Applied Mathematics & Optimization (2021) 84 (Suppl 2):S1821–S1863
https://doi.org/10.1007/s00245-021-09812-5

Fluid–Structure Interaction with Kelvin–Voigt Damping: Analyticity, Spectral Analysis, Exponential Decay

Rasika Mahawattege[1] · **Roberto Triggiani**[1]

Accepted: 9 July 2021 / Published online: 7 August 2021

Abstract
We consider a fluid–structure interaction model defined on a doughnut-like domain. It consists of the dynamic Stokes equations evolving on the exterior sub-domain, coupled with an elastic structure occupying the interior sub-domain. A key factor—a novelty over past literature—is that the structure equation includes a strong (viscoelastic) damping term of Kelvin–Voigt type at the interior. This affects the boundary conditions at the interface between the two media and accounts for a highly unbounded "perturbation". Results include: (i) analyticity of s.c semigroup of contractions defining the overall coupled system, (ii) its (uniform) exponential decay, along with (iii) sharp spectral properties of its generator. Some results are geometry-dependant.

Keywords Kelvin–Voigt damping · Analyticity · Exponential decay

1 Introduction and Statement of Main Results

1.1 The Coupled PDE Model

Throughout the paper, $\Omega_f \subseteq \mathbb{R}^d$, $d = 2$ or 3, will denote the bounded domain on which the fluid component of the coupled PDE system evolves. Its boundary will be denoted here as $\partial\Omega_f = \Gamma_s \cup \Gamma_f$, $\Gamma_s \cap \Gamma_f = \emptyset$, with each boundary component being sufficiently smooth. Moreover, the geometry Ω_s, immersed within Ω_f, will be the domain on which the structural component evolves with time. As configured then, the coupling between the two distinct fluid and elastic dynamics occurs across boundary interface $\Gamma_s = \partial\Omega_s$; see Fig. 1. In addition, the unit normal vector $\nu(x)$ will be directed away from Ω_f; thus on Γ_s, *toward* Ω_s. (This specification of the direction of ν will influence the computations to be done below.)

✉ Rasika Mahawattege
rmhwttge@memphis.edu

[1] Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA

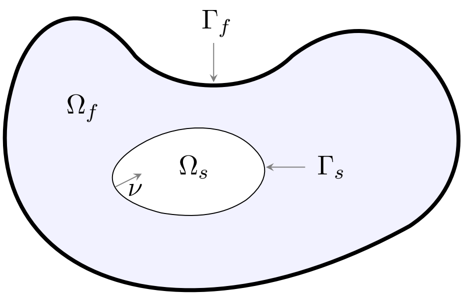


Fig. 1 The Fluid–Structure Interaction

On this geometry in Fig. 1, we thus consider the following fluid–structure PDE model in solution variables $u = [u_1(t, x), u_2(t, x), \ldots, u_d(t, x)]$ (the velocity field), and $w = [w_1(t, x), w_2(t, x), \ldots, w_d(t, x)]$ (the structural displacement field), while the scalar-valued variable p denotes the pressure:

$$\text{PDE} \begin{cases} u_t - \Delta u + \nabla p \equiv 0 \quad \text{in } (0, T] \times \Omega_f \equiv Q_f & (1.1\text{a}) \\ \operatorname{div} u \equiv 0 \quad \text{in } Q_f & (1.1\text{b}) \\ w_{tt} - \Delta w - \Delta w_t + bw \equiv 0 \quad \text{in } (0, T] \times \Omega_s \equiv Q_s & (1.1\text{c}) \end{cases}$$

$$\text{B.C} \begin{cases} u|_{\Gamma_f} \equiv 0 \quad \text{on } (0, T] \times \Gamma_f \equiv \Sigma_f & (1.1\text{d}) \\ u \equiv w_t \quad \text{on } (0, T] \times \Gamma_s \equiv \Sigma_s & (1.1\text{e}) \\ \dfrac{\partial u}{\partial \nu} - \dfrac{\partial (w + w_t)}{\partial \nu} = p\nu \quad \text{on } \Sigma_s & (1.1\text{f}) \end{cases}$$

$$\text{I.C.}\, u(0, \cdot) = u_0,\ w(0, \cdot) = w_0,\ w_t(0, \cdot) = w_1,\ \text{on } \Omega. \qquad (1.1\text{g})$$

The constant b in (1.1c) will take up either the value $b = 0$ or else the value $b = 1$. Accordingly, the space of well-posedness is taken to be the finite energy space [16]:

$$\mathcal{H}_b \begin{cases} (H^1(\Omega_s)/\mathbb{R})^d \times (L^2(\Omega_s))^d \times \tilde{H}_f, \quad b = 0; & (1.2\text{a}) \\ (H^1(\Omega_s))^d \times (L^2(\Omega_s))^d \times \tilde{H}_f, \quad b = 1, & (1.2\text{b}) \end{cases}$$

for the variable $[w, w_t, u]$, where

$$\tilde{H}_f = \{f \in (L_2(\Omega_f))^d : \operatorname{div} f \equiv 0 \text{ in } \Omega_f;\ f \cdot \nu \equiv 0 \text{ on } \Gamma_f\}. \qquad (1.3)$$

$\mathcal{H}_b$ is a Hilbert space with the following norm inducing inner product, where $(f, g)_\Omega \equiv \int_\Omega f \bar{g}\, d\Omega$:

$$\left(\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{f} \end{bmatrix} \right)_{\mathcal{H}_b} = \begin{cases} (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + (f, \tilde{f})_{\Omega_f}, b = 0; & (1.4\text{a}) \\ (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_1, \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + (f, \tilde{f})_{\Omega_f}, b = 1. & (1.4\text{b}) \end{cases}$$

In (1.2a), the space $(H^1(\Omega)/\mathbb{R})^d = (H^1(\Omega)/\text{ const })^d$ is endowed with the gradient norm. There are some noteworthy differences between the cases $b = 0$ and $b = 1$, as studied on their respective spaces in (1.4). The physical importance of the above model is described in [24,38].

Literature There is rather vast literature concerning fluid–structure interaction models, both linear and non-linear (full Navier–Stokes equations), with static interface (appropriate for small and rapid oscillations) [35, p. 53], [17, p. 53] as well as with moving interface. As we cannot be exhaustive, here we concentrate by necessity on works where the structure contains visco-elastic (Kelvin–Voigt) damping. While referring to [1–5,14,15,18,19] for linear/non-linear models with no Kelvin–Voigt damping. To gain insight, preliminary studies have focused on replacing the fluid equation with a heat equation, as the presence of the pressure is a source of additional complications. In this setting, a first work was [24] which in fact serves as a guidance for the present paper as well as [38]. Subsequent works [36,37] dealt with the structure being a plate with Kelvin–Voigt damping and physical interface conditions involving bending and stress boundary operators. [38] uses a variational approach in seeking to extend [24] to the Stokes case with pressure, while our present paper eliminates the pressure by expressing it explicitly in terms of other variables by solving a suitable elliptic problem. Of course, application of the Leray–Helmholtz projection to eliminate the pressure is out of question due to non-homogeneous interface conditions. Our present results are more precise than those in [38] and cover not only the case $b = 0$ but also the case $b = 1$, the two cases in effect having some important differences. In addition, we show that some results are geometric-dependent, an issue not present in [38]. [38] covers only the model $b = 0$. Once one establishes that the free-dynamics generates a s.c. (contraction) analytic semigroup, the next task is to seek to characterize the domains of fractional powers of the negative generator. This would then allow to establish optimal regularity results under the input of a control term acting either at the interface or else at the exterior boundary. Success in this will ultimately allow one to apply the abstract general optimal control theory with quadratic cost of [23] to the present problem, under the additional action of a boundary control. In the case with no pressure, this program was carried out in [32]. Its generalization to the present case will require extending the results of [25,33,34] on domain of fractional powers to the fluid case in (1.1a–g).

1.2 Main Results

We state here upfront some of the main results of the paper.

Theorem 1.1 (Semigroup well-posedness on $\mathcal{H}_b$, $b = 0$ and $b = 1$)
(i) *The PDE problem (1.1a–g) admits the following abstract model*

$$\dot{x} = \mathcal{A}_b x, \quad \textit{in } \mathcal{H}_b, \quad x = [w, w_t, f] \tag{1.5}$$

where the operator $\mathcal{A}_b : \mathcal{H}_b \supset \mathcal{D}(\mathcal{A}_b) \to \mathcal{H}_b$ is explicitly defined in Sect. 2 *by Eq.* (2.3) *with domain characterized in Proposition* 2.1.

(ii) *The operator $\mathcal{A}_b$ is dissipative on $\mathcal{H}_b$: for $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_b)$, we have:*

$$Re\left(\mathcal{A}_b\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}\right)_{\mathcal{H}_b} = -\int_{\Omega_s} |\nabla v_2|^2 d\Omega_f - \int_{\Omega_f} |\nabla f|^2 d\Omega_f \leq 0. \quad (1.6)$$

(iii) *The adjoint operator $\mathcal{A}_b^* : \mathcal{H}_b \supset \mathcal{D}(\mathcal{A}_b^*) \to \mathcal{H}_b$, which is explicitly defined by* (4.1)*, with domain $\mathcal{D}(\mathcal{A}_b^*)$ characterized by Proposition* 4.2 *, is likewise dissipative on $\mathcal{H}_b$: for $[v_1^*, v_2^*, f^*] \in \mathcal{D}(\mathcal{A}_b^*)$, we have:*

$$Re\left(\mathcal{A}_b^*\begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix}\right)_{\mathcal{H}_b} = -\int_{\Omega_s} |\nabla v_2^*|^2 \, d\Omega_s - \int_{\Omega_f} |\nabla f^*|^2 d\Omega_f. \quad (1.7)$$

. (iv) *The operators $\mathcal{A}_b$ and $\mathcal{A}_b^*$ generate s.c contraction semigroups $e^{\mathcal{A}_b t}$ and $e^{\mathcal{A}_b^* t}$ on $\mathcal{H}_b$. Thus the PDE-problem* (1.1a–g) *admits the following unique semigroup solution with respect to the abstract form* (1.5).

$$\begin{bmatrix} w(t) \\ w_t(t) \\ u(t) \end{bmatrix} = e^{\mathcal{A}_b t}\begin{bmatrix} w_0 \\ w_1 \\ u_0 \end{bmatrix} \in C\left([0, \infty); \mathcal{H}_b\right). \quad (1.8)$$

See Proposition 4.3 for the relationship between $\mathcal{D}(\mathcal{A}_b)$ and $\mathcal{D}(\mathcal{A}_b^*)$. Section 2 provides the explicit abstract model of the free dynamics (1.1a–g), $b = 0, b = 1$. Dissipativity of $\mathcal{A}_b$ is proved in Sect. 3. The adjoint $\mathcal{A}_b^*$ is defined and proved to be dissipative in Sect. 4. As a consequence of both $\mathcal{A}_b$ and $\mathcal{A}_b^*$ being dissipative, they are the generators of s.c contraction semigroups (Proposition 4.4) [6,27].

Theorem 1.2 *(Spectral properties of $\mathcal{A}_{b=0}$ and $\mathcal{A}_{b=0}^*$ on the imaginary axis $i\mathbb{R}$)*
(i) *For $b = 0$, the point $i\omega$, $0 \neq \omega \in \mathbb{R}$, belongs to the resolvent set of $\mathcal{A}_{b=0}$: $i\omega \in \rho(\mathcal{A}_{b=0}), \omega \neq 0$.*
(ii) *Let $b = 0$. Assume now, the generic geometric condition $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$ on Ω_s (see Remark 5.1). Then the point $\lambda = 0$ is in the resolvent set of $\mathcal{A}_{b=0} : 0 \in \rho(\mathcal{A}_{b=0})$. Hence there is a small open disk $\mathcal{S}_{r_0}$ in the complex plane centered at the origin and of small radius $r_0 > 0$, that it is all contained in $\rho(\mathcal{A}_{b=0}) : \mathcal{S}_{r_0} \in \rho(\mathcal{A}_{b=0})$.*
(iii) *Let $b = 0$. Assume now the condition $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$ (for symmetric regions Ω_s). Then $\lambda = 0$ is an eigenvalue of both the operator $\mathcal{A}_{b=0}$ and its adjoint $\mathcal{A}_{b=0}^*$ on $\mathcal{H}_{b=0}$ with corresponding common eigenvector $e_0 = [\eta_0, 0, 0]$, where η_0 is the unique solution of the following elliptic problem:*

$$\Delta\eta_0 = 0 \text{ in } \Omega_s, \left.\frac{\partial\eta_0}{\partial\nu}\right|_{\Gamma_s} = \nu \text{ on } \Gamma_s \quad (1.9)$$

Moreover, the point $\lambda = 0$ *is in the resolvent set* $\rho(\widehat{\mathcal{A}}_{b=0})$ *of the operator* $\widehat{\mathcal{A}}_{b=0} \equiv \mathcal{A}_{b=0}|_{\widehat{\mathcal{H}}_{b=0}}$*, where the space* $\widehat{\mathcal{H}}_{b=0}$ *of codimension 1 in* $\mathcal{H}_{b=0}$ *is given by*

$$\widehat{\mathcal{H}}_{b=0} = [Null(\mathcal{A}_{b=0})]^{\perp} = \mathcal{H}_{b=0}/[Null(\mathcal{A}_{b=0})] \\ = \left\{[v_1, v_2, f] \in \mathcal{H}_{b=0} : \int_{\Gamma_s} v_1 \cdot \nu \, d\Gamma_s = 0\right\}. \tag{1.10}$$

Hence there is a small open disk $\mathcal{S}_{r_0}$ *in the complex plane centered at the origin and of small radius* $r_0 > 0$*, that it is all contained in* $\rho(\widehat{\mathcal{A}}_{b=0}) : \mathcal{S}_{r_0} \in \rho(\widehat{\mathcal{A}}_{b=0})$*. Similar results holds for* $\widehat{\mathcal{A}}^*_{b=0} = \mathcal{A}^*_{b=0}|_{\mathcal{H}_{b=0}}$ *on* $\widehat{\mathcal{H}}_{b=0}$*:* $\mathcal{S}_{r_0} \in \rho(\widehat{\mathcal{A}}^*_{b=0})$*. The space* $\widehat{\mathcal{H}}_{b=0}$ *is invariant for* $e^{\mathcal{A}_{b=0}t}$ *or* $e^{\mathcal{A}^*_{b=0}t}$*.*

Theorem 1.2 (i) in its full strength is contained in Theorem 6.1 (for $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$) and in Theorem 7.1 (for $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$). That $i\omega \notin \sigma_p(\mathcal{A}_b), i\omega \notin \sigma_p(\mathcal{A}^*_b)$, hence $i\omega \notin \sigma_r(\mathcal{A}_b)$, is contained in Remark 4.1. The rest of Theorem 1.2 is proved in Proposition 5.1 through Proposition 5.3, with invariance established in Proposition 5.4. The main results of the paper are given in the next Theorem.

Theorem 1.3 *Let* $b = 0$. (i) *The generator* $\mathcal{A}_{b=0}$ *in* (2.3) *of the s.c. contraction semigroup* $e^{\mathcal{A}_{b=0}t}$ *asserted by Theorem* 1.1(iv) *satisfies the following resolvent condition for* $\omega \in \mathbb{R}$

$$\|(i\omega I - \mathcal{A}_{b=0})^{-1}\|_{\mathcal{L}(\mathcal{H}_{b=0})} = \|R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \frac{c}{|\omega|}. \\ \forall\, |\omega| \geq \text{ some } \omega_0 > 0 \text{ arbitrarily small,} \tag{1.11}$$

Hence, under the (generic) assumption $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$ *on* Ω_s*, the s.c. semigroup* $e^{\mathcal{A}_{b=0}t}$ *is analytic on the finite energy space* $\mathcal{H}_{b=0}$*,* $t > 0$*,* [23, Thm 3E.3, p 334].

(ii) *If, instead,* $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$ *(symmetric region* Ω_s*), then the s.c semigroup* $e^{\widehat{\mathcal{A}}_{b=0}t}$ *is analytic on the space* $\widehat{H}_{b=0}$ *where it is invariant; consequently* $e^{\mathcal{A}_{b=0}t}$ *is analytic on* $\mathcal{H}_{b=0}$ *also in this symmetric case.*

(iii) *More precisely, the resolvent operator* $R(\lambda, \mathcal{A}_{b=0}) = (\lambda I - \mathcal{A}_{b=0})^{-1}$ *of the generator* $\mathcal{A}_{b=0}$ *in* (2.3)*, satisfies the following estimate*

$$\|R(\lambda, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \frac{C}{|\lambda|}, \text{ for all } \lambda \in \mathbb{C}\backslash\mathcal{K} \tag{1.12a}$$

where $\mathcal{K}$ *is the (infinite) key-shaped set defined by (see Fig.* 2*)*

$$\mathcal{K} \equiv (-\infty, -2) \cup \{\mathcal{S}_{r=1}(x_0)\backslash\mathcal{S}_{r_0}\} \tag{1.12b}$$

with $\mathcal{S}_{r=1}(x_0)$ *the open disk centered at the point* $x_0 = \{-1, 0\}$ *and of radius* 1*; and* S_{r_0} *defined in Theorem* 1.2(ii). (iv) *The spectrum* $\sigma(\mathcal{A}_{b=0})$ *of* $\mathcal{A}_{b=0}$ *is confined within the set* $\mathcal{K}$*; in particular*

$$Re\,\sigma(\mathcal{A}_{b=0}) \subset (-\infty, -\delta], \quad \text{for some } \delta > 0. \tag{1.13}$$

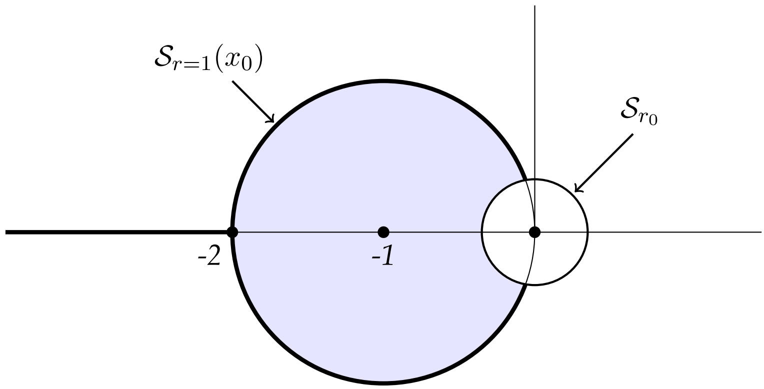


Fig. 2 The set $\mathcal{K}$. For $b = 0\,, \int_{\Gamma_s} \nu\, d\Gamma_s \neq 0$

(v) *The boundary (circumference) of the open disk* $\mathcal{S}_{r=1}(x_0)$ *either belongs to the resolvent set* $\rho(\mathcal{A}_{b=0})$ *of* $\mathcal{A}_{b=0}$*, or else belongs to the point spectrum* $\sigma_p(\mathcal{A}_{b=0})$ *of* $\mathcal{A}_{b=0}$*, according to as whether the over-determined elliptic problem*

$$\begin{cases} \Delta v_1 = 2\alpha v_1 \text{ in } \Omega_s \quad \alpha < 0 & \text{(1.14a)} \\ v_1|_{\Gamma_s} = 0, \ \left.\dfrac{\partial v_1}{\partial \nu}\right|_{\Gamma_s} = k\nu, \quad k = \text{ undetermined constant} & \text{(1.14b)} \end{cases}$$

implies $v_1 \equiv 0$ *in* Ω_s *and hence* $k = 0$*, or else implies* $v_1 \not\equiv 0$ *in* Ω_s*. Such outcome depends on the geometrical properties of* Ω_s*. See Remark 1.1.*

(vi) *Complementing* (1.12a) *we have that the resolvent* $R(\cdot, \mathcal{A}_{b=0})$ *is uniformly bounded on the imaginary axis*

$$\|R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq c, \quad \forall\, \omega \in \mathbb{R}. \tag{1.15}$$

Hence, the s.c. analytic semigroup $e^{\mathcal{A}_{b=0}t}$ *is uniformly exponentially stable on* $\mathcal{H}_{b=0}$*: there exist constants* $M \geq 1$*,* $\delta > 0$*, such that* [28]

$$\|e^{\mathcal{A}_{b=0}t}\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq Me^{-\delta t}, \quad t \geq 0. \tag{1.16}$$

Theorem 1.3 (analyticity) is proved in Theorem 6.1 for $\int_{\Gamma_s} \nu\, d\Gamma_s \neq 0$ and in Corollary 7.1 for $\int_{\Gamma_s} \nu\, d\Gamma_s = 0$.Invariance of $\widehat{\mathcal{H}}_{b=0}$ under the action of $e^{\mathcal{A}_{b=0}t}$ is established in Proposition 5.4. Theorem 1.3(v) is proved in Proposition 5.6. Theorem 1.3(vi) (uniform stabilization) is proved in Sect. 6.3 for $\int_{\Gamma_s} \nu\, d\Gamma_s \neq 0$. Refer to Theorem 1.2 (iii) for $\int_{\Gamma_s} \nu\, d\Gamma_s = 0$ whereby $0 \in \sigma_p(\mathcal{A}_b)$, $0 \in \sigma_p(\mathcal{A}_b^*)$.

