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A central limit theorem for the stochastic heat equation

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Abstract

We consider the one-dimensional stochastic heat equation driven by a multiplicative space–time white noise. We show that the spatial integral of the solution from $-R$ to R converges in total variance distance to a standard normal distribution as R tends to infinity, after renormalization. We also show a functional version of this central limit theorem.

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1. Introduction

We consider the one-dimensional stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sigma(u)\dot{W} \tag{1.1}$$

on $\mathbb{R}_+ \times \mathbb{R}$, where $\dot{W}$ is a space–time Gaussian white noise, with initial condition $u_0(x) = 1$. The coefficient σ is a Lipschitz function.

It is well-known (see, for instance, [12]) that this equation has a unique mild solution, which is adapted to the filtration generated by W, such that $\mathbb{E}[|u(t,x)|^2] < \infty$ and it satisfies the

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evolution equation

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)\,, \tag{1.2}$$

where in the right hand side the stochastic integral is in the sense of Walsh, $*$ denotes the spatial convolution and $p_t(x) = (2\pi t)^{-1/2}e^{-x^2/(2t)}$ is the heat kernel.

In this paper we are interested in the asymptotic behavior as R tends to infinity of the quantity

$$F_R(t) := \frac{1}{\sigma_R}\left(\int_{-R}^R u(t,x)dx - \mathbb{E}\int_{-R}^R u(t,x)dx\right), \tag{1.3}$$

where $R > 0$, $u(t,x)$ is the solution to (1.1) and $\sigma_R^2 = \mathrm{Var}\left(\int_{-R}^R u(t,x)dx\right)$.

From Eq. (1.2) and the properties of the heat kernel, it follows that the solution to Eq. (1.1) satisfies a localization property. This means that, for any fixed $t > 0$, the random variable $u(t,x)$ essentially depends on the noise in a small interval $[x-\epsilon, x+\epsilon]$. This property has been extensively used in the literature, see for example, [4–6].

In particular, for the parabolic Anderson model ($\sigma(u) = u$), it is shown in [5] that for each fixed $t > 0$, almost surely, the solution $u(t,x)$ develops high peaks along the x-axis. More precisely, it holds that, almost surely

$$0 < \limsup_{R\to\infty} \frac{\max_{|x|\le R} \log u(t,x)}{(\log R)^{2/3}} < \infty\,.$$

The basic idea in [5] to show this result is that one can define a "localized version" of Eq. (1.1) with solution $U(t,x)$, such that, whenever x_i and $x_j \in \mathbb{R}$ are far apart for $i \neq j$, $U(t,x_i)$, $i = 0, \pm 1, \pm 2, \ldots$, are i.i.d. random variables, and also $U(t,x)$ and $u(t,x)$ are close in certain sense. Since a rare event (high peak in this case) will happen with high probability if there are enough independent random variables, i.e., $U(t,x_i)$, $i = 0, \pm 1, \pm 2, \ldots$, one can see that $u(t,x)$, which is close to $U(t,x)$, also develops high peaks.

Following this idea, the spatial integral $\int_R^R u(t,x)dx$ is similar to a sum of i.i.d. random variables and we expect that certain central limit theorem holds in this case. To be more precise, our first result is the following quantitative central limit theorem:

Theorem 1.1. *Suppose that $u(t,x)$ is the mild solution to Eq. (1.1) and let $F_R(t)$ be given by (1.3). Suppose that $\sigma_R > 0$. Let d_{TV} denote the total variation distance and let $Z \sim \mathcal{N}(0,1)$. Then there exists a constant C, depending only on t, such that*

$$d_{TV}\left(F_R(t), Z\right) \le \frac{C}{\sqrt{R}}. \tag{1.4}$$

Remark. Condition $\sigma(1) \neq 0$ guarantees that $\sigma_R > 0$. Notice that this condition is not necessary. Taking into account that $\sigma_R = 0$ implies

$$\int_0^t \int_{\mathbb{R}} \mathbb{E}(\sigma^2(u(s,y)))\left(\int_{-R}^R p_{t-s}(x-y)dx\right)^2 dyds = 0,$$

a sufficient condition would be that $\sigma(u(s,y))$ is not identically zero on $[0,t]\times\mathbb{R}$ with positive probability.

We will show (see Proposition 3.1) that the variance σ_R^2 satisfies

$$\lim_{R\to\infty} \frac{\sigma_R^2}{R} = 2\int_0^t \xi(s)ds,$$

where $\xi(s) = \mathbb{E}[\sigma(u(s,y))^2]$. It turns out that $\mathbb{E}[\sigma(u(s,y))^2]$ does not depend on $y \in \mathbb{R}$ and is bounded on compact intervals. Then, we also prove the following functional version of Theorem 1.1 with a normalization by $1/\sqrt{R}$.

Theorem 1.2. *Suppose that $u(t,x)$ is the mild solution to Eq. (1.1). Set $\xi(s) = \mathbb{E}[\sigma(u(s,y))^2]$, $s \geq 0$. Then, for any $T > 0$,*

$$\left(\frac{1}{\sqrt{R}}\left(\int_{-R}^R u(t,x)dx - 2R\right)\right)_{t\in[0,T]} \to \left(\int_0^t \sqrt{2\xi(s)}dB_s\right)_{t\in[0,T]},$$

as R tends to infinity, where B is a Brownian motion and the convergence is in law on the space of continuous functions $C([0,T])$.

Theorem 1.1 is proved using a combination of Stein's method for normal approximations and Malliavin calculus, following the ideas introduced by Nourdin and Peccati in [8]. An innovative aspect of our methodology is to use the representation of $F_R(t)$ as a divergence, taking into account that the Itô–Walsh integral is a particular case of the Skorohod integral.

