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STATICIZATION AND ITERATED STATICIZATION*

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Abstract. Conservative dynamical systems propagate as stationary points of the action functional. Using this representation, it has previously been demonstrated that one may obtain fundamental solutions for two-point boundary value problems for some classes of conservative systems via a solution of an associated dynamic program. It is also known that the gravitational and Coulomb potentials may be represented as stationary points of cubicly parameterized quadratic functionals. Hence, stationary points of the action functional may be represented via iterated "staticization" of polynomial functionals, where the staticization operator (introduced and discussed in [*J. Differential Equations*, 264 (2018), pp. 525–549] and [Automatica J. IFAC, 81 (2017), pp. 56–67]) maps a function to the function value(s) at its stationary (i.e., critical) points. This leads to representations through operations on sets of solutions of differential Riccati equations. A key step in this process is the reordering of staticization operations. Conditions under which this reordering is allowed are obtained, and it is shown that the conditions are satisfied for an astrodynamics problem.

Key words. dynamic programming, stationary action, staticization, two-point boundary value problems, conservative dynamical systems

MSC Subject classifications. 49LXX, 93E20, 93C10, 60H10, 35G20, 35D40

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1. Introduction.

Staticization maps a function to its values at stationary points (i.e., critical points). More specifically, the set-valued "stat" operator has as its range the set of such values, and if there is such a unique value, then that value is the output of the (single-valued) stat operator. This operator is obviously a generalization of the minimization and maximization operators for appropriate classes of differentiable functionals and is also valid for functions with a range other than the reals, including complex-valued functionals. The stat operator is at the heart of a new approach to a solution of two-point boundary value problems (TPBVPs) in conservative dynamical systems [4, 5, 18, 19], as well as to a solution of the Schrödinger equation [14, 16, 17]. A key component in this development is the theory that allows one to reorder stat operators under certain conditions, and that theory is the focus of the effort here. In order to motivate the theory, first let us indicate application domains a bit further.

Recall that conservative dynamical systems propagate as stationary points of the action functional over the possible paths of the system. This stationary-action formulation has recently been found to be quite useful for generation of fundamental solutions to TPBVPs for conservative dynamical systems; cf. [4, 5, 18, 19]. To obtain a sense of this application domain, consider a finite-dimensional action functional formulation of such a TPBVP. Let the path of the conservative system be denoted by ξ_r for $r \in [0,t]$ with $\xi_0 = \bar{x}$, in which case the action functional, with an appended terminal cost, may take the form

$$J(t,\bar{x},u) \doteq \int_0^t T(u_r) - V(\xi_r)\,dr + \phi(\xi_t), \tag{1.1}$$

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where $\dot{\xi} = u$, $u \in \mathcal{U} \doteq L_2(0,t)$, $T(\cdot)$ denotes the kinetic energy associated to the momentum (specifically taken to be $T(v) \doteq \frac{1}{2}v^T\mathcal{M}v$ further below, with $\mathcal{M}$ positive-definite and symmetric), and $V(\cdot)$ denotes a potential energy field. If, for example, one takes $\phi(x) \doteq -\bar{v}^T\mathcal{M}x$, a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\dot{\xi}_t = \bar{v}$; if one takes ϕ to be a min-plus delta function centered at z, then a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\xi_t = z$; cf. [5]. In the early work of Hamilton, it was formulated as the least-action principle [8], which states that a conservative dynamical system follows the trajectory that minimizes the action functional. However, this is typically only the case for relatively short-duration cases; cf. [7] and the references therein. In such short-duration cases, optimization methods and semiconvex duality are quite useful [4, 5, 19]. However, in order to extend to indefinitely long duration problems, it becomes necessary to apply concepts of stationarity [18].

It is worth noting that if one defines $\operatorname{stat}_{x\in\mathcal{X}} \phi(x)$ to be the critical value of ϕ (defined rigorously in section 2.1), then a gravitational potential given as $V(x) = -\mu/|x|$ for $x \neq 0$ and constant $\mu > 0$ has the representation $V(x) = -(\frac{3}{2})^{\frac{3}{2}}\mu \operatorname{stat}_{\alpha>0}\{\alpha - \frac{\alpha^3|x|^2}{2}\}$, where we note that the argument of the stat operator is polynomial [9, 19]. The Schrödinger equation in the context of a Coulomb potential may be similarly addressed. In that case, it is particularly helpful to consider an extension of the space variable to a vector space over the complex field, say, $x \in \mathbb{C}^n$ rather than $x \in \mathbb{R}^n$. More specifically, for $x \in \mathbb{C}^n$, this representation takes a general form $V(x) = -(\frac{3}{2})^{\frac{3}{2}}\hat{\mu} \operatorname{stat}_{\alpha\in\mathcal{A}^R}\{\alpha - \frac{\alpha^3 x^T x}{2}\}$, where $\mathcal{A}^R \doteq \{\alpha = r[\cos(\theta) + i\sin(\theta)] \in \mathbb{C} \,|\, r \geq 0,\, \theta \in (\frac{-\pi}{2}, \frac{\pi}{2}]\}$ [3, 14]. In the simple one-dimensional case, the resulting function on $\mathbb{C}$ has a branch cut along the negative imaginary axis, and this generalizes to higher-dimensional cases in the natural way.

Although stationarity-based representations for gravitational and Coulomb potentials are inside the integral in (1.1), they may be moved outside through the introduction of α-valued processes; cf. [9, 19]. In particular, not only does one seek the stationary path for action J, but the action functional itself can be given as a stationary value of an integral of a polynomial, leading to an iterated-stat problem formulation for such TPBVPs. This may be exploited in the solution of TPBVPs in such systems (cf. [9, 18, 19]), which will be discussed further in section 5.

It has also been demonstrated that this stationary-action approach may be applied to TPBVPs for infinite-dimensional conservative systems described by classes of lossless wave equations; see, for example, [4, 5]. There, stat is used in the construction of fundamental solution groups for these wave equations by appealing to stationarity of action on longer horizons.

Lastly, it has recently been demonstrated that stationarity may be employed to obtain a Feynman–Kac type of representation for solutions of the Schrödinger initial value problem for certain classes of initial conditions and potentials [3, 17]. As with the conservative system cases above, these representations are valid for indefinitely long duration problems, whereas with only the minimization operation, such representations are valid only on time intervals such that the action remains convex, which is always a bounded duration and potentially zero.

In all of these examples, one obtains the stationary value of an action functional, where the action functional itself takes the form of a stationary value of a functional that is quadratic in the momentum (the $u_\cdot$ input in (1.1)) and cubic in the newly introduced potential energy parameterization variable (a time-dependent form of the α parameter above). That is, the overall stationary value is obtained from iterated staticization operations, where the outer stat is over a variable in which the functional is quadratic. Thus, if one can invert the order of the stat operations, then the inner stat operation results in a functional that is obtained as a solution of a differential Riccati equation (DRE). (It should be noted that this DRE must typically be propagated through past escape times, where this propagation may be efficiently performed through the use of what has been termed "stat duality"; cf. [15].) Hence, after inversion of the order of the iterated stat operations, the problem may be reduced to a single stat operation such that the argument takes the form of a linear functional operating on a set of DRE solutions. Consequently, an issue of fundamental importance regards conditions under which one may invert the order of stat operations in an iterated staticization.

In section 2, the stat operator will be rigorously defined, and a general problem class along with some corresponding notation will be indicated. Then, in section 3, a somewhat general condition will be indicated. Further, it will be shown that one may invert the order of staticization operations under that condition. This will be demonstrated by obtaining an equivalence between iterated staticization and full staticization over both variables together. Section 4 will present several classes of problems for which the general condition of section 3 holds. Finally, in section 5, a stationary-action application in astrodynamics will be discussed.

2. Problem and stationarity definitions.

Before the issue to be studied can be properly expressed, it is necessary to define stationarity and the stat operator.

2.1. Stationarity definitions.

As noted above, the motivation for this effort is the computation and propagation of stationary points of payoff functionals, which is unusual in comparison to the standard classes of problems in optimization (although one should note, for example, [6]). In analogy with the language for minimization and maximization, we will refer to the search for stationary points as "staticization," with these points being statica, in analogy with minima/maxima, and a single such point being a staticum in analogy with minimum/maximum. One might note here that the term staticization is being derived from a Latin root, staticus (presumably originating from the Greek statikós), in analogy with the Latin root maximus of "maximization." We note that Ekeland [6] employed the term "extremization" for what is largely the same notion that is being referred to here as staticization but with a very different focus. We make the following definitions. Let $\mathcal{F}$ denote either the real or complex field. Suppose $\mathcal{U}$ is a normed vector space (over $\mathcal{F}$) with $\mathcal{A} \subseteq \mathcal{U}$, and suppose $G : \mathcal{A} \to \mathcal{F}$. We will use the notation $|\cdot|$ for both modulus and appropriate norm, where in particular we will not subscript the norm by the space when it can be deduced from context. We say $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{A}} G(u) \doteq \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\}$ if $\bar{u} \in \mathcal{A}$, and either

$$\limsup_{u\to\bar{u},\, u\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(u)-G(\bar{u})|}{|u-\bar{u}|} = 0 \tag{2.1}$$

or there exists $\delta > 0$ such that $\mathcal{A} \cap B_\delta(\bar{u}) = \{\bar{u}\}$ (where $B_\delta(\bar{u})$ denotes the ball of radius δ around $\bar{u}$). If $\operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} \neq \emptyset$, we define the possibly set-valued stat^s operation by

$$\operatorname*{stat}^s_{u\in\mathcal{A}} G(u) \doteq \operatorname{stat}^s\{G(u) \,|\, u \in \mathcal{A}\} \doteq \left\{ G(\bar{u}) \,\middle|\, \bar{u} \in \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} \right\}. \tag{2.2}$$

If $\operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} = \emptyset$, then $\operatorname{stat}^s_{u\in\mathcal{A}} G(u)$ is undefined. Where applicable, we are also interested in a single-valued stat operation (note the absence of superscript s).

In particular, if there exists $a \in \mathcal{F}$ such that $\operatorname{stat}^s_{u\in\mathcal{A}} G(u) = \{a\}$, then $\operatorname{stat}_{u\in\mathcal{A}} G(u) \doteq a$; otherwise, $\operatorname{stat}_{u\in\mathcal{A}} G(u)$ is undefined. At times, we may abuse notation by writing $\bar u = \operatorname{argstat}\{G(u)\,|\, u\in\mathcal{A}\}$ in the event that the argstat is the single point $\{\bar u\}$.

In the case where $\mathcal{U}$ is a Banach space and $\mathcal{A}\subseteq\mathcal{U}$ is an open set, $G:\mathcal{A}\to\mathcal{F}$ is Fréchet differentiable at $\bar u\in\mathcal{A}$ with continuous, linear $DG(\bar u)\in\mathcal{L}(\mathcal{U};\mathcal{F})$ if

$$\lim_{w\to 0,\,\bar u+w\in\mathcal{A}\setminus\{\bar u\}} \frac{|G(\bar u+w)-G(\bar u)-[DG(\bar u)]w|}{|w|} = 0. \tag{2.3}$$

The following is immediate from the above definitions.

Lemma 2.1. *Suppose $\mathcal{U}$ is a Banach space, with open set $\mathcal{A}\subseteq\mathcal{U}$, and that G is Fréchet differentiable at $\bar u\in\mathcal{A}$. Then, $\bar u\in\operatorname{argstat}\{G(y)\,|\,y\in\mathcal{A}\}$ if and only if $DG(\bar u)=0$.*

2.2. Problem definition. Throughout, let $\mathcal{U},\mathcal{V}$ be Banach spaces. When $\mathcal{U}$ is also Hilbert, let the inner product be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ and similarly for $\mathcal{V}$. Let the inner product on $\mathcal{U}\times\mathcal{V}$ be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{U}\times\mathcal{V}}$. Let $\mathcal{A}\subseteq\mathcal{U}$ and $\mathcal{B}\subseteq\mathcal{V}$ be open. Throughout, we assume

$$G\in C^2(\mathcal{A}\times\mathcal{B};\mathcal{F}). \tag{A.1}$$

Let

(2.4)

$$\begin{aligned}
&\mathsf{Dom}(\bar G^1) \doteq \big\{u\in\mathcal{A}\,\big|\, \operatorname*{stat}_{v\in\mathcal{B}} G(u,v) \text{ exists}\big\}, && \mathsf{Dom}(\bar G^2) \doteq \big\{v\in\mathcal{B}\,\big|\, \operatorname*{stat}_{u\in\mathcal{A}} G(u,v) \text{ exists}\big\},\\
&\bar G^1(u) \doteq \operatorname*{stat}_{v\in\mathcal{B}} G(u,v)\ \ \forall\, u\in\mathsf{Dom}(\bar G^1), && \bar G^2(v) \doteq \operatorname*{stat}_{u\in\mathcal{A}} G(u,v)\ \ \forall\, v\in\mathsf{Dom}(\bar G^2),\\
&\mathcal{A}^1(u) \doteq \operatorname*{argstat}_{v\in\mathcal{B}} G(u,v), && \mathcal{A}^2(v) \doteq \operatorname*{argstat}_{u\in\mathcal{A}} G(u,v),\\
&\bar{\mathcal{A}}^1 \doteq \operatorname*{argstat}_{u\in\mathsf{Dom}(\bar G^1)} \bar G^1(u), && \bar{\mathcal{A}}^2 \doteq \operatorname*{argstat}_{v\in\mathsf{Dom}(\bar G^2)} \bar G^2(v).
\end{aligned}$$

We will discuss conditions under which

$$\operatorname*{stat}_{u\in\mathsf{Dom}(\bar G^1)} \bar G^1(u) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) = \operatorname*{stat}_{v\in\mathsf{Dom}(\bar G^2)} \bar G^2(v). \tag{2.5}$$

We will generally be concerned only with the left-hand equality in (2.5); obviously the right-hand equality would be obtained analogously. We refer to the left-hand object in (2.5) as an iterated stat operation, while the center object will be referred to as a full stat operation. Although in some results, the existence of both the iterated and full stat operations are obtained, many of the results will assume the existence of one or both of these objects. We list the two potential assumptions below. In each result to follow, we will indicate when one or both of these is utilized. The full stat assumption is as follows:

$$\text{Assume } \operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) \text{ exists.} \tag{A.2f}$$

Note that under assumption (A.2f), if $(\bar u,\bar v)\in\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$, then

$$\bar v\in\mathcal{A}^1(\bar u)\ \text{ and }\ \bar u\in\mathcal{A}^2(\bar v). \tag{2.6}$$

The iterated stat assumption is as follows:

$$\text{Assume } \operatorname{stat}_{u\in\mathsf{Dom}(\bar G^1)} \bar G^1(u) \text{ exists.} \tag{A.2i}$$

Note that under assumption (A.2i), if $\bar u \in \bar{\mathcal{A}}^1$, then

$$\text{there exists } \bar v \in \mathcal{A}^1(\bar u) \quad\text{and}\quad \operatorname*{stat}_{u\in \mathsf{Dom}(\bar G^1)} \bar G^1(u) = G(\bar u, \bar v). \tag{2.7}$$

We will first obtain (2.5) under some general assumptions. After that, we will demonstrate that these assumptions are satisfied under certain other sets of assumptions, where the latter sets describe more commonly noted classes of functions (specifically, quadratic, semiquadratic, and Morse functions). Again, we mainly address only the left-hand equality of (2.5); the right-hand equality is handled similarly.

