

CENTRAL LIMIT THEOREMS FOR STOCHASTIC WAVE EQUATIONS IN DIMENSIONS ONE AND TWO

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ABSTRACT. Fix $d \in \{1, 2\}$, we consider a d -dimensional stochastic wave equation driven by a Gaussian noise, which is temporally white and colored in space such that the spatial correlation function is integrable and satisfies Dalang's condition. In this setting, we provide quantitative central limit theorems for the spatial average of the solution over a Euclidean ball, as the radius of the ball diverges to infinity. We also establish functional central limit theorems. A fundamental ingredient in our analysis is the pointwise L^p -estimate for the Malliavin derivative of the solution, which is of independent interest. This paper is another addendum to the recent research line of averaging stochastic partial differential equations.

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1. INTRODUCTION

In this article, we fix $d \in \{1, 2\}$ and consider the stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \sigma(u)\dot{W}, \quad (1.1) \quad \{\text{2dSWE}\}$$

on $\mathbb{R}_+ \times \mathbb{R}^d$ with initial conditions $u(0, x) = 1$ and $\frac{\partial u}{\partial t}(0, x) = 0$, where Δ is Laplacian in space variables and \dot{W} is a centered Gaussian noise with covariance

$$\mathbb{E}[\dot{W}(t, x)\dot{W}(s, y)] = \delta_0(t - s)\gamma(x - y). \quad (1.2) \quad \{\text{cov}\}$$

Here \dot{W} is a distribution-valued field and will be formally introduced in Section 2.1.

Throughout this article, we fix the following conditions:

(C1) $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $L \in (0, \infty)$.

(C2) γ is a tempered nonnegative and nonnegative definite function, whose Fourier transform μ satisfies Dalang's condition:

$$\int_{\mathbb{R}^d} \frac{\mu(dz)}{1 + |z|^2} < \infty, \quad (1.3) \quad \{\text{DalangC}\}$$

where $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^d .

(C3) $\sigma(1) \neq 0$.

Conditions (C1) and (C2) ensure that equation (1.1) has a unique *random field solution*, which is adapted to the filtration generated by W , such that $\sup \{\mathbb{E}[|u(t, x)|^k] : (t, x) \in [0, T] \times \mathbb{R}^d\}$ is finite for all $T \in (0, \infty)$ and $k \geq 2$, and

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)\sigma(u(s, y))W(ds, dy), \quad (1.4) \quad \{\text{mild}\}$$

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where the above stochastic integral is defined in the sense of Dalang-Walsh and $G_{t-s}(x-y)$ denotes the fundamental solution to the corresponding deterministic wave equation, *i.e.*

$$G_t(x) := \begin{cases} \frac{1}{2} \mathbf{1}_{\{|x| < t\}}, & \text{if } d = 1 \\ \frac{1}{2\pi\sqrt{t^2 - |x|^2}} \mathbf{1}_{\{|x| < t\}}, & \text{if } d = 2; \end{cases} \quad (1.5) \quad \{\text{fundamentalS}\}$$

see [6, 7]. Condition **(C3)** excludes the trivial case where $u(t, x) = 1$, see Section 5.1.

We are interested in the behavior of the solution to (1.1) in the space variable, and the next result provides relevant stationarity and ergodicity properties.

{prop1}

Proposition 1.1. *Suppose that γ satisfies $\gamma \in L^1(\mathbb{R})$ if $d = 1$ and $\gamma \in L^1(\mathbb{R}^2) \cap L^\ell(\mathbb{R}^2)$ for some $\ell > 1$ if $d = 2$. Fix $t > 0$. Then, the random field $\{u(t, x) : x \in \mathbb{R}^d\}$ have the following properties:*

- (i) *it is strictly stationary: The finite-dimensional distributions of $\{u(t, x+y) : x \in \mathbb{R}^d\}$ does not depend on $y \in \mathbb{R}^d$;*
- (ii) *it is ergodic.*

We postpone the proof to Section 3.

We define for each $t \in [0, \infty)$,

$$F_R(t) = \int_{B_R} (u(t, x) - 1) dx, \quad \text{where } B_R = \{x \in \mathbb{R}^d : |x| \leq R\}. \quad (1.6)$$

It follows from Proposition 1.1 and the ergodic theorem that

$$F_R(t)/R^d \xrightarrow{R \rightarrow \infty} 0 \text{ almost surely and in } L^p(\Omega) \text{ for any } p \geq 1. \quad (1.7)$$

In this framework, it is natural to investigate second-order fluctuations. Indeed, when considering equation (1.1) driven by a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, it is easy to see that $u(t, x)$ is independent from $u(t, y)$ for $|x - y| > 2t$, see for instance [8, Page 3021]; in this case, the random variable $F_R(t)$ can be roughly understood as a sum of weakly dependent random variables. If the spatial correlation kernel γ is integrable, one deduce from Lemma 4.1 that the process $\{u(t, x) : x \in \mathbb{R}^d\}$ has short-range dependence so that it is natural to expect Gaussian fluctuations of $F_R(t)$, as $R \rightarrow \infty$.

Notation. We denote the standard Gaussian distribution by $\mathcal{N}(0, 1)$ and the $L^p(\Omega)$ -norm by $\|\cdot\|_p$ for any $p \geq 1$. Also, ω_d denotes the volume of the unit ball, that is, $\omega_d = 2$ for $d = 1$ and $\omega_d = \pi$ for $d = 2$. We put $a \lesssim b$ if $a \leq Cb$ for some positive constant C that does not depend on a, b .

In what follows, we present the main result of this article.

{MAIN}

Theorem 1.2. *Suppose that γ satisfies $\gamma \in L^1(\mathbb{R})$ if $d = 1$ and $\gamma \in L^1(\mathbb{R}^2) \cap L^\ell(\mathbb{R}^2)$ for some $\ell > 1$ if $d = 2$. Then the following statements hold:*

- (i) *The process $\{R^{-d/2}F_R(t) : t \geq 0\}$ converges in law to a centered continuous Gaussian process $\{\mathcal{G}_t : t \geq 0\}$, where*

$$\mathbb{E}[\mathcal{G}_{t_1}\mathcal{G}_{t_2}] = \omega_d \int_{\mathbb{R}^d} \text{Cov}(u(t_1, \xi), u(t_2, 0)) d\xi.$$

- (ii) *For any fixed $t > 0$,*

$$d_{\text{TV}}(F_R(t)/\sigma_R, \mathcal{N}(0, 1)) \lesssim R^{-d/2}, \quad (1.8)$$

{QCLT}

where $\sigma_R^2 := \text{Var}(F_R(t)) > 0$ for every¹ $R > 0$ is part of the conclusion and d_{TV} stands for the total variation distance.

Remark 1. (1) Here are two examples of integrable kernels:

$$\gamma(x) = \begin{cases} (4\pi)^{\alpha/2} \Gamma(\alpha/2) \int_0^\infty \mathbf{w}^{\frac{\alpha-d}{2}-1} e^{-\mathbf{w}} \exp\left(-\frac{|x|^2}{4\mathbf{w}}\right) d\mathbf{w} & \text{(Bessel kernel of order } \alpha > 0) \\ (4\pi\beta)^{-d/2} \exp(-|x|^2/(4\beta)) & \text{(heat kernel with parameter } \beta > 0) \end{cases}$$

and the corresponding spectral measures are given as $\mu(d\xi) = (1 + |\xi|^2)^{-\alpha/2} d\xi$ and $\mu(d\xi) = e^{-4\pi^2\beta|\xi|^2}$. Note that Dalang's condition (1.3) is satisfied for the Bessel kernel of order $\alpha \geq 2$.

(2) When $d = 1$, Theorem 1.2 and Theorem 1.3 below are still true under the assumption that γ is a finite measure. We leave the details to interested readers.

(3) Our rate of convergence and the limiting variance order depend on the correlation kernel γ . In [8], it was proved that for $d = 1$:

- when $\gamma = \delta_0$ (which is a finite measure on \mathbb{R}), the limiting variance is of order R and the rate of convergence is of order $R^{-1/2}$;
- when $\gamma(z) = |z|^{2H-2}$ for some $H \in (1/2, 1)$ (Riesz kernel), the limiting variance is of order R^{2H} and the rate of convergence is of order R^{H-1} .

In [2] it is proved that for $d = 2$, when $\gamma(z) = |z|^{-\beta}$ for some $\beta \in (0, 2)$, the limiting variance is of order $R^{4-\beta}$ and the rate of convergence is of order $R^{-\beta/2}$.

In the sequel, we sketch the “*usual proof strategy*” and highlight the key ingredients. The proof of the functional CLT consists in proving the *f.d.d.* convergence and tightness. We appeal to the tightness criterion of Komogorov-Chentsov (see *e.g.* [13]) and prove tightness by obtaining moment estimate of the increments $F_R(t) - F_R(s)$. For the *f.d.d.* convergence, we first derive the asymptotic variance and then apply the so-called Malliavin-Stein approach to show the *f.d.d.* convergence. More precisely, we need a multivariate Malliavin-Stein bound for this purpose, while the univariate Malliavin-Stein bound provides the rate for the marginal convergence that is described by the total-variation distance. It is worth remarking that as a tailor-made combination of Malliavin calculus and Stein's method initiated by Nourdin and Peccati, the Malliavin-Stein approach has proved to be a very useful toolkit in establishing Gaussian fluctuations in various frameworks, notably for the functionals over a Gaussian field, see the recent monograph [15] for a comprehensive treatment.

That being said, we will use the Malliavin calculus intensively for our computations and inevitably, we will encounter random variables of the form $D_{s,y}u(t, x)$. Note that $Du(t, x)$ denotes the Malliavin derivative of $u(t, x)$, which lives in the Hilbert space \mathfrak{H} associated to the noise W ; see section 2 for precise definitions. The space \mathfrak{H} may contain generalized functions, so to estimate L^p -norm of $D_{s,y}u(t, x)$, we shall first clarify that $D_{s,y}u(t, x)$ is indeed a real function in (s, y) . Moreover, we need to prove an estimate of the form

$$\|D_{s,y}u(t, x)\|_p \lesssim G_{t-s}(x - y)$$

in order to proceed with our computations for asymptotic variance and *f.d.d.* convergence. This is the content of the following theorem.

Theorem 1.3. *Let the assumptions in Theorem 1.2 prevail. For any $p \in [2, \infty)$ and any $t > 0$, the following estimates hold for almost all $(s, y) \in [0, t] \times \mathbb{R}^d$:*

$$G_{t-s}(x - y) \|\sigma(u(s, y))\|_p \leq \|D_{s,y}u(t, x)\|_p \leq C_{p,t,L,\gamma} \kappa_{p,t,L} G_{t-s}(x - y), \quad (1.9)$$

¹Note that σ_R depends on the parameter t and the conclusion “ $\sigma_R > 0$ for each $R > 0$ ” is ensured by condition (C3). The proof of this part is omitted here and can be done by following the same arguments as in [8, Lemma 3.4].

