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A Central Limit Theorem for the stochastic wave equation with fractional noise

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Abstract. We study the one-dimensional stochastic wave equation driven by a Gaussian multiplicative noise, which is white in time and has the covariance of a fractional Brownian motion with Hurst parameter $H \in [1/2, 1)$ in the spatial variable. We show that the normalized spatial average of the solution over $[-R, R]$ converges in total variation distance to a normal distribution, as R tends to infinity. We also provide a functional Central Limit Theorem.

Résumé. Nous étudions l'équation des ondes en une dimension, perturbée par un bruit gaussien multiplicatif, qui est blanc en temps et qui a la covariance d'un mouvement brownien fractionnaire avec paramètre de Hurst $H \in [1/2, 1)$ dans la variable d'espace. Nous démontrons que la moyenne spatiale normalisée de la solution sur un intervalle $[-R, R]$ converge, en la distance de la variation totale, vers une loi normale, quand R tend vers l'infini. Nous prouvons aussi un théorème central limite fonctionnel.

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1. Introduction

We consider the one-dimensional stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sigma(u)\frac{\partial^2 W}{\partial t\,\partial x}, \tag{1.1}$$

on $\mathbb{R}_+ \times \mathbb{R}$, where $W(t,x)$ is a Gaussian random field that is a Brownian motion in time and behaves as a fractional Brownian motion with Hurst parameter $H \in [1/2, 1)$ in the spatial variable. For $H = 1/2$, the random field W is just a two-parameter Wiener process on $\mathbb{R}_+ \times \mathbb{R}$. We assume $u(0,x) = 1$, $\frac{\partial}{\partial t}u(0,x) = 0$ and σ is a Lipschitz function with Lipschitz constant $L \in (0, \infty)$.

It is well-known (see, for instance, [4,17]) that equation (1.1) has a unique *mild solution*, which is adapted to the filtration generated by W, such that $\sup\{\mathbb{E}[|u(t,x)|^2] : x \in \mathbb{R}, t \in [0,T]\} < \infty$ and

$$u(t,x) = 1 + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{|x-y| \le t-s\}}\sigma\big(u(s,y)\big)W(ds,dy), \tag{1.2}$$

where the above stochastic integral is defined in the sense of Itô–Walsh.

In this paper, we are interested in the asymptotic behavior as R tends to infinity of the spatial averages

$$\int_{-R}^{R} u(t,x)\,dx, \tag{1.3}$$

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where $t>0$ is fixed and $u(t,x)$ is the solution to (1.1). We remark that, for each fixed $t>0$, the process $\{u(t,x), x\in\mathbb{R}\}$ is *strictly stationary*,[1] meaning that the finite-dimension distributions of the process $\{u(t,x+y), x\in\mathbb{R}\}$ do not depend on y. Furthermore, $u(t,x)$ is measurable with respect to the σ-field generated by the random variables $\{W(s,z): |x-z|\le t-s\}$. As a consequence,

(1) for $H=1/2$, the random variables $u(t,x)$ and $u(t,y)$ are independent if $|x-y|>2t$;
(2) for $H\in(1/2,1)$, $u(t,x)$ and $u(t,y)$ have a correlation that decays like $|x-y-2t|^{2H-2}$ when $|x-y|\to+\infty$, which is a consequence of Gebelein's inequality (see, for instance, [16]).

Therefore, we expect the Gaussian fluctuation of the spatial averages (1.3).

Our first goal is to apply the methodology of Malliavin–Stein to provide a quantitative Central Limit Theorem for (1.3), which will be described in total variation distance.

Define the normalized averages by

$$F_R(t) := \frac{1}{\sigma_R}\left(\int_{-R}^{R} u(t,x)\,dx - 2R\right), \tag{1.4}$$

where $u(t,x)$ is the solution to (1.1) and $\sigma_R^2 = \mathrm{Var}(\int_{-R}^R u(t,x)\,dx)$.

To avoid triviality, throughout this paper, we assume that $\sigma(1)\neq 0$, which guarantees that $\sigma_R>0$ for all $R>0$ and also that σ_R is of order R^H; see Lemma 3.4 and Propositions 3.2, 3.3 below.

Our first result is the following quantitative Central Limit Theorem.

Theorem 1.1. *Let d_{TV} denote the total variation distance (see (2.8)) and let $Z\sim\mathcal{N}(0,1)$. For any fixed $t>0$, there exists a constant $C=C_{t,H,\sigma}$, depending on t, H and σ, such that*

$$d_{\mathrm{TV}}\big(F_R(t), Z\big) \le C R^{H-1}.$$

Our second objective is to provide the functional version of Theorem 1.1.

Theorem 1.2. *For any $s>0$, we set $\eta(s)=\mathbb{E}[\sigma(u(s,y))]$ and $\xi(s)=\mathbb{E}[\sigma^2(u(s,y))]$, which do not depend on y due to the stationarity. Then, for any $T>0$, as $R\to+\infty$,*

(i) *if $H=1/2$, then*

$$\left\{\frac{1}{\sqrt{R}}\left(\int_{-R}^{R} u(t,x)\,dx - 2R\right)\right\}_{t\in[0,T]} \quad\Rightarrow\quad \left\{\sqrt{2}\int_0^t (t-s)\sqrt{\xi(s)}\,dB_s\right\}_{t\in[0,T]};$$

(ii) *if $H\in(1/2,1)$, then*

$$\left\{R^{-H}\left(\int_{-R}^{R} u(t,x)\,dx - 2R\right)\right\}_{t\in[0,T]} \quad\Rightarrow\quad \left\{2^H\int_0^t (t-s)\eta(s)\,dB_s\right\}_{t\in[0,T]}.$$

Here B is a standard Brownian motion and the above weak convergence takes place in the space of continuous functions $C([0,T])$.

Theorem 1.1 is proved using a combination of Stein's method for normal approximation and Malliavin calculus, following the ideas introduced by Nourdin and Peccati in [9]. The main idea is as follows. The total variation distance $d_{\mathrm{TV}}(F_R(t),Z)$ is bounded by $2\sqrt{\mathrm{Var}\langle DF_R(t), v_R\rangle_{\mathfrak{H}}}$, where D is the derivative in the sense of Malliavin calculus, $\mathfrak{H}$ is the Hilbert space associated to the noise W and v_R is an $\mathfrak{H}$-valued random variable such that $F_R(t)=\delta(v_R)$, δ being the adjoint of the derivative operator, called the divergence or the Skorohod integral. A key new ingredient in the application of this approach is to use the representation of $F_R(t)$ as a stochastic integral of v_R, taking into account that the Itô–Walsh integral is a particular case of the Skorohod integral.

A similar problem for the stochastic heat equation on $\mathbb{R}$ has been recently considered in [7], but only in the case of a space-time white noise. In this case, it was proved in [7] that the limiting process in the functional Central Limit Theorem is a martingale, which is not true for our wave equation. Moreover, in the colored case $H\in(1/2,1)$ considered here, we

[1] To see the strict stationarity, we fix $y\in\mathbb{R}$ and put $v(t,x)=u(t,x+y)$: It is clear that v solves the stochastic heat equation (1.1) driven by the shifted noise $\{W(t,x+y), t\in\mathbb{R}_+, x\in\mathbb{R}\}$, which has stationary increments in the spatial variable.

have found the surprising result that the square moment $\mathbb{E}[\sigma^2(u(s,y))]$ in the white noise case is replaced by the square of the first moment $(\mathbb{E}[\sigma(u(s,y))])^2$. Furthermore, the rate of convergence depends on the Hurst parameter H.

When $\sigma(u)=u$, the solution has an explicit Wiener chaos expansion. A natural question in this case is *whether the central limit is chaotic*, meaning that the projection on each Wiener chaos contributes to the limit. Such a phenomenon has been observed in other cases (see, for instance, [6]). We will show that for $H>1/2$ only the first chaos contributes to the limit, where as for $H=1/2$, we will see in Remark 1 that the first chaos is not the only contributor in the limit and to check whether or not this central limit is chaotic, one shall go through the usual arguments for chaotic Central Limit Theorem (see [11, Section 8.4]).

The rest of the paper is organized as follows. In Section 2 we recall some preliminaries on Malliavin calculus and Stein's method. Sections 3 and 4 are devoted to the proofs of our main theorems. We put the proof of a technical lemma (Lemma 2.2) in the Appendix. This lemma, which has an independent interest, states that the p-norm of the Malliavin derivative $D_{s,y}u(t,x)$ can be estimated, up to constant that depends on p and t, by the fundamental solution of the wave equation $\frac{1}{2}\mathbf{1}_{\{|x-y|\le t-s\}}$.

Along the paper we will denote by C a generic constant that might depend on the fixed time t, the Hurst parameter H and the non-linear coefficient σ, and it can vary from line to line.

2. Preliminaries

We denote by $W=\{W(t,x), t\ge 0, x\in\mathbb{R}\}$ a centered Gaussian family of random variables defined in some probability space $(\Omega,\mathcal{F},P)$, with covariance function given by

$$\mathbb{E}\big[W(t,x)W(s,y)\big]=\frac{s\wedge t}{2}\big(|x|^{2H}+|y|^{2H}-|x-y|^{2H}\big),$$

where $H\in[1/2,1)$.

Let $\mathfrak{H}_0$ be the Hilbert space defined as the completion of the set of step functions on $\mathbb{R}$ equipped with the inner product

$$\langle\varphi,\phi\rangle_{\mathfrak{H}_0}=\begin{cases}H(2H-1)\int_{\mathbb{R}^2}\varphi(x)\phi(y)|x-y|^{2H-2}\,dx\,dy & \text{if } H\in(1/2,1),\\ \int_{\mathbb{R}}\varphi(x)\phi(x)\,dx & \text{if } H=1/2.\end{cases}\tag{2.1}$$

Set $\mathfrak{H}=L^2(\mathbb{R}_+;\mathfrak{H}_0)$ and notice that

$$\mathbb{E}\big[W(t,x)W(s,y)\big]=\langle\mathbf{1}_{[0,t]\times[0,x]},\mathbf{1}_{[0,s]\times[0,y]}\rangle_{\mathfrak{H}},$$

where, by convention, $[0,x]=[-|x|,0]$ if x is negative. Therefore, the mapping $(t,x)\to W(t,x)$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian subspace of $L^2(\Omega)$ generated by W. We denote this isometry by $\varphi\longmapsto W(\varphi)$.

When $H=1/2$, the space $\mathfrak{H}$ is simply $L^2(\mathbb{R}_+\times\mathbb{R})$ and $W(\varphi)$ is the Wiener–Itô integral of φ:

$$W(\varphi)=\int_{\mathbb{R}_+\times\mathbb{R}}\varphi(t,x)W(dt,dx).$$

For $H\in(1/2,1)$, the space $L^{1/H}(\mathbb{R})$ is known to be continuously embedded into $\mathfrak{H}_0$; see [8,15].

For any $t\ge 0$, we denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s,x):0\le s\le t, x\in\mathbb{R}\}$. Then, for any adapted $\mathfrak{H}_0$-valued stochastic process $\{X(t),t\ge 0\}$ such that

$$\int_0^\infty\mathbb{E}\big[\|X(t)\|_{\mathfrak{H}_0}^2\big]\,dt<\infty,\tag{2.2}$$

the following stochastic integral

$$\int_0^\infty\int_{\mathbb{R}}X(s,y)W(ds,dy)\tag{2.3}$$

is well-defined and satisfies the isometry property

$$\mathbb{E}\bigg[\bigg(\int_0^\infty\int_{\mathbb{R}}X(s,y)W(ds,dy)\bigg)^2\bigg]=\mathbb{E}\bigg(\int_0^\infty\|X(t)\|_{\mathfrak{H}_0}^2\,dt\bigg).$$

We will make use of the following lemma and the notation $\alpha_H=H(2H-1)$.

Lemma 2.1. *For any* $H \in (1/2, 1)$, $s, t \geq 0$ *and* $x, \xi \in \mathbb{R}$, *we have*

$$\begin{aligned} 2\alpha_H \int_{\mathbb{R}^2} \mathbf{1}_{\{|x-y|\leq t\}} \mathbf{1}_{\{|\xi-z|\leq s\}} |y-z|^{2H-2}\, dy\, dz \\ = |x-\xi-t-s|^{2H} + |x-\xi+t+s|^{2H} \\ - |x-\xi+t-s|^{2H} - |x-\xi-t+s|^{2H}. \end{aligned} \tag{2.4}$$

Proof. Let B^H be a two-sided fractional Brownian motion with Hurst parameter H. That is, $B^H = \{B_t^H, t \in \mathbb{R}\}$ is a centered Gaussian process with covariance

$$\mathbb{E}\big[B_t^H B_s^H\big] = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \quad s, t \in \mathbb{R}.$$

Notice that both sides of (2.4) are equal to $2\mathbb{E}[(B^H_{x+t} - B^H_{x-t})(B^H_{\xi+s} - B^H_{\xi-s})]$, in view of (2.1) and the above covariance structure. So the desired equality follows immediately. □

The proof of our main theorems relies on a combination of Malliavin calculus and Stein's method. We will introduce these tools in the next two subsections.

2.1. *Malliavin calculus*

Now we recall some basic facts on Malliavin calculus associated with W. For a detailed account of the Malliavin calculus with respect to a Gaussian process, we refer to Nualart [10].

Denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let $\mathcal{S}$ be the space of simple functionals of the form

$$F = f\big(W(h_1), \dots, W(h_n)\big)$$

for $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}$, $1 \leq i \leq n$. Then, DF is the $\mathfrak{H}$-valued random variable defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\big(W(h_1), \dots, W(h_n)\big) h_i.$$

The derivative operator D is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathfrak{H})$ for any $p \geq 1$ and we let $\mathbb{D}^{1,p}$ be the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \big(\mathbb{E}\big[|F|^p\big] + \mathbb{E}\big[\|DF\|_{\mathfrak{H}}^p\big]\big)^{1/p}.$$

We denote by δ the adjoint of D given by the duality formula

$$\mathbb{E}\big(\delta(u)F\big) = \mathbb{E}\big(\langle u, DF\rangle_{\mathfrak{H}}\big)$$

for any $F \in \mathbb{D}^{1,2}$ and $u \in \operatorname{Dom}\delta \subset L^2(\Omega; \mathfrak{H})$, the domain of δ. The operator δ is also called the Skorohod integral, because in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod (see [5,12]). More generally, in the context of our Gaussian noise W, any adapted random field X that satisfies (2.2) belongs to the domain of δ and $\delta(X)$ coincides with the Dalang–Walsh-type stochastic integral (2.3):

$$\delta(X) = \int_0^\infty \int_{\mathbb{R}} X(s, y) W(ds, dy).$$

As a consequence, the mild formulation equation (1.2) can also be written as

$$u(t, x) = 1 + \frac{1}{2}\delta\big(\mathbf{1}_{\{|x-*|\leq t-\cdot\}}\sigma\big(u(\cdot, *)\big)\big). \tag{2.5}$$

It is known that for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, the solution $u(t,x)$ to equation (1.1) belongs to $\mathbb{D}^{1,p}$ for any $p \geq 2$ and the derivative satisfies the following linear stochastic integral differential equation for $t \geq s$,

$$D_{s,y}u(t,x) = \frac{1}{2}\mathbf{1}_{\{|x-y|\leq t-s\}}\sigma\big(u(s,y)\big) + \frac{1}{2}\int_s^t \int_{\mathbb{R}} \mathbf{1}_{\{|x-z|\leq t-r\}}\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz), \tag{2.6}$$

where $\Sigma(r,z)$ is an adapted process, bounded by the Lipschitz constant of σ (we refer to the Appendix for more details on the properties of the derivative). If σ is continuously differentiable, then $\Sigma(r,z) = \sigma'(u(r,z))$. This result is proved in [10, Proposition 2.4.4] in the case of the stochastic heat equation with Dirichlet boundary conditions on [0, 1] driven by a space-time white noise. Its proof can be easily extended to the wave equation on $\mathbb{R}$ driven by the colored noise W. We also refer to [1,13] for additional references, where this result is used for $\sigma \in C^1(\mathbb{R})$.

In the end of this subsection, we record a technical result that is essential for our arguments, and we postpone its proof to the Appendix.

Lemma 2.2. *For any $p \in [2,+\infty)$, $0 \leq t \leq T$ and $x \in \mathbb{R}$, we have for almost every $(s,y) \in [0,T] \times \mathbb{R}$,*

$$\big\| D_{s,y}u(t,x)\big\|_p \leq C\mathbf{1}_{\{|x-y|\leq t-s\}} \tag{2.7}$$

for some constant $C = C_{T,p,H,\sigma}$ that depends on T, p, H and the function σ.

2.2. *Stein's method*

Stein's method is a probabilistic technique that allows one to measure the distance between a probability distribution and a target distribution, notably the normal distribution. Recall that the total variation distance between two real random variables F and G is defined by

$$d_{\mathrm{TV}}(F,G) := \sup_{B\in\mathcal{B}(\mathbb{R})} \big|P(F\in B) - P(G\in B)\big|, \tag{2.8}$$

where $\mathcal{B}(\mathbb{R})$ is the collection of all Borel sets in $\mathbb{R}$.

The following theorem provides the well-known Stein's bound in the total variation distance; see [9, Chapter 3].

Theorem 2.3. *For $Z \sim \mathcal{N}(0,1)$ and for any integrable random variable F,*

$$d_{\mathrm{TV}}(F,Z) \leq \sup_{f\in\mathcal{F}_{\mathrm{TV}}} \big|\mathbb{E}\big[f'(F)\big] - \mathbb{E}\big[Ff(F)\big]\big|, \tag{2.9}$$

where $\mathcal{F}_{\mathrm{TV}}$ is the class of continuously differentiable functions $f: \mathbb{R} \to \mathbb{R}$ such that $\|f\|_\infty \leq \sqrt{\pi/2}$ and $\|f'\|_\infty \leq 2$.

For a proof of this theorem, see [9, Theorem 3.3.1]. Theorem 2.3 can be combined with Malliavin calculus to get a very useful estimate (see [7,11,14]).

Proposition 2.4. *Let $F = \delta(v)$ for some $\mathfrak{H}$-valued random variable $v \in \operatorname{Dom}\delta$. Assume $F \in \mathbb{D}^{1,2}$ and $\mathbb{E}[F^2] = 1$ and let $Z \sim \mathcal{N}(0,1)$. Then we have*

$$d_{\mathrm{TV}}(F,Z) \leq 2\sqrt{\operatorname{Var}\big[\langle DF, v\rangle_{\mathfrak{H}}\big]}. \tag{2.10}$$

In the course of proving Theorem 1.2, we also need the following lemma, which is a generalization of [9, Theorem 6.1.2]; see [7, Proposition 2.3].

Lemma 2.5. *Let $F = (F^{(1)},\dots,F^{(m)})$ be a random vector such that $F^{(i)} = \delta(v^{(i)})$ for $v^{(i)} \in \operatorname{Dom}\delta$ and $F^{(i)} \in \mathbb{D}^{1,2}$, $i = 1,\dots,m$. Let Z be an m-dimensional centered Gaussian vector with covariance $(C_{i,j})_{1\leq i,j\leq m}$. For any C^2 function $h: \mathbb{R}^m \to \mathbb{R}$ with bounded second partial derivatives, we have*

$$\big|\mathbb{E}\big[h(F)\big] - \mathbb{E}\big[h(Z)\big]\big| \leq \frac{m}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^m \mathbb{E}\big[\big(C_{i,j} - \big\langle DF^{(i)}, v^{(j)}\big\rangle_{\mathfrak{H}}\big)^2\big]},$$

where $\|h''\|_\infty := \sup\{|\frac{\partial^2}{\partial x_i\,\partial x_j}h(x)| : x \in \mathbb{R}^m, i,j = 1,\dots,m\}$.

3. Proof of Theorem 1.1

We begin with the asymptotic variance of $F_R(t)$, as R tends to infinity. We need some preliminary results and notation. We fix $t > 0$ and define

$$\varphi_R(s, y) = \frac{1}{2}\int_{-R}^{R} \mathbf{1}_{\{|x-y|\le t-s\}}\, dx.$$

Notice that $2\varphi_R(s, y)$ is the length of $[-R, R] \cap [y - t + s, y + t - s]$, so

$$\varphi_R(s, y) = \frac{1}{2}\big(\big[R \wedge (y + t - s)\big] - \big[(-R) \vee (y - t + s)\big]\big)_+.$$

As a consequence, we deduce that

$$\varphi_R(s, y) = 0, \quad \text{if } |y| \ge R + t - s; \quad \text{and} \quad \varphi_R(s, y) \le R \wedge (t - s).$$

Set $G_R = G_R(t) := \int_{-R}^{R} u(t, x)\, dx - 2R$. With this notation, we can write

$$G_R = \int_0^t \int_{\mathbb{R}} \varphi_R(s, y)\sigma\big(u(s, y)\big) W(ds, dy).$$

The next lemma provides a useful formula.

Lemma 3.1. *Let $0 < a \le b$ and define $\varphi_{a,R}(y) = \frac{1}{2}\int_{-R}^{R} \mathbf{1}_{\{|x-y|\le a\}}\, dx$, then we have, for any $R \ge 2b$,*

$$\int_{\mathbb{R}} \frac{1}{R}\varphi_{a,R}(y)\varphi_{b,R}(y)\, dy = 2ab - R^{-1}\left(\frac{1}{2}ab^2 + \frac{1}{6}a^3\right).$$

Therefore, $\lim_{R\to+\infty} \int_{\mathbb{R}} \frac{1}{R}\varphi_{a,R}(y)\varphi_{b,R}(y)\, dy = 2ab$.

Proof. We can write

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{R}\varphi_{a,R}(y)\varphi_{b,R}(y)\, dy \\
&\quad = \frac{1}{4R}\int_{\mathbb{R}}\int_{[-R,R]^2} \mathbf{1}_{\{|\tilde{x}-y|\le a\}}\mathbf{1}_{\{|x-y|\le b\}}\, d\tilde{x}\, dx\, dy \\
&\quad = \frac{1}{4R}\int_{[-R,R]^2} d\tilde{x}\, dx (\mathbf{1}_{\{|\tilde{x}-x|\le b-a\}} + \mathbf{1}_{\{b-a<|\tilde{x}-x|\le b+a\}}) \int_{\mathbb{R}} \mathbf{1}_{\{|\tilde{x}-y|\le a, |x-y|\le b\}}\, dy \\
&\quad = \frac{1}{4R}\int_{[-R,R]^2} \big\{\mathbf{1}_{\{|\tilde{x}-x|\le b-a\}}(2a) + \mathbf{1}_{\{b-a<|\tilde{x}-x|\le b+a\}}\big(a + b - |x - \tilde{x}|\big)\big\}\, dx\, d\tilde{x},
\end{aligned}$$

which is equal to $2ab - R^{-1}(\frac{1}{2}ab^2 + \frac{1}{6}a^3)$ for any $R \ge 2b$, as one can verify. □

The next result provides the asymptotic variance of $G_R(t)$ for $H = 1/2$.

Proposition 3.2. *Suppose $H = 1/2$. Denote $\xi(s) = \mathbb{E}[\sigma^2(u(s, x))]$, which does not depend on x as a consequence of stationarity. Then*

$$\lim_{R\to\infty} \frac{1}{R}\mathbb{E}\big[G_R^2\big] = 2\int_0^t (t - s)^2\xi(s)\, ds.$$

and $\mathbb{E}[G_R^2] \ge (\frac{5}{3}\int_0^t (t - s)^2\xi(s)\, ds)R$ for any $R \ge 2t$.

Proof. Thanks to the Itô isometry, we have

$$\mathbb{E}\big[G_R^2\big] = \int_0^t \int_{\mathbb{R}} \varphi_R^2(s, y)\mathbb{E}\big[\sigma^2\big(u(s, y)\big)\big]\, dy\, ds = \int_0^t \xi(s)\int_{\mathbb{R}} \varphi_R^2(s, y)\, dy\, ds.$$

If $R \geq 2t$, we can see from Lemma 3.1 that

$$\frac{1}{R}\int_{\mathbb{R}} \varphi_R^2(s,y)\,dy = 2(t-s)^2\left(1-\frac{t-s}{3R}\right) \in \left[\frac{5}{3}(t-s)^2, 2(t-s)^2\right]. \tag{3.1}$$

This leads easily to the results. □

Surprisingly, in the case $H > 1/2$, we obtain a different formula for the asymptotic variance of G_R.

