

NO PURE CAPILLARY SOLITARY WAVES EXIST IN 2D FINITE DEPTH*

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Abstract. We prove that the 2D finite depth capillary water wave equations admit no solitary wave solutions. This closes the existence/nonexistence problem for solitary water waves in 2D under the classical assumptions of incompressibility and irrotationality and with the physical parameters being gravity, surface tension, and the fluid depth.

Key words. capillary waves, solitary waves, water waves, holomorphic coordinates

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1. Introduction. Solitary water waves are localized disturbances of a fluid surface which travel at constant speed and with a fixed profile. Such waves were first observed by Russell in the mid-19th century [42] and are fundamental features of many water wave models. The objective of this paper is to settle the existence/nonexistence problem for the full irrotational water wave system in 2D, with the physical parameters being gravity, surface tension, and the fluid depth. Five of the six combinations have already been dealt with, and the results are summarized in Table 1; it is our intent to fill in the missing case.

TABLE 1
Existence of 2D solitary waves in irrotational fluids.

Gravity	Capillarity	Depth	Existence
Yes	Yes	Infinite	Yes
Yes	No	Infinite	No
No	Yes	Infinite	No
Yes	Yes	Finite	Yes
Yes	No	Finite	Yes
No	Yes	Finite	Unknown

In a nutshell, our result can be loosely formulated as follows.

THEOREM 1.1. *No solitary waves exist in finite depth for the pure capillary irrotational water wave problem in 2D, even without the assumption that the free surface is a graph.*

A more precise formulation of the result is given later in Theorem 4.1.

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Historical perspectives. The mathematical study of traveling waves has been a fundamental—and long-standing—problem in fluid dynamics. Perhaps the first rigorous construction of 2D finite depth pure gravity solitary waves occurred in [17, 32]; further refinements can be found in [7, 37]. Solitary waves with large amplitudes were first constructed by Amick and Toland [6] in 1981 using global bifurcation techniques, leading to the existence of a limiting extreme wave with an angled crest [3]; see also [5, 8, 44]. By now, a vast literature exists on this subject, including both results for gravity and for gravity-capillary waves [4, 11, 12, 13, 20, 21, 40, 41]. For water waves in deep water, solitary waves have been proved to exist provided that both gravity and surface tension are present; see [9, 10, 22, 28], following numerical work in [33, 34]. The forefront of current research on the mathematical theory of steady water waves is surveyed in [24].

The nonexistence of 2D pure gravity solitary waves in infinite depth was originally proved in [25] under certain decay assumptions. The proof uses conformal mapping techniques, and the decay assumptions ultimately stem from difficulties in estimating commutators involving the Hilbert transform. The decay assumptions were completely removed in [27], as the authors were able to effectively deal with the aforementioned commutator issues; see [27, Lemma 3.1].

The proof of our result is loosely based on the ideas of [27]. The key difference is that the Tilbert transform (see section 3 for the definition) does not enjoy the same commutator structure as the Hilbert transform. More precisely, we cannot simply replace Hilbert transforms with Tilbert transforms in [27, Lemma 3.1]. To circumvent this, we morally view the Tilbert transform as the Hilbert transform at high frequency and a derivative at low frequency and use these distinct regimes to close our argument.

For context, we mention that the problem we are considering in this article goes at least as far back as [18]. More specifically, in [18], it is noted that the systematic existence methods developed in [16, 17, 29] for the pure gravity problem in shallow water are unable to produce pure capillary solitary waves but can be modified to produce gravity-capillary solitary waves. One may contrast the question of the existence of solitary waves with that of the existence of periodic traveling waves. Indeed, for pure capillary irrotational waves in both finite and infinite depth, periodic traveling waves are known to exist. Most notably, one has the Crapper waves, which are quite explicit; see [14, 30] for the original results of Crapper and Kinnersley and also the survey in [39]. Interestingly, the free surfaces of the Crapper waves need not be graphs, which makes the lack of graph assumption in Theorem 1.1 essential. The reader is referred to [1, 15, 35, 36, 45] for further literature on pure capillary waves as well as gravity-capillary perturbations of these waves.

