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ON THE COMPLEXITY OF OPTIMAL LOTTERY PRICING AND RANDOMIZED MECHANISMS FOR A UNIT-DEMAND BUYER*

XI CHEN†, ILIAS DIAKONIKOLAS‡, ANTHI ORFANOU†, DIMITRIS PAPARAS§, XIAORUI SUN¶, AND MIHALIS YANNAKAKIS†

Abstract. We study the optimal lottery problem and the optimal mechanism design problem in the setting of a single unit-demand buyer with item values drawn from independent distributions. Optimal solutions to both problems are characterized by a linear program with exponentially many variables. For the menu size complexity of the optimal lottery problem, we present an explicit, simple instance with distributions of support size 2, and show that exponentially many lotteries are required to achieve the optimal revenue. We also show that, when distributions have support size 2 and share the same high value, the simpler scheme of item pricing can achieve the same revenue as the optimal menu of lotteries. The same holds for the case of two items with support size 2 (but not necessarily the same high value). For the computational complexity of the optimal mechanism design problem, we show that unless the polynomial-time hierarchy collapses (more exactly, $\mathrm{P}^{\mathrm{NP}} = \mathrm{P}^{\#\mathrm{P}}$), there is no efficient randomized algorithm to implement an optimal mechanism even when distributions have support size 3.

Key words. lottery pricing, optimal mechanism design, unit-demand buyer

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1. Introduction.

Optimal pricing problems have been studied intensively during the past decade, under various settings and from both algorithmic and complexity-theoretic perspectives. They are closely related to problems that arise from the area of optimal Bayesian multidimensional mechanism design; e.g., see [Tha04, GHK+05, HK05, BK06, DHFS06, BB06, MV06, CHK07, BBM08, Bri08, Pav10, CD11, CMS15, BCKW15, HN12, DW12, HN13, LY13, DDT13, DDT14a, WT14, BILW20, CDP+18, DDT14b, Yao15]. The latter is well understood in the single-parameter setting where Myerson's classic result shows that a simple, deterministic mechanism can achieve as much revenue as any sophisticated, randomized mechanism [Mye81]. The general case with multiple items, however, turns out to be more complex. Much effort has been devoted to understanding both the structure and complexity of optimal mechanisms, and to developing simple and computationally efficient mechanisms that are approximately optimal.

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†Department of Computer Science, Columbia University, New York, NY 10027 USA (xichen@cs.columbia.edu, anthi@cs.columbia.edu, mihalis@cs.columbia.edu).
‡Department of Computer Science, University of Wisconsin-Madison, Madison, WI 53706 USA (iliasdiakonikolas@gmail.com).
§Google Research, Mountain View, CA 94043 USA (dpaparas@google.com).
¶Computer Science Department, University of Illinois, Chicago, IL 60607 USA (xiaorui@uic.edu).

In this paper, we consider the following setting of monopolist lottery pricing where a buyer is interested in n heterogeneous items offered by a seller. We focus on the case when the buyer is *unit-demand* (i.e., only interested in obtaining at most one of the items) and *quasi-linear* (i.e., her utility is $v - p$ if she receives an item of value v to her and makes a payment of p to the seller). The seller is given full access to a probability distribution $\mathcal{D}$ from which the buyer's valuations $\mathbf{v} = (v_1, \ldots, v_n)$ for the items are drawn, and can exploit $\mathcal{D}$ to choose a *menu* (a set) M *of lotteries* that maximizes her expected revenue (i.e., payment from the buyer). Here a lottery is of the form $(\mathbf{x}, p)$, where $p \in \mathbb{R}$ is its price and $\mathbf{x} = (x_1, \ldots, x_n)$ is a nonnegative vector that sums to at most 1, with each x_i being the probability of the buyer receiving item i if this lottery is purchased (the buyer receives no item with probability $1 - \sum_i x_i$). After a menu M is chosen, the buyer draws a valuation vector $\mathbf{v}$ from $\mathcal{D}$ and receives a lottery that maximizes her expected utility $\sum_i x_i \cdot v_i - p$, or the empty lottery $(\mathbf{0}, 0)$ by default if every lottery in M has a negative utility.

Given $\mathcal{D}$, its optimal menus are characterized by a linear program in which we associate with each $\mathbf{v}$ in $D := \mathsf{supp}(\mathcal{D})$ a set of $n + 1$ variables to capture the lottery that the buyer receives at $\mathbf{v}$. We will refer to it as the *standard linear program* (see section 2.1) for the optimal lottery problem. In particular, for the case when $\mathcal{D}$ is correlated and given explicitly (i.e., given as a tuple of valuation vectors and their probabilities), one can find an optimal menu by solving the standard linear program in polynomial time [BCKW15].

We focus on the case when $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ is a product distribution and each v_i is drawn independently from $\mathcal{D}_i$. The standard linear program in this case has exponentially many variables in general (even when each $\mathcal{D}_i$ has support size 2), so one cannot afford to solve it directly. We are interested in the following two questions:

- Menu size complexity: *How many lotteries are needed to achieve the optimal revenue?*
- Computational complexity: *How difficult it is to compute*[1] *an optimal menu of lotteries?*

While much progress has been made when the buyer is additive (see discussions on related work later), both questions remain widely open for the unit-demand single-buyer setting being considered here. For example no explicit instance is known previously to require exponentially many lotteries for the optimal revenue. (A trivial upper bound on the menu size is $|D|$ since otherwise at least one lottery in the menu is never used.)

Our first result is an explicit, simple product distribution $\mathcal{D}$, for which exponentially many lotteries are needed ($\Omega(|D|)$ indeed) to achieve the optimal revenue. Let $\mathcal{D}'$ denote the distribution supported on $\{1, 2\}$, with probabilities $(1 - p, p)$, and let $\mathcal{D}''$ denote the distribution supported on $\{0, n+2\}$ with probabilities $(1-p, p)$, where $p = 1/n^2$. We prove the following theorem in section 4.

THEOREM 1.1. *When n is sufficiently large, any optimal menu for $\mathcal{D}^* = \mathcal{D}' \times \mathcal{D}' \times \cdots \times \mathcal{D}' \times \mathcal{D}''$ over n items must have $\Omega(2^n)$ many different lotteries.*

Note that all distributions in $\mathcal{D}^*$ are the same except one. We show that this is indeed necessary. Before stating our result, we review the optimal item pricing problem. The setup is the same, but now the seller can only assign a price $p_i \in \mathbb{R}$ to each item i. Once the prices are fixed, the buyer draws $\mathbf{v}$ from $\mathcal{D}$ and buys an item i that maximizes her utility $v_i - p_i$. The problem is to find a tuple of prices that

[1] See Theorem 1.4 for the exact meaning of "computing" an optimal menu here.

maximizes the seller's expected revenue. Equivalently, an item pricing is a menu in which each lottery is of the special form $(\mathbf{e}_i, p)$ for some unit vector $\mathbf{e}_i$ (so the menu size is at most n). In general lotteries can extract strictly higher revenue than the optimal item pricing, as shown in [Tha04] for two items with values drawn from $[5, 6]$ uniformly at random, which motivated much of the subsequent work.

We show that lotteries do not help when $\mathcal{D}_i$'s have support size 2 and share the same high value.

THEOREM 1.2. *If $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ and $\mathsf{supp}(\mathcal{D}_i) = \{a_i, b\}$ with $a_i < b$ for all $i \in [n]$, an optimal item pricing achieves the same expected revenue as that of an optimal menu of lotteries.*

Therefore, the exponential lower bound on the menu size in Theorem 1.1 cannot hold for support-size-2 distributions that share the same high value. The proof of Theorem 1.2 also implies that an optimal menu in this case can be computed in polynomial time.[2] For the special case of two items we show that the condition of $\mathcal{D}_1$ and $\mathcal{D}_2$ sharing the same high value can be dropped.

THEOREM 1.3. *If both $\mathcal{D}_1$ and $\mathcal{D}_2$ have support size 2, then an optimal item pricing for $\mathcal{D}_1 \times \mathcal{D}_2$ achieves the same expected revenue as that of an optimal menu of lotteries.*

In addition, we give examples of three-item support-size-2 and two-item support-size-3 instances where lotteries do achieve a strictly higher revenue than item pricings.

Now we describe our result on the problem of *computing* an optimal menu of lotteries. Although $\mathcal{D}^*$ in Theorem 1.1 trivially rules out any polynomial-time algorithm that lists explicitly all lotteries in an optimal menu, there is indeed a deterministic polynomial-time algorithm that, given any $\mathbf{v} \in D$, outputs a lottery $\ell_{\mathbf{v}}$ such that $\{\ell_{\mathbf{v}} : \mathbf{v} \in D\}$ is an optimal menu for $\mathcal{D}^*$ and $\ell_{\mathbf{v}}$ is the lottery bought by the buyer at $\mathbf{v}$ given this menu (see Corollary 4.5 and the remarks at the end of section 4). We are interested in the question of whether a *universal* efficient algorithm that computes an optimal menu in this fashion exists: given any product distribution $\mathcal{D}$ and any $\mathbf{v} \in D$, such an algorithm outputs a lottery $\ell_{\mathbf{v}}$ such that $\{\ell_{\mathbf{v}} : \mathbf{v} \in D\}$ is optimal for $\mathcal{D}$.

This question is motivated by a folklore connection between the lottery problem and the optimal mechanism design problem. Consider the same setting, where a unit-demand buyer with values drawn from $\mathcal{D}$ is interested in n items offered by a seller. Here a *mechanism* is a (possibly randomized) map $\mathcal{B}$ from the set D to $([n] \cup \{\text{nil}\}) \times \mathbb{R}$, where $\mathcal{B}(\mathbf{v}) = (b, p)$ means that the buyer is assigned item b (or no item if $b = \text{nil}$) and pays p to the seller. The optimal mechanism design problem is then to find an *individually rational* and *truthful* mechanism (see definitions in section 2.1) that maximizes the expected revenue of the seller.

Let $\overline{\mathcal{B}}(\mathbf{v}) = (\mathbf{x}(\mathbf{v}), \overline{p}(\mathbf{v}))$ denote the expected outcome of $\mathcal{B}$ on $\mathbf{v}$, i.e., $x_i(\mathbf{v})$ is the probability of $\mathcal{B}(\mathbf{v})$ assigning item i and $\overline{p}(\mathbf{v})$ is the expected payment. It follows from definitions of these two problems that, under the same $\mathcal{D}$, $\mathcal{B}$ is an optimal mechanism if and only if $(\overline{\mathcal{B}}(\mathbf{v}) : \mathbf{v} \in D)$ is an optimal menu (see section 2.1). Therefore, the standard linear program for the lottery problem also characterizes optimal mechanisms.

By exploring further ideas behind the proof of Theorem 1.1 we show that there exists no efficient universal algorithm to implement an optimal mechanism even when

[2]The proof of Theorem 1.2 gives an explicit list of $n+1$ item pricings and shows that at least one of them achieves the same revenue as an optimal lottery pricing. Computing the expected revenue of a given item pricing is in polynomial time; see [CDP+18].

$\mathcal{D}_i$'s have support size 3, unless $\mathrm{P}^{\mathrm{NP}} = \mathrm{P}^{\#\mathrm{P}}$, which by Toda's theorem [Tod91] would imply that the polynomial hierarchy collapses to the second level; this is considered unlikely.

THEOREM 1.4. *Unless* $\mathrm{P}^{\mathrm{NP}} = \mathrm{P}^{\#\mathrm{P}}$, *there exists no algorithm* $\mathcal{A}(\cdot,\cdot)$ *with the following two properties:*

1. $\mathcal{A}$ *is a randomized polynomial-time algorithm that always terminates in polynomial time.*
2. *Given any instance* $I = (n, \mathcal{D}_1, \ldots, \mathcal{D}_n)$ *to the optimal mechanism design problem, where each* $\mathcal{D}_i$ *has support size* 3, *and any* $\mathbf{v} \in \mathsf{supp}(\mathcal{D}_1) \times \cdots \times \mathsf{supp}(\mathcal{D}_n)$, $\mathcal{A}(I, \mathbf{v})$ *always outputs a pair in* $([n] \cup \{nil\}) \times \mathbb{R}$, *such that* $\mathcal{B}_I : \mathbf{v} \mapsto \mathcal{A}(I, \mathbf{v})$ *is an optimal mechanism for the instance* I.

We remark that the optimal solutions in the proofs of Theorems 1.1 and 1.4 have the property that they allocate with probability 1 some item for all valuations; such lotteries (or mechanisms) are called complete. Thus, the results hold also for the model where lotteries are required to be complete.

1.1. Related work. We briefly review related work in the language of the optimal mechanism design problem.

For the unit-demand, single-buyer setting considered here, Thanassoulis [Tha04] showed that unlike the single-parameter setting where the optimal mechanism is deterministic [Mye81], an optimal mechanism for two items drawn uniformly from $[5, 6]$ must involve randomization. In [BCKW15], Briest et al. showed that when $\mathcal{D}$ is correlated and given explicitly, one can solve the standard linear program to find an optimal mechanism in polynomial time. They also showed that the ratio of revenues obtained by an optimal randomized mechanism (lottery pricing) and an optimal deterministic mechanism (item pricing) can be unbounded in instances with three items. This was later improved to two items by Hart and Nisan in [HN13] (they also showed it for the additive setting, i.e., the model in which the buyer may buy any subset of items and the value of a subset is equal to the sum of values of items in the subset). In [DHN14], Dughmi, Han, and Nisan studied the sampling and representation complexity of mechanisms in a black box access model for the distribution $\mathcal{D}$, and showed that there is a correlated distribution for which any approximately revenue-maximizing mechanism requires exponential representation complexity. They also improved previous upper bounds on the menu size needed to extract at least $(1-\epsilon)$-fraction of the optimal revenue.

For the case of product distributions, Chawla, Malec, and Sivan [CMS15] showed that the ratio between the revenues of optimal randomized and deterministic mechanisms is at most 4. When there are two items drawn independently from distributions that meet certain conditions, Pavlov [Pav10] characterized optimal mechanisms under both unit-demand and additive settings. Recently, Kothari et al. [KSM+19] gave a quasi-polynomial-time algorithm for achieving $(1-\epsilon)$-fraction of the optimal revenue for any constant $\epsilon > 0$.

The problem of finding an optimal deterministic mechanism (or an optimal item pricing) in the unit-demand setting with a product distribution was shown to be NP-complete in [CDP+18], and this holds even when the item distributions have support size 3 or are identical. An optimal item pricing can be computed in polynomial time for support size 2 [CDP+18].

For the additive single-buyer setting, Manelli and Vincent [MV06] gave an example where randomization results in a strictly higher expected revenue. Much progress has been made on characterizing optimal mechanisms and developing simple, computationally efficient mechanisms that are approximately optimal (e.g., [HN12, LY13, DDT13, BILW20, WT14, DDT14b, Yao15, BGN17]). Most relevant to our work are the papers addressing the menu size and computational complexity of the problem in the additive setting. In [HN13], Hart and Nisan introduced the notion of *menu size*. They showed that there exists a (continuous) correlated distribution for which no mechanism of finite menu size can achieve a positive fraction of the optimal revenue. For product distributions with infinite support, an infinite menu may be required to achieve the optimal revenue [DDT13], but a constant fraction (1/6) of the optimal revenue can always be achieved by a small finite menu, using a simple deterministic mechanism that offers either the individual items (at their Myerson prices) or the grand bundle of all the items (at a suitable price) [BILW20]. Furthermore, any fraction $(1 - \epsilon) < 1$ of the optimal revenue can be achieved by some finite menu of sufficiently large size [BGN17]. In contrast, [BGN17] showed that $2^{\Omega(n)}$ menu size is needed to achieve $(1 - 1/n)$-fraction of the optimal revenue, even when the additive buyer has simple value support $\{0, 1\}$ for each item. Regarding the computational complexity, Daskalakis, Deckelbaum, and Tzamos showed in [DDT14a] that there cannot be an efficient universal algorithm that implements an optimal mechanism for product distributions, even when all items have support size 2, unless $\mathrm{P}^{\#\mathrm{P}} \subseteq \mathrm{ZPP}$. We compare our proof of Theorem 1.4 and the proof of [DDT14a] in section 1.2.

1.2. Ideas behind the proofs. The main difficulty in proving both Theorems 1.1 and 1.4 is to understand and characterize optimal solutions to the standard linear program (denoted by $\mathrm{LP}(I)$; see section 2.1) for certain input instances I. For Theorem 1.1 we need to show that every optimal solution to $\mathrm{LP}(I)$ with distribution $\mathcal{D}^*$ has exponentially many different lotteries; for Theorem 1.4 we need to embed an instance of a #P-hard problem in I and then show that every optimal solution to $\mathrm{LP}(I)$ helps us solve the #P-hard problem. However, characterizing optimal solutions to $\mathrm{LP}(I)$ is challenging due to its exponentially many variables and constraints, which result in a highly complex geometric object for which our current understanding is still very limited (e.g., compared to the literature on the additive setting). Compared with the optimal item pricing problem under the same setting where NP-completeness was established in [CDP+18], there is a significant difference in their underlying structures: the item pricing problem has a richer combinatorial flavor; characteristics of the lottery pricing problem are mostly "continuous," as suggested by its linear program formulation.