Remark 1.1 For the over-determined problem (1.14), we have $v_1 \not\equiv 0$, if Ω_s=2-D disk or 3-D ball. Instead, problem (1.14) implies $v_1 \equiv 0$ for many geometries; e.g if the boundary Γ_s of Ω_s is partially flat; or partially spherical; or partially parabola-like;

or partially hyperbola-like. This condition arose in the study of the fluid–structure interaction model of [1], [17, Chapter 2] with no visco-elastic damping [35, Chapter 2].

Theorem 1.4 *Let $b = 0, 1$.The point $x_0 = \{-1, 0\}$, center of the disk $\mathcal{S}_{r=1}(x_0)$, belongs to the continuous spectrum of $\mathcal{A}_b$, as well as of $\mathcal{A}_b^*$: $-1 \in \sigma_c(\mathcal{A}_b)$, $-1 \in \sigma_c(\mathcal{A}_b^*)$. Thus, $\mathcal{A}_b$ and $\mathcal{A}_b^*$ do not have compact resolvent on $\mathcal{H}_b$.*

Theorem 1.4 is proved in Proposition 4.5.

Remark 1.2 When Ω_s= 2-D disk,3-D ball, the point spectrum $\sigma_p(\mathcal{A}_{b=0})$ contains a branch of 'explicitly' known eigenvalues $\alpha_n^- \to -\infty$ monotonically and a branch of eigenvalues $\alpha_n^+ \nearrow -1$ (the point in the continuous spectrum $\sigma_c(\mathcal{A}_{b=0})$) monotonically. See Proposition 5.8. Such description is in line with the behavior of the eigenvalues in the abstract equation $\ddot{x} + A\dot{x} + Ax = 0$ studied in [10].

Theorem 1.5 ($b = 1$. Spectral properties of the origin, Semigroup generation)

(i) *Let $b = 1$. Then the origin $\lambda = 0$ is an eigenvalue of the operator $\mathcal{A}_{b=1}$ on $\mathcal{H}_{b=1}$, as well as the adjoint $\mathcal{A}_{b=1}^*$ with common corresponding eigenvector $e_1 = [\eta_1, 0, 0]$, where η_1 is the unique solution of the following elliptic problem:*

$$\Delta\eta_1 - \eta_1 = 0 \textit{ in } \Omega_s; \quad \left.\frac{\partial\eta_1}{\partial\nu}\right|_{\Gamma_s} = \nu \textit{ on } \Gamma_s. \tag{1.17}$$

Moreover, the point $\lambda = 0$ is in the resolvent set $\rho(\widehat{\mathcal{A}}_{b=1})$ of the operator $\widehat{\mathcal{A}}_{b=1} = \mathcal{A}_{b=1}|_{\widehat{\mathcal{H}}_{b=1}}$ where the space $\widehat{\mathcal{H}}_{b=1}$ of codimension 1 in $\mathcal{H}_{b=1}$ is given by

$$\begin{aligned} \widehat{\mathcal{H}}_{b=1} &= [Null(\mathcal{A}_{b=1}]^\perp = \mathcal{H}_{b=1}/[Null(\mathcal{A}_{b=1})] \\ &= \left\{[v_1, v_2, f] \in \mathcal{H}_{b=1} : \int_{\Gamma_s} v_1 \cdot \nu \, d\Gamma_s = 0\right\}. \end{aligned} \tag{1.18}$$

(ii) *The factor space $\widehat{\mathcal{H}}_{b=1} = [Null(\mathcal{A}_{b=1})]^\perp = [Null(\mathcal{A}_{b=1}^*)]^\perp$ is invariant under the action of the semigroups $e^{\mathcal{A}_{b=1}t}$ and $e^{\mathcal{A}_{b=1}^*t}$ on the space $\mathcal{H}_{b=1}$.*

(iii) *The operators $\widehat{\mathcal{A}}_{b=1}$ and $\widehat{\mathcal{A}}_{b=1}^*$ generate s.c contraction semigroups on the space $\widehat{\mathcal{H}}_{b=1}$ in (1.18), which, moreover, are uniformly stable on $\widehat{\mathcal{H}}_{b=1}$.*

(iv) *The operators $\mathcal{A}_{b=1}$ and $\mathcal{A}_{b=1}^*$ generate s.c contraction semigroups $e^{\mathcal{A}_{b=1}t}$ and $e^{\mathcal{A}_{b=1}^*t}$ on the space $\mathcal{H}_{b=1}$.*

Theorem 1.6 *The s.c semigroup $e^{\mathcal{A}_{b=1}t}$, as asserted by Theorem 1.5*(iv)*, is analytic on $\mathcal{H}_{b=1}$. More precisely, let*

$$\mathcal{H}_{b=1} = \mathcal{H}_{b=0} + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (H^1(\Omega_s)/\mathbb{R})^d \\ (L^2(\Omega_s))^d \\ \tilde{H}_f \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}, \quad c = constant \tag{1.19}$$

so that any $x_1 \in \mathcal{H}_{b=1}$ *can be written uniquely as the following direct sum*

$$x_1 = \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_1/\mathbb{R} \\ v_2 \\ f \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = x_0 + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \tag{1.20}$$

Then

$$e^{\mathcal{A}_{b=1}t}x_1 = e^{\mathcal{A}_{b=1}t}x_0 + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} \sin t. \tag{1.21}$$

Theorem 1.5 is proved in [26]. Theorem 1.6 is proved in Sect. 8.

Orientation on Anayticity Theorem 1.3 (i),(ii) for $b = 0$ on $\mathcal{H}_{b=0}$; and Theorem 1.6 for $b = 1$ on $\mathcal{H}_{b=1}$.

As noted in [24] in the case with no pressure (heat eqation rather than Stokes equations), analyticity per se is not surprising in view of the following motivating considerations.

A motivating result (a) *Analyticity* The following is *a very special case of a much more general result* for which we refer to [9–11], (see also [23, Appendix 3B of Chapter 3, pp 285-296], [12,13]). These references solve and improve upon the conjectures posted in [8]. Let A be a positive, self-adjoint operator on the Hilbert space Y. On it, consider the following abstract equation

$$\ddot{x} + Ax + A\dot{x} = 0; \quad \text{or } \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \mathbb{A}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}; \tag{1.22}$$

$$\mathbb{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & -A \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -A(x_1 + x_2) \end{bmatrix}; \tag{1.23a}$$

$$\mathcal{D}(\mathbb{A}) = \left\{[x_1, x_2] \in E \equiv \mathcal{D}(A^{\frac{1}{2}}) \times Y : \; x_2 \in \mathcal{D}(A^{\frac{1}{2}}), \; x_1 + x_2 \in \mathcal{D}(A)\right\}. \tag{1.23b}$$

The operator $\mathbb{A}$ is dissipative and with domain (1.23b) is closed and generates a s.c. contraction semigroup $e^{\mathbb{A}t}$ on the finite energy space $E \equiv \mathcal{D}(A^{\frac{1}{2}}) \times Y$, which moreover is analytic on E. Thus, the second-order dynamic (1.22) with strong 'structural' damping is parabolic-like.

(b) *The spectrum of* $\mathbb{A}$ Reference [11, Appendix A, Lemma A.1, p45] shows that the spectrum $\sigma(\mathbb{A})$ of the operator $\mathbb{A}$ defined in (1.22)–(1.23b) has the following features assuming that the positive self-adjoint operator A has compact resolvent on Y:

The spectrum of $\mathbb{A}$ consists of two branches of eigenvalues $\lambda_n^{+,-}$:

$$\lambda_n^{+,-} = -\frac{\mu_n}{2} \pm \frac{\mu_n}{2}\sqrt{\frac{\mu_n - 4}{\mu_n}}, \tag{1.24}$$

solutions of the algebraic equation $\lambda^2 + \mu_n\lambda + \mu_n = 0$, where $\{\mu_n\}_{n=1}^{\infty}$ are the eigenvalues of the positive self-adjoint operator $A : 0 < \mu_1 < \cdots < \mu_n \to +\infty$. The branch $\lambda_n^- \searrow -\infty$ monotonically. The branch $\lambda_n^+ \nearrow -1$ monotonically. Moreover, the point $\lambda = -1$ belongs to the continuous spectrum $\sigma_c(\mathbb{A})$ of the operator $\mathbb{A}$. The operator $\mathbb{A}$ does not have compact resolvent on the finite energy space E, even though A has compact resolvent on Y.

Regarding our original coupled problem (1.1a–g) even without pressure as in [24], the above abstract result for equation (1.23a) suggests, or makes one surmise, that the homogeneous problem (1.1a–g) is the coupling of 'two parabolic problems' and hence generates an analytic semigroup $e^{\mathcal{A}_b t}$ ($\mathcal{A}_b$ in (1.5)) on the finite energy space $\mathcal{H}_b$ in (1.4). Of course, the above considerations are purely indicative and qualitatively suggestive, as the Laplacian Δ in (1.1c) has coupled, high-level, non-homogeneous interface boundary conditions which constitute the crux of the matter to be resolved before making the assertion of analyticity of problem (1.1). At any rate analyticity cannot follow by a perturbation arguent.

Orientation on Spectral Properties It was already noted in Theorem 1.4 that the point -1 is a point in the continuous spectrum of the operator $\mathcal{A}_b$ as well as of the operator $\mathcal{A}_b^* : -1 \in \sigma_c(\mathcal{A}_b), -1 \in \sigma_c(\mathcal{A}_b^*)$, in line with what was noted in (b) above for the abstract operator $\mathbb{A}$ in (1.22), (1.23a). This result for $\mathcal{A}_b$, coupled with the location of its spectrum described by Theorem 1.3 (iv) make one expect that, qualitatively, the spectrum of $\mathcal{A}_b$ is like the spectrum of the operator $\mathbb{A}$ in (1.23a), with one branch of eigenvalues being negative and going to $-\infty$, and the other branch going to the point -1 of the continuous spectrum. In our present case, a perfect counter part of this behavioral property of the eigenvalues of $\mathcal{A}_b$ is offered by Proposition 5.8, at least for $b = 0$ and Ω_s being a 2-D disk or 3-D ball. The general case is unsettled yet.

Part I: Results valid for the pair $\{\mathcal{A}_b, \mathcal{H}_b\}$, $b = 0$ **and** $b = 1$

2 Abstract Model on $\mathcal{H}_b$ for the Free Dynamics (1.1 a–g), $b = 0$, $b = 1$

The Navier–Stokes (linear) part (1.1a) contains two unknowns: the velocity field and the pressure. In the present coupled case of problem (1.1), because of the (non-homogeneous) boundary coupling (1.1d–f), it is *not* possible to use the classical, standard idea of N-S problems with *no-slip boundary conditions* to eliminate the pressure: that is, by applying the Leray projector on the equation from $(L_2(\Omega))^d$ onto the classical space [7, p. 7] $\{f \in (L_2(\Omega))^d;\ \text{div } f \equiv 0 \text{ in } \Omega;\ f \cdot \nu = 0 \text{ on } \partial\Omega_f\}$. Accordingly, paper [1] (as well as paper [3], where the d-dimensional wave equation (1.1c) is replaced by the system of dynamic elasticity) eliminated the pressure by a completely different strategy. Following the idea of [31] (see also [23]), papers [1,3] identify a suitable elliptic problem for the pressure p, to be solved for p in terms of u, w and w_t.

Elimination of p, by expressing p in terms of u, w and w_t A key idea of [1], [17, Chapter 2], [31,35] is that the pressure $p(t, x)$ solves the following elliptic problem on Ω_f in x, for each t:

$$\begin{cases} \Delta p \equiv 0 \text{in } (0, T] \times \Omega_f \equiv Q_f; & \text{(2.1a)} \\ p = \dfrac{\partial u}{\partial \nu} \cdot \nu - \dfrac{\partial (w + w_t)}{\partial \nu} \cdot \nu \qquad \text{on } (0, T] \times \Gamma_s \equiv \Sigma_s; & \text{(2.1b)} \\ \dfrac{\partial p}{\partial \nu} = \Delta u \cdot \nu \text{on } (0, T] \times \Gamma_f \equiv \Sigma_f. & \text{(2.1c)} \end{cases}$$

In fact, (2.1aa) is obtained by taking the divergence div across Eq.(1.1a), and using div $u_t \equiv 0$ in Q_f by (1.1bb), as well as div $\Delta u = \Delta$ div $u \equiv 0$ in Q_f. Next, the B.C. (2.1b) on Γ_s is obtained by taking the inner product of Eq.(1.1f) with ν. Finally, the B.C. (2.1c) on Γ_f is obtained by taking the inner product of Eq. (1.1a) restricted on Γ_f, with ν, using $u|_{\Gamma_f} \equiv 0$ by (1.1d), so that on $\Gamma_f : \nabla p \cdot \nu = \frac{\partial p}{\partial \nu}|_{\Gamma_f}$. This then results in (2.1c).Through a technical argument based on elliptic theory, one then obtains that the original PDE problem (1.1a–g) can be written as

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A}_b \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}. \tag{2.2}$$

Here the operator $\mathcal{A}_b$ is given explicitly by

$$\mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - b v_1 \\ \Delta f - \nabla \pi \end{bmatrix} \equiv \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} \in \mathcal{H}_b, \quad \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \in \mathcal{D}(\mathcal{A}_b). \tag{2.3}$$

where the function π is defined by (compare with (2.1a–b–c) for the dynamic problem):

$$\begin{cases} \Delta \pi \equiv 0 \text{ in } \Omega_f; & \text{(2.4a)} \\ \pi = \dfrac{\partial f}{\partial \nu} \cdot \nu - \dfrac{\partial (v_1 + v_2)}{\partial \nu} \cdot \nu \in H^{-\frac{1}{2}}(\Gamma_s) \text{ on } \Gamma_s; ; \quad \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \in \mathcal{D}(\mathcal{A}_b). & \text{(2.4b)} \\ \dfrac{\partial \pi}{\partial \nu} = \Delta f \cdot \nu \in H^{-\frac{3}{2}}(\Gamma_f) \text{ on } \Gamma_f; & \text{(2.4c)} \end{cases}$$

This method of elimination the pressure,as the usual Leray projection is not possible due to the interface condition was introduced in [1], see also [17, Chapter 2],[35] in the case of no visco-elastic damping. It was labeld "novel" in the MathSciNet Review of [1]. Details in the present derivation of (2.3),(2.4) as well as of next Proposition 2.1 in the case of visco-elastic damping are given in [26].

Proposition 2.1 (a) *The domain $\mathcal{D}(\mathcal{A}_b)$ of the operator $\mathcal{A}_b : \mathcal{D}(\mathcal{A}_b) \subset \mathcal{H}_b \to \mathcal{H}_b$ in (2.3) is characterized as follows: $\{v_1, v_2, f\} \in \mathcal{D}(\mathcal{A}_b)$ if and only if the following properties hold true:*

(a_1)

$$\begin{aligned} & v_1, v_2 \in (H^1(\Omega_s))^d; \quad b=1 \quad v_1, v_2 \in (H^1(\Omega)/\mathbb{R})^d; \quad b=0, \\ & \quad \textit{such that } \Delta(v_1+v_2) \in (L^2(\Omega))^d \\ & \quad v_2|_{\Gamma_s} = f|_{\Gamma_s} \in (H^{\frac{1}{2}}(\Gamma_s))^d \quad [v_1+v_2]|_{\Gamma_s} \in (H^{\frac{1}{2}}(\Gamma_s))^d \\ & \quad \textit{so that } \left.\frac{\partial(v_1+v_2)}{\partial\nu}\right|_{\Gamma_s} = \left[\frac{\partial f}{\partial \nu} - \pi\nu\right]_{\Gamma_s} \in (H^{-\frac{1}{2}}(\Gamma_s))^d \end{aligned} \tag{2.5}$$

(a_2)

$$\begin{aligned} & f \in (H^1(\Omega_f))^d \cap \tilde{H}_f, \textit{ with } \Delta f - \nabla\pi \in \tilde{H}_f, \\ & \quad \textit{where } \pi(v_1, v_2, f) \in L_2(\Omega_f) \textit{ is the harmonic} \\ & \quad \textit{function defined by (2.4)}; \end{aligned} \tag{2.6}$$

(a_3)

$$\left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_s} \in (H^{-\frac{1}{2}}(\Gamma_s))^d \textit{ and } \pi|_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s); \tag{2.7}$$

(a_4)

$$f|_{\Gamma_f} = 0; \quad [\Delta f \cdot \nu]_{\Gamma_f} \in H^{-\frac{3}{2}}(\Gamma_f). \tag{2.8}$$

Remark 2.1 We note, more over that the divergence theorem applied to div $f \equiv 0$ implies as $f|_{\Gamma_f} \equiv 0$ by (2.8) and $f|_{\Gamma_s} = v_2|_{\Gamma_s}$ by (2.5)

$$0 = \int_{\Omega_f} \text{div } f \, d\Omega_f = \int_{\Gamma_s} f \cdot \nu \, d\Gamma_s = \int_{\Gamma_s} v_2 \cdot \nu \, d\Gamma_s = \int_{\Gamma_s} v_1^* \cdot \nu \, d\Gamma_s, \tag{2.9}$$

recalling (2.3), a necessary requirement on the image point $[v_1^*, v_2^*, f^*]$ to be in the range $\mathcal{R}(\mathcal{A}_b)$ of $\mathcal{A}_b$ (which defines a closed subspace on $(H^1(\Omega_s))^d$).

Remark 2.2 Henceforth, if we wish to emphasize the operator $\mathcal{A}_b$ in (2.3), for $b = 0$ on $\mathcal{H}_{b=0}$, or for $b = 1$ on $\mathcal{H}_{b=1}$, we shall accordingly use the notation $\mathcal{A}_{b=0}$ or $\mathcal{A}_{b=1}$, respectively. Instead, $\mathcal{A}_b$ will ordinary cover both cases $b = 0$ and $b = 1$.

3 The Operator $\mathcal{A}_b$ is Dissipative on $\mathcal{H}_b$, $b = 0$ and $b = 1$

In preparation for the well-posedness (semigroup generation) of Sect. 4, we here establish that the operator $\mathcal{A}_b$ is dissipative on $\mathcal{H}_b$, $b = 0$ and $b = 1$

Proposition 3.1 *Let $b = 0$ or $b = 1$. The operator $\mathcal{A}_b$ in* (2.3) *with domain described by Proposition* 2.1 *is dissipative on the space $\mathcal{H}_b$ defined in* (1.2). *More precisely, let $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_b) \subset \mathcal{H}_b$, then*

$$Re\left(\mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}\right)_{\mathcal{H}_b} = -\int_{\Omega_s} |\nabla v_2|^2 d\Omega_f - \int_{\Omega_f} |\nabla f|^2 d\Omega_f \leq 0. \tag{3.1}$$

Proof For $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_b)$, we return to identity (2.3) and obtain

$$\mathrm{Re}\left(\mathcal{A}_b\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}\right)_{\mathcal{H}_b} = \mathrm{Re}\left(\begin{bmatrix} v_2 \\ \Delta(v_1+v_2) - bv_1 \\ \Delta f - \nabla\pi \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}\right)_{\mathcal{H}_b} \tag{3.2a}$$

$$= \begin{cases} \mathrm{Re}\Big\{(v_2, v_1)_{(H^1(\Omega_s)/\mathbb{R})^d} + (\Delta(v_1+v_2), v_2)_{(L_2(\Omega_s))^d} + \ (\Delta f, f)_{\tilde{H}_f} - (\nabla\pi, f)_{\tilde{H}_f}\Big\}; & b=0 \\ \text{RHS of (3.2b)} \ + \mathrm{Re}\{(v_2, v_1) - (v_1, v_2)\} = \text{RHS of (3.2b)}; & b=1 \end{cases} \tag{3.2b}$$

$$= \mathrm{Re}\left\{\int_{\Omega_s} \nabla v_2 \cdot \nabla \bar{v}_1 d\Omega_s + (\Delta(v_1+v_2), v_2)_{(L_2(\Omega_s))^d} + \ (\Delta f, f)_{\tilde{H}_f} - (\nabla\pi, f)_{\tilde{H}_f}\right\}; b=0\,, b=1 \tag{3.2c}$$

Here π is defined by (2.4). Henceforth, properties of $\mathcal{D}(\mathcal{A}_b)$ listed in Proposition 2.1 will be invoked.

