Journal of Mathematical Physics

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Cite as: J. Math. Phys. **62**, 071507 (2021); https://doi.org/10.1063/5.0042321
Submitted: 29 December 2020 • Accepted: 02 July 2021 • Published Online: 29 July 2021

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Journal of Mathematical Physics | ARTICLE | scitation.org/journal/jmp

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Submitted: 29 December 2020 • Accepted: 2 July 2021 •
Published Online: 29 July 2021

View Online Export Citation CrossMark

Benjamin Dodson,[1] Avraham Soffer,[2,a)] and Thomas Spencer[3]

AFFILIATIONS

[1] Department of Mathematics, Johns Hopkins University, 3400 N. Charles St., Baltimore, Maryland 21218, USA
[2] Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, New Jersey 08854, USA
[3] School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540, USA

Note: This paper is part of the Special Issue on Celebrating the work of Jean Bourgain.
[a)] **Author to whom correspondence should be addressed:** soffer@math.rutgers.edu

ABSTRACT

In this paper, we continue our study [B. Dodson, A. Soffer, and T. Spencer, J. Stat. Phys. **180**, 910 (2020)] of the nonlinear Schrödinger equation (NLS) with bounded initial data which do not vanish at infinity. Local well-posedness on $\mathbb{R}$ was proved for real analytic data. Here, we prove global well-posedness for the 1D NLS with initial data lying in L^p for any $2 < p < \infty$, provided that the initial data are sufficiently smooth. We do not use the complete integrability of the cubic NLS.

I. INTRODUCTION

In this article, we continue the study[1] of the nonlinear Schrödinger equation (NLS) on the continuum,

$$iu_t + u_{xx} = |u|^2 u, \qquad u(0,x) = u_0(x), \qquad u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}. \tag{1.1}$$

Our analysis does not depend on the complete integrability of (1.1). A solution to (1.1) has a scaling symmetry. If $u(t,x)$ is a solution to (1.1), then for any $\lambda > 0$,

$$\lambda u(\lambda^2 t, \lambda x) \tag{1.2}$$

is a solution to (1.1) with initial data $\lambda u_0(\lambda x)$. Direct computation of (1.2) implies that (1.1) is $\dot{H}^{-1/2}$-critical since for any $s \in \mathbb{R}$,

$$\|\lambda u_0(\lambda x)\|_{\dot{H}^s(\mathbb{R})} = \lambda^{s+\frac{1}{2}} \|u_0\|_{\dot{H}^s(\mathbb{R})}, \tag{1.3}$$

so when $s = -\frac{1}{2}$, the norm of the initial data is invariant under the scaling.

Equation (1.3) also implies that (1.1) is L^2-subcritical and is also L^p-subcritical for any $p > 2$. Using now standard arguments (see, for example, Refs. 2–4) (1.1) is locally well-posed for initial data lying in L^2. Combining L^2 subcriticality of (1.1) with conservation of mass,

$$M(u(t)) = \int |u(t,x)|^2 dx = \int |u(0,x)|^2 = M(u(0)), \tag{1.4}$$

gives the global well-posedness of (1.1) with initial data in L^2.

For $u_0 \in L^p(\mathbb{R})$, $p > 2$, Hölder's inequality also implies that $u_0 \in L^2$ on any compact subset of $\mathbb{R}$. Therefore, the obstacle to well-posedness for $u_0 \in L^p$, $2 < p < \infty$, is that data which are initially spread out can move together. Finite propagation speed prevents this from happening for the nonlinear wave equation; see Ref. 1. However, for the nonlinear Schrödinger equation, the velocity is controlled by the frequency, and the nonlinearity may move the solution to higher frequencies. In Ref. 1, we studied (1.1) on a lattice or for a regularized nonlinearity on the continuum, which prevented the nonlinearity from moving the solution up to high frequencies. Global well-posedness was proved using a local energy argument. Local well-posedness was also established for (1.1) with bounded real analytic data.

In this paper, we prove global well-posedness for (1.1) with initial data u_0 lying in a sufficiently regular L^p-based Sobolev space but which may have infinite L^2 norm and infinite energy (1.7).

Theorem 1. *For any $n \in \mathbb{Z}$, $n \geq 0$, (1.1) is globally well-posed for initial data u_0 satisfying*

$$\|\langle\partial_x\rangle^{2n+2} u_0\|_{L_x^{4n+2}(\mathbb{R})} < \infty. \tag{1.5}$$

The norm (1.5) is defined below.

Definition 1. For any $p \in [1, \infty]$ and any positive integer n, define the norm

$$\|\langle\partial_x\rangle^n f\|_{L^p(\mathbb{R})} = \sum_{j=0}^{n} \|\partial_x^j f\|_{L^p(\mathbb{R})}. \tag{1.6}$$

Definition 2. The well-posedness definition is in the $L_x^{2n+2}(\mathbb{R})$ space. That is, $u \in C_t^0(\mathbb{R} : L_x^{2n+4}(\mathbb{R}))$ and the solution continuously depends on the initial data.

Remark 1. For example, Theorem 1 implies that $u_0 = [\cos(x) + \cos(\sqrt{2}x)](1 + |x|^2)^{-\alpha}$ is globally well posed for any $\alpha > 0$. When $\alpha = 0$, local existence was proved in Refs. 1 and 5; however, global existence is not known. In the Proof of Theorem 2 in Dodson, Soffer, Spencer, there was a missing factor of $(t - \tau)^{-1/2}$ in the Duhamel term (3.43). This factor could potentially improve the result.

For the KdV equation, another one dimensional dispersive PDE[6] (Partial Differential Equation) proved global well-posedness for almost periodic data by exploiting the completely integrable structure of the KdV equation. See also Refs. 7 and 8 for global results for the NLS in a subset of the almost periodic setting. It may be that the recent results of Ref. 9 could be useful in this direction.

