ASYMPTOTIC PROPERTIES OF THE STOCHASTIC HEAT EQUATION IN LARGE TIMES

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Abstract. In this article, we study the asymptotic behavior of the stochastic heat equation for large times.

1. Introduction and main results

Suppose that $W = \{W(t,x), t \geq 0, x \in \mathbb{R}\}$ is a two-parameter Wiener process. That is, W is a zero-mean Gaussian process with covariance function given by

$$\mathbb{E}(W(t,x)W(s,y)) = (s \wedge t)(|x| \wedge |y|)\mathbf{1}_{\{xy>0\}}.$$

Consider the stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \varphi(W(t,x))\frac{\partial^2 W}{\partial t \partial x}, \quad x \in \mathbb{R}, t \geq 0, \tag{1.1}$$ {ecu1}

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a given Borel measurable function such that for each $t \geq 0$ and $x \in \mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}^2} p_{t-s}^2(x-y)\varphi^2(z)p_{s|y|}(z)dydzds < \infty. \tag{1.2}$$ {ecu2}

Along the paper $p_t(x)$ denotes the one-dimensional heat kernel, that is, $p_t(x) = (2\pi t)^{-1/2}e^{-x^2/2t}$. The mild solution to equation (1.1) with initial condition $u(0,x) = 0$ is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\varphi(W(s,y))W(ds,dy),$$

where the stochastic integral is well defined in view of condition (1.2).

We are interested in the asymptotic behavior as $t \to \infty$ of $u(t,x)$ for $x \in \mathbb{R}$ fixed. Notice first that in the particular case where $\varphi(x) \equiv c$, then $u(t,x)$ is a centered Gaussian random variable with variance

$$\mathbb{E}(u(t,x)^2) = c^2 \int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y)dyds = c^2 \int_0^t p_{2(t-s)}^2(0)ds = \frac{c^2}{\sqrt{\pi}}\sqrt{t}.$$

Date: May 16, 2019.
The research of the second author was supported by KAKENHI grants 24340022 and 16H03642.
D. Nualart is supported by NSF Grant DMS 1811181.

Therefore $t^{-\frac{1}{4}}u(t,x)$ has the law $N(0,c^2/\sqrt{\pi})$. In the general case, using the change of variables $s\to ts$ and $y\to\sqrt{t}y$, we can write

$$
\begin{aligned}
u(t,x) &= \int_0^1\int_{\mathbb{R}} p_{t(1-s)}(x-y)\varphi(W(ts,y))W(tds,dy) \\
&= \frac{1}{\sqrt{t}}\int_0^1\int_{\mathbb{R}} p_{1-s}(\frac{x-y}{\sqrt{t}})\varphi(W(ts,y))W(tds,dy) \\
&= \frac{1}{\sqrt{t}}\int_0^1\int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}}-y)\varphi(W(ts,\sqrt{t}y))W(tds,\sqrt{t}dy). \qquad (1.3)
\end{aligned}
$$
{eq:1}

By the scaling properties of the two-parameter Wiener process it follows that $u(t,x)$ has the same law as

{ecu3}
$$
\widetilde{u}(t,x) = t^{1/4}\int_0^1\int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}}-y)\varphi(t^{3/4}W(s,y))W(ds,dy). \qquad (1.4)
$$

The asymptotic behavior of $u(t,x)$ will depend on the properties of the function φ. We will consider three classes of functions for which different behaviors appear. We are going to use the following notion of convergence, which is stronger that the convergence in distribution (see, for instance, [6, Chapter 4]).

{d:stable}
Definition 1.1. Let $\{F_n\}$ be a sequence of random variables defined on a probability space $(\Omega,\mathcal{F},P)$. Let F be a random variable defined on some extended probability space $(\Omega',\mathcal{F}',P')$. We say that F_n *converges stably* to F, written $F_n \overset{\text{stably}}{\longrightarrow} F$, if

{e:stable}
$$
\lim_{n\to\infty} E\left[Ge^{i\lambda F_n}\right] = E'\left[Ge^{i\lambda F}\right], \qquad (1.5)
$$

for every $\lambda\in\mathbb{R}$ and every bounded $\mathcal{F}$–measurable random variable G.

The first theorem deals with the case where φ is an homogeneous type function.

{thm1b}
Theorem 1.2. *Suppose that $\varphi:\mathbb{R}\to\mathbb{R}$, is a measurable and bounded on compacts function such that $\lim_{x\to\pm\infty}|x|^{-\alpha}\varphi(x) = c_\pm$ for some constants c_+, c_- and $\alpha\geq 0$. Then, as $t\to\infty$,*

$$
\begin{aligned}
t^{-\frac{3\alpha+1}{4}}u(t,x) \overset{\text{stably}}{\longrightarrow} c_- &\int_0^1\int_{\mathbb{R}} p_{1-s}(y)|\widehat{W}(s,y)|^\alpha \mathbf{1}_{\{\widehat{W}(s,y)<0\}}\widehat{W}(ds,dy) \\
&+ c_+\int_0^1\int_{\mathbb{R}} p_{1-s}(y)|\widehat{W}(s,y)|^\alpha \mathbf{1}_{\{\widehat{W}(s,y)>0\}}\widehat{W}(ds,dy),
\end{aligned}
$$

Note that in the case that $c_+ = c_-$ and $\alpha = 0$ then the limit is Gaussian. Note that one may also consider the case $\lim_{x\to-\infty}|x|^{\pm\alpha_\pm}\varphi(x) = c_\pm$ for some constants c_+, c_-, $\alpha_+,\alpha_-\geq 0$. In this case the renormalization factor is $t^{-\frac{3(\alpha_+\vee\alpha_-)+1}{4}}$ and the limit will only have contributions from the largest $\alpha_i = \alpha_+\vee\alpha_-$.

In the second theorem we consider the case were φ satisfies some integrability properties with respect to the Lebesgue measure on $\mathbb{R}$. The limit involves a weighted local time of the two-parameter Wiener process and the proof has been inspired by the work of Nualart and Xu [8] on the central limit theorem for an additive functional of the fractional Brownian motion.