The rest of the paper is organized as follows. In Section 2 we recall some preliminaries on Malliavin calculus and Stein's method. Sections 3 and 4 are devoted to the proofs of our main theorems. We put one technical lemma into the Appendix.

2. Preliminaries

Let us first introduce the white noise on $\mathbb{R}_+ \times \mathbb{R}$. We denote by $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R})$ the collection of Borel sets $A \subset \mathbb{R}_+ \times \mathbb{R}$ with finite Lebesgue measure, denoted by $|A|$. Consider a centered Gaussian family of random variables $W = \{W(A), A \in \mathcal{B}_b\}$, defined in a complete probability space $(\Omega, \mathcal{F}, P)$, with covariance

$$\mathbb{E}[W(A)W(B)] = |A \cap B|.$$

For any $t \geq 0$, we denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W([0,s] \times A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R})\}$. As proved in [12], for any adapted random field $\{X(s,y),\ (s,y) \in \mathbb{R}_+ \times \mathbb{R}\}$ that is jointly measurable and

$$\int_0^\infty \int_{\mathbb{R}} \mathbb{E}[X(s,y)^2]dyds < \infty, \tag{2.1}$$

the following stochastic integral

$$\int_0^\infty \int_{\mathbb{R}} X(s,y)W(ds,dy)$$

is well-defined.

The proof of the main theorems relies on Malliavin calculus and Stein's method. Next we will introduce the basic elements of these methodologies.

2.1. Malliavin calculus

In this subsection we recall some basic facts on the Malliavin calculus associated with W. We refer to [9] for a detailed account on the Malliavin calculus with respect to a Gaussian process. Consider the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}_+ \times \mathbb{R})$. The Wiener integral

$$W(h) = \int_0^\infty \int_{\mathbb{R}} h(t,x) W(dt,dx)$$

provides an isometry between the Hilbert space $\mathfrak{H}$ and $L^2(\Omega)$. In this sense $\{W(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process.

Denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let $\mathcal{S}$ be the space of simple random variables of the form

$$F = f(W(h_1), \dots, W(h_n))$$

for $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}$, $1 \le i \le n$. Then DF is the $\mathfrak{H}$-valued random variable defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i \,. \tag{2.2}$$

The derivative operator D is a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathfrak{H})$ for any $p \ge 1$. For any $p \ge 1$, let $\mathbb{D}^{1,p}$ be the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(\mathbb{E}|F|^p + \mathbb{E}\|DF\|_{\mathfrak{H}}^p\right)^{1/p}.$$

We denote by δ the adjoint of the derivative operator given by the duality formula

$$\mathbb{E}(\delta(u)F) = \mathbb{E}(\langle u, DF\rangle_{\mathfrak{H}}) \tag{2.3}$$

for any $F \in \mathbb{D}^{1,2}$, and any $u \in L^2(\Omega; \mathfrak{H})$ in the domain of δ, denoted by $\operatorname{Dom}\delta$. The operator δ is also called the Skorohod integral because in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod (see [7,10]). More generally, in the context of the space–time white noise W, any adapted random field X which is jointly measurable and satisfies (2.1) belongs to the domain of δ and $\delta(X)$ coincides with the Walsh integral:

$$\delta(X) = \int_0^\infty \int_{\mathbb{R}} X(s,y) W(ds,dy).$$

As a consequence, the mild Eq. (1.2) can also be written as

$$u(t,x) = 1 + \delta\left(p_{t-\cdot}(x-*)u(\cdot,*)\right). \tag{2.4}$$

It is known that for any (t,x) the solution $u(t,x)$ of Eq. (1.1) belongs to $\mathbb{D}^{1,p}$ for any $p \ge 2$ and the derivative satisfies the following linear stochastic integral differential equation for $t \ge s$,

$$\begin{aligned} D_{s,y}u(t,x) &= p_{t-s}(x-y)\sigma(u(s,y)) \\ &\quad + \int_s^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma(r,z) D_{s,y}u(r,z) W(dr,dz), \end{aligned} \tag{2.5}$$

where $\Sigma(r,z)$ is an adapted process, bounded by the Lipschitz constant of σ. If σ is continuously differentiable, then $\Sigma(r,z) = \sigma'(u(r,z))$. This result is proved in Proposition

2.4.4 of [9] in the case of Dirichlet boundary conditions on [0, 1] and the proof can be easily extended to equations on $\mathbb{R}$. We also refer to [2,11] for additional references where this result is used when σ is continuously differentiable.

2.2. Stein's method

Stein's method is a probabilistic technique which allows one to measure the distance between a probability distribution and normal distribution. The total variance distance between two random variables F and G is defined by

$$d_{TV}(F,G) := \sup_{B\in\mathcal{B}(\mathbb{R})} |P(F\in B)-P(G\in B)|, \tag{2.6}$$

where $\mathcal{B}(\mathbb{R})$ is the collection of all Borel sets in $\mathbb{R}$. We point out that $d_{TV}(F,G)$ only depends on the laws of F and G and it defines a metric on the set of probability measures on $\mathbb{R}$.

The following theorem provides an upper bound for the total variation distance between any random variable and a random variable with standard normal distribution.

Theorem 2.1. *For $Z\sim\mathcal{N}(0,1)$ and for any random variable F,*

$$d_{TV}(F,Z) \le \sup_{f\in\mathscr{F}_{TV}} |\mathbb{E}[f'(F)]-\mathbb{E}[Ff(F)]|, \tag{2.7}$$

where $\mathscr{F}_{TV}$ is the class of continuously differentiable functions f such that $\|f\|_\infty \le \sqrt{\pi/2}$ and $\|f'\|_\infty \le 2$.

See [8] for a proof of this theorem. Theorem 2.1 can be combined with Malliavin calculus to get the following estimate.