To explain the method of proof, we first note that it is done by successively increasing the p norm and regularity of the initial data. First observe that when $n = 0$, we can take $u_0 \in L^2$ only, and we do not need $\|\langle\partial_x\rangle^2 u_0\|_{L^2} < \infty$.

When $n = 1$, note that the choice for the Sobolev space is L^6 and with enough regularity. Then, although the conserved quantities of the equation are infinite, we note that $u_0^3 \in L^2$. Hence, the zeroth order iteration of the equivalent integral equation has its Duhamel term in L^2. The exploitation of the fact that the Duhamel term may live in a better space goes back at least to the ideas of Ref. 10, as well as in Ref. 11 in the context of random data. In Refs. 10 and 11, a better function space means functions with improved differentiability. Here, a better space means a Lebesgue space with a lower integrability index. For the cubic nonlinear Schrödinger equation on $\mathbb{R}$, this idea was also used in Ref. 12.

For larger n, one proves that under the linear flow, the L^p norms remain bounded if the data are sufficiently regular. The solution grows with time in L^p but only polynomially, which allows us to prove that the Picard iterations are such that the Duhamel term is in L^2 and in the local in time Strichartz norm. We can obtain a local solution by making the ansatz

$$u(t) = u^0(t) + u^1(t) + \ldots + u^{n-1}(t) + v(t), \quad \text{where} \quad u^0(t) = e^{it\partial_{xx}} u_0,$$

where $u^i(t)$ represents the ith Picard iterate and v is the remainder. The idea of using a higher order expansion and exploiting smoothing properties of higher order iterates appear in recent works (Refs. 13 and 14) on random data dispersive PDE. It is convenient to first rescale so that the initial data (1.5) are small. Then, by Picard iteration and stationary phase arguments, we prove the local well-posedness of (1.1) on $[-1, 1]$.

The next step is to observe that the equation is sub-critical in these L^p spaces, and therefore, it is possible to go from the local result to global. The fact that one can control the Duhamel nonlinear part of the solution in L^2 is a key fact that also allows us to use the conservation laws for the nonlinear terms. Indeed, (1.1) with initial data $v(1) \in H^1$ has a solution on $[1, \infty)$. We then prove that (1.1) with initial data $u(1)$ has a solution on $[1, \infty)$ by proving the global well-posedness of (1.1) with initial data $v(1)$ and treating $u(1) - v(1)$ as a perturbation. The analysis uses the conservation of mass (1.4) and energy

$$E(u(t)) = \frac{1}{2}\int |u_x(t,x)|^2 dx + \frac{1}{4}\int |u(t,x)|^4 dx$$
$$= \frac{1}{2}\int |u_x(0,x)|^2 dx + \frac{1}{4}\int |u(0,x)|^4 dx = E(u(0)). \tag{1.7}$$

One can then control higher L^p norms by the previous case. This argument has some similarity to the argument in Ref. 15 for the nonlinear wave equation with random initial data.

It should be pointed out that the decay at infinity of the initial data is crucial for the analysis. Hence, the case $p = \infty$ is left open. This is an indication that even though there is focusing that can produce large derivative and size locally, the decay of the solution at infinity allows for some dispersion.

Remark 2. Theorem 1 *is probably not sharp for any* $n > 0$.

Remark 3. Local well-posedness would hold equally well in the focusing case. While conservation of mass (1.4) would guarantee global well-posedness for both the focusing and defocusing problems in the case that u_0 *has finite mass, the fact that our proof of global well-posedness relies on the conservation of energy means that the global result only holds in the defocusing case.*

The local arguments would also work for

$$iu_t + u_{xx} = |u|^{2r} u \tag{1.8}$$

for some integer $r > 1$. *However, when* $r > 1$, *one cannot directly use the analog of (3.27) since the power of* $E(v)$ *will be larger than* 1 *in that case.*

Remark 4. The study of nonlinear Schrödinger equations of the form (1.1) has mostly focused on initial data in L^2*-based Sobolev spaces. Here, we study well-posedness questions for initial data in* L^p*-based Sobolev spaces with* $p > 2$. *See Refs. 16–18.*

II. LOCAL RESULT

We begin by proving a local version of Theorem 1.

Theorem 2. *For any* $n \in \mathbb{Z}$, $n \geq 1$, *there exists* $\epsilon(n) > 0$ *such that if*

$$\|\langle\partial_x\rangle^{2n+1} u_0\|_{L^{4n+2}(\mathbb{R})} \leq \epsilon(n), \tag{2.1}$$

then (1.1) has a local solution in $L^{4n+2}_{t,x}([-1,1]\times\mathbb{R})$ *on* $[-1,1]$.

Remark 5. Observe that the differentiablity assumption in Theorem 2 *is weaker than condition (1.5) for the global result.*

Proof. The case when $n = 0$ is already well-known, so start with $n = 1$.

Theorem 3. *There exists* $\epsilon > 0$ *such that if*

$$\|\langle\partial_x\rangle^2 u_0\|_{L^6(\mathbb{R})} < \epsilon, \tag{2.2}$$

then (1.1) has a local solution in $L^6_{t,x}([-1,1]\times\mathbb{R})$ *for* $\epsilon > 0$ *sufficiently small.*

Proof. We begin by proving an estimate on the operator $e^{it\partial_{xx}}$.