Finally, we mention a few recent directions that are somewhat outside the scope of this paper. The first is the study of steady water waves with vorticity, for which we refer the interested reader to the surveys [19, 43]. As mentioned, our nonexistence proof utilizes holomorphic coordinates, a technique which is not compatible with variable vorticity. However, such a restriction is quite natural, as heuristics dictate that one should expect solitary waves in problems with, say, constant nonzero vorticity. The other interesting direction—in situations where solitary waves are known to exist—is to determine which speeds are capable of sustaining solitary waves. Recently, it was shown in [31] that all finite depth, irrotational, pure gravity solitary waves must obey the inequality $c^2 > gh$. Here c is the speed, g the gravitational constant, and h the asymptotic depth. Heuristically, this result says that speeds that are precluded by the linearized problem are also precluded in the nonlinear problem. As a loose guide-

line, one expects solitary waves to travel at different speeds than the linear dispersive waves; the situations in Table 1 where solitary waves do not exist are exactly those in which the dispersion relation contains all speeds.

Our discussion above is fully confined to the 2D case, and that is for a good reason. All nonexistence results discussed above in 2D are essentially open problems in 3D, so the 3D case is left for the future.

2. The equations in Eulerian coordinates. We consider the incompressible, finite depth water wave equations in two space dimensions. The motion of the water is governed by the incompressible Euler equations with boundary conditions on the water surface and the flat, finite bottom. We emphasize that this section is purely for motivational purposes and is not the formulation we will use to prove our nonexistence result. In particular, for simplicity, this subsection assumes that $\Gamma(t)$ is a graph, but we will *not* assume this when working with the holomorphic formulation of our problem.

To describe the equations, denote the water domain at time t by $\Omega(t) \subseteq \mathbb{R}^2$ (we assume that $\Omega(t)$ has a flat finite bottom $\{y = -h\}$), and let $\eta(x, t)$ denote the height of the free surface as a function of the horizontal coordinate:

$$(2.1) \quad \Omega(t) = \{(x, y) \in \mathbb{R}^2 : -h < y < \eta(x, t)\}.$$

The free surface of the water at time t will be denoted by $\Gamma(t)$. As we are interested in solitary waves, we think of $\Gamma(t)$ as being asymptotically flat at infinity to $y \approx 0$. Since the 2D finite depth capillary water wave equations do permit periodic traveling waves, this decay at infinity will factor heavily into our proof, even though we do not impose any specific rate of decay.

We denote by u the fluid velocity and by p the pressure. The vector field u solves Euler's equations inside $\Omega(t)$,

$$(2.2) \quad \begin{cases} u_t + u \cdot \nabla u = -\nabla p - g e_2, \\ \operatorname{div} u = 0, \\ u(0, x) = u_0(x), \end{cases}$$

and the bottom boundary is impenetrable:

$$(2.3) \quad u \cdot e_2 = 0 \quad \text{when } y = -h.$$

On the upper boundary, the atmospheric pressure is normalized to zero, and we have the dynamic boundary condition

$$(2.4) \quad p = -\sigma \mathbf{H}(\eta) \quad \text{on } \Gamma(t),$$

and the kinematic boundary condition

$$(2.5) \quad \partial_t + u \cdot \nabla \text{ is tangent to } \bigcup \Gamma(t).$$

Here $g \geq 0$ represents the gravity,

$$(2.6) \quad \mathbf{H}(\eta) = \partial_x \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right)$$

is the mean curvature of the free boundary, and $\sigma > 0$ represents the surface tension coefficient.

We adhere to the classical assumption that the flow is irrotational, so we can write u in terms of a velocity potential ϕ as $u = \nabla\phi$. It is easy to see that ϕ is a harmonic function whose normal derivative is zero on the bottom. Thus, ϕ is determined by its trace $\psi = \phi|_{\Gamma(t)}$ on the free boundary $\Gamma(t)$. Under these assumptions, it is well known that the fluid dynamics can be expressed in terms of a 1D evolution of the pair of variables (η, ψ) via

$$(2.7) \quad \begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta - \sigma \mathbf{H}(\eta) + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2} \frac{(\nabla\eta \cdot \nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = 0. \end{cases}$$

Here G denotes the Dirichlet-to-Neumann map associated to the fluid domain. This operator is one of the main analytical obstacles in this formulation of the problem, and in the next subsection, we briefly discuss a change of coordinates that somewhat simplifies the analysis.

We now write down the solitary wave equations. We begin with (2.1)–(2.6) as well as the irrotationality condition and assume that the profile is uniformly translating in the horizontal direction with velocity c , i.e., $\phi(x, y, t) = \phi_0(x - ct, y)$, $\eta(x, y, t) = \eta_0(x - ct, y)$, and $p(x, y, t) = p_0(x - ct, y)$. This gives the steady water wave equations. To get to solitary waves (as opposed to, say, periodic waves), we impose some averaged decay on η_0 and u_0 , so that in the far field, the water levels out and is essentially still. Contrary to many works which use a frame of reference traveling with the localized disturbance, we choose a frame so that the fluid is at rest near infinity. This allows us to set to zero the integration constant in the Bernoulli equation; the price to pay is that there are terms with c in the equations below.

