Densely computable structures

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Abstract

In recent years, computability theorists have extensively studied generically and coarsely computable sets. This study of approximate computability was originally motivated by asymptotic density problems in combinatorial group theory. We generalize the notions of generic and coarse computability of sets, introduced by Jockusch and Schupp, to arbitrary structures by defining generically and coarsely computable and computably enumerable structures. There are two directions in which these notions could potentially trivialize: either all structures could have a densely computable copy or only those having a computable (or computably enumerable) copy. We show that some particular classes of structures realize each of these extremal conditions, while other classes realize neither of them. To further explore these concepts, we introduce a graded family of elementarity conditions for substructures, in which we require that the dense sets under consideration be 'strong' substructures of the original structure. Here, again, for a given class, the notion could trivialize in the same two directions and we show that both are possible. For each class that we investigate, there is some natural number n such that requiring Σ_n elementarity of substructures is enough to trivialize the class of generically or densely computable structures, witnessing the essentially structural character of these notions.

Keywords: Computability theory, computable model theory, injection structures, equivalence structures, asymptotic density, dense computability, generic computability, coarse computability

1 Introduction and preliminaries

Results about complexity of problems in computable structure theory typically depend on the behaviour of the hardest instances of the problem, which are not always common. One such motivating problem, the word problem, came from algebra. A standard construction of a finitely presented group with undecidable word problem [13] involves not just getting the right example of a group; the particular words within this group on which it is difficult to decide equality to the identity are very special words (and are even called by this term in some expositions).

The idea of studying generic properties of finitely presented groups goes back to Gromov's 1987 seminal paper [8] on hyperbolic groups. Gromov [7] further developed this idea by introducing the density model of random groups, where a certain density parameter controlled the number of defining relations put in a random group. Further research on generic group-theoretic properties was carried on by Olshanskii, Champetier, Arzhantseva, Zuk, P.-A. Cherix and others.

In 2003, Kapovich, Myasnikov, Schupp and Shpilrain [11] investigated generic-case complexity of decision problems in group theory. They showed that for a very large class of finitely generated

Vol. 32, No. 3,
Advance Access Publication on 24 September 2021 https://doi.org/10.1093/logcom/exab057

groups, the classical decision problems, such as the word problem, the conjugacy problem and the subgroup membership problem, have linear-time generic-case complexity.

In 2012, Jockusch and Schupp [10] introduced two notions of approximate computability, generically computable and coarsely computable sets. Roughly speaking, a generically computable set has a computable approximation that almost always gives an answer and is always correct; a coarsely computable set has a computable approximation that always gives an answer and is almost always correct. More precisely, a set is *generically computable* if there is a partial computable function with values in $\{0, 1\}$, the description of the set, with the domain of density one on which the set agrees with the function. A set is *coarsely computable* if there is a (total) computable function with values in $\{0, 1\}$ such that the set agrees with the function on a subset of density one. These two notions are distinct from each other. It was shown that they are incomparable even when restricted to computably enumerable sets. Moreover, Jockusch and Schupp showed that every nonzero Turing degree contains a set that is neither coarsely computable nor generically computable.

A structure is *computable* if its domain is a computable set and its relations and functions are uniformly computable. In this paper, we introduce and investigate generically computable and coarsely computable structures. In each case, the question is whether some 'large' substructure is relatively computable. Here, again, 'large' is in the natural density sense to be precisely defined below. Assume that all structures are countable with domain ω. As usual, we will abbreviate computably enumerable by c.e. As set of natural numbers is c.e. if and only if it is the domain of a partial computable function. Roughly speaking, a structure with an r-ary relation R is generically computable if there is a substructure with a c.e. universe D of density one, such that there exists a partial computable r-ary function ϕ with values in $\{0, 1\}$ such that ϕ agrees with R on D^r.

The *word problem* for a finitely presented group G, where

$$G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_m \rangle,$$

asks for an algorithm to decide whether a word w in $\{g_1, \ldots, g_k, g_1^{-1}, \ldots, g_k^{-1}\}$ represents the identity element of G. The classical theorem of Novikov and Boone establishes that there exists a finitely presented group with undecidable word problem. A group is computable if its domain is computable and the group operation is computable. It is not hard to prove that a finitely presented group has a computable isomorphic copy if and only if it has a decidable word problem. Kapovich, Myasnikov, Schupp and Shpilrain [11] established that a finitely presented group with undecidable word problem, given by Boone, has, in our sense, a generically computable copy.

There are, roughly, two extremal possibilities for structures in general.

(1) Every countable structure has a generically computable copy; or
(2) Every countable structure with a generically computable copy has a computable copy.

There are also analogous results for coarsely computable structures. We will show that each of these can be achieved in certain classes and that they do not exhaust all possibilities. We will also explore these conditions under the added hypothesis that the 'large' substructures in question be, in some weak sense, elementary. Again, we find that there are natural extremal possibilities and that both they and non-extremal cases are achieved. Finally, we find that as the elementarity hypotheses are strengthened, all known cases eventually (i.e. when we have Σ_n elementarity for sufficiently large n) trivialize. This demonstrates that these notions of dense computability are structural—they depend fundamentally on the semantics of the structure, and not only on the density or algorithmic features of the presentation.

It would be interesting to consider whether any appropriate class of structures (perhaps with bounded Scott rank or some similar condition) would trivialize at some level, but we do not yet have a solution for this general problem. To our thinking, this recalls the feature of computable categoricity by which every structure with a $\Pi_{\alpha+1}$ Scott sentence is Δ^0_α-categorical [12].

While we are concerned with initiating a systematic study of generic and coarse computability within computable structure theory, in computability theory in recent years, the study of generically and coarsely computable sets and related notions has led to a rich and interesting program of research; see [9] for a partial survey. Andrews, Astor, Cai, Cholak, Diamondstone, Downey, Hirschfeldt, Igusa, Jockusch, Kuyper, Lempp, McNicholl, Shupp and other researchers studied several computability-theoretic aspects of various notions of approximate computability.

In [6], Downey, Jockusch and Schupp studied some variations of the notions of generically and coarsely computable sets, e.g. whether a set contains a density one computable subset, i.e. has a computable approximation that always gives an answer, is almost always correct and is always correct when the answer is positive. In particular, they showed that every c.e. set can be approximated by a computable subset with arbitrarily close lower density. They also showed that a c.e. Turing degree is non-low (i.e. its jump is strictly above the Turing degree of the halting set) if and only if it contains a c.e. set of density one with no density one computable subset.

Downey, Jockusch, McNicholl and Schupp [5] classified the asymptotic densities of Δ^0_2 (limit computable) sets according to their levels in the Ershov hierarchy, i.e. according to the number of changes in their computable approximations. They showed that, with respect to density, the Ershov hierarchy collapses in certain sense to levels $0, 1, 2$ and ω.

As stated before, our goal is to distinguish which results in computable structure theory depend on 'special' (and potentially extremely rare) cases, and which are less sensitive to them. To achieve this goal in the context of decision problems on groups, Kapovich, Myasnikov, Schupp and Shpilrain [11] proposed using notions of asymptotic density to state whether a partial computable function could solve 'almost all' instances of a problem.

Jockusch and Schupp [10] generalized this approach to the broad context of computability theory in the following way.

DEFINITION 1.1
Let $S \subseteq \omega$.

(1) The density of S up to n, denoted by $\rho_n(S)$, is given by

$$\frac{|S \cap \{0, 1, 2, \dots, n\}|}{n+1}.$$

(2) The asymptotic density of S, denoted by $\rho(S)$, is given by $\lim_{n\to\infty} \rho_n(S)$.

A set S is *generically computable* if there is a partial computable function $\varphi : \omega \to \{0, 1\}$, such that $\varphi = c_S$ on the domain of φ, and such that the domain of φ has asymptotic density 1.

A set S is *coarsely computable* if there is a *total* computable function $h : \omega \to \{0, 1\}$ such that h agrees with c_S on a set of asymptotic density 1. Equivalently, there is a computable set H such that the symmetric difference $S \triangle H$ has asymptotic density 0.

We will need the following result from [10].

THEOREM 1.2 (Jockusch–Schupp).
There is a generically computable set that is not coarsely computable, and there is a coarsely computable set that is not generically computable.

For instance, the word problem for finitely generated groups is not generically computable, while it is coarsely computable.

We would like to extend the notions of being generically computable and coarsely computable from sets to structures. Assume that $\mathcal{A}$ is a given structure with universe ω and with finitely many functions $\{f_i : i \in I\}$, each f_i of arity p_i, and finitely many relations $\{R_j : j \in J\}$, each R_j of arity r_j. We would like to define what it means for $\mathcal{A}$ to be generically computable, or 'nearly computable' with respect to some other notion related to density. The idea is that $\mathcal{A}$ is *generically computable* if there is a substructure $\mathcal{D}$ with a c.e. universe D that has asymptotic density one, for which there exist partial computable functions $\{\phi_i : i \in I\}$ and $\{\psi_j : j \in J\}$ such that ϕ_i agrees with f_i on D^{p_i}, and ψ_j agrees with c_{R_j} on D^{r_j}. Similarly, a structure $\mathcal{A}$ is *coarsely computable* if there is a computable structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with universe D is a substructure of both $\mathcal{A}$ and of $\mathcal{E}$ and all relations and functions agree on D. A more interesting variant of this notion requires that $\mathcal{D}$ is a Σ_1 elementary submodel of $\mathcal{A}$, and, more generally, a Σ_n elementary submodel of $\mathcal{A}$. That is, if we say that $\mathcal{A}$ is 'nearly computable' when it has a dense substructure $\mathcal{D}$ that is c.e., then the substructure should be similar to $\mathcal{A}$ by some model-theoretic criterion.

To be precise, we say that $\mathcal{D}$ is a Σ_n *elementary substructure (submodel)* of $\mathcal{A}$ if for any infinitary Σ_n formula $\theta(x_1, \ldots, x_n)$ and any n-tuple of elements $d_1, \ldots, d_n \in D$, we have

$$\mathcal{A} \models \theta(d_1, \ldots, d_n) \Leftrightarrow \mathcal{D} \models \theta(d_1, \ldots, d_n).$$

We are aware of the slight tension in using the term 'elementary' to refer to $L_{\omega_1\omega}$ properties, but believe the term to be justified by its usage; at issue is the condition that the substructure should satisfy the same formulas as the superstructure.

We note that being a Σ_0 elementary substructure is the same as being a substructure since $\mathcal{B}$ is a substructure of $\mathcal{A}$ if and only if the domain of $\mathcal{B}$ is contained in the domain of $\mathcal{A}$, and $\mathcal{A}$ and $\mathcal{B}$ satisfy the same quantifier-free sentences with constants from $\mathcal{B}$. Moreover, the classical and infinitary notions of elementarity coincide at the Σ_1 level.

We say that the structure $\mathcal{A}$ is Σ_n*-generically c.e.* if there is an asymptotically dense c.e. set D such that:

(a) $\mathcal{D}$ is a Σ_n elementary substructure of $\mathcal{A}$;
(b) There exist partial computable functions $\{\phi_i : i \in I\}$ such that ϕ_i agrees with f_i on D^{p_i};
(c) Each R_j restricted to D^{r_j} is a c.e. relation.

We similarly define the notion of a Σ_n coarsely c.e. structure.

Example 1.3

Let $\mathcal{A} = (A, E)$ be a countable directed graph consisting of infinitely many finite chains of distinct lengths. Let $C(\mathcal{A})$ be the set of lengths of the chains. The structure $\mathcal{A}$ is c.e. if A is a c.e. set and E is a c.e. relation. For a c.e. structure $\mathcal{A}$, $C(\mathcal{A})$ will be a Σ^0_2 set. Then $\mathcal{A}$ is generically computable if there is an asymptotically dense c.e. set D such that a partial computable function agrees with c_E on D.

We will also be interested in the question of whether a structure $\mathcal{A}$ has a generically computable copy and, more generally, a Σ_n-generically c.e. copy. In this example, we will show that any such structure $\mathcal{A}$ will have a generically computable copy. Build the generically computable copy as follows. Let $D = \{d_0 < d_1 < \cdots\}$ be an asymptotically dense, co-infinite computable set and put edges from d_{2n} to d_{2n+1} for each n. Then use $\omega \setminus D$ to fill out the needed $(c_n - 1)$ vertices at the front of each chain to obtain a copy of $\mathcal{A}$.