Lemma 1. For any $2 \leq p \leq \infty$,

$$\|e^{it\partial_{xx}} u_0\|_{L^p} \lesssim (1 + t^{3/2})(\|\partial_{xx} u_0\|_{L^p} + \|\partial_x u_0\|_{L^p} + \|u_0\|_{L^p}). \tag{2.3}$$

Proof. Lemma 1 is proved by computing the stationary phase kernel

$$e^{it\partial_{xx}} u_0(x) = \frac{1}{Ct^{1/2}} \int e^{-i\frac{(x-y)^2}{4t}} u_0(y) dy. \tag{2.4}$$

Let χ be a smooth, compactly supported function, $\chi(y) = 1$ for $|y| \leq 1$, and χ is supported on $|y| \leq 2$. Integrating by parts,

$$\begin{aligned} &\frac{1}{Ct^{1/2}} \int e^{-i\frac{(x-y)^2}{4t}} (1-\chi(x-y))u_0(y)dy \\ &= \frac{1}{Ct^{1/2}} \int \frac{2it}{x-y} \frac{d}{dy}\left(e^{-i\frac{(x-y)^2}{4t}}\right)(1-\chi(x-y))u_0(y)dy \\ &= Ct^{1/2} \int \frac{d}{dy}\left(\frac{1}{x-y}(1-\chi(x-y))\right) \cdot e^{-i\frac{(x-y)^2}{4t}} u_0(y)dy \\ &\quad + Ct^{1/2} \int e^{-i\frac{(x-y)^2}{4t}} \frac{1}{x-y}(1-\chi(x-y))u_0'(y)dy. \end{aligned} \tag{2.5}$$

Since $\frac{d}{dy}(\frac{1}{x-y}(1-\chi(x-y))) \in L^1(\mathbb{R})$, Young's inequality implies that for any $1 \leq p \leq \infty$,

$$\|Ct^{1/2} \int \frac{d}{dy}\left(\frac{1}{x-y}(1-\chi(x-y))\right) \cdot e^{-i\frac{(x-y)^2}{4t}} u_0(y)dy\|_{L^p} \lesssim t^{1/2}\|u_0\|_{L^p}. \tag{2.6}$$

Making another integration by parts argument shows that the second term on the right hand side of (2.5) also has a bounded L^p norm,

$$\|Ct^{1/2} \int e^{-i\frac{(x-y)^2}{4t}} \frac{1}{x-y}(1-\chi(x-y))u_0'(y)dy\|_{L^p} \lesssim t^{3/2}\|\partial_x u_0\|_{L^p} + t^{3/2}\|\partial_{xx} u_0\|_{L^p}. \tag{2.7}$$

Now, by the fundamental theorem of calculus,

$$\begin{aligned} \chi(x-y)u_0(y) &= \chi(x-y)u_0(x) + \chi(x-y)(u_0(y) - u_0(x)) \\ &= \chi(x-y)u_0(x) + \chi(x-y)\int_x^y u_0'(s)ds. \end{aligned} \tag{2.8}$$

Since $\chi(y)$ is smooth and compactly supported, $\|\chi(x-y)u_0(x)\|_{H^1} \lesssim |u_0(x)|$, and therefore, by the Sobolev embedding theorem and the fact that $e^{it\Delta}$ is a unitary operator for L^2-based Sobolev spaces,

$$\|e^{it\partial_{xx}}(\chi(y)u_0(x))\|_{L^\infty} \lesssim |u_0(x)|. \tag{2.9}$$

In particular, this implies

$$|e^{it\partial_{xx}}(\chi(y)u_0(x))|(t,x) \lesssim |u_0(x)|. \tag{2.10}$$

Finally, as in (2.5),

$$\begin{aligned} &\frac{1}{Ct^{1/2}} \int e^{-i\frac{(x-y)^2}{4t}} \chi(x-y)(u_0(y) - u_0(x))dy \\ &= \frac{1}{Ct^{1/2}} \int \frac{2it}{x-y}\frac{d}{dy}\left(e^{-i\frac{(x-y)^2}{4t}}\right)\chi(x-y)(u_0(y) - u_0(x))dy \\ &= Ct^{1/2} \int \frac{d}{dy}\left(e^{-i\frac{(x-y)^2}{4t}}\right)\chi(y)\int_0^1 u_0'(x+sy)dsdy. \end{aligned} \tag{2.11}$$

Integrating by parts in y then implies

$$\|Ct^{1/2} \int \frac{d}{dy}\left(e^{-i\frac{(x-y)^2}{4t}}\right)\chi(y)\int_0^1 u_0'(sy)dsdy\|_{L^\infty} \lesssim t^{1/2}\|\partial_x u_0\|_{L^\infty} + t^{1/2}\|\partial_{xx} u_0\|_{L^\infty}. \tag{2.12}$$

Interpolating with the well-known unitary group bound $\|e^{it\partial_{xx}}u_0\|_{L^2} = \|u_0\|_{L^2}$ proves that for any $2 \le p \le \infty$,

$$\|e^{it\partial_{xx}}u_0\|_{L^p} \lesssim (1+t^{3/2})(\|\partial_{xx}u_0\|_{L^p} + \|\partial_x u_0\|_{L^p} + \|u_0\|_{L^p}). \tag{2.13}$$

□

Theorem 3 then follows directly from (2.3) by Picard iteration. Define a set

$$X = \{v : \|v\|_{L^6_{t,x}([0,1]\times\mathbb{R})} \lesssim \epsilon^3\}, \tag{2.14}$$

and define a sequence v_n recursively, where $v_0 = 0$ and

$$v_{n+1} = -i\int_0^t e^{i(t-\tau)\partial_{xx}}|e^{i\tau\partial_{xx}}u_0 + v_n|^2(e^{i\tau\partial_{xx}}u_0 + v_n)d\tau. \tag{2.15}$$

Recall the Strichartz estimates. See Refs. 4, 19, and 20 for more information.

Theorem 4. *Let (p_1, q_1) and (p_2, q_2) be admissible pairs in one dimension such that*

$$\frac{2}{p_i} = \frac{1}{2} - \frac{1}{q_i}, \qquad 4 \le p_i \le \infty, \qquad i = 1, 2. \tag{2.16}$$

If

$$u(t,x) = u^0(t,x) - i\int_0^t e^{i(t-\tau)\partial_{xx}}F(\tau,x)d\tau, \qquad u^0(t,x) = e^{it\partial_{xx}}u_0, \qquad u : I\times\mathbb{R}\to\mathbb{C}, \tag{2.17}$$

I is an interval containing 0, then

$$\|u\|_{L_t^{p_1}L_x^{q_1}(I\times\mathbb{R})} \lesssim \|u(0)\|_{L^2} + \|F\|_{L_t^{p_2'}L_x^{q_2'}(I\times\mathbb{R})}. \tag{2.18}$$

Remark 6. p' is the Lebesgue dual of p, $\frac{1}{p'} + \frac{1}{p} = 1$.