{thm2}

Theorem 1.3. *Suppose that* $\varphi \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ *for some* $p < 2$*. Then, as* $t \to \infty$*,*

$$t^{\frac{1}{8}} u(t,x) \stackrel{\text{stably}}{\longrightarrow} Z \left(\int_0^1 \int_{\mathbb{R}} p_{1-s}^2(y) \delta_0(\widehat{W}(s,y)) dy ds \right)^{\frac{1}{2}} \|\varphi\|_{L^2(\mathbb{R})},$$

where $\widehat{W}$ *is a two-parameter Wiener process independent,* Z *is a* $N(0,1)$ *random variable and* $\widehat{W}$*,* W *and* Z *are independent.*

In Theorem 1.3, $L_{0,1} := \int_0^1 \int_{\mathbb{R}} p_{1-s}^2(y) \delta_0(\widehat{W}(s,y)) dy ds$ is a weighted local time of the random field $\widehat{W}$, that can be defined (see Lemma 2.1 below) as the limit in $L^2(\Omega)$ of

$$L_{0,1}^{\varepsilon} := \int_0^1 \int_{\mathbb{R}} p_{1-s}^2(y) p_{\varepsilon}(\widehat{W}(s,y)) dy ds,$$

as ε tends to zero.

The paper is organized as follows. Section 2 contains the proofs of the above theorems and in Section 3 we discuss an extension of these results in the case where we consider also space averages on an interval $[-R, R]$ and both R and t tend to infinity.

2. Proofs

Proof of Theorem 1.2: We know that $t^{-\frac{3\alpha+1}{4}} u(t,x)$ has the same law as

$$v(x) := t^{-\frac{3\alpha}{4}} \int_0^1 \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) \varphi(t^{3/4} W(s,y)) W(ds,dy).$$

Therefore, we only have to show the stable convergence, with the limit defined as the stochastic integral with respect to an independent two-parameter Wiener process.

We divide the study of v into two parts according to the boundedness on compacts property for φ. In fact, for any compact K consider $\varphi_K(x) = \varphi(x) \mathbf{1}_{x \in K}$. Then we will prove that $v_K(x) \to 0$ in $L^2(\Omega)$ where

$$v_K(x) := t^{-\frac{3\alpha}{4}} \int_0^1 \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) \varphi_K(t^{3/4} W(s,y)) W(ds,dy).$$

In fact, as φ_K is bounded by a constant, say M, we have

$$\mathbb{E}[v_K^2(x)] \le M^2 t^{-\frac{3\alpha}{2}} \int_0^1 (1-s)^{-1/2} \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) \mathbb{P}(t^{3/4} W(s,y) \in K) ds dy.$$

The above quantity clearly converges to zero if one considers separately the cases $\alpha > 0$ and $\alpha = 0$.

Given the above result, we can assume without loss of generality that $\varphi(x) = f(x)|x|^{\alpha}$ with a bounded measurable function f such that $\lim_{x \to \pm\infty} f(x) = c_{\pm}$.

For this, we fix $t_0 > 0$ and compute the conditional characteristic function of $t^{-\frac{3\alpha+1}{4}} u(t,x)$ given $\mathcal{F}_{t_0}$, where $t > t_0$ and $\{\mathcal{F}_t, t \geq 0\}$ denotes the natural filtration of the two-parameter Wiener process used in the definition of u[1] W. For any $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}\left[e^{i\lambda t^{-\frac{3\alpha+1}{4}} u(t,x)} | \mathcal{F}_{t_0}\right] &= e^{i\lambda t^{-\frac{3\alpha+1}{4}} \int_0^{t_0} \int_{\mathbb{R}} p_{t-s}(x-y)\varphi(W(s,y))W(ds,dy)} \\ &\quad \times \mathbb{E}\left[e^{i\lambda t^{-\frac{3\alpha+1}{4}} \int_{t_0}^{t} \int_{\mathbb{R}} p_{t-s}(x-y)\varphi(W(s,y))W(ds,dy)} | \mathcal{F}_{t_0}\right] \\ &=: e^{i\lambda A_t} \times B_t.\end{aligned}$$

It is easy to show that $\lim_{t\to\infty} A_t = 0$ in $L^2(\Omega)$ as $t \to \infty$. In fact, using the rescaling properties we obtain that as $t \to \infty$

$$\begin{aligned}&t^{-\frac{3\alpha+1}{2}} \mathbb{E}\left[\int_0^{t_0} \int_{\mathbb{R}} p_{t-s}(x-y)^2 \varphi(W(s,y))^2 ds dy\right] \\ \leq &\|f\|_\infty^2 \mathbb{E}\left[\int_0^{t_0/t} \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y)^2 |W(s,y)|^{2\alpha} ds dy\right] \to 0.\end{aligned}$$

Now, we continue with the term B_t for which we will use the decomposition

$$\begin{aligned}W(s,y) &= W(s,y) - W(t_0,y) + W(t_0,y), \\ &=: \widehat{W}(s-t_0,y) + W(t_0,y).\end{aligned}$$

Then, we can write

$$B_t = \widehat{\mathbb{E}}\left[\exp\left(i\lambda t^{-\frac{3\alpha+1}{4}} \int_0^{t-t_0} \int_{\mathbb{R}} p_{t-t_0-s}(x-y)\varphi(\widehat{W}(s,y) + W(t_0,y))\widehat{W}(ds,dy)\right)\right],$$

where $\widehat{\mathbb{E}}$ denotes the mathematical expectation with respect to the two-parameter Wiener process $\widehat{W}$. By the same renormalization arguments as in (1.3) leading to (1.4), this gives

$$B_t = \widehat{\mathbb{E}}\left[\exp\left(i\lambda \left(\frac{t-t_0}{t}\right)^{\frac{3\alpha+1}{4}} \int_0^1 \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t-t_0}} - y) F(t,s,y)\widehat{W}(ds,dy)\right)\right].$$

Here

$$\begin{aligned}F(t,s,y) := f((t-s_0)^{3/4}(\widehat{W}(s,y) + (t-t_0)^{-\frac{3}{4}} W(t_0, \sqrt{t-t_0}y))) \\ \times |\widehat{W}(s,y) + (t-t_0)^{-\frac{3}{4}} W(t_0, \sqrt{t-t_0}y)|^\alpha.\end{aligned}$$

As $t \to \infty$, B_t converges almost surely to Probably more detailed should be added here

$$\widehat{\mathbb{E}}\left[\exp\left(i\lambda \int_0^1 \int_{\mathbb{R}} p_{1-s}(y) F_\alpha(\widehat{W}(s,y))\widehat{W}(ds,dy)\right)\right],$$

[1] That is, $\mathcal{F}_t$ is generated by $W(s,x), s \leq t, x \in \mathbb{R}$.

where $F_\alpha(x) = c_+\mathbf{1}_{x>0} + c_-\mathbf{1}_{x<0}$. Then the above formula is the characteristic function of $v(0)$. As a consequence, for every bounded $\mathcal{F}_{t_0}$ measurable random variable G we obtain

$$\lim_{t\to\infty} \mathbb{E}[Ge^{i\lambda t^{-\frac{3\alpha+1}{4}} u(t,x)}] = \mathbb{E}[G]\mathbb{E}[e^{i\lambda v(0)}].$$

This can be extended to any bounded random variable G measurable with respect to the two-parameter Wiener process W and this provides the desired stable convergence in the sense of Definition 1.1. □

For the proof of Theorem 1.3, we need the following lemma on the existence of the weighted local time $L_{0,r}$.