Suppose now that $\mathcal{D}$ is a Σ_1 elementary substructure of such a graph $\mathcal{A}$. Then for each $a \in A$, the chain containing a must be included in $\mathcal{D}$. For example, if a is in the chain $a_0 E a E a_2 E a_3$, then $\mathcal{A} \models (\exists x) x E a$. Thus $\mathcal{D} \models (\exists x) x E a$ and, therefore, $a_0 \in D$. Similarly, $\mathcal{A} \models (\exists y)(\exists z)(a E y \wedge y E z)$ and, therefore, a_2 and a_3 must be in $\mathcal{D}$. Thus, a structure $\mathcal{A}$ will be Σ_1-generically c.e. if there is an asymptotically dense set c.e. set D such that $\mathcal{D} = (D, E)$ is a Σ_1-elementary substructure of $\mathcal{A}$, and $E \cap (D \times D)$ is a c.e. relation of $\mathcal{A}$. Then the structure $\mathcal{A}$ will have a Σ_1-generically c.e. copy if and only if there exist $C \subseteq C(\mathcal{A})$ and a c.e. structure $\mathcal{D}$ with $C(\mathcal{D}) = C$.

Finally, suppose that $\mathcal{D} = (D, E)$ is a Σ_2 elementary substructure of $\mathcal{A}$. This will imply that $C(\mathcal{D}) = C(\mathcal{A})$ and hence $D = A$. It follows that $\mathcal{A}$ is Σ_2-generically c.e. if and only if $\mathcal{A}$ is a c.e. structure. Thus, a structure $\mathcal{A}$ has a Σ_2-generically c.e. copy if and only if it has a c.e. copy.

EXAMPLE 1.4

Fix a prime p, and consider a p-group $\mathcal{A} = \oplus_{n \in C} \mathbb{Z}(p^n)$ for some infinite set C. If $\mathcal{A}$ is computable, then C is a Σ_2^0 set and, furthermore, C has a computable Khisamiev s_1-function, the details of which are given below in Section 4. Conversely, for any Σ_2^0 set C with a computable s_1-function, there is such a computable structure $\mathcal{A}$ isomorphic to $\oplus_{n \in C} \mathbb{Z}(p^n)$.

Any such structure $\mathcal{A}$ will have a generically computable copy. Let $\mathcal{A} = \oplus_{i<\omega} \langle a_i \rangle$, where $o(a_i) = p^{n_i}$. Then consider the subgroup $\mathcal{B} = \oplus_{i<\omega} \langle p^{n_i - 1} a_i \rangle$, which is isomorphic to $\oplus_{i<\omega} \mathbb{Z}(p)$. We observe that $\mathcal{B}$ is not a Σ_1 elementary subgroup, since for each $n_i > 1$, the element $p^{n_i - 1}$ has height $(n_i - 1)$ in $\mathcal{A}$ but has height 1 in $\mathcal{B}$. The structure $\mathcal{B}$ has a computable copy, and we can construct a generically computable copy of $\mathcal{A}$ with the corresponding subgroup on an asymptotically dense set.

Suppose now that $\mathcal{D}$ is a Σ_1 elementary subgroup of $\mathcal{A}$. By $\chi(\mathcal{B})$ we denote the character of $\mathcal{B}$. Then $\chi(\mathcal{B}) \subseteq \chi(\mathcal{A})$. If $\mathcal{A}$ is Σ_1-generically c.e., then $\chi(\mathcal{A})$ has a Σ_2^0 subset that possesses a computable s_1-function. Thus if $\mathcal{A}$ has a Σ_1-generically c.e. copy, then C must have a Σ_2^0 subset with a computable s_1-function.

Finally, suppose that $\mathcal{B}$ is a Σ_2 elementary subgroup of $\mathcal{A}$. Then we claim that $\chi(\mathcal{B}) = \chi(\mathcal{A})$. To see this, let $n \in C$. Then in $\mathcal{A}$, there exists an a such that $o(a) = p^n$ and $\langle a \rangle$ is a pure subgroup of $\mathcal{A}$. However, this is a Σ_2^0 sentence, and therefore $\mathcal{B}$ also has such an element a. If $\{n_i : i < \omega\}$ is a set of distinct elements, then, in fact, $\mathcal{B} = \mathcal{A}$.

These notions prove quite interesting for certain families of structures. We will examine in some detail the notions of generically computable and coarsely computable structures, and the variations described above for injection structures and equivalence structures.

The outline of this paper is as follows. In Section 2, we present some background on asymptotic density and generalize the notion of generic computability from sets to structures. We show that a set $A \subseteq \omega$ has asymptotic density δ if and only if the set $A \times A$ has density δ^2 in $\omega \times \omega$. We show that there is a computable dense set $C \subseteq \omega \times \omega$ such that for any infinite c.e. set A, the product $A \times A$ is not a subset of C. These results guide us in our choice of the definition of a generically computable structure. We also introduce, for $n > 0$, a Σ_n-generically c.e. structure using the notion of a Σ_n elementary substructure.

In Section 3, we first present results about computable and c.e. injection structures including some complexity results about their orbits and characters. Then we establish results about injection structures that have generically computable and Σ_1-generically c.e. isomorphic copies. We show that an injection structure $\mathcal{A}$ has a generically computable copy if and only if it has an infinite orbit or its character has an infinite subset. We also show that an injection structure $\mathcal{A}$ has a Σ_1-generically c.e. copy if and only if it has a computable copy.

In Section 4, we present results about generically computable and Σ_n-generically c.e. equivalence structures. We obtain a surprising result that every equivalence structure $\mathcal{A}$ has a generically computable isomorphic copy. We further give a natural characterization of equivalence structures with Σ_n-generically c.e. isomorphic copies, in terms of the properties of their characters or of their infinite classes. In particular, we show that an equivalence structure $\mathcal{A}$ has a Σ_1-generically c.e. copy if and only if it has an infinite substructure that is isomorphic to a c.e. structure. We also show that $\mathcal{A}$ has a Σ_2-generically c.e. copy if and only if it has a c.e. copy. First, we extend an important lemma from [1] to show that any c.e. equivalence relation on a c.e. set, with no infinite equivalence classes and with unbounded character, has a computable Khisamiev s_1-function.

In Section 5, we introduce the notions of coarsely computable and Σ_n-coarsely c.e. structures. Our notion of a coarsely computable structure is a natural extension of the notion of a coarsely computable set. It follows that generically computable and coarsely computable structures are incomparable. While every generically computable injection structure has a coarsely computable copy, there is a generically computable injection structure that is not coarsely computable. We show that there are equivalence structures that have no Σ_1-coarsely c.e. copies and that there are injection structures that have no coarsely computable copies. For injection structures, a Σ_1-generically c.e. structure, or a Σ_1-coarsely c.e. structure, is always isomorphic to a computable structure. An equivalence structure is Σ_3-coarsely c.e. if and only if it has a c.e. copy.

2 Asymptotic density and generically computable structures

In this section, we provide some background on the notions of generically computable and coarsely computable sets. We extend these notions to structures by defining more general notions of generically computable and Σ_n-generically c.e. structures, and also of coarsely computable and Σ_n-coarsely c.e. structures. In subsequent sections, we will examine these notions when applied to injection structures and to equivalence structures.

The asymptotic density of a set $A \subseteq \omega$ is defined as follows.

DEFINITION 2.1

- The *upper asymptotic density* of A is $\limsup_n \frac{|(A \cap n)|}{n}$.
- The *lower asymptotic density* of A is $\liminf_n \frac{|(A \cap n)|}{n}$.
- The *asymptotic density* of A is $\lim_n \frac{|(A \cap n)|}{n}$, if this exists.
- We say that a set A is *dense* if its asymptotic density is 1.

It is easy to see that A has asymptotic density δ if and only if A has both upper and lower density δ; A has density 1 if and only if it has lower density 1; and A has density 0 if and only if it has upper density 0. Downey, Jockusch and Schupp [6] proved that there is a c.e. set of density 1 with no computable subset of nonzero density.

The following observation will be useful. Note that the set A has *upper density* 1 if and only if there is a sequence $n_0 < n_1 < \cdots$ such that $\lim_i \frac{|A \cap n_i|}{n_i} = 1$.

LEMMA 2.2
If A is a c.e. set with upper density 1, then A has a computable subset with upper density 1.

PROOF. Suppose that A is a c.e. set with upper density 1. Let $(A_s)_{s\in\omega}$ be a computable enumeration of A. Define computable sequences $n_0, n_1, n_2, \ldots$ and $s_0, s_1, s_2, \ldots$ as follows. Let $n_0 = s_0 = 0$.

Let s_1 be the least s such that, for some $n < s$, we have $|n \cap A_s| \geq \frac{1}{2}n$, and let n_1 be the least such n. Given n_k and s_k, let s_{k+1} be the least s such that, for some n with $n_k < n < s$, we have $|(n - n_k) \cap A_s| \geq \frac{2^{k+1}-1}{2^{k+1}}(n - n_k)$, and let n_{k+1} be the least such n. The computable dense set $B \subseteq A$ is defined so that, for each i, if $n_k \leq i < n_{k+1}$, then $i \in B \Leftrightarrow i \in A_{n_{k+1}}$. It follows from the construction that, for each k, the density of B in $\{i : i > n_k\}$ is at least $\frac{2^k-1}{2^k}$, so that B has upper density 1. □

In order to study binary relations and the corresponding structures, we need to look at notions such as generic computability for such relations.

LEMMA 2.3

Let $A \subseteq \omega$. Then A has asymptotic density δ if and only if $A \times A$ has asymptotic density δ^2 in $\omega \times \omega$. In particular, A is asymptotically dense in ω iff $A \times A$ is asymptotically dense in $\omega \times \omega$. More generally, if A has asymptotic density δ_A, and B has asymptotic density δ_B, then $A \times B$ has asymptotic density $\delta_A \cdot \delta_B$.

PROOF. Let $\delta_A(n) = \frac{|A \cap n|}{n}$ and let $\delta(n) = \frac{|(A \times A) \cap (n \times n)|}{n^2}$. Since $(A \times A) \cap (n \times n) = (A \cap n) \times (A \cap n)$, it follows that $|(A \times A) \cap (n \times n)| = |A \cap n|^2$ and hence $\delta(n) = \delta_A(n)^2$. If $\lim_n \delta_A(n) = \delta$ exists, then $\lim_n \delta(n) = \lim_n \delta_n(A)^2 = \delta^2$. Conversely, if $\lim_n \delta(n) = L = \delta^2$ exists, then $\lim_n \delta_A(n) = \lim_n \sqrt{\delta_n(A)} = \sqrt{L} = \delta$.

For the second part, let $\delta_A(n) = \frac{|A \cap n|}{n}$ and $\delta_B(n) = \frac{|B \cap n|}{n}$ and suppose that $\delta_A = \lim_n \delta_A(n)$ and $\delta_B = \lim_n \delta_n(B)$ both exist. Then $\delta(n) = |(A \times B) \cap (n \times n)| = \delta_A(n) \times \delta_B(n)$ so $\lim\limits_n \delta(n) = \delta_A \cdot \delta_B$ is the asymptotic density of $A \times B$. □

A similar result holds for the density of A^r in ω^r. On the other hand, we have the following result. Let $W_0, W_1, \ldots$ be an effective enumeration of all c.e. sets.

THEOREM 2.4

There is a computable dense set $C \subseteq \omega \times \omega$ such that for any infinite c.e. set $A \subseteq \omega$, the product $A \times A$ is not a subset of C.

PROOF. Define C as follows. For any pair (a, b) with $max\{a, b\} = m$, proceed as follows. For each $e < m$, look for the first element $n > 2^e$, which has been enumerated in W_e by stage m; call this n_e if it exists. Then put $(a, b) \in C$, unless either $a = n_e$ or $b = n_e$ for some $e < m$. If W_e is infinite, then it contains some element $n_e > 2^e$, which is the first to come into W_e at some stage s_e, and then there will be another $n \in W_e$ that is greater than s_e but (n_e, n) will not be in C. The set C is dense since there are at most i elements less than 2^i of the form n_e for any $e < i$, so C contains at least $(2^i - i)^2$ elements out of the 2^{2i} possible pairs up to 2^i. □

Considering Lemma 2.3 and Theorem 2.4, our definition of a generically computable structure with a binary relation calls for a dense set D in the domain so that the characteristic function on the relation agrees with some partial computable function restricted to $D \times D$, rather than with some partial computable function restricted to a dense set in $\omega \times \omega$. The most natural notion seems to require that the substructure with domain D resembles the given structure $\mathcal{A}$ by agreeing on certain sentences with constants from D, existential sentences in particular. Recall the following definition of elementary substructures.

DEFINITION 2.5
A substructure $\mathcal{B}$ of the structure $\mathcal{A}$ is said to be an *elementary* substructure, in symbols $\mathcal{B} \prec \mathcal{A}$, if for any first-order formula $\theta(x_1,\ldots,x_n)$ and any $b_1,\ldots,b_n \in \mathcal{B}$, we have $\mathcal{B} \models \theta(b_1,\ldots,b_n) \Leftrightarrow \mathcal{A} \models \theta(b_1,\ldots,b_n)$.