3. The general case. Given $\mathcal{C} \subseteq \mathcal{V}$ and $\hat v \in \mathcal{V}$, we let $d(\hat v, \mathcal{C}) \doteq \inf_{v\in\mathcal{C}} |v - \hat v|$, and use this distance notation more generally throughout. In addition to (A.1), we assume the following throughout this section.

(A.3) If (A.2f) is satisfied, then for any $(\bar u, \bar v) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$, there exist $\delta = \delta(\bar u, \bar v) > 0$ and $K = K(\bar u, \bar v) < \infty$ such that $d(\bar v, \mathcal{A}^1(u)) \le K\,|\bar u - u|\ \forall\ u \in \mathsf{Dom}(\bar G^1) \cap B_\delta(\bar u)$.

We note that (A.3) is trivially satisfied in the case that there exists $\delta > 0$ such that $B_\delta(\bar u) \cap \mathsf{Dom}(\bar G^1) = \emptyset$.

Lemma 3.1. *Assume* (A.2f)*, and let* $(\bar u, \bar v) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$*. If* $\bar u \in$ *$\mathsf{Dom}(\bar G^1)$, then* $\bar u \in \bar{\mathcal{A}}^1$ *and* $G(\bar u, \bar v) \in \operatorname{stat}^s_{u\in \mathsf{Dom}(\bar G^1)} \bar G^1(u)$.

Proof. Let $(\bar u, \bar v) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$, and let $R \doteq 1 \wedge d\big((\bar u, \bar v), (\mathcal{A}\times\mathcal{B})^c\big)$. By assumption (A.3), there exist $\delta \in (0, R/2)$ and $K < \infty$ such that for all $u \in \mathsf{Dom}(\bar G^1) \cap B_\delta(\bar u)$ and all $\epsilon \in (0, 1)$, there exists $v \in \mathcal{A}^1(u)$ such that

$$|v - \bar v| \le (K + \epsilon)|u - \bar u| \le (K + \epsilon)\delta. \tag{3.1}$$

Let $\tilde u \in \mathsf{Dom}(\bar G^1) \cap B_{\delta/(K+1)}(\bar u)$. By (2.6),

$$\big|\operatorname*{stat}_{v\in\mathcal{B}} G(\tilde u, v) - \operatorname*{stat}_{v\in\mathcal{B}} G(\bar u, v)\big| = \big|\operatorname*{stat}_{v\in\mathcal{B}} G(\tilde u, v) - G(\bar u, \bar v)\big|,$$

and by (3.1), there exists $\tilde v = \tilde v(\bar v) \in B_\delta(\bar v)$ such that this is

$$= |G(\tilde u, \tilde v) - G(\bar u, \bar v)|. \tag{3.2}$$

Let $f \in C^\infty\big((-3/2, 3/2); \mathcal{A}\times\mathcal{B}\big)$ be given by $f(\lambda) = \big(\bar u + \lambda(\tilde u - \bar u), \bar v + \lambda(\tilde v - \bar v)\big)$ for all $\lambda \in (-3/2, 3/2)$. Define $W^0(\lambda) = [G \circ f](\lambda)$ for all $\lambda \in (-3/2, 3/2)$, and note that by assumption (A.1) and standard results, $W^0 \in C^2\big((-3/2, 3/2); \mathcal{F}\big)$. Similarly, let $W^1(\lambda) = [(G_u, G_v) \circ f](\lambda) = \big(G_u(f(\lambda)), G_v(f(\lambda))\big)$. By assumption (A.1) and standard results, $W^1 \in C^1\big((-3/2, 3/2); \mathcal{U}' \times \mathcal{V}'\big)$, where $\mathcal{U}', \mathcal{V}'$ denote the dual spaces of $\mathcal{U}, \mathcal{V}$. Then, by a version of the mean value theorem [1, Theorem 12.6] (which is included in Appendix A for easy reference), there exists $\lambda_0 \in (0, 1)$ such that

$$\begin{aligned}|G(\tilde u, \tilde v) - G(\bar u, \bar v)| &= |W^0(1) - W^0(0)| \le \Big|\frac{dG}{d(u, v)}(f(\lambda_0))\Big|\,\Big|\frac{df}{d\lambda}(\lambda_0)\Big| \\ &= \big|\big(G_u(u_0, v_0), G_v(u_0, v_0)\big)\big|\,\big|(\tilde u - \bar u, \tilde v - \bar v)\big|,\end{aligned}$$

where $(u_0, v_0) \doteq f(\lambda_0)$ and which, by (3.1),

$$\le \big|\big(G_u(u_0, v_0), G_v(u_0, v_0)\big)\big|\,\sqrt{1 + (K + 1)^2}\,|\tilde u - \bar u|. \tag{3.3}$$

Again by the aforementioned mean value theorem, there exists $\lambda_1 \in (0, \lambda_0)$ such that

$$\begin{aligned}&\Big|\big(G_u(u_0,v_0), G_v(u_0,v_0)\big) - \big(G_u(\bar u,\bar v), G_v(\bar u,\bar v)\big)\Big| = |W^1(\lambda_0) - W^1(0)|\\ &\le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\,\Big|\frac{df}{d\lambda}(\lambda_1)\Big|\,|\lambda_1| \le \Big|\frac{d^2G}{d(u,v)^2}(u_1,v_1)\Big|\,\big|(u_1-\bar u, v_1-\bar v)\big|,\end{aligned}$$

where $(u_1, v_1) \doteq f(\lambda_1)$, and this is

$$\le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\sqrt{1+(K+1)^2}|\tilde u - \bar u|.$$

Recalling $(\bar u, \bar v) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$, this implies

$$\Big|\big(G_u(u_0,v_0), G_v(u_0,v_0)\big)\Big| \le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\sqrt{1+(K+1)^2}|\tilde u - \bar u|. \tag{3.4}$$

Combining (3.3) and (3.4) yields

$$|G(\tilde u,\tilde v) - G(\bar u,\bar v)| \le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\left[1+(K+1)^2\right]|\tilde u - \bar u|^2.$$

Let $K_1 \doteq \big|\frac{d^2G}{d(u,v)^2}(\bar u,\bar v)\big|$. By (A.1), there exists $\hat\delta \in (0, \delta/(K+1))$ such that for all $(u,v) \in B_{\hat\delta}(\bar u,\bar v)$, $\big|\frac{d^2G}{d(u,v)^2}(u,v)\big| \le K_1 + 1$. Hence, there exists $\bar C < \infty$ such that

$$|G(\tilde u,\tilde v) - G(\bar u,\bar v)| \le \bar C|\tilde u - \bar u|^2 \;\; \forall\, \tilde u \in \mathsf{Dom}(\bar G^1) \cap B_{\hat\delta/(K_1+1)}(\bar u). \tag{3.5}$$

Combining (3.2) and (3.5) and noting that $\tilde u \in \mathsf{Dom}(\bar G^1) \cap B_{\hat\delta/(K+1)}(\bar u)$ was arbitrary, one has $|\bar G^1(u) - \bar G^1(\bar u)|/|u-\bar u| \le \bar C|u-\bar u|$ for all $u \in [\mathsf{Dom}(\bar G^1) \cap B_{\hat\delta/(K_1+1)}(\bar u)] \setminus \{\bar u\}$, which implies $\bar u \in \bar{\mathcal{A}}^1$ by definition. The second assertion follows easily. ∎

THEOREM 3.2. *Assume* (A.2f)*, and let* $(\bar u,\bar v) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$*. Assume* (A.2i) *and that* $\bar u \in \mathsf{Dom}(\bar G^1)$*. Then*

$$\operatorname*{stat}_{u\in \mathsf{Dom}(\bar G^1)} \bar G^1(u) = G(\bar u,\bar v) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v).$$

Proof. The assertions follow directly from the assumption, (A.2f), and Lemma 3.1. ∎

4. Some specific cases. We examine several classes of functionals that fit within the general class above.

4.1. The quadratic case. Throughout this section, we take $\mathcal{A} = \mathcal{U}$ and $\mathcal{B} = \mathcal{V}$, where $\mathcal{U}, \mathcal{V}$ are Hilbert. Let

$$\begin{aligned}G(u,v) &= \tfrac{c}{2} + \langle w,u\rangle_{\mathcal{U}} + \langle y,v\rangle_{\mathcal{V}} + \tfrac12\langle \bar B_1 u,u\rangle_{\mathcal{U}} + \langle \bar B_2 v,u\rangle_{\mathcal{U}} + \tfrac12\langle \bar B_3 v,v\rangle_{\mathcal{V}}\\ &= \tfrac{c}{2} + \langle w,u\rangle_{\mathcal{U}} + \langle y,v\rangle_{\mathcal{V}} + \tfrac12\langle \bar B_1 u,u\rangle_{\mathcal{U}} + \langle \bar B_2' u,v\rangle_{\mathcal{V}} + \tfrac12\langle \bar B_3 v,v\rangle_{\mathcal{V}}\end{aligned} \tag{4.1}$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where $\bar B_1 \in \mathcal{L}(\mathcal{U};\mathcal{U})$, $\bar B_2 \in \mathcal{L}(\mathcal{V};\mathcal{U})$, $\bar B_3 \in \mathcal{L}(\mathcal{V};\mathcal{V})$, $w \in \mathcal{U}$, $y \in \mathcal{V}$, and $c \in \mathcal{F}$, where $\mathcal{L}(\cdot,\cdot)$ generically denotes a space of bounded linear operators and $\bar B_1, \bar B_3$ are self-adjoint and closed. We present results under both the cases of (A.2f) and (A.2i).

4.1.1. When the full staticization is known to exist. We suppose (A.2f), and let $(\bar{u},\bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$. This subcase is fully covered in [15], and hence here, we will mainly only indicate an additional approach. We begin by noting the following, which follows directly from (4.1) and Lemma 2.1.

LEMMA 4.1. *Let $\hat{u} \in \mathcal{A}$. Then $\hat{v} \in \mathcal{A}^1(\hat{u})$ if and only if $\bar{B}_2'\hat{u} + \bar{B}_3\hat{v} + y = 0$.*

Under condition (A.2f), the following is obtained in [15, Section 4.2], and those proofs are not repeated here. We note, however, that a proof of the second assertion of Lemma 4.2 is a subcase of the proof of the first assertion of Lemma 4.8 below, which covers a slightly more general class.

LEMMA 4.2. $\operatorname{stat}_{u\in\mathsf{Dom}(\bar{G}^1)} \bar{G}_1$ *exists (i.e.,* (A.2i) *is satisfied), and $\bar{u} \in \mathsf{Dom}(\bar{G}^1)$.*

LEMMA 4.3. *Assumption* (A.3) *is satisfied.*

Proof. We suppose $\mathsf{Dom}(\bar{G}^1) \neq \{\bar{u}\}$; otherwise the result is trivial. Let $\hat{u} \in \mathsf{Dom}(\bar{G}^1)\setminus\{\bar{u}\}$. By Lemma 4.1, $\hat{v} \in \mathcal{A}^1(\hat{u})$ if and only if $\bar{B}_2'\hat{u}+\bar{B}_3\hat{v}+y = 0$. However, by (2.6), $\bar{v} \in \mathcal{A}^1(\bar{u})$, and hence by Lemma 4.1, $\bar{B}_2'\bar{u} + \bar{B}_3\bar{v} + y = 0$. Combining these two inequalities, we see that $\hat{v} \in \mathcal{A}^1(\hat{u})$ if and only if $\bar{B}_2'(\hat{u}-\bar{u}) + \bar{B}_3(\hat{v}-\bar{v}) = 0$. We take $\hat{v} \doteq \bar{v} - \bar{B}_3^{\#}\bar{B}_2'(\hat{u}-\bar{u})$, where the # superscript indicates the Moore–Penrose pseudoinverse, where existence follows by the closedness of $\bar{B}_3$; cf. [2, 22]. Then $\hat{v} \in \mathcal{A}^1(\hat{u})$ and $|\hat{v}-\bar{v}| \le |\bar{B}_3^{\#}||\bar{B}_2'||\hat{u}-\bar{u}|$, where the induced norms on the operators are employed, which yields the desired assertion. ∎

By Lemmas 4.2 and 4.3 and Theorem 3.2, one has the following.

THEOREM 4.4. *Let $(\bar{u},\bar{v})$ denote any element of* $\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$*, and assume $\bar{u} \in \mathsf{Dom}(\bar{G}^1)$. Then*

$$\operatorname*{stat}_{u\in\mathsf{Dom}(\bar{G}^1)} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v). \tag{4.2}$$

4.1.2. When the iterated staticization is known to exist. We suppose (A.2i), and let $\bar{u} \in \bar{\mathcal{A}}^1$. We will find that $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists and obtain the equivalence between full and iterated staticization. We begin with a lemma (which is similar to Lemma 10 of [15]).

