

Stochastic Exponential Stability of Nonlinear Stochastic Model Predictive Control

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Abstract—In this work, we define and establish a selection of significant and new properties for nonlinear stochastic model predictive control (SMPC). First, we ensure that the underlying stochastic properties of the closed-loop stochastic system are indeed well-defined. We then define robust exponential stability in expectation (RESiE) and establish that nonlinear SMPC, under suitable assumptions, renders the origin of the closed-loop system RESiE. We conclude with a numerical example to demonstrate the implications of this analysis.

I. INTRODUCTION

For successful implementation of any control technique, some margin of robustness to disturbances is required. The inherent robustness of linear and nonlinear model predictive control (MPC) to small disturbances has been well-established [1]–[4]. To improve upon the inherent robustness of nominal MPC, SMPC uses a probabilistic description of uncertainty and optimizes the expected value of a cost function subject to both deterministic and probabilistic constraints on the state and input variables [5]–[7]. Solving and analyzing the closed-loop properties of the SMPC problem is considerably more difficult than the nominal MPC problem and therefore the majority of results for SMPC are restricted to linear systems. Although approximating and solving the nonlinear SMPC problem is a worthwhile pursuit in its own right, we instead focus this work on the closed-loop properties of nonlinear SMPC.

For linear SMPC with multiplicative disturbances, [8] establish that the origin is asymptotically stable with probability one if a *global* stochastic Lyapunov function is used for the terminal cost. For constrained linear SMPC with bounded, additive disturbances, we can construct terminal constraints and a terminal cost that ensure recursive feasibility and stability in expectation [9]. We can also establish that the minimal robust positive invariant set for the system is asymptotically stable with probability one [10].

For nonlinear SMPC, the closed-loop system is stable in expectation if a *global* stochastic Lyapunov function is used as the terminal cost [11]. For constrained nonlinear systems subject to bounded disturbances, these results extend to SMPC algorithms with properly formulated terminal costs and constraints [12]. Under certain viability and stochastic controllability assumptions, nonlinear SMPC without terminal conditions is also stabilizing, but these assumptions are difficult to verify for nonlinear systems [13].

There are, however, limitations to the current theory addressing nonlinear SMPC. Currently, there is little discussion of the existence of optimal solutions to the SMPC problem or measurability of the closed-loop trajectory. Both of these properties, although likely to hold under mild regularity assumptions, should be rigorously addressed. Furthermore, the established closed-loop properties of nonlinear SMPC are often limited to stability in expected value (no exponential decay of the effect of the initial condition) if disturbances do not vanish at the origin. In addition, there is no clear connection between the disturbance considered in the SMPC formulation and the closed-loop stability results. In this work, we address these limitations.

To analyze the closed-loop properties of nonlinear SMPC, we employ results derived from stochastic stability theory [14]. Specifically, we use the concepts of stochastic input-to-state stability (SISS) and SISS-Lyapunov functions [15]–[17]. While we use definitions and results from this literature as inspiration, we note that this literature often assumes that the stochastic disturbance vanishes at the origin (excluding additive disturbances) and does not address discontinuous closed-loop systems, such as nonlinear SMPC with a possibly discontinuous control law. Thus, we adjust these definitions and results to address nonlinear SMPC.

In Section 2, we introduce SMPC and some suitable assumptions. In Section 3, we discuss and establish the existence of optimal solutions for nonlinear SMPC problem and measurability of the closed-loop stochastic system. In Section 4, we define robust exponential stability in expectation (RESiE), provide a sufficient condition for this property in terms of an SISS-Lyapunov function, and establish that SMPC generates a closed-loop systems that is RESiE. We conclude with a nonlinear SMPC example.

Notation: Let $\mathbb{I}$ and $\mathbb{R}$ denote the integers and reals. Let superscripts and subscripts denote dimensions and restrictions (e.g., $\mathbb{R}_{\geq 0}^n$ denotes nonnegative reals of dimension n). We use $|\cdot|$ to denote Euclidean norm. Let $I_S(x)$ denote the indicator function for a set S , i.e., $I_S(x) = 1$ if $x \in S$ and zero otherwise. The function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. Let $\mathcal{P}(\Omega)$ denote the power set and $\mathcal{B}(\Omega)$ denote the Borel field of some set Ω . A set $F \subseteq \mathbb{R}^n$ is Borel measurable if $F \in \mathcal{B}(\mathbb{R}^n)$ and a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Borel measurable if for each open set $O \subseteq \mathbb{R}^m$, the set $f^{-1}(O) := \{x \in \mathbb{R}^n : f(x) \in O\}$ is Borel measurable. A set-valued mapping $S : X \rightrightarrows Y$ is Borel measurable if for every open set $O \subseteq Y$, the set $S^{-1}(O) := \{x \in X : S(x) \cap O \neq \emptyset\}$ is Borel measurable.

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II. STOCHASTIC MODEL PREDICTIVE CONTROL

A. The stochastic system

We consider the following discrete-time stochastic difference equation

$$x^+ = f(x, u, w) \quad f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \rightarrow \mathbb{X} \quad (1)$$

in which $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u \in \mathbb{U} \subseteq \mathbb{R}^m$ is the controlled input, and $w \in \mathbb{W} \subseteq \mathbb{R}^p$ is the disturbance (random variable). The successor state is denoted by x^+ .