Plugging the Strichartz estimates into (2.15), with $p_1 = q_1 = 6$ and $p_2' = 1, q_2' = 2$, we have

$$\|v_{n+1}\|_{L^6_{t,x}([-1,1]\times\mathbb{R})} \lesssim \|e^{it\partial_{xx}}u_0\|^3_{L^6_{t,x}([-1,1]\times\mathbb{R})} + \|v_n\|^3_{L^6_{t,x}([-1,1]\times\mathbb{R})} \lesssim \epsilon^3 + \|v_n\|^3_{L^6_{t,x}([-1,1]\times\mathbb{R})}. \tag{2.19}$$

Also by Strichartz estimates,

$$\|v_{n+1} - v_n\|_{L^6_{t,x}} \lesssim \epsilon^2\|v_n - v_{n-1}\|_{L^6_{t,x}} + (\|v_n\|^2_{L^6_{t,x}} + \|v_{n-1}\|^2_{L^6_{t,x}})\|v_n - v_{n-1}\|_{L^6_{t,x}}. \tag{2.20}$$

Then, by the contraction mapping principle, this proves that there is a unique $v \in L^6_{t,x}$ such that

$$v = -i\int_0^t e^{i(t-\tau)\partial_{xx}}|e^{i\tau\partial_{xx}}u_0 + v|^2(e^{i\tau\partial_{xx}}u_0 + v)d\tau. \tag{2.21}$$

This proves Theorem 3. □

Next, consider the case when $n = 2$.

Theorem 5. *There exists $\epsilon > 0$ such that if*

$$\|\langle\partial_x\rangle^5 u_0\|_{L^{10}} < \epsilon, \tag{2.22}$$

then (1.1) has a local solution on $[-1, 1]$.

Proof. The solution $u(t)$ is of the form

$$u(t) = u^0(t) + u^1(t) + v(t), \tag{2.23}$$

where

$$u^0(t) = e^{it\partial_{xx}} u_0 \tag{2.24}$$

and $u^1(t)$ is the next Picard iterate,

$$u^1(t) = \int_0^t e^{i(t-\tau)\partial_{xx}} |u^0(\tau)|^2 u^0(\tau) d\tau. \tag{2.25}$$

By Lemma 1, for $-1 \le t \le 1$,

$$\| \langle \partial_x \rangle^3 e^{it\partial_{xx}} u_0 \|_{L_x^{10}} \lesssim \epsilon, \tag{2.26}$$

and using the product rule, for $-1 \le t \le 1$,

$$\begin{aligned} \| \langle \partial_x \rangle u^1(t) \|_{L_x^{10/3}} &\lesssim \| \langle \partial_x \rangle \int_0^t e^{i(t-\tau)\partial_{xx}} |u^0(\tau)|^2 u^0(\tau) d\tau \|_{L^{10/3}} \\ &\lesssim \| \langle \partial_x \rangle^3 |u^0|^2 u^0 \|_{L_t^1 L_x^{10/3}} \lesssim \| \langle \partial_x \rangle^3 u^0 \|_{L_t^\infty L_x^{10}}^3 \lesssim \epsilon^3. \end{aligned} \tag{2.27}$$

Remark 7. Observe that by the Sobolev embedding theorem, $u^0, u^1 \in L_{t,x}^\infty$.

Then, as in Theorem 4, obtain $v(t)$ that solves

$$v(t) = \int_0^t e^{i(t-\tau)\partial_{xx}} |u|^2 u d\tau - u^1(t), \tag{2.28}$$

where u satisfies (2.23). We substitute (2.23) into (2.28). Then, since it is not too important to distinguish between u and $\bar{u}$,

$$\begin{aligned} \int_0^t e^{i(t-\tau)\partial_{xx}} |u|^2 u d\tau = &\int_0^t e^{i(t-\tau)\partial_{xx}} v^3 d\tau + 3 \int_0^t e^{i(t-\tau)\partial_{xx}} v^2 (u^0 + u^1) d\tau \\ &+ 3 \int_0^t e^{i(t-\tau)\partial_{xx}} v (u^0 + u^1)^2 d\tau + \int_0^t e^{i(t-\tau)\partial_{xx}} (u^0 + u^1)^3 d\tau. \end{aligned} \tag{2.29}$$

Then, by the Sobolev embedding theorem, (2.26), and (2.27),

$$\| u^0 \|_{L_{t,x}^\infty} + \| u^1 \|_{L_{t,x}^\infty} \lesssim \epsilon + \epsilon^3. \tag{2.30}$$

Let S^0 be the Strichartz space, $S^0([-1,1] \times \mathbb{R}) = L_t^\infty L_x^2([-1,1] \times \mathbb{R}) \cap L_t^4 L_x^\infty([-1,1] \times \mathbb{R})$. By Theorem 4, we can bound the first three terms on the right hand side of (2.29) by

$$\begin{aligned} &\| \int_0^t e^{i(t-\tau)\partial_{xx}} v^3 d\tau \|_{S^0([-1,1]\times\mathbb{R})} + \| \int_0^t e^{i(t-\tau)\partial_{xx}} v^2 (u^0 + u^1) d\tau \|_{S^0([-1,1]\times\mathbb{R})} \\ &+ \| \int_0^t e^{i(t-\tau)\partial_{xx}} v (u^0 + u^1)^2 d\tau \|_{S^0([-1,1]\times\mathbb{R})} \lesssim \| v \|_{S^0([-1,1]\times\mathbb{R})}^3 + \epsilon^2 \| v \|_{S^0([-1,1]\times\mathbb{R})}. \end{aligned} \tag{2.31}$$