{THM}

{IMP}

where the constant $\kappa_{p,t,L}$ is defined in (5.3) and the constant $C_{p,t,L,\gamma}$ is given by (5.16) in the 1D case and by (5.30) in the 2D case.

Remark 2. As one can read from the proof of the above theorem, the lower bound holds whenever the spatial correlation kernel γ satisfies Dalang's condition (1.3). Concerning the upper bound in the case $d = 2$, the assumption $\gamma \in L^1(\mathbb{R}^2)$ is not needed.

Let us compare our result with similar results in the literature. There have been several recent works on the application of the Malliavin-Stein approach to establish central limit theorems for spatial averages of stochastic partial differential equations and to derive quantitative error bound in the total variation distance. A fundamental ingredient in all these papers is an upper bound similar to (1.9). The works [10] and [11] deal with the stochastic heat equation with $d = 1$ and $\gamma = \delta_0$ and $d \geq 1$ and $\gamma(x) = |x|^{-\beta}$, $0 < \beta < \min(2, d)$ (Riesz kernel covariance), respectively. The case of an integrable kernel γ has been considered in [4, 5]. For the stochastic heat equation, an upper bound of the form (1.9), holds with G being the heat kernel. In this case, the proof relies heavily on the semigroup property of the heat kernel.

For the wave equation, the works [8, 2] establish the Gaussian fluctuation of spatial averages of stochastic wave equations in the following cases: $d = 1$ and $\gamma(x) = |x|^{2H-2}$, $H \in [1/2, 1)$, and $d = 2$ and $\gamma(x) = |x|^{-\beta}$, $0 < \beta < 2$, respectively. In the case $d = 1$, the proof of (1.9) is not very difficult because $G_{t-s}(x - y)$ is uniformly bounded. The case $d = 2$ is much more difficult due to the singularity within the fundamental solution (1.5). In the present article, we consider the integrable covariance kernel that requires novel technical estimates and as we can read from Theorem 1.2, the order of fluctuation in this case is $R^{d/2}$, which is the same as in the case of parabolic Anderson model driven by integrable covariance kernel [19]. Our paper can be viewed as another pixel, along with [2, 4, 8, 10, 11, 19], for completing the picture of averaging SPDEs. It is worth pointing out that the authors of [1] considered the 1D linear stochastic wave equation driven by space-time homogeneous Gaussian noise and they obtained a weaker result than (1.9). Their methodology is totally different than ours: Due to the linearity, one has the explicit chaos expansion of the solution, then obtaining the upper bound for $\|D_{s,y}u(t, x)\|_p$ reduces to explicit (but very complicated) computations. And in view of this reference, we believe our bounds in (1.9) could be very useful in establishing absolute continuity result for the solution to 2D stochastic wave equation.

The rest of this article is organized as follows: In Section 2 we present preliminary results for our proofs, and Sections 3 4, 5 are devoted to the proofs of Proposition 1.1, Theorem 1.2 and Theorem 1.3 respectively.

2. PRELIMINARIES

In this section we present some preliminaries on stochastic analysis, Malliavin calculus and the Stein-Malliavin approach to normal approximations.

2.1. Basic stochastic analysis. Let \mathfrak{H} be defined as the completion of $C_c(\mathbb{R}_+ \times \mathbb{R}^d)$ under the inner product

$$\langle f, g \rangle_{\mathfrak{H}} = \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} f(s, y) g(s, z) \gamma(y - z) dy dz ds = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^d} \mathcal{F}f(s, \xi) \mathcal{F}g(s, -\xi) \mu(d\xi) \right) ds,$$

where $\mathcal{F}f(s, \xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(s, x) dx$.

Consider an isonormal Gaussian process associated to the Hilbert space \mathfrak{H} , denoted by $W = \{W(\phi) : \phi \in \mathfrak{H}\}$. That is, W is a centered Gaussian family of random variables such that $\mathbb{E}[W(\phi)W(\psi)] = \langle \phi, \psi \rangle_{\mathfrak{H}}$ for any $\phi, \psi \in \mathfrak{H}$. As the noise W is white in time, a martingale structure naturally appears. First we define \mathcal{F}_t to be the σ -algebra generated by \mathbb{P} -negligible

sets and $\{W(\phi) : \phi \in C(\mathbb{R}_+ \times \mathbb{R}^d) \text{ has compact support contained in } [0, t] \times \mathbb{R}^d\}$, so we have a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in \mathbb{R}_+\}$. If $\{\Phi(s, y) : (s, y) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is an \mathbb{F} -adapted random field such that $\mathbb{E}[\|\Phi\|_{\mathfrak{H}}^2] < +\infty$, then

$$M_t = \int_{[0, t] \times \mathbb{R}^d} \Phi(s, y) W(ds, dy),$$

interpreted as the Dalang-Walsh integral ([6, 20]), is a square-integrable \mathbb{F} -martingale with quadratic variation

$$\langle M \rangle_t = \int_{[0, t] \times \mathbb{R}^{2d}} \Phi(s, y) \Phi(s, z) \gamma(y - z) dy dz ds.$$

Let us record a useful version of Burkholder-Davis-Gundy inequality (BDG for short); see *e.g.* [12, Theorem B.1].

Lemma 2.1. *If $\{\Phi(s, y) : (s, y) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is an adapted random field with respect to \mathbb{F} such that $\|\Phi\|_{\mathfrak{H}} \in L^p(\Omega)$ for some $p \geq 2$, then*

{BDG}

$$\left\| \int_{[0, t] \times \mathbb{R}^d} \Phi(s, y) W(ds, dy) \right\|_p^2 \leq 4p \left\| \int_{[0, t] \times \mathbb{R}^{2d}} \Phi(s, y) \Phi(s, z) \gamma(y - z) dy dz ds \right\|_{p/2}.$$

Now let us recall some basic facts on Malliavin calculus associated with W . For any unexplained notation and result, we refer to the book [16]. We denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let \mathcal{S} be the space of simple functionals of the form $F = f(W(h_1), \dots, W(h_n))$ for $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}$, $1 \leq i \leq n$. Then, the Malliavin derivative DF is the \mathfrak{H} -valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i.$$

The derivative operator D is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathfrak{H})$ for any $p \geq 1$ and we define $\mathbb{D}^{1,p}$ to be the completion of \mathcal{S} under the norm

$$\|F\|_{1,p} = (\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathfrak{H}}^p])^{1/p}.$$

The *chain rule* for D asserts that if $F \in \mathbb{D}^{1,2}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz, then $h(F) \in \mathbb{D}^{1,2}$ with

$$D[h(F)] = Y DF, \tag{2.1} \quad \{\text{cr}\}$$

where Y is some $\sigma\{F\}$ -measurable random variable bounded by the Lipschitz constant of h ; when h is additionally differentiable, we have $Y = h'(F)$.

We denote by δ the adjoint of D given by the duality formula

$$\mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF \rangle_{\mathfrak{H}}] \tag{2.2} \quad \{\text{D:delta}\}$$

for any $F \in \mathbb{D}^{1,2}$ and $u \in \text{Dom } \delta \subset L^2(\Omega; \mathfrak{H})$, the domain of δ . The operator δ is also called the Skorohod integral and in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod (see *e.g.* [9, 18]).

In our context, the Dalang-Walsh integral coincides with the Skorohod integral: Any adapted random field Φ that satisfies $\mathbb{E}[\|\Phi\|_{\mathfrak{H}}^2] < \infty$ belongs to the domain of δ and

$$\delta(\Phi) = \int_0^\infty \int_{\mathbb{R}^d} \Phi(s, y) W(ds, dy).$$

As a consequence, the mild formulation equation (1.4) can be written as

$$u(t, x) = 1 + \delta(G_{t-\bullet}(x - *)\sigma(u(\bullet, *))).$$

Using the two-parameter Clark-Ocone formula, we can represent $F \in \mathbb{D}^{1,2}$ as a stochastic integral; see *e.g.* [3, Proposition 6.3] for a proof.

Lemma 2.2 (Clark-Ocone formula). *Given $F \in \mathbb{D}^{1,2}$, we have almost surely*

$$F = \mathbb{E}[F] + \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mathbb{E}[D_{s,y}F | \mathcal{F}_s] W(ds, dy).$$

As a consequence of Lemma 2.2, we can derive the following *Poincaré inequality*: For any two random variables $F, G \in \mathbb{D}^{1,2}$, we have

$$|\text{Cov}(F, G)| \leq \int_0^\infty \int_{\mathbb{R}^{2d}} \|D_{s,y}F\|_2 \|D_{s,z}G\|_2 \gamma(y-z) dy dz ds. \quad (2.3)$$

Recall that the total variation distance between two probability measures μ and ν on \mathbb{R} is defined by

$$d_{\text{TV}}(\mu, \nu) = \sup_{B \in \mathcal{B}(\mathbb{R})} |\mu(B) - \nu(B)|,$$

where $\mathcal{B}(\mathbb{R})$ denotes the family of all Borel subsets on \mathbb{R} . As usual, $d_{\text{TV}}(F, G)$ will denote the total variation distance between the distribution measures of F and G ; and $d_{\text{TV}}(F, \mathcal{N}(0, 1))$ is the total variation distance between F and a standard Gaussian random variable.

The combination of Stein's method for normal approximations with Malliavin calculus leads to the following bound on the total variation distance, see [17, Theorem 8.2.1] for more details.

Proposition 2.3. *Suppose that $F = \delta(v) \in \mathbb{D}^{1,2}$ has unit variance for some $v \in \text{Dom}(\delta)$. Then*

$$d_{\text{TV}}(F, \mathcal{N}(0, 1)) \leq 2\sqrt{\text{Var}\langle DF, v \rangle_{\mathfrak{H}}}. \quad (2.4)$$

2.2. Basic formulas. We close this section with some basic relations for the fundamental solution $G_t(x)$. For $t > 0$ and $m > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^{2d}} G_t(y) G_t(z) \gamma(y-z) dy dz &= \int_{\mathbb{R}^d} \frac{\sin^2(t|\xi|)}{|\xi|^2} \mu(d\xi) \\ &= t^2 \int_{|\xi| \leq m} \frac{\sin^2(t|\xi|)}{t^2 |\xi|^2} \mu(d\xi) + \int_{|\xi| > m} \frac{\sin^2(t|\xi|)}{|\xi|^2} \mu(d\xi) \\ &\leq t^2 \mu(|\xi| \leq m) + \int_{|\xi| > m} \frac{\mu(d\xi)}{|\xi|^2}. \end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^{2d}} G_t(y) G_t(z) \gamma(y-z) dy dz \leq \inf_{m>0} \left(t^2 \mu(|\xi| \leq m) + \int_{|\xi| > m} |\xi|^{-2} \mu(d\xi) \right) =: \mathfrak{m}_t. \quad (2.5)$$

It is clear that \mathfrak{m}_t is nondecreasing in t .