Let $n \geq 0$. The substructure $\mathcal{B}$ is a Σ_n *elementary substructure*, in symbols $\mathcal{B} \prec_n \mathcal{A}$, if for any infinitary Σ_n formula $\theta(x_1,\ldots,x_n)$ any $b_1,\ldots,b_n \in \mathcal{B}$, we have $\mathcal{B} \models \theta(b_1,\ldots,b_n) \Leftrightarrow \mathcal{A} \models \theta(b_1,\ldots,b_n)$.

DEFINITION 2.6
A structure $\mathcal{D}$ for a finite language and with universe D is a *c.e. structure* if D is c.e., each relation is c.e. and each function is the restriction of a partial computable function to D (hence the partial computable function is total on D).

DEFINITION 2.7

(1) A structure $\mathcal{A}$ is *generically computable* if there is a substructure $\mathcal{D}$ with universe a c.e. dense set D such that for every k-ary function f and every k-ary relation R, both $f \restriction D^k$ and $c_R \restriction D^k$ are restrictions to D^k of some partial computable functions.
(2) Let $n \geq 0$. A structure $\mathcal{A}$ is Σ_n*-generically c.e.* if there is a c.e. dense set D such that the substructure $\mathcal{D}$ with universe D is a c.e. substructure and also a Σ_n elementary substructure of $\mathcal{A}$.

The condition that $\mathcal{A}$ is a Σ_0-generically c.e. structure is equivalent to the condition that $\mathcal{A}$ is generically computable. Clearly, any Σ_{n+1}-generically c.e. structure is Σ_n-generically c.e.

EXAMPLE 2.8
Consider a structure of the form $\mathcal{A} = (\omega, A)$, where A is a unary relation. By c_A we will denote the characteristic function of A. First, suppose that A is a generically computable set. Let ϕ be a partial computable function such that $D = dom(\phi)$ is a dense c.e. set and, for every $x \in D$, $\phi(x) = c_A(x)$. Then the substructure $\mathcal{D} = (D, A \cap D)$ can be considered a c.e. substructure of $\mathcal{A}$ since ϕ is total on the set D, and, therefore, $\mathcal{A}$ is a generically computable structure. On the other hand, suppose that $\mathcal{A}$ has a substructure $\mathcal{D} = (D, A \cap D)$ where D is a c.e. dense set such that there is a partial computable function ϕ that agrees with c_A on D. The restriction of ϕ to D gives the description of A establishing that the set A is generically computable. Similarly, we can show that $\mathcal{A}$ is Σ_1-generically c.e. if and only if there is a dense c.e. set D such that $A \cap D$ is an infinite c.e. set.

In the following sections, we will study generic computability of specific families of structures, such as injection structures and equivalence structures and will also investigate the notion of a coarsely computable structure.

3 Generically computable and Σ_1-generically c.e. injection structures

We will now focus on injection structures and their dense computability.

DEFINITION 3.1
An *injection structure* $\mathcal{A}$ is a set A together with a one-to-one function $f : A \to A$.

Without loss of generality, we may assume that $A \subseteq \omega$. Recall that $\mathcal{A} = (A,f)$ is computable if A is computable and f is computable, and $\mathcal{A}$ is c.e. if A is c.e. and f is the restriction of a partial computable function to A.

Let $a \in A$. The orbit $\mathcal{O}_f(a)$ of a under f is

$$\mathcal{O}_f(a) = \{x : (\exists n \in \omega)[x = f^{(n)}(a) \ \vee \ a = f^{(n)}(x)]\}.$$

Orbits are either finite or infinite. Infinite orbits may be of type $\mathbb{Z}$ where $\mathcal{O}_f(a) = \{\dots, f^{-2}(a), f^{-1}(a), a, f(a), f^2(a), \dots\}$ or of type ω where for some b not in the range of f, $\mathcal{O}_f(a) = \{b, f(b), f^{(2}(b), \dots\}$. The character of $\mathcal{A}$ is

$$\chi(\mathcal{A}) = \{(k,n) \in (\omega \setminus \{0\}) \times (\omega \setminus \{0\}) : \mathcal{A} \text{ has at least } n \text{ orbits of size } k\}.$$

DEFINITION 3.2
A set $K \subseteq (\omega \setminus \{0\}) \times (\omega \setminus \{0\})$ is said to be a *character* if, for all k and $n \geq 1$, $(k, n+1) \in K$ implies $(k,n) \in K$.

It is easy to see that K is a character if and only if $K = \chi(\mathcal{A})$ for some injection structure $\mathcal{A}$.

Computable and c.e. injection structures were investigated by the authors together with Morozov [2] and by Cenzer, Harizanov and Remmel [4], where the following results were shown. By $card(X)$ or $|X|$ we will denote the cardinality of X.

LEMMA 3.3
For any c.e. injection structure $\mathcal{A}$, we have the following properties.

(1) $\{(a,k) : a \in ran(f^{(k)})\}$ is a c.e. set.
(2) $\{(a,k) : card(\mathcal{O}_{f(a)}) \geq k\}$ is a c.e. set.
(3) $\{a : \mathcal{O}_f(a)$ is infinite$\}$ is the intersection of a Π^0_1 set with A.
(4) $\{a : \mathcal{O}_f(a)$ has type $\mathbb{Z}\}$ is a Π^0_2 set.
(5) $\{a : \mathcal{O}_f(a)$ has type $\omega\}$ is a Σ^0_2 set.
(6) $\chi(\mathcal{A})$ is a c.e. set.

PROPOSITION 3.4
For any c.e. character K, there is a computable injection structure $\mathcal{A} = (\omega, f)$ with character K and any specified finite or countably infinite number of orbits of types ω and $\mathbb{Z}$. Furthermore, the range of f is computable and $\{a : \mathcal{O}_f(a)$ is finite$\}$ is computable.

LEMMA 3.5
Any c.e. injection structure is isomorphic to a computable injection structure.

PROOF. Given an infinite c.e. set A and a partial computable function f that is an injection on A, let $A = \{\varphi(0), \varphi(1), \dots\} = ran(\varphi)$, where φ is a computable injection from ω onto A, and let $g(n) = \varphi^{-1}(f(\varphi(n)))$. Then φ is an isomorphism from the computable injection structure $\mathcal{E} = (\omega, g)$ to $\mathcal{A} = (A,f)$, since $\varphi(g(n)) = f(\varphi(n))$. □

PROPOSITION 3.6
Let $\mathcal{A} = (\omega, f)$ be an injection structure. Then $\mathcal{A}$ has a generically computable copy if and only if $\mathcal{A}$ has an infinite substructure that is isomorphic to a computable injection structure.

PROOF. Suppose first that $\mathcal{A} = (\omega, f)$ has a generically computable copy $\mathcal{C} = (\omega, g)$, and let $H : \mathcal{C} \to \mathcal{A}$ be an isomorphism. Now, by definition, there is a dense c.e. set D such that $\mathcal{D}$ is a c.e.

substructure of $\mathcal{C}$. The set D must be infinite since it is dense. Then the image $\mathcal{B} = (H(D), f)$ is an infinite substructure of $\mathcal{A}$, which is isomorphic to $\mathcal{D}$. The result now follows by Lemma 3.5.

Next, suppose that $\mathcal{A} = (\omega, f)$ has an infinite substructure $\mathcal{B} = (B, f)$, which is isomorphic to a computable injection structure with universe ω. We may assume that B is co-infinite since otherwise, $\mathcal{A}$ is a computable structure and hence also generically computable. We assume without loss of generality that (B, f) is itself computable (otherwise, we can simply introduce additional notation for the computable copy). Now, let D be a co-infinite dense computable set, enumerated as $D = \{d_n : n \in \mathbb{N}\}$. We will build a computable function g such that $(D, g) \cong (B, f)$. To do this, it suffices to define $g(d_n) = d_{f(n)}$.

We may extend the isomorphism $F : (B, f) \to (D, g)$ to a permutation of ω mapping $\omega \setminus B$ to $\omega \setminus D$. Then we may extend D to a generically computable injection structure $\mathcal{C} = (\omega, g)$ by defining $g(x)$ to be $F(f(F^{-1}(x)))$, so that F will be an isomorphism between $\mathcal{A}$ and $\mathcal{C}$. □

Note that in the proof of Proposition 3.6, we obtain a generically computable copy with a *computable* substructure $\mathcal{D}$ the domain of which is dense.

PROPOSITION 3.7
An injection structure $\mathcal{A} = (\omega, f)$ has a generically computable copy if and only at least one of the following holds:

(1) $\mathcal{A}$ has an infinite orbit;
(2) $\chi(\mathcal{A})$ has an infinite c.e. subset.

PROOF. Suppose that $\mathcal{A}$ has a generically computable copy. Then, by Proposition 3.6, $\mathcal{A}$ has an infinite substructure $\mathcal{D}$ that is isomorphic to a computable injection structure $\mathcal{C}$. There are two cases.

Case I. If $\mathcal{C}$ has an infinite orbit, then $\mathcal{D}$ has an infinite orbit $\mathcal{O}_f(a)$, and that orbit is also infinite in $\mathcal{A}$.

Case II. If $\mathcal{C}$ has no infinite orbits, then $\chi(\mathcal{C})$ is an infinite c.e. set and $\chi(\mathcal{C}) = \chi(\mathcal{D})$. Since every finite orbit in $\mathcal{D}$ is also an orbit in $\mathcal{A}$, it follows that $\chi(\mathcal{D})$ is an infinite c.e. subset of $\chi(\mathcal{A})$.

For the other direction, suppose first that $\mathcal{A}$ has an infinite orbit $\mathcal{O}_f(a)$. Then, by Proposition 3.4, there is a computable injection structure consisting of exactly one orbit of the same type as $\mathcal{O}_f(a)$. Thus, the orbit $\mathcal{O}_f(a)$ composes an infinite substructure of $\mathcal{A}$ isomorphic to a computable injection structure. It follows from Proposition 3.6 that $\mathcal{A}$ has a generically computable copy.

Next, suppose that $\mathcal{A}$ has no infinite orbits and that $\chi(\mathcal{A})$ has an infinite c.e. subset K. Then, again, by Proposition 3.4, there is a computable injection structure with character K. So it again follows from Proposition 3.6 that $\mathcal{A}$ has a generically computable copy. □

Next, we consider Σ_1-generically c.e. injection structures. We first characterize when $\mathcal{B}$ is a Σ_1 elementary substructure of an injection structure $\mathcal{A}$.

PROPOSITION 3.8
A structure $\mathcal{B}$ is a Σ_1 elementary substructure of the injection structure $\mathcal{A} = (\omega, f)$ if and only if:

(i) For all $b \in B$, the orbit of b in $\mathcal{B}$ equals the orbit of b in $\mathcal{A}$,
(ii) $\chi(\mathcal{A}) = \chi(\mathcal{B})$, and
(iii) If $\mathcal{A}$ has an infinite orbit, then either $\chi(\mathcal{B})$ is unbounded or $\mathcal{B}$ has an infinite orbit.

PROOF. Suppose that $\mathcal{B}$ is a Σ_1 elementary substructure of $\mathcal{A} = (\omega, f)$. Certainly, finite orbits and orbits of type ω are equal in $\mathcal{B}$ and in $\mathcal{A}$ since $\mathcal{B}$ is closed under the function f. Since $\mathcal{B} \prec_1 \mathcal{A}$, if $\mathcal{A} \models (\exists x)(f(x) = b)$, then $\mathcal{B} \models (\exists x)(f(x) = b)$, so $\mathcal{B}$ is also closed under f^{-1} and this preserves the orbits of type $\mathbb{Z}$. Since finite orbits are preserved, $\chi(\mathcal{B}) \subseteq \chi(\mathcal{A})$. The other inclusion follows from

$\mathcal{B} \prec_1 \mathcal{A}$. That is, let $\phi_k(x)$ be the first-order formula saying $[f^{(k)}(x) = x \ \wedge (\forall j < k)(f^{(j)}(x) \neq x)]$. Then $(k,n) \in \chi(\mathcal{A})$ if and only if $\mathcal{A}$ satisfies the sentence saying:

$$(\exists x_0,\ldots,x_{n-1})[(\forall i < n)\phi_k(x_i) \ \wedge \ (\forall i < j < n)(\forall t < k)(f^{(t)}(x_i) \neq x_j)].$$

Since this is a Σ_1 sentence, it follows that $(k,n) \in \chi(\mathcal{A})$ implies $(k,n) \in \chi(\mathcal{B})$.