Next, the last term on the right hand side of (2.29) is bounded by

$$\int_0^t e^{i(t-\tau)\partial_{xx}} |u^0 + u^1|^2 (u^0 + u^1) d\tau - u^1(t) = \int_0^t e^{i(t-\tau)\partial_{xx}} [|u^0 + u^1|^2 (u^0 + u^1) - |u^0|^2 u^0] d\tau. \tag{2.32}$$

Therefore,

$$\| \int_0^t e^{i(t-\tau)\partial_{xx}} |u^0 + u^1|^2 (u^0 + u^1) d\tau - u^1(t) \|_{S^0([-1,1]\times\mathbb{R})} \lesssim \|u^0\|_{L_t^\infty L_x^{10}}^2 \|u^1\|_{L_t^\infty L_x^{10/3}} + \|u^0\|_{L_t^\infty L_x^{10}} \|u^1\|_{L_t^\infty L_x^5}^2 + \|u^1\|_{L_t^\infty L_x^6}^3 \lesssim \epsilon^5. \tag{2.33}$$

Therefore,

$$\|v\|_{S^0([-1,1]\times\mathbb{R})} \lesssim \|v\|_{S^0([-1,1]\times\mathbb{R})}^3 + \epsilon^5, \tag{2.34}$$

which implies that $\|v\|_{S^0([-1,1]\times\mathbb{R})} \lesssim \epsilon^5$. As in the Proof of Theorem 4, we can prove Theorem 5 by a contraction mapping argument.

Remark 8. We can also take a derivative of v and make the same argument. Indeed, following (2.33),

$$\| \langle \partial_x \rangle \int_0^t e^{i(t-\tau)\partial_{xx}} |u^0 + u^1|^2 (u^0 + u^1) d\tau - u^1(t) \|_{S^0([-1,1]\times\mathbb{R})} \lesssim \|\langle \partial_x \rangle u^0\|_{L_t^\infty L_x^{10}}^2 \|\langle \partial_x \rangle u^1\|_{L_t^\infty L_x^{10/3}} + \|\langle \partial_x \rangle u^0\|_{L_t^\infty L_x^{10}} \|\langle \partial_x \rangle u^1\|_{L_t^\infty L_x^5}^2 + \|\langle \partial_x \rangle u^1\|_{L_t^\infty L_x^6}^3 \lesssim \epsilon^5. \tag{2.35}$$

Therefore, by the Sobolev embedding theorem, $v \in S^1 \subset L_{t,x}^6 \cap L_{t,x}^\infty \subset L_{t,x}^{4n+2}$ for any $n \geq 1$. □

Now, to prove Theorem 2 for a general n, define the sequence of functions

$$\begin{aligned} u^0(t) &= e^{it\partial_{xx}} u_0, \\ u^1(t) &= \int_0^t e^{i(t-\tau)\partial_{xx}} |u^0|^2 u^0(\tau) d\tau, \\ u^j(t) &= \int_0^t e^{i(t-\tau)\partial_{xx}} |\sum_{k=0}^{j-1} u^k|^2 \left(\sum_{k=0}^{j-1} u^k \right) d\tau - \sum_{k=0}^{j-1} u^k(t) \qquad \text{for any} \qquad 2 \leq j \leq n-1. \end{aligned} \tag{2.36}$$

Again by (2.5)–(2.12), for $-1 \leq t \leq 1$,

$$\|\langle \partial_x \rangle^{2n-1} e^{it\partial_{xx}} u_0\|_{L^{4n+2}} \lesssim \epsilon, \tag{2.37}$$

$$\|\langle \partial_x \rangle^{2n-3} u^1(t)\|_{L_x^{\frac{4n+2}{3}}} \lesssim \epsilon^3, \tag{2.38}$$

and arguing by induction, for any $0 \leq j \leq n-1$,

$$\|\langle \partial_x \rangle^{2(n-1-j)+1} u^j(t)\|_{L_x^{\frac{4n+2}{2j+1}}} \lesssim \epsilon^{2j+1}. \tag{2.39}$$

Remark 9. The implicit constants depend on n.

Then, let

$$v(t) = \int_0^t e^{i(t-\tau)\partial_{xx}} |u|^2 u d\tau - \sum_{j=1}^{n-1} u^j(t). \tag{2.40}$$

Following (2.29),

$$\begin{aligned} v(t) = \int_0^t e^{i(t-\tau)\partial_{xx}} |u|^2 u d\tau - \sum_{j=1}^{n-1} u^j(t) &= \int_0^t e^{i(t-\tau)\partial_{xx}} |v|^2 v d\tau \\ &+ 3\int_0^t e^{i(t-\tau)\partial_{xx}} v^2 \left(\sum_{j=0}^{n-1} u^j \right) d\tau + 3\int_0^t e^{i(t-\tau)\partial_{xx}} v \left(\sum_{j=0}^{n-1} u^j \right)^2 d\tau \\ &+ \int_0^t e^{i(t-\tau)\partial_{xx}} |\sum_{j=0}^{n-1} u^j|^2 \left(\sum_{j=0}^{n-1} u^j \right) d\tau - \sum_{j=1}^{n-1} u^j(t). \end{aligned} \tag{2.41}$$

Again, by Strichartz estimates,

$$\| \int_0^t e^{i(t-\tau)\partial_{xx}} |v|^2 v d\tau \|_{S^0([-1,1]\times\mathbb{R})} \lesssim \|v\|_{S^0([-1,1]\times\mathbb{R})}^3. \tag{2.42}$$