For any $s < r < t$ and $d = 1$, we have

$$\int_{\mathbb{R}} G_{t-r}(x-z) G_{r-s}(z-y) dz \leq \frac{1}{2}(t-s) G_{t-s}(x-y). \quad (2.6)$$

For any $t \in (0, \infty)$, we define

$$\varphi_{t,R}(s, y) = \int_{B_R} G_{t-s}(x-y) dx \quad (2.7)$$

where we recall that $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$. In the following lemma, we provide a useful estimate about $y \in \mathbb{R}^d \mapsto \varphi_{t,R}(s, y)$.

{lemfact}

Lemma 2.4. For $t_1, t_2 \in (0, \infty)$, the quantity

$$R^{-d} \int_{\mathbb{R}^{2d}} \varphi_{t_1, R}(s, y) \varphi_{t_2, R}(s, z) \gamma(y - z) dy dz$$

is uniformly bounded over $s \in [0, t_2 \wedge t_1]$ and $R > 0$, provided $\gamma \in L^1(\mathbb{R}^d)$.

Proof. It is clear that for any $d \in \{1, 2\}$ and $s < t$,

$$\varphi_{t, R}(s, y) \leq \int_{\mathbb{R}^d} G_{t-s}(x - y) dx = t - s, \quad (2.8) \quad \{\text{ineq:241}\}$$

$$\int_{\mathbb{R}^d} \varphi_{t, R}(s, y) dy = \int_{B_R} dx \int_{\mathbb{R}^d} dy G_{t-s}(x - y) \leq \omega_d(t - s) R^d. \quad (2.9) \quad \{\text{ineq:242}\}$$

Then, we can write

$$\begin{aligned} R^{-d} \int_{\mathbb{R}^{2d}} \varphi_{t_1, R}(s, y) \varphi_{t_2, R}(s, z) \gamma(y - z) dy dz &\leq (t_1 - s) R^{-d} \int_{\mathbb{R}^{2d}} \varphi_{t_2, R}(s, z) \gamma(y - z) dy dz \quad \text{by (2.8)} \\ &\leq (t_1 - s) \|\gamma\|_{L^1(\mathbb{R}^d)} R^{-d} \int_{\mathbb{R}^d} \varphi_{t_2, R}(s, z) dz \\ &\leq \omega_d(t_1 - s)(t_2 - s) \|\gamma\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

This gives us the desired uniform boundedness. \square

In the end of this section, we state two useful estimates from the paper [2].

{LEM51}

Lemma 2.5. When $d = 2$, $G_t(x) = \frac{1}{2\pi} [t^2 - |x|^2]^{-1/2} \mathbf{1}_{\{|x| < t\}}$. The following estimates hold:

(1) For $q \in (1/2, 1)$ and $s < t$, then

$$\int_s^t dr (G_{t-r}^{2q} * G_{r-s}^{2q})^{1/q}(z) \lesssim (t - s)^{\frac{1}{q}-1} G_{t-s}^{2-\frac{1}{q}}(z),$$

where the implicit constant only depends on q ; see Lemma 3.3 in [2].

(2) For any $p \in (0, 1)$ and $q \in (1/2, 1)$ such that $p + 2q \leq 3$, we have for $s < t$,

$$\int_s^t G_{t-r}^{2q} * G_{r-s}^p(z) dr \lesssim (t - s)^{3-p-2q} \mathbf{1}_{\{|z| < t-s\}},$$

where the implicit constant only depends on p and q ; see Lemma 3.4 in [2].

3. PROOF OF PROPOSITION 1.1

{SEC3}

The strict stationarity follows from two facts:

- (1) For each $y \in \mathbb{R}^d$, the random field $\{u(t, x + y) : x \in \mathbb{R}^d\}$ coincides almost surely with the random field u driven by the shifted noise W_y given by

$$W_y(\phi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \phi(s, x - y) W(ds, dx), \quad \phi \in \mathfrak{H}.$$

- (2) The noise W_y has the same distribution as W , which allows to conclude the proof of the stationarity property.

We refer readers to Lemma 7.1 in [3] and footnote 1 in [8] for similar arguments. Now let us prove the ergodicity and in view of [3, Lemma 7.2], it suffices to prove

$$V_R(t) := \text{Var} \left(R^{-d} \int_{B_R} \prod_{j=1}^k g_j(u(t, x + \zeta^j)) dx \right) \xrightarrow{R \rightarrow \infty} 0,$$

for any fixed $\zeta^1, \dots, \zeta^k \in \mathbb{R}^d$ and $g_1, \dots, g_k \in C_b(\mathbb{R})$ such that each g_j vanishes at zero and has Lipschitz constant bounded by 1.

Using the Poincaré inequality (2.3), we obtain

$$\begin{aligned} V_R(t) &\leq R^{-2d} \int_{B_R^2} dx dy |\text{Cov}(\mathcal{R}(x), \mathcal{R}(y))| \quad \text{with } \mathcal{R}(x) := \prod_{j=1}^k g_j(u(t, x + \zeta^j)) \\ \{EQ2\} \quad &\leq R^{-2d} \int_{B_R^2} \int_0^t \int_{\mathbb{R}^{2d}} \|D_{s,z} \mathcal{R}(x)\|_2 \|D_{s,z'} \mathcal{R}(y)\|_2 \gamma(z - z') dz dz' ds dx dy. \end{aligned} \quad (3.1)$$

By the chain rule (2.1),

$$|D_{s,z} \mathcal{R}(x)| \leq \mathbf{1}_{(0,t)}(s) \sum_{j_0=1}^k \left| \prod_{j=1, j \neq j_0}^k g_j(u(t, x + \zeta^j)) \right| |D_{s,y} u(t, x + \zeta^{j_0})|,$$

which implies, for any $s \in [0, t]$,

$$\{EQ1\} \quad \|D_{s,z} \mathcal{R}(x)\|_2 \leq \max_{1 \leq j \leq k} \sup_{a \in \mathbb{R}} |g_j(a)|^{k-1} \sum_{j_0=1}^k \|D_{s,y} u(t, x + \zeta^{j_0})\|_2 \lesssim \sum_{j_0=1}^k G_{t-s}(x - y + \zeta^{j_0}), \quad (3.2)$$

where in the second inequality we used Theorem 1.3. Plugging (3.2) into (3.1), yields

$$V_R(t) \lesssim R^{-2d} \sum_{j,\ell=1}^k \int_{B_R^2} \int_0^t \int_{\mathbb{R}^{2d}} G_{t-s}(x - z + \zeta^j) G_{t-s}(y - z' + \zeta^\ell) \gamma(z - z') dz dz' ds dx dy.$$

Using

$$\int_{B_R} dy G_{t-s}(y - z' + \zeta^\ell) \leq \int_{\mathbb{R}^d} dy G_{t-s}(y) \lesssim t - s \quad \text{and} \quad \int_{\mathbb{R}^d} dz' \gamma(z - z') < \infty,$$

we deduce that $V_R(t) \lesssim R^{-d}$. This finish the proof of Proposition 1.1. \square

4. PROOF OF THEOREM 1.2

The proof will be decomposed in several steps.

4.1. Asymptotic behavior of the covariance. For any $t_1, t_2 \geq 0$, in view of the stationarity of the random field $\{(u(t_1, x), u(t_2, x)) : x \in \mathbb{R}^d\}$, we have

$$\mathbb{E}[F_R(t_1) F_R(t_2)] = \int_{B_R^2} \mathbb{E}[(u(t_1, x - y) - 1)(u(t_2, 0) - 1)] dx dy.$$

By the dominated convergence theorem (see *e.g.* [19, page 27]), we obtain

$$\{R1\} \quad \lim_{R \rightarrow \infty} R^{-d} \mathbb{E}[F_R(t_1) F_R(t_2)] = \omega_d \int_{\mathbb{R}^d} \text{Cov}(u(t_1, x), u(t_2, 0)) dx, \quad (4.1)$$

provided $x \in \mathbb{R}^d \mapsto |\text{Cov}(u(t_1, x), u(t_2, 0))|$ is integrable. In the next lemma we show this integrability property.

Lemma 4.1. *For any $t_1, t_2 \geq 0$,*

$$\int_{\mathbb{R}^d} |\text{Cov}(u(t_1, x), u(t_2, 0))| dx < \infty.$$

Proof. Fix $t_1, t_2 \in [0, T]$. Using Poincaré inequality (2.3) and the estimate (1.9) yields

$$\begin{aligned} & \int_{\mathbb{R}^d} dx |\text{Cov}(u(t_1, x), u(t_2, 0))| \\ & \leq \int_{\mathbb{R}^d} dx \int_0^{t_1 \wedge t_2} \int_{\mathbb{R}^{2d}} dy dz ds \|D_{s,y} u(t_1, x)\|_2 \|D_{s,z} u(t_2, 0)\|_2 \gamma(y-z) dy dz ds \\ & \leq (C_{2,t,L,\gamma} \kappa_{2,t,L})^2 \int_{\mathbb{R}^d} dx \int_0^{t_1 \wedge t_2} \int_{\mathbb{R}^{2d}} ds G_{t_1-s}(x-y) G_{t_2-s}(z) \gamma(y-z) dy dz, \end{aligned}$$

which is finite, by integrating successively in the variables x, y, z and using the integrability of γ . This completes the proof. \square

{sec42}

4.2. Convergence of the finite-dimensional distributions. From the mild equation (1.4) satisfied by $u(t, x)$ and using the fact that the Dalang-Walsh integral coincides with the divergence operator, we can write

$$F_R(t) = \int_{B_R} (u(t, x) - 1) dx = \delta(V_{t,R}), \quad (4.2) \quad \{\text{EQ4}\}$$

with

$$V_{t,R}(s, y) = \varphi_{t,R}(s, y) \sigma(u(s, y)), \quad (4.3)$$

where $\varphi_{t,R}(s, y)$ has been defined in (2.7).

The next proposition is the basic ingredient for the convergence of the finite-dimensional distributions and also for the total variation bound in (1.8).

{prop42}

Proposition 4.2. *For any $t_1, t_2 \in [0, T]$,*

$$\text{Var}(\langle DF_R(t_1), V_{t_2,R} \rangle_{\mathfrak{H}}) \lesssim R^d. \quad (4.4) \quad \{\text{finalbdd}\}$$

Together with Proposition 2.3, the above estimate (4.4) leads us to the total variation bound in (1.8).