Proposition 2.2. *Let $F=\delta(v)$ for some $\mathfrak{H}$-valued random variable v which belongs to* Dom δ. *Assume $\mathbb{E}[F^2]=1$ and $F\in\mathbb{D}^{1,2}$. Let $Z\sim\mathcal{N}(0,1)$. Then we have*

$$d_{TV}(F,Z)\le 2\sqrt{\mathrm{Var}\langle DF,v\rangle_{\mathfrak{H}}}\,. \tag{2.8}$$

Proof. By our assumption on F, we have

$$\begin{aligned}\mathbb{E}[Ff(F)] &= \mathbb{E}[\delta(v)f(F)] = \mathbb{E}\langle v, D[f(F)]\rangle_{\mathfrak{H}}\\ &= \mathbb{E}\langle v, f'(F)DF\rangle_{\mathfrak{H}} = \mathbb{E}\left(f'(F)\langle v, DF\rangle_{\mathfrak{H}}\right).\end{aligned}$$

Thus, by Theorem 2.1,

$$\begin{aligned} d_{TV}(F,Z) &\le \sup_{f\in\mathscr{F}_{TV}} \left|\mathbb{E}[f'(F)-Ff(F)]\right|\\ &= \sup_{f\in\mathscr{F}_{TV}} \left|\mathbb{E}[f'(F)(1-\langle DF,v\rangle_{\mathfrak{H}})]\right|\\ &\le 2\mathbb{E}(|1-\langle DF,v\rangle_{\mathfrak{H}}|)\\ &\le 2\sqrt{\mathrm{Var}\langle DF,v\rangle_{\mathfrak{H}}}\,,\end{aligned}$$

where the last step follows from Cauchy–Schwarz inequality, (2.3) and

$$\mathbb{E}(\langle DF,v\rangle_{\mathfrak{H}}) = \mathbb{E}[F\delta(v)] = \mathbb{E}(F^2)=1. \quad \square$$

In proving Theorem 1.2 we also need the following proposition, which is a generalization of Theorem 6.1.2 in [8].

Proposition 2.3. *Let $F = (F^{(1)}, \dots, F^{(m)})$ be a random vector such that $F^{(i)} = \delta(v^{(i)})$ for $v^{(i)} \in \operatorname{Dom}\delta$, $i = 1, \dots, m$. Assume $F^{(i)} \in \mathbb{D}^{1,2}$ for $i = 1, \dots, m$. Let Z be an m-dimensional Gaussian centered vector with covariance matrix $(C_{i,j})_{1\le i,j\le m}$. For any C^2 function $h : \mathbb{R}^m \to \mathbb{R}$ with bounded second partial derivatives, we have*

$$|\mathbb{E}h(F_R) - \mathbb{E}h(Z)| \le \frac{1}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^m \mathbb{E}\left[(C_{i,j} - \langle DF^{(i)}, v^{(j)}\rangle_{\mathfrak{H}})^2\right]},$$

where

$$\|h''\|_\infty = \max_{1\le i,j\le m} \sup_{x\in\mathbb{R}^m} \left|\frac{\partial^2 h}{\partial x_i \partial x_j}(x)\right|.$$

Proof. The proof will follow the same ideas as those in the proof of Theorem 6.1.2 in [8]. Without loss of generality, we may assume that Z and F are independent. Let

$$\Phi(t) = \mathbb{E}\left[h\left(\sqrt{1-t}F + \sqrt{t}Z\right)\right].$$

Then

$$\mathbb{E}[h(Z)] - \mathbb{E}[h(F)] = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(t)dt,$$

with

$$\Phi'(t) = \sum_{i=1}^m \mathbb{E}\left(\frac{\partial h}{\partial x_i}\left(\sqrt{1-t}F + \sqrt{t}Z\right)\left[\frac{1}{2\sqrt{t}}Z^{(i)} - \frac{1}{2}\frac{1}{\sqrt{1-t}}F^{(i)}\right]\right). \tag{2.9}$$

The above expression is a sum of two expectations. For the first expectation, the proof of Theorem 6.1.2 in [8] already yields that

$$\mathbb{E}\left(\frac{\partial h}{\partial x_i}\left(\sqrt{1-t}F + \sqrt{t}Z\right)Z^{(i)}\right) = \sqrt{t}\sum_{j=1}^m C_{i,j}\mathbb{E}\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\left(\sqrt{1-t}F + \sqrt{t}Z\right)\right). \tag{2.10}$$

For the second expectation, let $\mathbb{E}_F$ be the expectation conditioned on Z, then we have

$$\begin{aligned}
&\mathbb{E}\left(\frac{\partial h}{\partial x_i}\left(\sqrt{1-t}F + \sqrt{t}Z\right)F^{(i)}\right) \\
=&\mathbb{E}\mathbb{E}_F\left(\frac{\partial h}{\partial x_i}\left(\sqrt{1-t}F + \sqrt{t}Z\right)\delta(v^{(i)})\right) \\
=&\mathbb{E}\mathbb{E}_F\left(\left\langle D\frac{\partial h}{\partial x_i}\left(\sqrt{1-t}F + \sqrt{t}Z\right), v^{(i)}\right\rangle_{\mathfrak{H}}\right) \\
=&\sqrt{1-t}\sum_{j=1}^m \mathbb{E}\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\left(\sqrt{1-t}F + \sqrt{t}Z\right)\langle DF^{(j)}, v^{(i)}\rangle_{\mathfrak{H}}\right).
\end{aligned}$$

Finally, combining the above calculation with (2.9) and (2.10) with an application of Cauchy–Schwarz inequality completes the proof. □

3. Proof of Theorem 1.1

We begin by computing the asymptotic covariance of $F_R(t)$ as R tends to infinity. This will be also relevant in the proof of Theorem 1.2.