{lem1}

Lemma 2.1. *For any $r \in [0,1]$, the limit in $L^2(\Omega)$, as ε tends to zero, of*

$$L^\varepsilon_{0,r} := \int_0^r \int_{\mathbb{R}} p^2_{1-s}(y) p_\varepsilon(W(s,y)) dy ds,$$

exists and will be denoted by $L_{0,r} := \int_0^r \int_{\mathbb{R}} p^2_{1-s}(y)\delta_0(W(s,y))dyds$.

Proof. Using the inverse Fourier transform formula for a Gaussian law, we have

$$L^\varepsilon_{0,r} = \frac{1}{2\pi} \int_0^r \int_{\mathbb{R}^2} p^2_{1-s}(y) e^{i\xi W(s,y) - \frac{\varepsilon^2}{2}\xi^2} d\xi dy ds.$$

Therefore,

$$\begin{aligned}\mathbb{E}(L^\varepsilon_{0,r} L^{\varepsilon'}_{0,r}) &= (2\pi)^{-2} \int_0^r \int_0^r \int_{\mathbb{R}^4} p^2_{1-s}(y) p^2_{1-s'}(y') \\ &\quad \times \mathbb{E}\left(e^{i\xi W(s,y) - \frac{\varepsilon^2}{2}\xi'^2 - i\xi' W(s',y') - \frac{\varepsilon'^2}{2}\xi'^2} \right) d\xi dy d\xi' dy'\ ds' ds \\ &= (2\pi)^{-2} \int_0^r \int_0^r \int_{\mathbb{R}^4} p^2_{1-s}(y) p^2_{1-s'}(y') e^{-\frac{\varepsilon^2}{2}\xi^2 - \frac{\varepsilon'^2}{2}\xi'^2} \\ &\quad \times e^{-\frac{1}{2}\mathbb{E}(|\xi W(s,y) - \xi' W(s',y')|^2)} d\xi dy d\xi' dy'\ ds' ds.\end{aligned}$$

As ε and ε' tend to zero we obtain the limit

$$\int_0^1 \int_0^1 \int_{\mathbb{R}^2} p^2_{1-s}(y) p^2_{1-s'}(y') f(s,y,s',y') dy dy'\ ds' ds,$$

where $f(s,y,s',y')$ is the density at $(0,0)$ of the random vector $(W(s,y), W(s',y'))$, which is bounded by

$$(2\pi)^{-1} \left(ss'|y||y'| - (s\wedge s')^2 (|y| \wedge |y'|)^2 \mathbf{1}_{\{yy'>0\}} \right)^{-\frac{1}{2}}.$$

Then, it is easy to check I do not see as so easy maybe a hint should be added here. In particular for the case $yy' > 0$ that

$$\int_0^1 \int_0^1 \int_{\mathbb{R}^2} p^2_{1-s}(y) p^2_{1-s'}(y') f(s,y,s',y') dy dy'\ ds' ds < \infty,$$

and this allows us to conclude the proof of the lemma. □

Proof of Theorem 1.3: Consider the random variable $\widetilde{u}(t,x)$ defined in (1.4). We can put $\widetilde{u}(t,x) = t^{-\frac{3}{8}} M_t(1,x)$, where for $r \in [0,1]$,

$$M_t(r,x) := t^{\frac{3}{8}} \int_0^r \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y)\varphi(t^{3/4}W(s,y))W(ds,dy).$$

Then $\{M_t(\cdot,x), t \geq 0\}$ is a family of continuous martingales in the time interval $[0,1]$. We will find the limit as $t \to \infty$ of the quadratic variation of these martingales. We have

$$\langle M_t(\cdot,x)\rangle_r = t^{\frac{3}{4}} \int_0^r \int_{\mathbb{R}} p_{1-s}^2(\frac{x}{\sqrt{t}} - y)\varphi^2(t^{\frac{3}{4}}W(s,y))dsdy.$$

The proof of the theorem will be done in several steps.

Step 1. In this step we prove that $\langle M_t(\cdot,x)\rangle_r$ converges in $L^1(\Omega)$ to the weighted local time $L_{0,r}$. First, we claim that

{ecu6} $$\lim_{t\to\infty} \mathbb{E}\left(\left|\langle M_t(\cdot,x)\rangle_r - t^{\frac{3}{4}} \int_0^r \int_{\mathbb{R}} p_{1-s}^2(y)\varphi^2(t^{\frac{3}{4}}W(s,y))dsdy\right|\right) = 0. \tag{2.1}$$

This follows from the fact that

$$\mathbb{E}\left(t^{\frac{3}{4}}\varphi^2(t^{\frac{3}{4}}W(s,y))\right) \leq \|\varphi\|_2^2(2\pi s|y|)^{-\frac{1}{2}}$$

and

$$\lim_{t\to\infty} \int_0^r \int_{\mathbb{R}} \left|p_{1-s}^2(\frac{x}{\sqrt{t}} - y) - p_{1-s}^2(y)\right| \frac{1}{\sqrt{s|y|}}dyds = 0.$$