Let $(\Omega, \mathcal{F}, P)$ be a probability space for the sequence $\mathbf{w}_\infty : \Omega \rightarrow \mathbb{W}^\infty$ of random variables, i.e., $\mathbf{w}_\infty := \{w_i\}_{i=0}^\infty$ for $w_i : \Omega \rightarrow \mathbb{W}$. We denote $\mathbf{w}_i : \Omega \rightarrow \mathbb{W}^i$ as the sequence $\mathbf{w}_i := (w_0, \dots, w_{i-1})$. Let $\{\mathcal{F}_i\}_{i=0}^\infty$ denote the natural filtration of the sequence $\{w_i\}_{i=0}^\infty$, that is $\mathcal{F}_i \subseteq \mathcal{F}$ is all sets of the form $\{\omega \in \Omega : \mathbf{w}_i(\omega) \in F\}$ for all $F \in \mathcal{B}(\mathbb{W}^i)$. By definition, $\mathbf{w}_i$ is measurable w.r.t. $\mathcal{F}_i$. For some Borel measurable function $g : \mathbb{W}^i \rightarrow \mathbb{R}$ we define expected value as

$$\mathbb{E}[g(\mathbf{w}_i)] := \int_{\Omega} g(\mathbf{w}_i(\omega)) dP(\omega)$$

We make the following assumption for the disturbance in SMPC. Note that we require bounded disturbances (i.e., compact support) because we allow state constraints in the problem formulation and do not require the terminal cost to be a global stochastic Lyapunov function.

Assumption 1: The disturbances $w_i : \Omega \rightarrow \mathbb{W}$ are independent and identically distributed, zero mean, random variables. Each random variable has a known and equivalent probability measure $\mu : \mathcal{B}(\mathbb{W}) \rightarrow [0, 1]$ defined such that $\mu(F) = P(\{\omega \in \Omega : w_i(\omega) \in F\})$ for all $F \in \mathcal{B}(\mathbb{W})$. The support $\mathbb{W}$ is compact and contains the origin. The second moment of $w(i)$ is finite and the covariance matrix is defined as $\Sigma = \mathbb{E}[w(i)w(i)']$ for all $i \in \mathbb{I}_{\geq 0}$.

For the random variables $(w_i, w_{i+1}, \dots, w_{i+N-1})$ and $N \in \mathbb{I}_{\geq 1}$, their joint distribution measure $\mu^N : \mathcal{B}(\mathbb{W}^N) \rightarrow [0, 1]$ is defined as $\mu^N(F) = \mu(F_i)\mu(F_{i+1})\dots\mu(F_{i+N-1})$ for all $F = (F_i, F_{i+1}, \dots, F_{i+N-1}) \in \mathcal{B}(\mathbb{W}^N)$. For any Borel measurable function $g : \mathbb{W}^{i+1} \rightarrow \mathbb{R}$ and all $i \in \mathbb{I}_{\geq 0}$, conditional expected value satisfies

$$\mathbb{E}[g(\mathbf{w}_{i+1}) | \mathcal{F}_i](\omega) = \int_{\mathbb{W}} g(w_0(\omega), \dots, w_{i-1}(\omega), w) d\mu(w)$$

B. SMPC formulation

In stochastic MPC, we typically use a predefined control policy $\pi : \mathbb{X} \times \mathbb{V} \rightarrow \mathbb{U}$ in which $v \in \mathbb{V}$ are the parameters.¹ Thus, we redefine the system of interest as

$$x^+ = f(x, \pi(x, v), w) \quad (2)$$

We denote the solution of (2) at time k , given the initial condition x , trajectory of parameters $\mathbf{v} \in \mathbb{V}^N$, and disturbance trajectory $\mathbf{w} \in \mathbb{W}^N$ as $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$.

We consider both hard constraints, i.e., $(x, u) \in \mathbb{Z}_h \subseteq \mathbb{X} \times \mathbb{U}$ and probabilistic constraints on the state defined as

$$\Pr(f(x, u, w) \in \tilde{\mathbb{X}}) \geq 1 - \varepsilon \quad (3)$$

for a set $\tilde{\mathbb{X}}$ and constant $\varepsilon \in [0, 1]$. We can reformulate this probabilistic constraint using the following function

$$G(x, u) := 1 - \int_{\mathbb{W}} I_{\tilde{\mathbb{X}}}(f(x, u, w)) d\mu(w)$$

and constraint $\tilde{\mathbb{Z}}_\varepsilon := \{(x, u) : G(x, u) \leq \varepsilon\}$. Then we define the combined hard and probabilistic constraints as

$$(x, u) \in \mathbb{Z} := \mathbb{Z}_h \cap \tilde{\mathbb{Z}}_\varepsilon$$

Calculating or approximating $\tilde{\mathbb{Z}}_\varepsilon$ is a difficult and important research problem that we somewhat obscure by this reformulation. However, we find this reformulation useful in subsequent analysis.

For SMPC with a horizon of $N \in \mathbb{I}_{\geq 1}$, the mixed constraint $\mathbb{Z}$, and terminal constraint $\mathbb{X}_f \subseteq \mathbb{X}$, we have the set of admissible $(x, \mathbf{v})$ pairs defined as

$$\begin{aligned} \mathcal{Z}_N := \{ & (x, \mathbf{v}) \in \mathbb{X} \times \mathbb{V}^N : \\ & (x(k), \pi(x(k), v(k))) \in \mathbb{Z} \quad \forall \mathbf{w} \in \mathbb{W}^N, \quad k \in \mathbb{I}_{[0, N-1]} \\ & x(N) \in \mathbb{X}_f \quad \forall \mathbf{w} \in \mathbb{W}^N \} \end{aligned}$$

in which $x(k) = \hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$. We define the set of admissible parameter trajectories and the set of admissible states as $\mathcal{V}_N(x) := \{\mathbf{v} \in \mathbb{V}^N : (x, \mathbf{v}) \in \mathcal{Z}_N\}$ and $\mathcal{X}_N := \{x \in \mathbb{X} : \exists \mathbf{v} \in \mathcal{V}_N(x)\}$.