Finally, suppose that $\mathcal{A}$ has an infinite orbit. Then for each k, $\mathcal{A}$ satisfies the sentence ψ_k saying:

$$(\exists x)(\forall i \leq k)(f^{(i)}(x) \neq x).$$

Then $\mathcal{B} \models \psi_k$ for each k as well. Now suppose that $\chi(\mathcal{B})$ was bounded below k_0. Then there is b such that $(\forall i \leq k_0)(f^{(i)}(b) \neq b)$ and, therefore, $\mathcal{O}_f(b)$ must be infinite.

For the other direction, suppose that $\mathcal{B}$ satisfies conditions (i)–(iii). Let $b_1,\ldots,b_m \in B$ and consider an arbitrary infinitary Σ_1 formula

$$\gamma(b_1,\ldots,b_m) = \bigvee_{i\in\mathbb{N}}\!\!\!\bigvee (\exists x_1,\ldots,x_n)\theta_i(b_1,\ldots,b_m,x_1,\ldots,x_n),$$

where each θ_i is quantifier-free. By distributing disjunctions in the usual way, we may assume without loss of generality that each θ_i is a conjunction of equalities and inequalities among some finite set of images $f^{(s)}(b_i)$ and $f^{(t)}(x_j)$. Since f is an injection, any equality of the form $f^{(s)}(b_i) = f^{(t)}(x_j)$ allows us to eliminate x_j from the formula. Now suppose that $\theta_i(b_1,\ldots,b_m,a_1,\ldots,a_n)$ holds. If any a_j is in the orbit of some b_i, then by (i), $a_j \in B$ and a_j may be eliminated from θ_i. Thus, the formula reduces to some $\theta_i'(a_1,\ldots,a_n)$. The equalities may be reduced to the form $a_h = f^{(t)}(a_j)$. If we have $a_j = f^{(t)}(a_j)$, then the orbit of a_j has type t. Since a_j is not in $\mathcal{O}_f(b_i)$ for any i, and $\chi(\mathcal{A}) = \chi(\mathcal{B})$, there must exist $c \in B$, with order type t, not in any of $\mathcal{O}_f(b_i)$ and that $c = c_j$ may be substituted for a_j. For the other equalities of the form $a_h = f^{(t)}(a_j)$, we need an orbit in $\mathcal{B}$ of size $\geq t$, and such an orbit exists by (iii). Thus, we can find c_h and c_j in B with $c_h = f^{(t)}(c_j)$. In the end, we have $c_1,\ldots,c_n \in B$ so that $\mathcal{B} \models \theta_i(b_1,\ldots,b_m,c_1,\ldots,c_n)$ and, therefore, $\mathcal{B} \models \gamma(b_1,\ldots,b_m)$. □

For injection structures, having a Σ_1-generically c.e. isomorphic copy has a simple characterization.

THEOREM 3.9
The following are equivalent for an injection structure $\mathcal{A} = (\omega,f)$.

(a) $\mathcal{A}$ has a Σ_1-generically c.e. copy.
(b) $\chi(\mathcal{A})$ is a c.e. set.
(c) $\mathcal{A}$ has a computable copy.
(d) $\mathcal{A}$ has a Σ_2-generically c.e. copy.

PROOF. The key is to show that (a) implies (b). Suppose that $\mathcal{A}$ has a Σ_1-generically c.e. copy $\mathcal{E} = (\omega,g)$, and let D be a dense c.e. set such that for some g, $\mathcal{D} = (D,g)$ is a c.e. structure and $\mathcal{D} \prec_1 \mathcal{E}$. Then $\chi(\mathcal{D})$ is a c.e. set and, by Proposition 3.8, $\chi(\mathcal{D}) = \chi(\mathcal{E})$. Since $\mathcal{A}$ is isomorphic to $\mathcal{E}$, it follows that $\chi(\mathcal{A})$ is a c.e. set. Proposition 3.4 shows that (b) implies (c). The implication from (c) to (d) is easy, since any computable structure is Σ_n-generically c.e. for any n. Any Σ_{n+1}-generically c.e. structure is Σ_n-generically c.e., so (d) implies (a). □

4 Generically computable and Σ_1- and Σ_2-generically c.e. equivalence structures

We will now focus on equivalence structures and their dense computability. Recall that an *equivalence structure* $\mathcal{A} = (A, E)$ is simply a set A with an equivalence relation E on A. Equivalence structures also have a character, defined as follows.

DEFINITION 4.1
The *character* of an equivalence structure $\mathcal{A} = (A, E)$ is

$$\chi(\mathcal{A}) = \{(k, n) \in (\omega \setminus \{0\}) \times (\omega \setminus \{0\}) : \mathcal{A} \text{ has at least } n \text{ equivalence classes of size } k\}.$$

We will sometimes just refer to the character of E when the set A is understood. As for injection structures, it is easy to see that a set K is a character if and only if $K = \chi(\mathcal{A})$ for some equivalence structure $\mathcal{A}$.

Computable and c.e. equivalence structures were studied by the authors and Morozov in [1] and by Cenzer, Harizanov and Remmel in [2], where the following results were shown. By $[a]_E$, or just $[a]$ when E is understood, we will denote the equivalence class of a. Let $Fin^{\mathcal{A}} = \{a : [a] \text{ is finite}\}$ and $Inf^{\mathcal{A}} = \{a : [a] \text{ is infinite}\}$.

LEMMA 4.2
For any c.e. equivalence structure $\mathcal{A}$ we have the following.

(1) $\{(a, k) : |[a]| \geq k\}$ is a c.e. set.
(2) $\{(a, k) : |[a]| = k\}$ is the difference of two c.e. sets.
(3) $Inf^{\mathcal{A}}$ is a Π^0_2 set.
(4) $\chi(\mathcal{A})$ is a Σ^0_2 set.

PROPOSITION 4.3
Let K be a Σ^0_2 character.

(1) There is a computable equivalence structure $\mathcal{A} = (\omega, E)$ with character K and with infinitely many infinite equivalence classes. Furthermore, $Inf^{\mathcal{A}}$ is a Π^0_1 set.
(2) For any finite $m \geq 1$, there is a c.e. equivalence structure $\mathcal{A} = (\omega, E)$ with character K and with exactly m infinite equivalence classes.

DEFINITION 4.4
The function $f : \omega^2 \to \omega$ is said to be an s_1-function if the following hold:

(1) For every i and $s, f(i, s) \leq f(i, s+1)$;
(2) For every i, the limit $m_i = \lim_{s\to\infty} f(i, s)$ exists;
(3) For every i , $m_i < m_{i+1}$.

The character K is said to *possess an s_1-function f* if it has an equivalence class of size m_i for each i. Here are some useful results about the characters of equivalence relations. The first result is a slight improvement of Lemma 2.1(c) of [3].

LEMMA 4.5
For any c.e. equivalence relation E on a c.e. set A, the character $\chi(E)$ is a Σ^0_2 set.

PROOF. The Lemma from [3] applies to a structure with universe ω. If E is only defined on the c.e. set A, just let $S(x,y) \Leftrightarrow (E(x,y) \ \vee \ x = y)$. This adds some classes of size 1 to the character, so that $\chi(S)$ is Σ^0_2 if and only if $\chi(E)$ is Σ^0_2. $\square$

The next lemma is part of Lemma 2.8 of [1].

LEMMA 4.6
For any Σ^0_2 character K, which is either bounded or possesses a computable s_1-function, there is a computable equivalence structure with character K and no infinite equivalence classes.

The next result is an improvement of Lemma 2.6 of [1]. It follows from Lemma 4.6 that it also holds for structures restricted to c.e. universes.

LEMMA 4.7
Let $\mathcal{A} = (\omega, E)$ be a c.e. equivalence structure with no infinite equivalence classes and an unbounded character. Then there is a computable s_1-function f such that $\mathcal{A}$ contains an equivalence class of size m_i for all i, where $m_i = lim_s f(i,s)$.

PROOF. Let E^p be the p^{th} stage in the enumeration of E so that $E = \cup_p E^p$. We will define a uniformly computable family a^s_i for $i \leq s$ in such a way that $a_i = lim_s a^s_i$ exists. We will also define a computable sequence p_s, and let

$$f(i,s) = \left|\{a \leq p_s : aE^{p_s} a^s_i\}\right| .$$

Hence, we will have

$$m_i = \lim_s \ \left(\left|\{a \leq p_s : aE^{p_s} a_i\}\right| = |[a_i]|\right) .$$

Stage 0. We set $p_0 = 0$ and $a^0_0 = 0$, so $f(0,0) = 1$. In fact, a^s_0 will equal 0 for all s.
Stage $s+1$. After stage s, we have p_s and $a^s_0, \ldots, a^s_s$ with $f(i,s)$ as above such that

$$f(0,s) < f(1,s) < \cdots < f(s,s).$$

At this stage, we define the least $p > p_s$ and the lexicographically least sequence $b_0, \ldots, b_{s+1}$ such that for all $i \leq s$, we have

$$f(i,s) \leq \left|\{a \leq p : aE^p b_i\}\right| < \left|\{a \leq p : aE^p b_{i+1}\}\right| ,$$

as follows. Let $b_0 = a_0 = 0$. Furthermore, $b_i = a^{s+1}_i$ whenever there do not exist a pair a,j with $j \leq i$, $aE^p a^s_j$ and $p_s < a \leq p$. Then we let $a^{s+1}_i = b_i$ for each i, and let $p_{s+1} = p$.

To see that such p exists at stage $s+1$, let m be the largest such that $[a^s_j] = \{a \leq p_s : aE^{p_s} a^s_j\}$ for all $j \leq m$, and let $b_i = a^s_i$ for all $i \leq m$. Then we use the assumption that $\chi(\mathcal{A})$ is unbounded to find $b_{m+1}, \ldots, b_{s+1}$ with

$$\left|[a^s_m]\right| < |[b_{m+1}]| < |[b_{m+2}]| < \cdots < |[b_{s+1}]| ,$$

and take p large enough so that $[b_i] = \{a \leq p : aE^p b_i\}$.

Finally, we verify that $a_i = lim_s a^s_i$ exists for each i. Since there is no $j < 0$, it follows from the construction that $a^s_0 = 0$ for all s. Given t such that $a_i = lim_s a^s_i$ has converged by stage t for all $i \leq k$, let $r \geq t$ be large enough so that

$$[a_i] = \{a < p_r : aE^{p_r} a_i\}$$

for all $i \leq k$. Here, we used the assumption that there are no infinite classes. It follows from the construction that $a^s_{i+1} = a^r_{i+1}$ for all $s > r$. □

PROPOSITION 4.8
If $\mathcal{A} = (\omega, E)$ is a c.e. equivalence structure with no infinite equivalence classes, then $\mathcal{A}$ is isomorphic to a computable structure.

PROOF. By Lemma 4.5, $\mathcal{A}$ has a Σ^0_2 character, and by Lemma 4.7, this character possesses a computable s_1-function. Then by Lemma 4.6, there is a computable structure with the same character and no infinite equivalence classes, and hence isomorphic to $\mathcal{A}$. □

This last result also holds for a c.e. equivalence structure $\mathcal{E} = (A, E)$.

We will now consider equivalence structures in the context of generic computability and the variants thereof.

THEOREM 4.9
If an equivalence structure $\mathcal{E} = (\omega, E)$ is generically computable, then there is some infinite computable $Y \subseteq \omega$ such that the restriction of E to $Y \times Y$ is computable.

PROOF. Let A be an asymptotically dense c.e. set and φ a binary partial computable function, given in the definition of a generically computable structure. Then, by Lemma 2.2, A has a computable subset Y with upper density 1 (hence infinite) with $Y \times Y \subseteq dom(\varphi)$. Then $c_E = \varphi$ on the computable set Y. □

Note that the set Y from the proof of Theorem 4.9 may not preserve the equivalence classes of $\mathcal{E}$.

Recall that a set $B \subseteq \omega$ is called *immune* if B is infinite and does not contain an infinite c.e. subset.

EXAMPLE 4.10
Let $K = \{(1, k) : k \in C\}$ where C has no infinite Σ^0_2 subset. Also, take an immune set B. Then define $\mathcal{E}$ so that B is one infinite class, and $\omega \setminus B$ has character K. Then, while $\mathcal{E}$ itself need not be computable, $\mathcal{E}$ has a generically computable copy, where the infinite class is a dense computable set. Now, let Y be an infinite computable subset of ω. Since B is immune, $Y \setminus B$ is infinite, so that Y has infinitely many elements with finite equivalence classes. If (Y, E) has a computable copy, then this copy has a Σ^0_2 character, which is a subset of C. Thus at least (Y, E) does not preserve equivalence classes.

The following result was unexpected.

PROPOSITION 4.11
Every equivalence structure $\mathcal{E} = (\omega, E)$ has a generically computable copy.