Second term of (3.2c) Since the unit normal on Γ_s is *inward* with respect to Ω_s (Fig. 1), we obtain by Green's first theorem, along with $\pi\nu - \frac{\partial f}{\partial \nu} = -\frac{\partial(v_1+v_2)}{\partial \nu}$ on Γ_s by (2.4b), and $f|_{\Gamma_s} = v_2|_{\Gamma_s}$ by (2.5); with ν inward to Ω_s:

$$\begin{aligned} &(\Delta(v_1+v_2), v_2)_{(L_2(\Omega_s))^d} \\ &\quad = \int_{\Gamma_s}\left(-\frac{\partial(v_1+v_2)}{\partial\nu}\right)\cdot \bar{v}_2 d\Gamma_s - \int_{\Omega_s} \nabla(v_1+v_2)\cdot\nabla\bar{v}_2\, d\Omega_s && (3.3) \\ &\quad = \int_{\Gamma_s} \pi\nu\cdot\bar{f}\, d\Gamma_s - \int_{\Gamma_s}\frac{\partial f}{\partial\nu}\cdot\bar{f}\, d\Gamma_s - \int_{\Omega_s}\nabla v_1\cdot\nabla\bar{v}_2\, d\Omega_s - \int_{\Omega_s}|\nabla v_2|^2\, d\Omega_s && (3.4) \end{aligned}$$

Third term of (3.2c) Recalling from (1.3) that $\tilde{H}_f$ is topologically $(L_2(\Omega_s))^d$, we compute with $\Gamma \equiv \Gamma_s \cup \Gamma_f = \partial\Omega_f$, via Green's first theorem, since $f|_{\Gamma_f} \equiv 0$ by (2.8):

$$(\Delta f, f)_{\tilde{H}_f} = \int_{\Omega_f}\Delta f\cdot\bar{f}\, d\Omega_f = \int_{\Gamma}\frac{\partial f}{\partial\nu}\cdot\bar{f}\, d\Gamma - \int_{\Omega_f}|\nabla f|^2 d\Omega_f \tag{3.5}$$

$$= \int_{\Gamma_s}\frac{\partial f}{\partial\nu}\cdot\bar{f}\, d\Gamma_s - \int_{\Omega_f}|\nabla f|^2 d\Omega_f. \tag{3.6}$$

Sum of first three terms of (3.2c) Summing up (3.4) and (3.6), and recalling the inner product (1.4a) for $(H^1(\Omega_s)/\mathbb{R})^d$, we obtain:

$$\begin{aligned} &\mathrm{Re}\left\{\int_{\Omega_s}\nabla v_2\cdot\nabla\bar{v}_1 d\Omega_s + (\Delta(v_1+v_2), v_2)_{(L_2(\Omega_s))^d} + (\Delta f, f)_{\tilde{H}_f}\right\} \\ &= \mathrm{Re}\left\{\int_{\Omega_s}\cancel{\nabla v_2\cdot\nabla\bar{v}_1 d\Omega_s} + \left[\int_{\Gamma_s}\pi\nu\cdot\bar{f}\, d\Gamma_s - \int_{\Gamma_s}\cancel{\frac{\partial f}{\partial\nu}\cdot\bar{f}\, d\Gamma_s} - \int_{\Omega_s}\cancel{\nabla v_1\cdot\nabla\bar{v}_2 d\Omega_s} - \int_{\Omega_s}|\nabla v_2|^2\, d\Omega_s\right]\right. \\ &\left. + \left[\int_{\Gamma_s}\cancel{\frac{\partial f}{\partial\nu}\cdot\bar{f}\, d\Gamma_s} - \int_{\Omega_f}|\nabla f|^2 d\Omega_f\right]\right\} = \mathrm{Re}\left(\int_{\Gamma_s}\pi\nu\cdot\bar{f}\, d\Gamma_s\right) - \int_{\Omega_s}|\nabla v_2|^2\, d\Omega_s - \int_{\Omega_f}|\nabla f|^2 d\Omega_f. \end{aligned} \tag{3.7}$$

Fourth term of (3.2c) By the divergence formula, with $\Gamma = \Gamma_f \cup \Gamma_s = \partial\Omega_f$, recalling from (1.3) that $\tilde{H}_f$ is topologized by the $(L_2(\Omega_s))^d$-norm and $\text{div}\, f \equiv 0$ and $f|_{\Gamma_f} \equiv 0$.

$$(\nabla\pi, f)_{\tilde{H}_f} = \int_{\Omega_f} \nabla\pi \cdot \bar{f}\, d\Omega_f = \int_{\Gamma} \pi \bar{f} \cdot \nu\, d\Gamma - \int_{\Omega_f} \cancel{\pi\, \text{div}\, \bar{f}}\, d\Omega_f = \int_{\Gamma_s} \pi \bar{f} \cdot \nu\, d\Gamma_s \tag{3.8}$$

Final identity of (3.2c) Summing up (3.7) and (3.8) yields

$$\begin{aligned}\text{RHS of (3.2)} &= \text{Re}\left\{(v_2, v_1)_{(\mathbb{H}^1(\Omega_s)/\mathbb{R})^d} + (\Delta(v_1+v_2), v_2)_{(L_2(\Omega_s))^d} + (\Delta f, f)_{\tilde{H}_f} - (\nabla\pi, f)_{\tilde{H}_f}\right\} \\ &= \text{Re}\left(\int_{\Gamma_s} \cancel{\pi\nu \cdot \bar{f}}\, d\Gamma_s\right) - \text{Re}\left(\int_{\Gamma_s} \cancel{\pi\bar{f} \cdot \nu}\, d\Gamma_s\right) - \int_{\Omega_s} |\nabla v_2|^2\, d\Omega_s - \int_{\Omega_f} |\nabla f|^2 d\Omega_f \end{aligned} \tag{3.9}$$

$$= -\int_{\Omega_s} |\nabla v_2|^2\, d\Omega_s - \int_{\Omega_f} |\nabla f|^2 d\Omega_f, \qquad [v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_b), \tag{3.10}$$

and (3.10) used in (3.2) proves (3.1), as desired, in both cases $b = 0$ and $b = 1$. □

We postpone to Remark 4.2 below an orientation on the spectral analysis of the imaginary axis for both the original operator $\mathcal{A}_b$ and its adjoint $\mathcal{A}_b^*$ on $\mathcal{H}_b$ as a consequence of each of them being dissipative.

Corollary 3.2 *Let $b = 0, 1$. Let $(\alpha + i\omega)$ be an eigenvalue of the operator $\mathcal{A}_b$ with normalized eigenvector $x = \{v_1, v_2, f\} \in \mathcal{H}_b$, $\|x\|_{\mathcal{H}_b} = 1$. Then, necessarily*

$$\alpha = -\|\nabla v_2\|^2 - \|\nabla f\|^2, \ \ (\|\nabla v_1\|^2)\alpha^2 + \alpha + (\omega^2\|\nabla v_1\|^2 + \|\nabla f\|^2) = 0 \tag{3.11}$$

If $b = 0$ and $\omega = 0$ (the case $\omega \neq 0$ is described in Propositions 5.6 and 5.7) then the eigenvalue $\alpha < 0$ is given by

$$\alpha = \frac{-1 \pm \sqrt{1 - 4\|\nabla v_1\|^2\|\nabla f\|^2}}{2\|\nabla v_1\|^2}, \quad \frac{4}{C_p^2}\|\nabla v_1\|^2\|f\|^2 \leq 4\|\nabla v_1\|^2\|\nabla f\|^2 \leq 1 \tag{3.12}$$

where C_p is the Poincare constant on Ω_f.

Proof Insert $\mathcal{A}_b x = (\alpha + i\omega)x$, with $\|x\|_{\mathcal{H}_b} = 1$ on the L.H.S of (3.1) and obtain (3.11) also by $v_2 = (\alpha + i\omega)v_1$. For $b = 0, \omega = 0$, estimate on the R.H.S of (3.12) uses the Poincare inequality $\|f\| \leq C_p\|\nabla f\|$ since $f|_{\Gamma_f} = 0$ by (2.8). Thus $\|\nabla v_1\|^2\|f\|^2 \leq \frac{C_p^2}{4}$, where $\|\nabla v_1\|^2 + \|v_2\|^2 + \|f\|^2 = 1$ □

4 The Adjoint Operator $\mathcal{A}_b^*$ ($b = 0, 1$) is Dissipativity on $\mathcal{H}_b$, $b = 0$ and $b = 1$

Proposition 4.1 (The adjoint operator $\mathcal{A}_b^*$ on $\mathcal{H}_b$) *Let $\mathcal{A}_b$ be the operator in (2.3), with domain $\mathcal{D}(\mathcal{A}_b)$ described in Proposition 2.1. Then, its $\mathcal{H}_b$-adjoint $\mathcal{A}_b^*$ is given by*

$$\mathcal{A}_b^* \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} = \begin{bmatrix} -v_2^* \\ \Delta(v_2^* - v_1^*) + bv_1^* \\ \Delta f^* - \nabla \pi^* \end{bmatrix} \in \mathcal{H}_b; \qquad \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} \in \mathcal{D}(\mathcal{A}_b^*), \tag{4.1}$$

where the function π^* is defined by

$$\begin{cases} \Delta\pi^* \equiv 0 \quad \text{in } \Omega_f; & \text{(4.2a)} \\ \pi^* = \dfrac{\partial f^*}{\partial\nu}\cdot\nu - \dfrac{\partial(v_2^* - v_1^*)}{\partial\nu}\cdot\nu \in H^{-\frac{1}{2}}(\Gamma_s) \quad \text{on } \Gamma_s; & \text{(4.2b)} \\ \dfrac{\partial\pi^*}{\partial\nu} = \Delta f^*\cdot\nu \in H^{-\frac{3}{2}}(\Gamma_f) \quad \text{on } \Gamma_f. & \text{(4.2c)} \end{cases}$$

The proof is a direct computation, given in [26], which extends the proof with no visco-elastic damping given in [24], as well as the proof of the next Proposition.

Proposition 4.2 *The point $\{v_1^*, v_2^*, f^*\} \in \mathcal{D}(\mathcal{A}_b^*) \subset \mathcal{H}_b$ in case,*

(a_1)

$$\begin{aligned} & v_1^*, v_2^* \in (H^1(\Omega_s))^d; \quad b = 1 \quad v_1^*, v_2^* \in (H^1(\Omega)/\mathbb{R})^d; \quad b = 0, \\ & \quad \text{such that } \Delta(v_2^* - v_1^*) \in (L^2(\Omega_s))^d \\ & v_2^*|_{\Gamma_s} = f^*|_{\Gamma_s} \in (H^{\frac{1}{2}}(\Gamma_s))^d \quad [v_2^* - v_1^*]|_{\Gamma_s} \in (H^{\frac{1}{2}}(\Gamma_s))^d \\ & \text{so that } \left.\frac{\partial(v_2^* - v_1^*)}{\partial\nu}\right|_{\Gamma_s} = \left[\frac{\partial f^*}{\partial\nu} - \pi^*\nu\right]_{\Gamma_s} \in (H^{-\frac{1}{2}}(\Gamma_s))^d \end{aligned} \tag{4.3}$$

(a_2)

$$\begin{aligned} & f^* \in (H^1(\Omega_f))^d \cap \tilde{H}_f, \text{ with } \Delta f^* - \nabla\pi^* \in \tilde{H}_f, \\ & \quad \text{where } \pi^*(v_1^*, v_2^*, f^*) \in L^2(\Omega_f) \text{ is the harmonic} \\ & \quad \text{function defined by (4.2)}; \end{aligned} \tag{4.4}$$

(a_3)

$$\left.\frac{\partial f^*}{\partial\nu}\right|_{\Gamma_s} \in (H^{-\frac{1}{2}}(\Gamma_s))^d \text{ and } \pi^*|_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s); \tag{4.5}$$

(a_4)

$$f^*|_{\Gamma_f} = 0; \quad [\Delta f^*\cdot\nu]_{\Gamma_f} \in H^{-\frac{3}{2}}(\Gamma_f). \tag{4.6}$$

Compare $\mathcal{D}(\mathcal{A}_b)$ in Proposition 2.1 with $\mathcal{D}(\mathcal{A}_b^*)$ in Proposition 4.2. Recall also [24, Section 1].

As in the latter reference, we have

Proposition 4.3 *On $\mathcal{H}_b$ the bounded, symmetric operator*

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} = \mathcal{T}^*, \quad \textit{has the properties } \mathcal{T}^2 = \textit{ identity on } \mathcal{H}_b \textit{ and} \tag{4.7}$$

(i)

$$\begin{aligned} &\mathcal{T} : \mathcal{D}(\mathcal{A}_b) \xrightarrow{onto} \mathcal{D}(\mathcal{A}_b^*) \qquad \mathcal{T} = \mathcal{T}^{-1} : \mathcal{D}(\mathcal{A}_b^*) \xrightarrow{onto} \mathcal{D}(\mathcal{A}_b) \\ &\mathcal{T}\mathcal{D}(\mathcal{A}_b) = \mathcal{D}(\mathcal{A}_b^*), \quad \mathcal{T}\mathcal{D}(\mathcal{A}_b^*) = \mathcal{D}(\mathcal{A}_b) \end{aligned} \tag{4.8}$$

(ii)

$$\begin{aligned} \mathcal{T}\mathcal{A}_b &= \mathcal{A}_b^*\mathcal{T} = \mathcal{A}_b^*\mathcal{T} = (\mathcal{T}\mathcal{A}_b)^* \textit{ on } \mathcal{D}(\mathcal{A}_b) \\ \mathcal{A}_b &= \mathcal{T}^{-1}\mathcal{A}_b^*\mathcal{T} \textit{ on } \mathcal{D}(\mathcal{A}_b) \textit{ (similarity)} \\ &\textit{and } \mathcal{T}\mathcal{A}_b \textit{ is self adjoint with domain } \mathcal{D}(\mathcal{A}_b) \end{aligned} \tag{4.9}$$

(iii)

$$(\mathcal{A}_b x, x)_{\mathcal{H}_b} = (\mathcal{A}_b^* x^*, x^*)_{\mathcal{H}_b}, \forall x \in \mathcal{D}(\mathcal{A}_b) \textit{ and } x^* = \mathcal{T}x \in \mathcal{D}(\mathcal{A}_b^*) \tag{4.10a}$$

$$Re(\mathcal{A}_b x, x)_{\mathcal{H}_b} = Re(\mathcal{A}_b^* x^*, x^*)_{\mathcal{H}_b}, \forall x \in \mathcal{D}(\mathcal{A}_b) \textit{ and } x^* = \mathcal{T}x \in \mathcal{D}(\mathcal{A}_b^*) \tag{4.10b}$$

(iv) *If λ is an eigenvalue of $\mathcal{A}_b$ with corresponding eigenvector e*

$$\mathcal{A}_b e = \lambda e \qquad 0 \neq e \in \mathcal{D}(\mathcal{A}_b) \tag{4.11}$$

then applying $\mathcal{T}$ on both sides and recalling $\mathcal{T}\mathcal{A}_b e = \mathcal{A}_b^*\mathcal{T}e$ by (4.9) yields

$$\mathcal{A}_b^*(\mathcal{T}e) = \lambda(\mathcal{T}e) \tag{4.12}$$

and λ is and eigenvalue of $\mathcal{A}_b^*$ with corresponding eigenvector $(\mathcal{T}e)$. And conversely. It follows that if λ is not an eigenvalue of $\mathcal{A}_b$, $\lambda \notin \sigma_p(\mathcal{A}_b)$, then it is not an eigenvalue of $\mathcal{A}_b^*$ either, $\lambda \notin \sigma_p(\mathcal{A}_b^*)$, thus $\lambda \notin \sigma_r(\mathcal{A}_b)$, the residual spectrum of $\mathcal{A}_b$ [29].

As a consequence of Propositions 4.3 and 3.1, we have:

Proposition 4.4 *Let $b = 0, b = 1$. The operator $\mathcal{A}_b^*$ in (4.1), with domain $\mathcal{D}(\mathcal{A}_b^*)$ defined by Proposition 4.1 is dissipative on the space $\mathcal{H}_b$ defined in (1.2): for $[v_1^*, v_2^*, f^*] \in \mathcal{D}(\mathcal{A}_b^*) \subset \mathcal{H}_b$, we have*

$$Re\left(\mathcal{A}_b^* \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix}\right)_{\mathcal{H}_b} = -\int_{\Omega_s} |\nabla v_2^*|^2 \, d\Omega_s - \int_{\Omega_f} |\nabla f^*|^2 d\Omega_f, \tag{4.13}$$

Hence $\mathcal{A}_b$ and $\mathcal{A}_b^*$ are the generators of s.c (C_0) semigroups $e^{\mathcal{A}_b t}$ and $e^{\mathcal{A}_b^* t}$ of contractions on the finite energy space $\mathcal{H}_b$.

For the last statement we recall in [27, Corollary 4.4, p 15] or in [6, Theorem 4.3.3, p 188]. Proposition 4.4 can also be shown directly from $\mathcal{A}_b^*$ in (4.1), Proposition 4.2.

Remark 4.1 Trivially, each operator $\mathcal{A}_{b=1}$ and $\mathcal{A}_{b=1}^*$ is a generator of a s.c (non-contraction) semigroup $e^{\mathcal{A}_{b=1} t}$ and $e^{\mathcal{A}_{b=1}^* t}$ on the space $\mathcal{H}_{b=0}$, each being on such space a bounded perturbation of $\mathcal{A}_{b=0}$ or $\mathcal{A}_{b=0}^*$, respectively. Similarly, the analyticity of $e^{\mathcal{A}_{b=0} t}$ and $e^{\mathcal{A}_{b=0}^* t}$ on $\mathcal{H}_{b=0}$ (Theorem 1.3(i)) implies at once analyticity of $e^{\mathcal{A}_{b=1} t}$ and $e^{\mathcal{A}_{b=1}^* t}$ on $\mathcal{H}_{b=0}$ as well. We shall however mostly focus on the *contraction* case: $\mathcal{A}_{b=0}$ and $\mathcal{A}_{b=0}^*$ on $\mathcal{H}_{b=0}$; and $\mathcal{A}_{b=1}$ and $\mathcal{A}_{b=1}^*$ on $\mathcal{H}_{b=1}$.

Remark 4.2 **Orientation on the spectral analysis of the imaginary axis for both the operators $\mathcal{A}_b$ and its adjoint $\mathcal{A}_b^*$ on $\mathcal{H}_b$.** For $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_b) \subset \mathcal{H}_b$ and $\omega \in \mathbb{R}$, let by (2.3)

$$\mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - b v_1 \\ \Delta f - \nabla \pi \end{bmatrix} = i\omega \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \tag{4.14}$$

so that,

$$\begin{aligned} \text{Re}\left(\mathcal{A}_b \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \right)_{\mathcal{H}_b} &= \text{Re}\left\{ (i\omega) \left\| \begin{matrix} v_1 \\ v_2 \\ f \end{matrix} \right\|_{\mathcal{H}_b}^2 \right\} \\ &= - \int_{\Omega_s} |\nabla v_2|^2 d\Omega_f - \int_{\Omega_f} |\nabla f|^2 d\Omega_f = 0. \end{aligned} \tag{4.15}$$

so that $\nabla v_2 \equiv 0$ in Ω_s, $\nabla f \equiv 0$ in Ω_f, hence $f \equiv 0$ in Ω_f, since $f|_{\Gamma_s} \equiv 0$ by (2.8); hence $\nabla \pi \equiv 0$ and $\pi = \text{const} = -k$ in Ω_f; finally, $v_2 \equiv 0$ in Ω_s, since $v_2|_{\Gamma_s} = f|_{\Gamma_s} = 0$ by (2.5).Thus, so far, as a consequence of dissipativity of $\mathcal{A}_b$ on $\mathcal{H}_b$, we have $\{v_2 = 0, f = 0\}$. Next, (4.14) implies $v_2 = (i\omega)v_1$, hence $v_1 \equiv 0$ if $\omega \neq 0$. We conclude that: *the points* $(i\omega)$ *are not eigenvalues* of $\mathcal{A}_b$ on $\mathcal{H}_b$, $b = 0, 1$, $0 \neq \omega \in \mathbb{R}$: $i\omega \notin \sigma_p(\mathcal{A}_b)$. Similarly the dissipativity identity (4.13) for the adjoint operator $\mathcal{A}_b^*$ in (4.1) will likewise yield that: the points $(i\omega)$ are not eigenvalues of $\mathcal{A}_b^*$, $0 \neq \omega \in \mathbb{R}$, $b = 0, 1$. Hence by [29], the points $(i\omega)$, $0 \neq \omega \in \mathbb{R}$, *are not in the residual spectrum* of $\mathcal{A}_b$ on $\mathcal{H}_b$: $(i\omega) \notin \sigma_r(\mathcal{A}_b)$, $\omega \in \mathbb{R}\backslash\{0\}$. When $\omega = 0$ (the origin) as to the corresponding first component v_1, the conclusion is more complex and depends on the geometrical conditions of the domain Ω_s (see Remark 5.1 below).

