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PROPERTIES OF THE CHARACTERISTIC FUNCTION GENERATOR OF THE TWO-STATE CAUCHY ESTIMATOR*

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Abstract. In order to better capture impulsive noises in dynamic systems, a state estimator in the presence of Cauchy distributed process and measurement noises has been studied in recent years. The Cauchy estimator is determined by expressing in closed form the characteristic function (CF) of the unnormalized conditional probability density functions of the states given measurement history. The CF is comprised of a sum of exponentials multiplied by a coefficient, both being nonlinear functions of the measurements and the spectral variable. In this paper, we uncover important properties of the exponential terms in the CF for the two-state Cauchy estimator. These properties can be used to simplify the estimator structure significantly.

Key words. Cauchy probability density function, nonlinear estimation with heavy tailed noises, characteristic functions

AMS subject classifications. 93E11, 62L12

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1. Introduction.

In many engineering, economic, telecommunications, and science applications, the underlying random processes or noises have significant volatility, which are not captured by Gaussian distributions [8]. In fact, heavy-tailed distributions were shown to better represent such phenomena. For linear systems with additive Gaussian noises the Kalman filter has been the main estimation paradigm in many fields of engineering. Its appeal in part is due to its analytic structure. However, in many practical systems, measurement and process uncertainties that have impulsive character are better described by heavy-tailed densities. Rather than light-tailed Gaussian distributions, heavy-tailed distributions have been shown to better capture these volatile random fluctuations. Examples from the practical engineering world include radar and sonar sensor noises [6], air turbulent environment noise [7], and adversarial motion.

Although no physical process is explicitly Cauchy distributed, since their tails over-bound other realistic densities, estimators that are based on the Cauchy probability density functions (pdfs) that have very heavy tails, are hypothesized to be robust to unknown physical densities. We use robustness in the statistical sense [1], where it means that the estimator achieves adequate performance when faced with outliers or unexplained events, which may arise either as large measurement errors, large process deviations, or due to misspecification of the dynamic model.

For linear dynamic systems, besides additive Gaussian noise, the general recursive and analytical structure of the conditional mean state estimator was developed

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assuming Cauchy distributed process and measurement noises [5], where the initial condition, and measurement and process noises are Cauchy with infinite variance. Since there appeared to be no direct method for generating the conditional pdf (cpdf) of the system state given the measurement history for the general multivariable state space, our novel approach to this estimation problem is to propagate the characteristic function (CF) of the unnormalized cpdf. From this CF, the finite conditional mean and finite conditional error variance can be analytically constructed, both of which are complicated nonlinear functions of the measurement history.

The essential computational difficulty associated with this estimator is that the above mentioned CF is expressed as a sum of terms that grows at each measurement update. It has been observed that many of these terms can be combined. However, the structural properties of the algorithm that produce these combinations are not clear for the general multivariable case. To gain insight into the structure of the characteristic function, the two-state dynamic systems are studied here. This work reveals several important properties of the terms comprising the characteristic function and in turn allows identifying analytically the terms that can be combined. These properties and simplifications allow an efficient combination of terms that express the CF, hence, reducing significantly the memory and computational burdens of the Cauchy filter.

1.1. Problem statement. This paper studies the two-state linear state space system

$$x_{k+1} = \Phi x_k + \Gamma w_k, \; z_k = H x_k + v_k, \tag{1}$$

where k is the discrete stage time. The state vector is $x_k \in \mathbb{R}^2$ and z_k is a scalar measurement. The state transition matrix $\Phi \in \mathbb{R}^{2\times 2}$, the process noise matrix $\Gamma \in \mathbb{R}^{2\times 1}$, and the measurement matrix $H \in \mathbb{R}^{1\times 2}$ are known. A single input process noise and a single output measurement restrictions are made to only simplify the subsequent analysis. The results presented here can be easily generalized to vector-valued process and measurement signals case. The process noise w_k and the measurement noise v_k are assumed to be independent Cauchy distributed random variables with a zero median and scale parameters β and γ, respectively. Thus, the characteristic functions of the pdfs of those noise signals are given by $\phi_W(\bar{\nu}) = \exp(-\beta|\bar{\nu}|)$ and $\phi_V(\bar{\nu}) = \exp(-\gamma|\bar{\nu}|)$, respectively, where $\bar{\nu}$ is a scalar spectral variable. The elements of the initial state vector x_1 are assumed to be independent and Cauchy distributed. The characteristic function of the pdf of the initial state vector is

$$\phi_{X_1}(\nu) = \exp\left(-\sum_{i=1}^{2} \alpha_i \left|e_i \nu\right| + j\bar{x}_1^T \nu\right), \tag{2}$$

where $\nu \in \mathbb{R}^2$ is the spectral vector, $e_1 = [1 \quad 0]$, $e_2 = [0 \quad 1]$, $\bar{x}_1 \in \mathbb{R}^2$ is the median of the distribution of the initial state x_1, and $\alpha_i \in \mathbb{R}$, $i = 1, 2$, are the scale parameters of the pdfs. The goal is to construct a minimum conditional variance estimate of x_k given the measurement history $y_k = \{z_1, \ldots, z_k\}$.

1.2. State estimation using the CF approach. The traditional estimator derivation approach for this problem did not directly generate a recursive and analytic form for the conditional pdf of the state given the measurements, except for the scalar problem [4]. However, a closed form recursive Cauchy estimator in [5] was formulated by propagating and updating the CF of the unnormalized cpdf (ucpdf) of the state

x_k given the measurement history $y_k = \{z_1, \ldots, z_k\}$ at step k, instead of updating the ucpdf directly.

Using Bayes's theorem, the cpdf of the state given the first measurement is expressed as

$$f_{X_1|Z_1}(x_1|z_1) = \frac{f_V(z_1 - Hx_1) f_{X_1}(x_1)}{f_{Z_1}(z_1)}, \tag{3}$$

where $f_{X_1}(\cdot)$ and $f_V(\cdot)$ are the pdfs of x_1 and v_1, respectively, and

$$f_{Z_1}(z_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_V(z_1 - Hx_1) f_{X_1}(x_1) dx_1. \tag{4}$$

The CF of this cpdf is given by

$$\begin{aligned} \phi_{X_1|Z_1}(\nu) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1|Z_1}(x_1|z_1) e^{j\nu^T x_1} dx_1 \\ &= \frac{1}{f_{Z_1}(z_1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_V(z_1 - Hx_1) f_{X_1}(x_1) e^{j\nu^T x_1} dx_1. \end{aligned} \tag{5}$$

In [5] the Cauchy estimator was derived when propagating the CF of the ucpdf defined by

$$\bar{\phi}_{X_1|Z_1}(\nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_V(z_1 - Hx_1) f_{X_1}(x_1) e^{j\nu^T x_1} dx_1. \tag{6}$$

Using the Fourier transform properties, the above can be expressed as [5, Appendix A]

$$\bar{\phi}_{X_1|Z_1}(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{X_1}(\nu - H^T\eta) \phi_V(-\eta) e^{jz_1\eta} d\eta, \tag{7}$$

which is then solved analytically [5, Appendix B].

The CF of the ucpdf was then time propagated and measurement updated at every step in a recursive manner. Specifically, starting with $\bar{\phi}_{X_k|Y_k}(\nu)$ at step k, where Y_k is the random variable of the measurement history and y_k is its realization, using the state transition equation in (1) and assuming that $\det(\Phi) \neq 0$, the time propagated CF is given by [5, Appendix C]

$$\bar{\phi}_{X_{k+1}|Y_k}(\nu) = \bar{\phi}_{X_k|Y_k}(\Phi^T\nu) \phi_W(\Gamma^T\nu). \tag{8}$$

Using the measurement equation in (1), the measurement update of the CF at time step $k+1$ is given by

$$\bar{\phi}_{X_{k+1}|Y_{k+1}}(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\phi}_{X_{k+1}|Y_k}(\nu - H^T\eta) \phi_V(-\eta) e^{jz_{k+1}\eta} d\eta, \tag{9}$$

which is solved analytically [5, Appendix B].

The conditional mean and second moment of the state were then evaluated by taking the first and second derivatives of the measurement-updated CF when evaluated at the origin of the spectral variable. Due to the continuity of the CF, those evaluations were performed by choosing a priori a fixed direction $\hat{\nu}$, setting $\nu = \epsilon\hat{\nu}$, and letting $\epsilon \to 0$. As shown in [5], the conditional mean and second moment of the state x_k given the measurement sequence y_k are given by

$$\hat{x}_k = \mathbf{E}\left[x_k|y_k\right] = \frac{1}{jf_{Y_k}(y_k)}\left(\frac{\partial\bar{\phi}_{X_k|Y_k}(\epsilon\hat{\nu})}{\partial(\nu)}\right)^T\Bigg|_{\epsilon=0}, \tag{10}$$

$$\mathbf{E}\left[x_k x_k^T|y_k\right] = \frac{1}{j^2 f_{Y_k}(y_k)}\left(\frac{\partial^2\bar{\phi}_{X_k|Y_k}(\epsilon\hat{\nu})}{\partial(\nu)\partial(\nu)^T}\right)^T\Bigg|_{\epsilon=0}, \tag{11}$$

where $f_{Y_k}(y_k)$ is determined by evaluating the CF at the origin, i.e., $f_{Y_k}(y_k) = \bar{\phi}_{X_k|Y_k}(\epsilon\hat{\nu})|_{\epsilon=0}$. The variance of the estimation error $\tilde{x}_k = x_k - \hat{x}_k$ is then given by $\mathbf{E}[\tilde{x}_k\tilde{x}_k^T|y_k] = \mathbf{E}[x_k x_k^T|y_k] - \hat{x}_k\hat{x}_k^T$.

1.3. Structure of the CF. In [5] it was shown that the measurement updated CF, $\bar{\phi}_{X_k|Y_k}(\nu)$, obtained by solving (9), is expressed as a sum of terms that are a product of an exponential and a coefficient, both being functions of the measurement sequence and the spectral variable, i.e.,

$$\bar{\phi}_{X_k|Y_k}(\nu) = \sum_{i=1}^{N_t^{k|k}} G_i^{k|k}(\nu)\,\mathcal{E}_i^{k|k}(\nu). \tag{12}$$

The exponential function is given by

$$\mathcal{E}_i^{k|k}(\nu) = \exp\left(-\sum_{l=1}^{N_{ei}^{k|k}} p_{i,l}^{k|k}\left|a_{i,l}^{k|k}\nu\right| + jb_i^{k|k}\nu\right). \tag{13}$$

The coefficient is expressed as

$$\begin{aligned} G_i^{k|k}(\nu) &= g_i^{k|k}\left(y_{gi}^{k|k}(\nu)\right) \\ &= \frac{1}{2\pi}\left[\frac{g_{r_i^{k|k}}^{k-1|k-1}\left(y_{gi1}^{k|k}(\nu) + h_i^{k|k}\right)}{jc_i^{k|k} + d_i^{k|k} + y_{gi2}^{k|k}(\nu)} - \frac{g_{r_i^{k|k}}^{k-1|k-1}\left(y_{gi1}^{k|k}(\nu) - h_i^{k|k}\right)}{jc_i^{k|k} - d_i^{k|k} + y_{gi2}^{k|k}(\nu)}\right] \end{aligned} \tag{14}$$

with

$$y_{gi}^{k|k}(\nu) = \sum_{l=1}^{N_{ei}^{k|k}} q_{il}^{k|k}\,\mathrm{sgn}\left(a_{i,l}^{k|k}\nu\right) \in \mathbb{R}^k \tag{15}$$

and

$$g_i^{1|1}\left(y_{gi}^{1|1}(\nu)\right) = \frac{1}{2\pi}\left[\left(jc_i^{1|1} + d_i^{1|1} + y_{gi}^{1|1}(\nu)\right)^{-1} - \left(jc_i^{1|1} - d_i^{1|1} + y_{gi}^{1|1}(\nu)\right)^{-1}\right]. \tag{16}$$

All the parameters in (12)–(16) are fully defined in a recursive form in [5], where only $b_i^{k|k}$ and $c_i^{k|k}$ are linear functions of the measurement z_k, while the others can be precomputed a priori. We note that at time step k the CF is constructed of $N_t^{k|k}$ terms, where each such term involves a sum of $N_{ei}^{k|k}$ elements (see (13) and (15)). The coefficient functions $g_i^{k|k}(\cdot)$ are determined by the parameters $c_i^{k|k}$, $d_i^{k|k}$, the offsets

$h_i^{k|k}$, and the index $r_i^{k|k}$ of the parent term $g_{r_i^{k|k}}^{k-1|k-1}(\cdot)$ in a layered fractional form (14). $y_{gi}^{k|k}(\cdot)$ and $q_{il}^{k|k}$ are k-dimensional vectors. $y_{gi}^{k|k}(\cdot)$ is partitioned into two parts: the first $k-1$ components of $y_{gi}^{k|k}(\cdot)$ construct $y_{gi1}^{k|k}(\cdot)$, while the last component of $y_{gi}^{k|k}(\cdot)$ comprises the scalar $y_{gi2}^{k|k}(\cdot)$.