PROOF. The proof is by cases. If $\chi(\mathcal{E})$ is bounded, then $\mathcal{E}$ has a computable isomorphic copy. If $\mathcal{E}$ has an infinite equivalence class, let B be such a class, and let D be a computable dense set. Then we can define a generically computable copy $\mathcal{A} = (\omega, R)$ of $\mathcal{E}$ so that D is an infinite equivalence class and $(\omega \setminus B, E)$ is isomorphic to $(\omega \setminus B, R)$.

Next, suppose that $\mathcal{E}$ has no infinite equivalence class and $\chi(\mathcal{E})$ is unbounded. Then there must be infinitely many different k such that $\mathcal{E}$ has an equivalence class of size k. Choose one such class B_k for each k, and let $B \subseteq \omega$ consist of exactly one element from each class B_k. Then the substructure (B, E) consists of infinitely many classes of size 1. Note that $\omega \setminus B$ is infinite. Now, let $D \subseteq \omega$ be a computable, co-infinite set of asymptotic density one, and let f be a permutation of ω mapping D onto B, and thus mapping $\omega \setminus D$ onto $\omega \setminus B$. Then we may define a generically computable copy (ω, R) of $\mathcal{E}$ by letting $xRy \Leftrightarrow f(x)Ef(y)$. Then for a computable dense set D, the relation R restricted to $D \times D$ is computable since for $x, y \in D$, we have $xRy \Leftrightarrow x = y$. $\square$

For equivalence structures, the Σ_1-generically c.e. structures have a nice characterization. Note that any substructure $\mathcal{B}$ of an equivalence structure $\mathcal{A}$ is also an equivalence structure since the definitions of reflexive, symmetric and transitive relations are all universal.

PROPOSITION 4.12
A structure $\mathcal{B}$ with domain B is a Σ_1 elementary substructure of the equivalence structure $\mathcal{A} = (\omega, E)$ if and only if:

(1) For all $b \in B$, if $[b]_{\mathcal{A}}$ is finite, then $[b]_{\mathcal{A}} = [b]_{\mathcal{B}}$, and if $[b]_{\mathcal{A}}$ is infinite, then $[b]_{\mathcal{B}}$ is infinite; and
(2) For all $k, n \geq 1$, if $\mathcal{A}$ has at least n classes of size $\geq k$, then $\mathcal{B}$ has at least n classes of size $\geq k$.

PROOF. One direction is immediate from the definition of a Σ_1 elementary substructure. For example, if $[b]_{\mathcal{A}} = \{a, b, c\}$, then

$$\mathcal{A} \models (\exists x)(\exists y)[b \neq x \wedge b \neq y \wedge x \neq y \wedge bEx \wedge bEy \wedge xEy].$$

Thus, $\mathcal{B}$ must also satisfy this formula, so $[b]_{\mathcal{B}}$ has at least 3 elements and, therefore, $[b]_{\mathcal{B}} = \{a, b, c\} = [b]_{\mathcal{A}}$.

For the other direction, suppose that $\mathcal{B}$ satisfies the two conditions in the statement of the theorem. Let $b_1, \ldots, b_m \in B$ and consider an arbitrary Σ_1 formula

$$\gamma(b_1, \ldots, b_m) = \bigvee\!\!\!\!\bigvee_{i \in \mathbb{N}} (\exists x_1, \ldots, x_n)\theta(b_1, \ldots, b_m, x_1, \ldots, x_n),$$

where each θ_i is quantifier-free. By distributing disjunctions in the usual way, we may assume, without loss of generality, that θ_i describes a partition of the set

$$\{b_1, \ldots, b_m, x_1, \ldots, x_n\}.$$

Suppose now that

$$\mathcal{A} \models \theta_i(b_1, \ldots, b_m, a_1, \ldots, a_n)$$

and consider a particular equivalence class (i.e., a particular part of the partition given by θ_i)

$$\{b_{j_1}, \ldots, b_{j_\ell}, a_{i_1}, \ldots, a_{i_k}\}.$$

If necessary, simplify the formula so that no two elements are equal. Let $b = b_{j_1}$. There are three cases to consider.

(1) Suppose that $[b]_{\mathcal{A}}$ is finite. Then, by condition (1), $[b]_{\mathcal{B}} = [b]_{\mathcal{A}}$, so that $a_{i_1}, \ldots, a_{i_k}$ belong to $[b]_{\mathcal{B}}$.
(2) Suppose that $[b]_{\mathcal{A}}$ is infinite. Then, by condition (2), $[b]_{\mathcal{B}}$ is also infinite, hence there are $b_{i_1}, \ldots, b_{i_k}$ such that the set $\{b_{j_1}, \ldots, b_{j_\ell}, a_{i_1}, \ldots, a_{i_k}\}$ may be replaced by the set $\{b_{j_1}, \ldots, b_{j_\ell}, b_{i_1}, \ldots, b_{i_k}\}$ in the partition described by θ.

(3) Finally, suppose $\ell = 0$ so that the equivalence class is just $\{a_{i_1}, \ldots, a_{i_k}\}$. Then, by condition (3), there is an equivalence class in $\mathcal{B}$ with at least k elements, which is disjoint from $\{b_1, \ldots, b_m\}$ and we may choose $\{b_{i_1}, \ldots, b_{i_k}\}$ from such a class.

It follows that elements $b'_1, \ldots, b'_n$ may be chosen so that

$$\mathcal{B} \models \theta_i(b_1, \ldots, b_m, b'_1, \ldots, b'_n)$$

and, therefore, $\mathcal{B} \models \gamma(b_1, \ldots, b_m)$. □

THEOREM 4.13
An equivalence structure $\mathcal{A} = (\omega, E)$ has a Σ_1-generically c.e. copy if and only if at least one of the following conditions holds:

(a) $\chi(\mathcal{A})$ is bounded.
(b) $\chi(\mathcal{A})$ has a Σ^0_2 *subcharacter* (i.e., a subset that is a character) K with a computable s_1-function.
(c) $\mathcal{A}$ has an infinite class and $\chi(\mathcal{A})$ has a Σ^0_2 subcharacter K.
(d) $\mathcal{A}$ has infinitely many infinite classes.

PROOF. If $\mathcal{A}$ has a Σ_1-generically c.e. copy, then it has a Σ_1 elementary substructure $\mathcal{B}$ that is isomorphic to a c.e. structure. Thus, one of the cases in the statement of the theorem must hold. We see this as follows. Suppose that $\mathcal{B}$ has bounded character and does not have infinitely many infinite classes. Then it follows from Proposition 4.12 that $\mathcal{A}$ has a bounded character. Thus, we may suppose that $\chi(\mathcal{B})$ is unbounded. By Lemma 4.2, $K = \chi(\mathcal{B})$ is a Σ^0_2 set, and is a subset of $\chi(\mathcal{A})$ by Proposition 4.12. If K does not have a computable s_1-function, then $\mathcal{B}$ has an infinite class by Lemma 4.7.

We prove the other direction by considering the four cases.

(a) If $\chi(\mathcal{A})$ is bounded, then $\mathcal{A}$ has a computable copy.

In cases (b) and (c), we will assume that $\chi(\mathcal{A})$ is unbounded and show that there is $\mathcal{B} \prec_1 \mathcal{A}$, which is isomorphic to a c.e. structure $\mathcal{D}$, then build a copy $\mathcal{C}$ of $\mathcal{A}$ with a dense c.e. substructure $\mathcal{D}$ and fill out the rest of $\mathcal{C}$ to make it isomorphic to $\mathcal{A}$.

(b) In this case, $\mathcal{A}$ has a substructure $\mathcal{B}$ with unbounded character K and no infinite classes, which will, therefore, be a Σ_1 elementary substructure. By Lemma 4.6, there is a computable structure with character K isomorphic to $\mathcal{B}$, and we may define a structure $\mathcal{D} = (D, R)$ on a computable dense set D with $|\omega \setminus D| = |\omega \setminus B|$. Let ψ be a set isomorphism from $\omega \setminus D$ to $\omega \setminus B$ and extend R to $\omega \setminus D$ by letting $xRy \Leftrightarrow \psi(x)E\psi(y)$. Then ψ will extend the isomorphism of $\mathcal{D}$ and $\mathcal{B}$ to an isomorphism of $\mathcal{A}$ and (ω, R). The structure (ω, R) is Σ_1-generically c.e. since it has a dense c.e. Σ_1 elementary substructure $\mathcal{D}$.

(c) This case is similar to part (b) except that $\mathcal{B}$ now has an infinite class as well. It is important to note that we define a c.e. structure $\mathcal{D} = (D, R)$ on a *computable* dense set D, although the relation R is c.e. and may not be computable.

(d) In this case, the substructure $\mathcal{B}$ consisting of the infinite classes will be a Σ_1 elementary substructure, and we proceed as in (b) to define a c.e. dense structure $\mathcal{D}$ with infinitely many infinite classes and extend this to a Σ_1-generically c.e. structure (ω, R), which is isomorphic to $\mathcal{A}$. □

We observe that the argument above also proves that $\mathcal{A}$ is Σ_1-generically c.e. if and only if it has an infinite substructure $\mathcal{B}$ that is isomorphic to a c.e. structure.

Equivalence structures with Σ_2-generically c.e. isomorphic copies have a simple characterization.

PROPOSITION 4.14
Let $\mathcal{B}$ be a Σ_2 elementary substructure of an equivalence structure $\mathcal{A} = (\omega, E)$. Then $\chi(\mathcal{A}) = \chi(\mathcal{B})$, and if $\mathcal{A}$ has an infinite class, either $\mathcal{B}$ has an infinite class or $\chi(\mathcal{B})$ is unbounded.

PROOF. Let $\mathcal{B} = (B, E)$ be a Σ_2 elementary substructure of an equivalence structure $\mathcal{A} = (\omega, E)$. Then $\chi(\mathcal{B}) = \chi(\mathcal{A})$. This holds because there is a Σ_2 formula $\psi_{n,k}$ that states that $(n,k) \in \chi(\mathcal{A})$.

Next, suppose that $\mathcal{A}$ has an infinite equivalence class, but $\mathcal{B}$ does not have an infinite class. Then for each k, $\mathcal{A}$ has a class of size at least k, i.e. $\mathcal{A} \models \psi_{1,k}$. It follows that $\chi(\mathcal{B})$ is unbounded. □

The following result strengthens Proposition 4.14.

THEOREM 4.15
Let $\mathcal{B} = (B, E)$ be a substructure of the equivalence structure $\mathcal{A} = (A, E)$ such that $\mathcal{B}$ is closed under E. Suppose that $\chi(\mathcal{A}) = \chi(\mathcal{B})$ and that, if $\mathcal{A}$ has any infinite classes, then either $\mathcal{B}$ has an infinite class or $\chi(\mathcal{B})$ is unbounded. Then $\mathcal{B}$ is an elementary substructure of $\mathcal{A}$.

PROOF. We use the following result from the proof of Theorem 5.2 in [4]. □

LEMMA 4.16
Let the formula $\gamma_k(x)$ state that the equivalence class of x has at least k elements. Then the expanded language of equivalence relations with $\{\gamma_k(x) : k \in \omega\}$ has quantifier elimination, i.e. every formula is logically equivalent to a quantifier-free formula.

Now suppose that $b_1, \ldots, b_n \in B$ and that $\mathcal{A} \models (\exists x)\varphi(x, b_1, \ldots, b_n)$; we need to show that $\mathcal{B} \models (\exists x)\varphi(x, b_1, \ldots, b_n)$.

So let $\varphi(a, b_1, \ldots, b_n)$ where by Lemma 4.16 we may assume that φ is quantifier-free in the expanded language. As usual, we may also assume that φ is a conjunction of literals of three forms, where terms s and t come from $a, b_1, \ldots, b_n$:

(i) $s = t$ or $\neg s = t$;
(ii) sEt or $\neg sEt$;
(iii) $\gamma_k(t)$ or $\neg\gamma_k(t)$

Simplifying further, for each t, there is a single value of k such that we either have $\gamma_k(t) \wedge \neg\gamma_{k+1}(t)$, which says that $[t]$ has exactly k elements, or we have just $\gamma_k(t)$.

If $a \in B$, then we are done. Otherwise, we have two cases.

Case 1: φ states that $[a]$ has exactly k elements.

In this case, since $\chi(\mathcal{A}) = \chi(\mathcal{B})$, there must be infinitely many equivalence classes of size k, so there is an element b not equivalent to any of $b_1, \ldots, b_n$ with an equivalence class of size k, and then $\varphi(b, b_1, \ldots, b_n)$.

Case 2: φ states that $[a]$ has at least k elements.

In this case, since $\chi(\mathcal{A})$ is unbounded, there must be an element b not equivalent to any of $b_1, \ldots, b_n$ with an equivalence class of size $\geq k$, and then $\varphi(b, b_1, \ldots, b_n)$.