We are thus interested in states (η, ϕ) satisfying the following equations:

$$(2.8) \quad \Delta\phi = 0 \quad \text{in } \Omega = \{(x, y) \in \mathbb{R}^2 : -h < y < \eta(x)\},$$

$$(2.9) \quad -c\phi_x + \frac{1}{2}|\nabla\phi|^2 + g\eta - \sigma\partial_x \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right) = 0 \quad \text{on } \Gamma = \{(x, y) \in \mathbb{R}^2 : y = \eta(x)\},$$

$$(2.10) \quad \phi_y = 0 \quad \text{when } y = -h,$$

$$(2.11) \quad -c\eta_x + \phi_x\eta_x = \phi_y \quad \text{on } \Gamma.$$

We prove that in the case $g = 0$ and $\sigma > 0$, the above equations admit no nontrivial solutions with appropriate (averaged) decay at infinity. Such a claim, of course, presupposes certain regularity requirements on the solutions, but this will not play a major role due to ellipticity. Indeed, the above system can be shown to be locally elliptic whenever (η, ϕ) is above critical regularity, which corresponds to $\eta \in H_{loc}^{\frac{3}{2}+}$.

3. The equations in holomorphic coordinates. As mentioned, one of the main difficulties of (2.7) is the presence of the Dirichlet-to-Neumann operator $G(\eta)$, which depends on the free boundary. For this reason, we will reformulate the equations in holomorphic coordinates, which, in some sense, diagonalizes $G(\eta)$. We will

only highlight briefly the procedure of changing coordinates; full details can be found in [23]. Moreover, although (2.7) assumes that $\Gamma(t)$ is a graph, the formulation below does not require this, which is another advantage of this approach. As we will see, making the solitary wave ansatz in holomorphic coordinates leads to remarkable simplifications, ultimately allowing us to derive (4.5), which we show admits no nontrivial solutions in appropriate function spaces. Equation (4.5) is very similar to the equation analyzed in the original paper of Crapper [14, equation (15)], though in that paper, they are viewed in infinite depth and in different function spaces.

The conditions we require on $\Gamma(t)$ are the same (or weaker; see the discussion below) as those listed in section 2.3 of [23], namely, that $\Gamma(t)$ can be parametrized to have sufficient Sobolev regularity, has no degeneracies or self-intersections, and never touches the bottom boundary. These assumptions are used in [23, Theorem 3] to justify the existence of the conformal map we refer to below.

In the holomorphic setting, the coordinates are denoted by $\alpha + i\beta \in S := \mathbb{R} \times (-h, 0)$, and the fluid domain is parameterized by the conformal map

$$z : S \rightarrow \Omega(t),$$

which takes the bottom $\mathbb{R} - ih$ into the bottom and the top $\mathbb{R}$ into the top $\Gamma(t)$. The restriction of this map to the real line is denoted by Z , i.e., $Z(\alpha) := z(\alpha - i0)$, and can be viewed as a parametrization of the free boundary $\Gamma(t)$. We will work with the variables $W(\alpha) = Z(\alpha) - \alpha$ and the trace $Q(\alpha)$ of the holomorphic velocity potential on the free surface. W and Q are traditionally called holomorphic functions, which in this terminology means that they can be realized as the trace on the upper boundary $\beta = 0$ of holomorphic functions in the strip S which are purely real on the lower boundary $\beta = -h$. The space of holomorphic functions is a real algebra but is not a complex algebra.

In terms of regularity, we note that the existence of the conformal map is guaranteed by the Riemann mapping theorem for any simply connected fluid domain. In order to have an equivalence between Sobolev norms, it suffices to assume that the free surface Γ has critical Besov regularity $B_{2,1}^{\frac{3}{2}}$. This, in particular, guarantees that Γ is a graph outside of a compact set. The conformal map, then, has the matching property $\Im(W) \in B_{2,1}^{\frac{3}{2}}$, and in particular, $\Im(W)$ and W_α are bounded. For more details, we refer the reader to both [23, section 2] and the stronger results in [2] as well as the more general local results of [38].