It was shown in [5] that for systems with two and more states, $N_t^{k|k}$ grows rapidly with k, consequently increasing the processing and storage requirement. This motivates exploring the properties of the CF so as to simplify the estimator structure analytically. Some of these properties, determined heuristically, have been presented in [2] for two-state systems, addressing mainly a simplified structure of the coefficient functions $g_i^{k|k}(\cdot)$. This structure reduced substantially the number of terms in (12). Moreover, the estimator was computationally simplified by truncating the measurement sequence by a "sliding window" approximation, allowing an unlimited number of measurements to be processed while maintaining good performance. In addition, stochastic robustness was demonstrated where the Cauchy conditional mean and conditional error variance performed very similarly to the Kalman filter in a Gaussian simulation, but dramatically outperformed the Kalman filter in a Cauchy simulation. Furthermore, the two-state estimator of [2] was compared in computation time with the Gaussian sum and particle filters in [3] and was shown to be impressively superior. In the current study we uncover many additional interesting properties of the CF for two-state systems and make rigorous the heuristic observations of [2]. Finally, the current study indicates a methodology that should generalize to higher-order systems.

1.4. Contribution of this paper. This paper focuses on the properties of the exponential terms of the CF for two-state systems to yield three main contributions. First, it is shown in section 2 that, after a measurement update, all but one of the directions $a_{i,l}^{k|k}, 1 \leq l \leq N_{ei}^{k|k}$, of the term i coalign. The elements with coaligned directions, easily identified by the element-combination rule presented in section 2, can be combined to yield terms with only two elements.[1] Some of these two-element terms are shown to have an identical exponential. Two term combination rules are determined and shown generically to be both unique and complete in section 3. Based on these term combination rules, an indexing scheme that is independent of the system parameters is introduced in sections 4 and 5 to efficiently identify the terms that can be combined, reducing the computation burden and memory requirements of the estimator.

2. Combining elements in a term.

In this section we will show that after a measurement update many directions in the argument of the real part of the exponential terms in (13) coalign, allowing the respective elements to be combined. To determine analytically which directions coalign, we examine the relation of the directions at time step k to those at $k+1$.

Suppose that at time step k the measurement updated CF is given by (12), with exponential and coefficient functions expressed in (13) and (14), respectively. Moreover, we assume that the directions in the real part of exponentials do not coalign with each other, i.e.,

$$\det\left[a_{i,l}^{k|k^T} \quad a_{i,m}^{k|k^T}\right] = a_{i,l}^{k|k}\mathbf{A}a_{i,m}^{k|k^T} \neq 0 \quad \forall \quad l \neq m, \quad (l,m) \in \left[1,\ldots,N_{ei}^{k|k}\right], \tag{17}$$

[1] Note that elements with coaligned directions have to be combined to avoid singularity in the implementation of the recursive estimator [5].

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{18}$$

The first equality in (17) can be easily verified for any two vectors in $\mathbb{R}^{1\times 2}$. In addition, we quote here several properties of the skew-symmetric matrix $\mathbf{A}$ that will be used throughout this work. Specifically,

$$\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = I, \tag{19}$$

$$\mathbf{A}\mathbf{A} = -I. \tag{20}$$

Moreover, for any c_1 and c_2 in $\mathbb{R}^{1\times 2}$ and $\Phi \in \mathbb{R}^{2\times 2}$,

$$c_1\mathbf{A}c_1^T = 0, \tag{21}$$

$$c_1^Tc_2 - c_2^Tc_1 = \left(c_1\mathbf{A}c_2^T\right)\mathbf{A}, \tag{22}$$

$$\Phi^T\mathbf{A}\Phi = \Phi\mathbf{A}\Phi^T = \det(\Phi)\mathbf{A}. \tag{23}$$

We will concentrate on the ith term in (12), which will be referred to as the parent term. According to (8), after time propagation to step $k+1$, the exponential of this term becomes

$$\mathcal{E}_i^{k+1|k}(\nu) = \exp\left(-\sum_{l=1}^{N_{ei}^{k|k}+1} p_{i,l}^{k+1|k}\left|a_{i,l}^{k+1|k}\nu\right| + jb_i^{k+1|k}\nu\right), \tag{24}$$

where: $p_{i,l}^{k+1|k} = p_{i,l}^{k|k}$ and $a_{i,l}^{k+1|k} = a_{i,l}^{k|k}\Phi^T$ for $1 \le l \le N_{ei}^{k|k}$; $p_{i,N_{ei}^{k|k}+1}^{k+1|k} = \beta$; $a_{i,N_{ei}^{k|k}+1}^{k+1|k} = \Gamma^T$; and $b_i^{k+1|k} = b_i^{k|k}\Phi^T$. First, using (17) and (23), we note that

$$\begin{aligned} a_{i,l}^{k+1|k}\mathbf{A}a_{i,m}^{k+1|k\,T} &= a_{i,l}^{k|k}\Phi^T\mathbf{A}\Phi a_{i,m}^{k|k\,T} \\ &= \det(\Phi)a_{i,l}^{k|k}\mathbf{A}a_{i,m}^{k|k\,T} \neq 0 \quad \forall \quad l \neq m, \quad (l,m) \in \left[1,\dots,N_{ei}^{k|k}\right], \end{aligned} \tag{25}$$

i.e., the first $N_{ei}^{k|k}$ elements in (24) do not coalign. Moreover, since generally those directions will not coalign with the new $a_{i,N_{ei}^{k|k}+1}^{k+1|k} = \Gamma^T$, the elements of the time updated CF in (24) will not combine.

Remark 2.1. In rare occasions, $a_{i,N_{ei}^{k|k}+1}^{k+1|k} = \Gamma^T$ may coalign with one of the other directions in the time propagated term i. In this case, the corresponding elements will combine and the number of elements in this term will reduce by one. Consequently, it will reduce the number of terms that will be generated from this parent term during the measurement update. However, it will not affect the analysis and derivation of the rules for combining elements addressed in this section or the rules for combining terms discussed in section 3. Hence, for the remainder of the paper we will assume that no elements are combined after the time propagation step.

A measurement update of the CF at $k+1$ is determined by (7). It was shown in [5] that the associated integral can be solved analytically and expressed in the form of (12), where each parent term i with the exponential as in (24) produces $N_{ei}^{k|k}+2$ child terms.

Remark 2.2. Note that the analytical solution is based on the assumption that all directions $a_{i,l}^{k+1|k}$ and, in particular, Γ^T, are not orthogonal to H, i.e.,

$$a_{i,l}^{k+1|k} H^T \neq 0 \quad \forall \quad 1 \leq l \leq N_{ei}^{k|k} + 1. \tag{26}$$

If some directions are orthogonal to H, the solution can also be obtained, but will have a slightly different form. The special case of orthogonality is discussed in [5].

Assuming (26) holds, the exponential of the first $N_{ei}^{k|k} + 1$ measurement updated child terms have the form [5]

$$\begin{aligned} &\mathcal{E}_{i,l}^{k+1|k+1}(\nu) \\ &= \exp\Bigg(- \sum_{\substack{m=1 \\ m\neq l}}^{N_{ei}^{k|k}+1} \frac{p_{i,m}^{k+1|k}}{\left|a_{i,l}^{k+1|k} H^T\right|} \left| H \left(a_{i,l}^{k+1|k^T} a_{i,m}^{k+1|k} - a_{i,m}^{k+1|k^T} a_{i,l}^{k+1|k} \right) \nu \right| - \gamma \left| \frac{a_{i,l}^{k+1|k} \nu}{a_{i,l}^{k+1|k} H^T} \right| \\ &\qquad + j \frac{z_{k+1} a_{i,l}^{k+1|k} + H \left(a_{i,l}^{k+1|k^T} b_i^{k+1|k} - b_i^{k+1|k^T} a_{i,l}^{k+1|k} \right)}{a_{i,l}^{k+1|k} H^T} \nu \Bigg) \end{aligned} \tag{27}$$

for $1 \leq l \leq N_{ei}^{k|k} + 1$. The exponential of the last child is given by

$$\mathcal{E}_{i,N_{ei}^{k|k}+2}^{k+1|k+1}(\nu) = \exp\left(- \sum_{l=1}^{N_{ei}^{k|k}+1} p_{i,l}^{k+1|k} \left| a_{i,l}^{k+1|k} \nu \right| + j b_i^{k+1|k} \nu \right). \tag{28}$$

This exponential is identical to that of the time propagated one in (24) and thus this term is called an *old* term or an old child. As before, since the directions in (24) are not coaligned, the elements of this last child will not combine.

Now we turn to the child terms of (27). Using (22), the mth element in the sum of (27) is manipulated as

$$\begin{aligned} p_{i,l,m}^{k+1|k+1} \left| a_{i,l,m}^{k+1|k+1} \nu \right| &= \frac{p_{i,m}^{k+1|k}}{\left| a_{i,l}^{k+1|k} H^T \right|} \left| H \left(a_{i,l}^{k+1|k^T} a_{i,m}^{k+1|k} - a_{i,m}^{k+1|k^T} a_{i,l}^{k+1|k} \right) \nu \right| \\ &= \frac{p_{i,m}^{k+1|k} \left| a_{i,l}^{k+1|k} \mathbf{A} a_{i,m}^{k+1|k^T} \right|}{\left| a_{i,l}^{k+1|k} H^T \right|} \left| H \mathbf{A} \nu \right|. \end{aligned} \tag{29}$$

Similarly, the imaginary component in (27) becomes

$$\begin{aligned} b_{i,l}^{k+1|k+1} &= \frac{z_{k+1} a_{i,l}^{k+1|k} + H \left(a_{i,l}^{k+1|k^T} b_i^{k+1|k} - b_i^{k+1|k^T} a_{i,l}^{k+1|k} \right)}{a_{i,l}^{k+1|k} H^T} \\ &= \frac{z_{k+1} a_{i,l}^{k+1|k} + \left(a_{i,l}^{k+1|k} \mathbf{A} b_i^{k+1|k^T} \right) H \mathbf{A}}{a_{i,l}^{k+1|k} H^T}. \end{aligned} \tag{30}$$

The result in (29) implies that the first $N_{ei}^{k|k}$ elements in any term with an exponential as in (27) coalign along common directions $H\mathbf{A}$ and hence can be combined. Moreover, testing its relation to the remaining direction $a_{i,l}^{k+1|k}$ in (27), while using (19) and (26), reveals that

$$a_{i,l}^{k+1|k}\mathbf{A}(H\mathbf{A})^T = a_{i,l}^{k+1|k}H^T \neq 0. \tag{31}$$

This implies that $H\mathbf{A}$ does not coalign with $a_{i,l}^{k+1|k}$ and thus is called a *new* direction. Consequently, the exponential in (27), after combining the elements with co-aligned directions, becomes

$$\mathcal{E}_{i,l}^{k+1|k+1}(\nu) = \exp\left(-p_{i,l}^{k+1|k+1}\left|H\mathbf{A}\nu\right| - \frac{\gamma}{\left|a_{i,l}^{k+1|k}H^T\right|}\left|a_{i,l}^{k+1|k}\nu\right| + jb_{i,l}^{k+1|k+1}\nu\right), \tag{32}$$

where we defined

$$p_{i,l}^{k+1|k+1} = \sum_{\substack{m=1\\ m\neq l}}^{N_{ei}^{k|k}+1} \frac{p_{i,m}^{k+1|k}\left|a_{i,l}^{k+1|k}\mathbf{A}a_{i,m}^{k+1|k^T}\right|}{\left|a_{i,l}^{k+1|k}H^T\right|}. \tag{33}$$

Since the exponential above involves the new direction $H\mathbf{A}$, the associated term will be referred to as a *new* term or a new child.