By the Sobolev embedding theorem and (2.39),

$$\| \int_0^t e^{i(t-\tau)\partial_{xx}} v^2 \left(\sum_{j=0}^{n-1} u^j\right) d\tau \|_{S^0([-1,1]\times\mathbb{R})} \lesssim \epsilon \|v\|_{S^0([-1,1]\times\mathbb{R})}^2 \tag{2.43}$$

and

$$\| \int_0^t e^{i(t-\tau)\partial_{xx}} v \left(\sum_{j=0}^{n-1} u^j\right)^2 d\tau \|_{S^0([-1,1]\times\mathbb{R})} \lesssim \epsilon^2 \|v\|_{S^0([-1,1]\times\mathbb{R})}. \tag{2.44}$$

Finally, compute

$$\begin{aligned} &\int_0^t e^{i(t-\tau)\partial_{xx}} |\sum_{j=0}^{n-1} u^j|^2 \left(\sum_{j=0}^{n-1} u^j\right) d\tau - \sum_{j=1}^{n-1} u^j(t) \\ &= \int_0^t e^{i(t-\tau)\partial_{xx}} \left[|\sum_{j=0}^{n-1} u^j|^2 \left(\sum_{j=0}^{n-1} u^j\right) - |\sum_{j=0}^{n-2} u^j|^2 \left(\sum_{j=0}^{n-2} u^j\right)\right] d\tau. \end{aligned} \tag{2.45}$$

Therefore,

$$\begin{aligned} &\| \int_0^t e^{i(t-\tau)\partial_{xx}} \left[|\sum_{j=0}^{n-1} u^j|^2 \left(\sum_{j=0}^{n-1} u^j\right) - |\sum_{j=0}^{n-2} u^j|^2 \left(\sum_{j=0}^{n-2} u^j\right)\right] d\tau \|_{S^0([-1,1]\times\mathbb{R})} \\ &\lesssim \|u^{n-1}\|_{L_t^\infty L_x^{\frac{4n+2}{2n-1}}} \left(\sum_{j=0}^{n-1} \|u^j\|_{L_t^\infty L_x^{4n+2}}\right)^2 \lesssim \epsilon^{2n+1}. \end{aligned} \tag{2.46}$$

Therefore,

$$\|v\|_{S^0([-1,1]\times\mathbb{R})} \lesssim \epsilon^{2n+1} + \|v\|_{S^0([-1,1]\times\mathbb{R})}^3, \tag{2.47}$$

which proves Theorem 2 for a general n. □

III. A GLOBAL RESULT

The local results in Sec. II may be extended to global results for a slightly smaller subset of initial data. First, consider the case when $n = 1$.

Theorem 6. *Equation (1.1) is globally well-posed for*

$$\|\langle\partial_x\rangle^4 u_0\|_{L^6} < \infty. \tag{3.1}$$

Proof. Using the scaling symmetry,

$$u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda x), \tag{3.2}$$

it is possible to rescale the initial data so that (3.1) $\le \epsilon$. Then, by Theorem 2, (1.1) has a solution on the interval $[-1,1]$, which is of the form

$$u(t) = e^{it\partial_{xx}} u_0 - i\int_0^t e^{i(t-\tau)\partial_{xx}} |u(\tau)|^2 u(\tau) d\tau = u^0(t) + v(t), \tag{3.3}$$

where $\|v(t)\|_{L^2} \lesssim \epsilon^3$ for all $t \in [-1,1]$.

Furthermore, if (1.1) has a solution on the maximal interval $[0,T)$, $T < \infty$, then

$$\lim_{t\nearrow T} \|v(t)\|_{L^2} = +\infty. \tag{3.4}$$

Indeed, suppose that there exists $t_0 < T$ such that

$$\|v(t_0)\|_{L^2} < \infty. \tag{3.5}$$

Then, by Strichartz estimates, there exists some $\delta(\|v(t_0)\|_{L^2}) > 0$ such that

$$\|e^{i(t-t_0)\partial_{xx}} v(t_0)\|_{L_t^3 L_x^6([t_0,t_0+\delta]\times\mathbb{R})} \leq \epsilon. \tag{3.6}$$

In addition, by (2.5)–(2.13), for $\delta(T) > 0$ sufficiently small,

$$\|e^{it\partial_{xx}} u_0\|_{L_t^3 L_x^6([t_0,t_0+\delta]\times\mathbb{R})} \leq \epsilon. \tag{3.7}$$

Following the Proof of Theorem 3, (1.1) is locally well-posed on the interval $[t_0, t_0 + \delta]$. Since δ is a function of $\|v(t_0)\|_{L^2}$ and T only, if there exists a sequence $t_n \nearrow T$ for which

$$\lim_{n\to\infty} \|v(t_n)\|_{L^2} < \infty, \tag{3.8}$$

then the solution of (1.1) can be continued past T.

Compute the energy and mass of v,

$$M(v) + E(v) = \frac{1}{2}\int |v|^2 + \frac{1}{2}\int |\partial_x v|^2 + \frac{1}{4}\int |v|^4. \tag{3.9}$$

Lemma 2. For any T, there exists a bound

$$\sup_{t\in[-T,T]} M(v(t)) + E(v(t)) \lesssim_T 1. \tag{3.10}$$

Proof. This lemma will be proved using a Gronwall-type argument. In general, it will be convenient to relabel

$$u_l = \sum_{j=0}^{n-1} u^j, \tag{3.11}$$

where u_l denotes the linear part. In this case, since $n = 1$, $u_l = u^0$.