The high-level approach behind proofs of Theorems 1.1 and 1.4 is similar to that of [DDT14a]. We simplify the problem by relaxing the standard linear program $\mathrm{LP}(I)$ to a smaller linear program denoted by $\mathrm{LP}'(I)$ on the same set of variables $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in D)$ but only subject to a subset of carefully picked constraints of $\mathrm{LP}(I)$. (Here $\mathbf{q}(\mathbf{v})$ is a tuple of n variables with $q_i(\mathbf{v})$ being the probability of the buyer receiving item i in the lottery; $u(\mathbf{v})$ is the utility of the buyer at $\mathbf{v}$ to replace the role of price of the lottery.) Then we focus on a highly restricted family of instances I and characterize optimal solutions to $\mathrm{LP}'(I)$, taking advantage of the relaxed, simplified $\mathrm{LP}'(I)$ as well as special structures of I. Finally we show that every optimal solution to $\mathrm{LP}'(I)$ is a feasible and, thus, optimal solution to the standard linear program $\mathrm{LP}(I)$ as well, and always satisfies the desired properties (e.g., has exponentially many different lotteries, for the purpose of Theorem 1.1, or can be used to solve the #P-hard instance embedded in it, for the purpose of Theorem 1.4).

The similarity between our proof techniques and those of [DDT14a], however, stops here due to a subtle but crucial difference between the two linear programs. In our standard LP(I), the allocation variables $\mathbf{q}(\mathbf{v})$ must sum to at most 1 because the buyer is unit-demand. For the additive setting, on the other hand, there is no such constraint on the sum of $q_i(\mathbf{v})$; the only constraint is that $q_i(\mathbf{v}) \in [0,1]$ for all i. It turns out that this difference requires a completely different set of ideas and techniques to carry out the plan described above for the unit-demand setting, which we sketch below.

Recall the two distributions $\mathcal{D}'$ and $\mathcal{D}''$ used in the statement of Theorem 1.1, supported on $D' = \{1,2\}$ and $D'' = \{0, n+2\}$, respectively. Consider the independent and identically distributed (i.i.d.) instance I of n items drawn from $\mathcal{D}'$ each. We make the following observation: an i.i.d. instance as I always has a "symmetric" optimal solution in which $q_i(\mathbf{v})$ only depends on the value of v_i and the number of 2's in $\mathbf{v}$,[3] and such a solution tends to have many different lotteries. For example, if in such an optimal solution, $q_i(\mathbf{v}) \neq q_j(\mathbf{v})$ when $v_i = 2$, $v_j = 1$, and $\mathbf{v}$ has $(n/2)$ many 2's, then all such exponentially many $\mathbf{v}$'s have distinct lotteries. Inspired by this, we analyze LP(I) (by a careful relaxation) and obtain a complete characterization of its optimal solutions. Each optimal solution is (almost) uniquely determined by $\mathbf{q}(\mathbf{1})$ of the all-1 vector. Moreover, there are exponentially many different lotteries when $\mathbf{q}(\mathbf{1})$ has full support. However, $\mathbf{q}(\mathbf{1})$ does not necessarily have full support; indeed any $\mathbf{q}(\mathbf{1})$ that sums to 1 results in the same optimal expected revenue. In fact, by Theorem 1.2, there is an optimal item pricing, i.e., an optimal solution with only n lotteries (this solution is not symmetric).

Our next idea is to add another item, drawn from $\mathcal{D}''$ (which breaks the symmetry of the instance), to enforce full support of $\mathbf{q}(\mathbf{1})$ in every optimal solution to LP(I'), where I' denotes the new instance with $n+1$ items. This is done by defining a relaxation LP$'(I')$ of LP(I'): LP$'(I')$ has the same set of variables and the same objective function as LP(I') but only a carefully picked subset of constraints of LP(I'). We then give a complete characterization of all optimal solutions to LP$'(I')$. We show that every optimal solution to LP$'(I')$ satisfies all the constraints of the original LP(I'), which implies that they share the same set of optimal solutions. Furthermore, our characterization for LP$'(I')$ shows that any of its optimal solutions must have $\mathbf{q}(\mathbf{1})$ being the uniform distribution over the first n items and almost all valuations are assigned a different lottery. This finishes the proof of Theorem 1.1.

The proof of Theorem 1.4 is based on similar ideas but is more delicate and involved. The goal here is to embed a subset-sum-type #P-hard problem in I. Let $g_1, \ldots, g_n$ denote the input integers of the #P-hard problem. Roughly speaking, we are given a subset H of $\{g_1, \ldots, g_n\}$ of size $n/2$, and are asked to decide whether the sum $\sum_{i \in H} g_i$ is at least as large as the median of $\binom{n}{n/2}$ many such sums derived from all subsets of $\{g_1, \ldots, g_n\}$ of size $n/2$ (note that the exact definition of the problem is more involved; see section 5.2).

We consider an instance I with $n+2$ items, where item i is supported on $\{a_i, \ell_i, h_i\}$ for each $i \in [n]$ with $a_i \approx 1$, $\ell_i < h_i$, and $\ell_i \approx h_i \approx 2$. The other two items $n+1$ and $n+2$ are supported on $\{0, s\}$ and $\{0, t\}$, respectively, for some s and t that satisfy $t \gg s \gg 1$. The probabilities of item i taking values a_i, ℓ_i, and h_i

[3]Indeed any optimal solution can be "symmetrized" into such a solution without any loss in revenue; a general symmetrization procedure can be found in [DW12]. However, there may be an optimal nonsymmetric solution that is much more compact than its symmetrization. This is the case, for example, with this instance of n items drawn from $\mathcal{D}'$ each, before we add the additional $(n+1)$th item.

are $1-p-r, p$, and r, respectively, for each $i \in [n]$; item $n+1$ takes value 0 with probability $1-\delta$, and s with probability δ; item $n+2$ takes value 0 with probability $1-\delta^2$, and t with probability δ^2. The parameters p, r, and δ satisfy $1 \gg p \gg r \gg \delta$. For now, we do not pin down exact values of the parameters a_i, s, h_i, t, but only assume that they satisfy certain mild conditions; the rest of the parameters ℓ_i, p, r, δ, on the other hand, are assigned specific values.

The first step of our proof is to characterize the set of optimal solutions of a carefully chosen relaxation $\text{LP}'(I)$ of $\text{LP}(I)$, assuming that parameters a_i, s, h_i, t satisfy the conditions specified. To this end we partition the set D of all valuation vectors into four types T_1, T_2, T_3, and T_4, where T_i denotes the set of type-i vectors: T_1 consists of vectors $\mathbf{v}$ with $v_{n+1} = v_{n+2} = 0$, T_2 consists of $\mathbf{v}$ with $v_{n+1} = s$ and $v_{n+2} = 0$, T_3 consists of $\mathbf{v}$ with $v_{n+2} = t$ and $v_{n+1} = 0$, and T_4 consists of $\mathbf{v}$ with $v_{n+1} = s$ and $v_{n+2} = t$. The following four smallest vectors in each of the four T_i's play a crucial role in the chracterization:

$$\mathbf{a} = (a_1, \ldots, a_n, 0, 0), \quad \mathbf{c}_2 = (a_1, \ldots, a_n, s, 0), \quad \text{and}$$
$$\mathbf{c}_3 = (a_1, \ldots, a_n, 0, t), \quad \mathbf{c}_4 = (a_1, \ldots, a_n, s, t).$$

We also let $\rho : T_2 \cup T_3 \cup T_4 \to T_1$ denote the map with $\rho(\mathbf{v}) = (v_1, \ldots, v_n, 0, 0)$.

Then our characterization shows that any optimal solution of $\text{LP}'(I)$ is (almost) uniquely determined by $\mathbf{q}(\mathbf{a}), u(\mathbf{c}_2), u(\mathbf{c}_3)$, and $u(\mathbf{c}_4)$. This is done by a sequence of lemmas, each imposing a condition called *Condition-Type-i* on type-i vectors in optimal solutions of $\text{LP}'(I)$. They are established in reverse order: we start by proving the condition on type-2 and -4 vectors first, followed by type-3 vectors, and finally type-1 vectors. The proof of Condition-Type-1 is among the technically most challenging parts of the paper. In particular, Condition-Type-i for $i = 2, 3, 4$ requires that

$$u(\mathbf{v}) = \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_i)\big\} \quad \text{for each } \mathbf{v} \in T_i. \tag{1.1}$$

Given the characterization, we start pinning down parameters a_i, s, h_i, t. By setting a_i and s carefully, we show that in any optimal solution to $\text{LP}'(I)$, the first n entries of $\mathbf{q}(\mathbf{a})$ sum to 1 and are almost uniform, i.e., $q_i(\mathbf{a}) \approx 1/n$ for $i \in [n]$. Next by setting h_i to encode the input integer g_i of the #P-hard problem, Condition-Type-1 implies that utilities of type-1 vectors (more exactly, a carefully chosen subset of type-1 vectors) encode the desired sums of $(n/2)$-subsets of $\{g_1, \ldots, g_n\}$, in every optimal solution to $\text{LP}(I)$. Finally, $u(\mathbf{c}_3)$ is tightly controlled by our choice of t and we can set it to an appropriate value so that $u(\mathbf{c}_3)$ encodes the median of sums obtained from all $(n/2)$-subsets of $\{g_1, \ldots, g_n\}$. Combining these with (1.1) we conclude that the #P-hard problem can be solved by comparing $u(\mathbf{c}_3)$ with $u(\mathbf{v})$, in any optimal solution of $\text{LP}'(I)$, at a specific type-3 vector $\mathbf{v} \in T_3$ such that $u(\rho(\mathbf{v}))$ encodes $\sum_{i \in H} g_i$ for the input set H.

In the last step of the proof, we follow our characterization to show that every optimal solution of the relaxation $\text{LP}'(I)$ must also be a feasible and, thus, optimal solution to the standard linear program $\text{LP}(I)$. This shows that they share the same set of optimal solutions and finishes the proof of Theorem 1.4.

For Theorems 1.2 and 1.3, our method for showing that randomization does not help in these settings, is by identifying suitable convex combinations of the revenues of item pricings which upper bound the revenues of all lotteries. Note that this proof method is not only sound, but also complete in the pricing problem in all cases where randomization does not help, by the properties of linear programming; the problem is to show the existence of suitable coefficients for the convex combinations.

Organization. We first give formal definitions of both problems and present the standard linear program in section 2. We prove Theorem 1.2 in section 3, showing that lotteries do not help when all distributions have support size 2 and share the same high value. We prove Theorem 1.1 in section 4, showing that the menu size complexity is exponential. We prove Theorem 1.4 in section 5, showing the computational hardness of the problem. In the appendix we prove Theorem 1.3, showing that lotteries do not help for two items and support size 2. We also present two small examples there showing that lotteries can help when either the number of items is 3 or the support has size 3.

2. Preliminaries.

We give formal definitions of the optimal lottery problem as well as the optimal mechanism design problem (with a single unit-demand buyer), and present the standard linear program that characterizes their optimal solutions in section 2.1. As a warm-up, we prove a few basic properties about the standard linear program in section 2.2.

2.1. Problem definitions and the standard linear program.

Consider an instance $I = (n, \mathcal{D}_1, \ldots, \mathcal{D}_n)$, where a seller offers n items, indexed by $[n] = \{1, \ldots, n\}$, to a unit-demand buyer, whose valuation $v_1, \ldots, v_n$ of items is drawn from n independent discrete distributions $\mathcal{D}_i$, $i \in [n]$. Each distribution $\mathcal{D}_i$ is given explicitly in I, including both its support $D_i = \mathsf{supp}(\mathcal{D}_i)$ and the probability of each value in D_i. Let $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ and $D = D_1 \times \cdots \times D_n$.

A solution to the optimal lottery problem is a menu (a set) M of lotteries $(\mathbf{x}, p)$, where each lottery consists of a nonnegative *allocation vector* $\mathbf{x} = (x_1, \ldots, x_n)$ that satisfies $\sum_i x_i \le 1$ and a *price* $p \in \mathbb{R}$.[4] Here x_i denotes the probability of the buyer receiving item i so with valuation $\mathbf{v} = (v_1, \ldots, v_n)$, the expected utility of the buyer purchasing a lottery $(\mathbf{x}, p)$ is $\sum_{i\in[n]} x_i \cdot v_i - p$. (Note that $\mathbf{x}$ does not necessarily sum to 1, and the buyer receives no item with probability $1 - \sum_i x_i$; we refer to a lottery $(\mathbf{x}, p)$ as a *complete* lottery if $\sum_i x_i = 1$.) We always assume that the empty lottery $(\mathbf{0}, 0)$ is in M as a potential choice when all other lotteries in M have a negative utility. (This corresponds to the buyer choosing to buy nothing.)

Given a menu M of lotteries, the buyer draws her valuation $\mathbf{v}$ of items from $\mathcal{D}$ and then receives a lottery that maximizes her utility with respect to $\mathbf{v}$ (so if there is a tie, the seller can assign the buyer, among those that maximize the buyer's utility, a lottery with the maximum price[5]). Let $\Pr[\mathbf{v}] = \prod_{i\in[n]} \Pr_{\mathcal{D}_i}[v_i]$ denote the probability of valuation $\mathbf{v} \sim \mathcal{D}$, and let $\textsc{Rev}_M(\mathbf{v})$ denote the price of the lottery that the buyer receives. Then the goal of the optimal lottery problem is to find a menu M of lotteries that maximizes the expected revenue of the seller: $\textsc{Rev}(M) = \sum_{\mathbf{v}\in D} \Pr[\mathbf{v}] \cdot \textsc{Rev}_M(\mathbf{v})$.

We now give the first (not the standard one) linear program characterization of optimal solutions to the optimal lottery problem. For each $\mathbf{v} \in D$ we introduce $n + 1$ variables $\mathbf{q}(\mathbf{v}) = (q_1(\mathbf{v}), \ldots, q_n(\mathbf{v}))$ and $p(\mathbf{v})$ to denote the allocation vector and price of the lottery that the buyer receives at $\mathbf{v}$. Then the menu is given by $M = \{(\mathbf{q}(\mathbf{v}), p(\mathbf{v})) : \mathbf{v} \in D\}$. The only conditions are to make sure the utility of the buyer is always nonnegative and that $(\mathbf{q}(\mathbf{v}), p(\mathbf{v}))$ is a lottery in M that maximizes

[4]Notice that we do not require that $p \ge 0$ in the problem definition. It will become clear after we define the objective function that the seller has no incentive to offer a lottery with a negative price so it is the same problem whether $p \ge 0$ is required or not.

[5]As in the case of deterministic pricing [CDP+18], the supremum achievable revenue is independent of the tie-breaking rule. Furthermore, the maximum price (equivalently, maximum expected value) rule tie-breaking has the property that the supremum can be achieved.

the utility of the buyer. This gives us a linear program characterization of optimal solutions over variables $(p(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in D)$:

$$\text{maximize} \sum_{\mathbf{v}\in D} \Pr[\mathbf{v}] \cdot p(\mathbf{v}) \text{ subject to}$$

$$q_i(\mathbf{v}) \geq 0 \ \text{ and } \sum_{i\in[n]} q_i(\mathbf{v}) \leq 1 \ \text{ for all } \mathbf{v} \in D \text{ and } i \in [n],$$

$$\sum_{i\in[n]} v_i \cdot q_i(\mathbf{v}) - p(\mathbf{v}) \geq 0 \ \text{ for all } \mathbf{v} \in D,$$

$$\sum_{i\in[n]} w_i \cdot q_i(\mathbf{w}) - p(\mathbf{w}) \geq \sum_{i\in[n]} w_i \cdot q_i(\mathbf{v}) - p(\mathbf{v}) \ \text{ for all } \mathbf{v}, \mathbf{w} \in D.$$

To obtain the standard linear program, we use instead of the price variables $p(\mathbf{v})$, variables $u(\mathbf{v})$ for the utilities of the buyer at the valuations $\mathbf{v}$, replacing $p(\mathbf{v})$ by the expression $\sum_i v_i \cdot q_i(\mathbf{v}) - u(\mathbf{v})$:

$$\text{maximize} \sum_{\mathbf{v}\in D} \Pr[\mathbf{v}] \cdot \left(\sum_{i\in[n]} v_i \cdot q_i(\mathbf{v}) - u(\mathbf{v}) \right) \text{ subject to}$$

$$u(\mathbf{v}) \geq 0, \ q_i(\mathbf{v}) \geq 0, \text{ and } \sum_{i\in[n]} q_i(\mathbf{v}) \leq 1 \ \text{ for all } \mathbf{v} \in D \text{ and } i \in [n],$$

$$u(\mathbf{v}) - u(\mathbf{w}) \leq \sum_{i\in[n]} (v_i - w_i) \cdot q_i(\mathbf{v}) \ \text{ for all } \mathbf{v}, \mathbf{w} \in D. \tag{2.1}$$

We will refer to it as the standard linear program that characterizes optimal solutions to the lottery problem and denote it by LP(I). When an optimal solution $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in D)$ to LP(I) is given, we refer to the number of lotteries in the menu it induces as its *menu size*.