Proposition 3.3. *Suppose $H \in (1/2, 1)$. Denote $\eta(s) = \mathbb{E}[\sigma(u(s,x))]$, which does not depend on x as a consequence of stationarity. Then*

$$\lim_{R\to\infty} R^{-2H}\mathbb{E}\big[G_R^2\big] = 2^{2H}\int_0^t (t-s)^2\eta^2(s)\,ds.$$

Proof. Thanks to the Itô isometry, we have

$$\mathbb{E}\big[G_R^2\big] = \alpha_H \int_0^t \int_{\mathbb{R}^2} \varphi_R(s,y)\varphi_R(s,z)\mathbb{E}\big[\sigma\big(u(s,y)\big)\sigma\big(u(s,z)\big)\big]|y-z|^{2H-2}\,dy\,dz\,ds,$$

where $\alpha_H = H(2H-1)$. Keeping in mind that $\{\sigma(u(t,x)), x \in \mathbb{R}\}$ is stationary, we write $\mathbb{E}[\sigma(u(s,y))\sigma(u(s,z))] =: \Psi(s, y-z)$. Then,

$$\mathbb{E}\big[G_R^2\big] = \alpha_H \int_0^t \int_{\mathbb{R}^2} \varphi_R(s,\xi+z)\varphi_R(s,z)\Psi(s,\xi)|\xi|^{2H-2}\,d\xi\,dz\,ds.$$

We claim that

$$\lim_{|\xi|\to+\infty} \sup_{0\leq s\leq t} \big|\Psi(s,\xi) - \eta^2(s)\big| = 0. \tag{3.2}$$

In order to show (3.2), we apply a two-parameter version of the Clark–Ocone formula (see *e.g.* [2, Proposition 6.3]). We can write

$$\sigma\big(u(s,y)\big) = \mathbb{E}\big[\sigma\big(u(s,y)\big)\big] + \int_0^s \int_{\mathbb{R}} \mathbb{E}\big[D_{r,\gamma}\big(\sigma\big(u(s,y)\big)\big)|\mathcal{F}_r\big]W(dr,d\gamma)$$

and

$$\sigma\big(u(s,z)\big) = \mathbb{E}\big[\sigma\big(u(s,z)\big)\big] + \int_0^s \int_{\mathbb{R}} \mathbb{E}\big[D_{r,\beta}\big(\sigma\big(u(s,z)\big)\big)|\mathcal{F}_r\big]W(dr,d\beta).$$

As a consequence,

$$\mathbb{E}\big[\sigma\big(u(s,y)\big)\sigma\big(u(s,z)\big)\big] = \eta^2(s) + T(s,y,z), \tag{3.3}$$

where

$$\begin{aligned} T(s,y,z) = {} & \int_0^s \int_{\mathbb{R}^2} \mathbb{E}\big\{\mathbb{E}\big[D_{r,\gamma}\big(\sigma\big(u(s,y)\big)\big)|\mathcal{F}_r\big]\mathbb{E}\big[D_{r,\beta}\big(\sigma\big(u(s,z)\big)\big)|\mathcal{F}_r\big]\big\} \\ & \times |\gamma-\beta|^{2H-2}\,d\gamma\,d\beta\,dr. \end{aligned} \tag{3.4}$$

By the chain-rule for the derivative operator (see [10, Proposition 1.2.4]),

$$D_{r,\gamma}\big(\sigma\big(u(s,y)\big)\big) = \Sigma(s,y)D_{r,\gamma}u(s,y)$$

and

$$D_{r,\beta}\big(\sigma\big(u(s,z)\big)\big) = \Sigma(s,z)D_{r,\beta}u(s,z)$$

with $\Sigma(s,y)$ an adapted random field uniformly bounded by the Lipschitz constant of σ, denoted by L. This implies, using (2.7),

$$\begin{aligned}&\left|\mathbb{E}\left\{\mathbb{E}\left[D_{r,\gamma}\left(\sigma\left(u(s,y)\right)\right)|\mathcal{F}_r\right]\mathbb{E}\left[D_{r,\beta}\left(\sigma\left(u(s,z)\right)\right)|\mathcal{F}_r\right]\right\}\right|\\&\quad\le L^2\left\|D_{r,\gamma}u(s,y)\right\|_2\left\|D_{r,\beta}u(s,z)\right\|_2\le C\mathbf{1}_{\{|\gamma-y|\le s-r\}}\mathbf{1}_{\{|\beta-z|\le s-r\}},\end{aligned}\tag{3.5}$$

for some constant C. Therefore, substituting (3.5) into (3.4), we can write

$$\left|T(s,y,z)\right|\le C\int_0^s\int_{\mathbb{R}^2}\mathbf{1}_{\{|\gamma-y|\le s-r\}}\mathbf{1}_{\{|\beta-z|\le s-r\}}|\gamma-\beta|^{2H-2}\,d\gamma\,d\beta\,dr.\tag{3.6}$$

If $|y-z|>2s$, we have

$$\mathbf{1}_{\{|\gamma-y|\le s-r\}}\mathbf{1}_{\{|\beta-z|\le s-r\}}|\gamma-\beta|^{2H-2}\le\mathbf{1}_{\{|\gamma-y|\le s-r\}}\mathbf{1}_{\{|\beta-z|\le s-r\}}\left(|y-z|-2s\right)^{2H-2}$$

and therefore deduce from (3.6) that (for $|y-z|>2s$)

$$\begin{aligned}\left|T(s,y,z)\right|&\le C\int_0^s\int_{\mathbb{R}^2}\mathbf{1}_{\{|\gamma-y|\le s-r\}}\mathbf{1}_{\{|\beta-z|\le s-r\}}\left(|y-z|-2s\right)^{2H-2}\,d\gamma\,d\beta\,dr\\&\le 4Ct^3\left(|y-z|-2t\right)^{2H-2}\xrightarrow{|y-z|\to+\infty}0.\end{aligned}$$

Thus, claim (3.2) is established in view of formula (3.3).

Let us continue our proof of Proposition 3.2. We first show that the quantity

$$\frac{1}{R^{2H}}\int_0^t\int_{\mathbb{R}^2}\varphi_R(s,\xi+z)\varphi_R(s,z)\left[\Psi(s,\xi)-\eta^2(s)\right]|\xi|^{2H-2}\,d\xi\,dz\,ds\tag{3.7}$$

converges to zero, as $R\to+\infty$.

By (3.2), we can find $K=K_\varepsilon>0$ for any given $\varepsilon>0$ such that

$$\sup\left\{\left|\Psi(s,\xi)-\eta^2(s)\right|:s\in[0,t],|\xi|>K\right\}<\varepsilon.$$

Now we divide the above integration domain into two parts $|\xi|\le K$ and $|\xi|>K$.

Case (i): On the region $|\xi|\le K$, by Cauchy–Schwarz inequality and (3.1), we get for $R\ge 2t$

$$\begin{aligned}\int_{\mathbb{R}}\varphi_R(s,\xi+z)\varphi_R(x,z)\,dz&\le\left(\int_{\mathbb{R}}\varphi_R^2(s,\xi+z)\,dz\right)^{1/2}\left(\int_{\mathbb{R}}\varphi_R^2(s,z)\,dz\right)^{1/2}\\&=\int_{\mathbb{R}}\varphi_R^2(s,z)\,dz=2R(t-s)^2\left(1-\frac{t-s}{3R}\right).\end{aligned}$$

Since $\Psi(s,y-z)-\eta^2(s)=T(s,y,z)$ is uniformly bounded for $(s,y,z)\in[0,t]\times\mathbb{R}^2$,

$$\begin{aligned}&R^{-2H}\int_0^t\int_{\mathbb{R}^2}\varphi_R(s,\xi+z)\varphi_R(s,z)\left|\Psi(s,\xi)-\eta^2(s)\right|\mathbf{1}_{\{|\xi|\le K\}}|\xi|^{2H-2}\,d\xi\,dz\,ds\\&\quad\le CR^{-2H}\int_0^t\int_{-K}^K\left(\int_{\mathbb{R}}\varphi_R(s,\xi+z)\varphi_R(x,z)\,dz\right)|\xi|^{2H-2}\,d\xi\,ds\\&\quad\le CR^{1-2H}\int_0^t(t-s)^2\int_{-K}^K|\xi|^{2H-2}\,d\xi\,ds\xrightarrow{R\to+\infty}0.\end{aligned}$$

Case (ii): On the region $|\xi|>K$, we know $|\Psi(s,\xi)-\eta^2(s)|<\varepsilon$ for $s\le t$. Thus,

$$\begin{aligned}&R^{-2H}\int_0^t\int_{\mathbb{R}^2}\varphi_R(s,\xi+z)\varphi_R(s,z)\left|\Psi(s,\xi)-\eta^2(s)\right|\mathbf{1}_{\{|\xi|>K\}}|\xi|^{2H-2}\,d\xi\,dz\,ds\\&\quad\le\frac{\varepsilon}{R^{2H}}\int_0^t\int_{\mathbb{R}^2}\varphi_R(s,\xi+z)\varphi_R(s,z)|\xi|^{2H-2}\,d\xi\,dz\,ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varepsilon}{4R^{2H}} \int_{-R}^{R}\int_{-R}^{R} dx\,dx' \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{|x-z|\le t-s\}} \mathbf{1}_{\{|x'-z'|\le t-s\}} \big|z-z'\big|^{2H-2}\,ds\,dz\,dz' \\
&\le \frac{\varepsilon t}{4R^{2H}} \int_{-R}^{R}\int_{-R}^{R} dx\,dx' \int_{\mathbb{R}^2} \mathbf{1}_{\{|x-z|\le t\}} \mathbf{1}_{\{|x'-z'|\le t\}} \big|z-z'\big|^{2H-2}\,dz\,dz' \\
&=: \varepsilon t^3 \mathcal{A}(R).
\end{aligned}
$$

We can rewrite $\mathcal{A}(R)$, after a change of variables and supposing $R > t$, in the following form

$$
\begin{aligned}
\mathcal{A}(R) &= \int_{[-1,1]^2} \int_{\mathbb{R}^2} \frac{R}{2t}\mathbf{1}_{\{|x-z|\le tR^{-1}\}} \frac{R}{2t}\mathbf{1}_{\{|x'-z'|\le tR^{-1}\}} \big|z-z'\big|^{2H-2}\,dz\,dz'\,dx\,dx' \\
&= \int_{[-1,1]^2} (\zeta_R * g_2)\big(x,x'\big)\,dx\,dx' \xrightarrow{R\to+\infty} \|g_1\|_{L^1(\mathbb{R}^2)},
\end{aligned}
$$

where $g_m(z,z') := |z-z'|^{2H-2}\mathbf{1}_{\{z\neq z'\}}\mathbf{1}_{\{z,z'\in[-m,m]\}}$, for $m = 1, 2$, are integrable functions on $\mathbb{R}^2$ and

$$
\left\{ \zeta_R\big(x,x'\big) = \frac{R}{2t}\mathbf{1}_{\{|x|\le tR^{-1}\}} \frac{R}{2t}\mathbf{1}_{\{|x'|\le tR^{-1}\}},\ R > t \right\}
$$

defines an approximation of the identity. This leads to the asymptotic negligibility of the quantity (3.7), as $\varepsilon > 0$ is arbitrary.

Therefore, it suffices to show that

$$
\begin{aligned}
&\lim_{R\to\infty} \frac{\alpha_H}{R^{2H}} \int_0^t \eta^2(s) \int_{\mathbb{R}^2} \varphi_R(s,\xi+z)\varphi_R(s,z)|\xi|^{2H-2}\,d\xi\,dz\,ds \\
&\quad = 2^{2H} \int_0^t (t-s)^2\eta^2(s)\,ds. \qquad (3.8)
\end{aligned}
$$

The previous computations imply that

$$
\frac{1}{R^{2H}} \int_{\mathbb{R}^2} \varphi_R(s,\xi+z)\varphi_R(s,z)|\xi|^{2H-2}\,d\xi\,dz \le 4t^2\mathcal{A}(R)
$$

is uniformly bounded over $s \in [0,t]$. Moreover, we can get

$$
\begin{aligned}
&\lim_{R\to+\infty} \frac{\alpha_H}{R^{2H}} \int_{\mathbb{R}^2} \varphi_R(s,\xi)\varphi_R(s,z)|\xi-z|^{2H-2}\,d\xi\,dz \\
&\quad = (t-s)^2\alpha_H\|g_1\|_{L^1(\mathbb{R}^2)} = 2^{2H}(t-s)^2, \qquad (3.9)
\end{aligned}
$$

where the last equality follows from Lemma 2.1. Hence (3.8) follows by the dominated convergence theorem and this concludes our proof. □

It follows from the above two propositions that for fixed $t > 0$, the variance of $G_R(t)$, denoted by σ_R^2, is $O(R^{2H})$. The next lemma states that R^{2H} is the exact order under our standing assumption $\sigma(1) \neq 0$, which is also a necessary condition to have this order. Moreover, $\sigma(1) \neq 0$ is equivalent to $\sigma_R > 0$ for all $R > 0$.

Lemma 3.4. *The following four conditions are equivalent*:

(i) $\sigma(1) = 0$.
(ii) $\sigma_R = 0$ *for all* $R > 0$.
(iii) $\sigma_R = 0$ *for some* $R > 0$.
(iv) $\lim_{R\to\infty} \sigma_R^2 R^{-2H} = 0$.

Proof. If $\sigma(1) = 0$, then writing the solution as the limit of the Picard iterations starting with the constant solution 1, we obtain that $u(t,x) = 1$ for all (t,x). As a consequence, $\sigma_R = 0$ for all $R > 0$ and (i) implies (ii). Clearly (ii) implies (iii) and (iv). Now suppose that (iv) holds. Then Propositions 3.2 and 3.3 imply that for almost every $s \in [0,t]$,

(a) $\mathbb{E}[\sigma^2(u(s,y))] = 0$ in the case $H = 1/2$,
(b) $\mathbb{E}[\sigma(u(s,y))] = 0$ in the case $H \in (1/2, 1)$.

By the $L^2(\Omega)$-continuity of the process $(s,y) \in \mathbb{R}_+ \times \mathbb{R} \longmapsto u(s,y)$ (see *e.g.* [3, Theorem 13]), letting s tend to 0, we deduce that $\sigma(1) = 0$ in both cases $H = 1/2$ and $H \in (1/2, 1)$.