Given a stage cost $\ell : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}_{\geq 0}$ and terminal cost $V_f : \mathbb{X} \rightarrow \mathbb{R}_{\geq 0}$, we define the function

$$J_N(x, \mathbf{v}, \mathbf{w}) := \sum_{k=0}^{N-1} \ell(x(k), \pi(x(k), v(k))) + V_f(x(N))$$

in which $x(k) = \hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$. In SMPC, we define the cost function based on the expected value of $J_N(\cdot)$, i.e., we define

$$V_N(x, \mathbf{v}) := \int_{\mathbb{W}^N} J_N(x, \mathbf{v}, \mathbf{w}) d\mu^N(\mathbf{w})$$

We define the SMPC problem for any $x \in \mathcal{X}_N$ as

$$\mathbb{P}_N(x) : \quad V_N^0(x) := \min_{\mathbf{v} \in \mathcal{V}_N(x)} V_N(x, \mathbf{v})$$

and the optimal solutions for a given initial state are defined by the set-valued mapping $\mathbf{v}^0 : \mathcal{X}_N \rightrightarrows \mathbb{V}^N$.

We require the following basic regularity assumptions.

Assumption 2: The function $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \rightarrow \mathbb{X}$, $\pi : \mathbb{X} \times \mathbb{V} \rightarrow \mathbb{U}$, $\ell : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}_{\geq 0}$, and $V_f : \mathbb{X}_f \rightarrow \mathbb{R}_{\geq 0}$ are continuous. Furthermore, we have that $f(0, 0, 0) = 0$, $\ell(0, 0) = 0$, and $V_f(0) = 0$.

Assumption 3: The set $\mathbb{Z}_h$ and $\tilde{\mathbb{X}}$ are closed and contain the origin. The sets $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{X}_f \subseteq \mathbb{X}$ are compact and contain the origin.

To ensure we achieve certain closed-loop performance and stability objectives, we require the terminal conditions to satisfy the following assumption.

Assumption 4: There exists a continuous terminal control law $\kappa_f : \mathbb{X}_f \rightarrow \mathbb{U}$ such that for all $x \in \mathbb{X}_f$, $f(x, \kappa_f(x), w) \in \mathbb{X}_f$ for all $w \in \mathbb{W}$ and

$$V_f(f(x, \kappa_f(x), 0)) \leq V_f(x) - \ell(x, \kappa_f(x))$$

¹In many cases, we define $\pi(x, v) = Kx + v$.

Furthermore, $(x, \kappa_f(x)) \in \mathbb{Z}_h$, $\mathbb{X}_f \subseteq \tilde{\mathbb{X}}$, and $\pi(x, 0) = \kappa_f(x)$ for all $x \in \mathbb{X}_f$. The functions $V_f(\cdot)$ and $f(\cdot)$ are Lipschitz continuous on the sets $\mathbb{X}_f$ and $\mathbb{X}_f \times \mathbb{U} \times \mathbb{W}$, respectively.

Note that we require $V_f(\cdot)$ to satisfy a cost decrease condition only for the nominal system, i.e., $w = 0$. The Lipschitz continuity of $V_f(\cdot)$ and $f(\cdot)$ are not typically required in closed-loop analysis of MPC, but this restriction is minor. Indeed, most MPC formulations require some assumptions of Lipschitz continuity to ensure that the problem can be solved by standard nonlinear optimization methods.

We also require bounds on the stage and terminal costs detailed in the following assumption. The extra requirements placed on the sets $\mathbb{X}_f$ and $\mathcal{X}_N$ are used to construct an upper bound for the optimal cost.

Assumption 5: There exists $a \geq 1$, $c_1, c_f > 0$ such that $\ell(x, u) \geq c_1|x|^a$ for all $(x, u) \in \mathbb{Z}_h$ and $V_f(x) \leq c_f|x|^a$ for all $x \in \mathbb{X}_f$. The set $\mathbb{X}_f$ contains the origin in its interior and $\mathcal{X}_N$ is bounded.

III. BASIC PROPERTIES OF SMPC

A. Existence of optimal solutions

To discuss the closed-loop properties of SMPC, we must first establish that the optimization problem $\mathbb{P}_N(x)$ is indeed well-defined for all $x \in \mathcal{X}_N$. Due to space limitations, all proofs of the results in this section are reported in [18]. We begin with the following result for the set $\mathbb{Z}$.

Lemma 1: Let Assumptions 1-3 hold. Then the set $\mathbb{Z} = \mathbb{Z}_h \cap \tilde{\mathbb{Z}}_\varepsilon$ is closed.

Since the probabilistic constraint can be reformulated as a closed set we can apply an approach similar to that in [19, Proposition 2.4] and Lebesgue's dominated convergence theorem to establish the following result.

Proposition 2: Let Assumptions 1-3 hold. Then for each $x \in \mathcal{X}_N$, the function $V_N(x, \cdot) : \mathbb{V}^N \rightarrow \mathbb{R}_{\geq 0}$ is continuous, the set $\mathcal{V}_N(x)$ is compact, and a solution to $\mathbb{P}_N(x)$ exists.

Thus, we guarantee that the nonlinear SMPC optimization problem is well-defined, even though solving such a problem may be challenging or even intractable with current optimization techniques.

B. Measurability of the closed-loop state trajectory

We define the control law mapping for SMPC as $K_N(x) := \pi(x, v^0(0; x))$ in which $v^0(0; x)$ is the first parameter vector in $\mathbf{v}^0(x)$. If there are multiple solutions to $\mathbb{P}_N(x)$, then $K_N(x)$ may be a set-valued mapping. However, we typically assume there exists some selection rule $\Psi : (\mathcal{P}(\mathbb{U}) \setminus \emptyset) \rightarrow \mathbb{U}$ that defines a single-valued control law $\kappa_N : \mathcal{X}_N \rightarrow \mathbb{U}$ such that $\kappa_N(x) := \Psi(K_N(x)) \in K_N(x)$ for all $x \in \mathcal{X}_N$. The resulting closed-loop stochastic system is then

$$x^+ = f_{cl}(x, w) := f(x, \kappa_N(x), w) \quad (4)$$

We define the solution to (4) at time $k \in \mathbb{I}_{\geq 0}$ given the initial condition x and disturbance sequence $\mathbf{w}_k$ as $\phi(k; x, \mathbf{w}_k)$.