For the optimal mechanism design problem (with a single unit-demand buyer), the setting is exactly the same (and so are the input instances I). A randomized mechanism is a randomized algorithm $\mathcal{B}$ that, given $\mathbf{v} \in D$, returns a pair (a, p), where $a \in [n] \cup \{\text{nil}\}$ is the item assigned to the buyer (or no item is assigned if $a = \text{nil}$) and $p \in \mathbb{R}$ is the payment from the buyer. Given $\mathcal{B}$, let $\overline{\mathcal{B}}(\mathbf{v}) = (\mathbf{x}(\mathbf{v}), \overline{p}(\mathbf{v}))$ denote the expected outcome of $\mathcal{B}$ on $\mathbf{v}$: $x_i(\mathbf{v})$ is the probability that $\mathcal{B}(\mathbf{v})$ assigns item i and $\overline{p}(\mathbf{v})$ is the expected payment.

We say $\mathcal{B}$ is *individually rational* if the buyer always has a nonnegative utility if she reports truthfully:

$$\sum_{i\in[n]} v_i \cdot x_i(\mathbf{v}) - \overline{p}(\mathbf{v}) \geq 0 \quad \text{for all } \mathbf{v} \in D.$$

We say $\mathcal{B}$ is *truthful* if the buyer has no incentive to misreport:

$$\sum_{i\in[n]} v_i \cdot x_i(\mathbf{v}) - \overline{p}(\mathbf{v}) \geq \sum_{i\in[n]} v_i \cdot x_i(\mathbf{w}) - \overline{p}(\mathbf{w}) \quad \text{for any } \mathbf{v}, \mathbf{w} \in D.$$

The goal of the optimal mechanism design problem is then to find an individually rational and truthful mechanism $\mathcal{B}$ that maximizes expected revenue $\sum_{\mathbf{v}\in D}\Pr[\mathbf{v}]\cdot\overline{p}(\mathbf{v})$. By the definitions, $\mathcal{B}$ is an optimal mechanism if and only if the set of lotteries $\{\overline{\mathcal{B}}(\mathbf{v}) : \mathbf{v}\in D\} = \{(\mathbf{x}(\mathbf{v}), \overline{p}(\mathbf{v})) : \mathbf{v}\in D\}$ is an optimal solution to the lottery problem, that is, $\mathcal{B}$ is an optimal mechanism if and only if the tuple $(u(\mathbf{v}), \mathbf{x}(\mathbf{v}) : \mathbf{v}\in D)$ it induces is an optimal solution to the standard LP(I), where we similarly replace $\overline{p}(\mathbf{v})$ by the utility $u(\mathbf{v})$ of the buyer.

2.2. Properties of optimal solutions to LP(I). Given an instance $I = (n, \mathcal{D}_1, \ldots, \mathcal{D}_n)$, we let $\mathbf{a}\in D$ denote the valuation vector with a_i being the lowest value in the support of $\mathcal{D}_i$ for each $i\in[n]$. Then we have the following lemma.

LEMMA 2.1. *$u(\mathbf{a}) = 0$ in any optimal solution $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v}\in D)$ to* LP(I).

Proof. Note that in any feasible solution to LP(I) we have $u(\mathbf{v}) \geq u(\mathbf{a})$ for all $\mathbf{v}\in D$ by (2.1). If $u(\mathbf{a}) > 0$, replace $u(\mathbf{v})$ by $u(\mathbf{v}) - u(\mathbf{a})$ for all $\mathbf{v}\in D$, which results in a feasible solution with a higher revenue. ∎

We assume from now on that $u(\mathbf{a}) = 0$ is fixed and $u(\mathbf{a})$ is no longer a variable of LP(I).

LEMMA 2.2. *In any feasible solution $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v}\in D)$ to* LP(I)*, the utility function is monotonically nondecreasing, i.e., for any two valuations $\mathbf{v}, \mathbf{w}$, if $\mathbf{v}\leq\mathbf{w}$, then $u(\mathbf{v})\leq u(\mathbf{w})$.*

Proof. If $\mathbf{v}\leq\mathbf{w}$ then constraint (2.1) implies that $u(\mathbf{v}) - u(\mathbf{w}) \leq 0$. ∎

The allocation function $\mathbf{q}$ is not in general monotonic, but if only one entry of the valuation changes, $\mathbf{q}$ changes monotonically in that coordinate. Given $\mathbf{v}\in D$ and $b\in D_j = \mathsf{supp}(\mathcal{D}_j)$, we use $(\mathbf{v}_{-j}, b)$ to denote the vector obtained from $\mathbf{v}$ by replacing v_j with b. The following lemma shows that if $b > v_j$, then we must have $q_j(\mathbf{v}_{-j}, b) \geq q_j(\mathbf{v})$.

LEMMA 2.3. *Let $\mathbf{v}\in D$ and $v_j < b\in D_j$. Then any feasible solution to LP(I) satisfies $q_j(\mathbf{v}_{-j}, b)\geq q_j(\mathbf{v})$.*

Proof. Let $\mathbf{w} = (\mathbf{v}_{-j}, b)$. Applying (2.1) on both $(\mathbf{v}, \mathbf{w})$ and $(\mathbf{w}, \mathbf{v})$, we get

$$u(\mathbf{v}) - u(\mathbf{w}) \leq \sum_{i\in[n]} (v_i - w_i)\cdot q_i(\mathbf{v}) \quad \text{and} \quad u(\mathbf{w}) - u(\mathbf{v}) \leq \sum_{i\in[n]} (w_i - v_i)\cdot q_i(\mathbf{w}).$$

The lemma follows by summing them up and using $v_i = w_i$ for all $i\neq j$. ∎

The lotteries of an optimal menu are not necessarily complete. However, they are complete for those valuations that are in the upper boundary of the domain D, i.e., have the maximum value in some coordinate (and this value is positive). In particular, if all the item supports have size 2, then all the lotteries in the optimal menu are complete, except possibly for the allocation $\mathbf{q}(\mathbf{a})$ for the valuation $\mathbf{a}$ where all the items have the minimum value.

LEMMA 2.4. *Let $\mathbf{v}\in D$ be a vector in which $v_i > 0$ is the largest value in D_i for some coordinate i. Then any optimal solution $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v}\in D)$ to* LP(I) *satisfies $\sum_{j\in[n]} q_j(\mathbf{v}) = 1$.*

Proof. Suppose that $\sum_{j\in[n]} q_j = 1 - c$ with $c > 0$. Increase the value of $q_i(\mathbf{v})$ by c. Then the value of the objective function strictly increases (by $\Pr[\mathbf{v}]\cdot v_j\cdot c$). The new solution is feasible: note that in (2.1), $q_i(\mathbf{v})$ appears on the right-hand side always with a nonnegative coefficient since $v_i\geq w_i$ for all $\mathbf{w}\in D$. ∎

3. Distributions with support $\{a_i, b\}$. In this section we prove Theorem 1.2. Assume that the n items $i = 1, \dots, n$ have distributions with support $\{a_i, b\}$ of size 2, where $0 \le a_i < b$, with the same high value b. Let q_i denote the probability that item i has value $v_i = a_i$ (and $1 - q_i$ that it has value $v_i = b$). We will show that lotteries do not offer any advantage over deterministic item pricing. A consequence of course is that in this case we can compute the optimal solution in polynomial time.

Fix an optimal set of lotteries L^*. Let N denote the set of all items $\{1, \dots, n\}$. For each subset $S \subseteq N$ of items we let $\mathbf{v}(S)$ be the valuation in which items in S have value b and the rest have value a_i. Let $\Pr(S)$ be the probability of $\mathbf{v}(S)$. Let L_S be the lottery of L^* that the buyer buys for valuation $\mathbf{v}(S)$, and let p_S be the price of L_S. Let $L_\emptyset = (x_1, \dots, x_n, p_\emptyset)$ be the lottery for the valuation $\mathbf{v}(\emptyset)$. Notice that $\sum_{i\in N} x_i \le 1$, and $p_\emptyset \le \sum_{i\in N} a_i x_i$ as the utility is nonnegative. For each subset $S \subseteq N$ of items let $x(S) = \sum_{i\in S} x_i$.

Let R^* be the expected revenue of the optimal set of lotteries L^*. We will show that R^* is bounded from above by a convex combination of the revenues of a set of $n+1$ item pricings. This implies that R^* is no greater than the revenue of the optimal item pricing.

Consider a valuation $\mathbf{v}(S)$ for a subset $S \ne \emptyset$. The utility of lottery $L_\emptyset$ for valuation $\mathbf{v}(S)$ is

$$\sum_{i\notin S} a_i x_i + b\sum_{i\in S} x_i - p_\emptyset \ge \sum_{i\notin S} a_i x_i + b\sum_{i\in S} x_i - \sum_{i\in N} a_i x_i = \sum_{i\in S}(b - a_i)x_i.$$

The utility of the lottery L_S that is bought under $\mathbf{v}(S)$ must be at least as large as that of $L_\emptyset$. The value of the lottery L_S is at most b, thus $b - p_S \ge \sum_{i\in S}(b - a_i)x_i$, hence $p_S \le b - \sum_{i\in S}(b - a_i)x_i$. Therefore, the total optimal expected revenue R^* is

$$\begin{aligned} R^* &= \sum_{\emptyset\ne S\subseteq N} p_S \Pr(S) + p_\emptyset \Pr(\emptyset) \\ &\le \sum_{\emptyset\ne S\subseteq N} [b - \sum_{i\in S}(b - a_i)x_i] \Pr(S) + \sum_{i\in N} a_i x_i \Pr(\emptyset) \\ &= b(1 - \Pr(\emptyset)) - \sum_{i\in N}(b - a_i)x_i(1 - q_i) + \sum_{i\in N} a_i x_i \Pr(\emptyset). \end{aligned}$$

Consider now the following set of $n + 1$ item pricings: pricing π_0 assigns price b to all items; for each $i \in N$, pricing π_i assigns price a_i to item i and b to all the other items. The expected revenue R_0 of π_0 is $b(1-\Pr(\emptyset))$. Under the pricing π_i, the revenue is b if $v_i = a_i$ and $v_j = b$ for some $j \ne i$, and is a_i in all other cases (i.e., if $v_i = b$ or if all $v_j = a_j$). So the expected revenue R_i of π_i is $b(q_i - \Pr(\emptyset)) + a_i(1 - q_i + \Pr(\emptyset))$.

Let $x_0 = 1 - x(N)$, and consider the convex combination $\sum_{i=0}^n x_i R_i$ of the expected revenues of the $n + 1$ pricings $\pi_i, i = 0, \dots, n$. We have

$$\begin{aligned} \sum_{i=0}^n x_i R_i &= x_0 b(1 - \Pr(\emptyset)) + b\sum_{i\in N} x_i(q_i - \Pr(\emptyset)) + \sum_{i\in N} a_i x_i(1 - q_i + \Pr(\emptyset)) \\ &= b\sum_{i=0}^n x_i(1 - \Pr(\emptyset)) - b\sum_{i\in N} x_i(1 - q_i) + \sum_{i\in N} a_i x_i(1 - q_i) + \sum_{i\in N} a_i x_i \Pr(\emptyset) \\ &= b(1 - \Pr(\emptyset)) - \sum_{i\in N}(b - a_i)x_i(1 - q_i) + \sum_{i\in N} a_i x_i \Pr(\emptyset). \end{aligned}$$

Thus, $R^* \le \sum_{i=0}^n x_i R_i$ and, hence, $R^* \le R_i$ for at least one π_i. This finishes the proof of Theorem 1.2.

4. A support-2 instance with exponentially many lotteries. We consider an instance I of the lottery problem with $n+1$ items, $[n+1] = \{1,\ldots,n,n+1\}$, where each item i has 2 possible values drawn according to the probability distribution D_i. In our instance the first n items will be identical: item $i \in [n]$ is supported over $\{1,2\}$ with probabilities $(1-p,p)$. The "special" item $n+1$ is supported over $\{0,s\}$ with probabilities $(1-p,p)$, where the above parameters are

$$p = 1/n^2 \quad \text{and} \quad s = 2 + 1/(np) = 2 + n. \tag{4.1}$$

As there are only two possible (high and low) values for each item, there is a natural bijection $S \mapsto \mathbf{v}(S)$ between $S \subseteq [n+1]$ (items that receive their high values) and valuation vectors: for each $S \subseteq [n+1]$ and $i \in [n]$, $v_i(S) = 2$ if $i \in S$ and $v_i(S) = 1$ otherwise; $v_{n+1}(S) = s$ if $n+1 \in S$ and $v_{n+1}(S) = 0$ otherwise. Now the standard linear program $\mathrm{LP}(I)$ for the lottery problem on input I over allocation variables $q_i(S)$ and utility variables $u(S)$, $S \subseteq [n+1]$, can be written as follows:

(4.2)

$$\text{maximize} \quad \sum_{S\subseteq[n+1]} \Pr[S] \cdot \left(\sum_{i\in[n+1]} v_i(S)\cdot q_i(S) - u(S) \right) \quad \text{subject to}$$

$$u(S) \ge 0,\ q_i(S) \ge 0,\ \sum_{i\in[n+1]} q_i(S) \le 1 \quad \text{for } S \subseteq [n+1] \text{ and } i \in [n+1],$$

$$u(S) - u(T) \le \sum_{i\in[n+1]} (v_i(S) - v_i(T))\cdot q_i(S) \quad \text{for all } S, T \subseteq [n+1]. \tag{4.3}$$

We use $\mathbf{q}(S)$ to denote the vector $(q_1(S),\ldots,q_{n+1}(S))$, and $\textsc{Rev}(S)$ to denote the expected revenue from S, i.e., $\textsc{Rev}(S) = \sum_{i\in[n+1]} v_i(S)\cdot q_i(S) - u(S)$, given $\mathbf{q}(S)$ and $u(S)$. By Lemma 2.1, we have $u(\emptyset) = 0$ in any optimal solution to $\mathrm{LP}(I)$.

We will define a relaxation $\mathrm{LP}'(I)$ of the standard $\mathrm{LP}(I)$, characterize its optimal solutions, and then show that an optimal solution to $\mathrm{LP}'(I)$ must also be feasible and, thus, optimal, for $\mathrm{LP}(I)$.

For the purposes of our analysis, we consider below two different types of sets $S \subseteq [n+1]$ depending on whether they include the special item $n+1$. We consider a set S to be of *type* 1 if $n+1 \notin S$; otherwise we call it a set of *type* 2. Correspondingly we use T_1 and T_2 to denote the set of type-1 and type-2 sets, respectively. We consider the partial order among the subsets of $[n+1]$ and its Hasse diagram (transitive reduction) G. For two sets $S, T \subseteq [n+1]$, we say that S is *above* T if $T \subset S$; the set S is a *parent* of T, and T a *child* of S if they are also adjacent in G, i.e., $|S| = |T| + 1$ and $T \subset S$.

We consider the following relaxed linear program $\mathrm{LP}'(I)$ of $\mathrm{LP}(I)$, which contains only (some of the) constraints (4.3) between adjacent sets, and between type-1 sets and $\emptyset$. $\mathrm{LP}'(I)$ has the same set of variables and maximizes the same objective function as $\mathrm{LP}(I)$ in (4.2) subject to the following constraints:

Part 1: Same constraints on $u(S)$ and $\mathbf{q}(S)$ as in $\mathrm{LP}(I)$:

$$u(S) \ge 0,\ q_i(S) \ge 0,\ \sum_{j\in[n+1]} q_j(S) \le 1 \quad \text{for all } S \subseteq [n+1] \text{ and } i \in [n+1].$$

Part 2: Constraints between (some of the) adjacent sets:

2*a*. $q_i(S) \ge u(S) - u(S \setminus \{i\})$ for all $S \subseteq [n+1]$, $i \in S$ and $i \ne n+1$, 2*b*. $u(S) \geq u(S \setminus \{i\})$ for all $S \subseteq [n+1]$ and $i \in S$.