Proof of Proposition 4.2. Note that

$$D_{s,y} F_R(t) = \varphi_{t,R}(s, y) \sigma(u(s, y)) + \int_s^t \int_{\mathbb{R}^d} \varphi_{t,R}(r, z) \Sigma_{r,z} D_{s,y} u(r, z) W(dr, dz),$$

where $\{\Sigma_{r,z} : (r, z) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is an adapted random field that is uniformly bounded by L , the Lipschitz constant of σ ; see condition **(C1)**. Then, for $t_1, t_2 \in [0, T]$, we can write $\langle DF_R(t_1), V_{t_2,R} \rangle_{\mathfrak{H}} = A_1 + A_2$, with

$$A_1 = \langle V_{t_1,R}, V_{t_2,R} \rangle_{\mathfrak{H}} = \int_0^{t_1 \wedge t_2} ds \int_{\mathbb{R}^{2d}} dy dz \gamma(y-z) \varphi_{t_1,R}(s, y) \varphi_{t_2,R}(s, z) \sigma(u(s, y)) \sigma(u(s, z))$$

and

$$A_2 = \int_0^{t_1 \wedge t_2} ds \int_{\mathbb{R}^{2d}} dy dy' \gamma(y-y') V_{t_2,R}(s, y') \left(\int_s^{t_1} \int_{\mathbb{R}^2} \varphi_{t_1,R}(r, z) \Sigma_{r,z} D_{s,y} u(r, z) W(dr, dz) \right).$$

It is clear that $\text{Var}(\langle DF_R(t_1), V_{t_2,R} \rangle_{\mathfrak{H}}) \lesssim \text{Var}(A_1) + \text{Var}(A_2)$. So in the sequel, we need to prove

$$\text{Var}(A_j) \lesssim R^d \text{ for } j = 1, 2.$$

Following the same strategy as in [2, Section 4.2], we only need to prove

$$\sup_{s \leq t_1 \wedge t_2} (\mathcal{T}_s + \mathcal{U}_s) \lesssim R^d, \quad (4.5) \quad \{\text{TU:suf}\}$$

where for $s \in (0, t_1 \wedge t_2]$,

$$\begin{aligned} \mathcal{T}_s = \int_0^s dr \int_{\mathbb{R}^{6d}} & \varphi_{t_1, R}(s, y) \varphi_{t_1, R}(s, y') \varphi_{t_2, R}(s, z) \varphi_{t_2, R}(s, z') G_{s-r}(z - \xi) G_{s-r}(z' - \xi') \\ & \times \gamma(\xi - \xi') \gamma(y - z) \gamma(y' - z') d\xi d\xi' dy dz dy' dz' \end{aligned}$$

and

$$\begin{aligned} \mathcal{U}_s = \int_s^{t_1} dr \int_{\mathbb{R}^{6d}} & dz d\tilde{z} dy dy' d\tilde{y} d\tilde{y}' \gamma(y - y') \gamma(\tilde{y} - \tilde{y}') \gamma(z - \tilde{z}) \\ & \times \varphi_{t_2, R}(s, y') \varphi_{t_2, R}(s, \tilde{y}') \varphi_{t_1, R}(r, z) \varphi_{t_1, R}(r, \tilde{z}) G_{r-s}(y - z) G_{r-s}(\tilde{y} - \tilde{z}). \end{aligned}$$

In what follows, we only prove $\sup \{\mathcal{T}_s : s \leq t_1 \wedge t_2\} \lesssim R^d$ and we omit the other part because \mathcal{U}_s has the similar-type expression as \mathcal{T}_s .

Using (2.8), we can write

$$\begin{aligned} \mathcal{T}_s \leq (t_1 - s)(t_2 - s) \int_0^s dr \int_{\mathbb{R}^{6d}} & \varphi_{t_1, R}(s, y) \varphi_{t_2, R}(s, z) \gamma(y - z) G_{s-r}(z - \xi) \\ & \times G_{s-r}(z' - \xi') \gamma(\xi - \xi') \gamma(y' - z') d\xi d\xi' dy dz dy' dz', \end{aligned}$$

then we perform integration with respect to $dy', dz', d\xi', d\xi$ successively to write

$$\mathcal{T}_s \lesssim \int_0^s dr \int_{\mathbb{R}^{2d}} \varphi_{t_1, R}(s, y) \varphi_{t_2, R}(s, z) \gamma(y - z) dy dz \lesssim R^d,$$

where the last estimate follows from Lemma 2.4. This leads to the bound (4.4). \square

We are ready to show the convergence of the finite-dimensional distributions. Let us choose $m \geq 1$ points $t_1, \dots, t_m \in (0, \infty)$. Consider the random vector $\Phi_R = (F_R(t_1), \dots, F_R(t_m))$ and let $\mathbf{G} = (\mathcal{G}_1, \dots, \mathcal{G}_m)$ denote a centered Gaussian random vector with covariance matrix $(\mathcal{C}_{i,j})_{1 \leq i,j \leq m}$, where

$$\mathcal{C}_{i,j} := \omega_d \int_{\mathbb{R}^d} \text{Cov}(u(t_i, \xi), u(t_j, 0)) d\xi.$$

Recall from (4.2) that $F_R(t_i) = \delta(V_{t_i, R})$ for all $i = 1, \dots, m$. Then, by a generalization of a result of Nourdin and Peccati (see *e.g.* [15, Theorem 6.1.2]), we can write

$$\{ \text{equa7} \} \quad |\mathbb{E}(h(R^{-d/2} \Phi_R)) - \mathbb{E}(h(\mathbf{G}))| \leq \frac{m}{2} \|h''\|_\infty \sqrt{\sum_{i,j=1}^m \mathbb{E} \left(|\mathcal{C}_{i,j} - R^{-d} \langle DF_R(t_i), V_{t_j, R} \rangle_{\mathfrak{H}}|^2 \right)} \quad (4.6)$$

for every $h \in C^2(\mathbb{R}^m)$ with bounded second partial derivatives, where

$$\|h''\|_\infty = \max_{1 \leq i,j \leq m} \sup_{x \in \mathbb{R}^m} \left| \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \right|;$$

see also Proposition 2.3 in [10]. Thus, in view of (4.6), in order to show the convergence in law of $R^{-d/2} \Phi_R$ to \mathbf{G} , it suffices to show that for any $i, j = 1, \dots, m$,

$$\{ \text{h6} \} \quad \lim_{R \rightarrow \infty} \mathbb{E} \left(\left| \mathcal{C}_{i,j} - R^{-d} \langle DF_R(t_i), V_{t_j, R} \rangle_{\mathfrak{H}} \right|^2 \right) = 0. \quad (4.7)$$

Notice that, by the duality relation (2.2) and the convergence (4.1), we have

$$\{ \text{h7} \} \quad R^{-d} \mathbb{E} \left(\langle DF_R(t_i), V_{t_j, R} \rangle_{\mathfrak{H}} \right) = R^{-d} \mathbb{E} \left(F_R(t_i) \delta(V_{t_j, R}) \right) = R^{-d} \mathbb{E} \left(F_R(t_i) F_R(t_j) \right) \xrightarrow{R \rightarrow +\infty} \mathcal{C}_{i,j}. \quad (4.8)$$

Therefore, the convergence (4.7) follows immediately from (4.8) and (4.4). Hence the finite-dimensional distributions of $\{R^{-d/2} F_R(t) : t \in \mathbb{R}_+\}$ converge to those of \mathcal{G} as $R \rightarrow \infty$. \square

{sec43}

4.3. Tightness via the criterion of Chentsov-Kolmogorov. In what follow, we appeal to the tightness criterion of Chentsov-Kolmogorov (see *e.g.* [13]) and we only need to obtain the following moment estimate

Lemma 4.3. *Let the conditions (C1)-(C3) hold. If additionally $\gamma \in L^1(\mathbb{R}^d)$, then for any $p \geq 2$ and $0 \leq s < t \leq T \leq R$, we have*

$$R^{-d/2} \|F_R(t) - F_R(s)\|_p \lesssim (t-s)^{1/d}, \quad (4.9) \quad \text{{lem_tight}} \quad \text{{tight}}$$

where the implicit constant does not depend on (R, s, t) .

Proof. Recall that

$$F_R(t) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi_{t,R}(r, y) \sigma(u(r, y)) W(dr, dy).$$

Then by BDG inequality (see Lemma 2.1) and Minkowski's inequality,

$$\begin{aligned} \|F_R(t) - F_R(s)\|_p^2 &\lesssim \left\| \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} (\varphi_{t,R}(r, y) - \varphi_{s,R}(r, y)) (\varphi_{t,R}(r, y') - \varphi_{s,R}(r, y')) \right. \\ &\quad \left. \times \sigma(u(r, y)) \sigma(u(r, y')) \gamma(y - y') dy dy' dr \right\|_{p/2} \\ &\lesssim \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} |\varphi_{t,R}(r, y) - \varphi_{s,R}(r, y)| |\varphi_{t,R}(r, y') - \varphi_{s,R}(r, y')| \\ &\quad \times \|\sigma(u(r, y)) \sigma(u(r, y'))\|_{p/2} \gamma(y - y') dy dy' dr \\ &\lesssim \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} \left(|\varphi_{t,R}(r, y) - \varphi_{s,R}(r, y)|^2 + |\varphi_{t,R}(r, y') - \varphi_{s,R}(r, y')|^2 \right) \gamma(y - y') dy dy' dr \quad (4.10) \quad \text{{s:app}} \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} |\varphi_{t,R}(r, y) - \varphi_{s,R}(r, y)|^2 \gamma(y - y') dy dy' dr, \end{aligned}$$

where we have used the following two facts to obtain (4.10):

- (i) $\|u(r, y)\|_p$ is uniformly bounded on $[0, T] \times \mathbb{R}^d$, (ii) $|ab| \leq \frac{1}{2}(a^2 + b^2)$ for any $a, b \in \mathbb{R}$.

Integrating first with respect to dy' yields,

$$\|F_R(t) - F_R(s)\|_p^2 \lesssim \int_{\mathbb{R}_+ \times \mathbb{R}^d} |\varphi_{t,R}(r, y) - \varphi_{s,R}(r, y)|^2 dy dr.$$

By direct computation (see also [2, Section 4.3] for the 2D case and [8, Equation (4.2)] for the 1D case),

$$|\varphi_{t,R}(r, y) - \varphi_{s,R}(r, y)|^2 \lesssim (t-s)^{2/d} \mathbf{1}_{\{|y| \leq R+t\}} \leq (t-s)^{2/d} \mathbf{1}_{\{|y| \leq 2R\}}$$

from which we have

$$\|F_R(t) - F_R(s)\|_p^2 \lesssim \int_0^t \int_{\mathbb{R}^d} (t-s)^{2/d} \mathbf{1}_{\{|y| \leq 2R\}} dy dr \lesssim R^d (t-s)^{2/d}.$$

This gives us the desired tightness. \square

Combing the results from Sections 4.1, 4.2 and 4.3, we can complete the proof of Theorem 1.2.

5. PROOF OF THEOREM 1.3

{BECaPd} of 12

5.1. Moment estimates for Picard approximations. We define $u_0(t, x) = 1$ and for $n \geq 0$,

$$u_{n+1}(t, x) = 1 + \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x - y) \sigma(u_n(s, y)) W(ds, dy).$$

It is a classic result that $u_n(t, x)$ converges in $L^p(\Omega)$ to $u(t, x)$ uniformly in $x \in \mathbb{R}^d$ for any $p \geq 2$; see *e.g.* [7, Theorem 4.3]. If $\sigma(1) = 0$, we will end up in the trivial case where $u(t, x) \equiv 1$, in view of the above iteration, which explains the imposed condition **(C3)**.