The 2D finite depth gravity-capillary water wave equations in holomorphic coordinates can be written as follows:

$$(3.1) \quad \begin{cases} W_t + F(1 + W_\alpha) = 0, \\ Q_t + FQ_\alpha - g\mathcal{T}_h[W] + \mathbf{P}_h \left[\frac{|Q_\alpha|^2}{J} \right] + \sigma \mathbf{P}_h \left[i \left(\frac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1+\overline{W_\alpha})} \right) \right] = 0, \end{cases}$$

where

$$(3.2) \quad J = |1 + W_\alpha|^2$$

and

$$(3.3) \quad F = \mathbf{P}_h \left[\frac{Q_\alpha - \overline{Q_\alpha}}{J} \right].$$

As before, g and σ are nonnegative parameters, at least one of which is nonzero. $\mathcal{T}_h$ denotes the Tilbert transform, which is the Fourier multiplier with symbol $-i \tanh(h\xi)$,

and arises in order to characterize what it means to be a holomorphic function. Precisely, holomorphic functions are described by the relation

$$(3.4) \quad \Im(u) = -\mathcal{T}_h \Re(u).$$

It is important to note that the Tilbert transform takes real-valued functions to real-valued functions and satisfies the following product rule:

$$(3.5) \quad u\mathcal{T}_h[v] + \mathcal{T}_h[u]v = \mathcal{T}_h[uv - \mathcal{T}_h[u]\mathcal{T}_h[v]].$$

Finally, $\mathbf{P}_h$ is the projection onto the space of holomorphic functions. In terms of $\mathcal{T}_h$, it can be written as

$$(3.6) \quad \mathbf{P}_h u = \frac{1}{2} [(1 - i\mathcal{T}_h)\Re(u) + i(1 + i\mathcal{T}_h^{-1})\Im(u)].$$

In the case of no surface tension, the equations in (3.1) were derived in [23]. We begin with a brief outline of how the surface tension term arises, as we are particularly interested in the case when $g = 0$ and $\sigma > 0$.

Following [23], we arrive at the Bernoulli equation

$$(3.7) \quad \phi_t + \frac{1}{2} |\nabla \phi|^2 + gy + p = 0.$$

We then evaluate this equation on the top boundary and apply the dynamic boundary condition to replace p by $-\sigma \mathbf{H}$. We then pass to the strip S (so the equations are now defined on $\{\beta = 0\}$), rewrite the equations in terms of the holomorphic variables, clear common factors of 2, and project. Running this procedure explicitly for the term with σ , we begin by parameterizing $\Gamma(t)$ by, say, $s \mapsto (\gamma_1(s), \gamma_2(s))$ and write $-\sigma \mathbf{H}$ in the standard parametric way. We then use the relations

$$\gamma_1(s) = \Re(Z(\alpha)), \quad \gamma_2(s) = \Im(Z(\alpha))$$

and formal calculations to write the capillary expression in terms of the holomorphic variables as

$$\sigma i \left(\frac{W_{\alpha\alpha}}{J^{1/2}(1 + W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1 + \overline{W_\alpha})} \right),$$

which after projecting gives us the capillary term in (3.1).

Remark 3.1. Before proceeding, we would like to point out some inherent ambiguities of the above equations which have to be properly interpreted. The first stems from the horizontal translation symmetry of the strip, which causes some arbitrariness in the choice of conformal mapping; precisely, $\Re(W)$ is only determined up to constants. A related issue is in the definition of the inverse Tilbert transform, as the Tilbert transform does not see constants. These ambiguities are built into the function spaces of [23] and play a much less significant role in our analysis than in the dynamic problem. Of course, a related but easily resolved ambiguity is that Q (and ϕ) are only defined up to addition of a real constant.

Remark 3.2. There are a few additional properties of z that we will note, all of which have been essentially verified in the proof of [23, Theorem 3]. The first is that the parameterization essentially moves “from left to right,” or, more specifically, the parameterization on top satisfies $\frac{d\alpha}{ds} > 0$. This was implicitly used above in the derivation of the capillary term. Next, since z is holomorphic and a diffeomorphism,

$|z_\alpha| > 0$ on S , which combined with the asymptotics at infinity implies that there is a $\delta > 0$ such that $|1 + W_\alpha| = |Z_\alpha| \geq \delta$ on top. Note that we only require positivity conditions on $|1 + W_\alpha|$; the boundary being a graph would assume positivity of $1 + \Re(W_\alpha)$.