The results above constitute the element combination rule for a two-state system that is summarized in the following theorem.

THEOREM 2.3. *In a two-state system, generically,*

1. *elements of the terms in the CF may combine only after a measurement update;*
2. *each term of a measurement updated CF with N_e elements at time k produces N_e+2 terms with N_e+1 elements after a measurement update at $k+1$, where the first N_e+1 terms of this CF are new and the last one is old;*
3. *the first N_e elements of the new terms are coaligned along a common direction $H\boldsymbol{A}$ and thus can be combined.*

Hence, each parent term with N_e elements at time step k produces N_e+1 two-element new terms and one old term with N_e+1 elements at $k+1$.

Once the elements with coaligning directions are combined, the exponentials of the terms of the CF can be compared empirically to test if those terms can be combined. It turns out that the term combination property can be described analytically in closed form by two rules, which are presented in the next section.

3. Term combination rules. It was observed numerically that many exponential terms of (12) are identical after a measurement update. In this section we determine analytically which terms have identical exponentials and thus can be combined. It will be shown that for two-state systems combining terms are determined by only two simple rules.

The rules for combining terms were derived by comparing their exponential functions analytically. Starting with a parent term i with the exponential $\mathcal{E}_i^{k|k}(\nu)$ at step k, the combination rules study the exponential $\mathcal{E}_{i,l,r}^{k+2|k+2}(\nu)$ of the grandchild terms at step $k+2$. Note the triple indexing in the subscript of $\mathcal{E}_{i,l,r}^{k+2|k+2}(\nu)$. It represents the exponential of the rth grandchild term at step $k+2$ generated from the lth child term with the exponential $\mathcal{E}_{i,l}^{k+1|k+1}(\nu)$ at step $k+1$. The child terms with $\mathcal{E}_{i,l}^{k+1|k+1}(\nu)$ for a fixed i do not combine because they are generated from the same parent term, while some child terms may combine across different i. The way they combine follows the two combination rules presented below.

3.1. Combination rule for the first grandchild terms. Suppose, as before, that at time step k the measurement updated CF is given by (12), with exponential and coefficient functions expressed in (13) and (14), respectively. Moreover, we assume that the directions in the real part of the exponential do not coalign (i.e., the elements with coaligned directions have been combined) and that no terms can be combined (i.e., terms with identical exponentials were already combined.) As shown in section 2, each parent term i produces $N_{ei}^{k|k}+1$ new two-element terms after a measurement update at $k+1$, given in (32). We will show analytically that the exponentials of the first grandchild terms at $k+2$ generated by *all* new child terms at $k+1$ are identical and thus the respective terms can be combined.

Using (8) and (32), the exponential of a new term indexed (i,l), after time propagation to $k+2$, is given by

$$\begin{aligned}(34)\quad \mathcal{E}_{i,l}^{k+2|k+1}(\nu) = \exp\Bigg(&-p_{i,l}^{k+1|k+1}\left|H\mathbf{A}\Phi^T\nu\right| \\ &-\frac{\gamma}{\left|a_{i,l}^{k+1|k}H^T\right|}\left|a_{i,l}^{k+1|k}\Phi^T\nu\right| - \beta\left|\Gamma^T\nu\right| + jb_{i,l}^{k+1|k+1}\Phi^T\nu\Bigg).\end{aligned}$$

Following the result in (27), the exponential of the first grandchild term generated after a measurement update at $k+2$ by the new child term (i,l) at $k+1$ is given by

$$\begin{aligned}(35)\quad &\mathcal{E}_{i,l,1}^{k+2|k+2}(\nu) \\ &= \exp\Bigg(-\frac{\gamma}{\left|a_{i,l}^{k+1|k}H^T\right|\left|H\mathbf{A}\Phi^TH^T\right|}\left|H\Phi\left(\mathbf{A}^TH^Ta_{i,l}^{k+1|k}-\Phi a_{i,l}^{k+1|k^T}H\mathbf{A}\right)\Phi^T\nu\right| \\ &\qquad -\frac{\beta}{\left|H\mathbf{A}\Phi^TH^T\right|}\left|H\left(\Phi\mathbf{A}^TH^T\Gamma^T-\Gamma H\mathbf{A}\Phi^T\right)\nu\right| - \gamma\left|\frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^TH^T}\right| \\ &\qquad + j\frac{z_{k+2}H\mathbf{A}\Phi^T+H\Phi\left(\mathbf{A}^TH^Tb_{i,l}^{k+1|k+1}-b_{i,l}^{k+1|k+1^T}H\mathbf{A}\right)\Phi^T}{H\mathbf{A}\Phi^TH^T}\nu\Bigg).\end{aligned}$$

Using (22), with the associations $c_1 = H\mathbf{A}$ and $c_2 = a_{i,l}^{k+1|k}$, together with (20) and (23), the ν-dependant term in the first line of (35) can be simplified as

$$\begin{aligned}(36)\quad \left|H\Phi\left(\mathbf{A}^TH^Ta_{i,l}^{k+1|k}-a_{i,l}^{k+1|k^T}H\mathbf{A}\right)\Phi^T\nu\right| &= \left|\left(H\mathbf{A}\mathbf{A}a_{i,l}^{k+1|k^T}\right)H\Phi\mathbf{A}\Phi^T\nu\right| \\ &= \left|\det(\Phi)\right|\left|a_{i,l}^{k+1|k}H^T\right|\left|H\mathbf{A}\nu\right|.\end{aligned}$$

Similarly, using $c_1 = H\mathbf{A}\Phi^T$ and $c_2 = \Gamma^T$ in (22), the first ν-dependant term in the second line of (35) can be simplified as

$$\left|H\left(\Phi\mathbf{A}^T H^T \Gamma^T - \Gamma H \mathbf{A}\Phi^T\right)\nu\right| = \left|H\mathbf{A}\Phi^T\mathbf{A}\Gamma\right|\left|H\mathbf{A}\nu\right|. \tag{37}$$

For the imaginary part, using $c_1 = H\mathbf{A}$ and $c_2 = b_{i,l}^{k+1|k+1}$ in (22), together with (20), (21), (23), and (30), the last term in the last line of (35) is manipulated as

$$\begin{aligned}
&H\Phi\left(\mathbf{A}^T H^T b_{i,l}^{k+1|k+1} - b_{i,l}^{k+1|k+1^T} H\mathbf{A}\right)\Phi^T \\
&= \left(H\mathbf{A}\mathbf{A}b_{i,l}^{k+1|k+1^T}\right) H\Phi\mathbf{A}\Phi^T = -\det(\Phi)\left(b_{i,l}^{k+1|k+1} H^T\right) H\mathbf{A} \\
&= -\frac{\det(\Phi)}{a_{i,l}^{k+1|k} H^T}\left(z_{k+1} a_{i,l}^{k+1|k} H^T + \left(a_{i,l}^{k+1|k}\mathbf{A}b_i^{k+1|k^T}\right) H\mathbf{A}H^T\right) H\mathbf{A} \\
&= -z_{k+1}\det(\Phi) H\mathbf{A}.
\end{aligned} \tag{38}$$

Substituting (36)–(38) into (35), while combining the first two elements, $\mathcal{E}_{i,l,1}^{k+2|k+2}(\nu)$ is restated as

$$\begin{aligned}
\mathcal{E}_{i,l,1}^{k+2|k+2}(\nu) = \exp\Bigg(&-\frac{\gamma\left|\det(\Phi)\right| + \beta\left|H\mathbf{A}\Phi^T\mathbf{A}\Gamma\right|}{\left|H\mathbf{A}\Phi^T H^T\right|}\left|H\mathbf{A}\nu\right| - \gamma\left|\frac{H\mathbf{A}\Phi^T\nu}{H\mathbf{A}\Phi^T H^T}\right| \\
&+ j\frac{z_{k+2} H\mathbf{A}\Phi^T - z_{k+1}\det(\Phi) H\mathbf{A}}{H\mathbf{A}\Phi^T H^T}\nu\Bigg).
\end{aligned} \tag{39}$$

Equation (39) indicates that this exponential does not depend on the specific parameters of the parent term i given in (13). It is determined in a closed form by the system parameters Φ, Γ, H, β, and γ and the last two measurements z_{k+1} and z_{k+2}, and this persists for all future k. Consequently, the exponential part of *all* first grandchild terms of *all* new two-element child terms are identical, i.e.,

$$\begin{aligned}
\mathcal{E}_{i,l,1}^{k+2|k+2}(\nu) = \mathcal{E}_{p,q,1}^{k+2|k+2}(\nu) \quad \forall \quad &(i,p) \in \left[1,\ldots,N_t^{k|k}\right], \\
&l \in \left[1,\ldots,N_{ei}^{k|k}+1\right],\ q \in \left[1,\ldots,N_{ep}^{k|k}+1\right],
\end{aligned} \tag{40}$$

and hence can be combined. This constitutes the first term combination rule.

3.2. Second term combination rule. The second term combination rule considers grandchild terms from the same parent i and shows analytically that the grandchild term indexed $(i,l,2)$ has the same exponential as the one indexed $(i, N_{ei}^{k|k}+2, l)$, i.e., that $\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu) = \mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2}(\nu)$ for each $1 \le l \le N_{ei}^{k|k}+1$.

Starting with $\mathcal{E}_{i,l}^{k+2|k+1}(\nu)$ of (34), while following the result in (27), the exponential of the second grandchild term generated after a measurement update at $k+2$ by the new child term (i,l) at $k+1$ is given by

$$\begin{aligned}
&\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu)\\
&= \exp\left(-\frac{p_{i,l}^{k+1|k+1}}{\left|a_{i,l}^{k+1|k}\Phi^T H^T\right|}\left|H\Phi\left({a_{i,l}^{k+1|k}}^T H\mathbf{A} - \mathbf{A}^T H^T a_{i,l}^{k+1|k}\right)\Phi^T\nu\right|\right.\\
&\quad - \frac{\beta}{\left|a_{i,l}^{k+1|k}\Phi^T H^T\right|}\left|H\left(\Phi {a_{i,l}^{k+1|k}}^T\Gamma^T - \Gamma a_{i,l}^{k+1|k}\Phi^T\right)\nu\right| - \gamma\left|\frac{a_{i,l}^{k+1|k}\Phi^T\nu}{a_{i,l}^{k+1|k}\Phi^T H^T}\right|\\
&\quad \left. + j\frac{z_{k+2}a_{i,l}^{k+1|k}\Phi^T + H\Phi\left({a_{i,l}^{k+1|k}}^T b_{i,l}^{k+1|k+1} - {b_{i,l}^{k+1|k+1}}^T a_{i,l}^{k+1|k}\right)\Phi^T}{a_{i,l}^{k+1|k}\Phi^T H^T}\nu\right).
\end{aligned} \tag{41}$$

We note that the term in the first line of (41) can be simplified by using the result in (36). Associating $c_1 = a_{i,l}^{k+1|k}\Phi^T$ and $c_2 = \Gamma^T$ in (22), the ν-dependant term in the second line of (41) can be simplified as

$$\left|H\left(\Phi {a_{i,l}^{k+1|k}}^T\Gamma^T - \Gamma a_{i,l}^{k+1|k}\Phi^T\right)\nu\right| = \left|a_{i,l}^{k+1|k}\Phi^T\Gamma\right|\left|H\mathbf{A}\nu\right|. \tag{42}$$