Proposition 3.1. *Denote $\xi(r) = \mathbb{E}[\sigma(u(r,x))^2]$ and set*

$$G_R(t) = \int_{-R}^{R} u(t,x)dx - 2R.$$

Then, for any $s,t \geq 0$,

$$\lim_{R\to\infty} \frac{1}{R}\mathrm{Cov}(G_R(t), G_R(s)) = 2\int_0^{s\wedge t} \xi(r)dr.$$

Proof. Thanks to the Itô isometry we have

$$\begin{aligned}
\mathbb{E}[u(t,x)u(s,x')] &= 1 + \int_0^{s\wedge t}\int_{\mathbb{R}} p_{t-r}(x-y)p_{s-r}(x'-y)\mathbb{E}[\sigma(u(r,y))^2]dydr \\
&= 1 + \int_0^{s\wedge t}\int_{\mathbb{R}} \xi(r)p_{t-r}(x'-y)p_{s-r}(x-y)dydr \\
&= 1 + \int_0^{s\wedge t} \xi(r)p_{t+s-2r}(x-x')dr,
\end{aligned}$$

where in the last line we have used the semigroup property

$$\int_{\mathbb{R}} p_t(x'-y)p_s(y-x)dy = p_{t+s}(x'-x). \tag{3.1}$$

Since

$$\mathbb{E}\left(\int_{-R}^{R} u(t,x)dx\right) = 2R,$$

we obtain

$$\begin{aligned}
\mathrm{Cov}(G_R(t), G_R(s)) &= \int_{-R}^{R}\int_{-R}^{R}\int_0^{s\wedge t} \xi(r)p_{t+s-2r}(x-x')drdxdx' \\
&= 2\int_0^{s\wedge t}\xi(r)\int_0^{2R} p_{t+s-2r}(z)(2R-z)dzdr.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\lim_{R\to\infty}\frac{1}{R}\mathrm{Cov}(G_R(t), G_R(s)) &= \lim_{R\to\infty} 2\int_0^{s\wedge t}\xi(r)\int_0^{2R} p_{t+s-2r}(z)(2-\frac{z}{R})dzdr \\
&= 2\int_0^{s\wedge t}\xi(r)dr.
\end{aligned}$$

This concludes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.2, we know that for any $F \in \mathbb{D}^{1,2}$ such that $\mathbb{E}(F^2) = 1$ and $F = \delta(v)$,

$$d_{TV}(F, Z) \leq 2\sqrt{\mathrm{Var}(\langle DF, v\rangle_{\mathfrak{H}})},$$

where v is such that $F = \delta(v)$. Recall that in our case we have, applying Fubini's theorem,

$$\begin{aligned} F_R(t) &= \frac{1}{\sigma_R}\left(\int_{-R}^{R} u(t,x)dx - 2R\right) \\ &= \frac{1}{\sigma_R}\left(\int_{-R}^{R}\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)dx\right) \\ &= \int_0^t\int_{\mathbb{R}}\left(\frac{1}{\sigma_R}\int_{-R}^{R} p_{t-s}(x-y)\sigma(u(s,y))dx\right)W(ds,dy). \end{aligned}$$

As a consequence, taking into account Eq. (2.4), we have, for any fixed $t \geq 0$, $F_R(t) = \delta(v_R)$, where

$$v_R(s,y) = \mathbf{1}_{[0,t]}(s)\frac{1}{\sigma_R}\int_{-R}^{R} p_{t-s}(x-y)\sigma(u(s,y))dx.$$

Moreover,

$$D_{s,y}F_R = \mathbf{1}_{[0,t]}(s)\frac{1}{\sigma_R}\int_{-R}^{R} D_{s,y}u(t,x)dx.$$

Therefore,

$$\langle DF_R(t), v_R\rangle_{\mathfrak{H}} = \frac{1}{\sigma_R^2}\int_0^t\int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} p_{t-s}(x-y)\sigma(u(s,y))D_{s,y}u(t,x')dxdx'dyds.$$

From (2.5), we know that

$$\begin{aligned} D_{s,y}u(t,x') =& p_{t-s}(x'-y)\sigma(u(s,y)) \\ &+ \int_s^t\int_{\mathbb{R}} p_{t-r}(x'-z)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz), \end{aligned} \tag{3.2}$$

where $\Sigma(r,z)$ is a bounded and adapted random field. This produces the decomposition

$$\begin{aligned} \langle DF_R(t), v_R\rangle_{\mathfrak{H}} =& \frac{1}{\sigma_R^2}\int_0^t\int_{\mathbb{R}}\left(\int_{-R}^{R} p_{t-s}(x-y)dx\right)^2 \sigma^2(u(s,y))dyds \\ &+ \frac{1}{\sigma_R^2}\int_0^t\int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} p_{t-s}(x-y)\sigma(u(s,y)) \\ &\times\left(\int_s^t\int_{\mathbb{R}} p_{t-r}(\tilde{x}-z)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz)\right)dxd\tilde{x}dyds. \end{aligned} \tag{3.3}$$