3.1. The solitary wave equations. In search for solitary wave solutions, we fix a speed c and make the ansatz $(Q(\alpha, t), W(\alpha, t)) = (Q(\alpha - ct), W(\alpha - ct))$. The first equation in (3.1) then becomes

$$(3.8) \quad -cW_\alpha + F(1 + W_\alpha) = 0,$$

while the second equation becomes

$$(3.9) \quad -cQ_\alpha + FQ_\alpha - g\mathcal{T}_h[W] + \mathbf{P}_h \left[\frac{|Q_\alpha|^2}{J} \right] + \sigma \mathbf{P}_h \left[i \left(\frac{W_{\alpha\alpha}}{J^{1/2}(1 + W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1 + \overline{W_\alpha})} \right) \right] = 0.$$

We rewrite the first equation as

$$(3.10) \quad F = \mathbf{P}_h \left[\frac{Q_\alpha - \overline{Q_\alpha}}{J} \right] = \frac{cW_\alpha}{1 + W_\alpha}.$$

This gives that

$$(3.11) \quad \Im \left[\mathbf{P}_h \left[\frac{Q_\alpha - \overline{Q_\alpha}}{J} \right] \right] = c\Im \left(\frac{W_\alpha}{1 + W_\alpha} \right) = \frac{c}{J} \Im (W_\alpha(1 + \overline{W_\alpha})) = \frac{c}{J} \frac{W_\alpha - \overline{W_\alpha}}{2i}.$$

Recalling (3.6) and that the Tilbert transform maps real-valued functions to real-valued functions, we have

$$(3.12) \quad \Im(\mathbf{P}_h u) = \frac{1}{2} [\Im(u) - \mathcal{T}_h \Re(u)].$$

Therefore,

$$(3.13) \quad \Im \left[\mathbf{P}_h \left[\frac{Q_\alpha - \overline{Q_\alpha}}{J} \right] \right] = \frac{1}{2} \Im \left(\frac{Q_\alpha - \overline{Q_\alpha}}{J} \right) = \frac{Q_\alpha - \overline{Q_\alpha}}{2iJ}.$$

The equation we end up with is, then,

$$(3.14) \quad \frac{Q_\alpha - \overline{Q_\alpha}}{2J} = \frac{c}{2} \frac{(W_\alpha - \overline{W_\alpha})}{J},$$

which simplifies to

$$(3.15) \quad \Im(Q_\alpha) = c\Im(W_\alpha),$$

so that

$$(3.16) \quad Q_\alpha = cW_\alpha$$

because Q and W are holomorphic. Note that, formally, this argument only tells us that $Q_\alpha = cW_\alpha$ up to addition of a real constant. However, the decay properties of (W_α, Q_α) at infinity require the constant to vanish.

We now begin to simplify the second water wave equation. Beginning with (3.9), substituting (3.16) and the definition of F gives

$$(3.17) \quad -c^2 W_\alpha + \frac{c^2 W_\alpha^2}{1 + W_\alpha} - g \mathcal{T}_h[W] + c^2 \mathbf{P}_h \left[\frac{|W_\alpha|^2}{J} \right] + \sigma \mathbf{P}_h \left[i \left(\frac{W_{\alpha\alpha}}{J^{1/2}(1 + W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1 + \overline{W_\alpha})} \right) \right] = 0.$$

Before continuing, we note a few things. First, we have

$$(3.18) \quad \mathbf{P}_h \left[\frac{|W_\alpha|^2}{J} \right] = \frac{1}{2} \left[(1 - i \mathcal{T}_h) \frac{|W_\alpha|^2}{J} \right].$$

This implies that

$$(3.19) \quad \Re \left(\mathbf{P}_h \left[\frac{|W_\alpha|^2}{J} \right] \right) = \frac{1}{2} \frac{|W_\alpha|^2}{J}.$$

Therefore, taking real part of (3.17) and then using the fact that holomorphic functions satisfy $\mathcal{T}_h[\Re(u)] = -\Im(u)$, we obtain

$$(3.20) \quad -c^2 \Re(W_\alpha) + c^2 \Re \left(\frac{W_\alpha^2}{1 + W_\alpha} \right) + g \Im(W) + \frac{c^2}{2} \frac{|W_\alpha|^2}{J} + \frac{\sigma}{2} i \left(\frac{W_{\alpha\alpha}}{J^{1/2}(1 + W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1 + \overline{W_\alpha})} \right) = 0,$$

which can be rewritten as

$$(3.21) \quad -c^2 \Re(W_\alpha) + c^2 \Re \left(\frac{W_\alpha^2}{1 + W_\alpha} \right) + g \Im(W) + \frac{c^2}{2} \frac{|W_\alpha|^2}{J} + \frac{i\sigma}{1 + W_\alpha} \partial_\alpha \left(\frac{1 + W_\alpha}{|1 + W_\alpha|} \right) = 0.$$