The second term in the imaginary part of (41) is simplified using (19), (21)–(23), and (30) as

$$\begin{aligned}
&H\Phi\left({a_{i,l}^{k+1|k}}^T b_{i,l}^{k+1|k+1} - {b_{i,l}^{k+1|k+1}}^T a_{i,l}^{k+1|k}\right)\Phi^T\\
&\quad = \det(\Phi)\left(a_{i,l}^{k+1|k}\mathbf{A}{b_{i,l}^{k+1|k+1}}^T\right)H\mathbf{A}\\
&\quad = \det(\Phi)\left(\frac{z_{k+1}a_{i,l}^{k+1|k} + \left(a_{i,l}^{k+1|k}\mathbf{A}{b_i^{k+1|k}}^T\right)H\mathbf{A}}{a_{i,l}^{k+1|k}H^T}\mathbf{A}^T{a_{i,l}^{k+1|k}}^T\right)H\mathbf{A}\\
&\quad = \det(\Phi)\left(a_{i,l}^{k+1|k}\mathbf{A}{b_i^{k+1|k}}^T\right)H\mathbf{A}.
\end{aligned} \tag{43}$$

Using (36), (42), and (43), while combining the first two elements in (41), the latter can be simplified as

$$\text{(44)}\quad \mathcal{E}_{i,l,2}^{k+2|k+2}(\nu) = \exp\left(-\frac{p_{i,l}^{k+1|k+1}\left|\det(\Phi)\right|\left|a_{i,l}^{k+1|k}H^T\right| + \beta\left|a_{i,l}^{k+1|k}\Phi^T\Gamma\right|}{\left|a_{i,l}^{k+1|k}\Phi^T H^T\right|}\left|H\mathbf{A}\nu\right| - \gamma\left|\frac{a_{i,l}^{k+1|k}\Phi^T\nu}{a_{i,l}^{k+1|k}\Phi^T H^T}\right| + j\frac{z_{k+2}a_{i,l}^{k+1|k}\Phi^T + \det(\Phi)\left(a_{i,l}^{k+1|k}\mathbf{A}b_i^{k+1|k^T}\right)H\mathbf{A}}{a_{i,l}^{k+1|k}\Phi^T H^T}\nu\right).$$

To determine $\mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2}(\nu)$, we start with $\mathcal{E}_{i,N_{ei}^{k|k}+2}^{k+1|k+1}(\nu)$ given in (28). Using (8) it is time propagated to $k+2$ as

$$\text{(45)}\quad \mathcal{E}_{i,N_{ei}^{k|k}+2}^{k+2|k+1}(\nu) = \exp\left(-\sum_{m=1}^{N_{ei}^{k|k}+1} p_{i,m}^{k+1|k}\left|a_{i,m}^{k+1|k}\Phi^T\nu\right| - \beta\left|\Gamma^T\nu\right| + jb_i^{k+1|k}\Phi^T\nu\right).$$

Similarly to the expression in (27), the exponential of the lth grandchild generated after a measurement update at $k+2$ by the above child term is given by

$$\text{(46)}\quad \begin{aligned}\mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2}(\nu) = \exp\Bigg(&-\sum_{\substack{m=1\\m\neq l}}^{N_{ei}^{k|k}+1}\frac{p_{i,m}^{k+1|k}}{\left|a_{i,l}^{k+1|k}\Phi^T H^T\right|}\left|H\Phi\left(a_{i,l}^{k+1|k^T}a_{i,m}^{k+1|k} - a_{i,m}^{k+1|k^T}a_{i,l}^{k+1|k}\right)\Phi^T\nu\right|\\ &- \frac{\beta}{\left|a_{i,l}^{k+1|k}\Phi^T H^T\right|}\left|H\left(\Phi a_{i,l}^{k+1|k^T}\Gamma^T - \Gamma a_{i,l}^{k+1|k}\Phi^T\right)\nu\right| - \gamma\left|\frac{a_{i,l}^{k+1|k}\Phi^T\nu}{a_{i,l}^{k+1|k}\Phi^T H^T}\right|\\ &+ j\frac{z_{k+2}a_{i,l}^{k+1|k}\Phi^T + H\Phi\left(a_{i,l}^{k+1|k^T}b_i^{k+1|k} - b_i^{k+1|k^T}a_{i,l}^{k+1|k}\right)\Phi^T}{a_{i,l}^{k+1|k}\Phi^T H^T}\nu\Bigg)\end{aligned}$$

for $1 \le l \le N_{ei}^{k|k}+1$. Using (22) and (23), the ν-dependant term in the first line of (46) can be simplified as

$$\text{(47)}\quad \left|H\Phi\left(a_{i,l}^{k+1|k^T}a_{i,m}^{k+1|k} - a_{i,m}^{k+1|k^T}a_{i,l}^{k+1|k}\right)\Phi^T\nu\right| = \left|\det(\Phi)\right|\left|a_{i,l}^{k+1|k}\mathbf{A}a_{i,m}^{k+1|k^T}\right|\left|H\mathbf{A}\nu\right|.$$

Using this result and the definition in (33), the sum in the first line of (46) can be manipulated as

$$
\begin{aligned}
&\sum_{\substack{m=1\\ m\neq l}}^{N_{ei}^{k|k}+1} p_{i,m}^{k+1|k}\left|H\Phi\left(a_{i,l}^{k+1|k^T}a_{i,m}^{k+1|k}-a_{i,m}^{k+1|k^T}a_{i,l}^{k+1|k}\right)\Phi^T\nu\right|\\
&=|\det(\Phi)|\left|a_{i,l}^{k+1|k}H^T\right|\left(\sum_{\substack{m=1\\ m\neq l}}^{N_{ei}^{k|k}+1}\frac{p_{i,m}^{k+1|k}\left|a_{i,l}^{k+1|k}\mathbf{A}a_{i,m}^{k+1|k^T}\right|}{\left|a_{i,l}^{k+1|k}H^T\right|}\right)|H\mathbf{A}\nu|\\
&=p_{i,l}^{k+1|k+1}|\det(\Phi)|\left|a_{i,l}^{k+1|k}H^T\right||H\mathbf{A}\nu|.
\end{aligned}\tag{48}
$$

The first term in the second line of (46) is identical to the one in the second line of (41), and was already simplified in (42). Similarly to the manipulation in (47), the second term in the imaginary part of (46) is expressed as

$$
H\Phi\left(a_{i,l}^{k+1|k^T}b_i^{k+1|k}-b_i^{k+1|k^T}a_{i,l}^{k+1|k}\right)\Phi^T=\det(\Phi)\left(a_{i,l}^{k+1|k}\mathbf{A}b_i^{k+1|k^T}\right)H\mathbf{A}.\tag{49}
$$

Using (42), (48), and (49), while combining the elements with the common $|H\mathbf{A}\nu|$ expression, (46) is simplified as

$$
\begin{aligned}
\mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2}(\nu)=\exp\Bigg(&-\frac{p_{i,l}^{k+1|k+1}|\det(\Phi)|\left|a_{i,l}^{k+1|k}H^T\right|+\beta\left|a_{i,l}^{k+1|k}\Phi^T\Gamma\right|}{\left|a_{i,l}^{k+1|k}\Phi^TH^T\right|}|H\mathbf{A}\nu|\\
&-\gamma\left|\frac{a_{i,l}^{k+1|k}\Phi^T\nu}{a_{i,l}^{k+1|k}\Phi^TH^T}\right|\\
&+j\frac{z_{k+2}a_{i,l}^{k+1|k}\Phi^T+\det(\Phi)\left(a_{i,l}^{k+1|k}\mathbf{A}b_i^{k+1|k^T}\right)H\mathbf{A}}{a_{i,l}^{k+1|k}\Phi^TH^T}\nu\Bigg).
\end{aligned}\tag{50}
$$

Equations (44) and (50) clearly indicate that the exponentials of the two respective grandchild terms are identical, thus establishing the second term combination rule

$$
\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu)=\mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2}(\nu)\quad\text{for each}\quad 1\le l\le N_{ei}^{k|k}+1.\tag{51}
$$

3.3. Noncombining terms. Terms that are not covered by the two combination rules discussed above are the third grandchild term at step $k+2$ from the new child terms at step $k+1$, the $(N_{ei}^{k|k}+2)^{nd}$ grandchild term at step $k+2$ from the old child term at step $k+1$, and all the old grandchild terms at step $k+2$. These terms are shown to be distinct from each other and thus generically could not be combined.

First, we consider the third grandchild term at step $k+2$ from the new child terms at step $k+1$, $\mathcal{E}_{i,l,3}^{k+2|k+2}$, all of which have two elements. Since the new child terms are expressed by (34), then their third grandchild term generated after an update at

$k+2$ is

$$(52)\quad \mathcal{E}_{i,l,3}^{k+2|k+2}(\nu) = \exp\left(-\frac{p_{i,l}^{k+1|k+1}\left|\Gamma^T \mathbf{A}\Phi^T \mathbf{A} H^T\right|}{\left|\Gamma^T H^T\right|}\left|H\mathbf{A}\nu\right| - \frac{\gamma}{\left|\Gamma^T H^T\right|}\left|\Gamma^T\nu\right| + j\left(z_{k+2}\Gamma^T\nu + \frac{b_{i,l}^{k+1|k+1}\Phi^T\Gamma}{H\Gamma}H\mathbf{A}\nu\right)\right)$$

for $1 \geq l \geq N_{ei}^{k|k}$. Therefore, the third grandchild terms at step $k+2$ do not combine generically, since the parameters $p_{i,l}^{k+1|k+1}$ are analytically different for each l.

Similarly, the two-element $(N_{ei}^{k|k}+2)^{nd}$ grandchild terms at step $k+2$ generated by old child terms at step $k+1$ have the form

$$(53)\quad \mathcal{E}_{i,N_{ei}^{k|k}+2,N_{ei}^{k|k}+2}^{k+2|k+2}(\nu) = \exp\left(-p_{i,N_{ei}^{k|k}+2}^{k+1|k+1}\left|H\mathbf{A}\nu\right| - \frac{\gamma}{\left|\Gamma^T H^T\right|}\left|\Gamma^T\nu\right| + j\left(z_{k+2}\Gamma^T\nu + \frac{b_{i,N_{ei}^{k|k}+2}^{k+1|k+1}\Phi^T\Gamma}{H\Gamma}H\mathbf{A}\nu\right)\right),$$

where $p_{i,N_{ei}^{k|k}+2}^{k+1|k+1}$ is defined in (33) for $l = N_{ei}^{k|k}+2$. Generically, this form does not combine with any other two-element terms derived within the context of the two term combination rules discussed earlier or the third grandchild terms in (52).

Due to the second condition of Theorem 2.3, old terms have between three elements, if they are the child of a two element term, to $k+2$ if they are generated from the initial old term. In addressing the old grandchild terms at step $k+2$ we assume that all the identical child terms at $k+1$ were already combined, i.e., all the terms at $k+1$ have distinct fundamental directions. When time propagated to time $k+2$ those directions are rotated by the nonsingular transition matrix Φ^T and Γ is introduced as an additional direction to each term. Hence, also after time propagation all terms are distinct. After a measurement updated at time $k+2$ the old grandchild terms have the same fundamental directions as the terms after time propagation. Since the latter are distinct, so are the old grandchild terms.

The analysis above verifies that generically, for the two-state systems addressed in this study, terms can be combined only using the two combination rules presented earlier in this section. This can be summarized in the following theorem.

THEOREM 3.1. *For two-state systems, suppose that at step k the CF of the ucpdf is given by* (12). *Then, at $k+2$, terms combine based on only one of the following two rules:*

(54)
$$\mathcal{E}_{i,l,1}^{k+2|k+2}(\nu) = \mathcal{E}_{p,q,1}^{k+2|k+2}(\nu) \quad \forall \quad (i,p) \in \left[1,\dots,N_t^{k|k}\right], \quad l \in \left[1,\dots,N_{ei}^{k|k}+1\right],\ q \in \left[1,\dots,N_{ep}^{k|k}+1\right],$$

(55)
$$\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu) = \mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2}(\nu) \quad \textit{for each}\ \ i \in \left[1,\dots,N_t^{k|k}\right], l \in \left[1,\dots,N_{ei}^{k|k}+1\right].$$

Remark 3.2. Theorem 3.1 was derived by examining only grandchild terms of parents at step k, disregarding possible term combinations of child terms at $k+1$. These child terms may combine due to the combination rules of Theorem 3.1 when applied to the grandparent terms at time $k-1$. If those child terms were not combined, they would have produced additional grandchild terms that would combine through Theorem 3.1 at $k+2$ when considering their respective parents. A scheme that tracks the locations of all terms that combine at $k+2$ and accounts for term combinations at $k+1$ is presented in sections 4 and 5.