LEMMA 4.5. *Given any $\tilde{u} \in \mathcal{A}$, $\mathcal{A}^1(\tilde{u})$ is an affine subspace, and further, if $\tilde{u} \in \mathsf{Dom}(\bar{G}^1)$, then $\mathcal{A}^1(\tilde{u})$ is nonempty.*

Proof. By Lemma 4.1 $v \in \mathcal{A}^1(\tilde{u})$ if and only if $\bar{B}_2'\hat{u} + \bar{B}_3 v + y = 0$, which yields the assertions. ∎

We remark that, by definition, for any $\tilde{u} \in \mathsf{Dom}(\bar{G}^1)$, $G(\tilde{u},\cdot)$ is constant on the affine subspace $\mathcal{A}^1(\tilde{u})$.

THEOREM 4.6. *Assume* (A.2i)*, and suppose $\bar{u} \in \bar{\mathcal{A}}^1$. Let $\bar{v}$ be as given in* (2.7)*. Then,* $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ *exists, and* $\operatorname{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} G(u,v) = G(\bar{u},\bar{v}) = \operatorname{stat}_{u\in\mathsf{Dom}(\bar{G}^1)} \bar{G}^1(u)$.

Proof. Assume (A.2i), and let $\bar{u} \in \bar{\mathcal{A}}^1$. Let $\bar{v}$ be as given in (2.7). First, note that the assertion that $G(\bar{u},\bar{v}) = \operatorname{stat}_{u\in\mathsf{Dom}(\bar{G}^1)} \bar{G}^1(u)$ will follow from the other assertions and (2.7). By Lemma 4.1, $v \in \mathcal{A}^1(\bar{u})$ if and only if $\bar{B}_2'\bar{u} + \bar{B}_3 v + y = 0$. For $u \in \mathsf{Dom}(\bar{G}^1)$, let

$$\breve{v}(u) \doteq \bar{v} - \bar{B}_3^{\#}\big[\bar{B}_2'u + y - (\bar{B}_2'\bar{u} + y)\big], \tag{4.3}$$

and note that

$$\breve{v}(\bar{u}) = \bar{v}. \tag{4.4}$$

Let $\tilde{v} \doteq -\bar{B}_3^{\#}[\bar{B}_2'\bar{u} + y]$, and note that as $\bar{v}$ and $\tilde{v}$ are both in $\mathcal{A}^1(\bar{u})$, by Lemma 4.1,

$$0 = \bar{B}_3[\bar{v} - \tilde{v}] = \bar{B}_3\big[\bar{v} + \bar{B}_3^{\#}(\bar{B}_2'\bar{u} + y)\big]. \tag{4.5}$$

Then using (4.3) and (4.5), we see that for $u \in \mathsf{Dom}(\bar{G}^1)$,

$$\begin{aligned}\bar{B}_3\breve{v}(u) + \bar{B}_2'u + y &= \bar{B}_3\big[\bar{v} - \bar{B}_3^{\#}(\bar{B}_2'u - \bar{B}_2'\bar{u})\big] + \bar{B}_2'u + y \\ &= \bar{B}_3\big[-\bar{B}_3^{\#}(\bar{B}_2'u + y)\big] + \bar{B}_2'u + y,\end{aligned}$$

which, by definition of the pseudoinverse and the fact that $\bar{B}_2'\bar{u} + y \in \mathrm{Range}(\bar{B}_3)$ for $u \in \mathsf{Dom}(\bar{G}^1)$,

$$= 0.$$

Hence, $\breve{v}(u) \in \mathcal{A}^1(u)\ \ \forall\, u \in \mathsf{Dom}(\bar{G}^1)$, and consequently,

$$\bar{G}^1(u) = G(u, \breve{v}(u))\ \ \forall\, u \in \mathsf{Dom}(\bar{G}^1). \tag{4.6}$$

Then, by (A.2i) and the choice of $\bar{u}$,

$$0 = \frac{d\bar{G}^1}{du}(\bar{u}),$$

which by (4.6), (A.1) and the chain rule,

$$= G_u(\bar{u}, \breve{v}(\bar{u})) + G_v(\bar{u}, \breve{v}(\bar{u}))\frac{d\breve{v}}{du}(\bar{u}),$$

which, by (4.4) and our choice of $\bar{v}$,

$$= G_u(\bar{u}, \bar{v}) + G_v(\bar{u}, \bar{v})\frac{d\breve{v}}{du}(\bar{u}) = G_u(\bar{u}, \bar{v}).$$

From this and the choice of $\bar{v}$, we see that

$$(\bar{u}, \bar{v}) \in \operatorname*{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \quad \text{and} \quad G(\bar{u}, \bar{v}) \in \operatorname*{stat^s}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v). \tag{4.7}$$

Now suppose there exists $(\hat{u}, \hat{v}) \in \mathrm{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}\, G(u, v) \setminus \{(\bar{u}, \bar{v})\}$. This implies

$$G_u(\hat{u}, \hat{v}) = 0 \quad \text{and} \quad G_v(\hat{u}, \hat{v}) = 0, \tag{4.8}$$

and consequently,

$$\hat{v} \in \mathcal{A}^1(\hat{u}) \quad \text{and} \quad \bar{G}^1(\hat{u}) = G(\hat{u}, \hat{v}). \tag{4.9}$$

Let

$$\breve{v}\,'(u) \doteq \hat{v} - \bar{B}_3^{\#}\big[\bar{B}_2'u + y - (\bar{B}_2'\hat{u} + y)\big]\ \ \forall\, u \in \mathsf{Dom}(\bar{G}^1), \tag{4.10}$$

and note that

$$\breve{v}\,'(\hat{u}) = \hat{v}. \tag{4.11}$$

Let $\hat{\hat{v}} \doteq -\bar{B}_3^{\#}(\bar{B}_2'\hat{u} + y)$, and note that $\hat{v}, \hat{\hat{v}} \in \mathcal{A}^1(\hat{u})$. Similar to the above, we see that

$$0 = \bar{B}_3(\hat{v} - \hat{\hat{v}}) = \bar{B}_3\big[\hat{v} + \bar{B}_3^{\#}(\bar{B}_2'\hat{u} + y)\big]. \tag{4.12}$$

Then, again similar to the above, using (4.12), the definition of the pseudoinverse, and $\bar{B}_2'\bar{u}+y \in \mathrm{Range}(\bar{B}_3)$, we see that

$$\begin{aligned}\bar{B}_3\breve{v}'(u)+\bar{B}_2'u+y &= \bar{B}_3\big[\hat{v}-\bar{B}_3^{\#}\big(\bar{B}_2'u+y-(\bar{B}_2'\hat{u}+y)\big)\big]+\bar{B}_2'u+y\\ &= \bar{B}_3\big[\hat{v}-\bar{B}_3^{\#}(\bar{B}_2'u+y)\big]+\bar{B}_2'u+y=0,\end{aligned}$$

which implies that $\breve{v}'(u)\in\mathcal{A}^1(u)$ for all $u\in\mathsf{Dom}(\bar{G}^1)$. Hence,

$$\bar{G}^1(u)=G(u,\breve{v}'(u)) \;\; \forall\, u\in\mathsf{Dom}(\bar{G}^1). \tag{4.13}$$

By (4.10), (4.13), (A.1), and the chain rule,

$$\frac{d\bar{G}^1}{du}(\hat{u})=G_u(\hat{u},\breve{v}'(\hat{u}))+G_v(\hat{u},\breve{v}'(\hat{u}))\frac{d\breve{v}'}{du}(\hat{u}),$$

which, by (4.8) and (4.11),

$$=G_u(\hat{u},\hat{v})+G_v(\hat{u},\hat{v})\frac{d\breve{v}'}{du}(\hat{u})=0,$$

which implies that $\hat{u}\in\bar{\mathcal{A}}^1$. Using this, (4.9), and (A.2i), we see that $G(\hat{u},\hat{v})=G(\bar{u},\bar{v})$. As $(\hat{u},\hat{v})\in\mathrm{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}\,G(u,v)\setminus\{(\bar{u},\bar{v})\}$ was arbitrary, we have the desired result. ∎

4.2. The semiquadratic case. Throughout this section, we take $\mathcal{A}\subseteq\mathcal{U}$ and $\mathcal{B}=\mathcal{V}$, with $\mathcal{V}$ being Hilbert. Let

$$G(u,v)\doteq f_1(u)+\langle f_2(u),v\rangle_{\mathcal{V}}+\tfrac{1}{2}\langle\bar{B}_3(u)v,v\rangle_{\mathcal{V}} \tag{4.14}$$

for all $u\in\mathcal{A}$ and $v\in\mathcal{V}$, where $f_1\in C^2(\mathcal{A};\mathcal{F})$, $f_2\in C^2(\mathcal{A};\mathcal{V})$, and $\bar{B}_3\in C^2(\mathcal{A};\mathcal{L}(\mathcal{V},\mathcal{V}))$ and $\bar{B}_3(u)$ is self-adjoint and closed for all $u\in\mathcal{A}$. For each $u\in\mathcal{A}$, let $\bar{B}_3^{\#}(u)\doteq[\bar{B}_3(u)]^{\#}$ denote the Moore–Penrose pseudoinverse of $\bar{B}_3(u)$ (where the existence of such follows from the closedness of $\bar{B}_3(u)$). Assume that there exists a constant $D>0$ such that $|\bar{B}_3^{\#}(u)|\le D$ for all $u\in\mathsf{Dom}(\bar{G}^1)$. Similar to Lemma 4.1, the next lemma follows directly from (4.14) and Lemma 2.1.

LEMMA 4.7. *Let $\hat{u}\in\mathcal{A}$. Then $\hat{v}\in\mathcal{A}^1(\hat{u})$ if and only if $f_2(\hat{u})+\bar{B}_3(\hat{u})\hat{v}=0$.*

4.2.1. When the full staticization is known to exist.

LEMMA 4.8. *Assume* (A.2f)*, and let $(\bar{u},\bar{v})\in\mathrm{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}\,G(u,v)$. Then $\bar{u}\in$ $\mathsf{Dom}(\bar{G}^1)$, and assumption* (A.3) *is satisfied.*

Proof. We begin with the first assertion. Let $(\bar{u},\bar{v})\in\mathrm{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}\,G(u,v)$. Then, by definition of stat,

$$\bar{B}_3(\bar{u})\bar{v}+f_2(\bar{u})=0. \tag{4.15}$$

For any $v\in\mathcal{V}$,

$$G(\bar{u},v)-G(\bar{u},\bar{v})=\langle f_2(\bar{u}),v-\bar{v}\rangle_{\mathcal{V}}+\tfrac{1}{2}\langle\bar{B}_3(\bar{u})v,v\rangle_{\mathcal{V}}-\tfrac{1}{2}\langle\bar{B}_3(\bar{u})\bar{v},\bar{v}\rangle_{\mathcal{V}},$$

and by the self-adjointness of $\bar{B}_3(\bar{u})$ and (4.15), one finds

$$\begin{aligned}&=\langle f_2(\bar{u}),v-\bar{v}\rangle_{\mathcal{V}}+\langle\bar{B}_3(\bar{u})\bar{v},v-\bar{v}\rangle_{\mathcal{V}}+\tfrac{1}{2}\langle\bar{B}_3(\bar{u})(v-\bar{v}),v-\bar{v}\rangle_{\mathcal{V}}\\ &=\tfrac{1}{2}\langle\bar{B}_3(\bar{u})(v-\bar{v}),v-\bar{v}\rangle_{\mathcal{V}}. \qquad (4.16)\end{aligned}$$

Now suppose there exists $\hat{v} \neq \bar{v}$ such that $\hat{v} \in \operatorname{argstat}_{v\in\mathcal{V}} G(\bar{u},v)$. This implies $\bar{B}_3(\bar{u})\hat{v} + f_2(\bar{u}) = 0$, and similar to the case for $\bar{v}$, one sees that for all $v \in \mathcal{V}$,

$$G(\bar{u},v) - G(\bar{u},\hat{v}) = \tfrac{1}{2}\langle \bar{B}_3(\bar{u})(v-\hat{v}), v-\hat{v}\rangle_{\mathcal{V}}. \tag{4.17}$$

Taking $v = \hat{v}$ in (4.16) and $v = \bar{v}$ in (4.17) yields $G(\bar{u},\bar{v}) = G(\bar{u},\hat{v})$. As $\hat{v} \in \mathcal{V}$ was arbitrary, we have the first assertion.