After straightforward manipulation of the terms with c^2 , we arrive at

$$(3.22) \quad -\frac{c^2}{2} \frac{(W_\alpha + \overline{W_\alpha} + W_\alpha \overline{W_\alpha})}{|1 + W_\alpha|^2} + g \Im(W) + \frac{i\sigma}{1 + W_\alpha} \partial_\alpha \left(\frac{1 + W_\alpha}{|1 + W_\alpha|} \right) = 0.$$

As it turns out, these are exactly the same equations as the infinite depth case considered in [27]. However, the function spaces are different, which plays a key role. In particular, as mentioned in the introduction, there are no infinite depth pure gravity solitary waves, but there are finite depth pure gravity solitary waves.

As a consistency check, we leave it as an exercise to show that (2.8)–(2.11) imply (3.22).

3.2. Notation for function spaces. The function spaces we use are standard and similar to [26]. However, to set notation, we recall a few facts.

Consider a standard dyadic Littlewood–Paley decomposition

$$1 = \sum_{k \in \mathbb{Z}} P_k,$$

where the projectors P_k select functions with frequencies $\approx 2^k$. We will place our (hypothetical) solutions in the critical Besov space $B_{2,1}^{\frac{1}{2}}$ defined via

$$\|u\|_{B_{2,1}^{\frac{1}{2}}} := \sum_{k \geq 1} 2^{\frac{k}{2}} \|P_k u\|_{L^2} + \|P_{\leq 0} u\|_{L^2}.$$

Our proof also makes use of the space $B_{2,1}^{\frac{3}{2}}$, which has the same norm as $B_{2,1}^{\frac{1}{2}}$ but with $2^{\frac{k}{2}}$ replaced by $2^{\frac{3k}{2}}$. Finally, we note the embedding of $B_{2,1}^{\frac{1}{2}}$ into L^∞ and the following Moser estimate.

LEMMA 3.3. Let $u \in B_{2,1}^{\frac{1}{2}}$, and suppose G is a smooth function with $G(0) = 0$. Then we have the Moser estimate

$$(3.23) \quad \|G(u)\|_{B_{2,1}^{\frac{1}{2}}} \lesssim C(\|u\|_{L^\infty}) \|u\|_{B_{2,1}^{\frac{1}{2}}}.$$

Proof. This is a standard result. For example, it follows from [26, Lemma 2.2] together with the analogous Moser estimate on the level of L^2 . $\square$

4. No solitary waves when only surface tension is present. We are now able to state our main theorem. The result is stated in the low regularity function space $B_{2,1}^{\frac{1}{2}}$ defined above. However, part of the proof involves upgrading potential solutions to sufficient regularity to justify basic computations. Comparing with the infinite depth results in [27], our function space requires more regularity for W_α at low frequency, but this is to be expected, as the same happens in the dynamic problem [23]. From a technical standpoint, the issue is that $\mathcal{T}_h^{-1}$ does not have good mapping properties (it is not even bounded on L^2) compared to the Hilbert transform, which satisfies $H^{-1} = -H$. For justification of the other assumption—and conclusion—of Theorem 4.1, recall Remarks 3.1 and 3.2.

THEOREM 4.1. Suppose $W_\alpha \in B_{2,1}^{\frac{1}{2}}$ is holomorphic and solves (3.22) with $g = 0$ and $\sigma > 0$, $|1 + W_\alpha| > \delta > 0$ on the top and its extension does not vanish on $\bar{S}$. Then $W_\alpha = 0$.

Proof. We work with the equation

$$(4.1) \quad i\sigma\partial_\alpha \left(\frac{1 + W_\alpha}{|1 + W_\alpha|} \right) = c^2 \left[W_\alpha + \frac{\overline{W_\alpha}}{1 + \overline{W_\alpha}} \right],$$

which holds on the top and is just a rescaling of (3.22) with $g = 0$.

For what follows, we slightly abuse notation by not distinguishing, notationally, between $1 + W_\alpha$ and its extension to the strip. First, note that since $1 + W_\alpha$ is nonvanishing on the simply connected domain S , it admits a holomorphic logarithm. However, one has to be a little careful to ensure that it is real on the bottom boundary. To see this, note that since on the bottom $1 + W_\alpha$ is real and nonvanishing and has limit 1 at infinity, it is positive on the bottom.