Remark 3.3. It should be noted that the combination rules of Theorem 3.1 imply that only new, two-element terms can be combined. This will be utilized in the next subsection that addresses the addition of the coefficients $G_i^{k|k}(\nu)$ of the terms with identical exponentials.

Remark 3.4. Following the conclusions of Theorem 2.3 we note that $\mathcal{E}_{i,l}^{k+1|k+1}(\nu)$, $1 \le l \le N_{ei}^{k|k}+1$, correspond to new child terms at $k+1$. Hence the exponentials $\mathcal{E}_{i,l,2}^{k+2|k+2}(\nu)$ on the left-hand side of (55) correspond to the second grandchild terms generated by new child terms at $k+1$. Similarly, $\mathcal{E}_{i,N_{ei}^{k|k}+2}^{k+1|k+1}(\nu)$ correspond to old child terms at $k+1$. Consequently, $\mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2}(\nu), 1 \le l \le N_{ei}^{k|k}+1$, on the right-hand side of (55) correspond to new grandchild terms generated at $k+2$ from an old parent at $k+1$. Therefore, the combination rule in (55) involves pairs of grandchild terms, one being the second grandchild of a new child term and the other being a new grandchild term generated by an old parent. Moreover, it implies that all combining new grandchild terms generated from old child terms combine with one of the second grandchild terms of a new child term.

3.4. Combining the terms. Once the terms with identical exponential functions are identified through Theorem 3.1, those terms can be combined by adding their respective coefficients. In other words, (12) can be rewritten as

$$\bar{\phi}_{X_k|Y_k}(\nu) = \sum_{i=1}^{\tilde{N}_t^{k|k}} \left[\left(\sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} G_{i,l}^{k|k}(\nu) \right) \mathcal{E}_i^{k|k}(\nu) \right] = \sum_{i=1}^{\tilde{N}_t^{k|k}} G_i^{k|k}(\nu)\mathcal{E}_i^{k|k}(\nu), \tag{56}$$

where $\tilde{N}_t^{k|k}$ is the number of distinct exponential terms, and $\tilde{N}_{t,i}^{k|k}$ is the number of coefficient terms associated with each distinct exponential $\mathcal{E}_i^{k|k}(\nu)$. As noted in Remark 3.3, the two combination rules only involve terms with two elements in the argument of the exponential. It was shown in [2] that the coefficients $G_{i,l}^{k|k}(\nu)$ of those two element terms have a simple form of a weighted sum of sign functions, i.e.,

$$\begin{aligned} G_{i,l}^{k|k}(\nu) = A_{i,l}^{k|k} &+ B_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,l,1}^{k|k}\nu\right) \operatorname{sgn}\left(a_{i,l,2}^{k|k}\nu\right) \\ &+ jC_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,l,1}^{k|k}\nu\right) + jD_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,l,2}^{k|k}\nu\right), \end{aligned} \tag{57}$$

where $a_{i,l,1}^{k|k}$ and $a_{i,l,2}^{k|k}$ are the two directions in the exponent of those terms, and $A_{i,l}^{k|k}$, $B_{i,l}^{k|k}$, $C_{i,l}^{k|k}$, $D_{i,l}^{k|k}$ are functions of the system parameters and the measurements y_k. For terms with identical exponentials, the directions in all these terms are identical with those in the coefficient functions, i.e., $a_{i,l,1}^{k|k} = a_{i,1}^{k|k}$ and $a_{i,l,2}^{k|k} = a_{i,2}^{k|k}$ for all

$1 \leq l \leq \tilde{N}_{t,i}^{k|k}$. Therefore, when combining terms, the resulting coefficient function can be obtained simply as

$$
\begin{aligned}
G_i^{k|k}(\nu) &= \sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} G_{i,l}^{k|k}(\nu) \\
&= \sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} \Big[A_{i,l}^{k|k} + B_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,1}^{k|k}\nu\right) \operatorname{sgn}\left(a_{i,2}^{k|k}\nu\right) \\
&\qquad + jC_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,1}^{k|k}\nu\right) + jD_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,2}^{k|k}\nu\right) \Big] \\
&= \sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} A_{i,l}^{k|k} + \sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} B_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,1}^{k|k}\nu\right) \operatorname{sgn}\left(a_{i,2}^{k|k}\nu\right) \\
&\quad + j \sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} C_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,1}^{k|k}\nu\right) + j \sum_{l=1}^{\tilde{N}_{t,i}^{k|k}} D_{i,l}^{k|k} \operatorname{sgn}\left(a_{i,2}^{k|k}\nu\right) \\
&= A_i^{k|k} + B_i^{k|k} \operatorname{sgn}\left(a_{i,1}^{k|k}\nu\right) \operatorname{sgn}\left(a_{i,2}^{k|k}\nu\right) + jC_i^{k|k} \operatorname{sgn}\left(a_{i,1}^{k|k}\nu\right) \\
&\quad + jD_i^{k|k} \operatorname{sgn}\left(a_{i,2}^{k|k}\nu\right),
\end{aligned} \tag{58}
$$

where $A_i^{k|k}$, $B_i^{k|k}$, $C_i^{k|k}$, $D_i^{k|k}$ are sums of the respective parameters as given above.

3.5. Number of terms. In this subsection, the term combination rules derived above are utilized to determine the number of distinct exponential terms at each time step k. It is assumed that at each step terms with identical exponential parts are combined before being propagated to the next step. The total number of distinct terms was denoted earlier by $\tilde{N}_t^{k|k}$. Among these terms we assume that there are $\tilde{N}_{t,new}^{k|k}$ new distinct two-element terms and $\tilde{N}_{t,old}^{k|k}$ old distinct terms with three and more elements, i.e.,

$$
\tilde{N}_t^{k|k} = \tilde{N}_{t,new}^{k|k} + \tilde{N}_{t,old}^{k|k}. \tag{59}
$$

Figure 1 illustrates how a general term with m elements at step k produces child and grandchild terms at steps $k+1$ and $k+2$, respectively. In this figure, the parent index is dropped from the notation for simplicity. According to what was discussed before, at $k+1$ the child terms numbered 1 through $m+1$ are new two-element terms. The $(m+2)^{nd}$ child is an old terms with $m+1$ elements. Each new child term produces four grandchildren at $k+2$. Among them, for each $1 \leq j \leq m+1$, terms indexed $(j,l), l = 1,2,3$, are new two-element terms, and $(j,4)$ is old with three elements. Similarly, the old child at $k+1$ produces $m+3$ grandchild terms: the first $m+2$ of them are new with two elements, and the last one, numbered $(m+2, m+3)$, is old and has $m+2$ elements.

It was shown in subsection 3.3 that each distinct child term at $k+1$ produces one distinct old grandchild term at $k+1$, depicted by a full black circle in Figure 1. Using the notation presented above, this is expressed as

$$
\tilde{N}_{t,old}^{k+2|k+2} = \tilde{N}_t^{k+1|k+1}. \tag{60}
$$

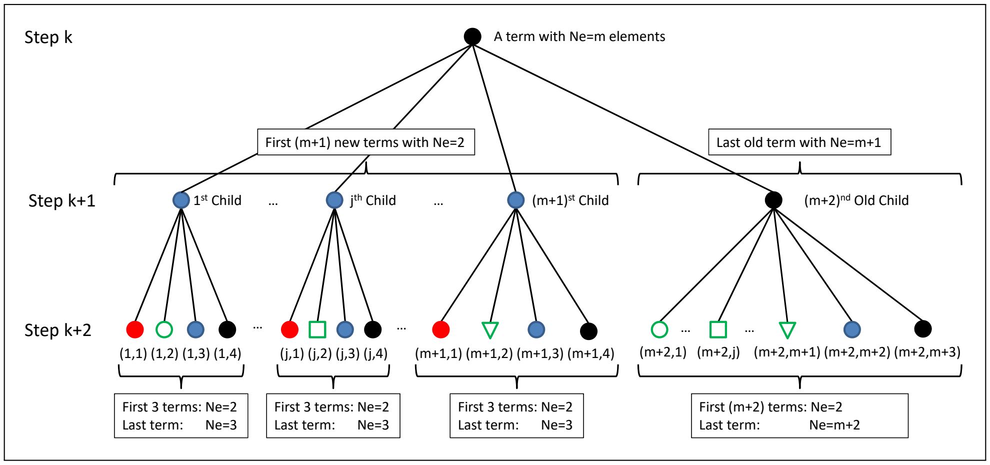

FIG. 1. *Term combination rules for two-state systems.*

Now we examine the number of distinct new terms at $k+2$ determined by the two term combination rules presented in Theorem 3.1. Here we distinguish between three different groups of grandchild terms.

1. The first term combination rule in (54) states that the first grandchild terms of *all* new child terms have the same exponential and hence are combined into one distinct new term at $k+2$. This is depicted by the red circles in Figure 1.
2. The second term combination rule of (55) states that the second grandchild of a new child combines with one of the new grandchild terms of an old child, as marked by various green symbols in Figure 1. This also implies that each one of the first $m+1$ grandchild terms of an old child will combine with one of the second grandchild terms of a new child. Since at time $k+1$ there are $\tilde{N}_{t,new}^{k+1|k+1}$ new child terms, this will also be the number of such combined terms at time $k+2$.
3. In subsection 3.3 it was shown that the one before last grandchild term of each child, i.e., the third grandchild of a new child and $(m+2)$nd grandchild of an old term, depicted in blue in Figure 1, are distinct. Hence, at time $k+2$ there will be $\tilde{N}_t^{k+1|k+1}$ such distinct terms.

Combining the three conclusions above we deduce that the number of new terms at time $k+2$ is given by

$$\tilde{N}_{t,new}^{k+2|k+2} = 1 + \tilde{N}_{t,new}^{k+1|k+1} + \tilde{N}_t^{k+1|k+1}. \tag{61}$$

The total number of distinct terms at $k+2$ is a sum of the new terms in (61) and the old ones give in (60), i.e.,

$$\tilde{N}_t^{k+2|k+2} = \tilde{N}_{t,new}^{k+2|k+2} + \tilde{N}_{t,old}^{k+2|k+2} = 1 + \tilde{N}_{t,new}^{k+1|k+1} + 2\tilde{N}_t^{k+1|k+1}. \tag{62}$$

Equations (61) and (62) can be expressed in a discrete linear system form as

$$\begin{Bmatrix} \tilde{N}_t^{k+2|k+2} \\ \tilde{N}_{t,new}^{k+2|k+2} \end{Bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \tilde{N}_t^{k+1|k+1} \\ \tilde{N}_{t,new}^{k+1|k+1} \end{Bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{63}$$

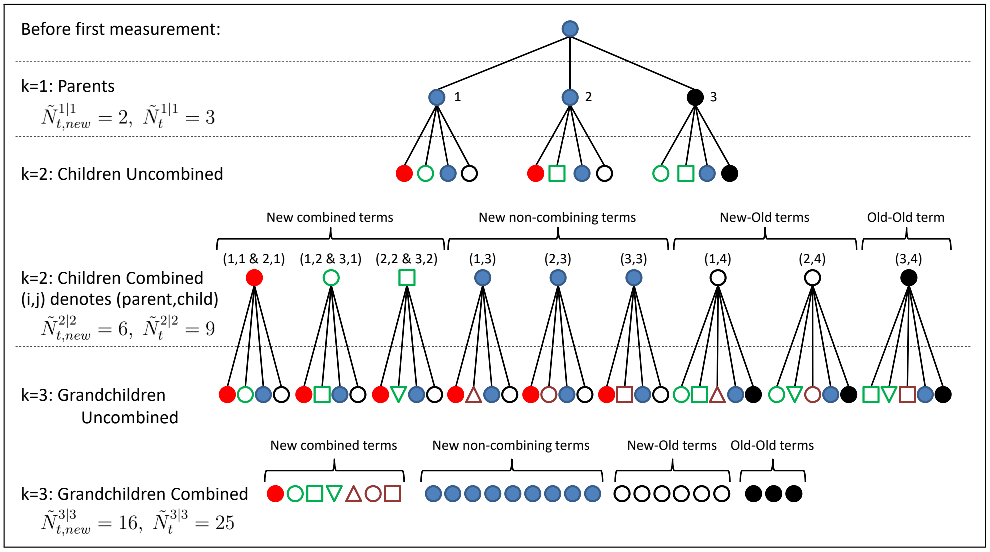

FIG. 2. *Propagation and combination of terms for $k = 1, 2,$ and 3.*

The initialization of (63) depends on when the first measurement update is performed. In this study we assume that the CF is initiated at $k = 1$ according to (2), i.e., with one term that has two elements. Moreover, it is assumed that a measurement update is performed at $k = 1$ using z_1. Following the conclusion of item 1 in Theorem 2.3, the measurement updated CF will have a total of $\tilde N^{1|1}_t = 3$ terms, two of which are new, i.e., $\tilde N^{1|1}_{t,new} = 2$, and one old term. All the terms will have two elements. This is depicted schematically at the top of Figure 2.