Next, suppose $\mathsf{Dom}(\bar{G}^1) \neq \{\bar{u}\}$; otherwise the result is trivial. Choose any $\delta > 0$ such that $\mathsf{Dom}(\bar{G}^1) \cap (B_\delta(\bar{u})\backslash\{\bar{u}\}) \neq \emptyset$. Let $\hat{u} \in [\mathsf{Dom}(\bar{G}^1) \cap B_\delta(\bar{u})] \setminus \{\bar{u}\}$. Let $\hat{v} = \bar{v} - \bar{B}_3^{\#}(\hat{u})f_2(\hat{u}) - \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}$. Note that as $f_2(\hat{u}) \in \operatorname{Range}(\bar{B}_3(\hat{u}))$,

$$\begin{aligned}
\bar{B}_3(\hat{u})\hat{v} + f_2(\hat{u}) &= \bar{B}_3(\hat{u})[\bar{v} - \bar{B}_3^{\#}(\hat{u})f_2(\hat{u}) - \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}] + f_2(\hat{u}) \\
&= \bar{B}_3(\hat{u})\bar{v} - f_2(\hat{u}) - \bar{B}_3(\hat{u})\bar{v} + f_2(\hat{u}) = 0.
\end{aligned}$$

Therefore, $\hat{v} \in \mathcal{A}^1(\hat{u})$ by Lemma 4.7. We have

$$|\hat{v} - \bar{v}| = \big|\bar{B}_3^{\#}(\hat{u})f_2(\hat{u}) + \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}\big|,$$

and noting that by Lemma 4.7, $\bar{B}_3(\bar{u})\bar{v} + f_2(\bar{u}) = 0$, this is

$$\begin{aligned}
&= \big|\bar{B}_3^{\#}(\hat{u})[f_2(\hat{u}) - f_2(\bar{u}) - \bar{B}_3(\bar{u})\bar{v} + \bar{B}_3(\hat{u})\bar{v}]\big| \\
&\leq |\bar{B}_3^{\#}(\bar{u})|\,|f_2(\hat{u}) - f_2(\bar{u}) + (\bar{B}_3(\hat{u}) - \bar{B}(\bar{u}))\bar{v}|,
\end{aligned}$$

and letting $K_f \doteq \max_{\lambda\in[0,1]} \big|\frac{df_2}{du}(\lambda\hat{u} + (1-\lambda)\bar{u})\big|$ and $K_B \doteq \max_{\lambda\in[0,1]} \big|\frac{d\bar{B}_3}{du}(\lambda\hat{u} + (1-\lambda)\bar{u})\big|$ and using the mean value theorem [1, Theorem 12.6] (see also Appendix A), we see that this is

$$\leq D\big[K_f|\hat{u} - \bar{u}| + K_B|\bar{v}||\hat{u} - \bar{u}|\big],$$

which yields (A.3). ∎

THEOREM 4.9. *Assume* (A.2f)*, and let* $(\bar{u},\bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$. *Also assume* (A.2i). *Then*

$$\operatorname*{stat}_{u\in \mathsf{Dom}(\bar{G}^1)} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v).$$

Proof. This follows immediately from Lemma 4.8 and Theorem 3.2. ∎

In order to remove the assumption in Theorem 4.9 that $\operatorname{stat}_{u\in\mathsf{Dom}(\bar{G}^1)} \bar{G}^1(u)$ exists (i.e., (A.2i)), we will use an assumption that is more easily verified. The following lemma and theorem perform that replacement.

LEMMA 4.10. *Suppose* $f_2(u) \in \operatorname{Range}[\bar{B}_3(u)]$ *for all* $u \in \mathsf{Dom}(\bar{G}^1)$. *Suppose* $\hat{u} \in \bar{\mathcal{A}}^1$*, and let* $\hat{v} \in \mathcal{A}^1(\hat{u})$. *Then* $G_u(\hat{u},\hat{v}) = 0$.

Proof. By assumption and Lemma 4.7,

$$G_v(\hat{u},\hat{v}) = 0. \tag{4.18}$$

Suppose

$$G_u(\hat{u},\hat{v}) \neq 0. \tag{4.19}$$

Then there exists $\epsilon > 0$, sequence $\{u_n\}$ with elements $u_n \in \mathcal{A} \setminus \{\hat{u}\}$ and $u_n \to \hat{u}$, and $\tilde{n} = \tilde{n}(\epsilon) \in \mathbb{N}$ such that

$$|G(u_{\tilde{n}},\hat{v}) - G(\hat{u},\hat{v})| > \epsilon|u_n - \hat{u}| \;\; \forall\, n \geq \tilde{n}. \tag{4.20}$$

Let

$$v_n \doteq \hat{v} - \bar{B}_3^{\#}(u_n)[f_2(u_n) + \bar{B}_3(u_n)\hat{v}] \;\; \forall\, n \in \mathbb{N}. \tag{4.21}$$

Then using Lemma 4.7,

$$|v_n - \hat{v}| \le |\bar{B}_3^{\#}(u_n)| \left| f_2(u_n) + \bar{B}_3(u_n)\hat{v} - f_2(\hat{u}) - \bar{B}_3(\hat{u})\hat{v} \right|,$$

which, by assumption,

$$\le D\big(|f_2(u_n) - f_2(\hat{u})| + |\bar{B}_3(u_n) - \bar{B}_3(\hat{u})||\hat{v}|\big). \tag{4.22}$$

Now, by mean value theorem [1, Theorem 12.6] (see also Appendix A), for each $n \in \mathbb{N}$, there exist $\lambda_n, \hat{\lambda}_n \in [0,1]$ such that

$$|f_2(u_n) - f_2(\hat{u})| \le \left| \frac{df_2}{du}(\lambda_n u_n + (1-\lambda_n)\hat{u}) \right| |u_n - \hat{u}|,$$

$$|\bar{B}_3(u_n) - \bar{B}_3(\hat{u})| \le \left| \frac{d\bar{B}_3}{du}(\hat{\lambda}_n u_n + (1-\hat{\lambda}_n)\hat{u}) \right| |u_n - \hat{u}|,$$

and hence by the smoothness of $f_2, \bar{B}_3$ and (4.22), there exist $K < \infty$ and $\hat{n} \in \mathbb{N}$ such that

$$|v_n - \hat{v}| \le DK(1 + |\hat{v}|)|u_n - \hat{u}| \;\; \forall\, n \ge \hat{n}. \tag{4.23}$$

Also, using (4.21),

$$\bar{B}_3(u_n)v_n + f_2(u_n) = \bar{B}_3(u_n)\big[\hat{v} - \bar{B}_3^{\#}(u_n)f_2(u_n) - \bar{B}_3^{\#}(u_n)\bar{B}_3(u_n)\hat{v}\big] + f_2(u_n),$$

which, by assumption and the properties of the pseudoinverse,

$$= \bar{B}_3(u_n)\hat{v} - f_2(u_n) - \bar{B}_3(u_n)\hat{v} + f_2(u_n) = 0. \tag{4.24}$$

By (4.24) and Lemma 4.7, $v_n \in \mathcal{A}^1(u_n)$ for all $n \in \mathbb{N}$. Using this, recalling that we took $\hat{v} \in \mathcal{A}^1(\hat{u})$, and noting the semiquadratic form, we see that

$$|G(u_n, v_n) - G(\hat{u}, \hat{v})| = |\bar{G}^1(u_n) - \bar{G}^1(\hat{u})|,$$

and by the assumption that $\hat{u} \in \bar{\mathcal{A}}^1$, there exists $\bar{n} = \bar{n}(\epsilon)$ such that for all $n \ge \bar{n}$,

$$< \frac{\epsilon}{2}|u_n - \hat{u}|,$$

which implies

$$|G(u_n, v_n) - G(u_n, \hat{v}) + G(u_n, \hat{v}) - G(\hat{u}, \hat{v})| < \frac{\epsilon}{2}|u_n - \hat{u}| \;\; \forall\, n \ge \bar{n},$$

and hence

$$|G(u_n, \hat{v}) - G(\hat{u}, \hat{v})| < \frac{\epsilon}{2}|u_n - \hat{u}| + |G(u_n, v_n) - G(u_n, \hat{v})| \;\; \forall\, n \ge \bar{n}. \tag{4.25}$$

Now by (4.14),

$$G(u_n, \hat{v}) - G(u_n, v_n) = \langle f_2(u_n), \hat{v} - v_n \rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3(u_n)\hat{v}, \hat{v} \rangle_{\mathcal{V}} - \tfrac{1}{2}\langle \bar{B}_3(u_n)v_n, v_n \rangle_{\mathcal{V}},$$

which, by (4.21),

$$= \langle f_2(u_n), \bar{B}_3^{\#}(u_n)[f_2(u_n) + \bar{B}_3(u_n)\hat{v}] \rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3(u_n)\hat{v}, \hat{v} \rangle_{\mathcal{V}} - \tfrac{1}{2}\langle \bar{B}_3(u_n)v_n, v_n \rangle_{\mathcal{V}},$$

and by Lemma 4.7 and the self-adjointness of $\bar{B}_3$, this is

$$= \langle -\bar{B}_3(u_n)v_n, \bar{B}_3^{\#}(u_n)\bar{B}_3(u_n)(\hat{v} - v_n)] \rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3(u_n)\hat{v}, \hat{v} \rangle_{\mathcal{V}} - \tfrac{1}{2}\langle \bar{B}_3(u_n)v_n, v_n \rangle_{\mathcal{V}},$$

$$= \langle \bar{B}_3(u_n)(\hat{v} - v_n), (\hat{v} - v_n) \rangle_{\mathcal{V}}. \tag{4.26}$$

Applying (4.23) in (4.26), we see that there exists $K_1 < \infty$ such that $|G(u_n, \hat{v}) - G(u_n, v_n)| \leq K_1 |u_n - \hat{u}|^2$ for all $n \geq \hat{n}$, and consequently, there exists $\bar{n}_1 = \bar{n}_1(\epsilon) \in (\hat{n}, \infty)$ such that

$$|G(u_n, \hat{v}) - G(u_n, v_n)| < \frac{\epsilon}{2} |u_n - \hat{u}| \;\; \forall\, n \geq \bar{n}_1. \tag{4.27}$$

By (4.25) and (4.27),

$$|G(u_n, \hat{v}) - G(\hat{u}, \hat{v})| < \epsilon |u_n - \hat{u}| \;\; \forall\, n \geq \bar{n} \wedge \bar{n}_1. \tag{4.28}$$

However, (4.28) contradicts (4.20), and consequently, $G_u(\hat{u}, \hat{v}) = 0$. ∎

THEOREM 4.11. *Assume* (A.2f)*, and let* $(\bar{u}, \bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$*. Also assume* $f_2(u) \in \operatorname{Range}[\bar{B}_3(u)]$ *for all* $u \in \mathit{Dom}(\bar{G}^1)$*. Then*

$$\operatorname*{stat}_{u\in \mathit{Dom}(\bar{G}^1)} \bar{G}^1(u) = G(\bar{u}, \bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v).$$

Proof. Suppose $\hat{u} \in \bar{\mathcal{A}}^1$, and let $\hat{v} \in \mathcal{A}^1(\hat{u})$. Then $G_v(\hat{u}, \hat{v}) = 0$, and by Lemma 4.10, $G_u(\hat{u}, \hat{v}) = 0$. These imply $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{V}} G(u,v)$, and hence, by the assumption of the subsection, $G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v})$. By this and the choice of $\hat{v}$, $\bar{G}^1(\hat{u}) = G(\bar{u}, \bar{v})$. As $\hat{u} \in \bar{\mathcal{A}}^1$ was arbitrary, $\operatorname{stat}_{u\in\mathrm{Dom}(\bar{G}^1)} \bar{G}^1(u)$ exists. The assertion then follows by Theorem 4.9. ∎

4.2.2. When the iterated staticization is known to exist. The case where the iterated staticization is known to exist appears to also require an additional assumption.

THEOREM 4.12. *Assume* (A.2i)*, and let* $\bar{u} \in \bar{\mathcal{A}}^1$*. Also assume that* $f_2(u) \in \operatorname{Range}[\bar{B}_3(u)]$ *for all* $u \in \mathit{Dom}(\bar{G}^1)$*. Then* $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ *exists, and*

$$\operatorname*{stat}_{u\in \mathit{Dom}(\bar{G}^1)} \bar{G}^1(u) = G(\bar{u}, \bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v).$$

Proof. Note that by assumption and Lemma 4.7, $\mathrm{Dom}(\bar{G}^1) = \mathcal{A}$. Let $\bar{u} \in \operatorname{argstat}_{u\in\mathrm{Dom}(\bar{G}^1)} \bar{G}^1(u)$, and let $\bar{v}$ be as in (2.7). By definition and Lemma 4.10, $G_v(\bar{u}, \bar{v}) = 0$ and $G_u(\bar{u}, \bar{v}) = 0$, which implies

$$(\bar{u}, \bar{v}) \in \operatorname*{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) \quad \text{and} \quad G(\bar{u}, \bar{v}) \in \operatorname*{stat^s}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v).$$

Now suppose there exists $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) \setminus \{(\bar{u}, \bar{v})\}$. This implies

$$G_u(\hat{u}, \hat{v}) = 0, \qquad G_v(\hat{u}, \hat{v}) = 0, \qquad \hat{v} \in \mathcal{A}^1(\hat{u}), \quad \text{and} \;\; \bar{G}^1(\hat{u}) = G(\hat{u}, \hat{v}). \tag{4.29}$$

Let

$$\breve{v}(u) \doteq \hat{v} - \bar{B}_3^{\#}(u) f_2(u) - \bar{B}_3^{\#}(u) \bar{B}_3(u) \hat{v} \;\; \forall\, u \in \mathcal{A}, \tag{4.30}$$

and note that $\breve{v}(\hat{u}) = \hat{v}$. Also note that by (4.30), the assumptions, and properties of the pseudoinverse,

$$\bar{B}_3(u)\breve{v}(u) + f_2(u) = \bar{B}_3(u)\hat{v} - f_2(u) - \bar{B}_3(u)\hat{v} + f_2(u) = 0,$$

which implies that $\breve{v}(u) \in \mathcal{A}^1(u)$ for all $u \in \mathcal{A}$. Hence, $\bar{G}^1(u) = G(u, \breve{v}(u))$ for all $u \in \mathcal{A}$. Note that

$$|\bar{G}^1(u) - \bar{G}^1(\hat{u})| = |G(u, \breve{v}(u)) - G(\hat{u}, \hat{v})| \le |G(u, \breve{v}(u)) - G(u, \hat{v})| + |G(u, \hat{v}) - G(\hat{u}, \hat{v})|,$$

and note that by (4.29), given $\epsilon > 0$, there exists $\hat{\delta}_1 = \hat{\delta}_1(\epsilon) > 0$ such that, for all $|u - \hat{u}| < \hat{\delta}_1$,

$$\le \frac{\epsilon}{2}|u - \hat{u}| + |G(u, \breve{v}(u)) - G(u, \hat{v})|. \tag{4.31}$$

Also, similar to the estimate in the proof of Lemma 4.10, we find that there exists $\hat{\delta}_2 = \hat{\delta}_2(\epsilon) > 0$ such that

$$|G(u, \breve{v}(u)) - G(u, \hat{v})| < \frac{\epsilon}{2}|u - \hat{u}| \quad \forall\, |u - \hat{u}| < \hat{\delta}_2.$$

Using this in (4.31), we see that

$$|\bar{G}^1(u) - \bar{G}^1(\hat{u})| < \epsilon|u - \hat{u}| \quad \forall\, |u - \hat{u}| < \hat{\delta}_1 \wedge \hat{\delta}_2. \tag{4.32}$$

Hence, $\frac{d\bar{G}^1}{du}(\hat{u}) = 0$, which implies that $\hat{u} \in \bar{\mathcal{A}}^1$. Using this and (A.2i), we see that $G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v})$. As $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \setminus \{(\bar{u}, \bar{v})\}$ was arbitrary, we have the desired result. ◻

4.3. The uniformly locally Morse case. Throughout this section, we will assume that G is uniformly locally Morse in v in the following sense. We assume that for all $(\hat{u}, \hat{v}) \in \mathcal{A} \times \mathcal{B}$ such that $G_v(\hat{u}, \hat{v}) = 0$, there exist $\tilde{\epsilon} = \tilde{\epsilon}(\hat{u}, \hat{v}) > 0$ and $\tilde{K} = \tilde{K}(\hat{u}, \hat{v}) < \infty$ such that $G_{vv}(u, v)$ is invertible with $\left|[G_{vv}(u, v)]^{-1}\right| \le \tilde{K}$ for all $(u, v) \in B_{\tilde{\epsilon}}(\hat{u}, \hat{v})$. We also assume that $G_{uv}(u, v)$ is bounded on bounded sets.