Together with Proposition 4.14, this implies that $\mathcal{B}$ is a Σ_2 elementary submodel of $\mathcal{A}$ if and only if it is an elementary substructure.

In particular, if $\mathcal{A}$ has one or more infinite classes, and $\mathcal{B}$ is the finite part of $\mathcal{A}$, then $\mathcal{B}$ is a Σ_3 elementary submodel of $\mathcal{A}$,

Thus, if $\mathcal{B}$ has an unbounded character with no s_1-function, then it distinguishes the definition of Σ_3-generically c.e. we actually used from a potential alternate definition using only finitary Σ_3 formulas.

THEOREM 4.17
The following are equivalent for any equivalence structure $\mathcal{A} = (\omega, E)$.

(a) $\mathcal{A}$ has a Σ_2-generically c.e. copy.
(b) $\mathcal{A}$ has a c.e. copy.
(c) $\mathcal{A}$ has a Σ_3-generically c.e. copy.

PROOF. To show that (a) implies (b), suppose that $\mathcal{A}$ has a Σ_2-generically c.e. copy $\mathcal{B} = (\omega, E)$. Let $\mathcal{D}$ be a dense c.e. set such that $\mathcal{D} = (D, E)$ is a c.e. structure and also a Σ_2 elementary substructure of $\mathcal{B}$. Then $\chi(\mathcal{D})$ is a Σ_2^0 set, since $\mathcal{D}$ is c.e. and $\chi(\mathcal{D}) = \chi(\mathcal{A})$ because $\mathcal{D}$ is a Σ_2 elementary substructure of $\mathcal{B}$. If $\chi(\mathcal{A})$ is bounded, then $\mathcal{A}$ has a computable copy. So suppose that $\mathcal{A}$ is unbounded. If $\mathcal{D}$ has no infinite classes, then $\chi(\mathcal{D})$ has a computable s_1-function. Thus, $\mathcal{A}$ has a computable copy, whether or not it has infinite classes. If $\mathcal{D}$ has an infinite class, then $\mathcal{A}$ also has an infinite class and, therefore, has a c.e. copy. The implication from (b) to (c) is easy, since any c.e. structure is Σ_n-generically c.e. Any Σ_{n+1}-generically c.e. structure is Σ_n-generically c.e., so (c) implies (a). □

5 Coarsely computable and Σ_n-coarsely c.e. structures

The results on Σ_n-generically c.e. structures lay down a baseline for the deeper results on coarsely computable injection structures. We will show, in particular, that not every Σ_1-coarsely c.e. injection structure has a generically computable copy and that there are injection structures that do not have coarsely computable copies.

In this section, we define the notions of coarsely computable and Σ_n-coarsely c.e. structures. We investigate these notions for equivalence structures and for injection structures.

DEFINITION 5.1

(1) A structure $\mathcal{A}$ is *coarsely computable* if there are a computable structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with universe D is a substructure of both $\mathcal{A}$ and $\mathcal{E}$ and all relations and functions agree on D.
(2) Let $n \geq 0$. A structure $\mathcal{A}$ is Σ_n*-coarsely c.e.* if there are a c.e. structure $\mathcal{E}$ and a dense set D such that the substructure $\mathcal{D}$ with universe D is a Σ_n elementary substructure of both $\mathcal{A}$ and of $\mathcal{E}$ and all relations and functions agree on D.

A Σ_0-coarsely c.e. structure is also called a *coarsely c.e.* structure. Clearly, every Σ_{n+1}-coarsely c.e. structure is Σ_n-coarsely c.e., and every coarsely computable structure is coarsely c.e.

PROPOSITION 5.2
Consider a structure $\mathcal{A} = (\omega, A)$, where A is a unary relation. Then

(1) $\mathcal{A}$ is coarsely computable if and only if A is coarsely computable.
(2) $\mathcal{A}$ is coarsely c.e. if and only if there is a dense c.e. set D such that $A \cap D$ is c.e.
(3) $\mathcal{A}$ is Σ_1-coarsely c.e. if and only if there is a dense c.e. set such that $A \cap D$ is an infinite c.e. set.

PROOF. Assume that A is coarsely computable and let $f : \omega \to \{0, 1\}$ be a total computable function, let $E = \{x : f(x) = 1\}$, and let D be a dense set such that f agrees with c_A on D. Let $\mathcal{E} = (\omega, E)$. This is, in fact, a computable structure. Then $A \cap D = E \cap D$, so that $\mathcal{D} = (D, A \cap D)$ is a substructure of both $\mathcal{A}$ and $\mathcal{E}$. Thus, $\mathcal{A}$ is a coarsely computable structure. On the other hand, suppose that there is a dense set D and a computable structure $\mathcal{E} = (\omega, E)$ such that $\mathcal{E}$ agrees with $\mathcal{A}$ on the set D, i.e. $A \cap D = E \cap D$. Then c_A agrees with the total computable function $f = c_E$ on the dense set D, and hence A is coarsely computable. The remaining two points follow by similar reasoning. □

Recall from Theorem 1.2 that the notions of generically computable and coarsely computable sets are incomparable. This implies that the same is true for structures.

PROPOSITION 5.3
There is a generically computable structure that is not coarsely computable, and there is a coarsely computable structure that is not generically computable.

PROOF. First, let A be a set that is generically computable but not coarsely computable. Then, by Example 2.8 and Proposition 5.2, the structure (ω, A) is generically computable but not coarsely computable. A similar argument works when the set A is coarsely computable but not generically computable. □

It is easy to see that the structure (ω, A) is coarsely c.e. if and only if there is a c.e. set E and a dense set D such that $A \cap D = E \cap D$.

We will compare and contrast coarsely computable, c.e. and Σ_n-coarsely c.e. structures with generically computable and Σ_n-generically c.e. structures.

PROPOSITION 5.4
Any generically computable injection structure has a coarsely computable copy.

PROOF. Let $\mathcal{A} = (\omega, f)$ be a generically computable injection structure. As noted after Proposition 3.6 above, we may assume that $\mathcal{A}$ has a dense computable substructure $\mathcal{D} = (D, f)$. We may extend $\mathcal{D}$ to a computable structure $\mathcal{C} = (\omega, g)$ by defining $g(x) = f(x)$ for $x \in D$ and $g(x) = x$ for $x \notin D$. Then $\mathcal{D}$ is a dense computable substructure of both $\mathcal{A}$ and $\mathcal{C}$, so that $\mathcal{A}$ is coarsely computable. □

It is natural to ask whether any generically computable structure actually is coarsely computable. The next result gives a negative answer to this question. The proof is based on the fact that each infinite orbit in a computable injection structure is a c.e. set, and the set of elements with finite orbits is also a c.e. set. This is seen by Lemma 3.3.

Recall that a set is *simple* if it is c.e. and its complement is immune.

THEOREM 5.5
There is a generically computable injection structure that is not coarsely computable.

PROOF. Let D be an asymptotically dense simple (hence c.e.) set. This is easily constructed by adding elements to a simple set as follows. Recall that the usual construction produces a c.e. set A, which contains at most n elements that are $< 2^n$ for each n, with a single element $i > 2^e$ entering A for each e, when it enters the e^{th} c.e. set W_e. Just take an arbitrary dense computable set B that contains exactly $(2^n - 2n)$ elements $< 2^n$ for each $n > 3$, and then $D = A \cup B$ will be a dense simple set.

Now, let $D = \{a_0, a_1, \ldots\}$ be a computable one-to-one enumeration, and define the function f on D so that $f(a_i) = a_{i+1}$. Then f is a partial computable function that is total on the set D. That is, given $a \in D$, just enumerate D until you see that $a = a_i$ and then output a_{i+1}. Let $\mathcal{D} = (D, f)$ and extend f arbitrarily to an injection structure $\mathcal{A} = (\omega, f)$. We claim that $\mathcal{A} = (\omega, f)$ cannot be coarsely computable.

Suppose, by way of contradiction, that $\mathcal{C} = (C, g)$ is a computable injection structure and E is a dense set such that $\mathcal{E} = (E, f) = (E, g)$. First, we show that $D = E$ modulo finite sets. Observe that $D \cap E$ must be asymptotically dense and, therefore, E contains some element a_i. It follows that $\{a_i, a_{i+1}, \ldots\} \subseteq E$ and thus $D \setminus E$ is finite. Now, suppose that $E \setminus D$ were infinite. There are three cases to consider. In the first case, $E \setminus D$ might contain an infinite orbit. Since g is computable, this orbit would be an infinite c.e. subset of $E \setminus D$. In the second case, E might extend D to an orbit of type $\mathbb{Z}$. In this case, $\{x : (\exists n) g^{(n)}(x) = a_0\}$ is an infinite c.e. subset of $E \setminus D$. In the third case, when neither of the first two cases apply, $E \setminus D$ must contain an infinite set of finite orbits. Then $\{x \in \omega : (\exists n)(g^{(n)}(x) = x)\}$ is an infinite c.e. subset of $\omega \setminus D$. In each case, we arrive at a conclusion that contradicts the fact that D is a simple set.

Thus, we may assume, without loss of generality, that $D = E$. Similarly, as above, we can show that $\omega \setminus D$ must be finite, contradicting the assumption that D is co-infinite. □

It would be interesting to generalize these results by giving exact conditions on structures under which generic and coarse computability either coincide or fail to coincide.

The situation is somewhat different for equivalence structures. Of course, we know that every equivalence structure has a generically computable copy.

PROPOSITION 5.6
Any generically computable equivalence structure is coarsely c.e.

PROOF. Let $\mathcal{A} = (\omega, E)$ be an equivalence structure, and let D be a dense c.e. set such that $\mathcal{D} = (D, E)$ is a c.e. substructure of $\mathcal{A}$. Then we may extend E to a c.e. equivalence relation R on ω by letting xRy if and only if $(x = y \vee (x, y \in D \wedge xEy))$. Thus for $x \in D$, we have $[x]_R = [x]$, and for $x \notin D$, we have $[x]_R = \{x\}$. □

Let $\mathcal{E} = (\omega, E)$ be the canonical equivalence structure with one class of every finite size k. The equivalence classes of (ω, E) are $\{0\}, \{1, 2\}, \{3, 4, 5\}, \ldots$ The first k classes have $1 + 2 + \cdots + k = k(k+1)/2$ elements. Thus, the class $[\frac{k(k+1)}{2}]$ has $k + 1$ elements. Let K be any set, and let $A_K = \bigcup_{k \in K} [\frac{k(k-1)}{2}]$ be the union of the classes of size k for $k \in K$, under E.

Similarly, let $\mathcal{C} = (\omega, f)$ be the injection structure with orbits $\{0\}, \{1, 2\}, \{3, 4, 5\}, \ldots$, so that $f(0) = 0, f(1) = 2$ and $f(2) = 1$, and so on. The first k orbits have $1 + 2 + \cdots + k = k(k+1)/2$ elements.

LEMMA 5.7
If K is a dense set, then A_K is also a dense set.

PROOF. Suppose that the complement of K contains m out of the first n positive numbers. Then the classes of size k with $k \in K \cap \{1, 2, \ldots, n\}$ contain at most $n + (n-1) + \cdots + (n - m + 1) = m(2n - m + 1)/2$ elements out of a total of $1 + 2 + \cdots + n = n(n+1)/2$ elements. Then the ratio is $\frac{m}{n} \cdot \frac{2n-m+1}{n+1} \leq 2m/n$. Thus, if $\omega \setminus K$ has density zero, then A_K will have density 1. □

THEOREM 5.8
For any dense co-infinite set $K \subseteq \omega - \{0\}$, there is a Σ_1-coarsely c.e. equivalence structure $\mathcal{A}$ with character $\{(k, i) : k \in K \wedge 1 \leq i \leq 2\}$ and no infinite classes.

PROOF. Let $\mathcal{E} = (\omega, E)$ be the canonical computable equivalence structure described above, with one class of every finite size k. Let A_K be the dense subset of ω, which will have character $\{(k, 1) : k \in K\}$ under E. Then take $\omega \setminus A_K$ and partition it into exactly one class of size k for $k \in K$ to create the structure $\mathcal{A}$. Then $\mathcal{A}$ agrees with $\mathcal{E}$ on the dense subset A_K. It follows from Proposition 4.12 that (A_K, E) is a Σ_1 elementary substructure of both $\mathcal{E}$ and $\mathcal{A}$ since $\chi(A_K) = \{(k, 1) : k \in K\}$ is unbounded. Thus, $\mathcal{A}$ is Σ_1-coarsely c.e. □

THEOREM 5.9
For any dense co-infinite set $K \subseteq \omega - \{0\}$, there is a coarsely computable injection structure with character $\{(k,i) : k \in K \wedge 1 \leq i \leq 2\}$ and no infinite orbits.

