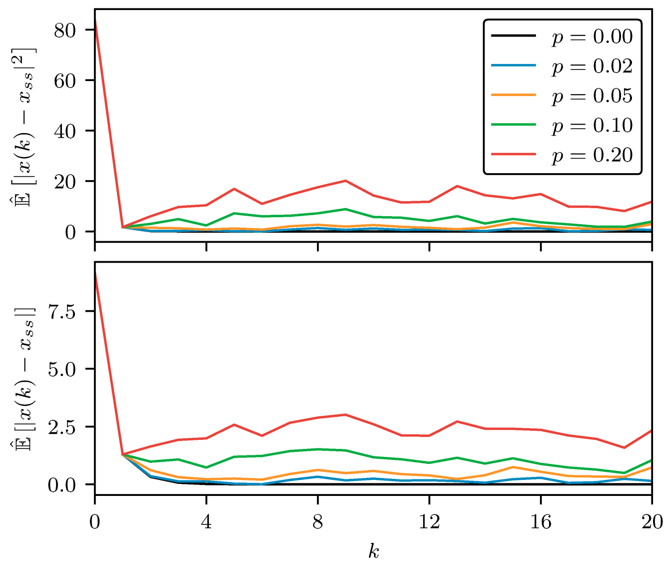

Fig. 3. Trajectory of the sample mean and mean-squared distance of the closed-loop trajectory to the steady state for different values of p.

plotted in Figure 3. In both plots, we observe an initial decrease of the sample mean (mean-squared) towards zero and then the sample mean (sample mean-squared) remains below some upper bound for all future k. This bound increases with increasing p (i.e., $\varepsilon = p^2 + 2p(1-p)$ in RESiE definition) and decreases towards zero as $p \to 0$ (i.e., $\varepsilon \to 0$). Thus, the results in Figure 3 are consistent with the bounds in (12) and (17).

5. CONCLUSIONS

We established that under typical assumptions for tracking MPC and an additional assumption of robust recursive feasibility, *nominal* MPC renders the origin robustly exponentially stable in expectation for the closed-loop system subject to discrete and infrequent disturbances. This robustness is solely the result of feedback as the nominal MPC algorithm does not consider disturbances in the optimization problem. Therefore, we can confidently apply MPC to problems in which discrete disturbances, such as breakdowns, are relevant. This extension is particularly meaningful for higher-level, scheduling problems in which discrete decisions and discrete disturbance are ubiquitous. We can also extend many of these results to include asymptotic stability and time-varying systems (McAllister and Rawlings, 2020).

To ensure these robustness properties, however, we require the MPC problem to be robustly recursively feasible by design. Thus, an important direction for future research is develop terminal conditions that guarantee robust recursively feasibility for relevant problems. In addition, we may want to combine these results with typical robustness results for small, persistent disturbances to construct a robustness result for MPC subject to both types of disturbances.

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