On the other hand, for any fixed t, we have

{ecu5} $$\lim_{\varepsilon\to 0} J_{\varepsilon,t} = 0, \tag{2.2}$$

where

$$\begin{aligned} J_{\varepsilon,t} = \mathbb{E}\Big(\Big|&t^{\frac{3}{4}} \int_0^r \int_{\mathbb{R}} p_{1-s}^2(y)\varphi^2(t^{\frac{3}{4}}W(s,y))dsdy \\ &- \int_0^r \int_{\mathbb{R}} p_{1-s}^2(y) \int_{\mathbb{R}} \varphi^2(\xi)p_{\varepsilon t^{-\frac{3}{4}}}(W(s,y) - t^{-\frac{3}{4}}\xi)d\xi dsdy\Big|\Big). \end{aligned}$$

Notice that

$$\begin{aligned} &\int_0^r \int_{\mathbb{R}} p_{1-s}^2(y) \int_{\mathbb{R}} \varphi^2(\xi)p_{\varepsilon t^{-\frac{3}{4}}}(W(s,y) - t^{-\frac{3}{4}}\xi)d\xi dsdy \\ &\quad = \int_0^r \int_{\mathbb{R}} p_{1-s}^2(y) \int_{\mathbb{R}} \varphi^2(\xi)p_{\varepsilon t^{-\frac{3}{4}}}(W(s,y) - t^{-\frac{3}{4}}\xi)d\xi dsdy \\ &\quad = t^{\frac{3}{4}} \int_0^r \int_{\mathbb{R}} p_{1-s}^2(y) \int_{\mathbb{R}} \varphi^2(\xi)p_{\varepsilon}(t^{\frac{3}{4}}W(s,y) - \xi)d\xi dsdy \\ &\quad = t^{\frac{3}{4}} \int_0^r \int_{\mathbb{R}} p_{1-s}^2(y)(\varphi^2 * p_{\varepsilon})(t^{\frac{3}{4}}W(s,y))dsdy. \end{aligned}$$

Therefore

$$
\begin{aligned}
J_{\varepsilon,t} &= \mathbb{E}\left(\left|t^{\frac{3}{4}}\int_0^r\int_{\mathbb{R}} p^2_{1-s}(y)(\varphi^2-\varphi^2*p_\varepsilon)(t^{\frac{3}{4}}W(s,y))dsdy\right|\right)\\
&\le t^{\frac{3}{4}}\int_0^r\int_{\mathbb{R}} p^2_{1-s}(y)\mathbb{E}(|(\varphi^2-\varphi^2*p_\varepsilon)(t^{\frac{3}{4}}W(s,y))|)dsdy\\
&\le \|\varphi^2-\varphi^2*p_\varepsilon\|_{L^1(\mathbb{R})}\int_0^r\int_{\mathbb{R}} p^2_{1-s}(y)(2\pi s|y|)^{-\frac{1}{2}}dyds,
\end{aligned}
$$

which converges to zero as ε tends to zero because $\int_0^r\int_{\mathbb{R}} p^2_{1-s}(y)(2\pi s|y|)^{-\frac{1}{2}}dyds<\infty$ and $\varphi\in L^2(\mathbb{R})$.

We also claim that

$$
\lim_{t\to\infty}\sup_{\varepsilon>0} I_{\varepsilon,t}=0, \tag{2.3}
$$

{ecu4}

where

$$
\begin{aligned}
I_{\varepsilon,t}=\mathbb{E}\Big(\Big|\int_0^r\int_{\mathbb{R}^2} & p^2_{1-s}(y)\varphi^2(\xi)p_{\varepsilon t^{-\frac{3}{4}}}(W(s,y)-t^{-\frac{3}{4}}\xi)d\xi dyds\\
&-\|\varphi\|_2^2\int_0^r\int_{\mathbb{R}} p^2_{1-s}(y)p_\varepsilon(W(s,y))dyds\Big|^2\Big).
\end{aligned}
$$

To show (2.3), we write

$$
\begin{aligned}
I_{\varepsilon,t} &= (2\pi)^{-2}\mathbb{E}\left(\left|\int_0^r\int_{\mathbb{R}^2} p^2_{1-s}(y)\varphi^2(\xi)\int_{\mathbb{R}} e^{i\eta W(s,y)-\frac{\varepsilon^2t^{-\frac{3}{2}}}{2}\eta^2}(e^{i\eta t^{-\frac{3}{4}}\xi}-1)d\eta d\xi dyds\right|^2\right)\\
&= (2\pi)^{-2}\int_{[0,r]^2}\int_{\mathbb{R}^4} p^2_{1-s}(y)p^2_{1-s'}(y')\varphi^2(\xi)\varphi^2(\xi')\\
&\quad\times\int_{\mathbb{R}^2} e^{-\frac{1}{2}\mathbb{E}(|\eta W(s,y)-\eta' W(s',y')|^2)}e^{-\frac{\varepsilon^2t^{-\frac{3}{2}}}{2}(\eta^2+\eta'^2)}\\
&\quad\times(e^{i\eta t^{-\frac{3}{4}}\xi}-1)(e^{-i\eta' t^{-\frac{3}{4}}\xi'}-1)d\eta d\eta' d\xi d\xi' dydy'dsds',
\end{aligned}
$$

which leads to the estimate

$$
\begin{aligned}
\sup_{\varepsilon>0} I_{\varepsilon,t} \le (2\pi)^{-2} & \int_{[0,r]^2}\int_{\mathbb{R}^4} p^2_{1-s}(y)p^2_{1-s'}(y')\varphi^2(\xi)\varphi^2(\xi')\\
&\times\int_{\mathbb{R}^2} e^{-\frac{1}{2}\mathbb{E}(|\eta W(s,y)-\eta' W(s',y')|^2)}\left(|\eta\xi\eta'\xi'|^\beta t^{-\frac{3}{4}\beta}\wedge 4\right)d\eta d\eta' d\xi d\xi' dydy'dsds',
\end{aligned}
$$

for any $\beta\in[0,1]$. Then, by the dominated convergence theorem, the limit (2.3) follows from

$$
\int_{[0,r]^2}\int_{\mathbb{R}^4} p^2_{1-s}(y)p^2_{1-s'}(y')\int_{\mathbb{R}^2} e^{-\frac{1}{2}\mathbb{E}(|\eta W(s,y)-\eta' W(s',y')|^2)}d\eta d\eta' dydy'dsds'<\infty,
$$

which can be easily proved using the fact that $(2\pi)^{-2}\int_{\mathbb{R}^2} e^{-\frac{1}{2}\mathbb{E}(|\eta W(s,y)-\eta' W(s',y')|^2)}d\eta d\eta'$ is the density at $(0,0)$ of the random vector $(W(s,y),W(s',y'))$.