$(i1)$ If $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$, then the point $\lambda = 0$ is not an eigenvalue of $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$ (Proposition 5.1(a)), nor is an eigenvalue of the adjoint operator $\mathcal{A}_{b=0}^*$ on $\mathcal{H}_{b=0}$, so that $0 \notin \sigma_r(\mathcal{A}_{b=0})$, the residual spectrum of $\mathcal{A}_{b=0}$ [[29], P 282]. In fact, for $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$, the point $\lambda = 0$ is in the resolvent set of $\mathcal{A}_{b=0}$: $0 \in \rho(\mathcal{A}_{b=0})$ on $\mathcal{H}_{b=0}$(Proposition 5.2). More conclusively, $i\mathbb{R} \in \rho(\mathcal{A}_{b=0})$ if $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$ (Theorem 6.1).

$(i2)$ On the other hand, if $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$, then $\lambda = 0$ is an eigenvalue of $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$ as well as of $\mathcal{A}^*_{b=0}$ on $\mathcal{H}_{b=0}$, with a one dimensional <u>common</u> eigenspace, explicitly identified (Proposition 5.1(b)). One then works on the space $\widehat{\mathcal{H}}_{b=0} = \mathcal{H}_{b=0} \backslash [Null(\mathcal{A}_{b=0})]$ of co-dimension one in $\mathcal{H}_{b=0}$, Eq (5.4), which is invariant for both $e^{\mathcal{A}_{b=0}t}$ and $e^{\mathcal{A}^*_{b=0}t}$ (Proposition 5.4). One has, $0 \in \rho(\widehat{\mathcal{A}}_{b=0})$, where $\widehat{\mathcal{A}}_{b=0} = \mathcal{A}_{b=0}|_{\widehat{\mathcal{H}}_{b=0}}$ (Proposition 5.3). More conclusively, $i\mathbb{R} \in \rho(\widehat{\mathcal{A}}_{b=0})$ if $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$ (Theorem 7.1, Corollary 7.1).

$(ii1)$ If $b = 1$, the the point $\lambda = 0$ is an eigenvalue of the operator $\mathcal{A}_{b=1}$ and $\mathcal{A}^*_{b=1}$ on $\mathcal{H}_{b=1}$ with, again a one-dimensional <u>common</u> eigenspace, explicitly identified (Theorem 1.5). Again, one then works on the space $\widehat{\mathcal{H}}_{b=1} = \mathcal{H}_{b=1} \backslash [Null(\mathcal{A}_{b=1}]$ of co-dimesion one in (1.18), which is invariant for both $e^{\mathcal{A}_{b=1}t}$ and $e^{\mathcal{A}^*_{b=1}t}$. Then the origin $\lambda = 0$ belongs to the resolvent set of $\widehat{\mathcal{A}}_{b=1} = \mathcal{A}_{b=1}|_{\widehat{\mathcal{H}}_{b=1}}$ (Theorem 1.5).

Proposition 4.5 *Let $b = 0$, $b = 1$*

(i) *The point -1 belongs neither to the point spectrum of $\mathcal{A}_b$, $-1 \notin \sigma_p(\mathcal{A}_b)$, nor to the point spectrum of $\mathcal{A}^*_b$, $-1 \notin \sigma_p(\mathcal{A}^*_b)$; thus [[29], P 282], -1 does not belongs to the residual spectrum of $\mathcal{A}_b$, $-1 \notin \sigma_r(\mathcal{A}_b)$. See also Proposition 4.3(iv).*

(ii) *In fact, -1 belongs to the continuous spectrum of $\mathcal{A}_b$, as well as of $\mathcal{A}^*_b$: $-1 \in \sigma_c(\mathcal{A}_b)$, $-1 \in \sigma_c(\mathcal{A}^*_b)$.*

Proof We consider only the operator $\mathcal{A}_b$, as the analysis for $\mathcal{A}^*_b$ is similar.

(i) For $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_b)$ defined by Proposition 2.1, let via (2.3),

$$(I + \mathcal{A}_b) \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ \Delta(v_1 + v_2) - bv_1 + v_2 \\ \Delta f + f - \nabla \pi \end{bmatrix} = 0 \tag{4.16}$$

Thus, $[v_1 + v_2] = 0$ and then $\Delta(0) - bv_1 + v_2 = 0$. This means $v_2 = 0$ for $b = 0$ and $[-v_1 + v_2] = 0$ for $b = 1$. In both cases,$v_1 = v_2 = 0$ in $(L^2(\Omega_s))^d$. Hence, $v_2|_{\Gamma_s} = 0 = f|_{\Gamma_S}$ by (2.5). Thus the third equation of (4.16) yields the problem

$$\begin{cases} \Delta f + f - \nabla \pi = 0 \text{ in } \Omega_f \\ \operatorname{div} f \equiv 0 \text{ in } \Omega_f \\ f|_{\Gamma_f} = 0; \, f|_{\Gamma_s} = 0 \text{ on } \partial\Omega_f \end{cases} \tag{4.17}$$

and (4.17) implies $f = 0$ in $L^2(\Omega_f)$. Thus $[v_1, v_2, f] = 0$ and $-1 \notin \sigma_p(\mathcal{A}_b)$. In fact, taking the inner product of the first equation with f yields

$$\begin{aligned} (\Delta f, f)_{\Omega_f} + \|f\|^2 &= \int_{\partial\Omega_f} \frac{\partial f}{\partial \nu} \cdot \bar{f} \, d(\partial\Omega_f) - \|\nabla f\|^2 + \|f\|^2 \\ &= \int_{\Omega_f} \nabla \pi \cdot \bar{f} \, d\Omega_f = \int_{\partial\Omega_f} \pi \bar{f} \cdot \nu \, d(\partial\Omega_f) \\ &\quad - \int_{\Omega_f} \pi \operatorname{div} \bar{f} \, d\Omega_f = 0 \end{aligned} \tag{4.18}$$

Thus, $f = 0$ as claimed. The proof of $-1 \notin \sigma_p(\mathcal{A}_b^*)$ is exactly the same using now (4.1) for $\mathcal{A}_b^*$.

(ii) We shall actually show that $-1 \in \sigma_{p,app}(\mathcal{A}_b)$, the approximate point spectrum of $\mathcal{A}_b$. In fact after step (i) we are left with alternative that either $-1 \in \rho(\mathcal{A}_b)$ or else $-1 \in \sigma_c(\mathcal{A}_b)$. We then recall the general result that [29]: $\sigma_c(\mathcal{A}_b) \cup \sigma_p(\mathcal{A}_b) \subset \sigma_{p,app}(\mathcal{A}_b) \subset \sigma(\mathcal{A}_b)$. Then, in fact $-1 \in \sigma_c(\mathcal{A}_b)$ Thus, we shall construct a sequence $x_n = \{v_{1n}, v_{2n}, f_n\}$ in $\mathcal{D}(\mathcal{A}_b) \subset \mathcal{H}_b$, such that

$$\|x_n\|_{\mathcal{H}_b} \equiv 1, \text{ yet } (I + \mathcal{A}_b)x_n \to 0 \text{ in } \mathcal{H}_b.$$

To this end, returning to the expression of $(I + \mathcal{A}_b)x_n$ given by (4.16), we pick in $\mathcal{D}(\mathcal{A}_b)$, two smooth sequences $\{v_{1n}\} \in (H^2(\Omega_s))^d$, $\{v_{2n}\} \in (H^2(\Omega_s))^d$ such that,

$$\begin{cases} v_{1n} \to 0 \text{ and } v_{2n} \to 0 \text{ in } (H^2(\Omega_s))^d, \text{ thus} \\ \Delta(v_{1n} + v_{2n}) - bv_{1n} + v_{2n} \to 0 \text{ in } (L^2(\Omega_s))^d \\ v_{2n}|_{\Gamma_s} \to 0 \text{ in } (H^{3/2}(\Gamma_s))^d, \ \left.\frac{\partial(v_{1n}+v_{2n})}{\partial\nu}\right|_{\Gamma_s} \to 0 \text{ in } (H^{1/2}(\Gamma_s))^d. \end{cases} \tag{4.19}$$

Next, pick corresponding sequences $\{f_n\} \in (H^2(\Omega_f))^d, \pi_n \in H^1(\Omega_f)$, with $\{v_{1n}, v_{2n}, f_n\} \in \mathcal{D}(\mathcal{A}_b)$, such that $f_n \to 0$ in $(H^2(\Omega_f))^d$, $\pi_n \to 0$ in $H^1(\Omega_f)$, div $f_n \equiv 0$ in $(L^2(\Omega_f))^d$ yielding therefore the problem

$$\begin{cases} \Delta f_n + f_n - \nabla\pi_n = r_n \to 0 \text{ in } (L^2(\Omega_f))^d \\ \operatorname{div} f_n \equiv 0 \text{ in } (L^2(\Omega_f))^d \\ f_n|_{\Gamma_f} = 0; \ f_n|_{\Gamma_s} = v_{2n}|_{\Gamma_s} \to 0 \text{ in } (H^{3/2}(\Gamma_s))^d \end{cases} \tag{4.20}$$

It then follows from [30] repeated also in [17, Chapter 2, Appendix A] that

$$\|f_n\|_{(H^2(\Omega_f))^d} + \|\pi_n\|_{H^1(\Omega_f)} \le C\left[\|r_n\|_{(L^2(\Omega_f))^d} + \|v_{2n}\|_{(H^{3/2}(\Gamma_s))^d}\right] \to 0 \text{ as } n \to \infty \tag{4.21}$$

Hence $f_n \to 0$ in $(H^2(\Omega_f))^d$. We have therefore obtained a nonzero sequence $\{v_{1n}, v_{2n}, f_n\} \in \mathcal{D}(\mathcal{A}_b)$ such that

$$(I + \mathcal{A}_b)\begin{bmatrix} v_{1n} \\ v_{2n} \\ f_n \end{bmatrix} \to 0 \text{ in } \mathcal{H}_b. \tag{4.22}$$

Finally, we normalize the sequence $x_n = \{v_{1n}, v_{2n}, f\}$ to obtain $\|x_n\|_{\mathcal{H}_b} \equiv 1$. Thus $-1 \in \sigma_{p,app}(\mathcal{A}_b)$. □

One can also give a direct proof that $-1 \in \sigma_c(\mathcal{A}_b)$ in the style of [24], See [26].

Part II: Results valid for $b = 0$: $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$

5 Spectral Analysis for $b = 0$

Remark 5.1 In the present Sect. 5 as well as in Sect. 6, we shall consider two cases regarding the domain of Ω_s; that is,

$$\text{whether} \quad (a) \int_{\Gamma_s} \nu \, d\Gamma_s \neq 0 \quad \text{or whether} \quad (b) \int_{\Gamma_s} \nu \, d\Gamma_s = 0 \tag{5.1}$$

where ν is the unit normal vector to Γ_s. Given Ω_s, which case arises depends on its geometrical conditions. Case (5.1)(a) was first pointed out in [21, p 66], where it was taken as an assumption. In this paper we shall allow each case to hold true. As noted in [21, p 66], the geometric condition (5.1)(a) is *"generically true. One is always able to achieve this condition by a sufficiently small pertubation of the boundary Γ_s. Condition (5.1) is related to [a property of] symmetry (or lack thereof) of the domain* Ω_S". For instance, (5.1)(b) holds true if Ω_s is a 2-D disk or a 3-D ball, or Γ_s is a 2-D ellipse, or a 3-D ellipsoid, etc. As noted, the generic condition (5.1)(a) is an assumption for the main strong stability result of the non-linear fluid–structure interaction model (with no viscoelastic damping) [21, Theorem 1.4, p 66] . In our present paper, the two geometrical cases in (5.1) have implication on the spectral properties of the point $\lambda = 0$ for $\mathcal{A}_{b=0}$ or $\mathcal{A}^*_{b=0}$ on $\mathcal{H}_{b=0}$. We also refer to [20,22].

Proposition 5.1 *Let $b = 0$*

(a) *Under the assumption $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$, the origin $\lambda = 0$ is not an eigenvalue of the operator $\mathcal{A}_{b=0}$ in (2.3) on $\mathcal{H}_{b=0}$; nor of the operator $\mathcal{A}^*_{b=0}$ in (4.1) (in line with Proposition 4.3(iv)). Thus, $0 \notin \sigma_p(\mathcal{A}_{b=0})$ and $0 \notin \sigma_r(\mathcal{A}_{b=0})$. [The susequent Proposition 5.2 will establish that $0 \in \rho(\mathcal{A}_{b=0})$].*

(b) *If $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$, then $\lambda = 0$ is an eigenvalue of the operator $\mathcal{A}_{b=0}$ as well as of the adjoint $\mathcal{A}^*_{b=0}$ on the space $\mathcal{H}_{b=0}$ with corresponding common eigenvector $e_0 = [\eta_0, 0, 0]$, where $\eta_0 \in (H^1(\Omega_s)/\mathbb{R})^d$ is the unique solution of the following elliptic problem:*

$$\Delta \eta_0 = 0 \text{ in } \Omega_s \,, \quad \left.\frac{\partial \eta_0}{\partial \nu}\right|_{\Gamma_s} = \nu \text{ on } \Gamma_s \quad \eta_0 = N_0 \nu \tag{5.2}$$

$N =$ Neumann map, so that the null space of $\mathcal{A}_{b=0}$ in $\mathcal{H}_{b=0}$ is one dimensional

$$Null(\mathcal{A}_{b=0}) = span\{[\eta_0, 0, 0]\} \text{ in } \mathcal{H}_{b=0}. \tag{5.3}$$

Moreover, when we define the space of codimension 1:

$$\widehat{\mathcal{H}}_{b=0} \equiv [Null(\mathcal{A}_{b=0})]^{\perp} = \mathcal{H}_{b=0} \backslash [Null(\mathcal{A}_{b=0})] = [Null(\mathcal{A}^*_{b=0})]^{\perp} \tag{5.4}$$

$$= \left\{ \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \in \mathcal{H}_{b=0} : \left(\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \eta_0 \\ 0 \\ 0 \end{bmatrix} \right)_{\mathcal{H}_{b=0}} = (v_1, \eta_0)_{(H^1(\Omega_s)/\mathbb{R})^d} = 0 \right\}. \tag{5.5}$$

then the following characterization of $\widehat{\mathcal{H}}_{b=0}$ holds true.

$$\widehat{\mathcal{H}}_{b=0} = \left\{ [v_1, v_2, f] \in \mathcal{H}_{b=0} : \int_{\Gamma_s} v_1 \cdot \nu \, d\Gamma_s = 0 \right\}. \tag{5.6}$$

Proof (a) For $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_{b=0})$, let via (2.3) with $b = 0$

$$\mathcal{A}_{b=0} \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta f - \nabla \pi \end{bmatrix} = 0 \implies v_2 = 0 \text{ in } \Omega_s \ , v_2|_{\Gamma_s} = f|_{\Gamma_s} = 0 \text{ on } \Gamma_s. \tag{5.7}$$

by the B.C in (2.5). Next, recalling (2.8)

$$\begin{cases} \Delta f - \nabla \pi = 0 \text{ in } \Omega_f & (5.8a) \\ \operatorname{div} f \equiv 0 \text{ in } \Omega_f; \qquad \implies f \equiv 0 \text{ in } \Omega_f \ , \pi \equiv \text{ constant} = c \text{ in } \Omega_f & (5.8b) \\ f|_{\Gamma_f} = 0; \ f|_{\Gamma_s} = 0 \text{ on } \partial\Omega_f & (5.8c) \end{cases}$$

as it follows by taking the inner product of Eq (5.8a) with f and using Greens's theorems

$$\int_{\partial\Omega_f} \frac{\partial f}{\partial \nu} \cdot \bar{f} \, d(\partial\Omega_f) - \int_{\Omega_f} |\nabla f|^2 \, d\Omega_f - \left[\int_{\partial\Omega_f} \pi \, \bar{f} \cdot \nu \, d(\partial\Omega_f) - \int_{\Omega_f} \pi \operatorname{div} f \, d\Omega_f \right] = 0 \tag{5.9}$$

and $f \equiv 0$ in Ω_f by the B.C $f|_{\Gamma_s} = 0$. Since $v_2 \equiv 0$ in Ω_s by (5.7) and $f \equiv 0$ by (5.8), we obtain from Eqn (5.7) that v_1 solves the following elliptic problem and corresponding B.C by (2.5):

$$\begin{cases} \Delta v_1 = 0 \text{ in } \Omega_s & (5.10a) \\ \left. \dfrac{\partial v_1}{\partial \nu} \right|_{\Gamma_s} = -\pi \nu = c\nu \text{ on } \Gamma_s & (5.10b) \end{cases}$$

When we take the inner product of Eqn (5.10a) with 1, considering inward unit normal vector ν we obtain

$$0 = \int_{\Omega_s} \Delta v_1 \cdot 1 \, d\Omega_s = -\int_{\Gamma_s} \frac{\partial v_1}{\partial \nu} \, d\Gamma_s - (\nabla v_1, \nabla 1) = \int_{\Gamma_s} \frac{\partial v_1}{\partial \nu} = -c \int_{\Gamma_s} \nu \, d\Gamma_s \tag{5.11}$$

Since $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$ (under present assumption), we conclude $c = 0$, which implies $v_1 \equiv$ constant $\implies v_1 = 0$ in $(H^1(\Omega_s)/\mathbb{R})^d$. Hence $[v_1, v_2, f] \equiv 0$. Part (a) is established.

(b) If $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$, according to (5.11), c can be any constant. For each $c \neq 0 \in \mathbb{R}$,there exists a unique solution $v_1(c) \neq 0$ of problem (5.10) and the map $c\nu \to v_1(c)$ is linear, allows us to write $v_1(c) = c\eta_0$ with η_0 solution of (5.10) for $c = 1$;that is of (5.2).Thus , in this case, the eigenvector of the eigenvalue $0 \in \sigma_p(\mathcal{A}_{b=0})$ on $\mathcal{H}_{b=0}$

is $c[\eta_0, 0, 0]$ as claimed in (5.3). For the next claim under the definition (5.4), we compute via Green's Theorem, with ν inward, via (5.2)

$$\begin{aligned}(v_1, \eta_0)_{(H^1(\Omega_s)/\mathbb{R})^d} &= (\nabla v_1, \nabla \eta_0)_{(L^2(\Omega_s))^d} \\ &= -(v_1, \nabla \eta_0 \cdot \nu)_{(L^2(\Gamma_s))^d} - (v_1, \text{div}\nabla \eta_0)_{(L^2(\Omega_s))^d} = -(v_1, \nu)_{(L^2(\Gamma_s))^d}\end{aligned}$$

after recalling $\nabla \eta_0 \cdot \nu = \frac{\partial \eta_0}{\partial \nu} = \nu$ and div $\nabla \eta_0 = \Delta \eta_0 = 0$ from (5.2) with ν inward in Γ_s Thus we have

$$(v_1, \eta_0)_{(H^1(\Omega_s)/\mathbb{R})^d} = 0 \text{ if and only if } (v_1, \nu)_{(L^2(\Gamma_s))^d} = 0. \tag{5.12}$$

□

The improvement of Proposition 5.1 part (a) is the following.

Proposition 5.2 *Let $b = 0$. Under the assumption $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$, the point $\lambda = 0$ is in the resolvent set $\rho(\mathcal{A}_{b=0})$ of the operator$\mathcal{A}_{b=0}$. Hence there is a small open disk $\mathcal{S}_{r_0}$ in the complex plane centered at the origin and of small radius $r_0 > 0$, that is all contained in $\rho(\mathcal{A}_{b=0})$: $\mathcal{S}_{r_0} \subset \rho(\mathcal{A}_{b=0})$.*

Remark 5.1 For $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$, Theorem 6.1 establishes that $i\mathbb{R} \in \rho(\mathcal{A}_{b=0})$.