PROOF. The proof is similar to the proof Theorem 5.8. To obtain the coarsely computable injection structure, define an injection g that agrees with the canonical function f on the set A_K, and extend this function on $\omega \setminus A_K$ to add one additional orbit of each size k for $k \in K$. Again, this structure agrees with the computable structure $\mathcal{C}$ on the dense set A_K. We note that (A_K, f) will not be a Σ_1 elementary substructure of $\mathcal{A}$ since the character is different from $\chi(\mathcal{A})$, as it has only one orbit of size k for $k \in K$, whereas $\mathcal{A}$ has two. □

The standard construction of a simple set (i.e., a c.e. set with immune complement) may be modified to construct a simple set of density zero. Then the complement is a Π^0_1 set of density one. The following lemma also gives a relativized version.

LEMMA 5.10
There is an infinite Π^0_1 set of asymptotic density 1 with no infinite c.e. subset, and an infinite Π^0_2 dense set K with no infinite Σ^0_2 subset. There is also an infinite Σ^0_1 set of asymptotic density 0 with no infinite c.e. subset and an infinite Π^0_2 set K of density 0 with no infinite Σ^0_2 subset.

PROOF. The notion of an immune set that is a Π^0_1 set (co-c.e.) is well-studied and easily generalized. The standard construction of an immune Π^0_2 set may be modified as follows to obtain a dense set. Let $S_1, S_2, \ldots$ be an enumeration of all Σ^0_2 sets. Define K to omit the least member of S_i, which is greater than 2^i. Then K must contain at least $(2^i - i)$ many of the first 2^i numbers and hence has density 1. For the second part, consider the set $C = \{2^n - 1, 2^n - 2 : n > 0\}$. The set C contains $2i$ many of the first 2^i numbers, for each $i > 0$. Thus, Π^0_1 set $K \cap C$ contains between i and $2i$ of the first 2^i numbers and is, therefore, infinite and has asymptotic density 0. Since $K \cap C \subseteq K$, it has no infinite c.e. subset. A similar argument works for the Π^0_2 set. □

The authors, together with Morozov, constructed in [1] an unbounded Δ^0_2 character K with no computable s_1-function. (In fact, the set K is D^0_1, i.e. the difference of two c.e. sets.) This result may be improved to obtain a set of asymptotic density zero.

PROPOSITION 5.11
There is an infinite Δ^0_1 set D of asymptotic density 1 such that $D \times \{1\}$ has no computable s_1-function, and, therefore, there is no computable structure with character $D \times \{1\}$. There is also an infinite Δ^0_1 set D of asymptotic density 0 such that $D \times \{1\}$ has no computable s_1-function, and, therefore, there is no computable structure with character $D \times \{1\}$.

PROOF. Let $\mathcal{C}_e := (\omega, S_e)$ be the e^{th} c.e. equivalence structure. That is, for the e^{th} c.e. set W_e, let S_e be the reflexive, symmetric and transitive closure of $\{(x,y) : \langle x,y \rangle \in W_e\}$. Let $[x]_e$ denote the equivalence class of x in $\mathcal{C}_e$. Define the c.e. relation R by

$$R(e,x) \Leftrightarrow card([x]_e) > 2^e.$$

Then, by a standard uniformization theorem for c.e. relations [see 14, p. 29], there is a partial computable *selector* function φ such that, for every e,

$$(\exists x)R(e,x) \Rightarrow R(e,\varphi(e)).$$

Now, define D as follows:

$$k \in D \Leftrightarrow (\forall e)(2^e < k \Rightarrow card([\varphi(e)]_e) \neq k).$$

Then D is a D^0_1 set by Lemma 4.2. Now suppose that $k < 2^i$ and $k \notin D$. Then for some e with $2^e < k$, $card([\varphi(e)]_e) = k$. It follows that $e < i$, so that

$$card(\{k < 2^i : k \notin D\}) \leq i.$$

It follows that K is asymptotically dense. Now, suppose that $\mathcal{A} = \mathcal{A}_e$ has unbounded character and no infinite equivalence class. Since $\chi(\mathcal{A})$ is unbounded, there exists a such that $R(e, a)$. Let $a = \varphi(e)$. Since $\mathcal{A}$ has no infinite classes, $card([\varphi(e)]_e) = k > 2^e$. Then, by definition, $(k, 1) \in \chi(\mathcal{A})$ but $k \notin D$. Hence $\chi(\mathcal{A}) \neq K$. It follows from Lemma 4.6 that K has no computable s_1-function.

As in the proof of Lemma 5.10, it can be shown that there is also such a set with asymptotic density 0. □

PROPOSITION 5.12

(1) There is a coarsely computable injection structure with no generically computable copy.
(2) There is a Σ_1-coarsely c.e. equivalence structure with no Σ_1-generically c.e. copy.

PROOF. Let $K \subseteq \omega - \{0\}$ be a dense immune set, and let $\mathcal{A}$ be the injection structure with character $\{(k, i) : k \in K \wedge 1 \leq i \leq 2\}$ from Theorem 5.9 with no infinite orbits. If $\mathcal{B}$ were a generically computable copy of $\mathcal{A}$, then $\mathcal{B}$ has no infinite orbits, and thus $\chi(\mathcal{B}) = \chi(\mathcal{A})$ must have an infinite c.e. subset C by Proposition 3.6. Then $\{k : (k, 1) \in C \vee (k, 2) \in C\}$ is an infinite c.e. subset of K, which is a contradiction.

The result for equivalence structures follows similarly from Lemma 5.10 and Theorem 4.13. □

Next, we will show that there are equivalence structures that do not have Σ_1-coarsely c.e. copies and injection structures that have no coarsely computable copies.

THEOREM 5.13
There is an infinite Π^0_4 set $K \subseteq \omega$ such that if $\mathcal{C} = (\omega, R)$ is a c.e. equivalence structure such that $\{x : |[x]_R| = k\}$ has asymptotic density 0 for any k, and such that if D is a set of asymptotic density 1, then D is not a subset of $\{x : |[x]_R| \in K\}$. Thus, any equivalence structure $\mathcal{A}$ with character $\chi(\mathcal{A}) \subseteq K \times \{1\}$ cannot be Σ_1-coarsely c.e.

PROOF. As before, let $\mathcal{C}_e := (\omega, S_e)$ be the e^{th} c.e. equivalence structure. That is, for the e^{th} c.e. set W_e, let S_e be the reflexive, symmetric, transitive closure of $\{(x, y) : \langle x, y\rangle \in W_e\}$. Let $[x]_e$ denote the equivalence class of x in $\mathcal{C}_e$. We need to meet the following requirements.

Requirement R_e: If $\{x : |[x]_e| = k\}$ has asymptotic density 0 for all k , then $\{x : |[x]_e| \in K\}$ does not have asymptotic density 1.

We begin the construction with $K^0 = \omega$ and remove numbers at certain stages to satisfy the requirements. At the same time, we need to ensure that K is infinite. So the construction will preserve an element of K each time that it removes an infinite number of elements. We may assume for the construction that $\{x : [x]_e \text{ is infinite}\}$ has upper density 0; otherwise, the conclusion is immediate.

We will show how to satisfy an individual requirement for the case $e = 0$. Let $\mathcal{C} = (\omega, S_0)$, let $S = S_0$, and consider the four sets $A_i = \{x : |[x]_S| = i \bmod 4\}$ for $i = 0, 1, 2, 3$. Since the union of the sets equals ω, at least one of the sets, say A_j, must have upper asymptotic density at least $1/4$. Let us suppose that $\{x : |[x]_S| = k\}$ has asymptotic density 0 for all k, so that we need to take action on requirement R_0. Then we will ensure that $K \cap \{i : i = j \ mod \ 4\} = \{4 + j\}$; i.e. we let $K^1 = \{4 + j\} \cup \{k : k \neq j \ mod \ 4\}$ and maintain $K \cap \{i : i = j \ mod \ 4\} = \{4 + j\}$ throughout the construction. Then $\{x : |[x]_S| \in K\}$ must have density at most $3/4$, so that it cannot contain any set D has asymptotic density 1.

The general construction of K is in stages. After stage e, we will have designated, for certain $i \leq e$, a value $j(i)$ and corresponding set $A_i = \{x : |[x]_i| = j(i) \ mod \ 2^{i+2}\}$, so that for $i \neq h$, we have $A_i \cap A_h = \varnothing$. We will have removed $K_i = \{m : m = j(i) \ mod \ 2^{i+2}\}$ from K, except for $2^{i+2} + j(i)$, for such i. Note that we will have removed at most one set K_i mod 2^{i+2} for each $i \leq e$, for a total of at most $2^e + 2^{e-1} + \cdots + 1 < 2^{e+1}$ classes mod 2^{e+2}, resulting in the set K^e. Thus, there remain 2^{e+1} classes mod 2^{e+2} to work with, each disjoint from the previous classes. At stage $e + 1$, we will ensure Requirement R_e (if necessary) by removing a set of class sizes from K. If there exists k such that $\{x : |[x]_{e+1}| = k\}$ has positive density, then we take no action. If not, then we select $j = j(e + 1) < 2^{e+3}$ such that $A_{e+1} = \{x : |[x]|_{e+1} = j \ mod \ 2^{e+3}\}$ has upper density at least 2^{-e-3} and we let $K_{e+1} = \{m : m = j(e + 1) \ mod \ 2^{e+3}\}$. If K_{e+1} meets one of the previous classes K_i, then in fact $K_{e+1} \subseteq K_i$, so that we have already removed all but one element of K_{e+1} from K. Otherwise, we remove $K_{e+1} = \{m : m = j \ mod \ 2^{e+3}\}$ from K^e, except for $2^{e+3} + j$, to obtain K^{e+1}.

Let $K = \cap_s K^s$. It remains to check that K satisfies each Requirement R_e and that K is an infinite set.

First, we show that action is taken infinitely often. Suppose, by way of contradiction, that no action is taken after stage e. Then K will consist of a finite number of equivalence classes modulo 2^{e+2} plus a finite set. Thus K will be computable. Hence there is some i such that $\mathcal{C}_i$ consists of exactly one class of size k for each $k \in K$. Thus, at stage i, when we select j such that $\{x : |[x]_i| = j \ mod \ 2^{i+2}\}$ has positive upper density in $\mathcal{C}_i$, and consider $K_i = \{m : m = j \ mod \ 2^{i+2}\}$, we would have $K_i \subseteq K \subseteq K^{i+1}$. But then we would have taken action and removed all but one value of K_i from K.

Next we need to check that K is infinite. Since action was taken infinitely often, we have preserved in K an element $2^{i+2} + j(i)$ of K_i for infinitely many i. Since the sets $\{K_i : i \in \omega\}$ are disjoint, this element is never removed at any later stage. Hence K is infinite.

Now, suppose that $\{x : |[x]_e| = k\}$ has asymptotic density 0 for all k, and suppose, by way of contradiction, that $\{x : |[x]_e| \in K\}$ has asymptotic density 1. Then at stage e of the construction, we will have selected $j < 2^{e+2}$ such that $A_j = \{x : |[x]|_e = j \ mod \ 2^{e+2}\}$ has upper density at least 2^{-e-2}, and defined

$$K_e = \{m : m = j \ mod \ 2^{e+2}\}.$$

Since $K \subseteq K^{e-1}$, it follows that K_e is disjoint from all previous K_i. So we will remove all but one element of K_e from K at stage e. It follows that $\{x : |[x]_e| \in K\}$ has lower density at most $1 - 2^{-e-2}$.

Finally, suppose that $\mathcal{A} = (\omega, S)$ has character $\chi(\mathcal{A}) \subseteq K \times \{1\}$ and is Σ_1-coarsely c.e. Let $\mathcal{C} = (\omega, R)$ be a computable equivalence structure, say $R = S_e$. Let D be a set of density 1 such that the structure $\mathcal{D} = (D, R) = (D, S)$ is a Σ_1 substructure of both $\mathcal{A}$ and $\mathcal{C}$. Since $\mathcal{D} \prec_1 \mathcal{A}$, we have $D \subseteq \{x : |[x]|_S \in K\}$. Since R and S agree on D, and $\mathcal{D} \prec_1 \mathcal{C}$, it follows that $D \subseteq \{x : |[x]|_e \in K\}$. By the assumption on $\mathcal{C}$, this means that $\{x : [x]_e = k\}$ has density 0 for each k. It follows from Requirement R_e that $\{x : |[x]_e| \in K\}$ does not have asymptotic density 1. However, this contradicts the fact that the subset D has density 1.