By Lemma 2.1 below, $\int_0^r \int_{\mathbb{R}} p_{1-s}^2(y) p_\varepsilon(W(s,y))dyds$ converges in $L^2(\Omega)$ as $t \to \infty$ to the weighted local time $L_{0,r}$. As a consequence, from (2.1), (2.2), (2.3) and Lemma 2.1 we deduce that $\langle M_t(\cdot,x)\rangle_r$ converges in $L^1(\Omega)$ to the weighted local time $L_{0,r} = \int_0^r \int_{\mathbb{R}} p_{1-s}^2(y)\delta_0(W(s,y))dyds$. □

Step 2. Fix an orthonormal basis $\{e_i, i \geq 1\}$ of $L^2(\mathbb{R})$ formed by bounded functions and consider the martingales

$$M^i(r) = \int_0^r \int_{\mathbb{R}} e_i(y) W(ds,dy), \quad r \in [0,1].$$

We claim that the joint quadratic variation $\langle M_t(\cdot,x), M^i\rangle_r$ converges to zero in $L^1(\Omega)$ as $t \to \infty$. Indeed,

$$\langle M_t(\cdot,x), M^i\rangle_r = t^{\frac{3}{8}} \int_0^r \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) e_i(y) \varphi(t^{\frac{3}{4}} W(s,y))dyds.$$

Then,

$$\begin{aligned}
\mathbb{E}(|\langle M_t(\cdot,x), M^i\rangle_r|) &\leq t^{\frac{3}{8}} \int_0^r \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) e_i(y) \mathbb{E}(|\varphi(t^{\frac{3}{4}} W(s,y))|)dyds \\
&\leq t^{\frac{3}{8}} \int_0^r \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) e_i(y) \int_{\mathbb{R}} |\varphi(z)| \frac{1}{\sqrt{2\pi t^{\frac{3}{2}} s|y|}} e^{-\frac{z^2}{2t^{\frac{3}{2}} s|y|}} dzdyds \\
&\leq t^{\frac{3}{8} - \frac{3}{4p}} \|e_i\|_\infty \|\varphi\|_{L^p(\mathbb{R})} \int_0^r \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y)(s|y|)^{-\frac{1}{2p}} dyds,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then, the claim follows because $\frac{3}{8} - \frac{3}{4p} < 0$ for $p \in (1,2)$ and

$$\int_0^r \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y)(s|y|)^{-\frac{1}{2p}} dyds < \infty.$$

Step 3. Given a sequence $t_n \uparrow \infty$, set $M_{0,r}^n = M_{t_n}(r,x)$ and $M_{i,r}^n = M^i(r)$ for $i \geq 1$. These martingales, after possibly enlarging the probability space, possess Dambis-Dubins-Schwarz Brownian motions β_i^n, such that

$$M_{0,r}^n = \beta_{0,\langle M_0^n\rangle_r}^n$$

and

$$M_{i,r}^n = \beta_{i,r\int_{\mathbb{R}} e_i(y)^2 dy}^n, \quad i \geq 1.$$

We have proved in Step 2 that $\sup_{r\in[0,1]} |\langle M_i^n, M_0^n\rangle_r| \to 0$ in probability as $n \to \infty$. Moreover, it is clear that for any $1 \leq i < j$, $\langle M_i^n, M_j^n\rangle_r = 0$. Then, by the asymptotic Ray-Knight theorem [9], we conclude that the Brownian motions $\beta_{i,y}^n$, $i \geq 0$, converge in law to a family of independent Brownian motions $\beta_{i,y}$, $i \geq 0$. Together with Step 1, we obtain that $M_{t_n}(r,x)$ converges weakly as $n \to \infty$ to $\beta_{0,L_{0,r}\|\varphi\|^2_{2(\mathbb{R})}}$, where the Brownian motion β_0 is independent of the stochastic integrals $\{\int_0^r e_i(y)W(ds,dy), r \in [0,1], i \geq 1\}$, that is, β_0 is independent of the white noise W on $[0,1] \times \mathbb{R}$. Thus, we have proved the convergence in law of $(\widehat{W}, t^{\frac{3}{8}}\widetilde{u}(t,x))$ to $(\widehat{W}, \beta_{0,L_{0,1}\|\varphi\|^2_{2(\mathbb{R})}})$ as $t \to \infty$, where $L_{0,1} = \int_0^1 \int_{\mathbb{R}} p^2_{1-s}(y)\delta_0(\widehat{W}(s,y))dyds$ and β_0 is independent of $\widehat{W}$. The role of $\widehat{W}$ has to be better explained It remains to show the independence of $(\beta_0, \widehat{W})$ and W. For this we use the method of characteristic functions as in the proof of Theorem 1.2.

Step 4. Fix $\lambda \in \mathbb{R}$ and $t_0 \geq 0$. We follow a similar argument as in the proof of Theorem 1.2. In fact, we can write

$$\begin{aligned}\mathbb{E}\left[e^{i\lambda t^{\frac{1}{8}}u(t,x)}\right] &= e^{i\lambda t^{\frac{1}{8}}\int_0^{t_0}\int_{\mathbb{R}} p_{t-s}(x-y)\varphi(W(s,y))W(ds,dy)} \\ &\quad \times \mathbb{E}\left[e^{i\lambda t^{\frac{1}{8}}\int_{t_0}^{t}\int_{\mathbb{R}} p_{t-s}(x-y)\varphi(W(s,y))W(ds,dy)}|\mathcal{F}_{t_0}\right] \\ &=: A_t \times B_t.\end{aligned}$$

As before, it is easy to show that $\lim_{t\to\infty} A_t = 1$ in $L^2(\Omega)$. On the other hand, with the decomposition

$$W(s,y) = W(s,y) - W(t_0,t) + W(t_0,y),$$

for the term B_t, we can write

$$B_t = \widehat{\mathbb{E}}\left[\exp\left(i\lambda t^{\frac{1}{8}}\int_0^{t-t_0}\int_{\mathbb{R}} p_{t-t_0-s}(x-y)\varphi(\widehat{W}(s,y) + W(t_0,y))\widehat{W}(ds,dy)\right)\right],$$

where $\widehat{\mathbb{E}}$ denotes the mathematical expectation with respect to the two-parameter Wiener process $\widehat{W}$. By the same arguments as before, this leads to

$$\begin{aligned}B_t = \widehat{\mathbb{E}}\Bigg[\exp\Bigg((i\lambda t^{\frac{1}{8}}(t-t_0)^{\frac{1}{4}}\int_0^1\int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t-t_0}} - y) \\ \times \varphi(\widehat{W}((t-t_0)s, \sqrt{t-t_0}y) + (t-t_0)^{-\frac{3}{4}}W(t_0, \sqrt{t-t_0}y))\widehat{W}(ds,dy)\Bigg)\Bigg].\end{aligned}$$