We note that $\kappa_N(\cdot)$ is not necessarily continuous or Borel measurable. Furthermore, if $\kappa_N(\cdot)$ is not Borel measurable,

then $\phi(k; x, \mathbf{w}_k)$ is not necessarily Borel measurable w.r.t. $\mathbf{w}_k$ and all stochastic properties of interest for the closed-loop system (e.g., expected value) may be undefined. Fortunately, basic regularity assumptions are sufficient to establish that $K_N(x)$ is Borel measurable.

Proposition 3: Let Assumptions 1-3 hold. Then the function $V_N^0 : \mathcal{X}_N \rightarrow \mathbb{R}$ is lower semicontinuous (Borel measurable), the set $\mathcal{X}_N$ is closed, and the set-valued mapping $K_N(x) : \mathcal{X}_N \rightrightarrows \mathbb{U}$ is Borel measurable.

The proof of this result can be found in [18] and relies (almost entirely) on Proposition 7.33 in [20]. We note that even if $K_N(x)$ is a Borel measurable function we can still (in theory) select a nonmeasurable selection rule. Accidentally constructing such a selection rule for a real system, however, is highly unlikely. Hence, we assume for the rest of this work that $\Psi(\cdot)$ is Borel measurable and have the following result. See Appendix A in [18] for further discussion of these topics.

Proposition 4: Let Assumptions 1-3 hold and assume $\mathcal{X}_N$ is robustly positive invariant for $f_{cl}(\cdot)$, i.e., $x \in \mathcal{X}_N$ implies $f_{cl}(x, w) \in \mathcal{X}_N$ for all $w \in \mathbb{W}$. Then the functions $\phi(k; x, \mathbf{w}_k)$ are well-defined and Borel measurable for all $k \in \mathbb{I}_{\geq 0}$. Furthermore, the integral

$$\int_{\Omega} g(\phi(k; x, \mathbf{w}_k(\omega))) dP(\omega)$$

is well-defined for all $x \in \mathcal{X}_N$, $k \in \mathbb{I}_{\geq 0}$, and any Borel measurable function $g : \mathcal{X}_N \rightarrow \mathbb{R}_{\geq 0}$.

We note that the function $V_N^0 : \mathcal{X}_N \rightarrow \mathbb{R}_{\geq 0}$ is Borel measurable (lower semicontinuous) from Proposition 3. Thus, the expected value of the optimal cost (and many other relevant properties) along the closed-loop trajectory is indeed well-defined for the stochastic system generated by SMPC.

IV. STOCHASTIC EXPONENTIAL STABILITY

A. Robust Exponential Stability in Expectation

We begin by defining the term robust exponential stability in expectation (RESiE).

Definition 1: The origin is robustly exponentially stable in expectation (RESiE) for the stochastic system $x^+ = f_{cl}(x, w)$; $w \in \mathbb{W}$ on the robustly positive invariant set $\mathcal{X}_N$ if there exists $\lambda \in (0, 1)$, $\rho > 0$, and $\gamma(\cdot) \in \mathcal{K}$ such that the closed-loop trajectory satisfies

$$\mathbb{E}[\|\phi(k; x, \mathbf{w}_k)\|] \leq \lambda^k \rho |x| + \gamma(\text{tr}(\Sigma)) \quad (5)$$

for all $x \in \mathcal{X}_N$ and $k \in \mathbb{I}_{\geq 0}$.

RESiE bounds the expected value of the norm of the closed-loop state based on the initial condition x and the covariance of the disturbance, i.e., $\text{tr}(\Sigma)$. Note that $\text{tr}(\Sigma) \rightarrow 0$ implies $\Sigma \rightarrow 0$ for positive semidefinite Σ . RESiE implies the typical notion of stability in expectation, but also ensures that the effect of the initial condition $|x|$ on the upper bound exponential decays to zero as $k \rightarrow \infty$. We also note the (intentional) similarities between the definition of RESiE and the definition of robust exponential stability typically used to characterize the inherent robustness of nominal MPC [2]. Furthermore, as $\text{tr}(\Sigma) \rightarrow 0$, i.e., nominal MPC applied to

a nominal system, we recover exponential stability of the origin.

To establish this property for a closed-loop system we use an (exponential) SISS-Lyapunov function defined as follows. Note that we do not require continuity of $f_{cl}(\cdot)$.

Definition 2: The Borel measurable function $V : \mathcal{X}_N \rightarrow \mathbb{R}_{\geq 0}$ is an (exponential) SISS-Lyapunov function on the robustly positive invariant set $\mathcal{X}_N$ for the stochastic system $x^+ = f_{cl}(x, w)$; $w \in \mathbb{W}$, if there exists $a \geq 1$, $c_1, c_2, c_3 > 0$, and $\sigma_2(\cdot), \sigma_3(\cdot) \in \mathcal{K}$ such that

$$c_1|x|^a \leq V(x) \leq c_2|x|^a + \sigma_2(\text{tr}(\Sigma)) \quad (6)$$

$$\int_{\mathbb{W}} V(f_{cl}(x, w)) d\mu(w) \leq V(x) - c_3|x|^a + \sigma_3(\text{tr}(\Sigma)) \quad (7)$$

for all $x \in \mathcal{X}_N$.

Proposition 5: If a system $x^+ = f_{cl}(x, w)$; $w \in \mathbb{W}$, with Borel measurable $f_{cl}(\cdot)$, admits an (exponential) SISS-Lyapunov function on the robustly positive invariant set $\mathcal{X}_N$, then the origin is RESiE.