Proof Let $[v_1^*, v_2^*, f] \in \mathcal{H}_{b=0}$ be arbitary. We seek $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_{b=0})$ which solves via (2.3) for $b = 0$

$$\mathcal{A}_{b=0} \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta f - \nabla \pi \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} \in \mathcal{H}_{b=0} \tag{5.13}$$

Then (5.13) yields

$$v_2 = v_1^* \in (H^1(\Omega_s)/\mathbb{R})^d; \tag{5.14}$$

Since the data $[v_1^*, v_2^*, f^*]$ is drawn from $\mathcal{H}_{b=0}$, then it satisfies the compatibility condition $\int_{\Gamma_s} v_1^* \cdot \nu \, d\Gamma_s = 0$ in (2.9). As a consequence, there exists a unique solution pair $\{f, \pi\} \in (H^1(\Omega_f))^d \times L^2(\Omega_f)/\mathbb{R}$ which solves the f-problem

$$\begin{cases} \Delta f - \nabla \pi = f^* \in \tilde{H}_f \subset (L_2(\Omega_f))^d; & (5.15a) \\ \text{div} f \equiv 0 \ \text{ in } \Omega_f; & (5.15b) \\ f|_{\Gamma_f} = 0; \ \ f|_{\Gamma_s} = v_2|_{\Gamma_s} = v_1^*|_{\Gamma_s} \in (H^{\frac{1}{2}}(\Gamma_s))^d, & (5.15c) \end{cases}$$

continuously in terms of the data $f^*, v_1^*|_{\Gamma_s}$, see [30, Thm. 2.4]. Thus, we have recovered the third component f (and the pressure π modulo a constant). Let π henceforth in this proof denote a definite pressure solution. For any constant C_0, then $\pi + C_0$ is also a viable pressure for the Stokes problem in (5.15a). We remark that by [3, p 436],[26, Lemma A.3, Appendix A], we have

$$\left\{ \frac{\partial f}{\partial \nu}\bigg|_{\partial\Omega_f}, \pi|_{\partial\Omega_f} \right\} \in (H^{-\frac{1}{2}}(\Gamma_s))^d \times (H^{-\frac{1}{2}}(\Gamma_s))^d, \tag{5.16}$$

continuously in terms of the data. Having recovered so far $v_2 = v_1^*$ in (5.14) and uniquely $\{f, \pi\}$ in $(H^1(\Omega_f))^d \times (L^2(\Omega_f))^d/\mathbb{R}$, we seek finally to recover also the first component v_1, as well as the unique constant C_0 defined by the coupled problem (5.15). To this end, we invoke v_1- problem in (5.13) with $v_2 = v_1^*$, then we obtain

$$\Delta(v_1 + v_1^*) = v_2^* \quad in\, \Omega_s \tag{5.17}$$

To recover v_1 explicitly, from (5.17) we define the following two operators.

$$\begin{cases} -A_{N_0}\phi = \Delta\phi \text{ in } \Omega_s\,;\ \mathcal{D}(A_{N_0}) = \left\{\phi \in (H^2(\Omega_s)/\mathbb{R})^d : \dfrac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_s\right\} & (5.18a) \\ A_{N_0} : \mathcal{D}(A_{N_0}) \subset (L^2(\Omega_s))^d \to (L^2(\Omega_s))^d; & (5.18b) \end{cases}$$

$$\begin{cases} N_0\mu = \psi \iff \left\{\Delta\psi = 0 \text{ in } \Omega_s \text{ and } \dfrac{\partial \psi}{\partial \nu} = \mu\right\} & (5.19a) \\ N_0 \in \mathcal{L}(H^{-\frac{1}{2}}(\Gamma_s))^d, (H^1(\Omega_s)/\mathbb{R})^d) & (5.19b) \end{cases}$$

for the Neumann map N_0. Thus, as usual, via (5.19) and (2.5),

$$\Delta\left[(v_1 + v_1^*) - N_0\left(\left(\frac{\partial f}{\partial \nu} - (\pi + C_0)\nu\right)_{\Gamma_s}\right)\right] = v_2^* \text{ in } \Omega_s \tag{5.20}$$

or by (5.18a):

$$-A_{N_0}\left[(v_1 + v_1^*) - N_0\left(\left(\frac{\partial f}{\partial \nu} - (\pi + C_0)\nu\right)_{\Gamma_s}\right)\right] = v_2^* \in (L^2(\Omega_s))^d \tag{5.21}$$

Hence v_1 is given by

$$v_1 = -A_{N_0}^{-1}v_2^* + N_0\left(\frac{\partial f}{\partial \nu} - (\pi + C_0)\nu\right)_{\Gamma_s} - v_1^* \in (H^1(\Omega_s)/\mathbb{R})^d \tag{5.22}$$

in terms of the data (via the prior recovery of $\{f, \pi\}$) where C_0 is still unknown. To identify C_0 in (5.22), we consider following integral on Γ_s of the Eq. (5.22).

$$\int_{\Gamma_s} \frac{\partial v_1}{\partial \nu}\, d\Gamma_s = \int_{\Gamma_s} \frac{\partial \left[-A_{N_0}^{-1} v_2^* + N_0 \left(\frac{\partial f}{\partial \nu} - (\pi + C_0)\nu \right)_{\Gamma_s} - v_1^* \right]}{\partial \nu}\, d\Gamma_s \tag{5.23}$$

Hence, we can recover C_0, since $\frac{\partial N_0 \nu}{\partial \nu} = \nu$ by definition, see (5.19)

$$C_0 = \frac{\int_{\Gamma_s} \frac{\partial v_1}{\partial \nu}\, d\Gamma_s - \int_{\Gamma_s} \frac{\partial \left[-A_{N_0}^{-1} v_2^* + N_0 \left(\frac{\partial f}{\partial \nu} - \pi \nu \right)_{\Gamma_s} \right]}{\partial \nu}\, d\Gamma_s + \int_{\Gamma_s} \frac{\partial v_1^*}{\partial \nu}\, d\Gamma_s}{-\int_{\Gamma_s} \nu\, d\Gamma_s} \tag{5.24}$$

where $\int_{\Gamma_s} \nu\, d\Gamma_s \neq 0$ by assumption. □

The improvement of the Proposition 5.1 part (b) is the following.

Proposition 5.3 *Let $b = 0$. Under the assumption $\int_{\Gamma_s} \nu\, d\Gamma_s = 0$, the point $\lambda = 0$ is in the resolvent set $\rho(\widehat{\mathcal{A}}_{b=0})$ of the operator $\widehat{\mathcal{A}}_{b=0} = \mathcal{A}_{b=0}|_{\widehat{\mathcal{H}}_{b=0}}$. Hence there is a small open disk $\mathcal{S}_{r_0}$ in the complex plane centered at the origin and of small radius $r_0 > 0$, that is all contained in $\rho(\widehat{\mathcal{A}}_{b=0})$: $\mathcal{S}_{r_0} \subset \rho(\widehat{\mathcal{A}}_{b=0})$ as in Proposition 5.2.*

Proof Let $[v_1^*, v_2^*, f] \in \widehat{\mathcal{H}}_{b=0}$ be arbitary. We seek $[v_1, v_2, f] \in \mathcal{D}(\widehat{\mathcal{A}}_{b=0}) \cap \widehat{\mathcal{H}}_{b=0}$ which solves via (2.3) for $b = 0$

$$\widehat{\mathcal{A}}_{b=0} \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta f - \nabla \pi \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} \in \widehat{\mathcal{H}}_{b=0} \tag{5.25}$$

Then (5.25) yields

$$v_2 = v_1^* \in (H^1(\Omega_s)/\mathbb{R})^d; \tag{5.26}$$

then the proof of this proposition follows as the proof of the Proposition 5.2, until the point in (5.22) where v_1 is explicitly written in terms of data, but the constant C_0 is still unknown.

$$v_1 = -A_{N_0}^{-1} v_2^* + N_0 \left(\frac{\partial f}{\partial \nu} - (\pi + C_0)\nu \right)_{\Gamma_s} - v_1^* \in (H^1(\Omega_s)/\mathbb{R})^d \tag{5.27}$$

To identify C_0 in (5.27), we impose on v_1 defined by (5.27) the required compatibility condition (5.6) in order to force $[v_1, v_2, f] \in \widehat{\mathcal{H}}_{b=0}$, that is $\int_{\Gamma_s} v_1 \cdot \nu\, d\Gamma_s = 0$, as required

$$\int_{\Gamma_s} v_1 \cdot \nu\, d\Gamma_s = \int_{\Gamma_s} \left[-A_{N_0}^{-1} v_2^* + N_0 \left(\frac{\partial f}{\partial \nu} - \pi \nu \right)_{\Gamma_s} \right] \cdot \nu\, d\Gamma_s - C_0 \int_{\Gamma_s} N_0 \nu \cdot \nu\, d\Gamma_s = 0 \tag{5.28}$$

since $\int_{\Gamma_s} v_1^* \cdot \nu \, d\Gamma_s = 0$ by (2.9). Next we recall by the definition ofN_0 in (5.19a), we have that $N_0 \nu = \eta_0$, where η_0 satisfies the elliptic problem (5.2). Integrating $(\Delta \eta_0, \eta_0) = 0$ from (5.2) yields by Green's First Theorem and $\eta_0 = N_0 \nu$.

$$-\int_{\gamma_s} \eta_0 \cdot \nu \, d\Gamma_s = -\int_{\Gamma_S} N_0 \nu \cdot \nu \, d\Gamma_s = -\int_{\Gamma_s} \frac{\partial \eta_0}{\partial \nu} \cdot \eta_0 \, d\Gamma_s = \|\nabla \eta_0\|^2 \neq 0 \quad (5.29)$$

Using (5.29) in (5.28) yields the sought-after constant C_0:

$$C_0 = \frac{\int_{\Gamma_s} \left[-A_{N_0}^{-1} v_2^* + N_0 \left(\frac{\partial f}{\partial \nu} - \pi \nu \right)_{\Gamma_s} \right] \cdot \nu \, d\Gamma_s}{\int_{\Gamma_s} N_0 \nu \cdot \nu \, d\Gamma_s} \quad (5.30)$$

where the denominator is different from zero by (5.29). □

Under asuumption $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$: Invariance of factor space $\widehat{\mathcal{A}}_{b=0} = [Null(\mathcal{A}_{b=0})]^{\perp} = [Null(\mathcal{A}_{b=0}^*)]^{\perp}$ under the action of semigroup $e^{\mathcal{A}_{b=0} t}$ and $e^{\mathcal{A}_{b=0}^* t}$

This dynamic property relies on the key feature that $\mathcal{A}_{b=0}$ and its adjoint $\mathcal{A}_{b=0}^*$ have a common finite dimensional null-space. In fact, one dimensional space spanned by the vector $[\eta_0, 0, 0]$ in (5.3).

Proposition 5.4 *The subspace $\widehat{\mathcal{H}}_{b=0} = [Null(\mathcal{A}_{b=0})]^{\perp} = [Null(\mathcal{A}_{b=0}^*)]^{\perp}$ of co-dimesion 1 in $\mathcal{H}_{b=0}$ is invariant under the action of the semigroups $e^{\mathcal{A}_{b=0} t}$ and $e^{\mathcal{A}_{b=0}^* t}$.*

Proof By Proposition 5.1(b)

$$\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \in \mathcal{H}_{b=0} \backslash [Null(\mathcal{A}_{b=0})] = \mathcal{H}_{b=0} \backslash [Null(\mathcal{A}_{b=0}^*)]$$
$$\Longleftrightarrow \left(\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \eta_0 \\ 0 \\ 0 \end{bmatrix} \right)_{\mathcal{H}_{b=0}} = 0 \Longleftrightarrow (v_1, \eta_0)_{(H^1(\Omega_s)/\mathbb{R})^d} = 0 \quad (5.31)$$

Of course, the common eigenspace $[Null(\mathcal{A}_{b=0})] = [Null(\mathcal{A}_{b=0}^*)]$ spanned by the eigenvector $[\eta_0, 0, 0]$ corresponding to the eigenvalue $\lambda = 0$, is invariant under both the semigroup $e^{\mathcal{A}_{b=0} t}$ and its adjoint $e^{\mathcal{A}_{b=0}^* t}$.

$$e^{\mathcal{A}_{b=1} t} \begin{bmatrix} \eta_0 \\ 0 \\ 0 \end{bmatrix} = e^{0t} \begin{bmatrix} \eta_0 \\ 0 \\ 0 \end{bmatrix} = e^{\mathcal{A}_{b=1}^* t} \begin{bmatrix} \eta_0 \\ 0 \\ 0 \end{bmatrix}, \quad t \geq 0 \quad (5.32)$$

Thus, the factor space $\mathcal{H}_{b=0}\backslash[Null(\mathcal{A}_{b=0})]$ is invariant under the action of the semigroup $e^{\mathcal{A}_{b=0}t}$:

$$\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \in \mathcal{H}_{b=0}\backslash[Null(\mathcal{A}_{b=0})] \implies e^{\mathcal{A}_{b=0}t}\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \in \mathcal{H}_{b=0}\backslash[Null(\mathcal{A}_{b=0})], \tag{5.33}$$

since

$$\left(e^{\mathcal{A}_{b=0}t}\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \eta_0 \\ 0 \\ 0 \end{bmatrix}\right)_{\mathcal{H}_{b=0}} = \left(\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, e^{\mathcal{A}^*_{b=0}t}\begin{bmatrix} \eta_0 \\ 0 \\ 0 \end{bmatrix}\right)_{\mathcal{H}_{b=0}}$$
$$= \left(\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \eta_0 \\ 0 \\ 0 \end{bmatrix}\right)_{\mathcal{H}_{b=0}} = 0 \tag{5.34}$$

by (5.31) and (5.32). Thus, the implication (5.33) follows. Similarly the factor space $\mathcal{H}_{b=0}\backslash[Null(\mathcal{A}^*_{b=1})] = \mathcal{H}_{b=0}\backslash[Null(\mathcal{A}_{b=0})]$ is invariant under the action of the adjoint semigroup $e^{\mathcal{A}^*_{b=0}t}$. □

Corollary 5.5 *Let $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$ as in Propositions 5.1(b), 5.3 and 5.4.*

(i) The operators $\widehat{\mathcal{A}}_{b=0} = \mathcal{A}_{b=0}|_{\widehat{\mathcal{H}}_{b=0}}$ and $\widehat{\mathcal{A}}^*_{b=0} = \mathcal{A}^*_{b=0}|_{\widehat{\mathcal{H}}_{b=0}}$ generate s.c contraction semigroups on the space $\widehat{\mathcal{H}}_{b=0} = [Null(\mathcal{A}_{b=0})]^{\perp} = [Null(\mathcal{A}^*_{b=0})]^{\perp}$, invariant for them.

(ii) The operators $\mathcal{A}_{b=0}$ and $\mathcal{A}^*_{b=0}$ generate s.c contraction semigroups $e^{\mathcal{A}_{b=0}t}$ and $e^{\mathcal{A}^*_{b=0}t}$ on the space $\mathcal{H}_{b=0}$ in (1.2b) (recovering Proposition 4.4).

Proof (i) (dissipativity) $\widehat{\mathcal{A}}_{b=0}$, respectively $\widehat{\mathcal{A}}^*_{b=0}$, are dissipative on $\widehat{\mathcal{H}}_{b=0}$, a fortiori from the dissipativity of $\mathcal{A}_{b=0}$ and $\mathcal{A}^*_{b=0}$ on $\mathcal{H}_{b=0}$ (Propositions 3.1 and 4.4)

Also, maximality follows from Proposition 5.3, as $0 \in \rho(\widehat{\mathcal{A}}_{b=0})$, $0 \in \rho(\widehat{\mathcal{A}}^*_{b=0})$.

(ii) If $x \in \mathcal{H}_{b=0}$, then $x = x|_{\widehat{\mathcal{H}}_{b=0}} + ae_0$, and $e^{\mathcal{A}_{b=0}t}x = e^{\widehat{\mathcal{A}}_{b=0}t}x|_{\widehat{\mathcal{H}}_{b=0}} + ae_0$ is the s.c contraction semigroup generated by $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$, via part (i), where $e_0 = [\eta_0, 0, 0]$ is the eigenvector of $\mathcal{A}_{b=0}$ corresponding to its eigenvalue $\lambda = 0$ and so $e^{\widehat{\mathcal{A}}_{b=0}t}e_0 = e^{0t}e_0 = e_0$. □

The next result will be much extended in Theorem 6.1 in Sect. 6. The proof of this significant special case here is simpler. It will suffice to consider $\mathbb{C}^- = \{\lambda \in \mathbb{C} : \text{Re}\lambda \leq 0\}$ by dissipativity of $\mathcal{A}_{b=0}$ and $\mathcal{A}^*_{b=0}$.

Proposition 5.6 *Let $b = 0$ in (1.1c).*

(i) *in $\mathbb{C}^-$, consider the closed disk $\overline{\mathcal{S}_{r=1}}(x_0)$, centered at the point $x_0 = \{-1, 0\}$ and of radius $r = 1$. Let $\overline{\mathcal{S}_{r=1}}^c(x_0)$ be its open complement in $\mathbb{C}^- = \{\lambda : \, Re \, \lambda < 0\}$. Then the operator $\mathcal{A}_{b=0}$ in* (2.3)*, (with $b = 0$) on $\mathcal{H}_{b=0}$ has no eigenvalue $\lambda + i\omega$ with $\omega \neq 0$ in $\overline{\mathcal{S}_{r=1}}^c(x_0) = \left\{(\alpha, \omega) : \alpha^2 + 2\alpha + \omega^2 > 0\right\}$.The same conclusion holds for the adjoint operator $\mathcal{A}^*_{b=0}$. Hence* [29, P 282]*, the points $\lambda + i\omega$, $\omega \neq 0$ in $\mathcal{S}^c_{r=1}(x_0)$ do not belong to the residual spectrum $\sigma_r(\mathcal{A}_{b=0})$ of $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$.*

(ii) *The points* $\lambda = \alpha + i\omega$ *with* $\omega \neq 0$ *on the circumference of* $\mathcal{S}_{r=1}(x_0)$, *i.e satisfying* $\alpha^2 + 2\alpha + \omega^2 = 0$ *may, or may not, belong to the point spectrum of* $\mathcal{A}_{b=0}$, *depending on geometrical conditions on* Ω_s. *More preciesly, such points* $\lambda = \alpha + i\omega$, $\omega \neq 0$ *do, respectively do not, belong to* $\sigma_p(\mathcal{A}_{b=0})$ *according to as whether the over-determined elliptic problem* (5.48) *below do not, respectively do, imply* $v_1 \equiv 0$ *on* Ω_s:

$$\{\alpha, \omega\}, \omega \neq 0 \textit{ in circumference of } \mathcal{S}_{r=1} \begin{cases} \in \sigma_p(\mathcal{A}_{b=0}) \\ \notin \sigma_p(\mathcal{A}_{b=0}) \end{cases}$$

$$\textit{iff } (5.49) \textit{ implies } \begin{cases} v_1 \not\equiv 0 \textit{ in } \Omega_s \\ v_1 \equiv 0 \textit{ in } \Omega_s. \end{cases}$$

Proof **Step 1** Consider the eigenvalue/eigenvector equation for $[v_1, v_2, f] \in \mathcal{D}(\mathcal{A}_{b=0})$ for $b = 0$ in (2.3)

$$\mathcal{A}_{b=0} \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta f - \nabla\pi \end{bmatrix} = (\alpha + i\omega) \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \tag{5.35}$$

where we are taking $\alpha < 0$, $\omega \in \mathbb{R}$; or explicitly,

$$\begin{cases} v_2 = (\alpha + i\omega)v_1, & (5.36a) \\ \Delta(v_1 + v_2) = (\alpha + i\omega)v_2, & (5.36b) \\ \Delta f - \nabla\pi = (\alpha + i\omega)f. & (5.36c) \end{cases}$$

Step 2 Multiply Eq. (5.36c) by $\bar{f}$ integrate over Ω_f, use Green's First Theorem on $\int_{\Omega_f} \Delta f \bar{f}\, d\Omega_f$ along with the B.C. $f|_{\Gamma_f} = 0$ in $\mathcal{D}(\mathcal{A}_{b=0})$ and use Divergence formula on $-\int_{\Omega_f} \nabla\pi \bar{f}\, d\Omega_f$ with $\partial\Omega_f \equiv \Gamma_s \cap \Gamma_f$ recalling div $f \equiv 0$ in Ω_f, to obtain (all norms are L^2-norms on their respective domains)

$$\int_{\Omega_f} (\Delta f - \nabla\pi) \cdot \bar{f}\, d\Omega_f = \int_{\Gamma_s} \left(\frac{\partial f}{\partial \nu} - \pi\nu \right) \cdot \bar{f}\, d\Gamma_s - ||\nabla f||^2 = (\alpha + i\omega)||f||^2 \tag{5.37}$$