Part 3: Constraints between type-1 sets and $\emptyset$:

$$u(S) \geq \sum_{i \in S} q_i(\emptyset) \quad \text{for all } S \in T_1.$$

4.1. Characterization of optimal solutions to $\mathbf{LP}'(I)$. Our goal is to give a complete characterization of optimal solutions to $\mathrm{LP}'(I)$. We do this in three steps. First we show how to determine the optimal allocations $\mathbf{q}(S)$, for all $S \neq \emptyset$ once the utilities $u(\cdot)$ are set (Lemma 4.2). Then we show how to determine the optimal utilities $u(\cdot)$ given $\mathbf{q}(\emptyset)$ (Lemma 4.3). Finally, we show that every optimal solution to $\mathrm{LP}'(I)$ must have $\mathbf{q}(\emptyset)$ being the uniform distribution over the first n items (Lemma 4.4). The characterization is summarized in Corollary 4.5.

We start with a few observations. First, note that the constraints of part 2b imply that $u(\cdot)$ is monotonic, i.e., $u(T) \leq u(S)$ for all $T \subset S$. Second, variables $\mathbf{q}(S)$ of each $S \neq \emptyset$ appear in $\mathrm{LP}'(I)$ only in parts 1 and 2a. Given a utility function $u : 2^{[n+1]} \to \mathbb{R}_{\geq 0}$ we define for each $S \neq \emptyset$ the following small linear program, denoted by $\mathrm{LP}(S : u)$ over $n+1$ variables $\mathbf{q} = (q_1, \ldots, q_{n+1})$:

$$\text{maximize} \sum_{j \in [n+1]} v_j(S) \cdot q_j - u(S) \;\text{ subject to} \tag{4.4}$$

$$q_i \geq 0 \;\text{ and } \sum_{j \in [n+1]} q_j \leq 1 \;\text{ for } i \in [n+1], \tag{4.5}$$

$$q_i \geq u(S) - u(S \setminus \{i\}) \;\text{ for } i \in S \cap [n]. \tag{4.6}$$

We emphasize that both $u(S)$ and $u(S \setminus \{i\})$ in $\mathrm{LP}(S : u)$ above are constants, instead of variables, given $u(\cdot)$. The following lemma shows that, given $u(\cdot)$, $\mathbf{q}(S)$ must be an optimal solution to $\mathrm{LP}(S : u)$.

LEMMA 4.1. *Let $(u(\cdot), \mathbf{q}(\cdot))$ be a solution to $LP'(I)$. Then for each $S \neq \emptyset$, $\mathbf{q}(S)$ satisfies all the constraints of $LP'(I)$ that involve $\mathbf{q}(S)$ if and only if it is a feasible solution to $LP(S : u)$. Moreover, if $(u(\cdot), \mathbf{q}(\cdot))$ is an optimal solution to* $\mathrm{LP}'(I)$, *then $\mathbf{q}(S)$ must be an optimal solution to* $\mathrm{LP}(S : u)$ *for all $S \neq \emptyset$.*

Proof. The first part follows from the fact that all the constraints in $\mathrm{LP}'(I)$ that involve $\mathbf{q}(S)$ are present in $\mathrm{LP}(S : u)$. The second part follows from the facts that the objective function of $\mathrm{LP}(S : u)$ is essentially the same as $\textsc{Rev}(S)$ and that $\Pr[S] > 0$ for all S. ∎

Given $u(\cdot)$, it is easy to characterize optimal solutions to $\mathrm{LP}(S : u)$.

LEMMA 4.2. *Suppose* $\mathrm{LP}(S : u)$ *is feasible for some utility function $u(\cdot)$. Then for each set $S \neq \emptyset$, the optimal solutions to* $\mathrm{LP}(S : u)$ *are characterized as follows:*

1. *If $S \in T_1$, then a solution $\mathbf{q}$ of* $\mathrm{LP}(S : u)$ *is optimal if and only if $q_i = 0$ for all $i \notin S$, $q_i \geq u(S) - u(S \setminus \{i\})$ for all $i \in S$, and $\sum_{i \in S} q_i = 1$.*
2. *If $S \in T_2$, then a solution $\mathbf{q}$ of* $\mathrm{LP}(S : u)$ *is optimal if and only if $q_i = 0$ for all $i \notin S$, $q_i = u(S) - u(S \setminus \{i\})$ for all $i \in S \cap [n]$, and $q_{n+1} = 1 - \sum_{i \in S \cap [n]} q_i$.*

Proof. When $S \in T_1$, the objective function is $\sum_{j \in S} 2q_j + \sum_{j \in [n] \setminus S} q_j - u(S)$. Note that all $q_i, i \notin S$, appear only in the constraints (4.5), while all $q_i, i \in S$, are also constrained by lower bounds in (4.6). The LP is feasible if and only if

$\sum_{i\in S}(u(S) - u(S \setminus \{i\})) \leq 1$. If a solution has $q_i > 0$ for some $i \notin S$, then we can obtain a better solution by setting $q_i = 0$ and incrementing q_j for some $j \in S$ by q_i. If $\sum_{i\in S} q_i < 1$ then we can improve the solution by incrementing any $q_i, i \in S$. Therefore, any optimal solution satisfies the claim. Conversely, all solutions that satisfy the claim have the same value.

For the case when $S \in T_2$, the objective function is $\sum_{j\in S\cap[n]} 2q_j + \sum_{j\notin S} q_j + sq_{n+1} - u(S)$. We can argue similarly that $q_i = 0$ for all $i \notin S$. As the coefficient s of q_{n+1} in the objective function is greater than coefficients of other variables q_i, and we have the constraint $\sum_j q_j(S) \leq 1$, it is clearly optimal to give as high a value as possible to q_{n+1}. Thus it is optimal to assign to each $q_i, i \in S \cap [n]$, the lower bound of (4.6) and give the rest of the probability mass to q_{n+1}. ∎

The next lemma tells us how to determine from given values for $\mathbf{q}(\emptyset)$, optimal values for all the utilities $u(S)$ (and from these we can determine optimal values for all $\mathbf{q}(S)$, $S \neq \emptyset$, by the above Lemma 4.2).

LEMMA 4.3. *In any optimal* $(u(\cdot), \mathbf{q}(\cdot))$ *of* $\mathrm{LP}'(I)$ *the following properties hold.*

1. *For every set S of type 1,* $u(S) = \sum_{i\in S} q_i(\emptyset)$. *For every set S of type 2,*

$$u(S) = \max\left(\sum_{i\in S\cap[n]} q_i(\emptyset),\ u(\{n+1\})\right). \tag{4.7}$$

2. $u(\{n+1\}) = \min_{i\in[n]} q_i(\emptyset)$. *Hence* $u(S) = \sum_{i\in S\cap[n]} q_i(\emptyset)$ *for all* $S \neq \{n+1\}$.

Proof. 1. Consider an optimal solution $(u(\cdot), \mathbf{q}(\cdot))$ to $\mathrm{LP}'(I)$. We know from the constraints of $\mathrm{LP}'(I)$ (part 3) that $u(S) \geq \sum_{i\in S} q_i(\emptyset)$ for all $S \in T_1$. For $S \in T_2$, as $u(S) \geq u(S \cap [n]) \geq \sum_{i\in S\cap[n]} q_i(\emptyset)$ and u is monotonic, $u(S) \geq u(\{n + 1\})$, and $u(S)$ is at least as large as the right-hand side (RHS) of (4.7). Call a set S *tight* if its inequality is tight (satisfied as equality, i.e., S satisfies the first property of Lemma 4.3), and *nontight* otherwise, and let R be the family of tight sets. If $u(\emptyset) > 0$, then we can subtract the amount $u(\emptyset)$ from $u(S)$ for all $S \subseteq [n+1]$ and get a feasible solution with strictly higher revenue. Therefore, an optimal solution must have $u(\emptyset) = 0$, and $\emptyset$ is a tight set. By definition, $\{n + 1\}$ is also tight.

Let $\epsilon > 0$ be the minimum difference between the left-hand side (LHS) and the RHS of the inequality over all nontight sets. Form another solution $\{u'(\cdot), q'(\cdot)\}$ by setting $u'(S) = u(S) - \epsilon$ if $S \notin R$, $u'(S) = u(S)$ if $S \in R$, and setting $\mathbf{q}'(S)$ to be a vector of an optimal solution to $\mathrm{LP}(S : u')$ for each $S \neq \emptyset$, while $\mathbf{q}'(\emptyset) = \mathbf{q}(\emptyset)$. We will argue that the new solution is feasible and yields strictly more revenue.

For feasibility note first that by our choice of ϵ the new utilities still satisfy $u'(S) \geq \sum_{i\in S} q_i(\emptyset)$ for $S \in T_1$, so constraints in part 3 are satisfied. For part 2b, letting $T = S \setminus \{i\}$, the only way to possibly get a violation is if T is tight. But then $u'(T) = u(T) = \sum_{i\in T} q_i(\emptyset)$ if $T \in T_1$ or $= \max(\sum_{i\in T\cap[n]} q_i(\emptyset), u(\{n + 1\})$ if $T \in T_2$; in either case we still have $u'(S) \geq u'(T)$. For the constraints in Parts 1 and 2a, we only need to check that all the $\mathrm{LP}(S : u')$ are feasible. We will do this next and also show that the optimal solutions $\mathbf{q}'(S)$ yield an increase in the total revenue.

Case 1: S is not tight. Then the old solution $\mathbf{q}(S)$ remains feasible to $\mathrm{LP}(S : u')$ and the contribution $\textsc{Rev}(S) \cdot \Pr(S)$ of the set S to the total revenue increases by $\epsilon \cdot \Pr(S)$, due to the decrease in $u(S)$.

Case 2: S is tight. If S is of type 1 then $u'(S) = u(S) = \sum_{i\in S} q_i(\emptyset)$ and all its children have $u'(S\setminus\{i\}) \geq \sum_{i\in S\setminus\{i\}} q_i(\emptyset)$. Thus, we can pick any values $q'_i(S) \geq q_i(\emptyset)$, $i \in S$, that sum to 1, and $\textsc{Rev}(S)$ does not decrease.

Suppose S is of type 2. If $u'(S) = u(S) = u(\{n+1\}) \geq \sum_{i\in S\cap[n]} q_i(\emptyset)$, then all its type-2 children must also have $u'(S\setminus\{i\}) = u(S\setminus\{i\}) = u(\{n+1\})$. Setting $q'_i(S) = 0$ for all $i \in [n]$ and $q'_{n+1}(S) = 1$ yields a feasible solution and $\textsc{Rev}(S)$ does not decrease.

Finally suppose that the type-2 set S has $u'(S) = u(S) = \sum_{i\in S\cap[n]} q_i(\emptyset) > u(\{n+1\})$. Then we set $q'_i(S) = u'(S) - u'(S\setminus\{i\})$ for each $i \in S\cap[n]$, $q'_i(S) = 0$ for $i \in [n]\setminus S$, and $q'_{n+1}(S) = 1 - \sum_{i\in[n]} q'_i(S)$. As $u'(S\setminus\{i\}) \geq \sum_{i\in(S\setminus\{i\})\cap[n]} q_i(\emptyset)$ for all $i \in [n]$, we have $q'_i(S) \leq q'_i(\emptyset)$ and, hence, $q'_{n+1}(S) \geq 0$, thus $\text{LP}(S : u')$ is feasible. Compared to the previous values of $\mathbf{q}(S)$ that were an optimal solution to $\text{LP}(S : u)$, for each nontight child $S\setminus\{i\}$, $i \in S\cap[n]$, the corresponding value $q'_i(S)$ increased by ϵ; on the other hand, the value of $q'_{n+1}(S)$ decreased by $k\cdot\epsilon$, where k is the number of such nontight children. The contribution $\textsc{Rev}(S)\cdot\Pr(S)$ of S to the total revenue decreased by $k\cdot\epsilon(s-2)\Pr(S)$. Charge this decrease in revenue for S equally to its k nontight type-2 children (their revenue increased by Case 1). It remains to verify that no set is overcharged, in fact, every nontight set still has a net positive increase in its revenue contribution.

Consider a nontight set S (of type 2) which is charged by some of its (tight) parents, say by t parents. Since $|S| \geq 2$ (recall $\{n+1\}$ is tight), we have $t \leq n-1$. Every parent has size $|S|+1$, hence its probability is $\Pr(S)\cdot(p/(1-p))$. The total charge from the parents is at most $(n-1)\epsilon(s-2)\Pr(S)p/(1-p)$, and it suffices to verify that this is strictly smaller than $\epsilon\cdot\Pr(S)$, in other words, that $(n-1)(s-2)p < 1-p$, i.e., that $(n-1)/n < 1-(1/n^2)$, which is true.

We conclude that in an optimal solution, all the sets must be tight.

2. Suppose that $u(\{n+1\}) > \min_{i\in[n]} q_i(\emptyset)$. Now we know that all sets satisfy the property of item 1. Call now a type-2 set S tight if $u(S) = \sum_{i\in S\cap[n]} q_i(\emptyset) \geq u(\{n+1\})$. Decrease the utility of all nontight sets, i.e., of $\{n+1\}$ and of all other type-2 sets S with $u(S) = u(\{n+1\}) > \sum_{i\in S\cap[n]} q_i(\emptyset)$, by a small amount $\epsilon > 0$. Using the same arguments as in item 1 this yields a new solution $\{u'(\cdot), \mathbf{q}'(\cdot)\}$ that is feasible and has strictly higher revenue. The only difference here is that the set $\{n+1\}$ is nontight, and this set has n parents. However, since $u(\{n+1\}) > \min_{i\in[n]} q_i(\emptyset)$, at least one of its parents (namely, the set $\{i, n+1\}$, where i has the minimum $q_i(\emptyset)$) is not tight, hence $\{n+1\}$ is charged again by at most $n-1$ of its parents, and the inequality of item 1 still holds.

Suppose $u(\{n+1\}) < \min_{i\in[n]} q_i(\emptyset)$. Then increase the utility of $\{n+1\}$ by a small amount $\epsilon > 0$ and keep the other utilities the same. The only sets that are affected are $\{n+1\}$ and its parents $\{i, n+1\}$, $i \in [n]$. The contribution $\textsc{Rev}(\{n+1\})\cdot\Pr(\{n+1\})$ of $\{n+1\}$ to the revenue decreases by $\epsilon\Pr(\{n+1\})$. For each parent $\{i, n+1\}$, the corresponding variable $q_i(\{i, n+1\})$ decreases by ϵ, and $q_{n+1}(\{i, n+1\})$ increases by ϵ, hence the contribution of $\{i, n+1\}$ to the revenue increases by $(s-2)\epsilon\cdot\Pr(\{i, n+1\})$. Since $\Pr(\{i, n+1\}) = \Pr(\{n+1\}p/(1-p)$ for each $i \in [n]$, the net effect on the total revenue is positive, provided that $n(s-2)p/(1-p) > 1$, i.e., $1/(1-p) > 1$, which is true.

We conclude that $u(\{n+1\}) = \min_{i\in[n]} q_i(\emptyset)$ in an optimal solution. Combined with item 1, this also implies that $u(S) = \sum_{i\in S\cap[n]} q_i(\emptyset)$ for all $S \neq \{n+1\}$. ∎

From the above lemmas, we know how to derive from $\mathbf{q}(\emptyset)$ all the u's and all the $\mathbf{q}(S)$, $S \neq \emptyset$, in an optimal solution, so we can calculate the revenue as a function of $\mathbf{q}(\emptyset)$. We can then determine the optimal value of $\mathbf{q}(\emptyset)$ that maximizes the revenue.

LEMMA 4.4. *Any optimal solution to* $\mathrm{LP}'(I)$ *satisfies* $q_{n+1}(\emptyset) = 0$ *and* $q_i(\emptyset) = 1/n$ *for all* $i \in [n]$.

Proof. We calculate the revenue of a solution that satisfies the properties of Lemmas 4.2 and 4.3.

For a type-1 set $S \neq \emptyset$, we have

$$\textsc{Rev}(S) = \sum_{i\in S} 2q_i(S) - u(S) = 2 - \sum_{i\in S} q_i(\emptyset) \quad \text{and}$$
$$\textsc{Rev}(\emptyset) = \sum_{i\in[n]} 1q_i(\emptyset) - u(\emptyset) = \sum_{i\in[n]} q_i(\emptyset).$$

Thus the contribution of type-1 sets to the revenue, denoted by $\textsc{Rev}_1$, is

$$\begin{aligned}\textsc{Rev}_1 &= \sum_{S\in T_1\setminus\emptyset} \Pr[S]\cdot\left(2 - \sum_{i\in S} q_i(\emptyset)\right) + \Pr[\emptyset]\cdot\left(\sum_{i\in[n]} q_i(\emptyset)\right)\\ &= 2\cdot\sum_{S\in T_1\setminus\emptyset}\Pr[S] + \sum_{i\in[n]} q_i(\emptyset)\cdot\left(\Pr[\emptyset] - \sum_{S\in T_1\,\mid\, i\in S}\Pr[S]\right).\end{aligned}$$

The first term is a constant, const_1, independent of $\mathbf{q}(\emptyset)$. For the coefficient of $q_i(\emptyset)$, $\Pr[\emptyset] = (1-p)^{n+1}$ while the total probability of all sets $S \in T_1$ that contain i is $(1-p)p$. Therefore, all $i \in [n]$ have the same coefficient $c_1 = (1-p)^{n+1} - (1-p)p \approx 1-(n+2)p \approx 1$. So the expression above is of the form $\textsc{Rev}_1 = \mathrm{const}_1 + c_1\cdot\sum_{i\in[n]} q_i(\emptyset)$.