The above provides the initial conditions for (63). In addition we note that although its derivation involved three generations of terms, the end result relates the number of terms at a given step based on the information at the previous step. Hence, the total number of terms and the number of new terms can be determined by

$$(64)\qquad \begin{Bmatrix} \tilde N^{k+1|k+1}_t \\ \tilde N^{k+1|k+1}_{t,new} \end{Bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \tilde N^{k|k}_t \\ \tilde N^{k|k}_{t,new} \end{Bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{Bmatrix} \tilde N^{1|1}_t \\ \tilde N^{1|1}_{t,new} \end{Bmatrix} = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}.$$

It provides a compact form to determine the number of distinct terms at every step. A sample case of the first three steps is depicted schematically in Figure 2. It can be easily verified that the number of terms shown in the figure matches the recursion in (64).

Remark 3.5. The results in (64) were already reported in [2], discovered by observing the empirical data. In this paper, it is determined from an analytic viewpoint.

4. Indexing matrix. In this section, an indexing matrix S is introduced to efficiently locate the terms that can be combined using the combination rules of Theorem 3.1. Since these term combination rules are independent of the system parameters, S can be precomputed and reused for any two-state system. It is determined below explicitly for the first four updates, demonstrating its recursive structure. This is then utilized in section 5 where S is constructed for any k.

Before explicitly constructing S, the process of repeatedly propagating and combining terms is illustrated by a tree structure depicted in Figure 2 for the first three updates. Since the combination rules are dictated by the properties of grandchild terms, the derivation here is valid for $k \geq 2$. As discussed in subsection 3.5, after the first measurement update, at $k = 1$ there are two new terms and one old one, i.e., $\tilde{N}_{t,new}^{1|1} = 2$, $\tilde{N}_{t,old}^{1|1} = 1$, $\tilde{N}_t^{1|1} = 3$, all of which have two elements. They produce four child terms each at step $k = 2$, three pairs of which combine. Red filled circles indicate two-element terms that combine due to the first rule. The green circle and square show the two combinations due to the second rule. The third terms, denoted by the blue filled circle, are new terms that do not combine. The black circles indicate the old child terms produced by new parents at $k = 1$. They are termed new-old terms in the figure. Similarly, the filled black circle shows the old term generated by the old parent, called here old-old term. It should be noted that both the new-old and old-old terms have three elements each. After the terms are combined, the remaining nine terms are shown in the different four groups mentioned above in the second line that corresponds to $k = 2$.

The distinct child terms at $k = 2$ produce grandchild terms at $k = 3$: the new two-element child terms produce four grandchild terms, while the old terms with three elements produce five grandchild terms. Here again, terms combine due to the two combination rules of Theorem 3.1: the terms denoted by filled red circles combine into one term due to rule one, while terms marked by the various green and purple symbols combine due to the second rule. It is important to point out that since, e.g., the first new child term at $k = 2$ was obtained after combining terms numbered in Figure 2 as (1,1) and (2,1), the second combination rule must be applied while accounting for the two different parent terms. Consequently, the terms marked with green symbols introduce a combination of three terms. After combining all the terms with identical exponentials, there is a total of 25 distinct terms, among which 16 are new. This is consistent with (64). Note that before combining at $k = 2$ and $k = 3$, the last two child terms of any parent are the only terms that do not combine. This holds for any step k depicted in Figure 1. Specifically, the last two grandchild terms at $k+2$ do not combine. They include the $m + 2$ new two-element noncombining grandchild terms depicted by blue circles, and $m + 2$ old grandchild terms depicted by black dots.

It is evident that the complex structure presented above, especially for $k = 3$, becomes even more involved for $k \geq 4$. The goal of constructing the indexing matrix S is to provide an effective method to identify the terms that can combine. The structure of S related to Figure 2 is presented in the remainder of this section. By considering the first four stages, the construction of S for any stage k becomes clearer and is presented in section 5.

To enhance the efficiency of combining terms, the latter are indexed such that terms with an identical index can be combined. Those indexes will be stored in S. The dimension of S^k, i.e., the matrix S at time step k, is determined by the number of parent terms at $k-1$ and the maximum number of child terms each of those parent terms generate at k. Specifically, we assume that at step $k - 1$ there are $\tilde{N}_t^{k-1|k-1}$ distinct terms in (56). Each term $i \in [1, \dots, \tilde{N}_t^{k-1|k-1}]$ is assumed to have $N_{ei}^{k-1|k-1}$ elements in the argument of its exponent. By Theorem 2.3, all the new terms have two elements in the exponential, while the old terms will have between three and k elements, i.e., $\max_i N_{ei}^{k-1|k-1} = k$. Each term i at $k-1$ produces $N_{ei}^{k-1|k-1} + 2$ child terms at k. Hence, to store the indexes of all those child terms, S^k is constructed to have $\tilde{N}_t^{k-1|k-1}$ rows and $\max_i(N_{ei}^{k-1|k-1} + 2) = k+2$ columns. The ith row in S^k will carry the indexes of the child terms produced by the ith parent term. For example, $S_{i,j}^k$, the (i, j) element of S^k stores the index of the jth child produced by the parent term i. If a parent has fewer than $k+2$ child terms, a zero value will be inserted in the respective location of S^k. If, by the term-indexing scheme presented below, $S^k_{i,j} = S^k_{p,q}$, the exponents $\mathcal{E}^{k|k}_{i,j}(\nu)$ and $\mathcal{E}^{k|k}_{p,q}(\nu)$ are equal and the respective terms can be combined.

The structure of $S^k, k = 1, 2, 3$, is illustrated in the following by using Figure 2. At $k = 1$, there are three terms after the first measurement update. The terms are all distinct and do not combine. Assigning $\tilde{N}_t^{k-1|k-1}$ at $k = 1$ the value of one to account for the fact that there was one parent that generated the measurement updated CF at $k = 1$, the corresponding indexing matrix is given by

$$S^1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}. \tag{65}$$

Each of the three, two-element terms in the CF at $k = 1$ will generate four child terms at $k = 2$. Hence the dimension of S^2 will be 3×4. Its entries will be determined by applying the term combination rules of Theorem 3.1 after a measurement update at $k = 2$. According to the first rule of (54), the first child terms of the first and second new parent terms should combine, i.e., $\mathcal{E}^{2|2}_{1,1}(\nu) = \mathcal{E}^{2|2}_{2,1}(\nu)$, as depicted by the red dots at $k = 2$ in Figure 2. Those terms will be indexed by 1, and thus the corresponding entries will be $S^2_{1,1} = S^2_{2,1} = 1$. Due to the second rule of (55), $\mathcal{E}^{2|2}_{1,2}(\nu) = \mathcal{E}^{2|2}_{3,1}(\nu)$ and $\mathcal{E}^{2|2}_{2,2}(\nu) = \mathcal{E}^{2|2}_{3,2}(\nu)$, i.e., in Figure 2, the green circles and green squares at $k = 2$, respectively. The first pair in the above (green circles) will be indexed as 2 and thus $S^2_{1,2} = S^2_{3,1} = 2$; similarly, the second pair (green squares) will be indexed by 3, i.e., $S^2_{2,2} = S^2_{3,2} = 3$. The rest of the terms at $k = 2$ do not combine and hence are numbered sequentially. Consequently,

$$S^2 = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 1 & 3 & 5 & 8 \\ 2 & 3 & 6 & 9 \end{bmatrix}. \tag{66}$$

It is important to point out that the indexes in the third column of S^2 represent the new, two-element noncombining terms, while the terms indexed in the fourth column are old and have three elements each.

The structure of S^3 is determined by the parents at $k = 2$ and the term combination rules. Here we note that some of those parents were obtained by combining pairs, as indicated in the previous paragraph and depicted in Figure 2 in the form $(i, j$ and $l, m)$ for the terms that were combined. This, in turn, will affect the terms that will combine at $k = 3$ due to the second rule of (55). Using a similar logic to that presented earlier, as will be explained next, S^3 is expressed as

$$S^3 = \begin{bmatrix} 1 & 2 & 8 & 17 & 0 \\ 1 & 3 & 9 & 18 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 7 & 13 & 22 & 0 \\ 2 & 3 & 5 & 14 & 23 \\ 2 & 4 & 6 & 15 & 24 \\ 3 & 4 & 7 & 16 & 25 \end{bmatrix}. \tag{67}$$

Due to the first combination rule of (54), the first grandchild terms of the six new child terms at $k = 2$ have identical exponentials, depicted by red dots in Figure 2. They are indexed by 1, i.e., $S^3_{i,1} = 1, i = 1, \ldots, 6$. Next, the second term combination rule of (55) is applied six times, each involving the second grandchild of the new child terms at $k = 2$. Since the first three child terms were obtained by combining pairs at $k = 2$, (55) will yield three combining grandchild triplets, indexed 2, 3, and 4. Specifically, the sets and their description in Figure 2 are $S^3_{1,2} = S^3_{7,1} = S^3_{8,1} = 2$ (green circles); $S^3_{2,2} = S^3_{9,1} = S^3_{7,2} = 3$ (green squares); and $S^3_{3,2} = S^3_{8,2} = S^3_{9,2} = 4$ (green triangles). Applying (55) to the second grandchild terms of the child terms 4, 5, and 6 results in three combining pairs, indexed and depicted in Figure 2 as follows: $S^3_{4,2} = S^3_{7,3} = 5$ (purple triangles); $S^3_{5,2} = S^3_{8,3} = 6$ (purple circles); and $S^3_{6,2} = S^3_{9,3} = 7$ (purple squares). The remaining grandchild terms, depicted by blue dots, black circles, and black dots in Figure 2, do not combine and are indexed sequentially: the nine two-element new terms (blue dots) are indexed 8 through 16; the six three-element old terms (black circles) are 17 through 22; and the three four-element old terms are 23, 24, and 25. Their location in S^3 of (67) can be matched with the lineage depicted in Figure 2. The zeros in the entries $S^3_{i,5}, i = 1, \ldots, 6$ indicate that the new child terms have only four grandchild terms, and no fifth one, as do the old child terms.