Next, multiply Eq.(5.36b) by $\bar{v_2}$, integrate over Ω_s, use the Green's First Theorem on $\int_{\Omega_s} \Delta(v_1 + v_2)\bar{v_2}\, d\Omega_s$, with unit normal vector ν inward to Ω_s to obtain by use the B.C (2.5).

$$\begin{aligned} (\Delta(v_1 + v_2), v_2) &= -\int_{\Gamma_s} \left(\frac{\partial f}{\partial \nu} - \pi\nu \right) \cdot \bar{f}\, d\Gamma_s - ||\nabla v_2||^2 - (\alpha - i\omega)||\nabla v_1||^2 \\ &= (\alpha + i\omega)||v_2||^2 \end{aligned} \tag{5.38}$$

To obtain (5.38), we have also invoked $\nabla v_2 = (\alpha + i\omega)\nabla v_1$ from (5.36a) on the term $(\nabla v_1, \nabla v_2) = (\alpha - i\omega)\|\nabla v_1\|^2$. Sum up (5.37) and (5.38) to obtain, after cancellation of the boundary terms

$$0 = ||\nabla v_2||^2 + ||\nabla f||^2 + \alpha\big(||\nabla v_1||^2 + ||v_2||^2 + ||f||^2\big) + i\omega\big(||v_2||^2 + ||f||^2 - ||\nabla v_1||^2\big) \tag{5.39}$$

Step 3 Taking the Real part and the Imaginary part of identity (5.39) yields (recall $\alpha < 0$)

$$||\nabla v_2||^2 + ||\nabla f||^2 + \alpha\left(||\nabla v_1||^2 + ||v_2||^2 + ||f||^2\right) = 0 \tag{5.40}$$

$$\omega\left(||v_2||^2 + ||f||^2 - ||\nabla v_1||^2\right) = 0 \tag{5.41}$$

(recall that we are taking $\alpha < 0$). For $\omega \neq 0$, we obtain from (5.41) and (5.36a),

$$\|\nabla v_1\|^2 = \|v_2\|^2 + \|f\|^2; \quad \|\nabla v_2\|^2 = (\alpha^2 + \omega^2)\|\nabla v_1\|^2, \quad \omega \neq 0 \tag{5.42}$$

which substituted in (5.40) yields

$$[\omega^2 + \alpha^2 + 2\alpha]\|\nabla v_1\|^2 + \|\nabla f\|^2 = 0, \quad \omega \neq 0. \tag{5.43}$$

First case Assume that

$$\omega^2 + \alpha^2 + 2\alpha > 0; \text{ i.e, } \lambda = \alpha + i\omega, \omega \neq 0 \text{ lies outside the closed disk } \overline{\mathcal{S}_{r=1}(x_0)} \tag{5.44}$$

with center $x_0 = \{-1, 0\}$ and radius $r = 1$. Then identity (5.43) implies

$$\begin{cases} \|\nabla v_1|\| = 0, \|\nabla f\| = 0, \text{ hence } f = \text{constant} = 0 \text{ in } \tilde{H}_f, \text{ since } f|_{\Gamma_f} = 0 \\ v_1 = \text{ constant } = 0 \text{ in } (H^1(\Omega_s)/\mathbb{R})^d \quad (b = 0) & (5.45a) \\ v_2 = (\alpha + i\omega)v_1 = 0 \text{ in } (L_2(\Omega_s))^d, \alpha + i\omega, \alpha < 0 & (5.45b) \end{cases}$$

In conclusion, we obtain $v_1 = v_2 = f = 0$ in $\mathcal{H}_{b=0}$ for $\lambda = \alpha + i\omega, \alpha < 0, \omega \neq 0$ outside the closed disk $\overline{\mathcal{S}_{r=1}(x_0)}$. Proposition 5.6 is established for $\mathcal{A}_{b=0}$. A similar argument applies to $\mathcal{A}^*_{b=0}$.

Second case. Assume next that $\lambda = \alpha + i\omega$, still $\omega \neq 0$, lies on the circumference of the disk $\mathcal{S}_{r=1}(x_0)$; that is, it satisfies

$$\alpha^2 + 2\alpha + \omega^2 = 0 \tag{5.46}$$

Then Eq. (5.43) gives $\|\nabla f\| \equiv 0$, hence $f \equiv 0$ since $f|_{\Gamma_f} = 0$, thus $\pi = \text{const} = c$, hence via (5.36a)

$$\left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_s} \equiv 0, \quad 0 \equiv f\big|_{\Gamma_s} = v_2\big|_{\Gamma_s}, \quad v_1\big|_{\Gamma_s} \equiv 0. \tag{5.47}$$

Returning to (5.36b) augmented by the relevant B.C in (2.5), we obtain via (5.36a) and (5.47)

$$\begin{cases} \Delta v_1 = \dfrac{(\alpha+i\omega)^2}{(1+\alpha)+i\omega} v_1 \\ [(1+\alpha)+i\omega]\dfrac{\partial v_1}{\partial \nu}\Big|_{\Gamma_s} = \Big[\dfrac{\partial f}{\partial \nu} - \pi \nu\Big]_{\Gamma_s} = -c\nu \end{cases} \quad \text{or} \quad \begin{cases} \Delta v_1 = 2\alpha v_1 \text{ in } \Omega_s \\ v_1|_{\Gamma_s} = 0, \ \dfrac{\partial v_1}{\partial \nu}\Big|_{\Gamma_s} = -c\nu \end{cases} \tag{5.48a}$$

since $\frac{(\alpha+i\omega)^2}{(1+\alpha)+i\omega} = 2\alpha$ by replaced by use of (5.46). The over-determined problem in (5.48) is precisely the one reported in [17, (2.4.4) p. 67, (2.4.13) p. 70, Appendix D],[35]. For $c = 0$, then $v_1 = 0$, hence $v_2 = 0$ by (5.36a) in which case the point (α, ω) on the circumference (5.46) is not an eigenvalue. For $c \neq 0$ and otherwise arbitrary the question arises as to whether the over-determined problem (5.48) implies $v_1 = 0$. The answer depends on geometrical conditions of Ω_s. For instance if Ω_s is a disk ($d = 2$) or a sphere ($d = 3$), the answer is negative, [17, R.T, Appendix D],[35] : and thus $\{v_1 \neq 0, 0, 0\}$ is an eigenvector of $\mathcal{A}_{b=0}$. On the other hand, many classes of geometries of Ω_s are given in [17,35] for which the over-determined problem (5.48) implies $v_1 = 0$, hence $v_2 = 0$ and then $\{\alpha, \omega\}$ satisfying (5.46) is not an eigenvalue. On the other hand, there are geometries for which non-zero solution $v_1 \neq 0$ exists that satisfies the over-determined problem (5.48). They include spheres($d = 2, 3$). See [17, Appendix D], [35].

One can also rerun the steps in the proof of Proportion 5.6 constructively as a necessary condition argument, assuming that $(\alpha + i\omega)$ is an eigenvalue of $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$.

Proposition 5.7 *Let $b = 0$. Let $\alpha + i\omega$, $\alpha < 0, \omega \neq 0$, be an eigenvalue of $\mathcal{A}_{b=0}$ in $\mathcal{H}_{b=0}$, with normalized eigenvector $\{v_1, v_2, f\}$, $\|\nabla v_1\|^2 + \|v_2\|^2 + \|f\|^2 = 1$. Then:*
(i)

$$\|\nabla v_1\|^2 = \frac{1}{2}, \quad \|v_2\|^2 + \|f\|^2 = \frac{1}{2}, \quad \alpha^2 + 2\alpha + (\omega^2 + 2\|\nabla f\|^2) = 0, \ \omega \neq 0 \tag{5.49}$$

$$\alpha = -1 \pm \sqrt{1 - (\omega^2 + 2\|\nabla f\|^2)}, \quad \|f\|^2 \leq C_p^2 \|\nabla f\|^2 \leq \frac{C_p^2}{2}(1 - \omega^2) \tag{5.50}$$

where C_p= Poicare constant of Ω_s; (ii)

$$\alpha^2 + \omega^2 + 2\alpha \leq 0, \quad \textit{i.e the point}(\alpha, \omega) \in \overline{\mathcal{S}_{r=1}(x_0)}, \ \omega \neq 0 \tag{5.51}$$

Proof (i) For $\omega \neq 0$, (5.41) yields $\|\nabla v_1\|^2 + \|v_2\|^2 + \|f\|^2 - 2\|v_1\|^2 = 1 - 2\|\nabla v_1\|^2 = 0$, as desired. Then (5.40) with $\|\nabla v_2\|^2 = (\alpha^2 + \omega^2)\|\nabla v_1\|^2 = (\alpha^2 + \omega^2)\frac{1}{2}$ and normalized eigenvector yields (5.49), (5.50) (Compare with (3.12)). Moreover (5.43) becomes $[\omega^2 + \alpha^2 + 2\alpha] + 2\|\nabla f\|^2 = 0$. If $\nabla f \neq 0$ (general case), then $\omega^2 + \alpha^2 + 2\alpha < 0$ and (α, ω), $\omega \neq 0$ belongs to the open disk $\mathcal{S}_{r=1}(x_0)$. It may happen that $\nabla f \equiv 0$, in which case (α, ω), $\omega \neq 0$, belongs to the boundary of such disk. If $\nabla f \equiv 0$ in Ω_f, hence $f \equiv 0$ in Ω_f and π = constant = $k \implies v_2|_{\Gamma_s} = f|_{\Gamma_s} = 0$, the v_2 problem becomes via (5.36a),(5.36b)

$$\begin{cases} \Delta v_2 = \dfrac{(\alpha + i\omega)^2}{(1+\alpha) + i\omega} v_2 & (5.52a) \\ v_2|_{\Gamma_s} = 0, \quad \dfrac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s} = \dfrac{\alpha + i\omega}{(1+\alpha) + i\omega} k\nu & (5.52b) \end{cases}$$

For many geometries of Ω_s, other than Ω_s= a disk or a ball, problem (5.52) implies $v_2 \equiv 0$, hence $v_1 \equiv 0$ via (5.36a), violating the assumption of $\{v_1, v_2, f\}$ being an eigenvector. □

In the next result we find exactly the (negative) eigenvalues α of the operator $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$, at least when Ω_s is a 2-D disk or a 3-D ball.

Proposition 5.8 *Let $b = 0$. Let Ω_s be a 2-D disk or 3-D ball. Let*

$$\Delta\phi_n = -\mu_n \phi_n \text{ in } \Omega_s, \quad \phi_n|_{\Gamma_s} = 0, \;\; n = 1, 2, \ldots \tag{5.53}$$

be the eigenvalue-vector problem of the Dirichlet Laplacian on Ω_s, where $0 < \mu_n \nearrow +\infty$. Consider the eigenvalue-vector problem (5.35) for the operator $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$ with focus on $\omega = 0$ (i.e on the negative real axis): Then, the eigenvalues of problem (5.35) with $\omega = 0$ are solutions of

$$\alpha_n^2 + \mu_n \alpha_n + \mu_n = 0 \tag{5.54}$$

(so that $\alpha_n^+ + \alpha_n^- = -\mu_n$; $\alpha_n^+ \alpha_n^- = \mu_n$) and are given by

$$\alpha_n^{+,-} = -1 \pm \sqrt{1 - 4\mu_n^2} = \text{ real negative, save possibly for finitely many } n. \tag{5.55}$$

We have that $\alpha_n^+ \nearrow -1$ monotonically, where $-1 \in \sigma_c(\mathcal{A}_b)$ by Proposition 4.5 and $\alpha_n^- \to -\infty$ monotonically. The corresponding normalized eigenvector is $\{v_{2n}/\alpha_n, v_{2n}, 0\}$, $\|v_{2n}\|^2 \equiv \frac{\alpha_n^2}{1+\alpha_n^2}$, where v_{2n} is identified in the proof below.

Proof The eigenvalue-vector problem (5.35) with focus on $\omega \equiv 0$ can be rewritten as

$$\begin{cases} \Delta v_2 = \dfrac{\alpha^2}{1+\alpha} v_2 \text{ in } \Omega_s; \quad \Delta f - \nabla \pi = \alpha f \text{ in } \Omega_f & (5.56a) \\ \quad \text{div } f \equiv 0 \text{ in } \Omega_f & (5.56b) \\ v_2|_{\Gamma_s} = f|_{\Gamma_s} f|_{\Gamma_f} = 0\,, \dfrac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s} = \dfrac{\alpha}{1+\alpha} \left[\dfrac{\partial f}{\partial \nu} - \pi \nu \right]_{\Gamma_s} & (5.56c) \end{cases}$$

Choose $f \equiv 0$ in Ω_f as a solution of (5.56a–b) in f, hence $\pi \equiv$ constant $= k$ in Ω_f, so that the v_2 problem becomes

$$\begin{cases} \Delta v_2 = \dfrac{\alpha^2}{1+\alpha} v_2 \text{ in } \Omega_s & (5.57a) \\ v_2|_{\Gamma_s} = 0, \ \dfrac{\partial v_2}{\partial \nu}\Big|_{\Gamma_s} = \dfrac{\alpha}{1+\alpha} k\nu\Big|_{\Gamma_s}. & (5.57b) \end{cases}$$

Since Ω_s= 2-D or 3-D ball, the overdetermined problem with $k \neq 0$ admits a non-zero solution $v_2 \neq 0$, which in fact, for each scalar component of problem (5.57) can be computed explicitly. [17,35]. Then by (5.36a), we have $v_1 \neq 0$ as well. On the other hand by (5.57a) and the zero Dirichlet condition $v_2|_{\Gamma_s} = 0$ we can appeal to (5.53) to obtain $\frac{\alpha_n^2}{1+\alpha_n} = -\mu_n$, $n = 1, 2, \ldots$ so that (5.54) follows. We are thus in the situation of [10, Lemma A.1 with $\alpha = 1, 2\rho = 1$], so that the two branches α_n^+ and α_n^- behave as stated in Proposition 5.8. Save for finitely many n, the values $\alpha_n^{+,-}$ are real negative and the stated form of the eigenvector $e_n^{+,-} = \{v_{2n}/\alpha_n^{+,-}, v_{2n}, 0\}$ follows. □

6 Case $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$. Analyticity, Location of the Spectrum $\sigma(\mathcal{A}_{b=0})$ Within the Set $\mathcal{K}$ of the Operator $\mathcal{A}_{b=0}$ on the Space $\mathcal{H}_{b=0}$, Exponential Decay

6.1 Orientation

For the clarity of presentation, this section is devoted to the operator $\mathcal{A}_{b=0}$ over the space $\mathcal{H}_{b=0}$ under the generic assumption $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$. The case $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$ will be handled in Sect. 7 via Proposition 5.3.

(a) We have already established that the operator $\mathcal{A}_{b=0}$, with the action as in (2.3), possesses the following two features: (i) it is the generator of a s.c. (C_0-) semigroup $e^{\mathcal{A}_{b=0}t}$ of contractions on the finite energy space $\mathcal{H}_{b=0}$ in (1.2a) (Proposition 4.4); (ii) Under the assumption $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$, $0 \in \rho(\mathcal{A}_{b=0})$, the resolvent set of $\mathcal{A}_{b=0}$, and hence there is a small open disk $\mathcal{S}_{r_0}$ in the complex plane centered at the origin and of small radius $r_0 > 0$, that is all contained in $\rho(\mathcal{A}_{b=0}) : \mathcal{S}_{r_0} \subset \rho(\mathcal{A}_{b=0})$.

Accordingly, to conclude that $e^{\mathcal{A}_{b=0}t}$ is, moreover, analytic on $\mathcal{H}_{b=0}$, all we need to show [23, Thm. 3E.3, p. 334] is that ($\mathcal{A}_{b=0}$ has no spectrum on the imaginary axis, and):

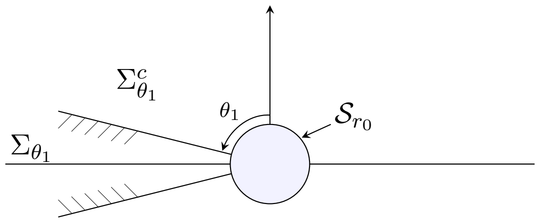
$$\|R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \frac{C}{|\omega|}, \quad \forall \, |\omega| \geq \text{ some } \omega_0 > 0. \tag{6.1}$$

Then, the proof in [23, p. 335] establishes that, in fact, for a suitable constant $M > 0$, we have

$$\|R(\lambda, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \frac{M}{|\lambda|}, \ \lambda \neq 0, \ \forall \, \lambda \in \Sigma_{\theta_1}^c, \tag{6.2a}$$

$$\Sigma_{\theta_1}^c = \left\{\lambda \in \mathbb{C} : \ 0 \leq |\arg \lambda| \leq \frac{\pi}{2} + \theta_1\right\}, \tag{6.2b}$$

Fig. 3 The Triangular Sector Σ_{θ_1} and its Complement $\Sigma^c_{\theta_1}$. The Disk $\mathcal{S}_{r_0} \subset \rho(\mathcal{A}_{b=0})$



where one may take the angle θ_1, $0 < \theta_1 < \frac{\pi}{2}$, such that $\tan\left(\frac{\pi}{2} - \theta_1\right) = \frac{C}{\rho}$, with C the constant in (6.1), for an arbitrary fixed constant $0 < \rho < 1$. We seek the 'largest' possible angle $\theta_1 < \frac{\pi}{2}$, at least after moving the vertex of the triangular sector in a nearby point. In our case, this nearby point will be $x_0 = \{-1, 0\}$; in which case, with vertex on $x_0 = \{-1, 0\}$ the angle θ_1 will be arbitrarily close to $\dfrac{\pi}{2}$. In this section, we shall establish a resolvent estimate such as (6.2a) for all $\lambda \in \mathbb{C}\backslash\mathcal{K}$, namely

$$\begin{cases} \|\lambda R(\lambda, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \text{const}, \quad \forall\, \lambda \in \mathbb{C}\backslash\mathcal{K}; & (6.3a) \\ \text{equivalently (since } \mathcal{A}_{b=0}R(\lambda, \mathcal{A}_{b=0}) = -I + \lambda R(\lambda, \mathcal{A}_{b=0})) \\ \|\mathcal{A}_{b=0}R(\lambda, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \text{const}, \quad \forall\, \lambda \in \mathbb{C}\backslash\mathcal{K}; & (6.3b) \end{cases}$$

$\mathcal{K}$ being the infinite key-shaped set defined in (6.7b) , see Fig. 2, whereby, moving the vertex of the triangular sector of analyticity to concide with the point $x_0 = \{-1, 0\}$, the corresponding angle θ_1 is arbitrarily close to $\dfrac{\pi}{2}$ (Fig. 3).

(b) Then (6.3b) and $\mathcal{S}_{r_0} \subset \rho(\mathcal{A}_{b=0})$ will imply that the real part of the spectrum $\sigma(\mathcal{A}_{b=0})$ of $\mathcal{A}_{b=0}$ is confined inside the negative axis $(-\infty, -\bar{r_0}]$, $0 < \bar{r_0} < r_0$. The direct passage from (6.3) to (6.2) is exhibited in Remark 6.2 below. Moreover, our proof below, once specialized with $\text{Re}\,\lambda = 0$, $\lambda = i\omega$, $\omega \in \mathbb{R}$, will yield (through simplified computations in Remark 6.1 below) the establishment of inequality (6.1) for *any* $\omega_0 > 0$. This result, combined with $\mathcal{S}_{r_0} \subset \rho(\mathcal{A}_{b=0})$ will allow us to conclude that

$$\|R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H})} \leq \text{const}, \ \omega \in \mathbb{R}. \tag{6.4}$$

Then, (6.4) will imply [28] uniform stabilization of the analytic semigroup $e^{\mathcal{A}_{b=0}t}$ on $\mathcal{H}_{b=0}$: there exists constants $M \geq 1$, $\delta > 0$ such that

$$\|e^{\mathcal{A}_{b=0}t}\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq Me^{-\delta t}, \ t \geq 0, \tag{6.5a}$$

and hence that

$$\text{Re}\,\sigma(\mathcal{A}_{b=0}) \in (-\infty; -\delta]. \tag{6.5b}$$

In conclusion, the present section establishes three results: (a) analyticity of the semigroup $e^{\mathcal{A}_{b=0}t}$; (b) location of the spectrum $\sigma(\mathcal{A}_{b=0})$ of $\mathcal{A}_{b=0}$, in Theorem 6.1, (i)–(iii); and (c) exponential stability (6.5a) in Theorem 6.1(iv). Analyticity and exponential stability require the assumption $\int_{\Gamma_s} \nu\, d\Gamma_s \neq 0$. The case $\int_{\Gamma_s} \nu\, d\Gamma_s = 0$ is analyzed in Sect. 7, still for $b = 0$.