4.3.1. When the full staticization is known to exist. We suppose (A.2f), and let $(\bar{u}, \bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$. We will find that (A.3) holds and that $\operatorname{stat}_{u\in\mathsf{Dom}(\bar{G}^1)} \bar{G}^1(u)$ exists. We will then obtain the equivalence between full and iterated staticization.

Lemma 4.13. *Assume* (A.2f)*, and let* $(\bar{u}, \bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$. *There exist* $\epsilon, \delta > 0$ *and* $\breve{v} \in C^1(B_\epsilon(\bar{u}); \mathcal{B} \cap B_\delta(\bar{v}))$ *such that* $\breve{v}(\bar{u}) = \bar{v}$, $G_v(u, \breve{v}(u)) = 0$, *and*

$$\frac{d\breve{v}}{du}(u) = -\left[G_{vv}(u, v)\big|_{(u,\breve{v}(u))}\right]^{-1} G_{uv}(u, v)\big|_{(u,\breve{v}(u))}$$

for all $u \in B_\epsilon(\bar{u})$.

Proof. The first two assertions are simply the implicit mapping theorem; cf. [12]. The final assertion then follows from an application of the chain rule; that is, noting that $G_v(u, \breve{v}(u)) = 0$ on $B_\epsilon(\bar{u})$,

$$0 = \frac{dG_v(u, \breve{v}(u))}{du} = G_{uv}(u, v)\big|_{(u,\breve{v}(u))} + G_{vv}(u, v)\big|_{(u,\breve{v}(u))} \frac{d\breve{v}}{du}(u) \;\; \forall\, u \in B_\epsilon(\bar{u}).$$ ◻

By Lemma 4.13 and the definition of $\mathsf{Dom}(\bar{G}^1)$,

$$\bar{G}^1(u) = \operatorname*{stat}_{v\in\mathcal{B}} G(u, v) = G(u, \breve{v}(u)) \;\; \forall\, u \in B_\epsilon(\bar{u}) \cap \mathsf{Dom}(\bar{G}^1). \tag{4.33}$$

Then, by (4.33), the chain rule, (A.1), and Lemma 4.13,

$$\bar{G}^1(\cdot) \in C^1(B_\epsilon(\bar{u}) \cap \mathsf{Dom}(\bar{G}^1); \mathcal{F}). \tag{4.34}$$

LEMMA 4.14. *Assume* (A.2f)*, and let* $(\bar u,\bar v)\in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$*. Then* (A.3) *is satisfied. That is, there exists* $K<\infty$ *and* $\delta\in(0,\epsilon)$ *such that* $|\breve v(u)-\breve v(\bar u)| = |\breve v(u)-\bar v|\le K|u-\bar u|$ *for all* $u\in B_\delta(\bar u)\cap \mathit{Dom}(\bar G^1)$*.*

Proof. By Lemma 4.13, $\frac{d\breve v}{du}(\cdot)$ is continuous on $B_\epsilon(\bar u)\cap \mathsf{Dom}(\bar G^1)$. Further, by the final assertion of Lemma 4.13, the uniformly locally Morse assumption, and the boundedness assumption of the lemma,

$$\Big|\frac{d\breve v}{du}(u)\Big| = \Big|\big[G_{vv}(u,v)\big|_{(u,\breve v(u))}\big]^{-1}\Big|\,\Big|G_{uv}(u,v)\big|_{(u,\breve v(u))}\Big| \le \tilde K\hat K,$$

where $\hat K$ is a bound on $|G_{uv}(u,\breve v(u))|_{(u,\breve v(u))}|$ over $B_\delta(\bar u)$. Hence, by an application of the mean value theorem, we obtain the asserted bound. ◻

By Lemma 4.14, we see that one may apply Theorem 3.2 if $\bar u\in \mathsf{Dom}(\bar G^1)$. This implies that the equivalence of stat and iterated stat holds under the assumption of existence of the latter. We proceed to obtain this existence.

LEMMA 4.15. *Assume* (A.2f)*, and let* $(\bar u,\bar v)\in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$*. Also assume that* $\bar u\in \mathit{Dom}(\bar G^1)$*. Then* $\operatorname{stat}_{u\in \mathit{Dom}(\bar G^1)} \bar G^1(u)$ *exists.*

Proof. Note first that by (4.33), (4.34), and the chain rule,

$$\frac{d}{du}\bar G^1(u)\big|_{u=\bar u} = \frac{d}{du}G(u,\breve v(u))\big|_{u=\bar u} = G_u(\bar u,\breve v(\bar u)) + G_v(\bar u,\breve v(\bar u))\frac{d\breve v}{du}(\bar u),$$

which, by (A.2f) and Lemma 4.13,

$$= 0.$$

Consequently,

$$\bar u\in \operatorname*{argstat}_{u\in\mathsf{Dom}(\bar G^1)} \bar G^1(u) \quad\text{and}\quad \bar G^1(\bar u)\in \operatorname*{stat^s}_{u\in\mathsf{Dom}(\bar G^1)} \bar G^1(u). \tag{4.35}$$

Suppose $\bar u\in\mathsf{Dom}(\bar G^1)$, with $\hat u\ne\bar u$, is such that

$$\hat u\in \operatorname*{argstat}_{u\in\mathsf{Dom}(\bar G^1)} \bar G^1(u). \tag{4.36}$$

Then, by (A.2f), there exists $\hat v\in\mathcal{A}^1(\hat u)$. Recalling that G is uniformly locally Morse in v and applying the implicit mapping theorem again, we find that there exists $\epsilon'>0$ and $\breve v'\in C^1(B_{\epsilon'}(\hat u);\mathcal{B})$ such that $B_{\epsilon'}(\hat u)\subseteq\mathsf{Dom}(\bar G^1)$ and

$$\breve v'(\hat u)=\hat v \quad\text{and}\quad G_v(u,\breve v'(u))=0\ \ \forall\, u\in B_{\epsilon'}(\hat u)\subseteq \mathsf{Dom}(\bar G^1). \tag{4.37}$$

Then, by (4.36), another application of the chain rule, and (4.37),

$$0=\frac{d}{du}\bar G^1(u)\big|_{u=\hat u} = G_u(\hat u,\breve v'(\hat u)) + G_v(\hat u,\breve v'(\hat u))\frac{d\breve v'}{du}(\hat u) = G_u(\hat u,\hat v). \tag{4.38}$$

By (4.37) and (4.38), $(\hat u,\hat v)\in\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$, and hence by (A.2f),

$$G(\hat u,\hat v) = G(\bar u,\bar v). \tag{4.39}$$

Recalling from (2.6) that $\hat v\in\operatorname{argstat}_{v\in\mathcal{B}} G(\hat u,v)$ and using (4.39), we have

$$\bar G^1(\hat u) = G(\hat u,\hat v) = G(\bar u,\bar v).$$

As $\hat u\in\operatorname{argstat}_{u\in\mathsf{Dom}(\bar G^1)} \bar G^1(u)\setminus\{\bar u\}$ was arbitrary, we have the desired result. ◻

THEOREM 4.16. *Assume* (A.2f)*, and let* $(\bar u,\bar v)\in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$*. Also assume* $\bar u\in \mathsf{Dom}(\bar G^1)$*. Then* $\operatorname{stat}_{u\in \mathsf{Dom}(\bar G^1)}\bar G^1(u)$ *exists, and*

$$\operatorname*{stat}_{u\in \mathsf{Dom}(\bar G^1)}\bar G^1(u)=G(\bar u,\bar v)=\operatorname*{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} G(u,v).$$

Proof. The assertion of the existence of $\operatorname{stat}_{u\in\mathsf{Dom}(\bar G^1)}\bar G^1(u)$ is simply Lemma 4.15. Then, noting that Lemma 4.14 implies that assumption (A.3) is satisfied, one may apply Theorem 3.2 to obtain the second assertion of the theorem. ∎

4.3.2. When the iterated staticization is known to exist. We suppose (A.2i), and let $\bar u\in\bar{\mathcal{A}}^1$. We will find that $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists and obtain the equivalence between full and iterated staticization.

LEMMA 4.17. *Assume* (A.2i)*, and let* $\bar u\in\bar{\mathcal{A}}^1$*. Then* $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ *exists.*

Proof. By (A.2i), (2.7), the uniform Morse property, and the implicit mapping theorem, there exists $\delta>0$ and $\breve v\in C^1(B_\delta(\bar u);\mathcal{B})$ such that $B_\delta\subseteq\mathsf{Dom}(\bar G^1)$,

$$\breve v(\bar u)=\bar v \quad\text{and}\quad G_v(u,\breve v(u))=0\ \ \forall\, u\in B_\delta(\bar u). \tag{4.40}$$

By the differentiability of $\breve v$, (A.1), and the chain rule,

$$\frac{d\bar G^1}{du}(\bar u)=G_u(\bar u,\breve v(\bar u))+G_v(\bar u,\breve v(\bar u))\frac{d\breve v}{du}(\bar u)=G_u(\bar u,\bar v)+G_v(\bar u,\bar v)\frac{d\breve v}{du}(\bar u).$$

Using (A.2i) and (2.7), this implies $0 = G_u(\bar u,\bar v)$, and hence $(\bar u,\bar v)\in\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$.

Now suppose there exists $(\hat u,\hat v)\in\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)\backslash\{(\bar u,\bar v)\}$, which implies

$$G_u(\hat u,\hat v)=0 \quad\text{and}\quad G_v(\hat u,\hat v)=0. \tag{4.41}$$

By (4.41), $(A.1)$, the uniform Morse property, and the implicit mapping theorem, there exists $\delta'>0$ and $\breve v\,'\in C^1(B_{\delta'}(\hat u);\mathcal{B})$ such that $B_{\delta'}(\hat u)\subseteq\mathsf{Dom}(\bar G^1)$,

$$\breve v\,'(\hat u)=\hat v \quad\text{and}\quad G_v(u,\breve v\,'(u))=0\ \ \forall\, u\in B_{\delta'}(\hat u). \tag{4.42}$$

Further, combining the definition of $\mathsf{Dom}(\bar G^1)$ and (4.42), we see that

$$\bar G^1(u)=\operatorname*{stat}_{v\in\mathcal{B}} G(u,v)=G(u,\breve v\,'(u))\ \ \forall\, uB_{\delta'}(\hat u). \tag{4.43}$$

Then, by (4.42), (4.43), (A.1), and the chain rule,

$$\frac{d\bar G^1(\hat u)}{du}(\hat u)=G_u(\hat u,\breve v\,'(\hat u))+G_u(\hat u,\breve v\,'(\hat u))\frac{d\breve v\,'}{du}(\hat u),$$

which, by (4.41) and the definition of $\breve v\,'(u)$,

$$=0;$$

that is, $\hat u\in\operatorname{argstat}_{u\in\mathsf{Dom}(\bar G^1)}\bar G^1(u)$, and using (A.2i), this implies $\bar G^1(\hat u)=\bar G^1(\bar u)$. Combining this with (4.42) and (4.43), we see that

$$G(\hat u,\hat v)=\bar G^1(\hat u)=\bar G^1(\bar u),$$

and then by the definition of $\bar G^1$ and (2.7), this is

$$=G(\bar u,\bar v).$$

As $(\hat u,\hat v)\in\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}$ was arbitrary, $G(\hat u,\hat v) = G(\bar u,\bar v)$ for all $(\hat u,\hat v)\in\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$. ∎

By Lemma 4.17 and Theorem 4.16 we have the following.

THEOREM 4.18. *Assume* (A.2i)*, and let* $\bar{u} \in \bar{\mathcal{A}}^1$*. Then* $\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ *exists, and*

$$\operatorname*{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} G(u,v) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{u\in Dom(\bar{G}^1)} \bar{G}^1(u).$$

5. Application to astrodynamics. As noted in the introduction, there are two classes of problems in dynamical systems that have motivated the above development. The first class consists of TPBVPs in astrodynamics, and we discuss that here. Specifically, one may obtain fundamental solutions to TPBVPs in astrodynamics through a stationary-action–based approach [9, 10, 18, 19]. We briefly recall the case of the n-body problem. In this case, the action functional with an appended terminal cost (cf. [19]) takes the form indicated in (1.1), where now $x = \big((x^1)^T, (x^2)^T, \ldots (x^n)^T\big)^T$, where each $x^j \in \mathbb{R}^3$ denotes a generic position of body j for $j \in \mathcal{N} \doteq \{1,2,\ldots n\}$, and $\xi_\cdot$, $u_\cdot$ of (1.1) are similarly constructed. The kinetic-energy term is $T(u_r) \doteq \frac{1}{2}\sum_{j=1}^n m_j |u_r^j|^2$, where m_j is the mass of the jth body.