Proof: To streamline notation, we define $d := \text{tr}(\Sigma)$ and note that this value is constant. We assume without loss of generality that $c_3 \leq c_2$.² From the upper bound on $V(\cdot)$, we have that $-|x|^a \leq -V(x)/c_2 + \sigma_2(d)/c_2$. Choose $x \in \mathcal{X}_N$ and let $x(k) := \phi(k; x, \mathbf{w}_k)$ for all $k \in \mathbb{I}_{\geq 0}$. We apply the definition of conditional expectation to give

$$\mathbb{E}[V(x(k+1)) | \mathcal{F}_k] \leq \tilde{\lambda}V(x(k)) + \sigma_4(d)$$

in which $\tilde{\lambda} := 1 - c_3/c_2 \in (0, 1)$ and $\sigma_4(d) := (c_3/c_2)\sigma_2(d) + \sigma_3(d)$. Note that $\sigma(\cdot) \in \mathcal{K}$. We apply the law of iterated expected value to give

$$\mathbb{E}[V(x(k+1))] \leq \tilde{\lambda}\mathbb{E}[V(x(k))] + \sigma_4(d)$$

By iteration and the fact that $\mathbb{E}[V(x)] = V(x)$ we have

$$\mathbb{E}[V(x(k))] \leq \tilde{\lambda}^k V(x(0)) + \sigma_4(d)/(1 - \tilde{\lambda})$$

We substitute the upper and lower bound for $V(x)$ to give³

$$\mathbb{E}[|x(k)|^a] \leq \tilde{\lambda}^k(c_2/c_1)|x|^a + (1/c_1)\sigma_2(d) + \frac{\sigma_4(d)}{c_1(1 - \tilde{\lambda})} \quad (8)$$

Since $a \geq 1$, we can apply Jensen's inequality and use the subadditivity of powers of $1/a \leq 1$ to give $\mathbb{E}[|x(k)|] \leq \lambda^k \rho |x| + \gamma(d)$ in which $\lambda := \tilde{\lambda}^{1/a}$, $\rho := (c_2/c_1)^{1/a}$, and

$$\gamma(d) := \left((1/c_1)\sigma_2(d) + \frac{\sigma_4(d)}{c_1(1 - \tilde{\lambda})} \right)^{1/a}$$

Note that $\gamma(\cdot) \in \mathcal{K}$ and $d = \text{tr}(\Sigma)$ to complete the proof. ■

B. RESiE of SMPC

To establish that the closed-loop system generated by SMPC is RESiE, we establish that the optimal cost is an (exponential) SISS-Lyapunov function. We construct the cost decrease condition by exploiting properties of the terminal control law.

²If this inequality does not hold we can simply redefine c_2 such that $c_3 \leq c_2$ with all the other constants remaining the same.

³If $a = 2$, (8) implies robust exponential stability in *mean-squared*.

Lemma 6: Let Assumptions 1-4 hold. Then there exists $\sigma(\cdot) \in \mathcal{K}$ such that

$$\int_{\mathbb{W}} V_f(f(x, \kappa_f(x), w)) d\mu(w) \leq V_f(x) - \ell(x, \kappa_f(x)) + \sigma(\text{tr}(\Sigma))$$

for all $x \in \mathbb{X}_f$.

Proof: To streamline notation we define $x^+ = f(x, \kappa_f(x), w)$ and $\hat{x}^+ = f(x, \kappa_f(x), 0)$. Since $V_f(\cdot)$ and $f(\cdot)$ are Lipschitz continuous on $\mathbb{X}_f$ and $\mathbb{X}_f \times \mathbb{U} \times \mathbb{W}$, respectively, and $\mathbb{X}_f$ is robustly positive invariant under the control law $\kappa_f(x)$, we have that

$$|V_f(x^+) - V_f(\hat{x}^+)| \leq L_{V_f}|x^+ - \hat{x}^+| \leq L_{V_f}L_f|w|$$

for all $x \in \mathbb{X}_f$ and $w \in \mathbb{W}$ in which L_{V_f} and L_f are the Lipschitz constants for $V_f(\cdot)$ and $f(\cdot)$, respectively. Thus, we have that

$$\int_{\mathbb{W}} V_f(x^+) d\mu(w) \leq V_f(\hat{x}^+) + \sigma(\mathbb{E}[|w|^2])$$

in which $\sigma(s) = L_{V_f}L_f s^{1/2}$ and $\sigma(\cdot) \in \mathcal{K}$. From Jensen's inequality we have $\mathbb{E}[|w|^2] \leq \mathbb{E}[|w|^2] = \text{tr}(\Sigma)$. We substitute in the nominal bound in Assumption 4 to complete the proof. ■

Lemma 7: Let Assumptions 1-4 hold. Then $\mathcal{X}_N$ is robustly positive invariant for the system $x^+ = f_{cl}(x, w)$; $w \in \mathbb{W}$ and there exists $\sigma(\cdot) \in \mathcal{K}$ such that

$$\int_{\mathbb{W}} V_N^0(x^+) d\mu(w) \leq V_N^0(x) - \ell(x, \kappa_N(x)) + \sigma(\text{tr}(\Sigma))$$

for all $x \in \mathcal{X}_N$.

Proof: Let $x \in \mathcal{X}_N$ and choose any $\mathbf{v} \in \mathbf{v}^0(x)$. We have that $x(N) = \phi(N; x, \mathbf{v}) \in \mathbb{X}_f$ for all $\mathbf{w} \in \mathbb{W}^N$ and $f(x(N), \kappa_f(x(N)), w) \in \mathbb{X}_f$ for all $w \in \mathbb{W}$ by Assumption 4. Thus, the candidate trajectory $\tilde{\mathbf{v}}^+ = (v(1), v(2), \dots, v(N-1), 0)$ satisfies $\tilde{\mathbf{v}}^+ \in \mathcal{V}_N(x^+)$ for $x^+ = f(x, \kappa_N(x), w)$ and all $w \in \mathbb{W}$. Since $\mathcal{V}_N(x^+) \neq \emptyset$, then $x^+ \in \mathcal{X}_N$ for all $w \in \mathbb{W}$ and $\mathcal{X}_N$ is robustly positive invariant. Thus, Proposition 4 holds for the closed-loop system.