Finally, suppose that (iii) holds and assume that $H \in (1/2, 1)$ (the proof in the case $H = 1/2$ is similar). By L^2-continuity, we can see that the function $\Psi(s,y) := \mathbb{E}[\sigma(u(s,0))\sigma(u(s,y))]$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$. Note that, for almost all $s \in [0,t]$,

$$\int_{\mathbb{R}^2} \varphi_R(s,y)\varphi_R(s,z)\sigma\big(u(s,y)\big)\sigma\big(u(s,z)\big)|y-z|^{2H-2}\,dy\,dz = 0 \tag{3.10}$$

almost surely. In the above integral, the variables y and z have support contained in the interval $[-R-t, R+t]$. If $\sigma(1) \neq 0$, there exists a sufficiently small $\delta > 0$ such that for $(s,y,z) \in [0,\delta] \times [-R-t, R+t]^2$

$$\mathbb{E}\big[\sigma\big(u(s,y)\big)\sigma\big(u(s,z)\big)\big] = \Psi(s, y-z) \geq \big|\sigma(1)\big|^2/2,$$

which is a contradiction to (3.10). Therefore, $\sigma(1) = 0$ and (iii) implies (i). □

Remark 1. It follows from Proposition 3.3 that, if $H \in (1/2, 1)$, the random variable G_R is *not chaotic* in the linear case. More precisely, when $\sigma(x) = x$, the above proposition gives us

$$\mathrm{Var}(G_R) \sim R^{2H} 2^{2H} \int_0^t (t-s)^2\,ds = \frac{4^H t^3}{3} R^{2H} \quad \text{as } R \to +\infty.$$

Due to linearity, one can obtain the Wiener-chaos expansion of G_R easily:

$$G_R = \int_0^t \int_{\mathbb{R}} \varphi_R(s,y) W(ds,dy) + \textit{ higher-order chaoses.}$$

Then, the variance of the first chaos is equal to

$$\int_0^t \alpha_H \int_{\mathbb{R}^2} \varphi_R(s,y)\varphi_R(s,z)|y-z|^{2H-2}\,dy\,dz\,ds \sim \frac{4^H t^3}{3} R^{2H} \quad \text{as } R \to +\infty,$$

which is a consequence of (3.9) and dominated convergence. This shows that only the first chaos contribute to the limit, that is, there is a non-chaotic behavior of the spatial average of the *linear* stochastic wave equation, when $H \in (1/2, 1)$.

For $H = 1/2$ and $\sigma(x) = x$, we obtain from Proposition 3.2

$$\mathrm{Var}(G_R) \sim 2R \int_0^t (t-s)^2 \mathbb{E}\big[u^2(s,x)\big]\,ds \quad \text{as } R \to +\infty,$$

whereas the variance of the projection on the first chaos is, using Lemma 3.1,

$$\int_0^t \int_{\mathbb{R}} \varphi_R^2(s,y)\,dy\,ds = \frac{2}{3} R t^3 - \frac{t^4}{6}.$$

Notice that $\mathbb{E}[u^2(s,x)] \geq (\mathbb{E}[u(s,x)])^2 = 1$ and the inequality is strict for all $s \in (0,t]$ (otherwise $u(s,x)$ would be a constant). This implies that the first chaos is not the only contributor to the limiting variance.

Before we give the proof of Theorem 1.1, by using the same argument as in the proof of Propositions 3.2 and 3.3, we obtain an asymptotic formula for $\mathbb{E}[G_R(t_i)G_R(t_j)]$ with $t_i, t_j \in \mathbb{R}_+$, which is a useful ingredient for our proof of functional Central Limit Theorem.

Remark 2. Suppose $t_i, t_j \in \mathbb{R}_+$. If $H = 1/2$, we have

$$\mathbb{E}\big[G_R(t_i)G_R(t_j)\big] = \int_0^{t_i \wedge t_j} \int_{\mathbb{R}} \varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,y)\xi(s)\,dy\,ds,$$

where $\varphi_R^{(i)}(s,y) = \frac{1}{2}\int_{-R}^{R} \mathbf{1}_{\{|x-y|\le t_i - s\}}\,dx$ and we obtain

$$\lim_{R\to+\infty} \frac{1}{R}\mathbb{E}\big[G_R(t_i)G_R(t_j)\big] = 2\int_0^{t_i\wedge t_j} (t_i - s)(t_j - s)\xi(s)\,ds.$$

In the case $H \in (1/2, 1)$, we have $\mathbb{E}[G_R(t_i)G_R(t_j)]$ equal to

$$\alpha_H \int_0^{t_i\wedge t_j} \int_{\mathbb{R}^2} \varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,z)\Psi(s,y-z)|y-z|^{2H-2}\,dy\,dz\,ds,$$

and we obtain

$$\begin{aligned}
&\lim_{R\to+\infty} R^{-2H}\mathbb{E}\big[G_R(t_i)G_R(t_j)\big] \\
&\quad= \lim_{R\to+\infty} \alpha_H \int_0^{t_i\wedge t_j} ds\,\eta^2(s) \int_{\mathbb{R}^2} \varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,z)|y-z|^{2H-2}\,dy\,dz \\
&\quad= 2^{2H} \int_0^{t_i\wedge t_j} (t_i - s)(t_j - s)\eta^2(s)\,ds.
\end{aligned}$$

Now let us prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.4, if $F = \delta(v) \in \mathbb{D}^{1,2}$ with $\mathbb{E}(F^2) = 1$, we have

$$d_{\mathrm{TV}}(F,Z) \le 2\sqrt{\mathrm{Var}\big[\langle DF, v\rangle_{\mathfrak{H}}\big]}.$$

Recall that in our case we have, as a consequence of Fubini's theorem, that

$$\begin{aligned}
F_R := F_R(t) &= \frac{1}{\sigma_R}\int_{-R}^{R} \big[u(t,x) - 1\big]\,dx \\
&= \frac{1}{\sigma_R}\int_0^t \int_{\mathbb{R}} \varphi_R(s,y)\sigma\big(u(s,y)\big)W(ds,dy).
\end{aligned}$$

Similarly as in (2.5), we can write, for any fixed $t > 0$, $F_R = \delta(v_R)$ with $v_R(s,y) = \sigma_R^{-1}\mathbf{1}_{[0,t]}(s)\varphi_R(s,y)\sigma(u(s,y))$. Moreover,

$$D_{s,y}F_R = \mathbf{1}_{[0,t]}(s)\frac{1}{\sigma_R}\int_{-R}^{R} D_{s,y}u(t,x)\,dx.$$

Then, it follows from (2.6) and Fubini's theorem that

$$\begin{aligned}
&\int_{-R}^{R} D_{s,y}u(t,x)\,dx \\
&\quad= \varphi_R(s,y)\sigma\big(u(s,y)\big) + \int_s^t \int_{\mathbb{R}} \varphi_R(r,z)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz).
\end{aligned}$$

In what follows, we separate our proof into two cases: $H = 1/2$ and $H > 1/2$.

Case $H = 1/2$. We write

$$\langle DF_R, v_R\rangle_{\mathfrak{H}} := B_1 + B_2,$$

where

$$B_1 = \frac{1}{\sigma_R^2}\int_0^t \int_{\mathbb{R}} \varphi_R^2(s,y)\sigma^2\big(u(s,y)\big)\,ds\,dy$$

and

$$B_2 = \frac{1}{\sigma_R^2} \int_0^t \int_{\mathbb{R}} \varphi_R(s,y)\sigma\big(u(s,y)\big) \times \left(\int_s^t \int_{\mathbb{R}} \varphi_R(r,z)\Sigma(r,z) D_{s,y}u(r,z) W(dr,dz) \right) dy\,ds.$$

Notice that for any process $\Phi = \{\Phi(s), s \in [0,t]\}$ such that $\sqrt{\operatorname{Var}(\Phi_s)}$ is integrable on $[0,t]$, it holds that

$$\sqrt{\operatorname{Var}\left(\int_0^t \Phi_s\,ds \right)} \le \int_0^t \sqrt{\operatorname{Var}(\Phi_s)}\,ds. \tag{3.11}$$

So we can write

$$\sqrt{\operatorname{Var}\big[\langle DF_R, v_R\rangle_{\mathfrak{H}}\big]} \le \sqrt{2}\big(\sqrt{\operatorname{Var}(B_1)} + \sqrt{\operatorname{Var}(B_2)}\big) \le \sqrt{2}(A_1 + A_2),$$

with

$$A_1 = \frac{1}{\sigma_R^2} \int_0^t \left(\int_{\mathbb{R}^2} \varphi_R^2(s,y)\varphi_R^2(s,y') \operatorname{Cov}\big[\sigma^2\big(u(s,y)\big), \sigma^2\big(u(s,y')\big)\big]\,dy\,dy' \right)^{1/2} ds$$

and

$$A_2 = \frac{1}{\sigma_R^2} \int_0^t \left(\int_{\mathbb{R}^3} \int_s^t \varphi_R^2(r,z)\varphi_R(s,y)\varphi_R(s,y') \times \mathbb{E}\big[\Sigma^2(r,z) D_{s,y}u(r,z) D_{s,y'}u(r,z)\sigma\big(u(s,y)\big)\sigma\big(u(s,y')\big)\big]\,dy\,dy'\,dz\,dr \right)^{1/2} ds.$$

Then the rest of the proof for this case ($H = 1/2$) consists in estimating A_2 and A_1. The proof will be done in two steps.

Step 1: Let us proceed with the estimation of A_2. As before, denote by L the Lipschitz constant of σ and for $p \ge 2$, as a consequence of stationarity, we write

$$K_p(t) = \sup_{0\le s\le t} \sup_{y\in\mathbb{R}} \big\|\sigma\big(u(s,y)\big)\big\|_p = \sup_{0\le s\le t} \big\|\sigma\big(u(s,0)\big)\big\|_p. \tag{3.12}$$

Then,

$$\big|\mathbb{E}\big(\Sigma^2(r,z) D_{s,y}u(r,z) D_{s,y'}u(r,z)\sigma\big(u(s,y)\big)\sigma\big(u(s,y')\big)\big)\big| \le K_4^2(t)L^2 \big\|D_{s,y}u(r,z)\big\|_4 \big\|D_{s,y'}u(r,z)\big\|_4 \le CK_4^2(t)L^2 \mathbf{1}_{\{|y-z|\le r-s\}} \mathbf{1}_{\{|y'-z|\le r-s\}},$$

where the last inequality follows from Lemma 2.2. This implies, together with Proposition 3.2, that, for any $R \ge 2t$,

$$A_2 \le \frac{C}{R} \int_0^t \left(\int_{\mathbb{R}^3} \int_s^t \varphi_R^2(r,z)\varphi_R(s,y)\varphi_R(s,y') \times \mathbf{1}_{\{|y-z|\le r-s\}} \mathbf{1}_{\{|y'-z|\le r-s\}}\,dy\,dy'\,dz\,dr \right)^{1/2} ds.$$

Using first $\varphi_R(s,y)\varphi_R(s,y') \le (t-s)^2$ and then integrating in y and y', we obtain

$$A_2 \le \frac{C}{R} \int_0^t \left(\int_s^t \int_{\mathbb{R}} \varphi_R^2(r,z)\,dz\,dr \right)^{1/2} ds \le \frac{C}{R} \int_0^t \left(\int_s^t 2R(t-r)^2\,dr \right)^{1/2} ds,$$

where the last inequality follows from (3.1). Therefore, we have $A_2 \le C/\sqrt{R}$ for any $R \ge 2t$.

Step 2: Consider now the term A_1. We begin with a bound for the covariance

$$\operatorname{Cov}\big[\sigma^2\big(u(s,y)\big), \sigma^2\big(u(s,y')\big)\big].$$

Using a version of Clark–Ocone formula for two-parameter processes, we write

$$\sigma^2\big(u(s,y)\big)=\mathbb{E}\big[\sigma^2\big(u(s,y)\big)\big]+\int_0^s\int_{\mathbb{R}}\mathbb{E}\big[D_{r,z}\big(\sigma^2\big(u(s,y)\big)\big)|\mathcal{F}_r\big]W(dr,dz).$$

Then, $\mathrm{Cov}[\sigma^2(u(s,y)),\sigma^2(u(s,y'))]$ is equal to

$$\int_0^s\int_{\mathbb{R}}\mathbb{E}\big\{\mathbb{E}\big[D_{r,z}\big(\sigma^2\big(u(s,y)\big)\big)|\mathcal{F}_r\big]\mathbb{E}\big[D_{r,z}\big(\sigma^2\big(u\big(s,y'\big)\big)\big)|\mathcal{F}_r\big]\big\}\,dz\,dr.$$

By the chain rule, we have $D_{r,z}(\sigma^2(u(s,y)))=2\sigma(u(s,y))\Sigma(s,y)D_{r,z}u(s,y)$, thus $\|\mathbb{E}[D_{r,z}(\sigma^2(u(s,y)))|\mathcal{F}_r]\|_2\le 2K_4(t)L\|D_{r,z}u(s,y)\|_4$. Then, using Lemma 2.2, we can write

$$\begin{aligned}
&\big|\mathrm{Cov}\big[\sigma^2\big(u(s,y)\big),\sigma^2\big(u\big(s,y'\big)\big)\big]\big|\\
&\quad\le C\int_0^s\int_{\mathbb{R}}\big\|D_{r,z}u(s,y)\big\|_4\big\|D_{r,z}u\big(s,y'\big)\big\|_4\,dz\,dr\\
&\quad\le C\int_0^s\int_{\mathbb{R}}\mathbf{1}_{\{|y-z|\le s-r\}}\mathbf{1}_{\{|y'-z|\le s-r\}}\,dz\,dr\le C\mathbf{1}_{\{|y-y'|\le 2s\}}.
\end{aligned}$$

This leads to the following estimate for A_1, for any $R\ge 2t$:

$$A_1\le\frac{C}{R}\int_0^t\left(\int_{\mathbb{R}^2}\varphi_R^2(s,y)\varphi_R^2\big(s,y'\big)\mathbf{1}_{\{|y-y'|\le 2s\}}\,dy\,dy'\right)^{1/2}ds.$$

Since $\varphi_R^2(s,y)\varphi_R^2(s,y')\le(t-s)^4\mathbf{1}_{\{|y|\vee|y'|\le R+t-s\}}$, we get $A_1\le C/\sqrt{R}$ for $R\ge 2t$. This concludes our proof for the case $H=1/2$.