We will first derive moment estimates for $u_n(t, x)$. By BDG and Minkowski's inequalities, we can write with $n \geq 1$,

$$\begin{aligned} & \|u_n(t, x)\|_p^2 \\ & \leq 2 + 8p \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G_{t-s}(x - y) G_{t-s}(x - y') \gamma(y - y') \|\sigma(u_{n-1}(s, y)) \sigma(u_{n-1}(s, y'))\|_{p/2} \\ & \leq 2 + 8p \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G_{t-s}(x - y) G_{t-s}(x - y') \gamma(y - y') \|\sigma(u_{n-1}(s, y))\|_p^2 \end{aligned}$$

since $\|\sigma(u_{n-1}(s, y)) \sigma(u_{n-1}(s, y'))\|_{p/2} \leq \frac{1}{2} (\|\sigma(u_{n-1}(s, y))\|_p^2 + \|\sigma(u_{n-1}(s, y'))\|_p^2)$;

$$\leq 2 + 8p \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G_{t-s}(x - y) G_{t-s}(x - y') \gamma(y - y') \left(2\sigma(0)^2 + 2L^2 \|u_{n-1}(s, y)\|_p^2 \right).$$

Then, it follows from the estimate (2.5) that

$$\text{\{eq11\}} \quad H_n(t) \leq c_1(t) + c_2(t) \int_0^t ds H_{n-1}(s), \quad (5.1)$$

where $H_n(t) = \sup_{x \in \mathbb{R}^d} \|u_n(t, x)\|_p^2$,

$$c_1(t) := 2 + 16p\sigma(0)^2 t \mathbf{m}_t \quad \text{and} \quad c_2(t) := 16pL^2 t \mathbf{m}_t.$$

Note that the functions $c_1(t), c_2(t)$ are nondecreasing in $t \in \mathbb{R}_+$. Therefore, by iterating the inequality (5.1) for $s \in [0, t]$ and taking into account that $H_0 \equiv 1$, yields

$$\text{\{Gronwall10\}} \quad H_n(s) \leq c_1(t) \exp(c_2(t)s), \text{ for all } s \in [0, t]. \quad (5.2)$$

Essentially we applied Gronwall's lemma here.

Now we deduce from (5.2) that

$$\|u_n(t, x)\|_p^2 \leq (2 + 16p\sigma(0)^2 t \mathbf{m}_t) \exp(16pL^2 t \mathbf{m}_t).$$

As a consequence,

$$\text{\{sigmap\}} \quad \|\sigma(u_n(t, x))\|_p \leq |\sigma(0)| + L \left(\sqrt{2} + 4\sqrt{p} |\sigma(0)| \sqrt{t \mathbf{m}_t} \right) \exp(8pL^2 t \mathbf{m}_t) =: \kappa_{p,t,L}. \quad (5.3)$$

5.2. Moment estimates for the derivative of Picard approximations. Now, let us derive moment estimates for the derivative of the Picard approximations. Our goal in this section is to establish that for $n \geq 4$,

$$\text{\{Goalsec\}} \quad \|D_{s,y} u_{n+1}(t, x)\|_p \leq C_{p,t,L,\gamma} \kappa_{p,t,L} G_{t-s}(x - y), \quad (5.4)$$

where the constant $\kappa_{p,t,L}$ is defined in (5.3) and the constant $C_{p,t,L,\gamma}$ is given by (5.16) in 1D case and by (5.30) in 2D case.

Proof of (5.4). It is known that for each $n \geq 0$, $u_n(t, x) \in \mathbb{D}^{1,p}$ with

$$D_{s,y}u_{n+1}(t, x) = G_{t-s}(x - y)\sigma(u_n(s, y)) + \int_s^t \int_{\mathbb{R}^d} G_{t-r}(x - z)\Sigma_{r,z}^{(n)} D_{s,y}u_n(r, z)W(dr, dz),$$

where $\{\Sigma_{s,y}^{(n)} : (s, y) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is an adapted random field that is uniformly bounded by L , for each n . Now finite iterations yield (with $r_0 = t, z_0 = x$)

$$\begin{aligned} D_{s,y}u_{n+1}(t, x) &= G_{t-s}(x - y)\sigma(u_n(s, y)) \\ &+ \int_s^t \int_{\mathbb{R}^d} G_{t-r_1}(x - z_1)\Sigma_{r_1,z_1}^{(n)} G_{r_1-s}(z_1 - y)\sigma(u_{n-1}(s, y))W(dr_1, dz_1) \\ &+ \sum_{k=2}^n \int_s^t \cdots \int_s^{r_{k-1}} \int_{\mathbb{R}^{2k}} G_{r_k-s}(z_k - y)\sigma(u_{n-k}(s, y)) \\ &\quad \times \prod_{j=1}^k G_{r_{j-1}-r_j}(z_{j-1} - z_j)\Sigma_{r_j,z_j}^{(n+1-j)}W(dr_j, dz_j) =: \sum_{k=0}^n T_k^{(n)}, \end{aligned} \quad (5.5) \quad \{\text{finiteit}\}$$

where $T_k^{(n)}$ denotes the k th item. For example, $T_0^{(n)} = G_{t-s}(x - y)\sigma(u_n(s, y))$ and

$$T_1^{(n)} = \int_s^t \int_{\mathbb{R}^d} G_{t-r_1}(x - z_1)\Sigma_{r_1,z_1}^{(n)} G_{r_1-s}(z_1 - y)\sigma(u_{n-1}(s, y))W(dr_1, dz_1).$$

We are going to estimate $\|T_k^{(n)}\|_p$ for each $k = 0, \dots, n$.

Case $k = 0$: It is clear that

$$\|T_0^{(n)}\|_p \leq \kappa_{p,t,L} G_{t-s}(x - y), \quad (5.6) \quad \{\mathbf{k}=0\}$$

where $\kappa_{p,t,L}$ is the constant defined in (5.3).

Case $k = 1$: Applying BDG and Minkowski's inequalities, we can write

$$\begin{aligned} \|T_1^{(n)}\|_p^2 &\leq 4p \left\| \int_s^t dr_1 \int_{\mathbb{R}^{2d}} dz_1 dz'_1 G_{t-r_1}(x - z_1) G_{t-r_1}(x - z'_1) G_{r_1-s}(z_1 - y) G_{r_1-s}(z'_1 - y) \right. \\ &\quad \left. \times \gamma(z_1 - z'_1)\Sigma_{r_1,z_1}^{(n)} \Sigma_{r'_1,z'_1}^{(n)} \sigma^2(u_{n-1}(s, y)) \right\|_{p/2} \leq 4pL^2 \kappa_{p,t,L}^2 \mathbf{K}_{s,t}(x, y), \end{aligned} \quad (5.7) \quad \{\mathbf{k}=1\}$$

where

$$\mathbf{K}_{s,t}(x, y) = \int_s^t dr \int_{\mathbb{R}^d} g_r(z)(g_r * \gamma)(z)dz \quad (5.8) \quad \{\text{eq:1w}\}$$

with the notation $g_r(z) = G_{t-r}(x - z)G_{r-s}(z - y)$.

Case $2 \leq k \leq n$: We can write

$$T_k^{(n)} = \int_s^t \int_{\mathbb{R}^d} G_{t-r_1}(x - z_1)\Sigma_{r_1,z_1}^{(n)} N_{r_1,z_1} W(dr_1, dz_1)$$

with

$$\begin{aligned} N_{r_1,z_1} &= \int_{s < r_k < \cdots < r_2 < r_1} \int_{\mathbb{R}^{kd-d}} G_{r_k-s}(z_k - y)\sigma(u_{n-k}(s, y)) \\ &\quad \times \prod_{j=2}^k G_{r_{j-1}-r_j}(z_{j-1} - z_j)\Sigma_{r_j,z_j}^{(n+1-j)}W(dr_j, dz_j), \end{aligned}$$

which is clearly \mathcal{F}_{r_1} -measurable. Then, by BDG inequality, we obtain

$$\begin{aligned}
\|T_k^{(n)}\|_p^2 &\leq 4p \left\| \int_s^t dr_1 \int_{\mathbb{R}^{2d}} G_{t-r_1}(x-z_1) \Sigma_{r_1, z_1}^{(n)} N_{r_1, z_1} G_{t-r_1}(x-z'_1) \Sigma_{r_1, z'_1}^{(n)} N_{r_1, z'_1} \right. \\
&\quad \left. \times \gamma(z'_1 - z_1) dz_1 dz'_1 \right\|_{p/2} \\
&\leq 4pL^2 \left\| \int_s^t dr_1 \int_{\mathbb{R}^{2d}} G_{t-r_1}(x-z_1) N_{r_1, z_1}^2 G_{t-r_1}(x-z'_1) \gamma(z'_1 - z_1) dz_1 dz'_1 \right\|_{p/2} \\
&\leq 4pL^2 \int_s^t dr_1 \int_{\mathbb{R}^{2d}} G_{t-r_1}(x-z_1) G_{t-r_1}(x-z'_1) \|N_{r_1, z_1}\|_p^2 \gamma(z'_1 - z_1) dz_1 dz'_1,
\end{aligned} \tag{5.9}$$

where we used $|ab| \leq \frac{a^2+b^2}{2}$ in the second inequality and we applied Minkowski's inequality in the last step.

Now we can iterate the above process to obtain

$$\begin{aligned}
\|T_k^{(n)}\|_p^2 &\leq (4pL^2)^{k-1} \int_s^t dr_1 \int_s^{r_1} \cdots \int_s^{r_{k-2}} dr_{k-1} \int_{\mathbb{R}^{2dk-2d}} dz_1 \cdots dz_{k-1} dz'_1 \cdots dz'_{k-1} \\
&\quad \times \left(\prod_{j=0}^{k-2} G_{r_j-r_{j+1}}(z_j - z_{j+1}) G_{r_j-r_{j+1}}(z_j - z'_{j+1}) \gamma(z_{j+1} - z'_{j+1}) \right) \|\hat{N}_{r_{k-1}, z_{k-1}}\|_p^2,
\end{aligned}$$

where $z_0 = x, r_0 = t$ and $\hat{N}_{r_{k-1}, z_{k-1}}$ is given by

$$\hat{N}_{r_{k-1}, z_{k-1}} := \int_{[s, r_{k-1}] \times \mathbb{R}^d} W(dr_k, dz_k) \sigma(u_{n-k}(s, y)) G_{r_{k-1}-r_k}(z_{k-1} - z_k) \Sigma_{r_k, z_k}^{(n+1-k)} G_{r_k-s}(z_k - y).$$

By the same arguments that led to (5.7), we have

$$\|\hat{N}_{r_{k-1}, z_{k-1}}\|_p^2 \leq 4pL^2 \kappa_{p, t, L}^2 \mathbf{K}_{s, r_{k-1}}(z_{k-1}, y),$$

which implies

$$\begin{aligned}
\|T_k^{(n)}\|_p^2 &\leq (4pL^2)^k \kappa_{p, t, L}^2 \int_s^t dr_1 \int_s^{r_1} \cdots \int_s^{r_{k-2}} dr_{k-1} \int_{\mathbb{R}^{2dk-2d}} dz_1 \cdots dz_{k-1} dz'_1 \cdots dz'_{k-1} \\
&\quad \times \left(\prod_{j=0}^{k-2} G_{r_j-r_{j+1}}(z_j - z_{j+1}) G_{r_j-r_{j+1}}(z_j - z'_{j+1}) \gamma(z_{j+1} - z'_{j+1}) \right) \mathbf{K}_{s, r_{k-1}}(x, y). \tag{5.10}
\end{aligned}$$

To complete the estimation of the quantities $\|T_k^{(n)}\|_p$ for $k = 1, \dots, n$, we consider separately the cases $d = 1$ and $d = 2$.