Combining all the relevant grandchild terms as shown at the bottom of Figure 2 and then generating new children at $k = 4$ will produce a complex diagram. The matrix S is used as a straightforward pointing process that summarizes these diagrams. In section 5, using the two combination rules, S^k is constructed for any measurement update k. In particular, at $k = 4$, where $\tilde{N}^{4|4}_{t,new} = 42$ and $\tilde{N}^{4|4}_{t} = 67$, S^4 is given by

$$S^4 = \begin{bmatrix} 1 & 2 & 18 & 43 & 0 & 0 \\ 1 & 3 & 19 & 44 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 17 & 33 & 58 & 0 & 0 \\ 2 & 3 & 9 & 34 & 59 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 8 & 14 & 39 & 64 & 0 \\ 3 & 4 & 6 & 15 & 40 & 65 \\ 3 & 5 & 7 & 16 & 41 & 66 \\ 4 & 5 & 8 & 17 & 42 & 67 \end{bmatrix}. \tag{68}$$

There are many similarities between S^4 and S^3 of (67). In particular, we note the zeros that first appear in the upper part of the right column in (67) and also in the upper parts of the two right columns of (68), which do not indicate a term but squares the matrix. Referring to the bottom of Figure 2, the first $\tilde{N}^{3|3}_{t,new} = 16$ new grandchild terms have only two elements and thus produce only four great-grandchild terms. Therefore, in (68), the fifth and six columns will be zeros through the top 16 rows. Next, note in Figure 2 that at $k = 2$ the $\tilde{N}^{2|2}_{t,new} = 6$ new-old child terms have three elements and thus produce five great-grandchild terms at $k = 4$. Therefore, in (68), the sixth column will be zeros through the top $\tilde{N}^{3|3}_{t} = 25$ rows. Finally, in Figure 2, the three old-old child terms at $k = 2$ have four elements and thus produce six great-grandchild terms at $k = 4$. This is represented by the full nonzero bottom three rows in (68). In the next section, the staircase structure of zeros in the upper right corner of the S^k will be revealed for any time step $k \geq 3$.

5. General structure of the indexing matrix. In this section, the indexing matrix S introduced in the previous section is constructed recursively for any time step

$k+2$, where the number of rows in S^{k+2} equals the number of parent terms at $k+1$, i.e., $\tilde{N}_t^{k+1|k+1}$, and the number of columns is the maximal number of grandchild terms those parent terms generate, i.e., $k+4$. The indexing matrix is constructed recursively by following the term combination rules of Theorem 3.1. This recursion contributes to the computational efficiency of the two-state Cauchy estimator, since it provides a priori information about which terms in the CF combine without the need of numerical comparison. We begin by forming the general pattern of the indexing matrix by decomposing S^{k+2} according to all of the previous generations that formed it.

5.1. Block structure of S^{k+2}. In the indexing matrix, we group grandchild terms that were generated by child terms with the same number of elements in blocks. In particular, the $\tilde{N}_{t,new}^{k+1|k+1}$ new two-element child terms at $k+1$ generate four grandchild terms at $k+2$. The indices of those grandchild terms are placed in the first $\tilde{N}_{t,new}^{k+1|k+1}$ rows and first four columns of S^{k+2}. Consequently, S^{k+2} will have an upper nonzero matrix block of dimension $\tilde{N}_{t,new}^{k+1|k+1} \times 4$, while the remaining columns in the corresponding rows will be filled with zeros to indicate no additional grandchild terms.

The other blocks in S^{k+2} will include indexes of the grandchild terms at $k+2$ generated by old child terms at $k+1$. Based on the conclusions drawn in section 3, at $k+1$ there will be k groups of old child terms with an equal number r of elements, where $r \in [3, \dots, k+2]$. Hence, for example, assuming that at $k+1$ there are $\tilde{N}_{t,old,r}^{k+1|k+1}$ old child terms with r elements, each of them will generate $r+2$ grandchild terms at $k+2$. Hence the corresponding $\tilde{N}_{t,old,r}^{k+1|k+1}$ rows in S^{k+2} will have $r+2$ nonzero columns, i.e., $\tilde{N}_{t,old,r}^{k+1|k+1} \times (r+2)$ nonzero subblock, while the remaining columns will be filled with zeros. In particular, the second block in S^{k+2}, populated by grandchild terms generated by three-element child terms at $k+1$, will have an $\tilde{N}_{t,old,3}^{k+1|k+1} \times 5$ nonzero subblock of 5 columns, and $k-1$ columns of zeros. The last block in S^{k+2} will be a fully populated block of dimension $\tilde{N}_{t,old,k+2}^{k+1|k+1} \times (k+4)$.

Consequently, S^{k+2} has the following staircase structure:

$$S^{k+2} = \begin{bmatrix} S^{k+2}_{\tilde{N}_{t,new}^{k+1|k+1} \times 4} & 0_{\tilde{N}_{t,new}^{k+1|k+1} \times (k)} \\ S^{k+2}_{\tilde{N}_{t,old,3}^{k+1|k+1} \times 5} & 0_{\tilde{N}_{t,old,3}^{k+1|k+1} \times (k-1)} \\ S^{k+2}_{\tilde{N}_{t,old,4}^{k+1|k+1} \times 6} & 0_{\tilde{N}_{t,old,4}^{k+1|k+1} \times (k-2)} \\ \vdots & \vdots \\ S^{k+2}_{\tilde{N}_{t,old,k+1}^{k+1|k+1} \times (k+3)} & 0_{\tilde{N}_{t,old,k+1}^{k+1|k+1} \times 1} \\ \multicolumn{2}{c}{S^{k+2}_{\tilde{N}_{t,old,k+2}^{k+1|k+1} \times (k+4)}} \end{bmatrix}, \tag{69}$$

where $S^{k+2}_{n \times m} \in \mathbb{R}^{n \times m}$ indicate nonzero subblocks, while $0_{p \times q}$ indicate subblocks of zero entries. Not surprisingly, for $k = 3, 4$, S^3 and S^4 given by (67) and (68), respectively, fit the structure of (69).

Using the results obtained in section 3 regarding the number of different terms, the number $\tilde{N}_{t,old,r}^{k+1|k+1}$ old child terms with r elements at $k+1$ can be determined analytically using a simple recursion. First we note that only the $\tilde{N}_{t,new}^{k|k}$ new two-element parent terms at k will generate $\tilde{N}_{t,new}^{k|k} = \tilde{N}_{t,old,3}^{k+1|k+1}$ three-element old child terms. Similarly, the number of four-element old child terms at $k+1$ can be determined by

$$\tilde{N}_{t,old,4}^{k+1|k+1} = \tilde{N}_{t,old,3}^{k|k} = \tilde{N}_{t,new}^{k-1|k-1}. \tag{70}$$

The recurrent relation in (70) can be restated for every $r \in [3, \ldots, k+2]$ as

$$\tilde{N}_{t,old,r}^{k+1|k+1} = \tilde{N}_{t,old,r-1}^{k|k} = \cdots = \tilde{N}_{t,old,3}^{k+4-r|k+4-r} = \tilde{N}_{t,new}^{k+3-r|k+3-r}. \tag{71}$$

Note that at $k = 1$, after the measurement update, there are three terms with two elements each, regardless if they are new or old. Therefore, there will be $\tilde{N}_{t,old,3}^{2|2} = \tilde{N}_t^{1|1} = 3$ old three-element terms at $k = 2$. Similarly, using (71), for any $k+1$, the number of child terms with $r = k+2$ elements will be

$$\tilde{N}_{t,old,k+2}^{k+1|k+1} = \tilde{N}_{t,new}^{1|1} = 3. \tag{72}$$

This means that the last full subblock in (69) will have three rows for all k. The number of rows in the subblocks $S^{k+2}_{\tilde{N}_{t,old,r}^{k+1|k+1} \times (r+2)}$ for $r \in [3, \ldots, k+1]$, i.e., $\tilde{N}_{t,old,r}^{k+1|k+1} = \tilde{N}_{t,new}^{k+3-r|k+3-r}$, can be easily determined by using (64).

5.2. Recursive structure of S^{k+2}. A recursion for constructing S^{k+2} from the indices of S^{k+1} is developed in this subsection. The process begins with the first term combination rule of (54), which states that at $k+2$ the first child terms generated by the $\tilde{N}_{t,new}^{k+1|k+1}$ new parent terms at step $k+1$ will combine. Those child terms will be indexed by ones. Hence, the first column in the top $\tilde{N}_{t,new}^{k+1|k+1}$ rows of S^{k+2} will be all ones. This is equivalent to stating that the first column of $S^{k+2}_{\tilde{N}_{t,new}^{k+1|k+1} \times 4}$, the left top block of S^{k+2} in (69), is populated by ones. For example, at step $k = 2$, the number of new terms is $\tilde{N}_{t,new}^{2|2} = 6$ and thus there are ones in the first column of the top six rows of S^3 in (67).

Next, we consider the second column of the top $\tilde{N}_{t,new}^{k+1|k+1}$ rows of S^{k+2}. The second combination rule of (55) implies that none of the second child terms from new parent terms will combine with each other, but with the children of old parent terms. Therefore, we assign the indices 2 through $\tilde{N}_{t,new}^{k+1|k+1} + 1$ in the second column of the first $\tilde{N}_{t,new}^{k+1|k+1}$ rows of S^{k+2}. For example, the second column in the $\tilde{N}_{t,new}^{2|2} = 6$ top rows of S^3 in (67) is populated by 2 through 7. These terms combine with children of old parent terms.

To proceed with deriving the recursive procedure for constructing the S matrix, we need the following lemma which relates terms that combined at $k+1$ to old terms that combine at $k+2$ and follows directly from results derived already in sections 2 and 3.

Lemma 5.1. *If, at step $k+1$, the new lth child term of the ith parent and the new pth child term of the mth parent combine, then, at step $k+2$, the new lth grandchild term of the old child term indexed $(i, N_{ei}^{k|k}+2)$ will combine with the new pth grandchild term of the old child term indexed $(m, N_{em}^{k|k}+2)$, i.e.,*

$$\mathcal{E}_{i,l}^{k+1|k+1} = \mathcal{E}_{m,p}^{k+1|k+1} \quad \Rightarrow \quad \mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2} = \mathcal{E}_{m,N_{em}^{k|k}+2,p}^{k+2|k+2}. \tag{73}$$

Proof. If $\mathcal{E}_{i,l}^{k+1|k+1}(\nu) = \mathcal{E}_{m,p}^{k+1|k+1}(\nu)$, then their coefficients $p_{i,l}^{k+1|k+1} = p_{m,p}^{k+1|k+1}$ and directions $a_{i,l}^{k+1|k} = a_{m,p}^{k+1|k}$ in (32) as defined in (33), where $p_{m,p}^{k+1|k+1}$ and $a_{m,p}^{k+1|k}$ are defined by an appropriate change of indices. Then, the coefficients and directions in $\mathcal{E}_{i,N_{ei}^{k|k}+2,l}^{k+2|k+2}(\nu)$ and $\mathcal{E}_{m,N_{em}^{k|k}+2,p}^{k+2|k+2}(\nu)$ are equal by inspection of (50), where the index variables of these coefficients are appropriately changed. ∎

Lemma 5.1 indicates that for all the new child terms that combine at step $k+1$, the corresponding new grandchild terms at step $k+2$ that are offsprings of old child terms will also combine. This is a central result that yields the recursive structure of the S^{k+2}. This recursion is derived by noting that the pattern of child terms that combine at $k+1$ repeats for grandchild terms of old child terms at $k+2$ as indicated in Figure 2. The pattern of terms that combine at $k=2$ is the same pattern for the three old terms at $k=3$, where the first two children at $k=2$ combine exactly as the first two children of old terms combine at $k=3$.

This pattern at $k+1$ is represented by a subarray of indices in S^{k+1} that covers all the new child terms at this time step and repeats as a subarray in S^{k+2}. As discussed earlier, in each block of the form $S^{k+2}_{\tilde{N}_{t,new}^{k+1|k+1}\times 4}$ and $S^{k+2}_{\tilde{N}_{t,old,r}^{k+1|k+1}\times r+2}$ in (69), all but the last column represent new child terms. Hence, the subarray of indices in S^{k+1} that covers all the new child terms, denoted by S_{new}^{k+1}, has the form

$$S_{new}^{k+1} = \begin{bmatrix} S^{k+1}_{\tilde{N}_{t,new}^{k|k}\times 3} \\ S^{k+1}_{\tilde{N}_{t,old,3}^{k|k}\times 4} \\ S^{k+1}_{\tilde{N}_{t,old,4}^{k|k}\times 5} \\ \vdots \\ S^{k+1}_{\tilde{N}_{t,old,k+1}^{k|k}\times k+1} \\ S^{k+1}_{\tilde{N}_{t,old,k+2}^{k|k}\times k+2} \end{bmatrix}. \tag{74}$$

Note that S_{new}^{k+1} has a staircase structure, and is not a full array or a matrix. Moreover, although the last child terms in each subblock of S_{new}^{k+1} do not combine at $k+1$, their new grandchild terms will combine at $k+2$. Hence all the new child terms at $k+1$ are included in S_{new}^{k+1}.