Therefore, using that for any process $\Phi = \{\Phi(s), s \in [0,t]\}$ such that $\sqrt{\mathrm{Var}(\Phi_s)}$ is integrable on $[0,t]$, we have

$$\sqrt{\mathrm{Var}\left(\int_0^t \Phi_s ds\right)} \leq \int_0^t \sqrt{\mathrm{Var}(\Phi_s)}ds,$$

we can write

$$\sqrt{\mathrm{Var}(\langle DF_R(t), v_R\rangle_{\mathfrak{H}})} \leq A_1 + A_2,$$

where

$$A_1 = \frac{1}{\sigma_R^2} \int_0^t \left(\int_{\mathbb{R}^2} \left(\int_{-R}^R p_{t-s}(x-y)dx \right)^2 \left(\int_{-R}^R p_{t-s}(x'-y')dx' \right)^2 \right.$$
$$\left. \times \mathrm{Cov}\left(\sigma^2(u(s,y)), \sigma^2(u(s,y'))\right) dydy' \right)^{\frac{1}{2}} ds$$

and

$$A_2 = \frac{1}{\sigma_R^2} \int_0^t \left(\int_{\mathbb{R}^2} \int_{[-R,R]^4} p_{t-s}(x-y)p_{t-s}(x'-y') \int_s^t \int_{\mathbb{R}} p_{t-r}(\tilde{x}-z)p_{t-r}(\tilde{x}'-z) \right.$$
$$\times \mathbb{E}\left(\sigma(u(s,y))\sigma(u(s,y'))\Sigma^2(r,z)D_{s,y}u(r,z)D_{s,y'}u(r,z)\right)$$
$$\left. \times dzdrdxdx'd\tilde{x}d\tilde{x}'dydy' \right)^{\frac{1}{2}} ds\,.$$

The proof will be done in two steps:

Step 1: Let us first estimate the term A_2. Denote by L the Lipschitz constant of σ and let, for $p \geq 2$,

$$K_p(t) = \sup_{0\leq s\leq t} \sup_{y\in\mathbb{R}} \|\sigma(u(s,y))\|_p. \tag{3.4}$$

Then,

$$|\mathbb{E}(\sigma(u(s,y))\sigma(u(s,y'))\Sigma^2(r,z)D_{s,y}u(r,z)D_{s,y'}u(r,z))|$$
$$\leq K_4^2(t)L^2\|D_{s,y}u(r,z)\|_4\|D_{s,y'}u(r,z)\|_4.$$

We need to estimate $\|D_{s,y}u(r,z)\|_p$ for any $p \geq 2$. According to (3.2), for any $s \in [0,r]$, applying Burkholder's inequality yields

$$\|D_{s,y}u(r,z)\|_p \leq p_{r-s}(z-y)K_p(t)$$
$$+ c_p \left(E\left(\left| \int_s^r \int_{\mathbb{R}} p_{r-r_1}^2(z-z_1)\Sigma^2(r_1,z_1)|D_{s,y}u(r_1,z_1)|^2 dr_1dz_1 \right|^{\frac{p}{2}} \right) \right)^{\frac{1}{p}}$$
$$\leq p_{r-s}(z-y)K_p(t)$$
$$+ Lc_p \left(\int_s^r \int_{\mathbb{R}} p_{r-r_1}^2(z-z_1)\|D_{s,y}u(r_1,z_1)\|_p^2 dr_1dz_1 \right)^{\frac{1}{2}},$$

which implies

$$\|D_{s,y}u(r,z)\|_p^2 \leq 2p_{r-s}^2(z-y)K_p^2(t)$$
$$+2L^2c_p^2 \int_s^r \int_{\mathbb{R}} p_{r-r_1}^2(z-z_1)\|D_{s,y}u(r_1,z_1)\|_p^2 dz_1dr_1\,.$$

By Lemma A.1, we have the estimate

$$\|D_{s,y}u(r,z)\|_p \leq Cp_{r-s}(z-y)\,, \tag{3.5}$$

where the constant C depends on t and p.

From (3.5) and Proposition 3.1, we derive the following estimate for the term A_2:

$$A_2 \leq \frac{C}{R}\int_0^t \Big(\int_{\mathbb{R}^2}\int_{[-R,R]^4}\int_s^t\int_{\mathbb{R}} p_{t-s}(x-y)p_{t-s}(x'-y')p_{t-r}(\tilde{x}-z) \times p_{t-r}(\tilde{x}'-z)p_{r-s}(z-y)p_{r-s}(z-y')dzdrdxdx'd\tilde{x}d\tilde{x}'dydy'\Big)^{\frac{1}{2}} ds\,.$$

Integrating $\tilde{x}, \tilde{x}'$ over $\mathbb{R}$, then integrating y', y over $\mathbb{R}$ and using the semigroup property, we obtain

$$\begin{aligned} A_2 &\leq \frac{C}{R}\int_0^t \left(\int_{[-R,R]^2}\int_s^t\int_{\mathbb{R}} p_{t+r-2s}(x-z)p_{t+r-2s}(x'-z)dzdrdxdx'\right)^{\frac{1}{2}} ds \\ &\leq \frac{C}{R}\int_0^t \left(\int_{[-R,R]^2}\int_s^t p_{2t+2r-4s}(x-x')drdxdx'\right)^{\frac{1}{2}} ds. \end{aligned}$$

Finally, integrating x over $\mathbb{R}$ and x' over $[-R, R]$, we get

$$A_2 \leq \frac{C}{\sqrt{R}}.$$

Step 2: To estimate the term A_1 we need a bound for the covariance

$$\operatorname{Cov}\left(\sigma^2(u(s,y)), \sigma^2(u(s,y'))\right).$$

Here, the main idea is to use a version of Clark–Ocone formula for two-parameter processes to write

$$\sigma^2(u(s,y)) = \mathbb{E}[\sigma^2(u(s,y))] + \int_0^s\int_{\mathbb{R}} \mathbb{E}[D_{r,z}(\sigma^2(u(s,y)))|\mathcal{F}_r]W(dr,dz).$$

Then,

$$\begin{aligned} &\operatorname{Cov}(\sigma^2(u(s,y)), \sigma^2(u(s,y'))) \\ &\quad = \int_0^s\int_{\mathbb{R}} \mathbb{E}\left[\mathbb{E}[D_{r,z}(\sigma^2(u(s,y)))|\mathcal{F}_r]\mathbb{E}[D_{r,z}(\sigma^2(u(s,y')))|\mathcal{F}_r]\right] dzdr. \end{aligned}$$