The proof for the other case is more involved but we can proceed in similar steps.

Case $H>1/2$. In this case, we write $\langle DF_R,v_R\rangle_{\mathfrak{H}}:=B_1+B_2$, where

$$B_1=\frac{\alpha_H}{\sigma_R^2}\int_0^t\int_{\mathbb{R}^2}\varphi_R(s,y)\varphi_R\big(s,y'\big)\sigma\big(u(s,y)\big)\sigma\big(u\big(s,y'\big)\big)\big|y-y'\big|^{2H-2}\,dy\,dy'\,ds$$

and

$$\begin{aligned}
B_2=\frac{\alpha_H}{\sigma_R^2}\int_0^t\int_{\mathbb{R}^2}&\left(\int_s^t\varphi_R(r,z)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz)\right)\\
&\times\varphi_R\big(s,y'\big)\sigma\big(u\big(s,y'\big)\big)\big|y-y'\big|^{2H-2}\,dy\,dy'\,ds.
\end{aligned}$$

This decomposition implies $\sqrt{\mathrm{Var}[\langle DF_R,v_R\rangle_{\mathfrak{H}}]}\le\sqrt{2}(A_1+A_2)$, with

$$\begin{aligned}
A_1=\frac{\alpha_H}{\sigma_R^2}\int_0^t\Bigg(&\int_{\mathbb{R}^4}\varphi_R(s,y)\varphi_R\big(s,y'\big)\varphi_R(s,\tilde{y})\varphi_R\big(s,\tilde{y}'\big)\big|y-y'\big|^{2H-2}\big|\tilde{y}-\tilde{y}'\big|^{2H-2}\\
&\times\mathrm{Cov}\big[\sigma\big(u(s,y)\big)\sigma\big(u\big(s,y'\big)\big),\sigma\big(u(s,\tilde{y})\big)\sigma\big(u\big(s,\tilde{y}'\big)\big)\big]\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\Bigg)^{1/2}ds
\end{aligned}$$

and

$$\begin{aligned}
A_2=\frac{\alpha_H^{3/2}}{\sigma_R^2}\int_0^t\Bigg(&\int_{\mathbb{R}^6}\int_s^t\varphi_R(r,z)\varphi_R(r,\tilde{z})\varphi_R\big(s,y'\big)\varphi_R\big(s,\tilde{y}'\big)\\
&\times\mathbb{E}\big\{\Sigma(r,z)D_{s,y}u(r,z)\Sigma(r,\tilde{z})D_{s,\tilde{y}}u(r,\tilde{z})\sigma\big(u\big(s,\tilde{y}'\big)\big)\sigma\big(u\big(s,y'\big)\big)\big\}\\
&\times\big|y-y'\big|^{2H-2}\big|\tilde{y}-\tilde{y}'\big|^{2H-2}|z-\tilde{z}|^{2H-2}\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\,dz\,d\tilde{z}\,dr\Bigg)^{1/2}ds.
\end{aligned}$$

The proof will be done in two steps:

Step 1: Let us first estimate the term A_2. Recall that L denotes the Lipschitz constant of σ and recall the notation $K_p(t)$ ($p \geq 2$) introduced in (3.12). We can write

$$\begin{aligned}&\left|\mathbb{E}\left\{\Sigma(r,z)D_{s,y}u(r,z)\Sigma(r,\tilde{z})D_{s,\tilde{y}}u(r,\tilde{z})\sigma\left(u\left(s,\tilde{y}'\right)\right)\sigma\left(u\left(s,y'\right)\right)\right\}\right|\\&\quad\leq K_4^2(t)L^2\left\|D_{s,y}u(r,z)\right\|_4\left\|D_{s,\tilde{y}}u(r,\tilde{z})\right\|_4\leq C\mathbf{1}_{\{|y-z|\leq r-s\}}\mathbf{1}_{\{|\tilde{y}-\tilde{z}|\leq r-s\}},\end{aligned}$$

where the last inequality follows from Lemma 2.2.

Now we derive from Proposition 3.3 the following estimate: For fixed $t>0$, there exists a constant R_t that depends on t such that for any $R \geq R_t$,

$$\begin{aligned}A_2\leq \frac{C}{R^{2H}}\int_0^t\biggl(\int_{\mathbb{R}^6}\int_s^t &\varphi_R(r,z)\varphi_R(r,\tilde{z})\varphi_R\left(s,y'\right)\varphi_R\left(s,\tilde{y}'\right)\mathbf{1}_{\{|y-z|\leq r-s,|\tilde{y}-\tilde{z}|\leq r-s\}}\\&\times\left|y-y'\right|^{2H-2}\left|\tilde{y}-\tilde{y}'\right|^{2H-2}|z-\tilde{z}|^{2H-2}\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\,dz\,d\tilde{z}\,dr\biggr)^{1/2}ds,\end{aligned}$$

where C is a constant that depends on t, p, H and σ.

The integral in the spatial variable term can be rewritten as

$$\begin{aligned}\mathbf{I}&:=\frac{1}{16}\int_{[-R,R]^4}\int_{\mathbb{R}^6}\mathbf{1}_{\{|x-z|\leq t-r,|\tilde{x}-\tilde{z}|\leq t-r,|x'-y'|\leq t-s,|\tilde{x}'-\tilde{y}'|\leq t-s,|y-z|\leq r-s,|\tilde{y}-\tilde{z}|\leq r-s\}}\\&\qquad\times\left|y-y'\right|^{2H-2}\left|\tilde{y}-\tilde{y}'\right|^{2H-2}|z-\tilde{z}|^{2H-2}\,dx\,d\tilde{x}\,dx'\,d\tilde{x}'\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\,dz\,d\tilde{z}\\&=\frac{R^{6H+4}}{16}\int_{[-1,1]^4}\int_{\mathbb{R}^6}\mathbf{1}_{\{|x-z|\leq\frac{t-r}{R},|\tilde{x}-\tilde{z}|\leq\frac{t-r}{R},|x'-y'|\leq\frac{t-s}{R},|\tilde{x}'-\tilde{y}'|\leq\frac{t-s}{R},|y-z|\leq\frac{r-s}{R},|\tilde{y}-\tilde{z}|\leq\frac{r-s}{R}\}}\\&\qquad\times\left|y-y'\right|^{2H-2}\left|\tilde{y}-\tilde{y}'\right|^{2H-2}|z-\tilde{z}|^{2H-2}\,dx\,d\tilde{x}\,dx'\,d\tilde{x}'\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\,dz\,d\tilde{z},\end{aligned}$$

where the second equality follows from a simple change of variables. Assuming $R \geq t$ and integrating in the variables $x, x', \tilde{x}, \tilde{x}' \in [-1,1]$, we have

$$\begin{aligned}\mathbf{I}\leq R^{6H}\int_{\mathbb{R}^6}&\mathbf{1}_{\{|y-z|\leq\frac{t}{R},|\tilde{y}-\tilde{z}|\leq\frac{t}{R}\}}\mathbf{1}_{[-2,2]}(z)\mathbf{1}_{[-2,2]}(\tilde{z})\mathbf{1}_{[-2,2]}\left(y'\right)\mathbf{1}_{[-2,2]}\left(\tilde{y}'\right)\\&\times\left|y-y'\right|^{2H-2}\left|\tilde{y}-\tilde{y}'\right|^{2H-2}|z-\tilde{z}|^{2H-2}\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\,dz\,d\tilde{z}.\end{aligned}$$

If $K=\sup_{y\in[-3,3]}\int_{-2}^2|y-y'|^{2H-2}\,dy'$, then for $R\geq R_t+t$,

$$\mathbf{I}\leq K^2R^{6H}\int_{\mathbb{R}^4}\mathbf{1}_{\{|y-z|\leq\frac{t}{R},|\tilde{y}-\tilde{z}|\leq\frac{t}{R}\}}\mathbf{1}_{[-2,2]}(z)\mathbf{1}_{[-2,2]}(\tilde{z})|z-\tilde{z}|^{2H-2}\,dy\,d\tilde{y}\,dz\,d\tilde{z}.$$

Finally, integrating in y and $\tilde{y}$, yields for $R \geq R_t+t$,

$$\mathbf{I}\leq 36K^2R^{6H-2}\int_{[-2,2]^2}|z-\tilde{z}|^{2H-2}\,dz\,d\tilde{z}.$$

As a consequence,

$$A_2\leq CR^{H-1}$$

for R big enough.

Step 2: It remains to estimate the term A_1. We will show $A_1 \leq CR^{H-1}$ for R big enough. We begin with a bound for the covariance

$$\operatorname{Cov}\left[\sigma\left(u(s,y)\right)\sigma\left(u\left(s,y'\right)\right),\sigma\left(u(s,\tilde{y})\right)\sigma\left(u\left(s,\tilde{y}'\right)\right)\right].$$

According to a version of Clark–Ocone formula for two-parameter processes, we write

$$\begin{aligned}\sigma\left(u(s,y)\right)\sigma\left(u\left(s,y'\right)\right)&=\mathbb{E}\left[\sigma\left(u(s,y)\right)\sigma\left(u\left(s,y'\right)\right)\right]\\&\quad+\int_0^s\int_{\mathbb{R}}\mathbb{E}\left[D_{r,z}\left(\sigma\left(u(s,y)\right)\sigma\left(u\left(s,y'\right)\right)\right)|\mathcal{F}_r\right]W(dr,dz).\end{aligned}$$

Then,

$$\begin{aligned}
&\operatorname{Cov}\bigl[\sigma\bigl(u(s,y)\bigr)\sigma\bigl(u\bigl(s,y'\bigr)\bigr),\sigma\bigl(u(s,\tilde{y})\bigr)\sigma\bigl(u\bigl(s,\tilde{y}'\bigr)\bigr)\bigr]\\
&\quad=\alpha_H\int_0^s\int_{\mathbb{R}^2}\mathbb{E}\bigl\{\mathbb{E}\bigl[D_{r,z}\bigl(\sigma\bigl(u(s,y)\bigr)\sigma\bigl(u\bigl(s,y'\bigr)\bigr)\bigr)|\mathcal{F}_r\bigr]\\
&\qquad\times\mathbb{E}\bigl[D_{r,z'}\bigl(\sigma\bigl(u(s,\tilde{y})\bigr)\sigma\bigl(u\bigl(s,\tilde{y}'\bigr)\bigr)\bigr)|\mathcal{F}_r\bigr]\bigr\}\bigl|z-z'\bigr|^{2H-2}\,dz\,dz'\,dr.
\end{aligned}$$

Applying the chain rule for Lipschitz functions (see [10, Proposition 1.2.4]), we have

$$\begin{aligned}
D_{r,z}\bigl(\sigma\bigl(u(s,y)\bigr)\sigma\bigl(u\bigl(s,y'\bigr)\bigr)\bigr)&=\sigma\bigl(u(s,y)\bigr)\Sigma\bigl(s,y'\bigr)D_{r,z}u\bigl(s,y'\bigr)\\
&\quad+\sigma\bigl(u\bigl(s,y'\bigr)\bigr)\Sigma(s,y)D_{r,z}u(s,y)
\end{aligned}$$

and therefore, $\|\mathbb{E}[D_{r,z}(\sigma(u(s,y))\sigma(u(s,y')))|\mathcal{F}_r]\|_2$ is bounded by

$$2K_4(t)L\bigl\{\bigl\|D_{r,z}u(s,y)\bigr\|_4+\bigl\|D_{r,z}u\bigl(s,y'\bigr)\bigr\|_4\bigr\}.$$

Applying Lemma 2.2, we get $|\operatorname{Cov}[\sigma(u(s,y))\sigma(u(s,y')),\sigma(u(s,\tilde{y}))\sigma(u(s,\tilde{y}'))]|$ bounded by

$$\begin{aligned}
&4L^2K_4^2(t)\int_0^s\int_{\mathbb{R}^2}\bigl(\bigl\|D_{r,z}u(s,y)\bigr\|_4+\bigl\|D_{r,z}u\bigl(s,y'\bigr)\bigr\|_4\bigr)\\
&\quad\times\bigl(\bigl\|D_{r,z'}u(s,\tilde{y})\bigr\|_4+\bigl\|D_{r,z'}u\bigl(s,\tilde{y}'\bigr)\bigr\|_4\bigr)\bigl|z-z'\bigr|^{2H-2}\,dz\,dz'\,dr\\
&\quad\le C\int_0^s\int_{\mathbb{R}^2}(\mathbf{1}_{\{|y-z|\le s-r\}}+\mathbf{1}_{\{|y'-z|\le s-r\}})\\
&\quad\times(\mathbf{1}_{\{|\tilde{y}-z'|\le s-r\}}+\mathbf{1}_{\{|\tilde{y}'-z'|\le s-r\}})\bigl|z-z'\bigr|^{2H-2}\,dz\,dz'\,dr.
\end{aligned}$$