Suppose $x^i \neq x^j$ for all $i \neq j$. Then, the additive inverse of the potential is given by

$$\begin{aligned} -V(x) &= \sum_{(i,j)\in\mathcal{I}^\Delta} \frac{\Gamma m_i m_j}{|x^i - x^j|} = \max_{\alpha\in\mathcal{M}_{(0,\infty)}} \sum_{(i,j)\in\mathcal{I}^\Delta} \left(\tfrac{3}{2}\right)^{\frac{3}{2}} \Gamma\, m_i m_j \left[\alpha_{i,j} - \frac{\alpha_{i,j}^3 |x^i - x^j|^2}{2}\right] \\ &\doteq \max_{\alpha\in\mathcal{M}_{(0,\infty)}} \big[-\tilde{V}(x,\alpha)\big] = -\tilde{V}(x,\bar{\alpha}), \end{aligned} \tag{5.1}$$

where Γ is the universal gravitational constant, $\mathcal{I}^\Delta \doteq \{(i,j) \in \mathcal{N}^2 \,|\, j > i\}$, $\mathcal{M}_{(0,\bar{a})}$ denotes the set of arrays indexed by $(i,j) \in \mathcal{I}^\Delta$ with elements in $(0,\bar{a})$, and $\bar{\alpha}_{i,j} = \bar{\alpha}_{i,j}(x) = \big[2/(3|x^i - x^j|^2)\big]^{1/2}$ for all $(i,j) \in \mathcal{I}^\Delta$; see [19]. Recalling the discussion in section 1, we note that solutions of stationary-action problems with these kinetic and potential energy functions will yield solutions of TPBVPs for the n-body dynamics. Letting $\mathcal{U}_{(0,t)} \doteq L_2\big((0,t);\mathbb{R}^{3n}\big)$, one finds that the problem becomes that of finding the stationary-action value function given by

$$W(t,x) = \operatorname*{stat}_{u\in\mathcal{B}} J^0(t,x,u), \tag{5.2}$$

where

$$\begin{aligned} J^0(t,x,u) &\doteq \int_0^t T(u_r) - V(\xi_r)\,dr + \phi(\xi_t) = \int_0^t T(u_r) + \max_{\alpha\in\mathcal{M}_{(0,\infty)}} \big[-\tilde{V}(x,\alpha)\big]\,dr \\ &\qquad + \phi(\xi_t), \end{aligned}$$

$$\mathcal{B} \subseteq \{u \in \mathcal{U}_{(0,t)} \,|\, |\xi_r^i - \xi_r^j| \neq 0 \ \ \forall\, (i,j) \in \mathcal{I}^\Delta,\ r \in [0,t]\}. \tag{5.3}$$

Remark 5.1. Throughout the discussion to follow, we assume that $W(t,x)$ given by (5.2) exists. In particular, we assume that $\mathcal{B}$ is open and that there exists $\bar{u} \in \mathcal{B}$ such that $\operatorname{argstat}_{u\in\mathcal{B}} J^0(t,x,u) = \{\bar{u}\}$. One may note that given $u \in \mathcal{B}$, there exists $\bar{\delta} > 0$ such that $|\xi_r^i - \xi_r^j| > \bar{\delta}$ for all $(i,j) \in \mathcal{I}^\Delta$ and $r \in [0,t]$, and consequently there exists an open ball, $B_\delta(u) \subseteq \mathcal{B}$, which implies that $\mathcal{B}$ has nonempty interior. In the case where the problem corresponds to a TPBVP, these conditions amount to an assumption that if there are multiple solutions to the TPBVP, then the solutions are isolated; cf. [9, 10, 19].

Let $\tilde{\mathcal{A}}^{\bar{a}}_{(0,t)} \doteq C\big((0,t);\mathcal{M}_{(0,\bar{a})}\big)$ and $\tilde{\mathcal{A}}^{\mathcal{B}}_{(0,t)} \doteq C\big((0,t);\mathcal{M}_{\mathbb{R}}\big)$, where $\mathcal{M}_{\mathbb{R}}$ denotes the set of arrays indexed by $(i,j) \in \mathcal{I}^\Delta$ with elements in $\mathbb{R}$ and where we note that the former is a subset of the latter, which is a Banach space.

LEMMA 5.2. *Let $x \in \mathbb{R}^{3n}$, $t \in (0,\infty)$, and $\mathcal{B} \subseteq \mathcal{U}_{(0,t)}$. Then*

$$W(t,x) = \operatorname*{stat}_{u\in\mathcal{B}} \operatorname*{stat}_{\tilde{\alpha}\in\tilde{\mathcal{A}}^\infty_{(0,t)}} J(t,x,u,\tilde{\alpha}),$$

where

$$J(t,x,u,\tilde{\alpha}) \doteq \int_0^t T(u_r) - \tilde{V}(\xi_r,\tilde{\alpha}_r)\,dr + \phi(\xi_t). \tag{5.4}$$

Further, if $\mathcal{A} \subset \tilde{\mathcal{A}}^\infty_{(0,t)}$ is open and such that $\bar{\tilde{\alpha}}^{i,j} \in \mathcal{A}$, where $\bar{\tilde{\alpha}}^{i,j}_r = \bar{\alpha}_{i,j}(\xi_r)$ for all $(i,j) \in \mathcal{I}^\Delta$ and a.e. $r \in (0,t)$, where $\xi_r = x + \int_0^r u_\rho\,d\rho$, then $W(t,x) = \operatorname{stat}_{u\in\mathcal{B}} \operatorname{stat}_{\tilde{\alpha}\in\mathcal{A}} J(t,x,u,\tilde{\alpha}) = \operatorname{stat}_{u\in\mathcal{B}} J(t,x,u,\bar{\tilde{\alpha}})$.

Proof. Let $x \in \mathbb{R}^{3n}$, $t \in (0,\infty)$, $u \in \mathcal{B} \subseteq \mathcal{U}_{(0,t)}$, and $\mathcal{A} = \tilde{\mathcal{A}}^\infty_{(0,t)}$. By [19, Theorem 4.7], we find $J^0(t,x,u) = \max_{\tilde{\alpha}\in\mathcal{A}} J(t,x,u,\tilde{\alpha})$, where $J(t,x,u,\tilde{\alpha})$ is given by (5.4). Noting that $J(t,x,u,\cdot)$ is differentiable and strictly concave then yields $J^0(t,x,u) = \operatorname{stat}_{\tilde{\alpha}\in\mathcal{A}} J(t,x,u,\tilde{\alpha})$. Combining this with (5.2) yields the first assertion. The second assertion then follows by noting the argmax of (5.1). ∎

If one is able to reorder the stat operations, then the stat representation of Lemma 5.2 may be decomposed as

$$W(t,x) \doteq \operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}} \tilde{W}(t,x,\tilde{\alpha}), \tag{5.5}$$

$$\tilde{W}(t,x,\tilde{\alpha}) \doteq \operatorname*{stat}_{u\in\mathcal{B}} \left\{ \int_0^t T(u_r) - \tilde{V}(\xi_r,\tilde{\alpha}_r)\,dr + \phi(\xi_t) \right\}. \tag{5.6}$$

Further, suppose ϕ is a quadratic form, say,

$$\phi(x) = \phi(x;z) \doteq \tfrac{1}{2}(x-z)^T P_0 (x-z) + \gamma_0, \tag{5.7}$$

where $z \in \mathbb{R}^{3n}$ and P_0 is symmetric, positive-definite. Then, the argument of stat in (5.6) will be quadratic in u, and we will have

$$\tilde{W}(t,x,\tilde{\alpha}) = \tfrac{1}{2}(x^T P^{\tilde{\alpha}}_t x + x^T Q^{\tilde{\alpha}}_t z + z^T Q^{\tilde{\alpha}}_t x + z^T R^{\tilde{\alpha}}_t z + \gamma^{\tilde{\alpha}}_t), \tag{5.8}$$

where $P^{\tilde{\alpha}}_\cdot, Q^{\tilde{\alpha}}_\cdot, R^{\tilde{\alpha}}_\cdot$ may be obtained from solution of $\tilde{\alpha}$-indexed DREs, and $\gamma^{\tilde{\alpha}}_t$ is obtained from an integral [15, 19]. It will now be demonstrated that in the case of quadratic ϕ, we may reorder the stat operators.

Remark 5.3. We remark that different forms of ϕ may be used such that payoffs (5.4) (which will be shown to be equivalent to (5.5)) correspond to different TPBVPs for the n-body problem; see section 1 and [19]. The means by which this may be utilized for efficient generation of fundamental solutions is indicated in [9, 10, 19].

Remark 5.4. It can be shown that for sufficiently short time intervals, $J^0(t,x,\cdot)$ is convex and coercive, and one then has $W(t,x) = \min_{u\in\mathcal{B}} \max_{\tilde{\alpha}\in\mathcal{A}} J(t,x,u,\tilde{\alpha})$ for appropriate $\mathcal{A},\mathcal{B}$. In that case, one also finds that $W(t,x) = \max_{\tilde{\alpha}\in\mathcal{A}} \min_{u\in\mathcal{B}} J(t,x,u,\tilde{\alpha})$, and one proceeds similarly to the case here. That is, one again has (5.8), where the coefficients satisfy DREs. See [19] for the details. Here, we will employ the reordering of iterated stat operations to obtain $W(t,x)$ in a similar form, i.e., in the form (5.5).

LEMMA 5.5. *Let $x \in \mathbb{R}^{3n}$, $t \in (0,\infty)$, and $\tilde\alpha \in \mathcal{A} \subseteq \tilde{\mathcal{A}}^{\infty}_{(0,t)}$. Suppose ϕ has the form* (5.7)*. Then*

$$J(t,x,u,\tilde\alpha) \doteq f_1(\tilde\alpha) + \langle f_2(\tilde\alpha), u\rangle_{\mathcal{U}_{(0,t)}} + \tfrac12 \langle \bar B_3(\tilde\alpha)u, u\rangle_{\mathcal{U}_{(0,t)}} \quad \forall\, u \in \mathcal{U}_{(0,t)},$$

where $f_1(\tilde\alpha) \in \mathbb{R}$, $f_2(\tilde\alpha) \in \mathcal{U}_{(0,t)}$, and $\bar B_3(\tilde\alpha) \in \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$. Further, if $\mathcal{A} \subseteq \tilde{\mathcal{A}}^{\bar a}_{(0,t)}$ with $\bar a < \infty$, then for $|P_0^{-1}|$ sufficiently small, $\mathrm{Range}[\bar B_3(u)] = \mathcal{U}_{(0,t)}$.

Proof. Using (5.1) and (5.4), we see that

$$J(t,x,u,\tilde\alpha) = \int_0^t \tfrac12 \sum_{j=1}^n m_j |u_r^j|^2 + \sum_{(i,j)\in\mathcal{I}^\Delta} \left(\tfrac32\right)^{\frac32} \Gamma\, m_i m_j \left[\tilde\alpha_r^{i,j} - \frac{\left(\tilde\alpha_r^{i,j}\right)^3 |\xi_r^i - \xi_r^j|^2}{2}\right] dr + \phi(\xi_t). \tag{5.9}$$

Note that for the kinetic-energy term, we have the Riesz representation

$$\int_0^t \tfrac12 \sum_{j=1}^n m_j |u_r^j|^2 dr = \tfrac12 \langle Q_1 u, u\rangle_{\mathcal{U}_{(0,t)}}, \tag{5.10}$$

where the operator $Q_1 \in \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$ is given by $[Q_1 u]_r \doteq \bar Q_1 u_r$ for all $r \ge 0$, and $\bar Q_1$ is the $3n \times 3n$ block-diagonal matrix with blocks $m_1 I_3, m_2 I_3, \ldots m_n I_3$.