Define

$$(4.2) \quad T := \log(1 + W_\alpha) := U + iV.$$

The unknowns $U + iV$ are closely related to the unknowns $\tau + i\theta$ in [14]. It is easy to see that T can be chosen to be holomorphic; in particular, it can be chosen to be real on the bottom.

Plugging into (4.1), we see that

$$(4.3) \quad -\sigma V_\alpha e^{iV} = c^2 \left[W_\alpha + \frac{\overline{W_\alpha}}{1 + \overline{W_\alpha}} \right] = c^2 (e^{U+iV} - e^{-U+iV}).$$

This implies that

$$(4.4) \quad -\sigma V_\alpha = 2c^2 \sinh(U).$$

Now we upgrade regularity. By (4.2), $|1 + W_\alpha| > \delta$, and Lemma 3.3, it follows that $U, V \in B_{2,1}^{\frac{1}{2}}$. Again by Moser, we obtain $\sinh(U) \in B_{2,1}^{\frac{1}{2}}$ which in turn implies that

$V_\alpha \in B_{2,1}^{\frac{1}{2}} \subseteq L^2$. From this, we get $P_{>0}U_\alpha = -P_{>0}\mathcal{T}_h^{-1}V_\alpha \in B_{2,1}^{\frac{1}{2}}$. But since $U \in L^2$, it follows that $U_\alpha \in B_{2,1}^{\frac{1}{2}}$. This will be enough regularity to justify the calculations below, though H^∞ regularity for U and V_α could be obtained by reiteration.

Rescaling again and using that $-V_\alpha = \mathcal{T}_h U_\alpha$, it suffices to show that the equation

$$(4.5) \quad \mathcal{T}_h U_\alpha = 2c^2 \sinh U$$

admits no nonzero $B_{2,1}^{\frac{3}{2}}$ solutions. For this, we let χ be a smooth function with $\chi = 0$ on $(-\infty, -1]$ and $\chi = 1$ on $[1, \infty)$ with $\chi' \sim 1$ on $(-\frac{1}{2}, \frac{1}{2})$. Define $\chi_r(\alpha) = \chi(\frac{\alpha}{r})$.

Next, we multiply (4.5) by $-\chi_r U_\alpha$ and obtain

$$(4.6) \quad -\chi_r U_\alpha \mathcal{T}_h U_\alpha = -2c^2 \chi_r U_\alpha \sinh U = -2c^2 \chi_r \partial_\alpha (\cosh(U) - 1).$$

An integration by parts yields the following identity:

$$(4.7) \quad -\int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha = \frac{2c^2}{r} \int_{\mathbb{R}} \chi'(\frac{\alpha}{r})(\cosh(U) - 1)d\alpha.$$

Now we treat the term on the left-hand side of (4.7). From the product rule for the Tilbert transform, we have

$$(4.8) \quad \chi_r \mathcal{T}_h U_\alpha = \mathcal{T}_h(\chi_r U_\alpha) - \mathcal{T}_h(\mathcal{T}_h \chi_r \mathcal{T}_h U_\alpha) - U_\alpha \mathcal{T}_h \chi_r.$$

Hence, using that the Tilbert transform is skew-adjoint and maps real-valued functions to real-valued functions,

$$(4.9) \quad \begin{aligned} -\int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha &= \int_{\mathbb{R}} U_\alpha \mathcal{T}_h(\mathcal{T}_h \chi_r \mathcal{T}_h U_\alpha) d\alpha + \int_{\mathbb{R}} |U_\alpha|^2 \mathcal{T}_h \chi_r d\alpha - \int_{\mathbb{R}} U_\alpha \mathcal{T}_h(\chi_r U_\alpha) d\alpha \\ &= \int_{\mathbb{R}} U_\alpha \mathcal{T}_h(\mathcal{T}_h \chi_r \mathcal{T}_h U_\alpha) d\alpha + \int_{\mathbb{R}} |U_\alpha|^2 \mathcal{T}_h \chi_r d\alpha + \int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha \\ &= -\int_{\mathbb{R}} |\mathcal{T}_h U_\alpha|^2 \mathcal{T}_h \chi_r d\alpha + \int_{\mathbb{R}} |U_\alpha|^2 \mathcal{T}_h \chi_r d\alpha + \int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha. \end{aligned}$$