Applying the chain rule for Lipschitz functions (see [9, Proposition 1.2.4]), we have

$$D_{r,z}(\sigma^2(u(s,y))) = 2\sigma(u(s,y))\Sigma(s,y)D_{r,z}u(s,y).$$

and

$$\left\|\mathbb{E}[D_{r,z}(\sigma^2(u(s,y)))|\mathcal{F}_r]\right\|_2 \leq 2K_4(t)L\left\|D_{r,z}u(s,y)\right\|_4.$$

Then, using (3.5), we can write

$$\begin{aligned} &\left|\operatorname{Cov}\left(\sigma^2(u(s,y)), \sigma^2(u(s,y'))\right)\right| \\ &\quad \leq 4L^2K_4^2(t)\int_0^s\int_{\mathbb{R}} \left\|D_{r,z}u(s,y)\right\|_4 \left\|D_{r,z}u(s,y')\right\|_4 dzdr \\ &\quad \leq C\int_0^s\int_{\mathbb{R}} p_{s-r}(z-y)p_{s-r}(z-y')dzdr \\ &\quad = C\int_0^s p_{2s-2r}(y-y')dr. \end{aligned}$$

Therefore,

$$
\begin{aligned}
A_1 \le & \frac{C}{R}\int_0^t \Bigg(\int_{\mathbb{R}^2}\left(\int_{-R}^{R} p_{t-s}(x-y)dx\right)^2\left(\int_{-R}^{R} p_{t-s}(x'-y')dx'\right)^2 \\
& \times \int_0^s p_{2s-2r}(y-y')drdydy'\Bigg)^{\frac{1}{2}} ds \\
\le & \frac{C}{R}\int_0^t \Bigg(\int_0^s \int_{\mathbb{R}^2}\int_{[-R,R]^4} p_{t-s}(x-y)p_{t-s}(\tilde{x}-y)p_{t-s}(x'-y')p_{t-s}(\tilde{x}'-y') \\
& \times p_{2s-2r}(y-y')dxd\tilde{x}dx'd\tilde{x}'dydy'dr\Bigg)^{\frac{1}{2}} ds.
\end{aligned}
$$

Again, integrate $\tilde{x}$ and $\tilde{x}'$ over $\mathbb{R}$, then integrate y and y' over $\mathbb{R}$ using the semigroup property, to obtain

$$
A_1 \le \frac{C}{R}\int_0^t \left(\int_0^s \int_{[-R,R]^2} p_{2t-2r}(x-x')dxdx'dr\right)^{\frac{1}{2}} ds.
$$

Finally, integrating x over $\mathbb{R}$ and x' from $-R$ to R, we obtain

$$
A_1 \le \frac{C}{\sqrt{R}}.
$$

This completes the proof of Theorem 1.1. □

With a slight modification of the above proof, we can extend Theorem 1.1 to more general initial conditions.

Corollary 3.2. *Assume that there are two positive constants c_1 and c_2 such that $0 < c_1 \le u_0(x) \le c_2 < \infty$ for all x, and also assume that σ^2 is nondecreasing. Then (1.4) still holds.*

Proof. The idea of the proof is as follows. Let u_1 be the solution of Eq. (1.2) with the constant initial condition c_1. According to the comparison principle [3], we have $u(t,x) \ge u_1(t,x)$ a.s. for all t and x. Thus,

$$
\begin{aligned}
\sigma_R^2 =& \mathrm{Var}(F_R(t)) \\
=& \int_0^t \int_{\mathbb{R}} \int_{-R}^{R}\int_{-R}^{R} p_{t-s}(x-y)p_{t-s}(\tilde{x}-y)\mathbb{E}\sigma(u(s,y))^2 dxd\tilde{x}dyds \\
\ge& \int_0^t \int_{\mathbb{R}} \int_{-R}^{R}\int_{-R}^{R} p_{t-s}(x-y)p_{t-s}(\tilde{x}-y)\mathbb{E}\sigma(u_1(s,y))^2 dxd\tilde{x}dyds \\
=& \int_0^t \mathbb{E}\sigma(u_1(s,0))^2 \int_{-R}^{R}\int_{-R}^{R} p_{2(t-s)}(x-\tilde{x})dxd\tilde{x}ds \\
\sim& 2R\int_0^t \mathbb{E}\sigma(u_1(s,0))^2 ds\,.
\end{aligned}
$$

Moreover, (3.5) still holds and the proof follows from the same argument. The reason we assume that $u_0(x) \le c_2$ is because we still need the finiteness of (3.4). The rest of the proof follows the same lines as the proof of Theorem 1.1. □

Remark 1. The assumption that u_0 is uniformly bounded away from 0 is important. For example, consider Eq. (1.1) and assume that $u_0 \in L^1(\mathbb{R})$ is bounded and $\sigma(u) = u$. Then, by [4], we know that the solution $u(t,\cdot) \in L^1(\mathbb{R})$ a.s. According to the mild formulation (1.2) we have

$$\int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx + \int_0^t \int_{\mathbb{R}} u(s,y)W(ds,dy).$$

We conclude that $\int_{\mathbb{R}} u(t,x)dx$ is not Gaussian since on the right-hand side $u(s,y)$ has an infinite chaos expansion, thus the integral $\int_0^t \int_{\mathbb{R}} u(s,y)W(ds,dy)$ also has an infinite chaos expansion.

4. Proof of Theorem 1.2

We begin with the following result that ensures tightness.