Case $d = 1$: In this case, $G_{t-r}(x-z) = \frac{1}{2} \mathbf{1}_{\{|x-z| < t-r\}}$, so that, using the integrability of γ and (2.6) yields

$$\begin{aligned}
\mathbf{K}_{s, t}(x, y) &\leq \frac{1}{4} \mathbf{1}_{\{|x-y| < t-s\}} \|\gamma\|_{L^1(\mathbb{R})} \int_s^t dr \int_{\mathbb{R}} dz G_{t-r}(x-z) G_{r-s}(z-y) \\
&\leq \frac{1}{8} \mathbf{1}_{\{|x-y| < t-s\}} \|\gamma\|_{L^1(\mathbb{R})} (t-s)^2 G_{t-s}(x-y) \\
&\leq \frac{t^2 \|\gamma\|_{L^1(\mathbb{R})}}{8} G_{t-s}(x-y).
\end{aligned} \tag{5.11}$$

Plugging this bound into (5.7) yields

$$\|T_1^{(n)}\|_p \leq tL\kappa_{p,t,L}\sqrt{p\|\gamma\|_{L^1(\mathbb{R})}}G_{t-s}(x-y). \quad (5.12)$$

For $k = 2, \dots, n$, from (5.10) and (5.11), we obtain

$$\begin{aligned} \|T_k^{(n)}\|_p^2 &\leq \frac{\|\gamma\|_{L^1(\mathbb{R})}}{8}(4pL^2)^k t^2 \kappa_{p,t,L}^2 \int_s^t dr_1 \cdots \int_s^{r_{k-2}} dr_{k-1} \\ &\quad \times \int_{\mathbb{R}^{2k-2}} dz_1 \dots dz_{k-1} dz'_1 \dots dz'_{k-1} G_{r_{k-1}-s}(z_{k-1}-y) \\ &\quad \times \left(\prod_{j=0}^{k-2} G_{r_j-r_{j+1}}(z_j-z_{j+1}) G_{r_j-r_{j+1}}(z_j-z'_{j+1}) \gamma(z_{j+1}-z'_{j+1}) \right). \end{aligned} \quad (5.13)$$

In the particular case $k = 2$, we obtain

$$\begin{aligned} &\int_s^t dr \int_{\mathbb{R}^2} dz dz' G_{t-r}(x-z) G_{t-r}(x-z') \gamma(z-z') G_{r-s}(z-y) \\ &\leq \frac{1}{4} \mathbf{1}_{\{|x-y|<t-s\}} \int_s^t dr \int_{\mathbb{R}^2} dz dz' G_{t-r}(x-z') \gamma(z-z') \leq \frac{\|\gamma\|_{L^1(\mathbb{R})}}{4} G_{t-s}(x-y)(t-s)^2, \end{aligned}$$

which yields

$$\|T_2^{(n)}\|_p \leq \left(t^2 \|\gamma\|_{L^1(\mathbb{R})} p L^2 \kappa_{p,t,L}\right) G_{t-s}(x-y). \quad (5.14)$$

For $3 \leq k \leq n$, we rewrite the spatial integral in (5.13) as

$$\begin{aligned} &\int_{\mathbb{R}^{2k-2}} dz_1 \dots dz_{k-1} dz'_1 \cdots dz'_{k-1} \left(\prod_{j=0}^{k-2} \mathbf{1}_{\left\{ \begin{array}{l} |z_j-z_{j+1}| < r_j-r_{j+1} \\ |z_j-z'_{j+1}| < r_j-r_{j+1} \end{array} \right\}} \right) \gamma(z_{j+1}-z'_{j+1}) \frac{\mathbf{1}_{\{|z_{k-1}-y|<r_{k-1}-s\}}}{2^{2k-1}} \\ &\leq \frac{\mathbf{1}_{\{|x-y|<t-s\}}}{2^{2k-1}} \int_{\mathbb{R}^{2k-4}} dz_2 dz'_2 \cdots dz_{k-1} dz'_{k-1} \mathbf{1}_{\left\{ \begin{array}{l} |x-z_2| < t-r_2 \\ |x-z'_2| < t-r_2 \end{array} \right\}} \left(\prod_{j=2}^{k-2} \mathbf{1}_{\left\{ \begin{array}{l} |z_j-z_{j+1}| < r_j-r_{j+1} \\ |z_j-z'_{j+1}| < r_j-r_{j+1} \end{array} \right\}} \right) \\ &\quad \times \gamma(z_{j+1}-z'_{j+1}) \gamma(z_2-z'_2) \int_{\mathbb{R}^2} dz_1 dz'_1 \gamma(z_1-z'_1) \mathbf{1}_{\{|x-z_1|<t-r_1\}} \end{aligned}$$

Note that $\int_{\mathbb{R}^2} dz_1 dz'_1 \gamma(z_1-z'_1) \mathbf{1}_{\{|x-z_1|<t-r_1\}} \leq 2t\|\gamma\|_{L^1(\mathbb{R})}$ and then we can iterate the above process to deduce that the spatial integral in (5.13) can be bounded by

$$\frac{\mathbf{1}_{\{|x-y|<t-s\}}}{2^{2k-1}} (2t\|\gamma\|_{L^1(\mathbb{R})})^{k-1} = \frac{\mathbf{1}_{\{|x-y|<t-s\}}}{2^k} (t\|\gamma\|_{L^1(\mathbb{R})})^{k-1}.$$

Thus, from (5.13),

$$\|T_k^{(n)}\|_p^2 \leq \frac{1}{8} \kappa_{p,t,L}^2 \frac{(2pL^2 t^2 \|\gamma\|_{L^1(\mathbb{R})})^k}{(k-1)!} \mathbf{1}_{\{|x-y|<t-s\}}.$$

That is,

$$\|T_k^{(n)}\|_p \leq \kappa_{p,t,L} \frac{(2pL^2 t^2 \|\gamma\|_{L^1(\mathbb{R})})^{k/2}}{\sqrt{(k-1)!}} G_{t-s}(x-y). \quad (5.15)$$

Now combining the estimates in (5.6), (5.12), (5.14) and (5.15) yields

$$\|D_{s,y}u_{n+1}(t,x)\|_p \leq \sum_{k=0}^n \|T_k^{(n)}\|_p \leq \kappa_{p,t,L} C_{p,t,L,\gamma} G_{t-s}(x-y),$$

with

$$C_{p,t,L,\gamma} := 1 + \sum_{k=1}^{\infty} \frac{(2pL^2t^2\|\gamma\|_{L^1(\mathbb{R})})^{k/2}}{\sqrt{(k-1)!}}. \quad (5.16)$$

Remark 3. In the case $d = 1$, one can give a simplified proof of the upper bound in (1.9). Indeed, note that $G_t^2(x) = \frac{1}{2}G_t(x)$ and one can use the relation (2.6) as a replacement for the semigroup property of the heat kernel so that the upper bound in (1.9) can be proved by using the same method as in the proof of [10, equation (5.1)].

Case $d = 2$: Recall $G_{t-r}(x-z) = \frac{1}{2\pi}[(t-r)^2 - |x-z|^2]^{-1/2}\mathbf{1}_{\{|x-z|<t-r\}}$ and

$$\mathbf{K}_{s,t}(x,y) = \int_s^t dr \int_{\mathbb{R}^d} g_r(z)(g_r * \gamma)(z)dz \quad (5.17)$$

with $g_r(z) = G_{t-r}(x-z)G_{r-s}(z-y)$, see (5.8). By Hölder's inequality and Young's inequality, we obtain

$$\int_{\mathbb{R}^d} g_r(z)(g_r * \gamma)(z)dz \leq \|g_r\|_{L^{2q}(\mathbb{R}^2)} \|g_r * \gamma\|_{L^{\frac{2q}{2q-1}}(\mathbb{R}^2)} \leq \|g_r\|_{L^{2q}(\mathbb{R}^2)}^2 \|\gamma\|_{L^\ell(\mathbb{R}^2)},$$

where $q := \frac{\ell}{2\ell-1} \in (1/2, 1)$. Therefore,

$$\begin{aligned} \mathbf{K}_{s,t}(x,y) &\leq \|\gamma\|_{L^\ell(\mathbb{R}^2)} \int_s^t dr (G_{t-r}^{2q} * G_{r-s}^{2q})^{1/q}(x-y) \\ &\leq C_\ell (t-s)^{(\ell-1)/\ell} \|\gamma\|_{L^\ell(\mathbb{R}^2)} G_{t-s}^{1/\ell}(x-y), \end{aligned} \quad (5.18)$$

where the last inequality follows from Lemma 2.5. Note that here and in the rest of the paper, C_ℓ will denote a generic constant that only depends on ℓ and may vary from line to line.

Then we deduce from (5.7) that

$$\|T_1^{(n)}\|_p \leq C_\ell L \kappa_{p,t,L} \sqrt{p\|\gamma\|_{L^\ell(\mathbb{R}^2)}} (t-s)^{\frac{\ell-1}{2\ell}} G_{t-s}^{\frac{1}{2\ell}}(x-y). \quad (5.19)$$

Note that $G_{t-s}^{\frac{1}{2\ell}}(x-y) \leq (2\pi)^{1-\frac{1}{2\ell}}(t-s)^{1-\frac{1}{2\ell}}G_{t-s}(x-y)$. Therefore, from (5.19) we can write

$$\|T_1^{(n)}\|_p \leq C_\ell L \kappa_{p,t,L} \sqrt{p\|\gamma\|_{L^\ell(\mathbb{R}^2)}} t^{\frac{3\ell-2}{2\ell}} G_{t-s}(x-y). \quad (5.20)$$

Consider now the case $k \in \{2, \dots, n\}$. We have, from (5.10) and (5.18)

$$\begin{aligned} \|T_k^{(n)}\|_p^2 &\leq (4pL^2)^k \kappa_{p,t,L}^2 (t-s)^{\frac{\ell-1}{\ell}} \|\gamma\|_{L^\ell(\mathbb{R}^2)} \int_s^t dr_1 \cdots \int_s^{r_{k-2}} dr_{k-1} \\ &\quad \times \int_{\mathbb{R}^{4k-4}} dz_1 \dots dz_{k-1} dz'_1 \dots dz'_{k-1} G_{r_{k-1}-s}^{1/\ell}(z_{k-1}-y) \\ &\quad \times \left(\prod_{j=0}^{k-2} G_{r_j-r_{j+1}}(z_j-z_{j+1}) G_{r_j-r_{j+1}}(z_j-z'_{j+1}) \gamma(z_{j+1}-z'_{j+1}) \right). \end{aligned} \quad (5.21)$$

For $k = 2$, we deduce from (5.21)

$$\|T_2^{(n)}\|_p^2 \leq (4pL^2)^2 \kappa_{p,t,L}^2 (t-s)^{\frac{\ell-1}{\ell}} \|\gamma\|_{L^\ell(\mathbb{R}^2)} \widehat{\mathbf{K}}_{s,t}(x-y), \quad (5.22)$$

with

$$\widehat{\mathbf{K}}_{s,t}(x-y) := \int_s^t dr \int_{\mathbb{R}^4} dz dz' G_{r-s}^{1/\ell}(z-y) G_{t-r}(x-z) G_{t-r}(x-z') \gamma(z-z'). \quad (5.23)$$

We can write, with $h_r(z) := G_{t-r}(x-z) G_{r-s}^{1/\ell}(z-y)$ and $q = \frac{\ell}{2\ell-1}$,

$$\begin{aligned} \widehat{\mathbf{K}}_{s,t}(x-y) &= \int_s^t dr \int_{\mathbb{R}^2} dz' G_{t-r}(x-z') (\gamma * h_r)(z') \\ &\leq \int_s^t dr \|G_{t-r}\|_{L^{2q}(\mathbb{R}^2)} \|\gamma\|_{L^\ell(\mathbb{R}^2)} \|h_r\|_{L^{2q}(\mathbb{R}^2)}, \end{aligned}$$

where the last inequality follows from Hölder's inequality and Young's convolution inequality.