6.2 Main Result for $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$

Theorem 6.1 *Let* $b = 0$. (i) *The generator* $\mathcal{A}_{b=0}$ *in* (2.3) *of the s.c. contraction semigroup* $e^{\mathcal{A}_{b=0}t}$ *asserted by Proposition* 4.4 *satisfies the following resolvent condition for* $\omega \in \mathbb{R}$

$$\|(i\omega I - \mathcal{A}_{b=0})^{-1}\|_{\mathcal{L}(\mathcal{H}_{b=0})} = \|R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \frac{c}{|\omega|}. \quad \forall\, |\omega| \geq \text{ some } \omega_0 > 0 \text{ arbitrarily small,} \tag{6.6}$$

Hence, under the assumption $\int_{\Gamma_s} \nu \, d\Gamma_s \neq 0$, *the s.c. semigroup* $e^{\mathcal{A}_{b=0}t}$ *is analytic on the finite energy space* $\mathcal{H}_{b=0}$, $t > 0$, [23, Thm 3E.3, p 334].

(ii) *More precisely, the resolvent operator* $R(\lambda, \mathcal{A}_{b=0}) = (\lambda I - \mathcal{A}_{b=0})^{-1}$ *of the generator* $\mathcal{A}_{b=0}$ *in (2.3), satisfies the following estimate*

$$\|R(\lambda, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \frac{C}{|\lambda|}, \text{ for all } \lambda \in \mathbb{C}\backslash\mathcal{K} \tag{6.7a}$$

where $\mathcal{K}$ *is the (infinite) key-shaped set defined in Fig.* 2

$$\mathcal{K} \equiv (-\infty, -2) \cup \{\mathcal{S}_{r=1}(x_0)\backslash\mathcal{S}_{r_0}\} \tag{6.7b}$$

with $\mathcal{S}_{r=1}(x_0)$ *the open disk centered at the point* $x_0 = \{-1, 0\}$ *and of radius* 1; *and* S_{r_0} *defined in Proposition* 5.2.

(iii) *The spectrum* $\sigma(\mathcal{A}_{b=0})$ *of* $\mathcal{A}_{b=0}$ *is confined within the set* $\mathcal{K}$; *in particular*

$$Re\, \sigma(\mathcal{A}_{b=0}) \subset (-\infty, -\delta], \quad \text{for some } \delta > 0. \tag{6.8}$$

(iv) *Complementing* (6.6) *we have that the resolvent* $R(\cdot, \mathcal{A}_{b=0})$ *is uniformly bounded on the imaginary axis*

$$\|R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq c, \quad \forall\, \omega \in \mathbb{R}. \tag{6.9}$$

Hence, the s.c. analytic semigroup $e^{\mathcal{A}_{b=0}t}$ *is uniformly exponentially stable on* $\mathcal{H}_{b=0}$: *there exist constants* $M \geq 1$, $\delta > 0$, *such that* [28]

$$\|e^{\mathcal{A}_{b=0}t}\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq Me^{-\delta t}, \quad t \geq 0. \tag{6.10}$$

Step 1 Given $\{v_1^*, v_2^*, f^*\} \in \mathcal{H}_{b=0}$, constants $\alpha < 0$ and $\omega \in \mathbb{R}\backslash\{0\}$, we seek to solve the equation

$$((\alpha + i\omega)I - \mathcal{A}_{b=0}) \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} \tag{6.11}$$

in terms of $\{v_1, v_2, f\} \in \mathcal{D}(\mathcal{A}_{b=0})$ uniquely, and establish, in fact, the analyticity estimate (6.3b). For $\lambda = \alpha + i\omega \in \rho(\mathcal{A}_{b=0})$, we have via (2.3)

$$\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = R(\lambda, \mathcal{A}_{b=0}) \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix}; \quad \mathcal{A}_{b=0} R(\lambda, \mathcal{A}_{b=0}) \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} = \mathcal{A}_{b=0} \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}$$
$$= \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta f - \nabla\pi \end{bmatrix}. \tag{6.12}$$

We see that the analyticity condition (6.3a) on $\mathcal{H}_{b=0}$ is equivalent to showing the following estimates (all norms are L_2-norms on the respective domains): there exists a constant $C > 0$ such that

$$\begin{cases} \|\nabla v_2\|^2_{\Omega_s} + \|\Delta(v_1 + v_2)\|^2_{\Omega_s} + \|(\Delta f - \nabla\pi)\|^2_{\Omega_f} \leq C\{\|\nabla v_1^*\|^2_{\Omega_s} + \|v_2^*\|^2_{\Omega_s} \\ \quad +\|f^*\|^2_{\Omega_f}\} \\ \text{for all } \lambda = \alpha + i\omega \text{ in } \rho(\mathcal{A}_{b=0})\backslash\mathcal{K}; \text{ that is, outside the set } \mathcal{K} \text{ defined in (6.7b)} \\ \quad \text{(Fig. 2)} \end{cases} \tag{6.13}$$

This is what we shall show below. Explicitly (6.11) is rewritten via (2.3) as

$$\begin{cases} (\alpha + i\omega)v_1 - v_2 = v_1^* \in (H^1(\Omega_s)/\mathbb{R})^d; & \text{(6.14a)} \\ (\alpha + i\omega)v_2 - \Delta(v_1 + v_2) = v_2^* \in (L_2(\Omega_s))^d; & \text{(6.14b)} \\ (\alpha + i\omega)f - (\Delta f - \nabla\pi) = f^* \in \tilde{H}_f \subset (L_2(\Omega_f))^d & \text{(6.14c)} \end{cases}$$

Step 2 Henceforth, to streamline the notation, $\|\cdot\|$, respectively $(\cdot,\cdot)$ will denote the $L_2(\cdot)$-norm, respectively the complex inner product on either the set Ω_s or the set Ω_f. No ambiguity is likely to occur. We take the $L_2(\Omega_f)$-inner product of Eq. (6.14c) against $(\Delta f - \nabla\pi)$, use Green's First Theorem to evaluate $\int_{\Omega_f} f\,\Delta\overline{f}\,d\Omega_f$, recall the B.C. $f|_{\Gamma_f} = 0$ (2.8), in $\mathcal{D}(\mathcal{A}_{b=0})$ and obtain

$$(\alpha + i\omega)\int_{\Gamma_s} f \cdot \frac{\partial\overline{f}}{\partial\nu}\,d\Gamma_s - (\alpha + i\omega)\|\nabla f\|^2 - (\alpha + i\omega)(f, \nabla\pi)$$
$$-\|(\Delta f - \nabla\pi)\|^2 = (f^*, (\Delta f - \nabla\pi)).$$
$$(\alpha + i\omega)\int_{\Gamma_s} f \cdot \frac{\partial\overline{f}}{\partial\nu}\,d\Gamma_s - (\alpha + i\omega)\|\nabla f\|^2 - (\alpha + i\omega)$$
$$\times\int_{\Gamma_s} \bar{\pi} f \cdot \nu\,d\Gamma_s - \|(\Delta f - \nabla\pi)\|^2 = (f^*, \Delta f - \nabla\pi). \tag{6.15}$$

by using Green'sTheorem with div $f \equiv 0$. Similarly, we take the $(L_2(\Omega_s))^d$-inner product of (6.14b) against $\Delta(v_1 + v_2)$, use Green's First Theorem to evaluate

$\int_{\Omega_s} v_2 \Delta(\overline{v}_1+\overline{v}_2) d\Omega_s$, recalling that the normal vector ν is inward w.r.t. Ω_s, and obtain

$$-(\alpha + i\omega)\int_{\Gamma_s} v_2 \cdot \frac{\partial(\overline{v}_1+\overline{v}_2)}{\partial\nu}\, d\Gamma_s - (\alpha+i\omega)(\nabla v_2, \nabla(v_1+v_2)) - \|\Delta(v_1+v_2)\|^2 = (v_2^*, \Delta(v_1+v_2)). \tag{6.16}$$

Invoking now the B.C. $f|_{\Gamma_s} = v_2|_{\Gamma_s}$ and $\left.\frac{\partial(v_1+v_2)}{\partial\nu}\right|_{\Gamma_s} = \left.\left(\frac{\partial f}{\partial\nu} - \pi\nu\right)\right|_{\Gamma_s}$ in $\mathcal{D}(\mathcal{A}_{b=0})$ (see (2.5)), we rewrite (6.16) as

$$\begin{aligned} &-(\alpha+i\omega)\int_{\Gamma_s} f \cdot \left(\frac{\partial\overline{f}}{\partial\nu} - \bar{\pi}\nu\right) d\Gamma_s - (\alpha+i\omega)\|\nabla v_2\|^2 - (\alpha+i\omega)(\nabla v_2, \nabla v_1) \\ &-\|\Delta(v_1+v_2)\|^2 \\ &= (v_2^*, \Delta(v_1+v_2)). \end{aligned} \tag{6.17}$$

Summing up (6.15) and (6.17) yields after a cancellation of the boundary terms

$$\begin{aligned} &-(\alpha+i\omega)[\|\nabla v_2\|^2 + \|\nabla f\|^2] \\ &\quad = \|\Delta(v_1+v_2)\|^2 + \|(\Delta f - \nabla\pi)\|^2 + (\alpha+i\omega)(\nabla v_2, \nabla v_1) \\ &\quad +(v_2^*, \Delta(v_1+v_2)) + (f^*, \Delta f - \nabla\pi) \end{aligned} \tag{6.18}$$

We now return to (6.14a), multiply by $(\alpha - i\omega) \neq 0$, and rewrite the result as $v_1 = [(\alpha - i\omega)/(\alpha^2+\omega^2)][v_2 + v_1^*]$, which introduced in the third term on the RHS of (6.18) yields

$$(\alpha+i\omega)(\nabla v_2, \nabla v_1) = \frac{(\alpha+i\omega)^2}{\alpha^2+\omega^2}[\|\nabla v_2\|^2 + (\nabla v_2, \nabla v_1^*)] \tag{6.19a}$$

$$= \frac{(\alpha^2-\omega^2) + i2\alpha\omega}{\alpha^2+\omega^2}\|\nabla v_2\|^2 + \frac{(\alpha+i\omega)^2}{\alpha^2+\omega^2}(\nabla v_2, \nabla v_1^*) \tag{6.19b}$$

Substituting (6.19b) into (6.18), we obtain the final identity

$$\begin{aligned} &\|\Delta(v_1+v_2)\|^2 + \|(\Delta f - \nabla\pi)\|^2 + i\left\{\omega\left[\left(1 + \frac{2\alpha}{\alpha^2+\omega^2}\right)\|\nabla v_2\|^2 + \|\nabla f\|^2\right]\right\} \\ &= \left[-\alpha + \frac{\omega^2-\alpha^2}{\alpha^2+\omega^2}\right]\|\nabla v_2\|^2 - \alpha\|\nabla f\|^2 - (v_2^*, \Delta(v_1+v_2)) \\ &- (f^*, \Delta f - \nabla\pi) - \frac{(\alpha+i\omega)^2}{\alpha^2+\omega^2}(\nabla v_2, \nabla v_1^*) \end{aligned} \tag{6.20}$$

Step 3 We take the real part of identity (6.20), thus obtaining the new identity:

$$\begin{aligned}
&\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\\
&=\left[-\alpha+\frac{\omega^2-\alpha^2}{\alpha^2+\omega^2}\right]\|\nabla v_2\|^2-\alpha\|\nabla f\|^2\\
&-\operatorname{Re}\left\{\frac{(\alpha+i\omega)^2}{\alpha^2+\omega^2}(\nabla v_2,\nabla v_1^*)\right\}\\
&-\operatorname{Re}(v_2^*,\Delta(v_1+v_2))-\operatorname{Re}(f^*,\Delta f-\nabla\pi) \qquad (6.21)
\end{aligned}$$

We estimate the RHS of (6.21), noticing that $\dfrac{|\omega^2-\alpha^2|}{\alpha^2+\omega^2}\le 1$, $\dfrac{|(\alpha+i\omega)^2|}{\alpha^2+\omega^2}\equiv 1$, thus obtaining the inequality

$$\begin{aligned}
&\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\le(|\alpha|+1+\epsilon_1)\|\nabla v_2\|^2+|\alpha|\|\nabla f\|^2\\
&\quad+\epsilon\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\right]+C_{\epsilon_1,\epsilon}\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\}
\end{aligned}$$

or

$$\begin{aligned}
&(1-\epsilon)\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\right]\le(|\alpha|+1+\epsilon_1)\,\|\nabla v_2\|^2+|\alpha|\|\nabla f\|^2\\
&\quad+C_{\epsilon_1,\epsilon}\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\}. \qquad (6.22)
\end{aligned}$$

We next take $|\alpha|>r_1>0$, with r_1 fixed but arbitrarily small, $1+\epsilon_1<\frac{1}{r_1}|\alpha|$ and setting $k_0=1+\frac{1}{r_1}$, we obtain

$$\begin{aligned}
&(1-\epsilon)\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\right]\le k_0|\alpha|\,\|\nabla v_2\|^2+|\alpha|\|\nabla f\|^2\\
&\quad+C_{\epsilon_1,\epsilon}\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\}. \qquad (6.23)
\end{aligned}$$

Next, with $\epsilon>0$ chosen above, we take $\epsilon|\alpha|<|\omega|$, thereby rewriting (6.23)

$$\begin{aligned}
&(1-\epsilon)\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\right]\\
&\le\frac{1}{\epsilon}\left[\epsilon k_0|\alpha|\|\nabla v_2\|^2+\epsilon|\alpha|\|\nabla f\|^2\right]+C_{\epsilon_1,\epsilon}\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\} \qquad (6.24)\\
&\le\frac{1}{\epsilon}\left[k_0|\omega|\|\nabla v_2\|^2+|\omega|\|\nabla f\|^2\right]+C_{\epsilon_1,\epsilon}\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\}
\end{aligned}$$

Step 4 We now take the imaginary part of identity (6.20), thus obtaining the new identity

$$\begin{aligned}
&\omega\left\{\left[1+\frac{2\alpha}{\alpha^2+\omega^2}\right]\|\nabla v_2\|^2+\|\nabla f\|^2\right\}=-\operatorname{Im}\left\{\frac{(\alpha+i\omega)^2}{\alpha^2+\omega^2}(\nabla v_2,\nabla v_1^*)\right\}\\
&-\operatorname{Im}\{(v_2^*,\Delta(v_1+v_2))+(f^*,(\Delta f-\nabla\pi))\}. \qquad (6.25)
\end{aligned}$$

With $\epsilon > 0$ arbitrary assume that

$$\begin{cases} 0 < \epsilon < \left[1 + \dfrac{2\alpha}{\alpha^2 + \omega^2} \right] \\ \text{that is, that the point } \{\alpha, \omega\} \text{ lies } \underline{\text{outside}} \text{ the disk } (\alpha + 1)^2 + \omega^2 = 1 \end{cases} \tag{6.26}$$

Then taking the absolute value of both sides of (6.25), using (6.26) as well as $\dfrac{|(\alpha + i\omega)^2|}{\alpha^2 + \omega^2} \equiv 1$, we obtain

$$\begin{aligned} \epsilon|\omega|\|\nabla v_2\|^2 + |\omega|\|\nabla f\|^2 &\leq \frac{\epsilon^2}{2}\|\nabla v_2\|^2 + C_{\epsilon^2}\|\nabla v_1^*\|^2 \\ &+ \epsilon^3 \left[\|\Delta(v_1 + v_2)\|^2 + \|(\Delta f - \nabla \pi)\|^2 \right] \\ &+ C_{\epsilon^3} \left[\|v_2^*\|^2 + \|f^*\|^2 \right] \end{aligned} \tag{6.27}$$

We then obtain from (6.27)

$$\begin{aligned} 0 < \frac{\epsilon}{2}|\omega|\|\nabla v_2\|^2 + |\omega|\|\nabla f\|^2 \leq \left[\epsilon|\omega| - \frac{\epsilon^2}{2} \right] \|\nabla v_2\|^2 + |\omega|\|\nabla f\|^2 \\ \leq \epsilon^3 \left[\|\Delta(v_1 + v_2)\|^2 + \|(\Delta f - \nabla \pi)\|^2 \right] + C_\epsilon \left[\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|f^*\|^2 \right] \end{aligned} \tag{6.28}$$

where the LHS of (6.28) is valid for all $|\omega|$ s.t.

$$\frac{\epsilon}{2}|\omega| \leq \epsilon|\omega| - \frac{\epsilon^2}{2}; \quad \text{or} \quad 0 < \epsilon < |\omega|. \tag{6.29}$$

From (6.28), we obtain

$$\begin{aligned} |\omega|\|\nabla f\|^2 &\leq \epsilon^3 \left[\|\Delta(v_1 + v_2)\|^2 + \|(\Delta f - \nabla \pi)\|^2 \right] \\ &+ C_\epsilon \left[\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|f^*\|^2 \right] \end{aligned} \tag{6.30}$$

and

$$\begin{aligned} |\omega|\|\nabla v_2\|^2 &\leq 2\epsilon^2 \left[\|\Delta(v_1 + v_2)\|^2 + \|(\Delta f - \nabla \pi)\|^2 \right] \\ &+ C_\epsilon \left[\|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|f^*\|^2 \right] \end{aligned} \tag{6.31}$$

Now invoke (6.30) and (6.31) in (6.24), we obtain

$$
\begin{aligned}
&(1-\epsilon)\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\right]\\
&\quad\le\frac{1}{\epsilon}\left\{2k_0\epsilon^2\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\right]\right\}\\
&\quad+\frac{1}{\epsilon}\left\{\epsilon^3\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\right]\right\}\\
&\quad+C_{\epsilon_1,\epsilon}\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\}.
\end{aligned}
\tag{6.32}
$$

Finally (6.32) implies

$$
\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\le C_{\epsilon,\epsilon_1,\epsilon_2}\left\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\right\}; \tag{6.33}
$$

Since $[1-\epsilon-2k_0\epsilon-\epsilon^2]>0$ by restricting further ϵ to have $\frac{2\epsilon}{r_1}<\epsilon_2$ with ϵ_2 arbitrarily small, so that recalling $2\epsilon k_0=2\epsilon+\frac{2\epsilon}{r_1}$ arbitrarily small.

Step 5 We return to estimate (6.28). On its LHS, we drop the positive term $|\omega|\|\nabla f\|^2$,while for this term we use: $\frac{\epsilon^2}{2}r_1\le\frac{\epsilon^2}{2}|\alpha|\le\frac{\epsilon}{2}|\omega|$ according to prior selections $\epsilon|\alpha|\le|\omega|$ and $|\alpha|>r_1$. On the RHS of (6.28), we invoke (6.33). We thus obtain, as desired

$$
\|\nabla v_2\|^2\le C_{\epsilon,\epsilon_1,\epsilon_2}\left\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\right\} \tag{6.34}
$$

Step 6 Summing up (6.33) and (6.34) we finally obtain

$$
\begin{aligned}
&\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2+\|\nabla v_2\|^2\right]\\
&\quad\le\text{const}_{\epsilon,r_1}\left\{\|\nabla v_1^*\|^2+\|v_2^*\|^2+\|f^*\|^2\right\}
\end{aligned}
\tag{6.35}
$$

for all points $\{\alpha,\omega\}$ satisfying (6.26) and (6.29) aslo the conditions $\epsilon|\alpha|\le|\omega|$. Since $\epsilon>0$ in these two relations is arbitrary, we conclude that (estimate (6.33) and hence the conclusive) estimate (6.35) hold(s) true for all points $\{\alpha,\omega\}$, $\alpha<0$, outside the disk: $(\alpha+1)^2+\omega^2=1$, with $\omega\ne0$. This is precisely the conclusion (6.3b); that is, conclusion (6.7aa). We have thus proved parts (ii), (iii) (location of the spectrum $\sigma(\mathcal{A}_{b=0})$) of Theorem 6.1 . □

Remark 6.1 (Specialization to the case $\alpha=0$) We specialize the above computations to the case $\alpha=0$, $\lambda=i\omega$, to obtain:

(a) The counterpart of identity (6.21) (real part) is

$$
\begin{aligned}
\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2&=\|\nabla v_2\|^2-\text{Re}(\nabla v_2,\nabla v_1^*)\\
&\quad-\text{Re}(v_2^*,\Delta(v_1+v_2))-\text{Re}(f^*,(\Delta f-\nabla\pi)),
\end{aligned}
\tag{6.36}
$$

which then yields

$$
(1-\epsilon)\left[\|\Delta(v_1+v_2)\|^2+\|(\Delta f-\nabla\pi)\|^2\right]
$$

$$\leq (1+\epsilon)\|\nabla v_2\|^2 + C_\epsilon \left\{ \|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|f^*\|^2 \right\}, \tag{6.37}$$

the counterpart of estimate (6.22).