As $t \to \infty$, B_t converges almost surely to

$$\widehat{\mathbb{E}}\left[\exp\left(i\lambda\beta_{0,L_{0,1}\|\varphi\|^2_{2(\mathbb{R})}}\right)\right].$$

As a consequence, for every bounded and $\mathcal{F}_{t_0}$ measurable random variable G we obtain

$$\lim_{t\to\infty}\mathbb{E}[G\exp\left(i\lambda t^{\frac{1}{8}}u(t,x)\right)] = \mathbb{E}[G]\mathbb{E}\left[\exp\left(i\lambda\beta_{0,L_{0,1}\|\varphi\|^2_{2(\mathbb{R})}}\right)\right].$$

This completes the proof. □

3. Large times and space averages

The asymptotic behavior of the spacial averages $\int_{-R}^{R} u(t,x)dx$ as $R\to\infty$ has been recently studied in the references [4, 5, 3]. In these papers $u(t,x)$ is the solution to a stochastic partial differential equation with initial condition $u(0,x)=1$ and a Lipschitz nonlinearity $\sigma(u)$. The solution process is stationary in $x\in\mathbb{R}$ and the limit is Gaussian with a proper normalization. In the case considered here, the lack of stationarity creates different limit behaviors. In order to have a more complete picture of the problem, we will consider the case where both R and t tend to infnity.

Set

$$u_R(t)=\int_{-R}^{R}\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\varphi(W(s,y))W(ds,dy)dx.$$

As before, $u_R(t)$ has the same law as

$$\widetilde{u}_R(t)=t^{\frac14}\int_{-R}^{R}\int_0^1\int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}}-y)\varphi(t^{\frac34}W(s,y))W(ds,dy)dx.$$

Consider first the case where φ is an homogeneous function.

{thm1a}

Theorem 3.1. *Suppose that $\varphi(x)=|x|^\alpha$ for some $\alpha>0$. Suppose that $t_R\to\infty$ as $R\to\infty$. Then, with $\widehat{W}$ is a two-parameter Wiener process independent of W, the following stable convergences hold true:*

(i) *If $\frac{R}{\sqrt{t_R}}\to c$, with $c\in(0,\infty)$,*

$$t_R^{-\frac34(\alpha+1)}u(t_R)\overset{\text{stably}}{\longrightarrow}\int_{-c}^{c}\int_0^1\int_{\mathbb{R}} p_{1-s}(x-y)|\widehat{W}(s,y))|^\alpha\widehat{W}(ds,dy)dx.$$

(ii) *If $\frac{R}{\sqrt{t_R}}\to 0$,*

$$R^{-1}t_R^{-\frac{3\alpha+1}{4}}u(t_R)\overset{\text{stably}}{\longrightarrow}2\int_0^1\int_{\mathbb{R}} p_{1-s}(y)|\widehat{W}(s,y))|^\alpha\widehat{W}(ds,dy)dx.$$

(iii) *If $\frac{R}{\sqrt{t_R}}\to\infty$,*

$$R^{-\frac{\alpha+1}{2}}t_R^{-\frac{\alpha+1}{2}}u(t_R)\overset{\text{stably}}{\longrightarrow}\int_{-1}^{1}\int_0^1|\widehat{W}(s,y)|^\alpha\widehat{W}(ds,dy).$$

Proof. We have, with the change of variable $\frac{x}{\sqrt{t_R}}\to x$,

$$\widetilde{u}_R(t_R)=t_R^{\frac{3\alpha+1}{4}+\frac12}\int_{-R/\sqrt{t_R}}^{R/\sqrt{t_R}}\int_0^1\int_{\mathbb{R}} p_{1-s}(x-y)|W(s,y)|^\alpha W(ds,dy)dx,$$

and (i) follows by letting $R\to\infty$. If $\frac{R}{\sqrt{t_R}}\to 0$, with the change of variable $x\to Rx$, we can write

{eq1}

$$\widetilde{u}_R(t_R)=Rt_R^{\frac{3\alpha+1}{4}}\int_{-1}^{1}\int_0^1\int_{\mathbb{R}} p_{1-s}(x\frac{R}{\sqrt{t_R}}-y)|W(s,y)|^\alpha W(ds,dy)dx, \tag{3.1}$$

which implies (ii). The proof of (iii) is more involved. Making the change of variable $y \to y\frac{R}{\sqrt{t_R}}$ in (3.1) yields

$$\begin{aligned}\widetilde{u}_R(t_R) &= R^{\frac{\alpha+3}{2}} t_R^{\frac{\alpha}{2}} \int_{-1}^{1}\int_0^1\int_{\mathbb{R}} p_{1-s}\Big(\frac{R}{\sqrt{t_R}}(x-y)\Big)|W(s,y)|^\alpha W(ds,dy)dx\\ &= R^{\frac{\alpha+1}{2}} t_R^{\frac{\alpha+1}{2}} \int_{-1}^{1}\int_0^1\int_{\mathbb{R}} p_{\frac{t_R}{R^2}(1-s)}(x-y)|W(s,y)|^\alpha W(ds,dy)dx.\end{aligned}$$

Finally the stochastic integral

$$\int_{-1}^{1}\int_0^1\int_{\mathbb{R}} p_{\frac{t_R}{R^2}(1-s)}(x-y)|W(s,y)|^\alpha W(ds,dy)dx$$

converges in $L^2(\Omega)$ as $R\to\infty$ to

$$\int_{-1}^{1}\int_0^1 |W(s,y)|^\alpha W(ds,dy).$$

The stable character of the convergence can be proved by the same arguments, based on the conditional characteristic function, as in the proof of Theorem 1.2. □

For a function which satisfies integrability conditions with respect to the Lebesgue measure, we have.