So the spatial integral in the expression of A_1 can be bounded by

$$\begin{aligned}
\mathbf{J}&:=C\int_0^s\int_{\mathbb{R}^6}\varphi_R(s,y)\varphi_R\bigl(s,y'\bigr)\varphi_R(s,\tilde{y})\varphi_R\bigl(s,\tilde{y}'\bigr)\\
&\quad\times\bigl|y-y'\bigr|^{2H-2}\bigl|\tilde{y}-\tilde{y}'\bigr|^{2H-2}\bigl|z-z'\bigr|^{2H-2}(\mathbf{1}_{\{|y-z|\le s-r\}}+\mathbf{1}_{\{|y'-z|\le s-r\}})\\
&\quad\times(\mathbf{1}_{\{|\tilde{y}-z'|\le s-r\}}+\mathbf{1}_{\{|\tilde{y}'-z'|\le s-r\}})\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\,dz\,dz'\,dr\\
&=C\int_0^s\int_{[-R,R]^4}\int_{\mathbb{R}^6}\mathbf{1}_{\{|x-y|\vee|x'-y'|\vee|\tilde{x}-\tilde{y}|\vee|\tilde{x}'-\tilde{y}'|\le t-s\}}\\
&\quad\times\bigl|y-y'\bigr|^{2H-2}\bigl|\tilde{y}-\tilde{y}'\bigr|^{2H-2}\bigl|z-z'\bigr|^{2H-2}(\mathbf{1}_{\{|y-z|\le s-r\}}+\mathbf{1}_{\{|y'-z|\le s-r\}})\\
&\quad\times(\mathbf{1}_{\{|\tilde{y}-z'|\le s-r\}}+\mathbf{1}_{\{|\tilde{y}'-z'|\le s-r\}})\,dx\,dx'\,d\tilde{x}\,d\tilde{x}'\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\,dz\,dz'\,dr\\
&\le 4Ct\int_{[-R,R]^4}\int_{\mathbb{R}^6}\mathbf{1}_{\{|x-y|\vee|x'-y'|\vee|\tilde{x}-\tilde{y}|\vee|\tilde{x}'-\tilde{y}'|\le t\}}\mathbf{1}_{\{|y-z|\le t\}}\mathbf{1}_{\{|\tilde{y}-z'|\le t\}}\\
&\quad\times\bigl|y-y'\bigr|^{2H-2}\bigl|\tilde{y}-\tilde{y}'\bigr|^{2H-2}\bigl|z-z'\bigr|^{2H-2}\,dx\,dx'\,d\tilde{x}\,d\tilde{x}'\,dy\,dy'\,d\tilde{y}\,d\tilde{y}'\,dz\,dz',
\end{aligned}$$

due to symmetry. Then, it follows from the exactly the same argument as in the estimation of $\mathbf{I}$ in the previous step that $\mathbf{J}$ is bounded by CR^{6H-2} for R big enough. This gives us the desired estimate for A_1 and finishes the proof. □

4. Proof of Theorem 1.2

We begin with the following result that ensures tightness.

Proposition 4.1. *Let $u(t,x)$ be the solution to equation (1.1). Then for any $0\le s<t\le T$ and any $p\ge 2$, there exists a constant $C_{p,T}$, depending on T and p, such that for any $R\ge T$,*

$$\mathbb{E}\biggl(\biggl|\int_{-R}^R u(t,x)\,dx-\int_{-R}^R u(s,x)\,dx\biggr|^p\biggr)\le C_{p,T}R^{pH}(t-s)^p. \tag{4.1}$$

Proof. Let us assume that $s < t$. We can write

$$2\int_{-R}^{R}\big[u(t,x)-u(s,x)\big]\,dx = \int_0^T\int_{\mathbb{R}}\big(\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big)\sigma\big(u(r,y)\big)W(dr,dy),$$

where $\varphi_{t,R}(r,y) = \mathbf{1}_{\{r\le t\}}\int_{-R}^{R}\mathbf{1}_{\{|x-y|\le t-r\}}\,dx$. The rest of our proof consists of two parts.

Step 1: Suppose that $H = 1/2$. Using Burkholder–Davis–Gundy inequality and Minkowski's inequality, we get, for some absolute constant $c_p \in (0,+\infty)$,

$$\begin{aligned}
&\mathbb{E}\bigg(\bigg|\int_{-R}^{R}u(t,x)\,dx - \int_{-R}^{R}u(s,x)\,dx\bigg|^p\bigg) \\
&\quad\le c_p\mathbb{E}\bigg[\bigg(\int_0^T\int_{\mathbb{R}}\big(\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big)^2\sigma^2\big(u(r,y)\big)\,dy\,dr\bigg)^{p/2}\bigg] \\
&\quad\le c_p\bigg(\int_0^T\int_{\mathbb{R}}\big(\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big)^2\big\|\sigma\big(u(r,y)\big)\big\|_p^2\,dy\,dr\bigg)^{p/2} \\
&\quad\le c_pK_p^p(T)\bigg(\int_0^T\int_{\mathbb{R}}\big(\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big)^2\,dy\,dr\bigg)^{p/2},
\end{aligned}$$

where $K_p(T)$ has been defined in (3.12). Now we notice that

$$\begin{aligned}
&\big|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big| \\
&\quad\le \mathbf{1}_{\{r\le s\}}\int_{-R}^{R}|\mathbf{1}_{\{|x-y|\le t-r\}}-\mathbf{1}_{\{|x-y|\le s-r\}}|\,dx + \mathbf{1}_{\{s<r\le t\}}\int_{-R}^{R}\mathbf{1}_{\{|x-y|\le t-r\}}\,dx \\
&\quad\le 2\big(t-s+(t-r)\mathbf{1}_{\{s<r\le t\}}\big)\mathbf{1}_{\{|y|\le R+t\}}\le 4(t-s)\mathbf{1}_{\{|y|\le R+t\}}.
\end{aligned}\tag{4.2}$$

This implies for $R \ge T$,

$$\int_0^T\int_{\mathbb{R}}\big(\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big)^2\,dy\,ds \le 64TR(t-s)^2, \quad \text{and thus establishes (4.1).}$$

Step 2: Suppose that $H \in (1/2,1)$. In the same way, we write

$$\begin{aligned}
&\mathbb{E}\bigg(\bigg|\int_{-R}^{R}u(t,x)\,dx - \int_{-R}^{R}u(s,x)\,dx\bigg|^p\bigg) \\
&\quad\le c_p\mathbb{E}\bigg[\bigg(\int_0^T\big\|\big(\varphi_{t,R}(r,\cdot)-\varphi_{s,R}(r,\cdot)\big)\sigma\big(u(r,\cdot)\big)\big\|_{\mathfrak{H}_0}^2\,dr\bigg)^{p/2}\bigg].
\end{aligned}\tag{4.3}$$

As mentioned in Section 2, for $H \in (1/2,1)$, the space $L^{1/H}(\mathbb{R})$ is continuously embedded into $\mathfrak{H}_0$. Consequently, there is a constant $C_H > 0$, depending on H, such that

$$\begin{aligned}
&\big\|\big(\varphi_{t,R}(r,\cdot)-\varphi_{s,R}(r,\cdot)\big)\sigma\big(u(r,\cdot)\big)\big\|_{\mathfrak{H}_0}^2 \\
&\quad\le C_H\bigg(\int_{\mathbb{R}}\big|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big|^{1/H}\big|\sigma\big(u(r,y)\big)\big|^{1/H}\,dy\bigg)^{2H}.
\end{aligned}\tag{4.4}$$

Substituting (4.4) into (4.3) and applying Hölder's and Minkowski's inequalities, we can write

$$\begin{aligned}
&\mathbb{E}\bigg(\bigg|\int_{-R}^{R}u(t,x)\,dx - \int_{-R}^{R}u(s,x)\,dx\bigg|^p\bigg) \\
&\quad\le c_pC_H^{p/2}T^{p/2-1}\int_0^T\mathbb{E}\bigg[\bigg(\int_{\mathbb{R}}\big|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big|^{1/H}\big|\sigma\big(u(r,y)\big)\big|^{1/H}\,dy\bigg)^{pH}\bigg]\,dr \\
&\quad\le c_pC_H^{p/2}T^{p/2-1}\int_0^T\bigg(\int_{\mathbb{R}}\big|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big|^{1/H}\big\|\sigma\big(u(r,y)\big)\big\|_p^{1/H}\,dy\bigg)^{pH}\,dr
\end{aligned}$$

$$\le c_p C_H^{p/2} T^{p/2-1} K_p^p(T) \int_0^T \left(\int_{\mathbb{R}} \left| \varphi_{t,R}(r,y) - \varphi_{s,R}(r,y) \right|^{1/H} dy \right)^{pH} dr.$$

Finally, from (4.2), which holds true for any $R \ge T$, we can write

$$\left(\int_{\mathbb{R}} \left| \varphi_{t,R}(r,y) - \varphi_{s,R}(r,y) \right|^{1/H} dy \right)^{pH} \le 4^{p(1+H)} (t-s)^p R^{pH}.$$

It is then straightforward to get (4.1). □

Proof of Theorem 1.2. We need to prove tightness and the convergence of the finite-dimensional distributions. Notice that tightness follows from Proposition 4.1 and the well-known criterion of Kolmogorov.

Let us now show the convergence of the finite-dimensional distributions. We fix $0 \le t_1 < \cdots < t_m \le T$ and consider

$$F_R(t_i) := \frac{1}{R^H} \left(\int_{-R}^{R} u(t_i, x)\, dx - 2R \right) = \delta\big(v_R^{(i)}\big) \quad \text{for } i = 1, \dots, m,$$

where

$$v_R^{(i)}(s,y) = \mathbf{1}_{[0,t_i]}(s) \frac{\sigma(u(s,y))}{R^H} \varphi_R^{(i)}(s,y) \quad \text{with } \varphi_R^{(i)}(s,y) = \frac{1}{2} \int_{-R}^{R} \mathbf{1}_{\{|x-y| \le t_i - s\}}\, dx.$$

Set $\mathbf{F}_R = (F_R(t_1), \dots, F_R(t_m))$ and let Z be a centered Gaussian vector on $\mathbb{R}^m$ with covariance $(C_{i,j})_{1 \le i,j \le m}$ given by

$$C_{i,j} := \begin{cases} 2 \int_0^{t_i \wedge t_j} (t_i - r)(t_j - r) \xi(r)\, dr & \text{if } H = 1/2; \\ 2^{2H} \int_0^{t_i \wedge t_j} (t_i - r)(t_j - r) \eta^2(r)\, dr & \text{if } H \in (1/2, 1). \end{cases}$$

We recall here that $\xi(r) = \mathbb{E}[\sigma^2(u(r,y))]$ and $\eta(r) = \mathbb{E}[\sigma(u(r,y))]$. Then, we need to show $\mathbf{F}_R$ converges in distribution to Z and in view of Lemma 2.5, it suffices to show that for each i, j, $\langle DF_R(t_i), v_R^{(j)} \rangle_{\mathfrak{H}}$ converges to $C_{i,j}$ in $L^2(\Omega)$, as $R \to +\infty$. The case $i = j$ has been tackled before and the other case can be dealt with by using arguments similar to those in the proof of Theorem 1.1. For the convenience of readers, we only sketch these arguments as follows.

We consider two cases: $H = 1/2$ and $H \in (1/2, 1)$. In each case, we need to show (i) $\mathbb{E}[F_R(t_i) F_R(t_j)] \to C_{i,j}$ and (ii) $\mathrm{Var}(\langle DF_R(t_i), v_R^{(j)} \rangle_{\mathfrak{H}}) \to 0$, as $R \to +\infty$. Point (i) has been established in *Remark* 2. To see point (ii) for the case $H = 1/2$, we begin with the decomposition $\langle DF_R(t_i), v_R^{(j)} \rangle_{\mathfrak{H}} = B_1(i,j) + B_2(i,j)$ with

$$B_1(i,j) := \frac{1}{R} \int_0^{t_i \wedge t_j} \int_{\mathbb{R}} \varphi_R^{(i)}(s,y) \varphi_R^{(j)}(s,y) \sigma^2\big(u(s,y)\big)\, ds\, dy$$

and

$$B_2(i,j) := \frac{1}{R} \int_0^{t_i \wedge t_j} \int_{\mathbb{R}} \varphi_R^{(j)}(s,y) \sigma\big(u(s,y)\big) \times \left(\int_s^{t_i} \int_{\mathbb{R}} \varphi_R^{(i)}(r,z) \Sigma(r,z) D_{s,y} u(r,z) W(dr,dz) \right) dy\, ds.$$

Then using (3.11) and going through the same lines as for the estimation of A_1, A_2, we can get

$$\begin{aligned} \sqrt{\mathrm{Var}\big(B_2(i,j)\big)} &\le \frac{1}{R} \int_0^{t_i \wedge t_j} ds \left(\int_{\mathbb{R}^3} \int_s^{t_i} \varphi_R^{(i)}(r,z)^2 \varphi_R^{(j)}(s,y) \varphi_R^{(j)}\big(s,y'\big) \right. \\ &\quad \left. \times \mathbb{E}\big[\Sigma^2(r,z) D_{s,y} u(r,z) D_{s,y'} u(r,z) \sigma\big(u(s,y)\big) \sigma\big(u\big(s,y'\big)\big) \big]\, dy\, dy'\, dz\, dr \right)^{1/2} \\ &\le \frac{C}{R} \int_0^{t_i \wedge t_j} \left(\int_{\mathbb{R}^3} \int_s^{t_i} \varphi_R^{(i)}(r,z)^2 \varphi_R^{(j)}(s,y) \varphi_R^{(j)}\big(s,y'\big) \right. \\ &\quad \left. \times \mathbf{1}_{\{|y-z| \vee |y'-z| \le r-s\}}\, dy\, dy'\, dz\, dr \right)^{1/2} ds \le \frac{C}{\sqrt{R}}. \end{aligned}$$