Let $\hat\Gamma \doteq \left(\frac32\right)^{32} \Gamma$. Similarly, we find that the potential term in J may be decomposed as

$$\begin{aligned}
&\hat\Gamma \sum_{(i,j)\in\mathcal{I}^\Delta} m_i m_j \int_0^t \left[\tilde\alpha_r^{i,j} - \left(\tilde\alpha_r^{i,j}\right)^3 \frac{|\xi_r^i - \xi_r^j|^2}{2}\right] dr \\
&= \hat\Gamma \sum_{(i,j)\in\mathcal{I}^\Delta} -m_i m_j \int_0^t \left[\left(\tilde\alpha_r^{i,j}\right)^3 \frac{|\int_0^r u_\rho^i d\rho|^2 + |\int_0^r u_\rho^j d\rho|^2 - 2\left(\int_0^r u_\rho^i d\rho\right)^T \int_0^r u_\tau^j d\tau}{2}\right] dr \\
&\quad + \hat\Gamma \sum_{(i,j)\in\mathcal{I}^\Delta} -m_i m_j \int_0^t \left[\left(\tilde\alpha_r^{i,j}\right)^3 \frac{2(x^i - x^j)^T \left(\int_0^r u_\rho^i d\rho\right) + 2(x^j - x^i)^T \left(\int_0^r u_\rho^j d\rho\right)}{2}\right] dr \\
&\quad + \hat\Gamma \sum_{(i,j)\in\mathcal{I}^\Delta} m_i m_j \int_0^t \left[\tilde\alpha_r^{i,j} - \left(\tilde\alpha_r^{i,j}\right)^3 \frac{|x^i|^2 + |x^j|^2 - 2(x^i)^T x^j}{2}\right] dr \qquad (5.11) \\
&\doteq \tfrac12 \langle Q_2(\tilde\alpha)u, u\rangle_{\mathcal{U}_{(0,t)}} + \langle R_2(\tilde\alpha), u\rangle_{\mathcal{U}_{(0,t)}} + S_2(\tilde\alpha) \quad \forall\, u \in \mathcal{U}_{(0,t)},
\end{aligned}$$

where we will obtain explicit expressions for $Q_2(\tilde\alpha) \in L(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$, $R_2(\tilde\alpha) \in \mathcal{U}_{(0,t)}$, and $S_2(\tilde\alpha) \in \mathbb{R}$. Considering a single generic component inside the first summation on the right-hand side of (5.11), note that

$$\begin{aligned}
&\int_0^t \left(\tilde\alpha_r^{i,j}\right)^3 \left(\int_0^r u_\rho^i d\rho\right)^T \int_0^r u_\tau^j d\tau\, dr \\
&= \int_0^t \int_0^t \int_0^t \mathcal{I}_{(0,r)}(\rho) \mathcal{I}_{(0,r)}(\tau) \left(\tilde\alpha_r^{i,j}\right)^3 \left(u_\rho^i\right)^T u_\tau^j \, d\rho\, d\tau\, dr,
\end{aligned}$$

where generically, $\mathcal{I}_{\mathcal{C}}$ denotes the indicator function on set $\mathcal{C}$, and this is

$$\begin{aligned}
&= \int_0^t \int_0^t \int_0^t \mathcal{I}_{(\rho,t)}(r) \mathcal{I}_{(\tau,t)}(r) \left(\tilde\alpha_r^{i,j}\right)^3 \left(u_\rho^i\right)^T u_\tau^j \, dr\, d\rho\, d\tau \\
&= \int_0^t \left(u_\rho^i\right)^T \left\{ \int_0^t \left[\int_{\rho\vee\tau}^t \left(\tilde\alpha_r^{i,j}\right)^3 dr\right] u_\tau^j \, d\tau \right\} d\rho.
\end{aligned}$$

Combining all these generic terms and rearranging our choice of dummy variables, we find that for all $u \in \mathcal{U}_{(0,t)}$, $[Q_2(\tilde{\alpha})u]_r = \int_0^t [\bar{Q}_2(\tilde{\alpha})](r,\tau)u_\tau \, d\tau$, where $[\bar{Q}_2(\tilde{\alpha})](r,\tau)$ is given as follows. For $i,j \in]1,n[$ such that $i \neq j$, let

$$[\hat{Q}_2(\tilde{\alpha})](r,\tau)]_{i,j} \doteq \hat{\Gamma} m_i m_j \int_{\tau\vee r}^t \left(\tilde{\alpha}_\sigma^{i,j}\right)^3 \, d\sigma,$$

and for $i \in]1,n[$, let

$$[\hat{Q}_2(\tilde{\alpha})](r,\tau)]_{i,i} \doteq - \sum_{j\in\,]1,n[,\, j\neq i} [\hat{Q}_2(\tilde{\alpha})](r,\tau)]_{i,j}.$$

Then $[\bar{Q}_2(\tilde{\alpha})](r,\tau) = [\hat{Q}_2(\tilde{\alpha})](r,\tau)] \otimes I_3$, where $\otimes$ denotes the Kronecker product here.

Proceeding similarly, we find that $R_2(\tilde{\alpha}) \in \mathcal{U}_{(0,t)}$ has the Riesz representation

$$R_2(\tilde{\alpha}) = \Big(([\hat{R}_2(\tilde{\alpha})(r)]_1)^T, ([\hat{R}_2(\tilde{\alpha})(r)]_2)^T, \ldots ([\hat{R}_2(\tilde{\alpha})(r)]_n)^T \Big)^T,$$

where for $i \in]1,n[$,

$$[\hat{R}_2(\tilde{\alpha})(r)]_i = -\hat{\Gamma} \sum_{j\neq i} m_i m_j \int_r^t \left(\tilde{\alpha}_\tau^{i,j}\right)^3 d\tau \, (x^i - x^j).$$

For the zeroth order in the expansion of the integral of the potential term, we have

$$S_2(\tilde{\alpha}) = \sum_{(i,j)\in\mathcal{I}^\Delta} \hat{\Gamma} m_i m_j \int_0^t \left[\tilde{\alpha}_r^{i,j} - \left(\tilde{\alpha}_r^{i,j}\right)^3\right] dr \, \frac{|x^i|^2 + |x^j|^2 - 2(x^i)^T x^j}{2}.$$

Now, we turn to the terminal cost. Recalling (5.7), we have

$$\begin{aligned}\phi(\xi_t) &= \tfrac{1}{2}\Big(\int_0^t u_\rho d\rho\Big)^T P_0 \Big(\int_0^t u_\rho d\rho\Big) + \big(x-z\big)^T P_0 \Big(\int_0^t u_\rho d\rho\Big) + \tfrac{1}{2}(x-z)^T P_0 (x-z) + \gamma_0 \\ &\doteq \tfrac{1}{2}\langle Q_3 u, u\rangle_{\mathcal{U}_{(0,t)}} + \langle R_3, u\rangle_{\mathcal{U}_{(0,t)}} + S_3,\end{aligned}$$

where $Q_3 \in \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$, $R_3 \in \mathcal{U}_{(0,t)}$, and $S_3 \in \mathbb{R}$. In particular, we have $[Q_3 u]_r = P_0 \int_0^t u_\rho \, d\rho$ and $[R_3]_r = P_0(x-z)$ for all $r \in (0,t)$ and $S_3 = \frac{1}{2}(x-z)^T P_0 (x-z) + \gamma_0$. Combining the terms, we have the asserted form for $J(t,x,u,\tilde{\alpha})$, where

$$f_1(\tilde{\alpha}) = S_2(\tilde{\alpha}) + S_3, \quad f_2(\tilde{\alpha}) = R_2(\tilde{\alpha}) + R_3, \text{ and } \bar{B}_3(\tilde{\alpha}) = Q_1 + Q_2(\tilde{\alpha}) + Q_3.$$

That $\bar{B}_3(\tilde{\alpha}) \in \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$ and $f_2(\tilde{\alpha}) \in \mathcal{U}_{(0,t)}$ is easily seen from the above expressions. The final assertion follows from the dominance of Q_3 when the minimal eigenvalue of P_0 is sufficiently large. ∎

THEOREM 5.6. *Let $t \in (0,\infty)$ and $x \in \mathbb{R}^{3n}$. Suppose $W(t,x)$ given by (5.2) exists. Let $\bar{\tilde{\alpha}}^{i,j} \in \tilde{\mathcal{A}}^{\bar{a}}_{(0,t)}$ be as in Lemma 5.2 for some $\bar{a} < \infty$, and let $D > |\bar{B}_3^{\#}(\bar{\tilde{\alpha}})|$. Let $\mathcal{A} \doteq \{\tilde{\alpha} \in \tilde{\mathcal{A}}^{\bar{a}}_{(0,t)} \,|\, |\bar{B}_3^{\#}(\tilde{\alpha})| < D\}$. Then*

$$W(t,x) = \operatorname*{stat}_{u\in\mathcal{B}} \operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}} J(t,x,u,\tilde{\alpha}) = \operatorname*{stat}_{(u,\tilde{\alpha})\in\mathcal{B}\times\mathcal{A}} J(t,x,u,\tilde{\alpha}) = \operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}} \operatorname*{stat}_{u\in\mathcal{B}} J(t,x,u,\tilde{\alpha}).$$

Proof. Note that for heuristic reasons, some technical derivative computations in this proof are delayed to Appendix B.

Fix $t \in (0,\infty)$ and $x \in \mathbb{R}^{3n}$. Note that by the conditions of Remark 5.1, $\mathcal{B}$ is open. By Lemma 5.5, $\bar{B}_3(\tilde{\alpha}) \in \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$ for all $\tilde{\alpha} \in \tilde{\mathcal{A}}^{\bar{a}}_{(0,t)}$, where this implies that all such $\bar{B}_3(\tilde{\alpha})$ are closed operators, and hence $[\bar{B}_3^{\#}(\tilde{\alpha})] \in \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$ exists for all $\tilde{\alpha} \in \tilde{\mathcal{A}}^{\bar{a}}_{(0,t)}$. Let $g : \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)}) \to \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$ be given by $g(B) \doteq B^{\#}$ for all $B \in \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)})$. Let D be as given and $\hat{D} \in (D,\infty)$. Let the open ball of radius D be denoted by $\mathcal{D}_D \doteq \{B \in \mathcal{L}(\mathcal{U}_{(0,t)};\mathcal{U}_{(0,t)}) \,|\, |B| < D\}$ and similarly for $\hat{D}$. Let $Q_D \doteq g^{-1}(\mathcal{D}_D)$ and $Q_{\hat{D}} \doteq g^{-1}(\mathcal{D}_{\hat{D}})$, and note that g is continuous on $Q_{\hat{D}}$ [11, 21]. Hence, Q_D is open, and as $\bar{B}_3(\cdot)$ is continuous, we find that $\mathcal{A} = (\bar{B}_3)^{-1}(Q_D)$ is open. The first asserted equality then follows from Lemma 5.2. Further, this implies that assumption (A.2i) is satisfied by the expression on the right-hand side of the first equality. Hence, if the conditions of section 4.3 are met, then Theorem 4.18 will yield the second equality. In this case here, the Morse condition of section 4.3 is that for all $(\tilde{\alpha},u) \in \mathcal{A}\times\mathcal{B}$, $D^2_{\tilde{\alpha}}J(t,x,u,\tilde{\alpha}) \in \mathcal{L}(\tilde{\mathcal{A}}^B_{(0,t)};\tilde{\mathcal{A}}^B_{(0,t)})$ is invertible with locally bounded inverse. From Lemma B.2, the differential $D^2_{\tilde{\alpha}}J(t,x,u,\tilde{\alpha})\gamma$ for $\gamma \in \mathcal{A}^B_{(0,t)}$ has representation with components given by

$$[\nabla^2_{\tilde{\alpha}}J(t,x,u,\tilde{\alpha})\gamma]^{i,j}_r = -3\hat{\Gamma}m_im_j\tilde{\alpha}^{i,j}_r|\xi^i_r-\xi^j_r|^2\gamma^{i,j}_r \ \ \forall\,(i,j)\in\mathcal{I}^\Delta, \text{ a.e. } r\in(0,t).$$

As $\tilde{\alpha}^{i,j}_r, |\xi^i_r-\xi^j_r| > 0$ for all $(i,j)\in\mathcal{I}^\Delta$ and $r \in (0,t)$, one finds that operator $D^2_{\tilde{\alpha}}J(t,x,u,\tilde{\alpha})$ is indeed invertible with locally bounded inverse for all $(\tilde{\alpha},u)\in\mathcal{A}\times\mathcal{B}$. Lastly, noting the representation given in Lemma B.3, one may easily show that $D^2_{u,\tilde{\alpha}}J(t,x,u,\tilde{\alpha})$ is bounded on bounded sets. Hence, the conditions of section 4.3 are met, and one may apply Theorem 4.18 to obtain the second equality.

Note that the second equality also implies that the expression on the right-hand side of that equality satisfies assumption (A.2f). If the conditions of Theorem 4.9 are satisfied, we will have the final equality. It is sufficient to show that, as a function of $(\tilde{\alpha},u)\in\mathcal{A}\times\mathcal{B}$, $J(t,x,u,\tilde{\alpha})$ satisfies the conditions of section 4.2. That is, suppressing the dependence on (t,x), we must have

$$J(t,x,u,\tilde{\alpha}) = f_1(\tilde{\alpha}) + \langle f_2(\tilde{\alpha}),u\rangle_{\mathcal{U}_{(0,t)}} + \tfrac{1}{2}\langle\bar{B}_3(\tilde{\alpha})u,u\rangle_{\mathcal{U}_{(0,t)}}$$

with $f_1, f_2, \bar{B}_3$ satisfying the conditions indicated there. From Lemma 5.5, we see that $f_1, f_2, \bar{B}_3$ are C^2 with Range$[\bar{B}_3(u)] = \mathcal{U}_{(0,t)}$ and that $\bar{B}_3^{\#}(\tilde{\alpha})$ exists and is uniformly bounded over $\mathcal{A}$. The result follows from Theorem 4.9. ∎

Remark 5.7. It should be noted that the assertions of Theorem 5.6 allow the staticization problem of (5.2) to be reduced to staticization over the set of DRE solutions and integrals, $\mathcal{P} \doteq \{(P^{\tilde{\alpha}}_t, Q^{\tilde{\alpha}}_t, R^{\tilde{\alpha}}_t, \gamma^{\tilde{\alpha}}_t)\,|\,\tilde{\alpha}\in\mathcal{A}\}$, as noted in (5.8). In cases where the terminal cost, ϕ (indexed by z), has been constructed so that the staticization problems correspond to TPBVPs, the set $\mathcal{P}$ provides a fundamental solution object for a set of TPBVPs. One may see [9, 10, 19] for more detailed discussions regarding the calculations.

Appendix A. A mean value theorem. For ease of reading, we recall a version of the mean value theorem from [1, Theorem 12.6].