{thm11}

Theorem 3.2. *Suppose that $\varphi(x)\in L^2(\mathbb{R})$. Suppose that $t_R\to\infty$ as $R\to\infty$. Then, with Z a $N(0,1)$ random variable and $\widehat{W}$ an independent two-parameter Wiener process such that $(Z,\widehat{W})$ are independent of W, the following stable convergences hold true:*

(i) *If $\frac{R}{\sqrt{t_R}}\to c$, with $c\in(0,\infty)$,*

$$t_R^{-\frac{3}{8}}u(t_R) \overset{\text{stably}}{\longrightarrow} Z\left(\|\varphi\|^2_{L^2(\mathbb{R})}\int_0^1\int_{\mathbb{R}}\left(\int_{-c}^{c}p_{1-s}(x-y)dx\right)^2\delta_0(W(s,y))dyds\right)^{\frac12}.$$

(ii) *If $\frac{R}{\sqrt{t_R}}\to 0$,*

$$R^{-1}t_R^{\frac12}u(t_R) \overset{\text{stably}}{\longrightarrow} Z\left(2\|\varphi\|^2_{L^2(\mathbb{R})}\int_0^1\int_{\mathbb{R}} p^2_{1-s}(y)\delta_0(W(s,y))dyds\right)^{\frac12}.$$

(iii) *If $\frac{R}{\sqrt{t_R}}\to \infty$,*

$$R^{-\frac12}t_R^{\frac14}u(t_R) \overset{\text{stably}}{\longrightarrow} Z\left(2\|\varphi\|^2_{L^2(\mathbb{R})}\int_0^1\int_{-1}^{1} p^2_{1-s}(y)\delta_0(W(s,y))dyds\right)^{\frac12}.$$

Proof. Let us prove first the case (i). We have, with the change of variable $\frac{x}{\sqrt{t_R}}\to x$,

$$\widetilde{u}_R(t_R) = t_R^{\frac34}\int_{-R/\sqrt{t_R}}^{R/\sqrt{t_R}}\int_0^1\int_{\mathbb{R}} p_{1-s}(x-y)\varphi(t_R^{\frac34}W(s,y))W(ds,dy)dx.$$

Consider the family of martingales

$$M_R(\cdot,x)=t_R^{\frac{3}{8}}\int_{-R/\sqrt{t_R}}^{R/\sqrt{t_R}}\int_0^{\cdot}\int_{\mathbb{R}}p_{1-s}(x-y)\varphi(t_R^{\frac{3}{4}}W(s,y))W(ds,dy)dx,$$

$r\in[0,1]$. We can write

$$\langle M_R(\cdot,x)\rangle_r=t_R^{\frac{3}{4}}\int_{[-\frac{R}{\sqrt{t_R}},\frac{R}{\sqrt{t_R}}]^2}\int_0^r\int_{\mathbb{R}}p_{1-s}(x-y)p_{1-s}(x'-y)\varphi^2(t_R^{\frac{3}{4}}W(s,y))dydsdxdx'.$$

Then, as in the proof of Theorem 1.3, we can show that $\langle M_R(\cdot,x)\rangle_r$ converges in $L^1(\Omega)$ as $R\to\infty$ to the weighted local time

$$\|\varphi\|^2_{L^2(\mathbb{R})}\int_0^r\int_{\mathbb{R}}\left(\int_{-c}^{c}p_{1-s}(x-y)dx\right)^2\delta_0(W(s,y))dyds.$$

This completes the proof of (i).

If $\frac{R}{\sqrt{t_R}}\to 0$, with the change of variable $x\to Rx$, we can write

{eq2}

$$\widetilde{u}_R(t_R)=Rt_R^{\frac{1}{4}}\int_{-1}^{1}\int_0^1\int_{\mathbb{R}}p_{1-s}(x\frac{R}{\sqrt{t_R}}-y)\varphi(t_R^{\frac{3}{4}}W(s,y))W(ds,dy)dx. \tag{3.2}$$

As before, the stochastic integral

$$t_R^{\frac{3}{4}}\int_{-1}^{1}\int_0^1\int_{\mathbb{R}}p_{1-s}(x\frac{R}{\sqrt{t_R}}-y)\varphi(t_R^{\frac{3}{4}}W(s,y))W(ds,dy)dx$$

converges in law to

$$Z\left(2\|\varphi\|^2_{L^2(\mathbb{R})}\int_0^1\int_{\mathbb{R}}p^2_{1-s}(y)\delta_0(W(s,y))dyds\right)^{\frac{1}{2}},$$

which implies (ii). To show (iii), we make the change of variable $y\to y\frac{R}{\sqrt{t_R}}$ in (3.2) to get

$$\begin{aligned}\widetilde{u}_R(t_R)&=R^{\frac{3}{2}}\int_{-1}^{1}\int_0^1\int_{\mathbb{R}}p_{1-s}(\frac{R}{\sqrt{t_R}}(x-y))\varphi(t_R^{\frac{3}{4}}W(s,y))W(ds,dy)dx\\&=R^{\frac{1}{2}}t_R^{\frac{1}{2}}\int_{-1}^{1}\int_0^1\int_{\mathbb{R}}p_{\frac{t_R}{R^2}(1-s)}(x-y)\varphi(t_R^{\frac{3}{4}}W(s,y))W(ds,dy)dx.\end{aligned}$$

Finally the stochastic integral

$$t_R^{\frac{3}{4}}\int_{-1}^{1}\int_0^1\int_{\mathbb{R}}p_{\frac{t_R}{R^2}(1-s)}(x-y)\varphi(t_R^{\frac{3}{4}}W(s,y))W(ds,dy)dx$$

converges in law as $R\to\infty$ to

$$Z\left(2\|\varphi\|^2_{L^2(\mathbb{R})}\int_0^1\int_{-1}^{1}p^2_{1-s}(y)\delta_0(W(s,y))dyds\right)^{\frac{1}{2}}.$$

The stable character of the convergence can be proved by the same arguments, based on the conditional characteristic function, as in the proof of Theorem 1.3. □

4. Case of a nonlinear coefficient σ

In this section we discuss the case of a nonlinear stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\frac{\partial^2 W}{\partial t \partial x}, \quad x \in \mathbb{R}, t \geq 0, \tag{4.1}$$ {ecu7}

with initial condition $u(0,z) = 1$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. The mild solution to equation (4.1) is given by