By direct computation, $\|G_{t-r}\|_{L^{2q}(\mathbb{R}^2)} = \left(\frac{(2\pi)^{1-2q}}{2-2q}\right)^{\frac{1}{2q}} (t-r)^{\frac{1-q}{q}}$. Then,

$$\begin{aligned} \widehat{\mathbf{K}}_{s,t}(x-y) &\leq \left(\frac{(2\pi)^{1-2q}}{2-2q}\right)^{\frac{1}{2q}} \|\gamma\|_{L^\ell(\mathbb{R}^2)} t^{\frac{1-q}{q}} \int_s^t dr \sqrt{(G_{t-r}^{2q} * G_{r-s}^{2q/\ell})^{1/q}(x-y)} \\ &\leq \left(\frac{(2\pi)^{1-2q}}{2-2q}\right)^{\frac{1}{2q}} \|\gamma\|_{L^\ell(\mathbb{R}^2)} t^{\frac{2-q}{2q}} \left(\int_s^t dr (G_{t-r}^{2q} * G_{r-s}^{2q/\ell})^{1/q}(x-y)\right)^{\frac{1}{2}}, \end{aligned}$$

where we used the Jensen's inequality for finite measure in the last estimate. Using $G_{r-s}^{2q/\ell}(z-y) \leq (2\pi)^{\frac{2\ell-2}{2\ell-1}} r^{\frac{2\ell-2}{2\ell-1}} G_{r-s}^{2q}(z-y)$ and applying Lemma 2.5, we obtain

$$\begin{aligned} \int_s^t dr (G_{t-r}^{2q} * G_{r-s}^{2q/\ell})^{1/q}(x-y) &\leq (2\pi)^{\frac{2\ell-2}{\ell}} t^{\frac{2\ell-2}{\ell}} \int_s^t dr (G_{t-r}^{2q} * G_{r-s}^{2q})^{1/q}(x-y) \\ &\leq C_\ell t^{\frac{3\ell-3}{\ell}} G_{t-s}^{1/\ell}(x-y). \end{aligned}$$

Therefore,

$$\widehat{\mathbf{K}}_{s,t}(x-y) \leq C_\ell \|\gamma\|_{L^\ell(\mathbb{R}^2)} t^{\frac{2-q}{2q}} \left(t^{\frac{3\ell-3}{\ell}} G_{t-s}^{1/\ell}(x-y)\right)^{\frac{1}{2}} = C_\ell \|\gamma\|_{L^\ell(\mathbb{R}^2)} t^{\frac{6\ell-5}{2\ell}} G_{t-s}^{\frac{1}{2\ell}}(x-y) \quad (5.24) \quad \{\text{EST:Khat}\}$$

from which, together with (5.22), we obtain

$$\begin{aligned} \|T_2^{(n)}\|_p &\leq C_\ell \left(t^{\frac{8\ell-7}{2\ell}} (4pL^2 \kappa_{p,t,L} \|\gamma\|_{L^\ell(\mathbb{R}^2)})^2 G_{t-s}^{\frac{1}{2\ell}}(x-y)\right)^{1/2} \\ &\leq C_\ell t^{\frac{3\ell-2}{\ell}} 4pL^2 \kappa_{p,t,L} \|\gamma\|_{L^\ell(\mathbb{R}^2)} G_{t-s}(x-y), \end{aligned} \quad (5.25) \quad \{\text{QQ1}\}$$

where we used $G_{t-s}^{\frac{1}{4\ell}}(x-y) \leq 2\pi t^{1-\frac{1}{4\ell}} G_{t-s}(x-y)$ to obtain the last estimate.

For $k \in \{3, \dots, n\}$, we first point out that the following integral

$$\begin{aligned} &\int_s^{r_{k-2}} dr_{k-1} \int_{\mathbb{R}^4} dz_{k-1} dz'_{k-1} G_{r_{k-1}-s}^{1/\ell}(z_{k-1}-y) G_{r_{k-2}-r_{k-1}}(z_{k-2}-z_{k-1}) \\ &\quad \times G_{r_{k-2}-r_{k-1}}(z_{k-2}-z'_{k-1}) \gamma(z_{k-1}-z'_{k-1}) \end{aligned}$$

is exactly $\widehat{\mathbf{K}}_{s,r_{k-2}}(z_{k-2}-y)$, see (5.23). This is bounded by $C_\ell \|\gamma\|_{L^\ell(\mathbb{R}^2)} t^{\frac{6\ell-5}{2\ell}} G_{r_{k-2}-s}^{\frac{1}{2\ell}}(z_{k-2}-y)$, in view of $r_{k-2} \leq t$ and (5.24).

Then, we have

$$\|T_k^{(n)}\|_p^2 \leq C_\ell (4pL^2)^k \kappa_{p,t,L}^2 t^{\frac{8\ell-7}{2\ell}} \|\gamma\|_{L^\ell(\mathbb{R}^2)}^2 \int_s^t dr_1 \cdots \int_s^{r_{k-3}} dr_{k-2} \int_{\mathbb{R}^{4k-8}} dz_1 \dots dz_{k-2} dz'_1 \dots dz'_{k-2}$$

$$\times \left(\prod_{j=0}^{k-3} G_{r_j-r_{j+1}}(z_j - z_{j+1}) G_{r_j-r_{j+1}}(z_j - z'_{j+1}) \gamma(z_{j+1} - z'_{j+1}) \right) G_{r_{k-2}-s}^{\frac{1}{2\ell}}(z_{k-2} - y).$$

Similar to the estimation of $\widehat{\mathbf{K}}_{s,t}$, we write with $\tilde{h}_r(z) := G_{t-r}(x - z) G_{r-s}^{\frac{1}{2\ell}}(z - y)$ and $q = \frac{\ell}{2\ell-1}$,

$$\begin{aligned} \tilde{\mathbf{K}}_{s,t}(x - y) &:= \int_s^t dr \int_{\mathbb{R}^2} G_{t-r}(x - z') (\gamma * \tilde{h}_r)(z') \leq \int_s^t dr \|G_{t-r}\|_{L^{2q}(\mathbb{R}^2)} \|\gamma\|_{L^\ell(\mathbb{R}^2)} \|\tilde{h}_r\|_{L^{2q}(\mathbb{R}^2)} \\ &\leq C_\ell t^{\frac{1-q}{q}} \|\gamma\|_{L^\ell(\mathbb{R}^2)} \int_s^t dr \left(\int_{\mathbb{R}^2} G_{t-r}^{2q}(x - z) G_{r-s}^{q/\ell}(z - y) dz \right)^{\frac{1}{2q}}. \end{aligned}$$

Since $2q + \frac{q}{\ell} = \frac{2\ell+1}{2\ell-1} < 3$, we can apply Lemma 2.5 to write

$$\left(\int_s^t dr \int_{\mathbb{R}^2} G_{t-r}^{2q}(x - z) G_{r-s}^{q/\ell}(z - y) dz \right)^{\frac{1}{2q}} \leq C_\ell t^{\frac{2\ell-2}{\ell}} \mathbf{1}_{\{|x-y|<t-s\}}.$$

Thus,

$$\tilde{\mathbf{K}}_{s,t}(x - y) \leq C_\ell t^{\frac{6\ell-5}{2\ell}} \|\gamma\|_{L^\ell(\mathbb{R}^2)} \mathbf{1}_{\{|x-y|<t-s\}}. \quad (5.26)$$

From this estimate, we deduce

$$\begin{aligned} \|T_3^{(n)}\|_p &\leq C_\ell \sqrt{(4pL^2)^3 \kappa_{p,t,L}^2 t^{\frac{8\ell-7}{2\ell}} \|\gamma\|_{L^\ell(\mathbb{R}^2)}^2 \tilde{\mathbf{K}}_{s,t}(x - y)} \\ &\leq C_\ell (4pL^2 \|\gamma\|_{L^\ell(\mathbb{R}^2)})^{3/2} \kappa_{p,t,L} t^{\frac{7\ell-6}{2\ell}} \mathbf{1}_{\{|x-y|<t-s\}} \\ &\leq C_\ell (4pL^2 \|\gamma\|_{L^\ell(\mathbb{R}^2)})^{3/2} \kappa_{p,t,L} t^{\frac{9\ell-6}{2\ell}} G_{t-s}(x - y), \end{aligned} \quad (5.27)$$

where we also used the fact $\mathbf{1}_{\{|x-y|<t-s\}} \leq 2\pi t G_{t-s}(x - y)$.