For each type-2 set $S \neq \{n+1\}$, by Lemma 4.3, we have $u(S) = \sum_{i\in S\cap[n]} q_i(\emptyset)$. Hence, for $|S| > 2$, we have $q_i(S) = q_i(\emptyset)$ for $i \in S\cap[n]$; $q_{n+1}(S) = 1-\sum_{i\in S\cap[n]} q_i(\emptyset)$; and $q_i(S) = 0$ for $i \notin S$. Thus,

$$\textsc{Rev}(S) = 2\sum_{i\in S\cap[n]} q_i(\emptyset) + s\left(1 - \sum_{i\in S\cap[n]} q_i(\emptyset)\right) - \sum_{i\in S\cap[n]} q_i(\emptyset) = s-(s-1)\sum_{i\in S\cap[n]} q_i(\emptyset).$$

Let $q_{\min} = \min_{i\in[n]} q_i(\emptyset) = u(\{n+1\})$. For $S_i = \{i, n+1\}$ we have $u(S_i) = q_i(\emptyset)$, $q_i(S_i) = q_i(\emptyset) - q_{\min}$, $q_{n+1}(S_i) = 1 - (q_i(\emptyset) - q_{\min})$, and $q_j(S_i) = 0$ for $j \notin S_i$. Thus,

$$\textsc{Rev}(\{i, n+1\}) = 2(q_i(\emptyset)-q_{\min}) + s(1-q_i(\emptyset)+q_{\min}) - q_i(\emptyset) = s-(s-1)q_i(\emptyset)+(s-2)q_{\min}.$$

For $S = \{n+1\}$, we have $u(\{n+1\}) = q_{\min}$, $q_{n+1} = 1$, and $q_i = 0$ for all $i \in [n]$. Thus, $\textsc{Rev}(\{n+1\}) = s - q_{\min}$. Therefore, the revenue from type-2 sets is

$$\begin{aligned}\textsc{Rev}_2 = s\sum_{S\in T_2}\Pr[S] - (s-1)\sum_{S\in T_2}\Pr[S]\sum_{i\in S\cap[n]} q_i(\emptyset)\\ + (s-2)q_{\min}\sum_{i\in[n]}\Pr[\{i, n+1\}] - q_{\min}\Pr[\{n+1\}].\end{aligned}$$

The first term is a constant, sp. The coefficient of each $q_i(\emptyset)$ is $-(s-1)$ times the sum of the probabilities of all the type-2 sets that contain i, thus, it is $-(s-1)p^2$, i.e., a constant $-c_2$ independent of i, which satisfies $c_2 = \Theta(1/n^3) \ll c_1$. The coefficient of $q_{\min}$ is

$$c_3 = (s-2)np^2(1-p)^{n-1} - p(1-p)^n = p^2(1-p)^{n-1} > 0.$$

So $\textsc{Rev}_2 = sp - c_2 \sum_{i\in[n]} q_i(\emptyset) + c_3 q_{\min}$. Combining with $\textsc{Rev}_1$, the total revenue is

$$\textsc{Rev} = \text{const} + (c_1 - c_2) \sum_{i\in[n]} q_i(\emptyset) + c_3 q_{\min}.$$

As $(c_1 - c_2) > 0$ and $c_3 > 0$, in order to maximize the revenue we want to maximize $\sum_{i\in[n]} q_i(\emptyset)$ and $q_{\min}$. Both these quantities are maximized simultaneously if we have $\sum_{i\in[n]} q_i(\emptyset) = 1$ and $q_{\min} = 1/n$, that is, if $q_i(\emptyset) = 1/n$ for $i \in [n]$. Setting $\mathbf{q}(\emptyset)$ to these values and then setting all the utilities $u(S)$ and allocations $\mathbf{q}(S)$ for all $S \neq \emptyset$ according to Lemmas 4.2 and 4.3, yields a feasible solution to $\mathrm{LP}(I')$ that achieves the maximum possible revenue. Therefore, any optimal solution to $\mathrm{LP}'(I)$ must achieve the same revenue and, hence, must also satisfy $q_i(\emptyset) = 1/n$ for all $i \in [n]$. ∎

Combining the previous lemmas yields the following characterization of optimal solutions to $\mathrm{LP}'(I)$.

COROLLARY 4.5. *Any optimal solution* $(u(\cdot), \mathbf{q}(\cdot))$ *to* $\mathrm{LP}'(I)$ *satisfies*

- $\mathbf{q}(\emptyset) = (1/n, \ldots, 1/n, 0)$;
- $u(S) = |S \cap [n]|/n$ *for all* $S \neq \{n+1\}$; $u(\{n+1\}) = 1/n$;
- *for* $S \in T_1 \setminus \{\emptyset\}$: $q_i(S) = 0$ *for* $i \notin S$; $q_i(S) \geq 1/n$ *for* $i \in S$; $\sum_{i\in S} q_i(S) = 1$;
- *for* $S \in T_2$ *and* $|S| \leq 2$: $\mathbf{q}(S) = (0, \ldots, 0, 1)$;
- *for* $S \in T_2$ *and* $|S| > 2$: $q_i(S) = 0$ *for* $i \notin S$; $q_i(S) = 1/n$ *for* $i \in S \cap [n]$;
$$q_{n+1}(S) = 1 - \frac{|S \cap [n]|}{n}.$$

4.2. Returning to the standard LP(I). Now that we have characterized the optimal solutions of the relaxed $\mathrm{LP}'(I)$ in Corollary 4.5, it is straightforward to show that they are also feasible and, hence, also optimal, in the full standard $\mathrm{LP}(I)$.

LEMMA 4.6. *Any optimal solution* $(u(\cdot), \mathbf{q}(\cdot))$ *to* $\mathrm{LP}'(I)$ *is a feasible (and optimal) solution to* $\mathrm{LP}(I)$.

Proof. Consider an optimal solution $(u(\cdot), \mathbf{q}(\cdot))$ to $\mathrm{LP}'(I)$. We need to show that (4.3) holds for any two subsets $T, S \subseteq [n+1]$. The case when $S = \emptyset$ is easy to check, since we have $u(S) - u(T) = -u(T)$,

$$\sum_{i\in[n+1]} (v_i(S) - v_i(T)) \cdot q_i(S) = -\frac{|T \cap [n]|}{n},$$

and $u(T) \geq |T \cap [n]|/n$ for all T. Below we assume that $S \neq \emptyset$.

We claim that it suffices to show (4.3) for $T, S \subseteq [n+1]$ that satisfy $T \subseteq S$ and $S \neq \emptyset$. To see this, consider any T, S with $S \neq \emptyset$, let $T' = T \cap S \subseteq S$, and suppose that T', S satisfy (4.3): $u(S) - u(T') \leq \sum_{i\in[n+1]}(v_i(S) - v_i(T'))q_i(S)$. Note that $u(T) \geq u(T')$ by the monotonicity of u, hence $u(S) - u(T) \leq u(S) - u(T')$. Further, $v_i(T)$ and $v_i(T')$ differ only on elements $i \in T \setminus T' = T \setminus S$, but $q_i(S) = 0$ for all such i since $S \neq \emptyset$ by Corollary 4.5. Therefore, we have

$$\sum_{i\in[n+1]} (v_i(S) - v_i(T))q_i(S) = \sum_{i\in[n+1]} (v_i(S) - v_i(T'))q_i(S),$$

and (4.3) holds for T, S as well.

Consider two sets $T \subset S \subseteq [n+1]$. If S is not one of the sets $\{n+1\}$ or $\{i, n+1\}, i \in [n]$, then the LHS of (4.3) is $u(S) - u(T) \le |(S \setminus T) \cap [n]|/n$ which is at most $\sum_{i\in[n+1]}(v_i(S) - v_i(T))q_i(S)$ because $q_i(S) \ge 1/n$ for all $i \in (S \setminus T) \cap [n]$. If S is $\{n+1\}$ or $\{i, n+1\}$, $i \in [n]$, then either $T = \emptyset$, in which case

$$u(S) - u(T) = \frac{1}{n} \le \sum_{i\in[n+1]} (v_i(S) - v_i(T))q_i(S) = s,$$

or $|T| = 1$, in which case $u(T) = 1/n$ and

$$u(S) - u(T) = 0 \le \sum_{i\in[n+1]} (v_i(S) - v_i(T))q_i(S).$$

Thus, (4.3) is satisfied in all cases. This finishes the proof of the lemma. □

It follows from Lemma 4.6 that $\{u(\cdot), \mathbf{q}(\cdot)\}$ is an optimal solution to LP(I) if and only if it is an optimal solution to LP′(I). Finally, we show that any optimal solution to LP(I) requires an exponential number of lotteries.

THEOREM 4.7. *Any optimal solution* $(u(\cdot), \mathbf{q}(\cdot))$ *to* I *has* $\Theta(2^n)$ *different lotteries.*

Proof. For all $S \subseteq [n+1]$, except for $\emptyset, [n+1]$ and $\{i, n+1\}$ for $i \in [n]$, the support of $\mathbf{q}(S)$ is equal to S, thus all these lotteries are different. Hence any optimal solution has $2^{n+1} - n - 2$ different lotteries. □

We remark that, even though an optimal solution has an exponential number of different lotteries, it follows from Corollary 4.5 that there is a deterministic polynomial-time algorithm, which takes as input a valuation $\mathbf{v} \in D$ and outputs an optimal lottery for $\mathbf{v}$: letting S be the set of items that have high value in the given valuation $\mathbf{v}$, we can obtain the corresponding allocation $\mathbf{q}(S)$ and utility $u(S)$ of the optimal solution of the LP from Corollary 4.5, and the price of the lottery can be easily derived from the allocation and the utility.

5. Hardness of optimal mechanism design.

In this section we prove Theorem 1.4. This is done by giving a polynomial-time reduction from a #P-hard problem called COMP. We delay its definition and proof of #P-hardness to section 5.2.3 because it is used only towards the end of the proof.

This section is organized as follows. In section 5.1, we characterize optimal solutions to a relaxation to the standard linear program LP(I) when parameters of the instance I satisfy certain conditions. We call the relaxed linear program LP′(I), and the characterization is summarized in section 5.1.7. In section 5.2 we pin down the rest of the parameters of I to embed the #P-hard problem COMP. More formally, one can construct an instance I to the lottery problem from an instance of COMP in polynomial time such that a specific entry of any optimal solution to LP′(I) can be used to answer COMP. Finally we show that for such instances I, any optimal solution to LP′(I) must be an optimal solution to LP(I) and, thus, LP′(I) and LP(I) share the same set of optimal solutions. Then an efficient universal algorithm for the optimal mechanism design problem implies $\mathrm{P}^{\mathrm{NP}} = \mathrm{P}^{\#\mathrm{P}}$. This finishes the proof of Theorem 1.4.

5.1. Linear program relaxation.

Let I denote an instance of $n+2$ items with the following properties. Each item $i \in [n]$ is supported over $D_i = \{a_i, \ell_i, h_i\}$ with $a_i < \ell_i < h_i$. Probabilities of a_i, ℓ_i and h_i are $1 - p - r$, p, and r, respectively, where

$$p = 1/2^{n^4} \quad \text{and} \quad r = p/2^{n^2}. \tag{5.1}$$

So p and r satisfy $p=(r/p)^{n^2}$. Let $\beta=1/2^n$. The support $\{a_i,\ell_i,h_i\}$ of item $i\in[n]$ satisfies

$$\ell_i=2+3(n-i)\beta,\quad \ell_i+\beta\le h_i\le \ell_i+\left(1+\frac{1}{2^{2n}}\right)\beta,\quad\text{and}\quad |a_i-1|=O(np). \tag{5.2}$$

Let $d_i=\ell_i-a_i\approx 1$ and $\tau_i=h_i-\ell_i$. Our choices of ℓ_i and h_i guarantee that $\tau_i\approx\beta$ as well as $\ell_i>h_{i+1}+\beta$ (or $\ell_i\approx h_{i+1}+2\beta$ more exactly) for all i from 1 to $n-1$. Item $n+1$ takes value 0 with probability $1-\delta$, and s with probability δ; item $n+2$ takes value 0 with probability $1-\delta^2$, and t with probability δ^2. So let $D_{n+1}=\{0,s\}$, $D_{n+2}=\{0,t\}$, and $D=D_1\times\cdots\times D_{n+2}$. We impose the following conditions on δ,s, and t throughout section 5.1:

$$\delta=\frac{1}{2^{n^6}},\quad s=\Theta\left(\frac{1}{pn}\right),\quad t=O\left(\frac{\beta}{r^{m+1}m}\right),\quad t=\Omega\left(\frac{\beta}{r^{m+1}m2^n}\right), \tag{5.3}$$

where $m=\lceil n/2\rceil$.

Note that $\delta\ll r\ll p$, and $t=2^{\Theta(n^5)}\gg s=2^{\Theta(n^4)}\gg 1$. Precise values of the a_i's, the h_i's, and s and t will be chosen later on in section 5.2 after we have analyzed the structure of the problem. In particular, the h_i's and t will be used to reflect the instance of the #P-hard problem that we will embed in I and LP(I). (5.1), (5.2), and (5.3) are sufficient for our analysis in section 5.1 of the relaxed LP to be described below.

We need some notation before describing the relaxation of LP(I). Given $\mathbf{v}\in D$, we use $S(\mathbf{v})$ to denote the set of $i\in[n]$ such that $v_i\in\{\ell_i,h_i\}$, $S^-(\mathbf{v})$ to denote the set of $i\in[n]$ such that $v_i=\ell_i$, and $S^+(\mathbf{v})$ to denote the set of $i\in[n]$ such that $v_i=h_i$. So we always have $S(\mathbf{v})=S^+(\mathbf{v})\cup S^-(\mathbf{v})\subseteq[n]$.

Next we partition D into T_1,T_2,T_3,T_4, where T_1 consists of vectors with $v_{n+1}=v_{n+2}=0$, T_2 consists of vectors with $v_{n+1}=s$, and $v_{n+2}=0$, T_3 consists of vectors with $v_{n+2}=t$ and $v_{n+1}=0$, and T_4 consists of vectors with $v_{n+1}=s$ and $v_{n+2}=t$. We call vectors in T_i *type-i* vectors. We denote the bottom vector $(a_1,\ldots,a_n,0,0)$ by $\mathbf{a}$, $(a_1,\ldots,a_n,s,0)$ by $\mathbf{c}_2$, $(a_1,\ldots,a_n,0,t)$ by $\mathbf{c}_3$, and $(a_1,\ldots,a_n,s,t)$ by $\mathbf{c}_4$ (so $\mathbf{c}_i$ is the *bottom* of type-i vectors for i from 2 to 4). By Lemma 2.1, we have $u(\mathbf{a})=0$ in any optimal solution to LP(I) so we fix it to be 0.

Given $\mathbf{v}\in D$, we write Block($\mathbf{v}$) to denote the set of $\mathbf{w}\in D$ with $S(\mathbf{w})=S(\mathbf{v})$, $w_{n+1}=v_{n+1}$, and $w_{n+2}=v_{n+2}$; we refer to Block($\mathbf{v}$) as the *block* that contains $\mathbf{v}$. It would also be helpful to view each T_i as a collection of (disjoint) blocks. We say $\mathbf{v}\in D$ is *essential* if $S^+(\mathbf{v})=\emptyset$ (here the intuition is that within each block, there is a unique essential vector with the largest mass of probability, given $r\ll p$ in (5.1)). We use D' to denote the set of essential vectors, and write $T_i'=T_i\cap D'$ and $T_i^*=T_i\setminus T_i'$ for each i. Given $\mathbf{v}\in D$, we use Lower($\mathbf{v}$) to denote the unique essential vector in Block($\mathbf{v}$), i.e., Lower($\mathbf{v}$) is the vector obtained by replacing each h_i in $\mathbf{v}$ by ℓ_i.

We let $\min(S(\mathbf{v}))$ denote the smallest index in $S(\mathbf{v})$ and let $S'(\mathbf{v})$ denote $S(\mathbf{v})\setminus\{\min(S(\mathbf{v}))\}$.

Given a vector $\mathbf{v}\in D$ we follow the convention and write $(\mathbf{v}_{-i},\alpha)$ to denote the vector obtained from $\mathbf{v}$ by replacing its ith entry v_i with α. We write $(\mathbf{v}_{[n]},\alpha,\alpha')$ to denote the vector obtained from $\mathbf{v}$ by replacing v_{n+1} with α and v_{n+2} with α'. We let $\rho: T_2\cup T_3\cup T_4\to T_1$ denote the map with $\rho(\mathbf{v})=(\mathbf{v}_{[n]},0,0)$.