An upper bound on the complexity of K may be determined as follows. First, we observe that $\{x : |[x]_i| = j\}$ is uniformly Σ^0_2, and thus $C(i,j,e) = \{x : |[x]_i| = j \ mod \ 2^e\}$ is also uniformly Σ^0_2.

Then the lower density $\delta(C(i,j,e)) \geq \frac{1}{4}$ if and only if

$$(\forall m)(\exists n \geq m)|C(i,j,e) \cap n| \geq \frac{n}{4}.$$

Thus, this test is Π^0_3. So the construction may be done using an oracle for $\varnothing'''$. So the set K_i is uniformly computable in $\varnothing'''$. Since K is the intersection of the sequence $(K_i)_{i\in\omega}$, it follows that K is a Π^0_4 set. □

Here is a corresponding result for injection structures.

THEOREM 5.14
There is an infinite set $K \subseteq \omega$ such that if $\mathcal{C} = (\omega,f)$ is a computable injection structure for which the set $\{x : |\mathcal{O}_f(x)| = k\}$ has asymptotic density 0 for any k, and if D is a set of asymptotic density 1, then D is not a subset of $\{x : |\mathcal{O}_f(x)| \in K\}$. Thus, any injection structure $\mathcal{A}$ with character $\chi(\mathcal{A}) \subseteq K \times \{1\}$ cannot be coarsely computable.

PROOF. Here we let $\mathcal{C}_e := (\omega,f_e)$ be the e^{th}*potential* computable injection structure. That is, for the e^{th} c.e. set W_e, let $f_e(x)$ be the least y such that $\langle x,y\rangle \in W_e$, if it exists. Let $\mathcal{O}_e(x)$ be the orbit of x under f_e, if defined. Then we need to meet the following requirements R_e for every $e \in \omega$.

Requirement R_e: If $\mathcal{C}_e$ is an injection structure and $\{x : |\mathcal{O}_e(x)| = k\}$ has asymptotic density 0 for all k, then $\{x : |\mathcal{O}_e(x)| \in K\}$ does not have asymptotic density 1.

We begin the construction with $K^0 = \omega$ and remove numbers at certain stages to satisfy the requirements. At the same time, we need to ensure that K is infinite. So the construction will preserve an element of K each time that it removes an infinite number of elements. We may assume for the construction that $\{x : \mathcal{O}_e(x)$ is infinite$\}$ has upper density 0; otherwise, the conclusion is immediate.

We will show how to satisfy an individual requirement for the case $e = 0$. Let $\mathcal{C} = (\omega,f_0)$, let $f = f_0$ and consider the four sets $A_i = \{x : |\mathcal{O}_0(x)| = i \ mod\ 4\}$ for $i = 0,1,2,3$. Since the union of the sets equals ω, at least one of the sets, say A_j, must have upper asymptotic density at least $1/4$. Let us suppose that $\{x : |\mathcal{O}_e(x)| = k\}$ has asymptotic density 0 for all k, so that we need to take action on requirement R_0. Then we will ensure that $K \cap \{i : i = j \ mod\ 4\} = \{4+j\}$; i.e., we let $K^1 = \{4+j\} \cup \{k : k \neq j \ mod\ 4\}$ and maintain $K \cap \{i : i = j \ mod\ 4\} = \{4+j\}$ throughout the construction. Then $\{x : |[x]_S| \in K\}$ must have density at most $3/4$, so that it cannot contain any set D that has asymptotic density 1.

The details of the construction are similar to those given in the proof of Theorem 5.13 and are, therefore, omitted here. An upper bound on the complexity of K may be determined as follows. First, we observe that $\{x : |\mathcal{O}_i(x)| = j\}$ is uniformly Σ^0_1, and, thus, $C(i,j,e) = \{x : |\mathcal{O}_i(x)| = j \ mod\ 2^e\}$ is also uniformly Σ^0_1. Then the lower density $\delta(C(i,j,e) \geq \frac{1}{4}$ if and only if

$$(\forall m)(\exists n \geq m)|C(i,j,e) \cap n| \geq \frac{n}{4}.$$

Thus, this test is Π^0_2. So the construction may be done using an oracle for $\varnothing''$, and it follows that the K is a Π^0_3 set. □

As was the case for Σ_1-generically c.e. structures, any Σ_1-coarsely c.e. injection structure is isomorphic to a computable structure.

PROPOSITION 5.15
The following are equivalent for any injection structure $\mathcal{A} = (\omega,f)$:

(a) $\mathcal{A}$ has a Σ_1-coarsely c.e. copy;

(b) $\chi(\mathcal{A})$ is a c.e. set;
(c) $\mathcal{A}$ has a computable copy.

PROOF. Suppose first that $\mathcal{A} = (\omega, f)$ is a Σ_1-coarsely c.e. injection structure. Let $\mathcal{B} = (\omega, g)$ be a c.e. structure and D be a dense set such that $f = g$ on the set D, and $\mathcal{D} = (D, f)$ is a Σ_1 elementary substructure of both $\mathcal{A}$ and $\mathcal{B}$. Then $\chi(\mathcal{A}) = \chi(\mathcal{D}) = \chi(\mathcal{B})$ and is, therefore, a c.e. set. The implication (b)⇒(c) follows from Proposition 3.4. □

For Σ_2-coarsely c.e. equivalence structures, the characterization is in two cases.

PROPOSITION 5.16
Let $\mathcal{A}$ be an equivalence structure with character K such that at least one of the following holds:

(i) $\mathcal{A}$ has an infinite equivalence class,
(ii) K is bounded,
(iii) K has a computable s_1-function.

Then the following are equivalent:

(1) $\mathcal{A}$ has a Σ_2-coarsely c.e. copy;
(2) K is Σ^0_2;
(3) $\mathcal{A}$ has a c.e. copy.

PROOF. Let $\mathcal{A}$ be an equivalence structure with character K, which satisfies one of the three conditions above.

Suppose first that $\mathcal{A} = (\omega, E)$ is Σ_2-coarsely c.e. Let $\mathcal{B} = (\omega, R)$ be a c.e. structure and D be a dense set such that $E = R$ on the set D, and $\mathcal{D} = (D, E)$ is a Σ_2 elementary substructure of both $\mathcal{A}$ and $\mathcal{B}$. Then by, Proposition 4.14, $\chi(\mathcal{A}) = \chi(\mathcal{D}) = \chi(\mathcal{B})$ and is, therefore, a Σ^0_2 set.

Next, suppose that K is Σ^0_2. Since $\mathcal{A}$ satisfies one of the three conditions, it follows from Proposition 4.3 that $\mathcal{A}$ is isomorphic to a c.e. structure.

Finally, suppose that $\mathcal{A}$ has c.e. copy. Then the copy is itself Σ_2-coarsely c.e., where the desired dense set is $D = \omega$. □

PROPOSITION 5.17
Let $\mathcal{A}$ be an equivalence structure with unbounded character K such that $\mathcal{A}$ has no infinite equivalence class and K does not have a computable s_1-function. Then the following are equivalent:

(1) $\mathcal{A}$ has a Σ_2-coarsely c.e. copy;
(2) K is Σ^0_2 and, for some finite k, $\mathcal{A}$ has infinitely many classes of size k.

PROOF. Let $\mathcal{A}$ be an equivalence structure with unbounded character K such that $\mathcal{A}$ has no infinite equivalence class and K does not have a computable s_1-function. Suppose first that $\mathcal{A} = (\omega, E)$ is Σ_2-coarsely c.e. Let $\mathcal{B} = (\omega, R)$ be a c.e. structure and D be a dense set such that $E = R$ on the set D, and $\mathcal{D} = (D, E)$ is a Σ_2 elementary substructure of both $\mathcal{A}$ and $\mathcal{B}$. Then, by Proposition 4.14, $\chi(\mathcal{A}) = \chi(\mathcal{D}) = \chi(\mathcal{B})$ and is, therefore, a Σ^0_2 set. Since $\mathcal{B}$ is c.e., it follows from Lemma 4.7 that $\mathcal{B}$ has an infinite class. Therefore, $\omega \setminus D$ is infinite. Now, consider $a \in \omega \setminus D$ and $k = |[a]_E|$. If $(k, n) \in K$, then $(k, n) \in \chi(\mathcal{D})$ and thus $(k, n+1) \in \chi(\mathcal{A})$, so that $(k, n+1) \in K$. It follows that $(k, m) \in K$ for all m.

Next, suppose that K is Σ^0_2 and, for some k, $\mathcal{A}$ has infinitely many classes of size k. By Proposition 4.3, there is a c.e. structure $\mathcal{C} = (\omega, E)$ with character K and an infinite class. Let D be a co-infinite computable dense set. Define $\mathcal{D} = (D, R)$ to consist of infinitely many classes of size k. Define R

on $\omega \setminus D$ to be a copy of $\mathcal{C}$. This defines a c.e. structure isomorphic to $\mathcal{B}$. Now, let $B = \{x \in \omega : |[x]_R| \text{ is finite}\}$. The set B is dense since it includes the dense set D. We have that $\mathcal{B} \prec_2 \mathcal{C}$ since K is unbounded. Now, $\mathcal{B} = (B, R)$ has no infinite classes and has character K, so it is isomorphic to $\mathcal{A}$. We will now build a copy (ω, E) of $\mathcal{A}$ by letting $E = R$ on B, and letting $(\omega \setminus B)$ consist of infinitely many classes of size k. □

Note, in particular, that if K is a Σ^0_2 set with no computable s_1-function and $\mathcal{A}$ consists of one class of size k for each $k \in K$, then no proper substructure of $\mathcal{A}$ can have the same character as $\mathcal{A}$ and, therefore, $\mathcal{A}$ cannot be Σ_2-coarsely c.e.

Here is a related result.

THEOREM 5.18
There is a Δ^0_2 set K with no computable s_1-function, which is asymptotically dense.

LEMMA 5.19
If $\mathcal{B}$ is a Σ_3 elementary substructure of the equivalence structure $\mathcal{A} = (\omega, E)$ and $\mathcal{A}$ has an infinite class, then $\mathcal{B}$ has an infinite class.

PROOF. Let $\mathcal{B} = (B, E)$ be a Σ_3 elementary substructure of an equivalence structure $\mathcal{A} = (\omega, E)$ and suppose that $\mathcal{A}$ has an infinite equivalence class. Then $\mathcal{A}$ satisfies the infinitary Σ_3 formula $(\exists x) \bigwedge\!\!\bigwedge_{n\in\mathbb{N}} \psi_n(x)$, where $\psi_n(x)$ is the Σ^0_1 formula

$$(\exists x_1)\cdots(\exists x_n) \bigwedge_{i<j\leq n} [x_i \neq x_j \ \wedge \ x_i E x].$$

It follows that $\mathcal{B} \models (\exists x) \bigwedge\!\!\bigwedge_n \psi_n(x)$ as well and, therefore, $\mathcal{B}$ has an infinite class. □

THEOREM 5.20
For any equivalence structure $\mathcal{A}$, $\mathcal{A}$ is Σ_3-coarsely c.e. if and only if $\mathcal{A}$ has a c.e. copy.

PROOF. Let $\mathcal{A} = (\omega, E)$ be a Σ_3-coarsely c.e. equivalence structure with character K. First, assume that one of the following conditions is satisfied: $\mathcal{A}$ has an infinite equivalence class, K is bounded or K has a computable s_1-function. Since $\mathcal{A}$ is also Σ_2-coarsely c.e, it follows from Proposition 5.16 that $\mathcal{A}$ has a c.e. copy.

Next, assume that $\mathcal{A}$ has no infinite equivalence class and character K is unbounded and does not have a computable s_1-function. Let $\mathcal{B} = (\omega, R)$ be a c.e. structure and D be a dense set such that $E = R$ on the set D, and $\mathcal{D} = (D, E)$ is a Σ_3 elementary substructure of both $\mathcal{A}$ and $\mathcal{B}$. Then, by Proposition 4.14, $\chi(\mathcal{A}) = \chi(\mathcal{D}) = \chi(\mathcal{B})$. By Proposition 5.19, $\mathcal{D}$ and hence $\mathcal{B}$ have no infinite classes. Thus, $\mathcal{A}$ is isomorphic to $\mathcal{B}$.

Now, assume that $\mathcal{A}$ has a c.e. copy with domain ω. Then the copy itself is Σ_3-coarsely c.e., where the desired dense set is $D = \omega$. □

Acknowledgements

This research was partially supported by the National Science Foundation SEALS grant DMS-1362273. The work was done partially while the latter two authors were visiting the Institute for Mathematical Sciences, National University of Singapore, in 2017. The visits were supported by the Institute. This material is partially based upon work supported by the National Science Foundation under grant DMS-1928930 while all three authors participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2020 semester. Harizanov was partially supported by the Simons Foundation grant 429466 and GW Dean's Research Chair award.

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Received 23 March 2021