Hence, we obtain

$$(4.10) \quad -\int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha = \frac{1}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \mathcal{T}_h \chi_r d\alpha.$$

Combining this with (4.7), we get

$$(4.11) \quad \frac{2c^2}{r} \int_{\mathbb{R}} \chi'(\frac{\alpha}{r})(\cosh(U) - 1)d\alpha = \frac{1}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \mathcal{T}_h \chi_r d\alpha.$$

The idea now is to use the fact that at low frequency, the Tilbert transform agrees with the multiplier $\xi \mapsto -h i \xi$ to third order. With this in mind, we rewrite the above equation as follows:

$$(4.12) \quad \begin{aligned} \frac{2c^2}{r} \int_{\mathbb{R}} \chi'(\frac{\alpha}{r})(\cosh(U) - 1)d\alpha &= \frac{1}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) (\mathcal{T}_h + h \partial_\alpha) \chi_r d\alpha \\ &\quad - \frac{h}{2r} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \chi'(\frac{\alpha}{r}) d\alpha. \end{aligned}$$

Equivalently, we have

$$(4.13) \quad \begin{aligned} 2c^2 \int_{\mathbb{R}} \chi' \left(\frac{\alpha}{r} \right) (\cosh(U) - 1) d\alpha + \frac{h}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \chi' \left(\frac{\alpha}{r} \right) d\alpha \\ = \frac{r}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) (\mathcal{T}_h + h\partial_\alpha) \chi_r d\alpha. \end{aligned}$$

We are now in a position to estimate the right-hand side of (4.13). Indeed, by Cauchy–Schwarz and Sobolev embedding, we have

$$(4.14) \quad \begin{aligned} \frac{r}{2} \left| \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) (\mathcal{T}_h + h\partial_\alpha) \chi_r d\alpha \right| &\leq Cr (\|U_\alpha\|_4^2 + \|\mathcal{T}_h U_\alpha\|_4^2) \|(\mathcal{T}_h + h\partial_\alpha) \chi_r\|_2 \\ &\leq Cr \|U\|_{B_{2,1}^{\frac{3}{2}}}^2 \|(\mathcal{T}_h + h\partial_\alpha) \chi_r\|_2. \end{aligned}$$

Using Plancherel’s theorem, we then obtain the simple estimate

$$(4.15) \quad \begin{aligned} r \|U\|_{B_{2,1}^{\frac{3}{2}}}^2 \|(\mathcal{T}_h + h\partial_\alpha) \chi_r\|_2 &= Cr \|U\|_{B_{2,1}^{\frac{3}{2}}}^2 \|(\tanh(h\xi) - h\xi) \widehat{\chi_r}\|_2 \\ &\leq \frac{C}{r} \|U\|_{B_{2,1}^{\frac{3}{2}}}^2 \left\| \frac{\tanh(h\xi) - h\xi}{\xi^2} \right\|_{L^\infty} \|\chi'' \left(\frac{\alpha}{r} \right)\|_2 \\ &\leq \frac{C}{r^{1/2}} \|U\|_{B_{2,1}^{\frac{3}{2}}}^2 \|\chi''\|_2. \end{aligned}$$

Hence, we obtain

$$(4.16) \quad 2c^2 \int_{\mathbb{R}} \chi' \left(\frac{\alpha}{r} \right) (\cosh(U) - 1) d\alpha + \frac{h}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \chi' \left(\frac{\alpha}{r} \right) d\alpha = \mathcal{O}_{\|U\|_{B_{2,1}^{\frac{3}{2}}}} (r^{-1/2}).$$

Letting $r \rightarrow \infty$, dominated convergence gives

$$2c^2 \int_{\mathbb{R}} (\cosh(U) - 1) d\alpha = -\frac{h}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) d\alpha = -\frac{h}{2} \int_{\mathbb{R}} |\xi|^2 |\widehat{U}|^2 \operatorname{sech}^2(h\xi) \leq 0.$$

Therefore, since $\cosh(U) - 1 \geq 0$, we have

$$\cosh(U) = 1,$$

so that $U \equiv 0$. Note that taking the limit is justified because $\cosh(U) - 1$ is integrable. This is thanks to the fact that U is bounded, vanishes at infinity, and belongs to L^2 .

We remark that if one assumes instead some stronger decay at infinity for U , then the above argument proving nonexistence of solutions for (4.5) can be simplified somewhat by working directly with the choice $\chi(\alpha) = \alpha$. This, of course, leads to a weaker result; the details are left for the reader. $\square$

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