THEOREM A.1. *Let $\mathcal{U},\mathcal{V}$ denote Banach spaces, and let $f:\mathcal{D}\to\mathcal{V}$ where $\mathcal{D}\subseteq\mathcal{U}$. Suppose $u_1,u_2\in\mathcal{D}$ are such that $\hat{u}(\lambda)\doteq\lambda u_1+(1-\lambda)u_2\in\mathcal{D}$ for all $\lambda\in[0,1]$. Suppose f is continuous at u for all $u\in\{\hat{u}(\lambda)\,|\,\lambda\in[0,1]\}$ and f is differentiable at u for all $u\in\{\hat{u}(\lambda)\,|\,\lambda\in(0,1)\}$. Then there exists $\bar{\lambda}\in(0,1)$ such that $|f(u^1)-f(u^1)|\le |Df(\hat{u}(\bar{\lambda}))|\,|u^1-u^2|$.*

Appendix B. Calculation of derivatives. We begin by indicating some notation and recalling standard results; cf. [1]. Let $f:\mathcal{U}_{(0,t)}\times\tilde{\mathcal{A}}^B_{(0,t)}\to\mathbb{R}$ satisfy

$f(u,\cdot) \in C^2(\tilde{\mathcal{A}}^B_{(0,t)};\mathbb{R})$, $f(\cdot,\tilde{\alpha}) \in C^2(\mathcal{U}_{(0,t)};\mathbb{R})$ for all $u \in \mathcal{U}_{(0,t)}$, and $\tilde{\alpha} \in \tilde{\mathcal{A}}^B_{(0,t)}$. Let $D_u f : \mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^B_{(0,t)} \to \mathcal{L}(\mathcal{U}_{(0,t)};\mathbb{R})$ and $D_{\tilde{\alpha}} f : \mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^B_{(0,t)} \to \mathcal{L}(\tilde{\mathcal{A}}^B_{(0,t)};\mathbb{R})$ denote the Fréchet derivatives with respect to u and $\tilde{\alpha}$, respectively. Note that we have $[D_u f(u,\tilde{\alpha})]\delta_u \in \mathbb{R}$, $[D_\alpha f(u,\tilde{\alpha})]\delta_{\tilde{\alpha}} \in \mathbb{R}$ $\forall \delta_u \in \mathcal{U}_{(0,t)}$, $\delta_{\tilde{\alpha}} \in \tilde{\mathcal{A}}^B_{(0,t)}$. By the Riesz representation theorem, for each $\hat{u} \in \mathcal{U}_{(0,t)}$ and $\hat{\tilde{\alpha}} \in \tilde{\mathcal{A}}^B_{(0,t)}$, there exists unique $\nabla_u f(\hat{u},\hat{\tilde{\alpha}}) \in \mathcal{U}_{(0,t)}$ such that $D_u f(\hat{u},\hat{\tilde{\alpha}})\delta_u = \langle \delta_u, \nabla_u f(\hat{u},\hat{\tilde{\alpha}})\rangle_{\mathcal{U}_{(0,t)}}$ $\forall\, \delta_u \in \mathcal{U}_{(0,t)}$.

For $L \in L_2((0,t);\mathcal{M}_{\mathbb{R}})$ and $\gamma \in \tilde{\mathcal{A}}^B_{(0,t)}$, define the continuous, bilinear functional $\langle L,\gamma\rangle_2 = \langle \gamma, L\rangle_2 \doteq \sum_{(i,j)\in\mathcal{I}^\Delta} \int_0^t L_r^{i,j}\gamma_r^{i,j}\,dr$. Note that $\nabla_{\tilde{\alpha}} f(\hat{u},\hat{\tilde{\alpha}}) : \mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^B_{(0,t)} \to \tilde{\mathcal{A}}^B_{(0,t)}$ is a representation of $D_\alpha f(\hat{u},\hat{\tilde{\alpha}})\delta_{\tilde{\alpha}}$ everywhere in $\mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^B_{(0,t)}$ if $\langle \nabla_{\tilde{\alpha}} f(\hat{u},\hat{\tilde{\alpha}}), \delta_{\tilde{\alpha}}\rangle_2 = D_\alpha f(\hat{u},\hat{\tilde{\alpha}})\delta_{\tilde{\alpha}}$ for all $\delta_{\tilde{\alpha}} \in \tilde{\mathcal{A}}^B_{(0,t)}$, $(\hat{u},\hat{\tilde{\alpha}}) \in \mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^B_{(0,t)}$.

Let $D^2_{\tilde{\alpha}} f : \mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^B_{(0,t)} \to \mathcal{L}(\tilde{\mathcal{A}}^B_{(0,t)}, \mathcal{L}(\tilde{\mathcal{A}}^B_{(0,t)},\mathbb{R}))$ denote the second Fréchet derivative with respect to $\tilde{\alpha}$. Note that for each $\delta_{\tilde{\alpha}} \in \tilde{\mathcal{A}}^B_{(0,t)}$ and pair $(\hat{u},\hat{\tilde{\alpha}})$, we have $D^2_{\tilde{\alpha}} f(\hat{u},\hat{\tilde{\alpha}})\delta_{\tilde{\alpha}} \in \mathcal{L}(\tilde{\mathcal{A}}^B_{(0,t)};\mathbb{R})$. Further, $D^2_{\tilde{\alpha}} f(\hat{u},\hat{\tilde{\alpha}})$ is the second Fréchet derivative with respect to $\tilde{\alpha}$ at $(\hat{u},\hat{\tilde{\alpha}})$ if $D^2_{\tilde{\alpha}} f(\hat{u},\hat{\tilde{\alpha}}) = D_{\tilde{\alpha}}[D_{\tilde{\alpha}} f](\hat{u},\hat{\tilde{\alpha}})$. Analogous definitions hold for second derivatives with respect to u.

We now proceed to obtain certain derivatives and Riesz representations employed in the proof of Theorem 5.6. Let $J : (0,t) \times \mathbb{R}^{3n} \times \mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^\infty_{(0,t)}$ be given by (5.4) with quadratic terminal cost (5.7).

Lemma B.1. *For any $t \in (0,\infty)$, $x \in \mathbb{R}^{3n}$, and $u \in \mathcal{U}_{(0,t)}$, $J(t,x,u,\cdot)$ is Fréchet differentiable over $\tilde{\mathcal{A}}^B_{(0,t)}$, and the Fréchet derivative has Riesz representation $\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$, where $\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$ acting on $\gamma \in \tilde{\mathcal{A}}^B_{(0,t)}$ is given by $\langle \nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}), \gamma\rangle_2$, and*

$$[\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})]_r^{i,j} = \hat{\Gamma} m_i m_j \Big[1 - \frac{3\big(\tilde{\alpha}_r^{i,j}\big)^2 |\xi_r^i - \xi_r^j|^2}{2}\Big] \;\; \forall\,(i,j) \in \mathcal{I}^\Delta, r \in (0,t). \tag{B.1}$$

Proof. Let $\gamma \in \tilde{\mathcal{A}}^B_{(0,t)}$, and let L denote the object indicated by the right-hand side of (B.1). With a small amount of algebra, one finds

$$\begin{aligned}
&|J(t,x,u,\tilde{\alpha}+\gamma) - J(t,x,u,\tilde{\alpha}) - \langle L,\gamma\rangle_2| \\
&= \left| \hat{\Gamma} \sum_{(i,j)\in\mathcal{I}^\Delta} \int_0^t \frac{-m_i m_j}{2} \big[3\tilde{\alpha}_r^{i,j}\big(\gamma_r^{i,j}\big)^2 + \big(\gamma_r^{i,j}\big)^3\big] |\xi_r^i - \xi_r^j|^2\,dr \right| \\
&\le \hat{\Gamma} \sum_{(i,j)\in\mathcal{I}^\Delta} \frac{m_i m_j}{2} \int_0^t \big(1 + 3\tilde{\alpha}_r^{i,j}\big) |\xi_r^i - \xi_r^j|^2\,dr \sup_{r\in(0,t)} \big[\big|\gamma_r^{i,j}\big|^2 + \big|\gamma_r^{i,j}\big|^3\big],
\end{aligned}$$

which, for appropriate choice of $K_0(t,x,u,\tilde{\alpha}) < \infty$ and $|\gamma| \le 1$,

$$\le K_0(t,x,u,\tilde{\alpha})|\gamma|^2,$$

which implies that the Fréchet derivative $D_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$ exists and has the indicated Riesz representation. ∎

Lemma B.2. *For any $t \in (0,\infty)$, $x \in \mathbb{R}^{3n}$, and $u \in \mathcal{U}_{(0,t)}$, the second-order Fréchet derivative $D^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$ exists for all $\tilde{\alpha} \in \mathcal{A}_{(0,t)}$, and the differential has representation $\nabla^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})\gamma$, which, for all $\gamma \in \tilde{\mathcal{A}}^B_{(0,t)}$, is given by*

$$[\nabla^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})\gamma]_r^{i,j} = -3\hat{\Gamma} m_i m_j \tilde{\alpha}_r^{i,j} |\xi_r^i - \xi_r^j|^2 \gamma_r^{i,j} \;\; \forall\,(i,j) \in \mathcal{I}^\Delta, \;\; a.e.\ r \in (0,t).$$

Proof. Recalling the above discussion, we obtain the second-derivative representation by examining the Fréchet derivative of $\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$. Let t,x,u be as specified, and take $\tilde{\alpha} \in \mathcal{A}_{(0,t)}$. Let $\gamma \in \tilde{\mathcal{A}}^B_{(0,t)}$, and let $[T\gamma]^{i,j}_r \doteq -3\hat{\Gamma} m_i m_j \tilde{\alpha}^{i,j}_r |\xi^i_r - \xi^j_r|^2 \gamma^{i,j}_r$ for all $i,j \in]1,n[$ and $r \in (0,t)$, where $\xi^i_r = x^i + \int_0^r u_\rho \, d\rho$. Note that

$$
\begin{aligned}
&|\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}+\gamma) - \nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}) - [T\gamma]| \\
&= \bigg[\sum_{(i,j)\in\mathcal{I}^\Delta} \int_0^t \bigg| [\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}+\gamma)]^{i,j}_r \\
&\quad - [\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})]^{i,j}_r + 3\hat{\Gamma} m_i m_j \tilde{\alpha}^{i,j}_r |\xi^i_r - \xi^j_r|^2 \gamma^{i,j}_r \bigg|^2 dr \bigg]^{\frac{1}{2}},
\end{aligned}
$$

which, by (B.1),

$$
\begin{aligned}
&= \bigg[\hat{\Gamma} \sum_{(i,j)\in\mathcal{I}^\Delta} \int_0^t \Big| \frac{-3}{2} m_i m_j \big[2\tilde{\alpha}^{i,j}_r \gamma^{i,j}_r + (\gamma^{i,j}_r)^2 \big] \big| \xi^i_r - \xi^j_r \big|^2 \\
&\qquad\qquad\qquad + 3 m_i m_j \tilde{\alpha}^{i,j}_r |\xi^i_r - \xi^j_r|^2 \gamma^{i,j}_r \Big|^2 dr \bigg]^{\frac{1}{2}} \\
&= \bigg[\hat{\Gamma} \sum_{(i,j)\in\mathcal{I}^\Delta} \frac{9}{4} m_i^2 m_j^2 \int_0^t \Big| (\gamma^{i,j}_r)^2 \big|\xi^i_r - \xi^j_r\big|^2 \Big|^2 dr \bigg]^{\frac{1}{2}} \\
&\le \hat{\Gamma} \sum_{(i,j)\in\mathcal{I}^\Delta} \frac{9}{4} m_i^2 m_j^2 \Big(\int_0^t \big|\xi^i_r - \xi^j_r\big|^4 \, dr \Big)^{\frac{1}{2}} \sup_{r\in(0,t)} \big|\gamma^{i,j}_r\big|^2 \le K_1 |\gamma|^2
\end{aligned}
$$

for appropriate choice of $K_1 = K_1(t,x,u) < \infty$, and this yields the result. ∎

The following is obtained in a similar manner to Lemma B.1, and the proof is not included.

Lemma B.3. *For any $t \in (0,\infty)$ and $x \in \mathbb{R}^{3n}$, $J(t,x,\cdot,\cdot) : \mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^\infty_{(0,t)} \to \mathbb{R}$ has a mixed second partial Fréchet derivative, and this derivative, evaluated at $(u,\tilde{\alpha}) \in \mathcal{U}_{(0,t)} \times \tilde{\mathcal{A}}^\infty_{(0,t)}$, $D^2_{u,\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$, has a representation comprised of the Riesz representations of the derivatives of $[\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})]^{i,j}_\cdot$ with respect to u for $(i,j) \in \mathcal{I}^\Delta$. More specifically, for $\delta_u \in \mathcal{U}_{(0,t)}$ and $\delta_{\tilde{\alpha}} \in \tilde{\mathcal{A}}^\infty_{(0,t)}$,*

$$
\begin{aligned}
\big[D^2_{u,\tilde{\alpha}} J(t,x,u,\tilde{\alpha}) \delta_{\tilde{\alpha}} \big] \delta_u &= \Big\langle \nabla^2_{u,\tilde{\alpha}} J(t,x,u,\tilde{\alpha}) \delta_{\tilde{\alpha}}, \delta_u \Big\rangle_{\mathcal{U}_{(0,t)}} \\
&= \sum_{k\in\mathcal{N}} \int_0^t \big[\nabla^2_{u,\tilde{\alpha}} J(t,x,u,\tilde{\alpha}) \delta_{\tilde{\alpha}} \big]^k_\rho [\delta_u]^k_\rho \, d\rho,
\end{aligned}
$$

where

$$
\big[\nabla^2_{u,\tilde{\alpha}} J(t,x,u,\tilde{\alpha}) \delta_{\tilde{\alpha}} \big]^k_\rho = \sum_{(i,j)\in\mathcal{I}^\Delta} \int_0^t \big[[\nabla_{\tilde{\alpha},u} J(t,x,u,\tilde{\alpha})]^{i,j}_r \big]^k_\rho [\delta_{\tilde{\alpha}}]^{i,j}_r \, dr \quad \forall\, k \in \mathcal{N},
$$

$\rho \in (0,t)$,

$$
\big[[\nabla_{\tilde{\alpha},u} J(t,x,u,\tilde{\alpha})]^{i,j}_r \big]^k_\rho \doteq \begin{cases} -3\hat{\Gamma} m_i m_j (\tilde{\alpha}^{i,j}_r)^2 (\xi^i_r - \xi^j_r) \mathcal{I}_{(0,r)}(\rho) & \text{if } k = i, \\ 3\hat{\Gamma} m_i m_j (\tilde{\alpha}^{i,j}_r)^2 (\xi^i_r - \xi^j_r) \mathcal{I}_{(0,r)}(\rho) & \text{if } k = j, \\ 0, & \text{otherwise} \end{cases}
$$

for all $r,\rho \in (0,t)$, $k \in \mathcal{N}$, and $(i,j) \in \mathcal{I}^\Delta$, and we recall that $\mathcal{I}_{(0,r)}(\cdot)$ denotes the indicator function on set $(0,r)$.

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