Lemma 5.1 suggests that S_{new}^{k+1} is placed in S^{k+2} in the lower blocks below the $\left[S^{k+2}_{\tilde{N}_{t,new}^{k+1|k+1}\times 4} \quad 0_{\tilde{N}_{t,new}^{k+1|k+1}\times(k-1)}\right]$ block in (69). Since the index one was already used above for terms that combine due to rule (54), the indexes of S_{new}^{k+1} are placed into S^{k+2} after adding one to each of its entries, an operation we denote by $S_{new}^{k+1}+[1]$. This also matches the indexes 2 through $\tilde{N}_{t,new}^{k+1|k+1}+1$ that were already placed in the second column of $S^{k+2}_{\tilde{N}_{t,new}^{k+1|k+1}\times 4}$. For example, the index 2 assigned to the $(1,2)$ entry of $S^{k+2}_{\tilde{N}_{t,new}^{k+1|k+1}\times 4}$ equals the index 2 assigned to the $(1,1)$ entry of the second subblock $S^{k+2}_{\tilde{N}_{t,old,3}^{k+1|k+1}\times 5}$, which can be easily verified to combine due to the second term combination rule in (55). The matching of all the entries in the second column of $S^{k+2}_{\tilde{N}_{t,new}^{k+1|k+1}\times 4}$ to the appropriate entries in $S_{new}^{k+1}+[1]$ can be verified similarly using (55) along with Lemma 5.1. Figure 2 for $k=3$ illustrates this matching of terms, which is consistent with S^3 of (67).

At this stage, S^{k+2} is filled except for the two last nonzero columns associated with each block in (69) that represent the terms that do not combine: the columns before the last indexing the new noncombining grandchild terms, and the last columns for old noncombining grandchild terms. For the new terms, the entries in the columns before last in each nonzero subblock that will staircase down from row one to $\tilde{N}_t^{k+1|k+1}$ will be indexed $\tilde{N}_{t,new}^{k+1|k+1}+2$ to $\tilde{N}_t^{k+1|k+1}+\tilde{N}_{t,new}^{k+1|k+1}+1$. This array forms S_{new}^{k+2} to be used in forming S^{k+3}. Now the remaining last columns in each nonzero block are indexed by continuation, i.e., $\tilde{N}_t^{k+1|k+1}+\tilde{N}_{t,new}^{k+1|k+1}+2$ to $2\tilde{N}_t^{k+1|k+1}+\tilde{N}_{t,new}^{k+1|k+1}+1$. It should be noted that the index assignments to the old grandchild terms (the last step above) could be skipped, since those indexes are never used in constructing S nor in checking if those terms combine—they don't. Consequently, those entries can be left unassigned (or assigned, e.g., zero). However, to emphasize that there are terms that correspond to those entries in S, they are indexed for consistency.

This completes the derivation of the recursive construction of S^{k+2} that can be summarized as follows. Assuming that we know S_{new}^{k+1}, $S^{k+2}\in\mathbb{R}^{\tilde{N}_t^{k+1|k+1}\times k+4}$ initialized to all zero entries is constructed using the following four simple steps:

1. Assign ones to the first column and 2 through $\tilde{N}_{t,new}^{k+1|k+1}+1$ to the second column of the first $\tilde{N}_{t,new}^{k+1|k+1}$ rows of S^{k+2}.
2. Place $S_{new}^{k+1}+[1]$ right below the the first $\tilde{N}_{t,new}^{k+1|k+1}$ rows of S^{k+2}.
3. Staircase indexes $\tilde{N}_{t,new}^{k+1|k+1}+2$ to $\tilde{N}_t^{k+1|k+1}+\tilde{N}_{t,new}^{k+1|k+1}+1$ in the columns before last in the nonzero subblocks of S^{k+2} (see (69) for its structure). Store the current nonzero staircase structure as S_{new}^{k+2} for constructing S^{k+3}.
4. Staircase indexes $\tilde{N}_t^{k+1|k+1}+\tilde{N}_{t,new}^{k+1|k+1}+2$ to $2\tilde{N}_t^{k+1|k+1}+\tilde{N}_{t,new}^{k+1|k+1}+1$ in the last columns of the nonzero subblocks of S^{k+2}.

As presented in section 4, the procedure is initiated with $S^1=\begin{bmatrix}1 & 2 & 3\end{bmatrix}$ and $S_{new}^1=\begin{bmatrix}1 & 2\end{bmatrix}$. All the counters in the above procedure, such as $\tilde{N}_{t,new}^{k+1|k+1}$ and the subblock sizes of S^{k+2} in (69) are determined using the recursive expression (64).

This recursive scheme is demonstrated by constructing recursively S^2, S^3, and S^4 that were constructed heuristically in section 4, (66)–(68). Using the initial S_{new}^1, S^2 is constructed as follows:

$$(75)\quad 0_{3\times 4}\xrightarrow{1}\begin{bmatrix} \mathit{1} & \mathit{2} & 0 & 0\\ \mathit{1} & \mathit{3} & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\xrightarrow{2}\begin{bmatrix} \mathit{1} & \mathit{2} & 0 & 0\\ \mathit{1} & \mathit{3} & 0 & 0\\ \mathbf{2} & \mathbf{3} & 0 & 0\end{bmatrix}$$
$$\xrightarrow{3}\left[\begin{array}{ccc|c} \mathit{1} & \mathit{2} & \mathit{4} & 0\\ \mathit{1} & \mathit{3} & 5 & 0\\ \mathbf{2} & \mathbf{3} & \mathit{6} & 0\end{array}\right]\xrightarrow{4}\left[\begin{array}{ccc|c} \mathit{1} & \mathit{2} & \mathit{4} & 7\\ \mathit{1} & \mathit{3} & 5 & 8\\ \mathbf{2} & \mathbf{3} & \mathit{6} & 9\end{array}\right]=S^2.$$

In the above, the numbers above the arrow indicate the step in the recursive procedure described above. Specifically, we note in the first step the subcolumn of ones (surrounded by the dotted line) that indicate terms that combine through rule (54), and the second column indicating new terms that combine only with terms generated by old parents. In step two we see the $S_{new}^1+[1]$ subblock depicted by bold-faced characters. Step three adds a column of new indexes, thus completing the S_{new}^2 subarray delimited by a solid line, which in this case is square. Clearly the result in (75) matches the one in (66).

The four-step procedure is used now to construct S^3, while using S^2_{new} that was obtained in the third step of (75), as follows:

(76)
$$0_{9\times 5} \xrightarrow{1} \begin{bmatrix} \mathit{1} & \mathit{2} & 0 & 0 & 0 \\ \mathit{1} & \mathit{3} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathit{1} & \mathit{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} \mathit{1} & \mathit{2} & 0 & 0 & 0 \\ \mathit{1} & \mathit{3} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathit{1} & \mathit{7} & 0 & 0 & 0 \\ \mathbf{2} & \mathbf{3} & \mathbf{5} & 0 & 0 \\ \mathbf{2} & \mathbf{4} & \mathbf{6} & 0 & 0 \\ \mathbf{3} & \mathbf{4} & \mathbf{7} & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{3} \begin{bmatrix} \mathit{1} & \mathit{2} & \mathit{8} & 0 & 0 \\ \mathit{1} & \mathit{3} & \mathit{9} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathit{1} & \mathit{7} & \mathit{13} & 0 & 0 \\ \mathbf{2} & \mathbf{3} & \mathbf{5} & \mathit{14} & 0 \\ \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathit{15} & 0 \\ \mathbf{3} & \mathbf{4} & \mathbf{7} & \mathit{16} & 0 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} \mathit{1} & \mathit{2} & \mathit{8} & 17 & 0 \\ \mathit{1} & \mathit{3} & \mathit{9} & 18 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathit{1} & \mathit{7} & \mathit{13} & 22 & 0 \\ \mathbf{2} & \mathbf{3} & \mathbf{5} & \mathit{14} & 23 \\ \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathit{15} & 24 \\ \mathbf{3} & \mathbf{4} & \mathbf{7} & \mathit{16} & 25 \end{bmatrix} = S^3.$$

It is interesting to point out the square 3×3 subblock $S^2_{new}+[1]$ depicted by bold-faced characters starting step two. In addition, S^3_{new} delimited by a solid line after step three has, for the first time, a staircase structure mentioned earlier in this section. Finally, as expected, the results in (67) and (76) are the same.

Finally, in (77) we demonstrate the construction of S^4 using the same steps and demarkation as before:

(77)
$$0_{25\times 6} \xrightarrow{1} \begin{bmatrix} \mathit{1} & \mathit{2} & 0 & 0 & 0 & 0 \\ \mathit{1} & \mathit{3} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathit{1} & \mathit{17} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} \mathit{1} & \mathit{2} & 0 & 0 & 0 & 0 \\ \mathit{1} & \mathit{3} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathit{1} & \mathit{17} & 0 & 0 & 0 & 0 \\ \mathbf{2} & \mathbf{3} & \mathbf{9} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{2} & \mathbf{8} & \mathbf{14} & 0 & 0 & 0 \\ \mathbf{3} & \mathbf{4} & \mathbf{6} & \mathbf{15} & 0 & 0 \\ \mathbf{3} & \mathbf{5} & \mathbf{7} & \mathbf{16} & 0 & 0 \\ \mathbf{4} & \mathbf{5} & \mathbf{8} & \mathbf{17} & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{3} \begin{bmatrix} \mathit{1} & \mathit{2} & \mathit{18} & 0 & 0 & 0 \\ \mathit{1} & \mathit{3} & \mathit{19} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathit{1} & \mathit{17} & \mathit{33} & 0 & 0 & 0 \\ \mathbf{2} & \mathbf{3} & \mathbf{9} & \mathit{34} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{2} & \mathbf{8} & \mathbf{14} & \mathit{39} & 0 & 0 \\ \mathbf{3} & \mathbf{4} & \mathbf{6} & \mathbf{15} & \mathit{40} & 0 \\ \mathbf{3} & \mathbf{5} & \mathbf{7} & \mathbf{16} & \mathit{41} & 0 \\ \mathbf{4} & \mathbf{5} & \mathbf{8} & \mathbf{17} & \mathit{42} & 0 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} \mathit{1} & \mathit{2} & \mathit{18} & 43 & 0 & 0 \\ \mathit{1} & \mathit{3} & \mathit{19} & 44 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathit{1} & \mathit{17} & \mathit{33} & 58 & 0 & 0 \\ \mathbf{2} & \mathbf{3} & \mathbf{9} & \mathit{34} & 59 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{2} & \mathbf{8} & \mathbf{14} & \mathit{39} & 64 & 0 \\ \mathbf{3} & \mathbf{4} & \mathbf{6} & \mathbf{15} & \mathit{40} & 65 \\ \mathbf{3} & \mathbf{5} & \mathbf{7} & \mathbf{16} & \mathit{41} & 66 \\ \mathbf{4} & \mathbf{5} & \mathbf{8} & \mathbf{17} & \mathit{42} & 67 \end{bmatrix} = S^4.$$

Also here, we observe a staircase structure of S^4_{new} delimited by a solid line after step three in (77).

6. Conclusion. This paper uncovers properties of the exponential terms of the characteristic function for two state systems with additive Cauchy uncertainties. First, we show that any two nonzero directions in the exponential at time step k produce a coaligned direction along $H\mathbf{A}$ at time step $k+1$, allowing two-element new terms (Theorem 2.3). Second, we prove that only terms with two-element exponentials can combine through the two term combination rules (Theorem 3.1). Third, an indexing scheme, called the S matrix, that is independent of system parameters, is proposed to indicate the terms that can be combined at each step. A procedure for constructing this S matrix is derived analytically while relying on the two-term combination rules. These properties of the exponential terms help one to understand and simplify significantly the structure of the characteristic function associated with the Cauchy estimation problem, allowing for reducing greatly its computational complexity.

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