That is, we have $\mathrm{Var}(B_2(i,j)) \to 0$, as $R \to +\infty$. We can also get

$$\begin{aligned}\sqrt{\mathrm{Var}\big(B_1(i,j)\big)} &\le \frac{1}{R}\int_0^{t_i\wedge t_j}\bigg(\int_{\mathbb{R}^2}\varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,y)\varphi_R^{(i)}\big(s,y'\big)\varphi_R^{(j)}\big(s,y'\big)\\ &\quad\times \mathrm{Cov}\big[\sigma^2\big(u(s,y)\big),\sigma^2\big(u\big(s,y'\big)\big)\big]\,dy\,dy'\bigg)^{1/2}ds\\ &\le \frac{C}{R}\int_0^{t_i\wedge t_j}\bigg(\int_{\mathbb{R}^2}\varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,y)\varphi_R^{(i)}\big(s,y'\big)\varphi_R^{(j)}\big(s,y'\big)\mathbf{1}_{\{|y-y'|\le 2s\}}\,dy\,dy'\bigg)^{1/2}ds\\ &\le \frac{C}{R}\int_0^{t_i\wedge t_j}\bigg(\int_{\mathbb{R}^2}(t_i+t_j)^4\mathbf{1}_{\{|y|\vee|y'|\le R+t_i+t_j\}}\mathbf{1}_{\{|y-y'|\le 2s\}}\,dy\,dy'\bigg)^{1/2}ds \le \frac{C}{\sqrt{R}}.\end{aligned}$$

That is, we have $\mathrm{Var}(B_1(i,j)) \to 0$, as $R \to +\infty$.

To see point (ii) for the case $H \in (1/2,1)$, one can begin with the same decomposition and then use (3.11) to arrive at similar estimations as those for $\mathbf{I}$ and $\mathbf{J}$. Therefore the same arguments ensure $\mathrm{Var}(\langle DF_R(t_i), v_R^{(j)}\rangle_{\mathfrak{H}}) \le CR^{2H-2}$. Now the proof of Theorem 1.2 is completed. □

Appendix: Proof of Lemma 2.2

This appendix provides the proof of our technical Lemma and it consists of two parts. The first part proceeds assuming

$$\mathfrak{L} := \sup_{(r,z)\in[0,t]\times\mathbb{R}}\big\|D_{s,y}u(r,z)\big\|_p < +\infty \quad \text{for almost every } (s,y)\in\mathbb{R}_+\times\mathbb{R} \tag{A.1}$$

and the second part is devoted to establishing the above bound. Note that *a priori*, we do not know whether $D_{s,y}u(r,z)$ is a function of (s,y) or not in the case where $H \in (1/2,1)$, so the assumption (A.1) also guarantees that $D_{s,y}u(r,z)$ is indeed a random function in (s,y); see Section A.2 for more explanation.

A.1. *Proof of Lemma 2.2 assuming (A.1)*

The proof will be done in two steps.

Step 1: Case $H = 1/2$. From (2.6), using Burkholder's and Minkowski's inequality, we can write

$$\begin{aligned}&\big\|D_{s,y}u(t,x)\big\|_p\\ &\quad\le \frac{K_p(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} + \frac{Lc_p}{2}\bigg(\int_s^t\int_{\mathbb{R}}\mathbf{1}_{\{|x-z|\le t-r\}}\big\|D_{s,y}u(r,z)\big\|_p^2\,dr\,dz\bigg)^{1/2}\end{aligned}$$

with c_p a constant that only depends on p. It follows from the elementary inequality $(a+b)^2 \le 2a^2+2b^2$ that

$$\big\|D_{s,y}u(t,x)\big\|_p^2 \le \frac{K_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} + \frac{L^2c_p^2}{2}\int_s^t\int_{\mathbb{R}}\mathbf{1}_{\{|x-z|\le t-r\}}\big\|D_{s,y}u(r,z)\big\|_p^2\,dr\,dz.$$

Iterating this inequality yields, for any positive integer M,

$$\begin{aligned}\big\|D_{s,y}u(t,x)\big\|_p^2 &\le \frac{K_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}}\\ &\quad+\frac{K_p^2(t)}{2}\sum_{N=1}^M\frac{c_p^{2N}L^{2N}}{2^N}\int_{\Delta_N(s,t)}\int_{\mathbb{R}^N}\bigg(\prod_{n=1}^N\mathbf{1}_{\{|z_{n-1}-z_n|\le r_{n-1}-r_n\}}\bigg)\mathbf{1}_{\{|z_N-y|\le r_N-s\}}\,d\mathbf{r}\,d\mathbf{z}\\ &\quad+\frac{c_p^{2M+2}L^{2M+2}}{2^{M+1}}\int_{\Delta_{M+1}(s,t)}\int_{\mathbb{R}^{M+1}}\bigg(\prod_{n=1}^M\mathbf{1}_{\{|z_{n-1}-z_n|\le r_{n-1}-r_n\}}\bigg)\mathbf{1}_{\{|z_M-z_{M+1}|\le r_M-r_{M+1}\}}\\ &\quad\times\big\|D_{s,y}u(r_{M+1},z_{M+1})\big\|_2^p\,d\mathbf{r}\,d\mathbf{z},\end{aligned}$$

where $\Delta_N(s,t) := \{(r_1,\dots,r_N) \in \mathbb{R}^N | s < r_N < r_{N-1} < \cdots < r_1 < t\}$, $d\mathbf{r} = dr_1 \cdots dr_N$, $d\mathbf{z} = dz_1 \cdots dz_N$ and with the convention $r_0 = t$ and $z_0 = x$.

Notice that if

$$\left(\prod_{n=1}^{N} \mathbf{1}_{\{|z_{n-1}-z_n| \le r_{n-1}-r_n\}}\right) \mathbf{1}_{\{|z_N - y| \le r_N - s\}} \neq 0,$$

then on $\Delta_N(s,t)$, $|x-y| = |z_0 - y| \le \sum_{n=1}^{N} |z_{n-1} - z_n| + |z_N - y| \le t-s$ and similarly on $\Delta_N(s,t)$, $|z_n - y| \le t-s$ for[2] $n = 1,\dots,N$.

Now we deduce from (A.1) that

$$\begin{aligned}\|D_{s,y}u(t,x)\|_p^2 &\le \mathbf{1}_{\{|x-y|\le t-s\}} \frac{K_p^2(t)}{2}\left(1 + \sum_{N=1}^{\infty} c_p^{2N} L^{2N} \frac{(t-s)^{2N}}{N!}\right) \\ &\le \mathbf{1}_{\{|x-y|\le t-s\}} \frac{K_p^2(t)}{2} \exp\bigl(c_p^2 L^2 t^2\bigr),\end{aligned}$$

which provides the desired estimate.

Step 2: Case $H \in (1/2, 1)$. Proceeding as before, and using the inequality

$$\|D_{s,y}u(r,z) D_{s,y}u(r,\tilde{z})\|_{p/2} \le \frac{1}{2}\bigl(\|D_{s,y}u(r,z)\|_p^2 + \|D_{s,y}u(r,\tilde{z})\|_p^2\bigr),$$

we obtain

$$\begin{aligned}\|D_{s,y}u(t,x)\|_p^2 &\le \frac{K_p^2(t)}{2} \mathbf{1}_{\{|x-y|\le t-s\}} \\ &\quad + \frac{c_p^2 L^2}{2} \alpha_H \int_s^t \int_{\mathbb{R}^2} \mathbf{1}_{\{|x-z|\le t-r\}} \mathbf{1}_{\{|x-\tilde{z}|\le t-r\}} \|D_{s,y}u(r,z)\|_p^2 |z-\tilde{z}|^{2H-2}\, dr\, dz\, d\tilde{z}.\end{aligned}$$

By iteration, this leads to the following estimate. For any positive integer M,

$$\begin{aligned}&\|D_{s,y}u(t,x)\|_p^2 \\ &\quad \le \frac{K_p^2(t)}{2} \mathbf{1}_{\{|x-y|\le t-s\}} + \frac{K_p^2(t)}{2} \sum_{N=1}^{M} \frac{c_p^{2N} L^{2N}}{2^N} \int_{\Delta_N(s,t)} \alpha_H^N \int_{\mathbb{R}^{2N}} \\ &\qquad \times \left(\prod_{n=1}^{N} \mathbf{1}_{\{|z_{n-1}-\tilde{z}_n| \vee |z_{n-1}-z_n| \le r_{n-1}-r_n\}} |z_n - \tilde{z}_n|^{2H-2}\right) \mathbf{1}_{\{|z_N - y|\le r_N - s\}}\, d\mathbf{r}\, d\mathbf{z}\, d\tilde{\mathbf{z}} \\ &\qquad + \frac{(Lc_p)^{2M+2}}{2^{M+1}} \int_{\Delta_{M+1}(s,t)} d\mathbf{r}\, \alpha_H^{M+1} \int_{\mathbb{R}^{2M+2}} d\mathbf{z}\, d\tilde{\mathbf{z}} \\ &\qquad \times \left(\prod_{n=1}^{M+1} \mathbf{1}_{\{|z_{n-1}-\tilde{z}_n| \vee |z_{n-1}-z_n| \le r_{n-1}-r_n\}} |z_n - \tilde{z}_n|^{2H-2}\right) \|D_{s,y}u(r_{M+1}, z_{M+1})\|_p^2\end{aligned}$$

with the same convention as before. Note that Lemma 2.1 implies that on $\Delta_N(s,t)$,

$$\begin{aligned}&\alpha_H \int_{\mathbb{R}^2} \mathbf{1}_{\{|z_{n-1}-z_n| \vee |z_{n-1}-\tilde{z}_n| \le r_{n-1}-r_n\}} |z_n - \tilde{z}_n|^{2H-2}\, dz_n\, d\tilde{z}_n \\ &\quad \le \alpha_H \int_{\mathbb{R}^2} \mathbf{1}_{\{|z_{n-1}-z| \vee |z_{n-1}-z'| \le t\}} \bigl|z - z'\bigr|^{2H-2}\, dz\, dz' \le 4^H t^{2H}; \qquad \text{(A.2)}\end{aligned}$$

[2]This in particular implies that the contribution of the integration with respect to dz_n is at most $2(t-s)$.

and note that we again have the following implication:

$$\prod_{n=1}^{N} \mathbf{1}_{\{|z_{n-1}-z_n|\le r_{n-1}-r_n\}} \mathbf{1}_{\{|z_N - y|\le r_N - s\}} \neq 0 \quad \Longrightarrow \quad |x-y|\le t-s,$$

which, together with (A.2), implies

$$\begin{aligned}\|D_{s,y}u(t,x)\|_p^2 &\le \frac{K_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} + \frac{K_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} \sum_{N=1}^{M} \frac{(Lc_p 2^H t^H)^{2N}}{2^N} \frac{t^N}{N!} \\ &\quad + \mathfrak{L}^2 \frac{(Lc_p 2^H t^H)^{2M+2}}{2^{M+1}} \frac{t^{M+1}}{(M+1)!} \quad (\mathfrak{L} \text{ is defined in (A.1)}).\end{aligned}$$

Letting $M \to +\infty$ leads to

$$\begin{aligned}\|D_{s,y}u(t,x)\|_p^2 &\le \frac{K_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} + \frac{K_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} \sum_{N=1}^{\infty} \frac{(Lc_p 2^H t^H)^{2N}}{2^N} \frac{t^N}{N!} \\ &\le \frac{K_p^2(t)}{2} \exp\big(2L^2 c_p^2 t^{2H+1}\big)\mathbf{1}_{\{|x-y|\le t-s\}}.\end{aligned}$$

This concludes our proof of Lemma 2.2 assuming (A.1).

A.2. *Proof of* (A.1)

The proof will be done in two steps.

Step 1: Case $H = 1/2$. It is well known in the literature that for any $p \ge 2$, $u(t,x) \in \mathbb{D}^{1,p}$ and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\big[\|Du(t,x)\|_{\mathfrak{H}}^p\big] < +\infty; \tag{A.3}$$

indeed, in the Picard iteration scheme (see *e.g.* (A.6)), one can first prove the iteration u_n converges to the solution u in $L^p(\Omega)$ uniformly in $[0,T]\times\mathbb{R}$, then we derive the uniform bounded for $\mathbb{E}[\|Du_n(t,x)\|_{\mathfrak{H}}^p]$, so that by standard Malliavin calculus argument, we can get the convergence of $Du_n(t,x)$ to $Du(t,x)$ with respect to the weak topology on $L^p(\Omega;\mathfrak{H})$ and hence the desired uniform bound (A.3). We omit the details for this case ($H = 1/2$) and refer to the arguments for the other case ($H > 1/2$).