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy).$$

We are interested in the asymptotic behavior of $u(t,x)$ as t tends to infinity. As before we consider different cases:

Case 1. Suppose that $\sigma(u) = u$. In this case, the solution has a Wiener chaos expansion given by

$$\begin{aligned} u(t,x) &= 1 + \sum_{n\geq 1} \int_{\mathbb{R}^n} \int_{\Delta_n(t)} \prod_{i=0}^{n-1} p_{s_i - s_{i+1}}(x_i - x_{i+1}) \; W(ds_1, dx_1) \cdots W(ds_n, dx_n) \\ &=: 1 + \sum_{n\geq 1} I_n(f_{t,x,n}), \end{aligned}$$

with $f_{t,x,n}(s_1,\dots,s_n,x_1,\dots,x_n) = \mathbf{1}_{\Delta_n(t)}(s_1,\dots,s_n)\prod_{i=0}^{n-1} p_{s_i-s_{i+1}}(x_i - x_{i+1})$. Here I_n denotes the multiple stochastic integral of order n with respect to the noise W. If we consider the projection of $u(t,x)$ on a fixed Wiener chaos, we can write with the change of variables $s_i \to ts_i$ and $x_i \to \sqrt{t}y_i$,

$$\begin{aligned} I_n(f_{t,x,n}) &= \int_{\Delta_n(1)} \int_{\mathbb{R}^n} p_{t(1-s_1)}(\frac{x}{\sqrt{t}} - x_1) \\ &\quad \times \prod_{i=1}^{n-1} p_{t(s_i - s_{i+1})}(x_i - x_{i+1}) W(tds_1, \sqrt{t}dx_1) \cdots W(tds_n, \sqrt{t}dx_n) \\ &= t^{-\frac{n}{2}} \int_{\Delta_n(1)} \int_{\mathbb{R}^n} p_{1-s_1}(\frac{x}{\sqrt{t}} - x_1) \\ &\quad \times \prod_{i=1}^{n-1} p_{s_i - s_{i+1}}(x_i - x_{i+1}) W(tds_1, \sqrt{t}dx_1) \cdots W(tds_n, \sqrt{t}dx_n). \end{aligned}$$

By the scaling properties of the two-parameter Wiener process it follows that $I_n(f_{t,x,n})$ has the same law as

$$\widetilde{I}_n(f_{t,x,n}) := t^{\frac{3n}{4}} \int_{\Delta_n(1)} \int_{\mathbb{R}^n} p_{1-s_1}(\frac{x}{\sqrt{t}} - x_1) \times \prod_{i=1}^{n-1} p_{s_i - s_{i+1}}(x_i - x_{i+1}) W(ds_1, dx_1) \cdots W(ds_n, dx_n).$$

As a consequence, $t^{-\frac{3}{4}n} I_n(f_{t,x,n})$ converges stably to

$$\int_{\Delta_n(1)} \int_{\mathbb{R}^n} \prod_{i=0}^{n-1} p_{s_i - s_{i+1}}(x_i - x_{i+1}) \widehat{W}(ds_1, dx_1) \cdots \widehat{W}(ds_n, dx_n),$$

where $\widehat{W}$ is a two-parameter Wiener process independent of W and with the convention $s_0 = 1$ and $x_0 = 0$.

Notice that the rate of convergence depends on the order of the Wiener chaos. This is consistent with the asymptotic behavior of $\log u(t,x)$, when $u(0,x) = \delta_0(x)$, obtained by Amir, Corwin and Quastel in [1].

Case 2. When σ is a Lipschitz function that belongs to $L^2(\mathbb{R}, \gamma)$, the problem is much more involved. We can write

$$u(t,x) = 1 + \frac{1}{\sqrt{t}} \int_0^1 \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) \sigma(u(ts, \sqrt{t}y)) W(tds, \sqrt{t}dy).$$

Furthermore,

$$\begin{aligned} u(ts, \sqrt{t}y) &= 1 + \int_0^{ts} \int_{\mathbb{R}} p_{ts-r}(\sqrt{y} - z) \sigma(u(r,z)) W(dr, dz) \\ &= 1 + \frac{1}{\sqrt{t}} \int_0^s \int_{\mathbb{R}} p_{s-r}(y-z) \sigma(u(tr, \sqrt{t}z)) W(tdr, \sqrt{t}dz). \end{aligned}$$

By the scaling properties of the two-parameter Wiener process, as a function of $\widehat{W}$, $u(ts, \sqrt{t}y)$ has the same law as

$$v^t(s,y) = 1 + t^{\frac{1}{4}} \int_0^s \int_{\mathbb{R}} p_{s-r}(y-z) \sigma(v^t(r,z))) \widehat{W}(dr, dz).$$

Therefore, $u(t,x)$ has the same law as

$$\widetilde{u}(t,x) = 1 + t^{\frac{1}{4}} \int_0^1 \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) \sigma(v^t(s,y)) \widehat{W}(ds, dy).$$

Then,

$$t^{-\frac{1}{6}} \widetilde{u}(t,x) = t^{-\frac{1}{6}} + t^{\frac{1}{12}} \int_0^1 \int_{\mathbb{R}} p_{1-s}(\frac{x}{\sqrt{t}} - y) \sigma(v^t(s,y)) \widehat{W}(ds, dy).$$

The quadratic variation of the martingale part of the above stochastic integral is

$$t^{\frac{1}{6}}\int_0^1\int_{\mathbb{R}} p_{1-s}^2(\frac{x}{\sqrt{t}}-y)\sigma^2(1+t^{\frac{1}{6}}Z^t(s,y))dsdy$$
$$=\int_0^1\int_{\mathbb{R}} p_{1-s}^2(\frac{x}{\sqrt{t}}-y)\int_{\mathbb{R}}\sigma^2(\xi)\delta_0(Z^t(s,y)+t^{-\frac{1}{6}}-\xi t^{-\frac{1}{6}})d\xi dsdy$$

where $Z^t(s,y)$ satisfies

$$Z^t(s,y)=t^{-\frac{1}{6}}+t^{\frac{1}{12}}\int_0^s\int_{\mathbb{R}} p_{s-r}(y-z)\sigma(1+t^{\frac{1}{6}}Z^t(r,z)))\widehat{W}(dr,dz).$$

We see that $t^{-\frac{1}{6}}$ seems to be the right normalization and the limit would satisfy an equationvinvolving a weighted local time of the solution.

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