Observe that v solves the nonlinear Schrödinger equation

$$i\partial_t v + \partial_{xx} v = |v|^2 v + 2|v|^2 u_l + v^2 \bar{u}_l + 2|u_l|^2 v + (u_l)^2 \bar{v} + |u_l|^2 u_l. \tag{3.12}$$

Therefore,

$$\frac{d}{dt} M(v) = 2(|v|^2 v, iu_l) + (|v|^2 \bar{v}, i\bar{u}_l) + (v^2, i(u_l)^2) + (v, i|u_l|^2 u_l), \tag{3.13}$$

where

$$(f, g) = \mathrm{Re}\int f(x)\overline{g(x)}dx. \tag{3.14}$$

First, by (2.13),

$$(v, i|u_l|^2 u_l) \lesssim \|v\|_{L^2} \|u_l\|_{L^6}^3 \lesssim_T M(v)^{1/2}. \tag{3.15}$$

Next,

$$(v^2, i(u_l)^2) \lesssim \|v\|_{L^4}^{4/3} \|v\|_{L^2}^{2/3} \|u_l\|_{L^6}^2 \lesssim_T M(v)^{1/3} E(v)^{1/3} \tag{3.16}$$

and

$$2(|v|^2 v, iu_l) + (|v|^2\bar{v}, i\bar{u}_l) \lesssim \|v\|_{L^2}^{1/3}\|v\|_{L^4}^{8/3}\|u_l\|_{L^6} \lesssim_T M(v)^{1/6}E(v)^{2/3}. \tag{3.17}$$

Now, compute the change of energy

$$\begin{aligned}\frac{d}{dt}E(v) &= (\partial_x\partial_t v, \partial_x v) + (\partial_t v, |v|^2 v)\\ &= (\partial_t v, -\partial_{xx} v + |v|^2 v)\\ &= (\partial_t v, i\partial_t v - (|u|^2 u - |v|^2 v))\\ &= (\partial_t v, |v|^2 v - |u|^2 u)\\ &= -(\partial_t v, |u_l|^2 u_l) - 2(\partial_t v, |u_l|^2 v) - (\partial_t v, u_l^2\bar{v}) - (\partial_t(|v|^2 v), u_l).\end{aligned} \tag{3.18}$$

By the product rule,

$$-(\partial_t v, |u_l|^2 u_l) = -\frac{d}{dt}(v, |u_l|^2 u_l) + (v, \partial_t(|u_l|^2 u_l)). \tag{3.19}$$

Integrating by parts,

$$\begin{aligned}(v, \partial_t(|u_l|^2 u_l)) &= 3(v, (i\partial_{xx}u_l)u_l^2) = -6(v, i(\partial_x u_l)^2 u_l) - 3(\partial_x v, i(\partial_x u_l)u_l^2)\\ &\lesssim \|\partial_x u_l\|_{L^6}^2\|u_l\|_{L^6}\|v\|_{L^2} + \|\partial_x v\|_{L^2}\|\partial_x u_l\|_{L^6}\|u_l\|_{L^6}^2 \lesssim_T M(v)^{1/2} + E(v)^{1/2}.\end{aligned} \tag{3.20}$$

Remark 10. Since $\|\langle\partial_x\rangle^4 u_0\|_{L^6} < \infty$, *(2.5)–(2.13) imply*

$$\|\langle\partial_x\rangle^2 e^{it\partial_{xx}}u_0\|_{L^6} \lesssim \|\langle\partial_x\rangle^4 u_0\|_{L^6} \lesssim_T 1. \tag{3.21}$$

Next,

$$-2(\partial_t v, |u_l|^2 v) = -(\partial_t|v|^2, |u_l|^2) = -\frac{d}{dt}(|v|^2, |u_l|^2) + (|v|^2, \partial_t|u_l|^2). \tag{3.22}$$

Again, by the product rule, (3.21), and integrating by parts,

$$\begin{aligned}(|v|^2, \partial_t|u_l|^2) &= 2(|v|^2, (i\partial_{xx}u_l)\bar{u}_l) = -4((\partial_x v)v, (i\partial_x u_l)\bar{u}_l)\\ &\lesssim \|\partial_x v\|_{L^2}\|v\|_{L^2}^{2/3}\|v\|_{L^4}^{4/3}\|\partial_x u_l\|_{L^6}\|u_l\|_{L^\infty} \lesssim_T E(v)^{2/3}M(v)^{1/6}.\end{aligned} \tag{3.23}$$

By a similar calculation,

$$-(\partial_t v, u_l^2\bar{v}) = -\frac{1}{2}(\partial_t(v^2), u_l^2) = -\frac{1}{2}\frac{d}{dt}(v^2, u_l^2) + (v^2, u_l(\partial_t u_l)). \tag{3.24}$$

Integrating by parts,

$$\begin{aligned}(v^2, u_l(\partial_t u_l)) &= -(v^2, u_l(i\partial_{xx}u_l)) = -(v^2, (\partial_x u_l)(i\partial_x u_l)) - 2(v(\partial_x v), u_l(i\partial_x u_l))\\ &\lesssim \|v\|_{L^2}^{2/3}\|v\|_{L^4}^{4/3}\|\partial_x u_l\|_{L^6}^2 + \|\partial_x v\|_{L^2}\|v\|_{L^4}^{2/3}\|v\|_{L^2}^{1/3}\|\partial_x u_l\|_{L^6}\|u_l\|_{L^\infty}\\ &\lesssim_T E(v)^{1/3}M(v)^{1/3} + E(v)^{2/3}M(v)^{1/6}.\end{aligned} \tag{3.25}$$

Finally,

$$-(\partial_t|v|^2 v, u_l) = -\frac{d}{dt}(|v|^2 v, u_l) + (|v|^2 v, \partial_t u_l). \tag{3.26}$$

Integrating by parts and using (3.21) to control $\|\partial_x u_l\|_{L^\infty}$,

$$(|v|^2 v, \partial_t u_l) = (|v|^2 v, i\partial_{xx} u_l) = (-\partial_x(|v|^2 v), i\partial_x u_l) \lesssim \|\partial_x v\|_{L^2} \|v\|_{L^4}^2 \|\partial_x u_l\|_{L^\infty} \lesssim_T E(v). \tag{3.27}$$