Consider an approximation of the identity ($M_\varepsilon, \varepsilon > 0$) in $L^1(\mathbb{R}_+\times\mathbb{R})$ satisfying $M_\varepsilon(s,y) = \varepsilon^{-2}M(s/\varepsilon, y/\varepsilon)$ for some nonnegative $M \in C_c(\mathbb{R}_+\times\mathbb{R})$. Taking into account that $(\omega, s, y) \to D_{s,y}u(t,x)$ belongs to $L^2(\mathbb{R}_+\times\mathbb{R}; L^2(\Omega))$, we deduce that the convolution $Du(t,x) * M_\varepsilon$ converges to $Du(t,x)$ in $L^2(\mathbb{R}_+\times\mathbb{R}; L^2(\Omega))$, as ε tends to zero. Therefore, there exist a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$ and $(Du(t,x) * M_{\varepsilon_n})(s,y)$ converges almost surely to $D_{s,y}u(t,x)$ for almost all $(s,y) \in \mathbb{R}_+\times\mathbb{R}$, as $n \to +\infty$. By Fatou's Lemma, this implies that for almost all $(s,y)\in\mathbb{R}_+\times\mathbb{R}$,

$$\|D_{s,y}u(t,x)\|_p \le \sup_{n\in\mathbb{N}} \big\|\big(Du(t,x) * M_{\varepsilon_n}\big)(s,y)\big\|_p. \tag{A.4}$$

Now we fix (s,y) that satisfies (A.4) and put for $\varepsilon > 0$

$$\begin{aligned}Q_\varepsilon(t) &:= \sup_{z\in\mathbb{R}} \big\|\big(Du(t,z) * M_\varepsilon\big)(s,y)\big\|_p^2 \\ &= \sup_{z\in\mathbb{R}} \bigg\| \int_{\mathbb{R}_+\times\mathbb{R}} D_{s',y'}u(t,z) M_\varepsilon\big(s'-s, y'-y\big)\, ds'\, dy' \bigg\|_p^2, \quad t\in[0,T].\end{aligned} \tag{A.5}$$

In the following,

(1) we will prove for each $\varepsilon > 0$, Q_ε is uniformly bounded on $[0,T]$;
(2) we will obtain an integral inequality for Q_ε;
(3) we will conclude with the classic Gronwall's lemma.

Recall from (2.6) and we can write

$$\big(Du(t,z)*M_\varepsilon\big)(s,y)=\frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}}\mathbf{1}_{\{|z-y'|\le t-s'\}}\sigma\big(u(s',y')\big)M_\varepsilon\big(s'-s,y'-y\big)\,ds'\,dy'$$
$$+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\mathbf{1}_{\{|z-\xi|\le t-a\}}\Sigma(a,\xi)\big(Du(a,\xi)*M_\varepsilon\big)(s,y)W(da,d\xi).$$

Then, using Burkholder's inequality and Minkowski's inequality in the same way as before, we can arrive at

$$\begin{aligned}Q_\varepsilon(t)&\le\frac{K_p^2(t)}{2}\bigg(\sup_{z\in\mathbb{R}}\int_{\mathbb{R}_+\times\mathbb{R}}\mathbf{1}_{\{|z-y'|\le t-s'\}}M_\varepsilon\big(s'-s,y'-y\big)\,ds'\,dy'\bigg)^2\\&\quad+L^2c_p^2t\int_0^t Q_\varepsilon(a)\,da\\&\le\frac{K_p^2(t)}{2}\|M\|^2_{L^1(\mathbb{R}_+\times\mathbb{R})}+L^2c_p^2t\int_0^t Q_\varepsilon(a)\,da,\quad t\in[0,T].\end{aligned}$$

We know from (A.3) and Cauchy–Schwarz inequality that

$$Q_\varepsilon(t)\le\|M_\varepsilon\|^2_{\mathfrak{H}}\sup_{z\in\mathbb{R}}\big(\mathbb{E}\big[\big\|Du(t,z)\big\|^p_{\mathfrak{H}}\big]\big)^{2/p},$$

which is uniformly bounded on $[0,T]$. Then it follows from Gronwall's lemma that

$$Q_\varepsilon(t)\le\frac{K_p^2(t)}{2}e^{L^2c_p^2t}\|M\|^2_{L^1(\mathbb{R}_+\times\mathbb{R})},\quad\forall t\in[0,T].$$

The above bound is independent of ε, thus we can further deduce that

$$\sup_{(r,z)\in[0,t]\times\mathbb{R}}\big\|D_{s,y}u(r,z)\big\|_p^2\le\frac{K_p^2(t)}{2}e^{L^2c_p^2t}\|M\|^2_{L^1(\mathbb{R}_+\times\mathbb{R})}<+\infty.$$

That is, claim (A.1) is established for the case $H=1/2$.

Step 2: Case $H\in(1/2,1)$. In this case we have first to show that $Du(t,x)$ is an element of $L^2(\Omega\times\mathbb{R}_+\times\mathbb{R})$ and for this we will use the Picard iterations. Let $u_0(t,x)=1$ and for $n\ge0$, set

$$u_{n+1}(t,x)=1+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\mathbf{1}_{\{|x-y|\le t-s\}}\sigma\big(u_n(s,y)\big)W(ds,dy).\tag{A.6}$$

It is routine to show that for any given $T\in\mathbb{R}_+$,

$$\lim_{n\to+\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}}\big\|u(t,x)-u_n(t,x)\big\|_p=0.\tag{A.7}$$

We know that for each $n\ge0$, $u_n(t,x)\in\mathbb{D}^{1,p}$ with

$$D_{s,y}u_{n+1}(t,x)=\frac{1}{2}\mathbf{1}_{\{|x-y|\le t-s\}}\sigma\big(u_n(s,y)\big)+\frac{1}{2}\int_s^t\int_{\mathbb{R}}\mathbf{1}_{\{|x-z|\le t-r\}}\Sigma_n(r,z)D_{s,y}u_n(r,z)W(dr,dz),$$

with $\Sigma_n(r,z)$ being an adapted process bounded by L. Thus, using Burkholder's inequality, Minkowski's inequality and the easy inequality

$$\|XY\|_{p/2}\le\frac{1}{2}\|X\|_p^2+\frac{1}{2}\|Y\|_p^2\tag{A.8}$$

for any $X, Y \in L^p(\Omega)$, we get $\|D_{s,y}u_{n+1}(t,x)\|_p^2$ bounded by

$$\frac{\widetilde{K}_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} + \frac{L^2c_p^2}{2}\int_s^t \alpha_H \int_{\mathbb{R}^2} \mathbf{1}_{\{|x-z|\vee|x-z'|\le t-r\}}\big|z-z'\big|^{2H-2} \times \big\|D_{s,y}u_n(r,z)\big\|_p^2\,dr\,dz\,dz',$$

where $\widetilde{K}_p(t) := \sup\{\|\sigma(u_n(s,x))\|_p : n \ge 0, (s,x) \in [0,t]\times\mathbb{R}\}$. Iterating this procedure gives us

$$\begin{aligned}
&\big\|D_{s,y}u_{n+1}(t,x)\big\|_p^2 \\
&\quad\le \frac{\widetilde{K}_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} + \frac{\widetilde{K}_p^2(t)}{2}\sum_{\ell=1}^n \frac{c_p^{2\ell}L^{2\ell}}{2^\ell}\int_{\Delta_\ell(s,t)} d\mathbf{r}\,\alpha_H^\ell \int_{\mathbb{R}^{2\ell}} \\
&\qquad\times\left(\prod_{k=0}^{\ell-1}\mathbf{1}_{\{|z_k-z_{k+1}|\vee|z_k-z'_{k+1}|\le r_{k-1}-r_k\}}\big|z_{k+1}-z'_{k+1}\big|^{2H-2}\right)\mathbf{1}_{\{|z_\ell-y|\le r_\ell-s\}}\,d\mathbf{z}'\,d\mathbf{z}.
\end{aligned}$$

Again, it is easy to see the following implication holds:

$$\mathbf{1}_{\{|z_\ell-y|\le r_\ell-s\}}\prod_{k=0}^{\ell-1}\mathbf{1}_{\{|z_k-z_{k+1}|\vee|z_k-z'_{k+1}|\le r_{k-1}-r_k\}} \ne 0 \quad\Longrightarrow\quad |x-y|\le t-s,$$

therefore

$$\begin{aligned}
&\big\|D_{s,y}u_{n+1}(t,x)\big\|_p^2 \\
&\quad\le \frac{\widetilde{K}_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} + \frac{\widetilde{K}_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}}\sum_{\ell=1}^n \frac{c_p^{2\ell}L^{2\ell}}{2^\ell}\int_{\Delta_\ell(s,t)} d\mathbf{r} \\
&\qquad\times\alpha_H^\ell\int_{\mathbb{R}^{2\ell}}\left(\prod_{k=0}^{\ell-1}\mathbf{1}_{\{|z_k-z_{k+1}|\vee|z_k-z'_{k+1}|\le r_{k-1}-r_k\}}\big|z_{k+1}-z'_{k+1}\big|^{2H-2}\right)d\mathbf{z}'\,d\mathbf{z} \\
&\quad\le \frac{\widetilde{K}_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}} + \frac{\widetilde{K}_p^2(t)}{2}\mathbf{1}_{\{|x-y|\le t-s\}}\sum_{\ell=1}^n \frac{c_p^{2\ell}L^{2\ell}}{2^\ell}\big(4^H t^{2H+1}\big)^\ell\frac{1}{\ell!},
\end{aligned}$$

where the last inequality is a consequence of (A.2). We conclude that

$$\big\|D_{s,y}u_{n+1}(t,x)\big\|_p^2 \le \frac{\widetilde{K}_p^2(t)}{2}e^{2t^{2H+1}c_p^2L^2}\mathbf{1}_{\{|x-y|\le t-s\}} =: C\mathbf{1}_{\{|x-y|\le t-s\}}. \tag{A.9}$$

It follows immediately from Minskowski's inequality and (A.9) that

$$\mathbb{E}\big[\big\|Du_n(t,x)\big\|_{L^2(\mathbb{R}_+\times\mathbb{R})}^p\big] \le \left(\int_{\mathbb{R}_+\times\mathbb{R}}\big\|D_{s,y}u_n(t,x)\big\|_p^2\,ds\,dy\right)^{p/2} \le \big(Ct^2\big)^{p/2}$$

uniformly in $n \ge 1$ and *uniformly* in $x \in \mathbb{R}$. In particular, $\{Du_n(t,x), n \ge 1\}$ is uniformly bounded in $L^p(\Omega; L^2(\mathbb{R}_+\times\mathbb{R}))$. Note that the convergence in (A.7) and standard Malliavin calculus arguments can lead us to the fact that up to some subsequence, $Du_n(t,x)$ converges to $Du(t,x)$ in the weak topology of $L^p(\Omega; L^2(\mathbb{R}_+\times\mathbb{R}))$, so we can conclude that $D_{s,y}u(t,x)$ is indeed a function in (s,y) and for any fixed $T \in \mathbb{R}_+$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}\mathbb{E}\big[\big\|Du(t,x)\big\|_{L^2(\mathbb{R}_+\times\mathbb{R})}^p\big] < +\infty.$$

Now we use the same approximation of the identity (M_ε) and obtain for almost every $(s,y) \in \mathbb{R}_+\times\mathbb{R}$,

$$\big\|D_{s,y}u(t,x)\big\|_p^2 \le \sup_{\varepsilon>0}\left\|\int_{\mathbb{R}_+\times\mathbb{R}} D_{s',y'}u(t,x)M_\varepsilon\big(s'-s, y'-y\big)\,ds'\,dy'\right\|_p^2.$$

Let $\varepsilon > 0$ be fixed and let $Q_\varepsilon(t)$ be defined as in (A.5), we have in this case, applying Lemma 2.1,

$$\begin{aligned} Q_\varepsilon(t) &\le \frac{1}{2} K_p^2(t) \|M\|^2_{L^1(\mathbb{R}_+\times\mathbb{R})} \\ &\quad + \frac{L^2 c_p^2}{2} \int_0^t \alpha_H \int_{\mathbb{R}^2} \mathbf{1}_{\{|x-z|\vee|x-z'|\le t-r\}} \big|z-z'\big|^{2H-2} Q_\varepsilon(r)\, dr\, dz\, dz' \\ &\le \frac{1}{2} K_p^2(t) \|M\|^2_{L^1(\mathbb{R}_+\times\mathbb{R})} + \frac{L^2 c_p^2 4^H t^{2H}}{2} \int_0^t Q_\varepsilon(r)\, dr. \end{aligned}$$

Similarly as in previous case, we have

$$Q_\varepsilon(t) \le \|M_\varepsilon\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} \big(\mathbb{E}\big[\big\|Du(t,x)\big\|^p_{L^2(\mathbb{R}_+\times\mathbb{R})}\big]\big)^{2/p}$$

so that the same application of Gronwall's lemma gives

$$\sup_{(r,z)\in[0,t]\times\mathbb{R}} \big\|D_{s,y} u(r,z)\big\|_p^2 \le \frac{K_p^2(t)}{2} e^{2L^2 c_p^2 t^{2H}} \|M\|^2_{L^1(\mathbb{R}_+\times\mathbb{R})} < +\infty.$$

That is, claim (A.1) is also established for the case $H \in (1/2, 1)$.

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