Proposition 4.1. *Let $u(t,x)$ be the solution to Eq. (1.1). Then for any $0 \le s < t \le T$ and any $p \ge 1$ there exists a constant $C = C(p,T)$ such that*

$$\mathbb{E}\left(\left|\int_{-R}^{R} u(t,x)dx - \int_{-R}^{R} u(s,x)dx\right|^p\right) \le CR^{\frac{p}{2}}(t-s)^{\frac{p}{2}}.$$

Proof. Let us assume that $s < t$. Recall that

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-r}(x-y)\sigma(u(r,y))W(dr,dy),$$

and thus

$$\begin{aligned}&\int_{-R}^{R} u(t,x)dx - \int_{-R}^{R} u(s,x)dx\\ &\quad= \int_0^T \int_{\mathbb{R}} \int_{-R}^{R} dx \left(p_{t-r}(x-y)\mathbf{1}_{\{r\le t\}} - p_{s-r}(x-y)\mathbf{1}_{\{r\le s\}}\right)\sigma(u(r,y))W(dr,dy).\end{aligned}$$

Moreover, recall that $\mathbb{E}(|u(s,y)|^p)$ is bounded on $s \le T$ and $y \in \mathbb{R}$ for any $p \ge 1$. Using Burkholder–Davis–Gundy inequality, we can write

$$\begin{aligned}&\mathbb{E}\left(\left|\int_{-R}^{R} u(t,x)dx - \int_{-R}^{R} u(s,x)dx\right|^p\right)\\ &\le c_p\mathbb{E}\left(\int_0^T \int_{\mathbb{R}} \left(\int_{-R}^{R} \left(p_{t-r}(x-y)\mathbf{1}_{\{r\le t\}} - p_{s-r}(x-y)\mathbf{1}_{\{r\le s\}}\right)dx\right)^2 \sigma(u(r,y))^2 dydr\right)^{\frac{p}{2}}\\ &\le c_p\left(\int_0^T \int_{\mathbb{R}} \left(\int_{-R}^{R} \left(p_{t-r}(x-y)\mathbf{1}_{\{r\le t\}} - p_{s-r}(x-y)\mathbf{1}_{\{r\le s\}}\right)dx\right)^2 \|\sigma(u(r,y))\|_p^2 dydr\right)^{\frac{p}{2}}\\ &\le C_{p,T}\left(\int_0^T \int_{\mathbb{R}} \left(\int_{-R}^{R} \left(p_{t-r}(x-y)\mathbf{1}_{\{r\le t\}} - p_{s-r}(x-y)\mathbf{1}_{\{r\le s\}}\right)dx\right)^2 dydr\right)^{\frac{p}{2}}.\end{aligned}$$

Thus it suffices to prove that

$$\int_0^T \int_{\mathbb{R}} \left(\int_{-R}^{R} \left(p_{t-r}(x-y)\mathbf{1}_{\{r\le t\}} - p_{s-r}(x-y)\mathbf{1}_{\{r\le s\}}\right)dx\right)^2 dydr \le CR(t-s). \tag{4.1}$$

Using Fourier transform we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left(\int_{-R}^{R} \left(p_{t-r}(x-y)\mathbf{1}_{\{r\le t\}} - p_{s-r}(x-y)\mathbf{1}_{\{r\le s\}} \right) dx \right)^2 dydr \\
=&C \int_0^T \int_{\mathbb{R}} \left(\int_{-R}^{R} e^{i\xi x} dx \right)^2 \left(e^{-\frac{t-r}{2}|\xi|^2}\mathbf{1}_{\{r\le t\}} - e^{-\frac{s-r}{2}|\xi|^2}\mathbf{1}_{\{r\le s\}} \right)^2 d\xi dr \\
=&C \int_0^T \int_{\mathbb{R}} \frac{\sin^2(R|\xi|)}{|\xi|^2} \left(e^{-\frac{t-r}{2}|\xi|^2}\mathbf{1}_{\{r\le t\}} - e^{-\frac{s-r}{2}|\xi|^2}\mathbf{1}_{\{r\le s\}} \right)^2 d\xi dr \\
=&C \int_0^s \int_{\mathbb{R}} \left(e^{-\frac{t-r}{2}|\xi|^2} - e^{-\frac{s-r}{2}|\xi|^2} \right)^2 \frac{\sin^2(R|\xi|)}{|\xi|^2} d\xi dr \\
&+ C \int_s^t \int_{\mathbb{R}} e^{-(t-r)|\xi|^2} \frac{\sin^2(R|\xi|)}{|\xi|^2} d\xi dr \\
:=&C(I_1 + I_2)\,.
\end{aligned}
$$

For I_1 we can write

$$
\begin{aligned}
I_1 &= \int_0^s \int_{\mathbb{R}} e^{r|\xi|^2} \left(e^{-\frac{t}{2}|\xi|^2} - e^{-\frac{s}{2}|\xi|^2} \right)^2 \frac{\sin^2(R|\xi|)}{|\xi|^2} d\xi dr \\
&= \int_{\mathbb{R}} \frac{1-e^{-s|\xi|^2}}{|\xi|^2} \left(e^{-\frac{t-s}{2}|\xi|^2} - 1 \right)^2 \frac{\sin^2(R|\xi|)}{|\xi|^2} d\xi\,.
\end{aligned}
$$

Using the bound $|1-e^{-a}| \le \sqrt{a}$ for all $a \ge 0$ in the above parenthesis, we obtain that

$$
I_1 \le \int_{\mathbb{R}} \frac{1}{|\xi|^2} |\xi|^2 |t-s| \frac{\sin^2(R|\xi|)}{|\xi|^2} d\xi \le CR|t-s|.
$$

For I_2, using the bound $1-e^{-a} \le a$ for any $a \ge 0$,

$$
I_2 = \int_{\mathbb{R}} \frac{1-e^{-(t-s)|\xi|^2}}{|\xi|^2} \frac{\sin^2(R|\xi|)}{|\xi|^2} d\xi \le (t-s) \int_{\mathbb{R}} \frac{\sin^2(R|\xi|)}{|\xi|^2} d\xi = CR(t-s).
$$

The proof is finished by combining I_1 and I_2. □

Proof of Theorem 1.2. It suffices to prove the convergence of the finite-dimensional distributions and tightness. However, the latter follows directly from Proposition 4.1.