Given two vectors $\mathbf{v},\mathbf{w}\in T_i$ *of the same type*, we write $\mathbf{v}\prec\mathbf{w}$ (or say that $\mathbf{v}$ lies below $\mathbf{w}$, or $\mathbf{w}$ lies above $\mathbf{v}$) if either $S(\mathbf{v})\subset S(\mathbf{w})$, or $S(\mathbf{v})=S(\mathbf{w})$ and $S^+(\mathbf{v})\subset S^+(\mathbf{w})$. By definition $\prec$ is transitive.

The linear program $\text{LP}'(I)$ is presented in Figure 5.1 which has the same objective function and variables $(u(\mathbf{v}),\mathbf{q}(\mathbf{v}):\mathbf{v}\in D)$ as $\text{LP}(I)$. We refer to $u(\mathbf{v})$ and $\mathbf{q}(\mathbf{v})$ as the utility and allocation variables of $\mathbf{v}\in D$, respectively. For convenience, we write $(u(\cdot),\mathbf{q}(\cdot))$ to denote a solution to $\text{LP}'(I)$, and call $u(\cdot):D\rightarrow\mathbb{R}_{\ge 0}$ a utility function. Constraints in Part 1 of $\text{LP}'(I)$ concerns variables of type-1 vectors; Part 2 concerns type-2 and type-1 vectors; Part 3 concerns type-3 and type-1 vectors; Part 4 concerns type-4, -3, and -1 vectors.

$$\text{Maximize } \sum_{\mathbf{v}\in D}\Pr[\mathbf{v}]\cdot\left(\sum_{i\in[n+2]} v_i\cdot q_i(\mathbf{v})-u(\mathbf{v})\right) \text{ subject to}$$

Part 0. Same constraints on $u(\mathbf{v})$ and $\mathbf{q}(\mathbf{v})$ as in $\text{LP}(I)$:

$$u(\mathbf{v})\ge 0,\ q_i(\mathbf{v})\ge 0,\ \text{and} \sum_{j\in[n+2]} q_j(\mathbf{v})\le 1 \quad \text{for } \mathbf{v}\in D \text{ and } i\in[n+2].$$

Part 1. Constraints on type-1 vectors:

$$u(\mathbf{v})\ge \sum_{i\in S(\mathbf{v})} d_i\cdot q_i(\mathbf{a}) \quad \text{for } \mathbf{v}\in T_1' \tag{5.4}$$

$$u(\mathbf{v})-u(\mathbf{w})\le \tau_i\cdot q_i(\mathbf{v}) \quad \text{for } \mathbf{v}\in T_1,\ i\in S^+(\mathbf{v}) \text{ and } \mathbf{w}=(\mathbf{v}_{-i},\ell_i), \tag{5.5}$$

$$u(\mathbf{v})-u(\mathbf{w})\ge \sum_{j\in S^+(\mathbf{v})} \tau_j\cdot q_j(\mathbf{w}) \quad \text{for } \mathbf{v}\in T_1,\ \mathbf{w}=\textsc{Lower}(\mathbf{v}), \tag{5.6}$$

$$u(\mathbf{v})\ge u(\mathbf{w}) \quad \text{for } \mathbf{v}\in T_1,\ i\in S(\mathbf{v}),\ \mathbf{w}=\textsc{Lower}(\mathbf{v}_{-i},a_i), \tag{5.7}$$

$$u(\mathbf{v})-u(\mathbf{w})\le \sum_{j\in[n]}(v_j-w_j)\cdot q_j(\mathbf{v}), \quad \text{for } \mathbf{v}\in T_1,\ i\in S(\mathbf{v}),\ \mathbf{w}\in\textsc{Block}(\mathbf{v}_{-i},a_i). \tag{5.8}$$

Part 2. Constraints on type-2 vectors:

$$u(\mathbf{v})\ge u(\rho(\mathbf{v})) \text{ and } u(\mathbf{v})\ge u(\mathbf{c}_2) \quad \text{for } \mathbf{v}\in T_2, \tag{5.9}$$

$$u(\mathbf{v})-u(\mathbf{w})\le \tau_i\cdot q_i(\mathbf{v}) \quad \text{for } \mathbf{v}\in T_2,\ i\in S^+(\mathbf{v}),\ \mathbf{w}=(\mathbf{v}_{-i},\ell_i), \tag{5.10}$$

$$u(\mathbf{v})-u(\mathbf{w})\le \sum_{j\in[n]}(v_j-w_j)\cdot q_j(\mathbf{v}) \quad \text{for } \mathbf{v}\in T_2,\ i\in S(\mathbf{v}),\ \mathbf{w}\in\textsc{Block}(\mathbf{v}_{-i},a_i). \tag{5.11}$$

Part 3: Constraints on type-3 vectors:

$$u(\mathbf{v})\ge u(\rho(\mathbf{v})) \text{ and } u(\mathbf{v})\ge u(\mathbf{c}_3) \quad \text{for } \mathbf{v}\in T_3, \tag{5.12}$$

$$u(\mathbf{v})-u(\mathbf{w})\le \tau_i\cdot q_i(\mathbf{v}) \quad \text{for } \mathbf{v}\in T_3,\ i\in S^+(\mathbf{v}),\ \mathbf{w}=(\mathbf{v}_{-i},\ell_i) \tag{5.13}$$

$$u(\mathbf{v})-u(\mathbf{w})\le \sum_{j\in[n]}(v_j-w_j)\cdot q_j(\mathbf{v}) \quad \text{for } \mathbf{v}\in T_3,\ i\in S(\mathbf{v}),\ \mathbf{w}\in\textsc{Block}(\mathbf{v}_{-i},a_i). \tag{5.14}$$

FIG. 5.1. *(continued)*

Part 4: Constraints on type-4 vectors:

$$u(\mathbf{c}_4) \ge u(\mathbf{c}_3) \text{ and } u(\mathbf{c}_4) - u(\mathbf{c}_3) \le s \cdot q_{n+1}(\mathbf{c}_4),$$

$$u(\mathbf{v}) \ge u(\rho(\mathbf{v})) \text{ and } u(\mathbf{v}) \ge u(\mathbf{c}_4) \quad \text{for } \mathbf{v} \in T_4, \tag{5.15}$$

$$u(\mathbf{v}) - u(\mathbf{w}) \le \tau_i \cdot q_i(\mathbf{v}) \quad \text{for } \mathbf{v} \in T_4,\ i \in S^+(\mathbf{v}),\ \mathbf{w} = (\mathbf{v}_{-i}, \ell_i), \tag{5.16}$$

$$u(\mathbf{v}) - u(\mathbf{w}) \le \sum_{j \in [n]} (v_j - w_j) \cdot q_j(\mathbf{v}) \quad \text{for } \mathbf{v} \in T_4,\ i \in S(\mathbf{v}),\ \mathbf{w} \in \textsc{Block}(\mathbf{v}_{-i}, a_i). \tag{5.17}$$

FIG. 5.1. *Relaxed linear program LP′(I).*

It is easy to check that $\text{LP}'(I)$ is a relaxation of $\text{LP}(I)$. Compared to the standard LP, we included in $\text{LP}'(I)$ the constraint between $\mathbf{v}$ and $\mathbf{w}$ as given in (2.1) for only a carefully picked subset of pairs. At a high level, we would like to keep as few constraints (2.1) of $\text{LP}(I)$ as possible to simplify the characterization of optimal solutions to $\text{LP}'(I)$; on the other hand, we need to keep enough constraints of $\text{LP}(I)$ so that at the end, optimal solutions to $\text{LP}'(I)$ can be shown to be feasible to $\text{LP}(I)$ and, thus, are optimal solutions to $\text{LP}(I)$ as well.

5.1.1. Properties of a small linear program. We start with the following lemma on $\mathbf{q}(\mathbf{c}_2)$, $\mathbf{q}(\mathbf{c}_3)$, and $\mathbf{q}(\mathbf{c}_4)$ in any optimal solution to $\text{LP}'(I)$.

LEMMA 5.1. *If* $(u(\cdot), \mathbf{q}(\cdot))$ *is an optimal solution to* $\text{LP}'(I)$*, then it satisfies*

$$q_{n+1}(\mathbf{c}_2) = 1, \quad q_{n+2}(\mathbf{c}_3) = 1, \quad q_{n+1}(\mathbf{c}_4) = \frac{u(\mathbf{c}_4) - u(\mathbf{c}_3)}{s}, \quad q_{n+2}(\mathbf{c}_4) = 1 - q_{n+1}(\mathbf{c}_4);$$

all other entries of the three vectors $\mathbf{q}(\mathbf{c}_2), \mathbf{q}(\mathbf{c}_3)$*, and* $\mathbf{q}(\mathbf{c}_4)$ *are* 0*.*

Proof. No constraint in $\text{LP}'(I)$ involves $\mathbf{q}(\mathbf{c}_2)$ or $\mathbf{q}(\mathbf{c}_3)$ other than those in Part 0. For $\mathbf{q}(\mathbf{c}_4)$, in additional to Part 0, there is a constraint in Part 4 that involves $q_{n+1}(\mathbf{c}_4)$: $s \cdot q_{n+1}(\mathbf{c}_4) \ge u(\mathbf{c}_4) - u(\mathbf{c}_3)$. (Note that we have $u(\mathbf{c}_4) \ge u(\mathbf{c}_3)$ by (5.15) in Part 4.) The lemma then follows from the objective function and that $t \gg s \gg 1$. ◻

Let $\hat{D} = T_2 \cup T_3 \cup T_4 \setminus \{\mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$. Vectors $\mathbf{q}(\mathbf{v})$ for $\mathbf{v} \in \hat{D}$ are more involved. Given a utility function $u : D \to \mathbb{R}_{\ge 0}$, we define for each $\mathbf{v} \in \hat{D}$ the following small linear program $\text{LP}(\mathbf{v} : u)$ over $n+2$ variables $\mathbf{q} = (q_1, \ldots, q_{n+2})$:

$$\text{maximize} \quad \sum_{j \in [n+2]} v_j \cdot q_j - u(\mathbf{v}) \quad \text{subject to}$$

$$q_i \ge 0 \quad \text{and} \quad \sum_{j \in [n+2]} q_j \le 1 \quad \text{for } i \in [n+2], \tag{5.18}$$

$$\tau_i \cdot q_i \ge u(\mathbf{v}) - u(\mathbf{w}) \quad \text{for } i \in S^+(\mathbf{v}) \text{ and } \mathbf{w} = (\mathbf{v}_{-i}, \ell_i), \tag{5.19}$$

$$\sum_{j \in [n]} (v_j - w_j) \cdot q_j \ge u(\mathbf{v}) - u(\mathbf{w}) \quad \text{for } i \in S(\mathbf{v}) \text{ and } \mathbf{w} \in \textsc{Block}(\mathbf{v}_{-i}, a_i). \tag{5.20}$$

Note that $\text{LP}(\mathbf{v} : u)$ uses utilities of $\mathbf{v}$ and $\mathbf{w}$ in blocks nearby $\mathbf{v}$ given by u (so the RHS of the constraints $u(\mathbf{v}) - u(\mathbf{w})$ and $u(\mathbf{v})$ in the objective function are all constants instead of variables), and that q_{n+1}, q_{n+2}, and q_i, $i \in [n] \setminus S(\mathbf{v})$, do not appear in constraints of $\text{LP}(\mathbf{v} : u)$ other than (5.18) and the objective function.

Comparing $\text{LP}'(I)$ and $\text{LP}(\mathbf{v} : u)$ gives us the following lemma.

LEMMA 5.2. *Given a utility function* $u(\cdot) : D \to \mathbb{R}_{\geq 0}$ *and* $\mathbf{v} \in \hat{D}$, $\mathbf{q}(\mathbf{v})$ *satisfies all constraints in* $\text{LP}'(\mathbf{v})$ *that involve* $\mathbf{q}(\mathbf{v})$ *if and only if it is a feasible solution to* $\text{LP}(\mathbf{v} : u)$. *Moreover, if* $(u(\cdot), \mathbf{q}(\cdot))$ *is an optimal solution to* $\text{LP}'(I)$, *then* $\mathbf{q}(\mathbf{v})$ *must be an optimal solution to* $\text{LP}(\mathbf{v} : u)$ *for all* $\mathbf{v} \in \hat{D}$.

Proof. The first part is trivial since we included in $\text{LP}(\mathbf{v} : u)$ every constraint in $\text{LP}'(I)$ that involves $\mathbf{q}(\mathbf{v})$. The second part follows directly from the first part, since the objective function of $\text{LP}(\mathbf{v} : u)$ is exactly $\text{REV}(\mathbf{v})$, the revenue at $\mathbf{v}$ (and we also know that $\Pr[\mathbf{v}] > 0$ for all $\mathbf{v} \in D$). ∎

Next we prove a few properties of optimal solutions to $\text{LP}(\mathbf{v} : u)$.

LEMMA 5.3. *Suppose that* $\text{LP}(\mathbf{v} : u)$ *is feasible for some utility function* $u : D \to \mathbb{R}_{\geq 0}$ *and* $\mathbf{v} \in \hat{D}$. *Then any optimal solution* $\mathbf{q} = (q_1, \ldots, q_{n+2})$ *to* $\text{LP}(\mathbf{v} : u)$ *satisfies* $q_i(\mathbf{v}) = 0$ *for all* $i \in [n] \setminus S(\mathbf{v})$ *and entries of* $\mathbf{q}$ *sum to* 1. *Moreover, we also have* $q_{n+2}(\mathbf{v}) = 0$ *if* $\mathbf{v} \in T_2$, *and* $q_{n+1}(\mathbf{v}) = 0$ *if* $\mathbf{v} \in T_3 \cup T_4$.

Proof. If any of the q_i's listed above is positive, replacing q_i by 0 and adding q_i to q_{n+1} if $\mathbf{v} \in T_2$ or adding q_i to q_{n+2} if $\mathbf{v} \in T_3 \cup T_4$ would result in a strictly better feasible solution. If the entries of $\mathbf{q}$ sum to $1 - c$ for some $c > 0$, adding c to either q_{n+1} or q_{n+2} would result in a strictly better feasible solution. ∎

In the proof sometimes we need to compare optimal solutions to $\text{LP}(\mathbf{v} : u)$ versus $\text{LP}(\mathbf{v} : u')$ for two utility functions u and u' that are entrywise close to each other. The following lemma comes in handy.

LEMMA 5.4. *Assume* $\text{LP}(\mathbf{v} : u)$ *and* $\text{LP}(\mathbf{v} : u')$ *are feasible for some* $\mathbf{v} \in \hat{D}$ *and utilities* $u, u' : D \to \mathbb{R}_{\geq 0}$. *Let* OPT *and* OPT' *denote optimal values of* $\text{LP}(\mathbf{v} : u)$ *and* $\text{LP}(\mathbf{v} : u')$, *respectively. Let* $\epsilon > 0$. *Then*

1. *if* $\mathbf{v} \in T_2$ *and* $|u(\mathbf{w}) - u'(\mathbf{w})| \leq \epsilon$ *for* $\mathbf{w} \in T_2$, *then* $|\text{OPT} - \text{OPT}'| = O(n\epsilon s/\beta)$.
2. *if* $\mathbf{v} \in T_3$ *(or* T_4*) and* $|u(\mathbf{w}) - u'(\mathbf{w})| \leq \epsilon$ *for all* $\mathbf{w} \in T_3$ *(or* T_4*), then we have* $|\text{OPT} - \text{OPT}'| = O(n\epsilon t/\beta)$.

Proof. We prove that $\text{OPT}' \geq \text{OPT} - O(n\epsilon s/\beta)$ when $\mathbf{v} \in T_2$. All other cases can be proved similarly. For this purpose, let $\mathbf{q}$ and $\mathbf{q}'$ denote an optimal solution to $\text{LP}(\mathbf{v} : u)$ and $\text{LP}(\mathbf{v} : u')$, respectively. We consider the following two cases.

Case 1: $q_{n+1} \geq 4n\epsilon/\beta$. Let $\mathbf{q}^*$ denote the following nonnegative vector obtained from $\mathbf{q}$:

$$q^*_{n+1} = q_{n+1} - |S(\mathbf{v})| \cdot \frac{4\epsilon}{\beta} \quad \text{and} \quad q^*_i = q_i + \frac{4\epsilon}{\beta} \quad \text{for each } i \in S(\mathbf{v}).$$

It is a feasible solution to $\text{LP}(\mathbf{v} : u')$, given (5.1) and (5.2). As a result, we have that $\text{OPT}' \geq \text{OPT} - O(n\epsilon s/\beta)$.