Therefore, we have proved that for all $t \in [0, T)$,

$$\frac{d}{dt}(M(v) + E(v)) \le C(T)(M(v)^{1/2} + E(v)^{1/2} + M(v)^{1/3} E(v)^{1/3} + M(v)^{1/6} E(v)^{2/3} + E(v)) - \frac{d}{dt} f(t), \tag{3.28}$$

where

$$f(t) = (v, |u_l|^2 u_l) + (|v|^2, |u_l|^2) + \frac{1}{2}(v^2, u_l^2) + (|v|^2 v, u_l). \tag{3.29}$$

Now, let

$$\mathcal{E}(t) = M(v)(t) + E(v)(t) + f(t). \tag{3.30}$$

By Hölder's inequality,

$$|f(t)| \lesssim_T \|v\|_{L^2} + \|v\|_{L^4}^{4/3} \|v\|_{L^2}^{2/3} + \|v\|_{L^2}^{1/3} \|v\|_{L^4}^{8/3} \lesssim_T M(v)^{1/2} + M(v)^{1/3} E(v)^{1/3} + M(v)^{1/6} E(v)^{2/3} \ll M(v)(t) + E(v)(t), \tag{3.31}$$

when $M(v) + E(v)$ is large. Plugging (3.31) into (3.30) implies

$$M(v) + E(v) \lesssim_T \mathcal{E}(t) + 1 \tag{3.32}$$

and

$$\frac{d}{dt}\mathcal{E}(t) = \frac{d}{dt}(M(t) + E(t) + f(t)) \lesssim_T (M + E) \lesssim_T \mathcal{E}(t) + 1. \tag{3.33}$$

The first inequality in (3.33) uses (3.28). By Gronwall's inequality, the proof is complete. □

Then, by (3.4)–(3.8), this proves Theorem 6. □

This argument can be generalized to prove the following theorem:

Theorem 7. *For any $n \in \mathbb{Z}$, $n \ge 1$, if*

$$\|\langle \partial_x \rangle^{2n+2} u_0\|_{L^{4n+1}(\mathbb{R})} \le \epsilon(n), \tag{3.34}$$

then (1.1) has a global solution.

Proof. In this case, let

$$u_l(t) = \sum_{j=0}^{n-1} u^j(t). \tag{3.35}$$

Then, v solves the equation

$$i\partial_t v + \partial_{xx} v = |u|^2 u - F(t) \tag{3.36}$$

and

$$i\partial_t u_l + \partial_{xx} u_l = F(t), \tag{3.37}$$

where

$$F(t) = |\sum_{j=0}^{n-2} u^j|^2 \left(\sum_{j=0}^{n-2} u^j \right). \tag{3.38}$$

By (2.37)–(2.39),

$$\|u_l\|_{L^\infty\cap L^{4n+2}} + \|\partial_x u_l\|_{L^\infty\cap L^{4n+2}} \lesssim \epsilon, \tag{3.39}$$

so using (3.37), all the terms in the Proof of Theorem 6 that have two or three v terms can be handled in exactly the same manner, after doing some algebra with the various L^p norms. The crucial fact is that

$$(|v|^2 v, \partial_t u_l) \tag{3.40}$$

is the only term which is bounded by some $C(T)E(v)$. All other terms are bounded by $C(T)M(v)^\alpha E(v)^{c-\alpha}$ for $\alpha > 0$ and $0 \le c < 1$.

Finally, using (2.46),

$$(v, |u_l|^2 u_l - F(t)) \lesssim_T M(v)^{1/2}. \tag{3.41}$$

To compute

$$(v, \partial_t(|u_l|^2 u_l - F(t))), \tag{3.42}$$

decompose

$$|u_l|^2 u_l - F(t) = u^{n-1} \cdot \sum_{j_1,j_2=0}^{n-1} c(j_1,j_2) u^{j_1} u^{j_2}. \tag{3.43}$$

By (2.37)–(2.39),

$$\|(\partial_t - i\partial_{xx})u^{n-1}\|_{L_x^{\frac{4n+2}{2n-1}}} \lesssim 1, \tag{3.44}$$

while integrating by parts,

$$\begin{aligned}\left(v, i\partial_{xx}u^{n-1}\cdot\sum_{j_1,j_2=0}^{n-1} c(j_1,j_2)u^{j_1}u^{j_2}\right) &\lesssim \|\partial_x v\|_{L^2}\|\partial_x u^{n-1}\|_{L^{\frac{4n+2}{2n-1}}}\sum_{j=0}^{n-1}\|u^j\|_{L^{4n+2}}^2 \\ &+\|v\|_{L^2}\|\partial_x u^{n-1}\|_{L^{\frac{4n+2}{2n-1}}}\sum_j\|u^j\|_{L^{4n+2}}\cdot\sum_j\|\partial_x u^j\|_{L^{4n+2}} \lesssim_T M(v)^{1/2} + E(v)^{1/2}.\end{aligned} \tag{3.45}$$

Therefore,

$$\left(v, (\partial_t u^{n-1})\cdot\sum_{j_1,j_2=0}^{n-1} c(j_1,j_2)u^{j_1}u^{j_2}\right) \lesssim_T M(v)^{1/2} + E(v)^{1/2}. \tag{3.46}$$

The contribution of $\partial_t\sum_{j_1,j_2=0}^{n-1} c(j_1,j_2)u^{j_1}u^{j_2}$ to (3.42) is similar. This completes the Proof of Theorem 7. □

DEDICATION

Dedicated to Jean Bourgain with admiration for his fundamental contributions to analysis.

ACKNOWLEDGMENTS

The authors thank J. Bourgain, P. Deift, J. Lebowitz, and W. Schlag for helpful discussions. B.D. gratefully acknowledges the support of NSF Grant Nos. DMS-1500424 and DMS-1764358. He also gratefully acknowledges support from the von Neumann Fellowship at the Institute for Advanced Study. A.S. was supported, in part, by NSF Grant No. DMS-160074.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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