(b) The counterpart of identity (6.25) (imaginary part) is

$$\omega \left[\|\nabla v_2\|^2 + \|\nabla f\|^2 \right] = -\mathrm{Im}(\nabla v_2, \nabla v_1^*) - \mathrm{Im}(v_2^*, \Delta(v_1+v_2)) - \mathrm{Im}(f^*, (\Delta f - \nabla\pi)). \tag{6.38}$$

Thus, (6.38) implies the estimate

$$\left[\|\nabla v_2\|^2 + \|\nabla f\|^2 \right] \leq \left(\frac{\epsilon}{|\omega| - \epsilon} \right) \left[\|\Delta(v_1+v_2)\|^2 + \|(\Delta f - \nabla\pi)\|^2 \right] + \left(\frac{C_\epsilon}{|\omega| - \epsilon} \right) \left\{ \|\nabla v_1^*\|^2 + \|v_2^*\|^2 + \|f^*\|^2 \right\}. \tag{6.39}$$

(c) Use of inequality (6.39) into the RHS of inequality (6.37) for the ∇v_2-term yields

$$\left[1 - \epsilon - \frac{\epsilon(1+\epsilon)}{|\omega| - \epsilon} \right] \left[\|\Delta(v_1+v_2)\|^2 + \|(\Delta f - \nabla\pi)\|^2 \right] \leq \left(C_\epsilon + \frac{(1+\epsilon)C_\epsilon}{|\omega| - \epsilon} \right) \left\{ \|\nabla v_1^*\|^2 + \|v_2^*)\|^2 + \|f^*\|^2 \right\}, \tag{6.40}$$

or taking $|\omega| - \epsilon \geq \omega_0 > 0$, hence $\frac{1}{2} < \left[1 - \epsilon - \frac{\epsilon(1+\epsilon)}{\omega_0} \right] < \left[1 - \epsilon - \frac{\epsilon(1+\epsilon)}{|\omega| - \epsilon} \right]$:

$$\|\Delta(v_1+v_2)\|^2 + \|\Delta f - \nabla\pi\|^2 \leq \mathrm{const}_{\epsilon,\omega_0} \left\{ \|\nabla v_1^*\|^2 + \|v_2^*)\|^2 + \|f^*\|^2 \right\}, \quad |\omega| > \omega_0 > 0 \tag{6.41}$$

which is the counterpart of (6.33).

(d) Finally, returning to (6.39) and using here (6.33), we obtain

$$\|\nabla v_2\|^2 + \|\nabla f\|^2 \leq \mathrm{const}_{\epsilon,\omega_0} \left\{ \|\nabla v_1^*\|^2 + \|v_2^*)\|^2 + \|f^*\|^2 \right\}, \tag{6.42}$$

which is the counterpart of estimate (6.34).

(e) Summing up (6.41) and (6.42), we obtain the counterpart of (6.35) = (6.3b) by (6.12), (6.13) for $\alpha = 0$, i.e.

$$\left\{ \begin{array}{ll} \|\mathcal{A}_{b=0} R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq C, \quad \forall\, |\omega| \geq \omega_0 > 0 \text{ arbitrary, equivalently} & (6.43a) \\ \|R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \dfrac{C}{|\omega|}, \quad \forall\, |\omega| \geq \omega_0 > 0 \text{ arbitrary,} & (6.43b) \end{array} \right.$$

Theorem 6.1 (i) is established.

Remark 6.2 (direct passage from (6.3) to (6.2)) Returning to Eqns. (6.14a–c), we obtain for $\lambda = \alpha + i\omega$:

$$|\lambda| \, \|\nabla v_1\| - \|\nabla v_2^*\| \leq \|\lambda \nabla v_1 - \nabla v_1^*\| = \|\nabla v_2\|; \tag{6.44}$$

$$|\lambda| \, \|v_2\| - \|v_2^*\| \leq \|\lambda v_2 - v_2^*\| = \|\Delta(v_1 + v_2)\|; \tag{6.45}$$

$$|\lambda| \, \|f\| - \|f^*\| \leq \|\lambda f - f^*\| = \|(\Delta f - \nabla \pi)\|. \tag{6.46}$$

Hence, summing up,

$$\begin{aligned} |\lambda|[\|\nabla v_1\| + \|v_2\| + \|f\|] &\leq \|\nabla v_2\| + \|\Delta(v_1 + v_2)\| + \|(\Delta f - \nabla \pi)\| \\ &\quad + \|\nabla v_2^*\| + \|v_2^*\| + \|f^*\| \\ (\text{by } (6.35)) &\leq C \left\{ \|\nabla v_1^*\| + \|v_2^*\| + \|f^*\| \right\}, \end{aligned} \tag{6.47}$$

for all λ satisfying (6.26) and (6.29). In short, in view of (6.11) and (1.2a), estimate (6.47) says that

$$\left\| \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\|_{\mathcal{H}} = \left\| R(\lambda, \mathcal{A}_{b=0}) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right\|_{\mathcal{H}} \leq \frac{C_{\omega_0}}{|\lambda|} \left\| \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right\|_{\mathcal{H}}, \tag{6.48}$$

for all such $\lambda = \alpha + i\omega$. Theorem 6.1(ii), Eq. (6.7a), is proved.

6.3 Exponential Stability

The resolvent bound (6.43b) combined with $\mathcal{A}_{b=0}^{-1} \in \mathcal{L}(\mathcal{H}_{b=0})$, hence $\mathcal{S}_{r_0} \subset \rho(\mathcal{A}_{b=0})$ by Proposition 2.1, Fig. 3 allows one to conclude that the resolvent is uniformly bounded on the imaginary axis $i\mathbb{R}$:

$$\|R(i\omega, \mathcal{A}_{b=0})\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq \text{const}, \tag{6.49}$$

as claimed in (6.9). Hence, [28] the s.c. analytic semigroup $e^{\mathcal{A}_{b=0}t}$ is, moreover. (uniformly) exponentially bounded: There exist constants $M \geq 1$, $\delta > 0$, such that

$$\|e^{\mathcal{A}_{b=0}t}\|_{\mathcal{L}(\mathcal{H}_{b=0})} \leq M e^{-\delta t}, \quad t \geq 0, \quad \text{any } \delta > r_0 \tag{6.50}$$

by (6.8). This proves Theorem 6.1(iii), Eq. (6.10). Theorem 6.1 is fully proved.

7 Case $b = 0$, $\int_{\Gamma_s} v \, d\Gamma_s = 0$: Analyticity, Location of the Spectrum Within the Set $\mathcal{K}$ (Within the Set $\overline{\mathcal{K} \cup \{0\}}$) of the Operator $\widehat{\mathcal{A}}_{b=0}$ on the Space $\widehat{\mathcal{H}}_{b=0}$ (of the Operator $\mathcal{A}_{b=0}$ on the Space $\widehat{\mathcal{H}}_{b=0}$)

We consider at first the operator $\widehat{\mathcal{A}}_{b=0}$ on the space $\widehat{\mathcal{H}}_{b=0} = [Null(\mathcal{A}_{b=0})]^{\perp}$, as claimed in (5.4). We rely on Propositions 5.1(b), 5.3, 5.4 and Corollary 5.5.

Theorem 7.1 *Let $b = 0$ and $\int_{b=0} \nu \, d\Gamma_s = 0$.* (i) *The perfect counterpart of Theorem 6.1 on analyticity, location of spectrum and uniform stability holds true for the operator $\widehat{\mathcal{A}}_{b=0}$ on the space $\widehat{\mathcal{H}}_{b=0}$.*

Proof The proof is exactly verbatim the one of Theroem 6.1 after replacing the operator $\mathcal{A}_{b=0}$ on the space $\mathcal{H}_{b=0}$ with the operator $\widehat{\mathcal{A}}_{b=0}$ in $\widehat{\mathcal{H}}_{b=0}$, taking into account Propositions 5.1(b), 5.3, 5.4 and Corollary 5.5. □

Corollary 7.1 *Let $b = 0$ and $\int_{\Gamma_s} \nu \, d\Gamma_s = 0$. The operator $\mathcal{A}_{b=0}$ generates a s.c analytic semigroup $e^{\mathcal{A}_{b=0}t}$ on the space $\mathcal{H}_{b=0}$. As to its spectrum, one has $\sigma(\mathcal{A}_{b=0}) = \sigma(\widehat{\mathcal{A}}_{b=0}) \cup \{0\}$ on $\mathcal{H}_{b=0}$, where $\sigma(\widehat{\mathcal{A}}_{b=0})$ in $\widehat{\mathcal{H}}_{b=0}$ is centered in the set $\mathcal{K}$ in 6.7b.*

Proof As in the proof of Corollary 5.5, if $x \in \mathcal{H}_{b=0}$, then $x = \hat{x} + ae_0$, with $\hat{x} \in \widehat{\mathcal{H}}_{b=0}$ and the eigenvector defined in Proposition (5.1)(b). Then $e^{\mathcal{A}_{b=0}t}x = e^{\widehat{\mathcal{A}}_{b=0}t}\hat{x} + ae_0$ defines the analytic contraction semigroup on $\mathcal{H}_{b=0}$. □

8 *b* = 1. Proof of Theorem 1.6

Proof **Step 1** We already know that $e^{\mathcal{A}_{b=1}t}$ is a s.c analytic semigroup on $\mathcal{H}_{b=0}$, as on this space the generator $\mathcal{A}_{b=1}$ is an innocuous bounded perturbation of the of the analytic semigroup generator $\mathcal{A}_{b=0}$ on $\mathcal{H}_{b=0}$.

Step 2 Via definition (2.3), we obtain

$$\mathcal{A}_{b=1}\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -c \\ -\nabla\pi \end{bmatrix} = \begin{bmatrix} 0 \\ -c \\ 0 \end{bmatrix};$$
$$\mathcal{A}^2_{b=1}\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \mathcal{A}_{b=1}\begin{bmatrix} 0 \\ -c \\ 0 \end{bmatrix} = \begin{bmatrix} -c \\ 0 \\ -\nabla\pi \end{bmatrix} = -\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \tag{8.1}$$

recalling from (2.4a) that in the present case π satisfies:

$$\Delta\pi = 0 \text{ in } \Omega_f, \quad \pi|_{\Gamma_s} = 0, \quad \left.\frac{\partial\pi}{\partial\nu}\right|_{\Gamma_s} = 0, \text{ so that } \pi \equiv 0 \text{ in } \Omega_f,$$

and similarly for π^* via (4.2). Thus $-1 \in \sigma_p(\mathcal{A}^2_{b=1})$, with eigenvector $\Phi = [c, 0, 0]$. [Recall that via Theorem 1.5(i), $0 \in \sigma_p(\mathcal{A}_{b=1})$, so we cannot deduce from (8.1) that $\mathcal{A}_{b=1}\Phi = i\Phi$, which in fact is incorrect]

Step 3 It follows from the RHS of (8.1) that

$$\mathcal{A}^{2n}_{b=1}\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = (-1)^n\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}, \quad n = 1, 2, 3 \ldots, \quad i.e\,(-1)^n \in \sigma_p(\mathcal{A}^{2n}_{b=1}) \tag{8.2}$$

while (8.2) for $n = 1$, and the LHS of (8.1) imply

$$\mathcal{A}_{b=1}^3 \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \mathcal{A}_{b=1}\mathcal{A}_{b=1}^2 \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = -\mathcal{A}_{b=1} \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} \tag{8.3}$$

Similarly, (8.2) for $n = 2$ and the LHS of (8.1) yield

$$\mathcal{A}_{b=1}^5 \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \mathcal{A}_{b=1}\mathcal{A}_{b=1}^4 \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \mathcal{A}_{b=1} \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -c \\ 0 \end{bmatrix}; \; \mathcal{A}_{b=1}^7 \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} \tag{8.4}$$

In general

$$\mathcal{A}_{b=1}^{2n-1} \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = (-1)^n \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}, \quad n = 1, 2, 3 \dots \tag{8.5}$$

Thus, invoking (8.5) and (8.1), we obtain, as desired in (1.21):

$$e^{\mathcal{A}_{b=1}t} \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{A}_{b=1}^k \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \tag{8.6}$$

$$= \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \mathcal{A}_{b=1}^{2n-1} \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} + \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \mathcal{A}_{b=1}^{2n} \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \tag{8.7}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n-1}}{(2n-1)!} \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} + \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \tag{8.8}$$

$$= \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \cos t. \tag{8.9}$$

□

Acknowledgements The authors wish to thank two referees for their much appreciated comments. Research partially supported by the National Science Foundation under Grant DMS-1713506. This paper was written while R.T was a MSRI Research Professor during the 2021 Spring Semester, “Mathematical problems in fluid dynamics”, held at the Mathematical Sciences Research Institute, University of California, Berkeley.

References

1. Avalos, G., Triggiani, R.: The coupled PDE-system arising in fluid-structure interaction. Part I: Explicit semigroup generator and its spectral properties (with G. Avalos). AMS Contemp. Math. Fluids Waves **440**, 15–55 (2007)
2. Avalos, G., Triggiani, R.: Uniform Stabilization of a coupled PDE system arising in fluid–structure interaction with boundary dissipation at the interface. Discret. Cont. Dyn. Syst. **22**, 817–833 (2008)

3. Avalos, G., Triggiani, R.: Boundary feedback stabilization of a coupled parabolic-hyperbolic Stokes–Lamé PDE system. J. Evol. Equ. **9**, 341–370 (2009)
4. Avalos, G., Lasiecka, I., Triggiani, R.: Higher regularity of a coupled parabolic-hyperbolic fluid–structure interaction system. Georgian Math. J. **15**, 403–437 (2008)
5. Avalos, G., Lasiecka, I., Triggiani, R.: Heat–wave interaction in 2–3 dimensions: optimal rational decay rate. J. Math. Anal. Appl. **437**(2), 782–815 (2016)
6. Balakrishnan, A.V.: Semigroups of Operators: Theory and Applications. Springer, Basel (1998)
7. Constantin, P., Foias, C.: Navier–Stokes Equations. University of Chicago Press, Chicago (1989)
8. Chen, G., Russell, D.L.: A mathematical model for linear elastic systems with structural damping. Quart. Appl. Math. **433–454** (1982)
9. Chen, S., Triggiani, R.: Proof of two conjectures of G. Chen and D.L. Russell on structural damping for elastic systems: the case $\alpha = 1/2$ (with S.Chen). Springer-Verlag Lecture Notes in Mathematics 1354 (1988), 234–256. Proceedings of Seminar on Approximation and Optimization, University of Havana, Cuba (January 1987)
10. Chen, S., Triggiani, R.: Proof of the extensions of two conjectures on structural damping for elastic system. Pac. J. Math. **136**, 15–55 (1989)
11. Chen, S., Triggiani, R.: Gevrey class semigroups arising from elastic systems with gentle dissipation: the case $0<\alpha<\frac{1}{2}$. Proc. Am. Math. Soc. **110**, 401–415 (1990)
12. Chen, S., Triggiani, R.: Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications. J. Differ. Equ. **88**, 279–293 (1990)
13. Chen, S., Triggiani, R.: Gevrey class semigroups arising from elastic systems with gentle perturbation. Proc. Am. Math. Soc. **110**, 401–415 (1990)
14. Ignatova, M., Kukavica, I., Lasiecka, I., Tuffaha, A.: On well-posedness for a free boundary fluid–structure model. J. Math. Phys. **53**, 15624 (2012)
15. Ignatova, M., Kukavica, I., Lasiecka, I., Tuffaha, A.: On wellposedness and small data global existence for an interface damped free boundary fluid–structure interaction model-. Nonlinearity **27**(3), 467–499 (2014)
16. Kesavan, S.: Topic in Functional Analysis and Applications. New Age International(P)Ltd, Chennai (1989)
17. Kaltenbacher, B., Kukavica, I., Lasiecka, I., Triggiani, R., Tuffaha, A., Webster, J.T.: Mathematical Theory of Evolutionary Fluid–Flow Structure Interactions. Oberwolfach Seminars, Volume 48, Birkhauser
18. Kukavica, I., Tuffaha, A., Ziane, M.: Strong solutions to a nonlinear fluid–structure interaction system. J. Differ. Equ. **247**, 1452–1478 (2009)
19. Kukavica, I., Tuffaha, A., Ziane, M.: Strong solutions to a nonlinear fluid–structure interaction system. Adv. Differ. Equ. **15**, 231–254 (2010)
20. Lu, Y.: Uniform decay rates for the energy in nonlinear fluid–structure interaction with monotone viscous damping. PJM **2**(2), 215–232 (2013)
21. Lasiecka, I., Lu, Y.: Asymptotic stability of finite energy in Navier–Stokes-elastic wave interaction. Semigroup forum **82**, 61–82 (2011)
22. Lasiecka, I., Lu, Y.: Interface feedback control stabilization to a nonlinear fluid–structure interaction model. Nonlinear Anal. **75**, 1449–1460 (2012)
23. Lasiecka, I., Triggiani, R.: Control Theory for Partial Differential Equations: Continuous and Approximation Theories, Vol. 1 (680 pp.), Vol. 2 (422 pp.). Encyclopedia of Mathematics and Its Applications Series, Cambridge University Press, Cambridge (January 2000)
24. Lasiecka, I., Triggiani, R.: Heat-structure interaction with viscoelastic damping: analyticity with sharp analytic sector, exponential decay, fractional powers. Commun. Pure Appl. Anal. **15**, 1515–1543 (2016)
25. Lasiecka, I.,Triggiani, R.: Domain of fractional powers of matrix-valued operators: a general approach, invited paper. Operator Semigroups meet Complex Analysis, Harmonic Analysis and Mathematical Physics, Birkauser, Vol. 250, dedicated to C.Batty (2015), pp 297–311
26. Mahawattege, R.: Analysis and control of fluid-structure interaction with visco-elastic Kelvin–Voigt damping. PhD thesis, University of Memphis
27. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, Berlin (1983)
28. Pruss, J.: On the spectrum of C_0 semigroups. Trans. Am. Math. Soc. **284**, 847–857 (1984)
29. Taylor, A., Lay, D.: Introduction to Functional Analysis, 2nd edn. Wiley, New York (1980)

30. Teman, R.: Navier–Stokes Equations and Non-linear Functional Analysis, North Holland, Amsterdam (1978). Differ. Int. Equ. **3**, 503–522 (1990) (Also, preliminary version in Proceedings INRIA Conference, Paris, France (June 1988), Springer-Verlag Lecture Notes)
31. Triggiani, R.: Finite rank, relatively bounded perturbations of semi-groups generators. III. A sharp result on the lack of uniform stabilization. Differ. Int. Equ. **3**, 503–522 (1990)
32. Triggiani, R.: A heat-structure interaction model with Neumann or Dirichlet boundary control at the interface: optimal regularity, control theoretic implications. Appl. Math. Optim. **73**, 571–594 (2016)
33. Triggiani, R.: A matrix-valued generator A with strong boundary coupling: a critical subspace of $D((-A)^{1/2})$ and $D((-A^*)^{1/2})$ and implications. Evol. Equ. Control Theory **5**(1), 185–199 (2016)
34. Triggiani, R.: Domain of fractional powers of the heat-structure operator with visco-elastic damping: regularity and control-theoretic implication. J. Evol. Equ. **17**(1), 573–597 (2017)
35. Triggiani, R.: Linear parabolic-hyperbolic fluid-structure interaction models. The case of static interface. Chapter 2 in "Mathematical Theory of Evolutionary Fluid-Flow Structure Interaction", pp. 53–172, Birkhauser (2018). Oberwolfach Seminars vol 48, B. Kaltenbacher, I. Kukavica, I. Lasiecka, R. Triggiani, A. Tuffaha, J. Webster
36. Triggiani, R.: Heat-viscoelastic plate interaction via bending moments and shear forces operators: analyticity, spectral analysis, exponential decay. Appl. Math Optim. **82**, 755–797 (2020)
37. Triggiani, R., Zhang, J.: Heat–viscoelastic plate interaction: analyticity, spectral analysis, exponential decay. Evol. Equ. Control Thoery **1**(1), 153–182 (2017)
38. Zhang, J.: The analyticity and exponential decay of a Stokes–Wave coupling system with visco-elastic damping in the variational frame work. Evol. Equ. Control Theory **6**(1), 135–154 (2017)