For $4 \leq k \leq n$, we write

$$\begin{aligned} \|T_k^{(n)}\|_p^2 &\leq C_\ell (4pL^2)^k \kappa_{p,t,L}^2 t^{\frac{8\ell-7}{2\ell}} \|\gamma\|_{L^\ell(\mathbb{R}^2)}^2 \int_s^t dr_1 \cdots \int_s^{r_{k-4}} dr_{k-3} \int_{\mathbb{R}^{4k-12}} dz_1 \cdots dz_{k-3} dz'_1 \cdots dz'_{k-3} \\ &\quad \times \left(\prod_{j=0}^{k-4} G_{r_j-r_{j+1}}(z_j - z_{j+1}) G_{r_j-r_{j+1}}(z_j - z'_{j+1}) \gamma(z_{j+1} - z'_{j+1}) \right) \tilde{\mathbf{K}}_{s,r_{k-3}}(z_{k-3} - y) \\ &\leq C_\ell (4pL^2)^k \kappa_{p,t,L}^2 t^{\frac{7\ell-6}{\ell}} \|\gamma\|_{L^\ell(\mathbb{R}^2)}^3 \mathbf{1}_{\{|x-y|<t-s\}} \int_s^t dr_1 \cdots \int_s^{r_{k-4}} dr_{k-3} \\ &\quad \times \int_{\mathbb{R}^{4k-12}} dz_1 \cdots dz_{k-3} dz'_1 \cdots dz'_{k-3} \left(\prod_{j=0}^{k-4} G_{r_j-r_{j+1}}(z_j - z_{j+1}) G_{r_j-r_{j+1}}(z_j - z'_{j+1}) \gamma(z_{j+1} - z'_{j+1}) \right) \end{aligned}$$

using (5.26). Now we can perform integration successively with respect to $dz_{k-3} dz'_{k-3}, \dots, dz_1 dz'_1$ and we get

$$\int_{\mathbb{R}^{4k-12}} dz_1 \cdots dz_{k-3} dz'_1 \cdots dz'_{k-3} \prod_{j=0}^{k-4} G_{r_j-r_{j+1}}(z_j - z_{j+1}) G_{r_j-r_{j+1}}(z_j - z'_{j+1}) \gamma(z_{j+1} - z'_{j+1}) \leq m_t^{k-3}$$

so that

$$\|T_k^{(n)}\|_p \leq C_\ell \sqrt{(4pL^2)^k \kappa_{p,t,L}^2 t^{\frac{7\ell-6}{\ell}} \|\gamma\|_{L^\ell(\mathbb{R}^2)}^3 \mathbf{1}_{\{|x-y|<t-s\}} \frac{(tm_t)^{k-3}}{(k-3)!}}$$

$$\{QQ3\} \leq C_{\ell}(4pL^2)^{k/2} \kappa_{p,t,L} t^{\frac{9\ell-6}{2\ell}} \|\gamma\|_{L^{\ell}(\mathbb{R}^2)}^{3/2} \sqrt{\frac{(t\mathbf{m}_t)^{k-3}}{(k-3)!}} G_{t-s}(x-y). \quad (5.28)$$

Combining (5.6), (5.20), (5.25), (5.27) and (5.28) yields

$$\|D_{s,y}u_{n+1}(t,x)\|_p \leq C_{p,t,L,\gamma} \kappa_{p,t,L} G_{t-s}(x-y), \quad (5.29) \quad \{\text{Conq1}\}$$

with

$$\begin{aligned} C_{p,t,L,\gamma} &:= 1 + C_{\ell} L t^{\frac{3\ell-2}{2\ell}} \sqrt{p \|\gamma\|_{L^{\ell}(\mathbb{R}^2)}} + C_{\ell} p L^2 t^{\frac{3\ell-2}{\ell}} \|\gamma\|_{L^{\ell}(\mathbb{R}^2)} \\ &\quad + C_{\ell} t^{\frac{9\ell-6}{2\ell}} (p L^2 \|\gamma\|_{L^{\ell}(\mathbb{R}^2)})^{3/2} \sum_{k=0}^{\infty} \frac{(4p L^2 t \mathbf{m}_t)^{k/2}}{\sqrt{k!}}. \end{aligned} \quad (5.30) \quad \{\text{def:CPTL2}\}$$

This concludes the proof of (5.4). \square

5.3. Proof of Theorem 1.3. We can now proceed with the proof of Theorem 1.3. We first apply Minkowski's inequality and (5.4) to obtain

$$\begin{aligned} \|Du_{n+1}(t,x)\|_{L^p(\Omega;\mathfrak{H})}^2 &\leq \int_0^t dr \int_{\mathbb{R}^{2d}} dz dy \gamma(z-y) \|D_{s,y}u_{n+1}(t,x) D_{s,z}u_{n+1}(t,x)\|_{p/2} \\ &\leq \int_0^t dr \int_{\mathbb{R}^{2d}} dz dy \gamma(z-y) \|D_{s,y}u_{n+1}(t,x)\|_p \|D_{s,z}u_{n+1}(t,x)\|_p \\ &\lesssim \int_0^t dr \int_{\mathbb{R}^{2d}} dz dy \gamma(z-y) G_{t-s}(x-y) G_{t-s}(x-z), \end{aligned}$$

which is uniformly bounded. Then standard Malliavin calculus arguments imply that up to a subsequence $Du_{n_k}(t,x)$ converges to $Du(t,x)$ with respect to the weak topology on $L^p(\Omega;\mathfrak{H})$; see e.g. [14]. Similarly, for any $q \in (1,2)$,

$$\begin{aligned} \|Du_{n+1}(t,x)\|_{L^p(\Omega;L^q(\mathbb{R}_+ \times \mathbb{R}^d))}^p &= \left\| \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} ds dy |D_{s,y}u_{n+1}(t,x)|^q \right\|_{p/q}^{p/q} \\ &\leq \left(\int_{\mathbb{R}_+ \times \mathbb{R}^d} ds dy \|D_{s,y}u_{n+1}(t,x)\|_p^q \right)^{p/q} \lesssim \left(\int_{\mathbb{R}_+ \times \mathbb{R}^d} ds dy G_{t-s}^q(x-y) \right)^{p/q} \lesssim 1. \end{aligned}$$

So $\{Du_{n_k}(t,x)\}$ has a further subsequence that converges to the same limit $Du(t,x)$ with respect to the weak topology on $L^p(\Omega;L^q(\mathbb{R}_+ \times \mathbb{R}^d))$ and as a result, for $1 < q < 2 \leq p < \infty$ and for any finite T ,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \left\| \int_{\mathbb{R}_+ \times \mathbb{R}^d} |D_{s,y}u_{t,x}|^q dy ds \right\|_{p/q} < \infty.$$

Therefore, following exactly the same lines in the proof of [2, Theorem 1.2] (step 4 therein), we can get the upper bound in (1.9). The lower bound is straightforward in light of the formula of Clark-Ocone (Lemma 2.2). \square

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REFERENCES

- [1] R. M. Balan, L. Quer-Sardanyons and J. Song (2019): Existence of density for the stochastic wave equation with space-time homogeneous Gaussian noise. *Electron. J. Probab.* 24 (2019), no. 106, 1-43. <https://doi.org/10.1214/19-EJP363>
- [2] R. Bolaños Guerrero, D. Nualart and G. Zheng (2020). Averaging 2D stochastic wave equation. (arXiv version 1) <https://arxiv.org/abs/2003.10346v1>
- [3] L. Chen, D. Khoshnevisan, D. Nualart and F. Pu: Spatial ergodicity for SPDEs via Poincaré-type inequalities. (2019) <https://arxiv.org/abs/1907.11553>
- [4] L. Chen, D. Ksoshnevisan, D. Nualart and F. Pu: Poincaré inequality, and central limit theorems for parabolic stochastic partial differential equations. (2019) <https://arxiv.org/abs/1912.01482>
- [5] L. Chen, D. Ksoshnevisan, D. Nualart and F. Pu: Central limit theorems for spatial averages of the stochastic heat equation via Malliavin-Stein method. (2020) <https://arxiv.org/pdf/2008.02408>
- [6] R. C. Dalang: Extending the Martingale Measure Stochastic Integral With Applications to Spatially Homogeneous S.P.D.E.'s. *Electron. J. Probab.* Volume 4 (1999), paper no. 6, 29 pp. <https://doi.org/10.1214/EJP.v4-43>
- [7] R. C. Dalang: The Stochastic wave equation. In: Khoshnevisan D., Rassoul-Agha F. (eds) *A Minicourse on Stochastic Partial Differential Equations*. Lecture Notes in Mathematics, vol 1962. Springer, Berlin, Heidelberg (2009) <https://doi.org/10.1007/978-3-540-85994-9>
- [8] F. Delgado-Vences, D. Nualart and G. Zheng: A Central Limit Theorem for the stochastic wave equation with fractional noise. *Ann. Inst. Henri Poincaré Probab. Stat.* 2020, Vol. 56, No. 4, 3020–3042. <https://doi.org/10.1214/20-AIHP1069>
- [9] B. Gaveau and P. Trauber: L'intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel. *J. Funct. Anal.* 46 (1982), 230-238. [https://doi.org/10.1016/0022-1236\(82\)90036-2](https://doi.org/10.1016/0022-1236(82)90036-2)
- [10] J. Huang, D. Nualart and L. Viitasaari: A central limit theorem for the stochastic heat equation. *Stochastic Processes and Their Applications*. 130, Issue 12, 7170–7184. <https://doi.org/10.1016/j.spa.2020.07.010>
- [11] J. Huang, D. Nualart, L. Viitasaari and G. Zheng: Gaussian fluctuations for the stochastic heat equation with colored noise. *Stoch PDE: Anal Comp* (2020) 8: 402–421 <https://doi.org/10.1007/s40072-019-00149-3>
- [12] D. Khoshnevisan. *Analysis of Stochastic Partial Differential Equations*. CBMS Regional Conference Series in Mathematics, 119. Published for the Conference Board of the Mathematical Sciences, Washington DC; by the American Mathematical Society, Providence, RI, 2014. viii+116 pp. MR-3222416 <https://doi.org/10.1090/cbms/119>
- [13] H. Kunita: Tightness of probability measures in $D([0, T]; C)$ and $D([0, T]; D)$. *J. Math. Soc. Japan*. Vol. 38, No. 2, 1986. <https://doi.org/10.2969/jmsj/03820309>
- [14] A. Millet and M. Sanz-Solé. A stochastic wave equation in two dimension: Smoothness of the law. *Ann. Probab.* 1999, Vol. 27, No. 2, 803-844. <https://doi.org/10.1214/aop/1022677387>
- [15] I. Nourdin and G. Peccati: Normal approximations with Malliavin calculus. From Stein's method to universality. Cambridge Tracts in Mathematics, 192. *Cambridge University Press*, Cambridge, 2012. xiv+239 pp. <https://doi.org/10.1017/CBO9781139084659>
- [16] D. Nualart: The Malliavin calculus and related topics. Second edition. Probability and its Applications (New York). *Springer-Verlag, Berlin*, 2006. xiv+382 pp. <https://doi.org/10.1007/3-540-28329-3>
- [17] D. Nualart and E. Nualart: *Introduction to Malliavin Calculus*. IMS Textbooks, Cambridge University Press, 2018. <https://doi.org/10.1017/9781139856485>
- [18] D. Nualart and E. Pardoux: Stochastic calculus with anticipating integrands. *Probab. Theory Re. Fields* 78 (1988), 535-581. <https://doi.org/10.1007/BF00353876>
- [19] D. Nualart and G. Zheng: Averaging Gaussian functionals. *Electron. J. Probab.* 25 (2020), no. 48, 1-54. <https://doi.org/10.1214/20-EJP453>
- [20] J. B. Walsh: *An Introduction to Stochastic Partial Differential Equations*. In: École d'été de probabilités de Saint-Flour, XIV—1984, 265–439. Lecture Notes in Math. 1180, Springer, Berlin, 1986. <https://doi.org/10.1007/BFb0074920>