Case 2: $q_{n+1} < 4n\epsilon/\beta$. This case is more involved. From Lemma 5.3 we have $q_{n+2} = q'_{n+2} = 0$. Let $c = \max_{i \in [n]}\{q_i - q'_i\}$. If $c \leq 8n\epsilon/\beta$, then we immediately have (using $q'_{n+1} \geq 0$)

$$\text{OPT}' \geq \text{OPT} - s \cdot (4n\epsilon/\beta) - n \cdot c \cdot O(1) \geq \text{OPT} - O(n\epsilon s/\beta),$$

since we assumed $s \gg n$ in (5.3). Otherwise ($c > 8n\epsilon/\beta$), let $k \in S(\mathbf{v})$ denote an index that achieves the maximum ($k \in S(\mathbf{v})$ since $q_i = q'_i = 0$ for all $i \in [n] \setminus S(\mathbf{v})$ by Lemma 5.3): $q_k - q'_k = c > 8n\epsilon/\beta$. As $\sum_{i \in S(\mathbf{v})} q_i = 1 - q_{n+1} > 1 - c$ and $\sum_{i \in S(\mathbf{v})} q'_i \leq 1$ we have $q_i \geq q'_i - (n+1)c$ for all $i \in S(\mathbf{v})$. Now let $\mathbf{q}^*$ denote the vector obtained from $\mathbf{q}$ by replacing

$$q_k^* = q_k - (|S(\mathbf{v})| - 1) \cdot \frac{4\epsilon}{\beta} \quad \text{and} \quad q_i^* = q_i + \frac{4\epsilon}{\beta} \quad \text{for all other } i \in S(\mathbf{v}).$$

One can verify that $\mathbf{q}^*$ is a feasible solution to $\text{LP}'(\mathbf{v})$. The only nontrivial case in verifying this is to show that $\sum_{j\in[n]}(v_j - w_j) \cdot q_j^* \geq u'(\mathbf{v}) - u'(\mathbf{w})$ for any $\mathbf{w} \in \textsc{Block}(\mathbf{v}_{-k}, a_k)$. For this, note that

$$\sum_{j\in[n]} (v_j - w_j) \cdot q_j^* - \sum_{j\in[n]} (v_j - w_j) \cdot q_j' \geq (v_k - a_k) \cdot \frac{c}{2} - n \cdot O(\beta) \cdot O(nc) = \Omega(c) > 0.$$

As a result, we have $\textsc{Opt}' \geq \textsc{Opt} - O(n\epsilon/\beta) \cdot O(n\beta) = \textsc{Opt} - O(n^2\epsilon)$.

The lemma follows by combining the two cases and the fact that $s/\beta \gg n$. ☐

5.1.2. Condition on utilities of type-2 vectors. We show that utilities of type-2 vectors in any optimal solution $(u(\cdot), \mathbf{q}(\cdot))$ to $\text{LP}'(I)$ must satisfy

Condition-Type-2: Each type-2 vector $\mathbf{v} \in T_2$ has utility

$$u(\mathbf{v}) = \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_2)\big\}.$$

Recall that $\rho(\mathbf{v}) = (\mathbf{v}_{-(n+1)}, 0)$ for type-2 vectors. By (5.9) of $\text{LP}'(I)$ in Part 2, $u(\mathbf{v})$ is at least as large as the RHS. So Condition-Type-2 requires that it is tight for every $\mathbf{v} \in T_2$ in an optimal solution.

We now prove Condition-Type-2.

Lemma 5.5. *Given* (5.1), (5.2), *and* (5.3), *any optimal solution to* $\text{LP}'(I)$ *satisfies* Condition-Type-2.

Proof. Let $(u(\cdot), \mathbf{q}(\cdot))$ denote an optimal solution to $\text{LP}'(I)$. Let R denote the set of $\mathbf{v} \in T_2$ that satisfies $u(\mathbf{v}) > \max\{u(\rho(\mathbf{v})), u(\mathbf{c}_2)\}$. Note that $\mathbf{c}_2 \notin R$, and we assume for contradiction that R is nonempty.

Our plan is to derive a solution $(u'(\cdot), \mathbf{q}'(\cdot))$ from $(u(\cdot), \mathbf{q}(\cdot))$, by modifying utilities and allocations of type-2 vectors only. We then get a contradiction by showing that $(u'(\cdot), \mathbf{q}'(\cdot))$ is feasible and has a strictly higher revenue than $(u(\cdot), \mathbf{q}(\cdot))$. (Because we only modify utilities and allocations of type-2 vectors, for the feasibility it suffices to verify constraints of $\text{LP}'(I)$ in Part 2.) We use $\textsc{Rev}(\mathbf{v})$ and $\textsc{Rev}'(\mathbf{v})$ to denote the revenue from $\mathbf{v}$ in the old and new solutions. By Lemma 5.3 $\textsc{Rev}(\mathbf{v})$ is the value of $\text{LP}(\mathbf{v} : u)$ for $\mathbf{v} \in \hat{D}$.

To define the new solution $(u'(\cdot), \mathbf{q}'(\cdot))$, let $\epsilon > 0$ denote the following parameter:

$$\epsilon = \min\Big\{\min_{\mathbf{v}\in R}\Big(u(\mathbf{v}) - \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_2)\big\}\Big), \text{ smallest positive entry in } \mathbf{q}(\mathbf{v}) \text{ among all } \mathbf{v} \in D\Big\}.$$

For each $\mathbf{v} \in T_2$, set $u'(\mathbf{v}) = u(\mathbf{v})$ if $\mathbf{v} \notin R$ and $u'(\mathbf{v}) = u(\mathbf{v}) - \epsilon$ if $\mathbf{v} \in R$. All other entries of u' remain the same as in u. Note that u' still satisfies (5.9) in Part 2. Given $u'(\cdot)$, we set $\mathbf{q}'(\mathbf{v})$ for each $\mathbf{v} \in T_2 \setminus \{\mathbf{c}_2\}$ to be an optimal solution to the linear program $\text{LP}(\mathbf{v} : u')$ (though it is not clear for now if $\text{LP}(\mathbf{v} : u')$ is still feasible or not; we will show that this is indeed the case for every $\mathbf{v} \in T_2 \setminus \{\mathbf{c}_2\}$) and all other allocations remain the same as those in $\mathbf{q}(\cdot)$. This finishes the description of $(u'(\cdot), \mathbf{q}'(\cdot))$.

By Lemma 5.2, to show that $(u'(\cdot), \mathbf{q}'(\cdot))$ is well-defined and feasible it suffices to show that $\text{LP}(\mathbf{v} : u')$ is feasible for all $\mathbf{v} \in T_2 \setminus \{\mathbf{c}_2\}$ (because $(u'(\cdot), \mathbf{q}'(\cdot))$ satisfies trivially all constraints of $\text{LP}'(I)$ except (5.10) and (5.11) in Part 2). To see this is the case we fix such a $\mathbf{v}$. If $\mathbf{v} \in R$ (and $u'(\mathbf{v}) = u(\mathbf{v}) - \epsilon$), every feasible solution to $\text{LP}(\mathbf{v} : u)$ is also feasible to $\text{LP}(\mathbf{v} : u')$. As a result, $\text{LP}(\mathbf{v} : u')$ is feasible as well. Furthermore, we also have $\text{Rev}'(\mathbf{v}) \geq \text{Rev}(\mathbf{v}) + \epsilon$ since $u'(\mathbf{v}) = u(\mathbf{v}) - \epsilon$ for any $\mathbf{v} \in R$.

If $\mathbf{v} \notin R$, then either $u'(\mathbf{v}) = u(\mathbf{v}) = u(\mathbf{c}_2)$ or $u'(\mathbf{v}) = u(\mathbf{v}) = u(\rho(\mathbf{v}))$. For the former case, setting $q_{n+1} = 1$ and $q_i = 0$ for all other i is a feasible solution to $\text{LP}(\mathbf{v} : u')$, since $u'(\mathbf{w}) \geq u(\mathbf{c}_2)$ for all $\mathbf{w} \in T_2$. For the latter case, $\mathbf{q} = \mathbf{q}(\rho(\mathbf{v}))$ is a feasible solution to $\text{LP}(\mathbf{v} : u')$ since constraints on $\mathbf{q}(\rho(\mathbf{v}))$ in $\text{LP}'(I)$ are at least as strong as those on $\mathbf{q}$ in $\text{LP}(\mathbf{v} : u)$ using $u'(\mathbf{v}) = u(\rho(\mathbf{v}))$ and $u'(\mathbf{w}) \geq u(\rho(\mathbf{w}))$ for $\mathbf{w} \in T_2$. More specifically, (5.19) of $\text{LP}(\mathbf{v} : u')$ follows from (5.5) of Part 1 in $\text{LP}'(I)$ over $\mathbf{q}(\rho(\mathbf{v}))$; (5.20) follows from (5.8) of Part 1 in $\text{LP}'(I)$ over $\mathbf{q}(\rho(\mathbf{v}))$. We conclude that $(u'(\cdot), \mathbf{q}'(\cdot))$ is well-defined, feasible to $\text{LP}'(I)$.

The only thing left to show that the expected revenue from $(u'(\cdot), \mathbf{q}'(\cdot))$ is strictly higher. By the definition of $(u'(\cdot), \mathbf{q}'(\cdot))$, we have $\text{Rev}'(\mathbf{v}) = \text{Rev}(\mathbf{v})$ for all $\mathbf{v}$ other than those in $T_2 \setminus \{\mathbf{c}_2\}$ since each such $\mathbf{v}$ receives the same allocation and utility as in $(u(\cdot), \mathbf{q}(\cdot))$. By Lemma 5.4, we also have

$$\text{Rev}'(\mathbf{v}) \geq \text{Rev}(\mathbf{v}) - O(n\epsilon s/\beta) \quad \text{for all } \mathbf{v} \in T_2 \setminus \{\mathbf{c}_2\}.$$

Moreover, if $\mathbf{v} \in T_2 \setminus R$ and there is no $\mathbf{w} \in R$ below $\mathbf{v}$ (or $\mathbf{w} \prec \mathbf{v}$) then $\text{LP}(\mathbf{v} : u')$ is exactly the same as $\text{LP}(\mathbf{v} : u)$ so $\text{Rev}'(\mathbf{v}) = \text{Rev}(\mathbf{v})$. This inspires us to define $R' \subseteq R$ as the *bottom* of R: $\mathbf{v} \in R'$ if there is no other vector in R below $\mathbf{v}$. (Since R is nonempty, R' is nonempty as well.) For each $\mathbf{v} \in R'$, we claim that $\text{Rev}'(\mathbf{v})$ from the new solution indeed has a much bigger advantage over $\text{Rev}(\mathbf{v})$:

$$\text{Rev}'(\mathbf{v}) \geq \text{Rev}(\mathbf{v}) + \Omega(\epsilon s). \tag{5.21}$$

To prove (5.21), we first show that $q_i(\mathbf{v}) > 0$ for some $i \in S(\mathbf{v})$. For this, setting $\mathbf{w} = \text{Lower}(\mathbf{v}_{-j}, a_j)$ for some $j \in S(\mathbf{v})$ in (5.11) of Part 2 in $\text{LP}'(I)$ (note that $\mathbf{v} \neq \mathbf{c}_2$ implies $S(\mathbf{v}) \neq \emptyset$), we have

$$\sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\mathbf{v}) \geq u(\mathbf{v}) - u(\mathbf{w}).$$

It follows from $\mathbf{v} \in R' \subseteq R$ that

$$u(\mathbf{v}) > \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_2)\big\} \quad \text{and} \quad u(\mathbf{w}) = \max\big\{u(\rho(\mathbf{w})), u(\mathbf{c}_2)\big\}.$$

By (5.8) of Part 1 in $\text{LP}'(I)$, we have $u(\rho(\mathbf{v})) \geq u(\rho(\mathbf{w}))$. It follows that $u(\mathbf{v}) > u(\mathbf{w})$ and, thus, $q_i(\mathbf{v}) > 0$ for some $i \in S(\mathbf{v})$. Let k be an index in $S(\mathbf{v})$ with $q_k(\mathbf{v}) > 0$. As a result, the following vector $\mathbf{q}^*$ (which is nonnegative because of our choice of ϵ): $q^*_{n+1} = q_{n+1}(\mathbf{v}) + \epsilon/2$, $q^*_k = q_k(\mathbf{v}) - \epsilon/2$, and $q^*_i = q_i(\mathbf{v})$ for all other i, must be a feasible solution to $\text{LP}(\mathbf{v} : u')$. (5.21) then follows from $s \gg 1$.

We say a type-2 vector is above R' if it is above one of the vectors in R'. Combining all cases together, to show that revenue from $(u'(\cdot), \mathbf{q}'(\cdot))$ is strictly higher than that from $(u(\cdot), \mathbf{q}(\cdot))$, it suffices to show that

$$\Pr[\text{vectors in } R'] \cdot \Omega(\epsilon s) \gg \Pr[(\text{type-2) vectors above } R'] \cdot O(n\epsilon s/\beta). \tag{5.22}$$

This follows from our choices of p and r in (5.1). Taking any $\mathbf{v} \in R'$, we have the following bound:

$$
\begin{aligned}
&\Pr[\text{vectors above } \mathbf{v}] \\
&\quad = \Pr[\text{vectors } \mathbf{w} \succ \mathbf{v},\, S(\mathbf{w}) = S(\mathbf{v})] + \Pr[\text{vectors } \mathbf{w} \succ \mathbf{v},\, S(\mathbf{v}) \subset S(\mathbf{w})] \\
&\quad = \left(O\left(\frac{nr}{p}\right) + O\left(\frac{np^{|S(\mathbf{v})|+1}}{r^{|S(\mathbf{v})|}}\right)\right) \cdot \Pr[\mathbf{v}] \\
&\quad = O\left(\frac{nr}{p} + np \cdot \left(\frac{p}{r}\right)^n\right) \cdot \Pr[\mathbf{v}] \\
&\quad \ll \frac{\beta}{n} \cdot \Pr[\mathbf{v}].
\end{aligned}
$$

Then (5.22) follows from a union bound. This finishes the proof of the lemma. ◻

Arguments used in Lemma 5.5 imply the following property. Suppose $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ satisfies all the constraints of $\mathrm{LP}'(I)$ in Parts 0 and 1. Given any nonnegative number u_2, we can extend it to T_2 by setting $u(\mathbf{c}_2) = u_2$ and $u(\mathbf{v}) = \max\{u(\rho(\mathbf{v})), u_2\}$ for each other $\mathbf{v}$ in T_2, and then setting $\mathbf{q}(\mathbf{c}_2)$ according to Lemma 5.1 and $\mathbf{q}(\mathbf{v})$ to be an optimal solution to $\mathrm{LP}(\mathbf{v} : u)$ for each other $\mathbf{v} \in T_2$. It is easy to show, by an argument similar to Lemma 5.5, that $\mathrm{LP}(\mathbf{v} : u)$ is feasible, and $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1 \cup T_2)$ now satisfies all the constraints of $\mathrm{LP}'(I)$ in Parts 0, 1, and 2.

5.1.3. Conditions on utilities of type-4 vectors. Next we show that utilities of type-4 vectors satisfy the following condition:

Condition-Type-4: Each type-4 vector $\mathbf{v} \in T_4$ has utility

$$u(\mathbf{v}) = \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_4)\big\}.$$

Lemma 5.6. *Given* (5.1), (5.2), *and* (5.3), *any optimal solution to* $\mathrm{LP}'(I)$ *satisfies* Condition-Type-4.

Proof. Let $(u(\cdot), \mathbf{q}(\cdot))$ be an optimal solution, and let R be the set of $\mathbf{v} \in T_4$ with $u(\mathbf{v}) > \{u(\rho(\mathbf{v})), u(\mathbf{c}_4)\}$ (so we have $\mathbf{c}_4 \notin R$). Assume for contradiction that R is nonempty. Our plan is to derive $(u'(\cdot), \mathbf{q}'(\cdot))$ from $(u(\cdot), \mathbf{q}(\cdot))$ by modifying utilities and allocations of vectors in $T_4 \setminus \{\mathbf{c}_4\}$ only. We reach a contradiction by showing that the new solution $(u'(\cdot), \mathbf{q}'(\cdot))$ is feasible and has a strictly higher revenue than $(u(\cdot), \mathbf{q}(\cdot))$.

To define the new solution $(u'(\cdot), \mathbf{q}'(\cdot))$, let $\epsilon > 0$ denote the following parameter:

$$
\begin{aligned}
\epsilon = \min\bigg\{ &\min_{\mathbf{v} \in R}\Big(u(\mathbf{v}) - \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_3)\big\}\Big), \\
&\qquad\qquad \text{smallest positive entry in } \mathbf{q}(\mathbf{v}) \text{ among all } \mathbf{v} \in D\bigg\}.
\end{aligned}
$$

First for each $\mathbf{v} \in T_4$ we set $u'(\mathbf{v}) = u(\mathbf{v})$ if $\mathbf{v} \notin R$, and $u'(\mathbf{v}) = u(\mathbf{v}) - \epsilon$ if $\mathbf{v} \in R$; all other entries of u' are the same as those in u. Note that $u'(\cdot)$ still satisfies (5.15) in Part 4 of $\mathrm{LP}'(I)$. Given $u'(\cdot)$ we set $\mathbf{q}'(\mathbf{v})$ for each $\mathbf{v} \in T_4 \setminus \{\mathbf{c}_4\}$ to be an optimal solution to the linear program $\mathrm{LP}(\mathbf{v} : u')$. With an argument similar to that used in Lemma 5.5, $\mathrm{LP}(\mathbf{v} : u')$ is feasible (if $\mathbf{v} \in R$, $\mathbf{q}(\mathbf{v})$ is feasible; otherwise $\mathbf{q}(\rho(\mathbf{v}))$ is feasible).