Let $\tilde{\mathbf{w}}^+ = (w_1, \dots, w_{N-1}, w_N)$ and using the definition of $J_N(\cdot)$ we obtain

$$\begin{aligned} J_N(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}}^+) \\ = J_N(x, \mathbf{v}, \mathbf{w}) - \ell(x, \kappa_N(x)) + \eta(x(N), w_N) \end{aligned} \quad (9)$$

in which

$$\eta(x, w) = -V_f(x) + \ell(x, \kappa_f(x)) + V_f(f(x, \kappa_f(x), w))$$

From Lemma 6 and the fact that $x(N) \in \mathbb{X}_f$, we have

$$\int_{\mathbb{W}^{N+1}} \eta(x(N), w_N) d\mu^N(\mathbf{w}) d\mu(w_N) \leq \sigma(\text{tr}(\Sigma))$$

We also have that

$$\int_{\mathbb{W}^{N+1}} J_N(x, \mathbf{v}^0(x), \mathbf{w}) d\mu^N(\mathbf{w}) d\mu(w_N) = V_N^0(x)$$

and by the principle of optimality we have

$$V_N^0(x^+) \leq \int_{\mathbb{W}^N} J_N(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}}^+) d\mu^N(\tilde{\mathbf{w}}^+)$$

We integrate (9) w.r.t. $\mathbb{W}^{N+1}$ and apply these inequalities to complete the proof. ■

Next we construct the upper bound for the optimal cost.

Lemma 8: Let Assumptions 1-4 hold. Then there exists $\sigma_2(\cdot) \in \mathcal{K}$ such that $V_N^0(x) \leq V_f(x) + \sigma_2(\text{tr}(\Sigma))$ for all $x \in \mathbb{X}_f$.

Proof: We choose $x \in \mathbb{X}_f$ and consider a trajectory generated by repeated application of the terminal control law, i.e., $x(k) := \hat{\phi}(k; x, \mathbf{0}, \mathbf{w})$. Note that $x(k) \in \mathbb{X}_f$ for all $k \in \mathbb{I}_{\geq 0}$ since $\mathbb{X}_f$ is robustly positive invariant for the terminal control law and therefore $\mathbf{0} \in \mathcal{V}_N(x)$. We define $d := \text{tr}(\Sigma)$. From Assumption 4 and Lemma 7 we have

$$\begin{aligned} & \int_{\mathbb{W}^N} (V_f(x(k+1)) - V_f(x(k))) d\mu^N(\mathbf{w}) \\ & \leq - \int_{\mathbb{W}^N} \ell(x(k), \kappa_f(x(k))) d\mu^N(\mathbf{w}) + \sigma(d) \end{aligned}$$

for all $k \in \mathbb{I}_{[0, N-1]}$. We sum both sides of the inequality from $k = 0$ to $k = N - 1$ to give

$$\begin{aligned} & \int_{\mathbb{W}^N} (V_f(x(k)) - V_f(x(0))) d\mu^N(\mathbf{w}) \\ & \leq - \int_{\mathbb{W}^N} \sum_{k=0}^{N-1} \ell(x(k), \kappa_f(x(k))) d\mu^N(\mathbf{w}) + N\sigma(d) \end{aligned}$$

By rearranging and using the definition of $J_N(\cdot)$, we have

$$\int_{\mathbb{W}^N} J_N(x, \mathbf{0}, \mathbf{w}) d\mu^N(\mathbf{w}) \leq V_f(x) + N\sigma(d) \quad (10)$$

for all $x \in \mathbb{X}_f$. Define $\sigma_2(d) := N\sigma(d) \in \mathcal{K}$. Because $V_N^0(x)$ is optimal and $\mathbf{0} \in \mathcal{V}_N(x)$ for all $x \in \mathbb{X}_f$, we have that

$$V_N^0(x) \leq \int_{\mathbb{W}^N} J_N(x, \mathbf{0}, \mathbf{w}) d\mu^N(\mathbf{w}) \quad (11)$$

Combine (10) and (11) to complete the proof. ■

We use these Lemmata to establish the main theorem of this work.

Theorem 9: Let Assumptions 1-5 hold. Then the origin is RESiE for the stochastic system $x^+ = f_{cl}(x, w)$; $w \in \mathbb{W}$ on the robustly positive invariant set $\mathcal{X}_N$.

Proof: From Assumption 5 we have that $c_1|x|^a \leq \ell(x, u) \leq V_N^0(x)$. From Lemma 7 and the lower bound on the stage cost we have the cost decrease condition in (7) with $\sigma_3(\cdot) = \sigma(\cdot)$ and $c_3 = c_1$. From Lemma 8, we have that $V_N^0(x) \leq c_f|x|^a + \sigma_2(\text{tr}(\Sigma))$ for all $x \in \mathbb{X}_f$.