In order to show the convergence of the finite-dimensional distributions, fix points $0 \le t_1 < \cdots < t_m \le T$ and consider the random variables

$$
F_R^{(i)} = \frac{1}{\sqrt{R}} \left(\int_{-R}^{R} u(t_i, x) dx - 2R \right),
$$

for $i = 1, \ldots, m$. We can write $F_R^{(i)} = \delta(v_R^{(i)})$, where

$$
v_R^{(i)}(s,y) = \mathbf{1}_{[0,t_i]}(s) \frac{1}{\sqrt{R}} \int_{-R}^{R} p_{t_i - s}(x-y)\sigma(u(s,y))dx.
$$

Set $F_R = (F_R^{(1)}, \ldots, F_R^{(m)})$ and let Z be an m-dimensional Gaussian centered vector with covariance

$$
C_{i,j} := \mathbb{E}[Z^i Z^j] = \int_0^{t_i \wedge t_j} \xi(r) dr,
$$

where we recall that $\xi(r) = \mathbb{E}[\sigma(u(r,x))^2]$. Then, applying Proposition 2.3, for any C^2 function $h : \mathbb{R}^m \to \mathbb{R}$ with bounded second partial derivatives, we have

$$|\mathbb{E}(h(F_R)) - \mathbb{E}(h(Z))| \leq \frac{1}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^m \mathbb{E}\left[\left(C_{i,j} - \langle DF_R^{(i)}, v_R^{(j)}\rangle_{\mathfrak{H}}\right)^2\right]}.$$

Then, it suffices to show that for each i, j, $\langle DF_R^{(i)}, v_R^{(j)}\rangle_{\mathfrak{H}}$ converges in L^2, as R tends to infinity to $C_{i,j}$. To be more precisely, similarly with (3.3), we have

$$\begin{aligned}
&\langle DF_R^{(i)}, v_R^{(j)}\rangle_{\mathfrak{H}} \\
=&\frac{1}{R}\int_0^{t_i\wedge t_j}\int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} p_{t_i-s}(x-y)p_{t_j-s}(\tilde{x}-y)\sigma^2(u(s,y))dxd\tilde{x}dyds \\
&+\frac{1}{R}\int_0^{t_i\wedge t_j}\int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} p_{t_j-s}(\tilde{x}-y)\sigma(u(s,y)) \\
&\quad\times\left(\int_s^{t_i}\int_{\mathbb{R}} p_{t_i-r}(x-z)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz)\right)dxd\tilde{x}dyds \\
:=&I_{1,i,j}(R) + I_{2,i,j}(R)\,.
\end{aligned} \tag{4.2}$$

Then we obtain that

$$\mathbb{E}\left[\left(C_{ij} - \langle DF_R^{(i)}, v_R^{(j)}\rangle_{\mathfrak{H}}\right)^2\right] \leq 2\mathbb{E}\left(C_{ij} - I_{1,i,j}(R)\right)^2 + 2\mathbb{E}\left(I_{2,i,j}(R)^2\right).$$

By noting that

$$C_{ij} = \lim_{R\to\infty}\mathbb{E}(I_{1,i,j}(R)),$$

and using arguments similar as those in the proof of Theorem 1.1, we can show that $\mathbb{E}\left[\left(C_{ij} - \langle DF_R^{(i)}, v_R^{(j)}\rangle_{\mathfrak{H}}\right)^2\right] \to 0$ as $R \to \infty$. The proof is finished. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Here we prove a technical lemma which is used in the proof of Theorem 1.1.

Lemma A.1. *Let* $\|D_{s,y}u(r,z)\|_p^2$ *satisfy*

$$\|D_{s,y}u(r,z)\|_p^2 \leq Cp_{r-s}^2(z-y) + C\int_s^r\int_{\mathbb{R}} p_{r-r_1}^2(z-z_1)\|D_{s,y}u(r_1,z_1)\|_p^2dz_1dr_1\,, \tag{A.1}$$

for any $0 < s < r \leq t$ *and* $y, z \in \mathbb{R}$, *for some constant* C *which depends on* t *and* p. *Then we have*

$$\|D_{s,y}u(r,z)\|_p \le C_{t,p}\,p_{r-s}(z-y)\,. \tag{A.2}$$

for some constant $C_{t,p}$ which depends on t and p.

Proof. Consider the stochastic heat equation

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + \sqrt{C}\,v\dot{W}, \quad t \ge s,$$

with initial condition $v(s,z) = \sqrt{C}\delta_y(z)$. The mild formulation of this equation is

$$v(r,z) = \sqrt{C}\,p_{r-s}(z-y) + \sqrt{C}\int_s^r \int_{\mathbb{R}} p_{r-r_1}(z-z_1)v(r_1,z_1)W(dr_1,dz_1)$$

for $r \ge s$ and $z \in \mathbb{R}$. Therefore, the moment of order 2 satisfies

$$\mathbb{E}[v^2(r,z)] = Cp_{r-s}^2(z-y) + C\int_s^r \int_{\mathbb{R}} p_{r-r_1}^2(z-z_1)\mathbb{E}[v^2(r_1,z_1)]dz_1dr_1.$$

which is exactly Eq. (A.1) with equality. Using the iteration method and the moment bounds in [1] (Theorem 2.4, Proposition 2.2 and Example 2.10) we conclude that (A.2) holds true. □

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