Since $\mathbb{X}_f$ contains the origin in its interior, there exists $r > 0$ such that $\{x \in \mathbb{R}^n : |x| \leq r\} \subseteq \mathbb{X}_f$. We define the set $\mathcal{C} := \{x \in \mathcal{X}_N : |x| \geq r\}$ and since $\mathcal{X}_N$ is compact (bounded by Assumption 5 and closed by Proposition 3) so is $\mathcal{C}$. Furthermore, because $J_N(\cdot)$ is continuous, the constant

$$\delta := \max \left\{ \frac{J_N(x, \mathbf{u}, \mathbf{v})}{|x|^a} : (x, \mathbf{v}, \mathbf{w}) \in \mathcal{C} \times \mathbb{V}^N \times \mathbb{W}^N \right\}$$

is well-defined and finite since $|x| > 0$ for all $x \in \mathcal{C}$. We define $c_2 := \max\{\delta, c_f\}$. Clearly, we have

$$V_N^0(x) \leq c_2|x|^a + \sigma_2(\text{tr}(\Sigma)) \quad (12)$$

for all $x \in \mathbb{X}_f$. Furthermore, if $x \in \mathcal{X}_N$ but $x \notin \mathbb{X}_f$, then $x \in \mathcal{C}$, and therefore $(V_N^0(x)/|x|^a) \leq \delta$ by optimality. Thus, (12) holds for all $x \in \mathcal{X}_N$. Therefore, $V_N^0(x)$ is an (exponential) SISS-Lyapunov function and by Proposition 5 the proof is complete. ■

We note that the $\mathcal{K}$ -function σ_2 that we have constructed from this analysis increases with increasing horizon length N . This feature suggests that the function $\gamma(\text{tr}(\Sigma))$ in (5) also increases with horizon length N for an otherwise equivalent SMPC problem. We argue, however, that this result is not a weakness of the analysis technique chosen, but an underlying feature of nonlinear SMPC (subject to these assumptions). We demonstrate this fact and other nonintuitive behaviors of nonlinear SMPC with the following numerical example.

V. NUMERICAL EXAMPLE

Consider the discrete-time stochastic system

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} x_1 \\ 1.1x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} (1-x_2)w \\ 0 \end{bmatrix}$$

in which $\Pr(w = 0.4) = \Pr(w = -0.4) = 0.3$ and $\Pr(w = 0) = 0.4$. We enforce the following constraints

$$\begin{bmatrix} -3 \\ -0.1 \end{bmatrix} \leq x \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -5 \\ -0.2 \end{bmatrix} \leq u \leq \begin{bmatrix} 5 \\ 0.2 \end{bmatrix}$$

and apply the stage cost $\ell(x, u) = x'Qx + u'Ru$ in which $Q = R = \text{diag}([1, 0.1])$. We note that the nominal system (i.e., $w = 0$) is linear. We use the LQR cost P and gain K for the nominal system to define the terminal cost $V_f(x) = x'Px$ and terminal control law $\kappa_f(x) = Kx$. We define the terminal constraint as $\mathbb{X}_f = \{x \in \mathbb{R}^n : |x_1| \leq 0.6, |x_2| \leq 0.05\}$ and confirm that Assumption 4 holds. We define the control policy as $\pi(x, v) := Kx + v$ and $\mathbb{V}$ such that for all $(x, u) \in \mathbb{Z}$ there exists $v \in \mathbb{V}$ such that $\pi(x, v) = u$. Assumption 5 is satisfied with $a = 2$.

We simulate the closed-loop response of this stochastic system subject to nominal MPC and nonlinear SMPC controllers with varying horizon lengths. We initialize the state at $x(0) = [-2, 0]'$. We solve the stochastic optimization problem by considering all possible disturbance realizations in the optimization horizon. Since the disturbance has finite support, all expected values are calculated exactly.

We plot the expected value of the state in Figure 1. The SMPC controller with a horizon of $N = 1$ produces a similar closed-loop trajectory to the nominal MPC controller, but with a slightly more aggressive control action. However, as we increase the horizon length of the SMPC controller, $\mathbb{E}[x_2(k)]$ is driven away from the origin. Nonlinear SMPC with $N \geq 2$ does not stabilize the terminal region (with probability one) as we might expected for linear SMPC or ISS nonlinear stochastic systems in general [10], [21]. We emphasize that this behavior is a result of the controller and not strictly necessary for the chosen stochastic system (as evident by the closed-loop trajectory of SMPC with $N = 1$).

In Figure 2, we plot the value of $\mathbb{E}[|x(k)|]$ for each closed-loop stochastic system. We note that these results are consistent with the bound in (5). For all horizon lengths of

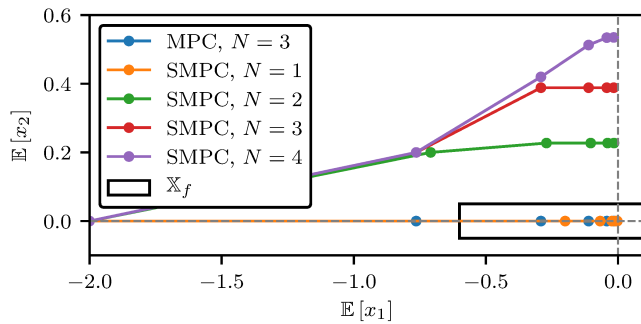


Fig. 1. The expected value of each state variable for the closed-loop stochastic system for different controllers.

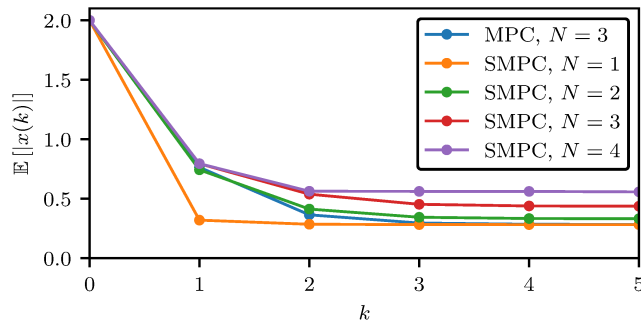


Fig. 2. The expected value of the norm of the state for the closed-loop stochastic system for different controllers.

SMPC, we observe an initial decay in the value of $\mathbb{E}[||x(k)||]$ as the effect of the initial condition vanishes. After $k = 5$ the value of $\mathbb{E}[||x(k)||]$ remains fairly constant. The value of $\mathbb{E}[||x(k)||]$ as $k \rightarrow \infty$, however, increases with increasing horizon length N for SMPC. This example suggests that the dependence of $\gamma(\cdot)$ on the horizon length N is not an artifact of the chosen analysis approach and is instead a property of nonlinear SMPC. Even quasi-linear systems, such as the one in this example, can produce closed-loop results that are distinctly different from linear systems.

This nonintuitive behavior occurs because the effect of the disturbance is attenuated by larger values of x_2 and we have selected smaller penalties for x_2, u_2 than x_1, u_1 . Thus, driving the system to a large value of x_2 is favorable in terms of expected cost. By increasing the horizon length, we allow the optimizer to explore operating points further away from the origin and terminal set, resulting in larger values of $\mathbb{E}[x_2(k)]$ as $k \rightarrow \infty$. In general, however, the effect of horizon length on the behavior of nonlinear SMPC remains an open question.

VI. CONCLUSIONS

In this work, we have established that SMPC, under suitable assumptions, produces a well-defined optimization problem and all stochastic properties of interest for the closed-loop system are indeed well-defined. We also establish that the closed-loop system generated by nonlinear SMPC is RESiE. This result provides a clear connection

between the behavior of nonlinear SMPC and the disturbance w that is often absent from closed-loop analysis of SMPC. Through a numerical example, we demonstrated both the implications of this analysis and the nonintuitive closed-loop behavior that may arise from nonlinear SMPC.

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REFERENCES

- [1] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, "Examples when nonlinear model predictive control is nonrobust," *Automatica*, vol. 40, pp. 1729–1738, 2004.
- [2] G. Pannocchia, J. B. Rawlings, and S. J. Wright, "Conditions under which suboptimal nonlinear MPC is inherently robust," *Sys. Cont. Let.*, vol. 60, pp. 747–755, 2011.
- [3] S. Yu, M. Reble, H. Chen, and F. Allgöwer, "Inherent robustness properties of quasi-infinite horizon nonlinear model predictive control," *Automatica*, vol. 50, no. 9, pp. 2269 – 2280, 2014.
- [4] D. A. Allan, C. N. Bates, M. J. Risbeck, and J. B. Rawlings, "On the inherent robustness of optimal and suboptimal nonlinear MPC," *Sys. Cont. Let.*, vol. 106, pp. 68 – 78, 2017.
- [5] D. Mayne, "Robust and stochastic model predictive control: Are we going in the right direction?" *Annual Rev. Control*, vol. 41, pp. 184 – 192, 2016.
- [6] A. Mesbah, "Stochastic model predictive control," *IEEE Ctl. Sys. Mag.*, pp. 30–44, Dec 2016.
- [7] M. Farina, L. Giulioni, and R. Scattolini, "Stochastic linear model predictive control with chance constraints—a review," *J. Proc. Cont.*, vol. 44, pp. 53–67, 2016.
- [8] J. A. Primbs and C. H. Sung, "Stochastic receding horizon control of constrained linear systems with state and control multiplicative noise," *IEEE Trans. Auto. Cont.*, vol. 54, no. 2, pp. 221–230, 2009.
- [9] B. Kouvaritakis, M. Cannon, S. V. Raković, and Q. Cheng, "Explicit use of probabilistic distributions in linear predictive control," *Automatica*, vol. 46, no. 10, pp. 1719 – 1724, 2010.
- [10] M. Lorenzen, F. Dabbene, R. Tempo, and F. Allgöwer, "Constraint-tightening and stability in stochastic model predictive control," *IEEE Trans. Auto. Cont.*, vol. 62, no. 7, pp. 3165–3177, 2016.
- [11] D. Chatterjee and J. Lygeros, "On stability and performance of stochastic predictive control techniques," *IEEE Trans. Auto. Cont.*, vol. 60, no. 2, pp. 509–514, 2014.
- [12] D. Q. Mayne and P. Falugi, "Stabilizing conditions for model predictive control," *Int. J. Robust and Nonlinear Control*, vol. 29, no. 4, pp. 894–903, 2019.
- [13] M. Lorenzen, M. A. Müller, and F. Allgöwer, "Stochastic model predictive control without terminal constraints," *Int. J. Robust and Nonlinear Control*, vol. 29, no. 15, pp. 4987–5001, 2019.
- [14] H. J. Kushner, *Stochastic Stability and Control*, ser. Mathematics in Science and Engineering. New York: Academic Press, 1967, vol. 33.
- [15] J. Tsiniias, "Stochastic input-to-state stability and applications to global feedback stabilization," *Int. J. Control*, vol. 71, no. 5, pp. 907–930, 1998.
- [16] M. Krstic and H. Deng, *Stabilization of nonlinear uncertain systems*. Springer-Verlag, 1998.
- [17] C. Tang and T. Basar, "Stochastic stability of singularly perturbed nonlinear systems," in *Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No. 01CH37228)*, vol. 1. IEEE, 2001, pp. 399–404.
- [18] R. D. McAllister and J. B. Rawlings, "Stochastic model predictive control: Existence and measurability," TWCCC Technical Report, Tech. Rep. 2021–01, March 2021. [Online]. Available: <https://sites.engineering.ucsb.edu/~jbraw/jbrweb-archives/tech-reports/twccc-2021-01.pdf>
- [19] J. B. Rawlings, D. Q. Mayne, and M. M. Diehl, *Model Predictive Control: Theory, Design, and Computation*, 2nd ed. Madison, WI: Nob Hill Publishing, 2020, 770 pages, ISBN 978-0-9759377-5-4.
- [20] D. P. Bertsekas and S. E. Shreve, *Stochastic Optimal Control: The Discrete Time Case*. New York: Academic Press, 1978.
- [21] D. Munoz-Carpintero and M. Cannon, "Convergence of stochastic nonlinear systems and implications for stochastic model predictive control," *IEEE Trans. Auto. Cont.*, 2020.