Given that $(u'(\cdot), \mathbf{q}'(\mathbf{v}))$ is well-defined and feasible, we next show that its expected revenue is strictly higher than that of $(u(\cdot), \mathbf{q}(\cdot))$. We follow the approach as in the proof of Lemma 5.5. Let R' be the bottom of R: R' contains $\mathbf{w} \in R$ if no other vector in R lies below $\mathbf{w}$. For each $\mathbf{v} \in T_4 \setminus R'$ with $\mathbf{w} \prec \mathbf{v}$ for some $\mathbf{w} \in R'$, we apply $\text{Rev}'(\mathbf{v}) \geq \text{Rev}(\mathbf{v}) - O(n\epsilon t/\beta)$ by Lemma 5.4. For each $\mathbf{v} \in T_4 \setminus R'$ that is not above any vector in R', we have $\text{Rev}'(\mathbf{v}) = \text{Rev}(\mathbf{v})$. Finally, for each $\mathbf{v} \in R'$, the same proof of (5.21) in Lemma 5.5 gives that $\text{Rev}'(\mathbf{v}) \geq \text{Rev}(\mathbf{v}) + \Omega(\epsilon t)$.

Combining all the cases and following the same argument used in Lemma 5.5, we have

$$\Pr[\text{vectors in } R'] \cdot \Omega(\epsilon t) \gg \Pr[(\text{type-4}) \text{ vectors above } R'] \cdot O(n\epsilon t/\beta).$$

This finishes the proof of the lemma. ∎

Arguments used in Lemma 5.6 further imply the following fact. Suppose that $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1 \cup T_2 \cup T_3)$ satisfies all the constraints of $\text{LP}'(I)$ in Parts 0, 1, 2, and 3. Given any nonnegative u_4 with $u(\mathbf{c}_3) \leq u_4 \leq u(\mathbf{c}_3) + s$, we can extend it to T_4 by setting $u(\mathbf{c}_4) = u_4$ and $u(\mathbf{v}) = \max\{u(\rho(\mathbf{v})), u_4\}$ for other $\mathbf{v}$ in T_4, and setting $\mathbf{q}(\mathbf{c}_4)$ according to Lemma 5.1 and $\mathbf{q}(\mathbf{v})$ to be an optimal solution to $\text{LP}(\mathbf{v} : u)$ for other $\mathbf{v} \in T_4$. By similar arguments in Lemma 5.6, $\text{LP}(\mathbf{v} : u)$ is feasible, and $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in D)$ is feasible to $\text{LP}'(I)$.

5.1.4. Condition on utilities of type-3 vectors. A similar condition holds for utilities of type-3 vectors in any optimal solution to $\text{LP}'(I)$:

CONDITION-TYPE-3: Each type-3 vector $\mathbf{v} \in T_3$ has utility

$$u(\mathbf{v}) = \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_3)\big\}.$$

LEMMA 5.7. *Given* (5.1), (5.2), *and* (5.3), *any optimal solution to* $\text{LP}'(I)$ *satisfies* CONDITION-TYPE-3.

Proof. Assume for contradiction that $(u(\cdot), \mathbf{q}(\cdot))$ is an optimal solution to $\text{LP}'(I)$ that violates CONDITION-TYPE-3. Let R denote the nonempty set of $\mathbf{v} \in T_3$ with $u(\mathbf{v}) > \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_3)\big\}$ (so $\mathbf{c}_3 \notin R$).

To reach a contradiction, we derive from $(u(\cdot), \mathbf{q}(\cdot))$ a new solution $(u'(\cdot), \mathbf{q}'(\cdot))$ by modifying utilities and allocations of $\mathbf{v} \in T_3 \setminus \{\mathbf{c}_3\}$ only. (All constraints are satisfied trivially except those in Part 3; note that only $u(\mathbf{c}_3)$ appears in Part 4 but it remains the same in $u'(\cdot)$.) We then show that $(u'(\cdot), \mathbf{q}'(\cdot))$ is better.

We define $(u'(\cdot), \mathbf{q}'(\cdot))$ from $(u(\cdot), \mathbf{q}(\cdot))$ as follows. Let $\epsilon > 0$ denote the following parameter:

$$\epsilon = \min\Big\{\min_{\mathbf{v}\in R}\Big(u(\mathbf{v}) - \max\big\{u(\rho(\mathbf{v})), u(\mathbf{c}_3)\big\}\Big), \\ \text{smallest positive entry in } \mathbf{q}(\mathbf{v}) \text{ among all } \mathbf{v} \in D\Big\}.$$

For each $\mathbf{v} \in T_3$ we set $u'(\mathbf{v}) = u(\mathbf{v})$ if $\mathbf{v} \notin R$ and $u'(\mathbf{v}) = u(\mathbf{v}) - \epsilon$ if $\mathbf{v} \in R$; all other entries remain the same. Note that the new u' satisfies (5.12) in Part 3 of $\text{LP}'(I)$. Then for each $\mathbf{v} \in T_3 \setminus \{\mathbf{c}_3\}$, we set $\mathbf{q}'(\mathbf{v})$ to be an optimal solution to $\text{LP}(\mathbf{v} : u')$. With an argument similar to the one used in the proof of Lemma 5.5, $\text{LP}(\mathbf{v} : u')$ is feasible (if $\mathbf{v} \in R$, $\mathbf{q}(\mathbf{v})$ is feasible; otherwise, $\mathbf{q}(\rho(\mathbf{v}))$ is feasible). All other entries of $\mathbf{q}'(\cdot)$ remain the same. It is clear now that $(u'(\cdot), \mathbf{q}'(\cdot))$ is a feasible solution to $\text{LP}'(I)$.

We compare the expected revenues from $(u(\cdot), \mathbf{q}(\cdot))$ and $(u'(\cdot), \mathbf{q}'(\cdot))$ and show that the latter is higher. Let R' denote the *bottom* of R: R' contains $\mathbf{v} \in R$ if no other vector in R lies below $\mathbf{v}$. For each $\mathbf{v} \in T_3 \setminus R'$ above a vector in R', we apply

$\text{Rev}'(\mathbf{v}) \geq \text{Rev}(\mathbf{v}) - O(n\epsilon t/\beta)$ by Lemma 5.4. For each vector $\mathbf{v} \in T_3 \setminus R'$ that is not above any vector in R', we have $\text{Rev}'(\mathbf{v}) = \text{Rev}(\mathbf{v})$. Finally, for each $\mathbf{v} \in R'$, we can show that $\text{Rev}'(\mathbf{v}) \geq \text{Rev}(\mathbf{v}) + \Omega(\epsilon t)$ with an argument similar to that in the proof of Lemma 5.5.

Combining all these bounds together and following the same argument used in Lemma 5.5, we have

$$\Pr[\text{vectors in } R'] \cdot \Omega(\epsilon t) \gg \Pr[(\text{type-3) vectors above } R'] \cdot O(n\epsilon t/\beta).$$

This finishes the proof of the lemma. ∎

Arguments used in Lemma 5.6 also imply the following property. Suppose that $(u(\mathbf{v}),\ \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1 \cup T_2)$ satisfies all the constraints of $\text{LP}'(I)$ in Parts 0, 1, and 2. Given a nonnegative number u_3, we can extend it to T_3 by setting $u(\mathbf{c}_3) = u_3$ and $u(\mathbf{v}) = \max\{u(\rho(\mathbf{v})), u_3\}$ for other $\mathbf{v}$ in T_3, and setting $\mathbf{q}(\mathbf{c}_3)$ according to Lemma 5.1 and $\mathbf{q}(\mathbf{v})$ to be an optimal solution to $\text{LP}(\mathbf{v} : u)$ for other $\mathbf{v} \in T_3$. By similar arguments used in Lemma 5.7, $\text{LP}(\mathbf{v} : u)$ is feasible, and $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1 \cup T_2 \cup T_3)$ now satisfies Parts 0, 1, 2, and 3.

5.1.5. Expected revenue from types-2, -3, and -4 vectors.

Before working on type-1 vectors, which is the most challenging part of the characterization, we summarize our progress so far. We need the following notation. Let $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ denote a partial solution that satisfies all constraints of $\text{LP}'(I)$ in Parts 0 and 1. Given $u_2, u_3, u_4 \geq 0$ that satisfy $u_3 \leq u_4 \leq u_3 + s$, we use $\mathsf{Ext}(u(\cdot), \mathbf{q}(\cdot); u_2, u_3, u_4)$ to denote the following set of solutions $\{u'(\mathbf{v}), \mathbf{q}'(\mathbf{v}) : \mathbf{v} \in D\}$ to $\text{LP}'(I)$:

1. $u'(\mathbf{v}) = u(\mathbf{v})$ and $\mathbf{q}'(\mathbf{v}) = \mathbf{q}(\mathbf{v})$ for all $\mathbf{v} \in T_1$.
2. $u'(\mathbf{c}_2) = u_2$, $u'(\mathbf{c}_3) = u_3$, and $u'(\mathbf{c}_4) = u_4$; $\mathbf{q}'(\mathbf{c}_2) = \mathbf{e}_{n+1}$ and $\mathbf{q}'(\mathbf{c}_3) = \mathbf{e}_{n+2}$.
3. All entries of $\mathbf{q}'(\mathbf{c}_4)$ are 0 except
$$q_{n+1}(\mathbf{c}_4) = (u_4 - u_3)/s \quad \text{and} \quad q_{n+2}(\mathbf{c}_4) = 1 - (u_4 - u_3)/s.$$
4. For each $\mathbf{v} \in T_2 \setminus \{\mathbf{c}_2\}$, $u'(\mathbf{v}) = \max\{u(\rho(\mathbf{v})), u_2\}$ and $\mathbf{q}'(\mathbf{v})$ is an optimal solution to $\text{LP}(\mathbf{v} : u')$.
5. For each $\mathbf{v} \in T_3 \setminus \{\mathbf{c}_3\}$, $u'(\mathbf{v}) = \max\{u(\rho(\mathbf{v})), u_3\}$ and $\mathbf{q}'(\mathbf{v})$ is an optimal solution to $\text{LP}(\mathbf{v} : u')$.
6. For each $\mathbf{v} \in T_4 \setminus \{\mathbf{c}_4\}$, $u'(\mathbf{v}) = \max\{u(\rho(\mathbf{v})), u_4\}$ and $\mathbf{q}'(\mathbf{v})$ is an optimal solution to $\text{LP}(\mathbf{v} : u')$.

By discussions at the end of sections 5.1.2, 5.1.3, and 5.1.4, $\mathsf{Ext}(u(\cdot), \mathbf{q}(\cdot) : u_2, u_3, u_4)$ is well-defined (and nonempty). The next two lemmas summarize our progress so far.

LEMMA 5.8. *Suppose that* $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ *satisfies all constraints of* $\text{LP}'(I)$ *in Parts* 0 *and* 1. *Given any* $u_2, u_3, u_4 \geq 0$, *where* $u_3 \leq u_4 \leq u_3 + s$, *solutions in* $\mathsf{Ext}(u(\cdot), \mathbf{q}(\cdot) : u_2, u_3, u_4)$ *are feasible to* $\text{LP}'(I)$ *and for each* $i = 1, 2, 3, 4$, *they all share the same expected revenue from type-i vectors.*

LEMMA 5.9. *Any optimal solution* $(u'(\cdot), \mathbf{q}'(\cdot))$ *to the linear program* $\text{LP}'(I)$ *belongs to* $\mathsf{Ext}(u(\cdot), \mathbf{q}(\cdot) : u_2, u_3, u_4)$, *where we set* $u_i = u'(\mathbf{c}_i)$ *for* $i = 2, 3, 4$ *and* $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ *to be the restriction of* $(u'(\cdot), \mathbf{q}'(\cdot))$ *on* T_1.

Let $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ and $(u'(\mathbf{v}), \mathbf{q}'(\mathbf{v}) : \mathbf{v} \in T_1)$ denote two partial solutions that satisfy Parts 0 and 1 of $\text{LP}'(I)$. The next lemma shows that if $|u(\mathbf{v}) - u'(\mathbf{v})|$ is small for all $\mathbf{v} \in T_1$, then expected revenues of $\mathsf{Ext}(u(\cdot), \mathbf{q}(\cdot) : u_2, u_3, u_4)$ and $\mathsf{Ext}(u'(\cdot), \mathbf{q}'(\cdot) : u_2, u_3, u_4)$ from types-2, -3, -4 vectors are also close.

LEMMA 5.10. *Suppose* $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ *and* $(u'(\mathbf{v}), \mathbf{q}'(\mathbf{v}) : \mathbf{v} \in T_1)$ *satisfy all constraints of* $\text{LP}'(I)$ *in Parts* 0 *and* 1 *and* $|u(\mathbf{v}) - u'(\mathbf{v})| \le \epsilon$ *for all* $\mathbf{v} \in T_1$. *Let* $u_2, u_3, u_4, u_2', u_3', u_4' \ge 0$ *with* $u_3 \le u_4 \le u_3 + s$, $u_3' \le u_4' \le u_3' + s$, *and* $|u_i - u_i'| \le \epsilon$ *for* $i = 2, 3, 4$. *Then we have*

$$\big|\textsc{Rev}_2 - \textsc{Rev}_2'\big| \le O\bigg(\frac{\delta n \epsilon s}{\beta}\bigg), \quad \big|\textsc{Rev}_3 - \textsc{Rev}_3'\big| \le O\bigg(\frac{\delta^2 n \epsilon t}{\beta}\bigg), \quad \textit{and}$$
$$\big|\textsc{Rev}_4 - \textsc{Rev}_4'\big| \le O\bigg(\frac{\delta^3 n \epsilon t}{\beta}\bigg),$$

where we let $\textsc{Rev}_i$ *and* $\textsc{Rev}_i'$ *denote revenues from type-i vectors in solutions of* $\mathsf{Ext}(u(\cdot), \mathbf{q}(\cdot) : u_2, u_3, u_4)$ *and solutions of* $\mathsf{Ext}(u'(\cdot), \mathbf{q}'(\cdot) : u_2', u_3', u_4')$, *respectively.*

Proof. We focus on $|\textsc{Rev}_4 - \textsc{Rev}_4'|$. The same argument applies to types-3 and -4 vectors. For convenience, we abuse the notation slightly and still write $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in D)$ and $(u'(\mathbf{v}), \mathbf{q}'(\mathbf{v}) : \mathbf{v} \in D)$ to denote two full feasible solutions of $\text{LP}'(I)$ after extension. By definition,

$$\big|\textsc{Rev}_4 - \textsc{Rev}_4'\big| = \Big| \sum_{\mathbf{v} \in T_4} \Pr[\mathbf{v}] \cdot \big(\textsc{Rev}(\mathbf{v}) - \textsc{Rev}'(\mathbf{v})\big) \Big|.$$

It is also clear that $|\textsc{Rev}(\mathbf{c}_4) - \textsc{Rev}'(\mathbf{c}_4)| \le O(\epsilon t/s)$. For other $\mathbf{v} \in T_4$, $\mathbf{q}(\mathbf{v})$ is optimal to $\text{LP}(\mathbf{v} : u)$ and $\mathbf{q}'(\mathbf{v})$ is optimal to $\text{LP}(\mathbf{v} : u')$, both of which are feasible. It follows from Lemma 5.4 and

$$|u(\mathbf{w}) - u'(\mathbf{w})| = \big|\max\{u(\rho(\mathbf{w})), u_4\} - \max\{u'(\rho(\mathbf{w})), u_4'\}\big| \le \epsilon \quad \text{for all } \mathbf{w} \in T_4,$$

that $|\textsc{Rev}(\mathbf{v}) - \textsc{Rev}(\mathbf{v}')| \le O(n\epsilon t/\beta)$. As $\sum_{\mathbf{v} \in T_4} \Pr[\mathbf{v}] < \delta^3$ we have $|\textsc{Rev}_4 - \textsc{Rev}_4'| \le O(\delta^3 n \epsilon t/\beta)$. ∎

5.1.6. Condition over type-1 vectors. Finally we show that any optimal solution $(u(\cdot), \mathbf{q}(\cdot))$ to $\text{LP}'(I)$ satisfies the following condition:

> CONDITION-TYPE-1: For each type-1 essential vector $\mathbf{v} \in T_1'$ and $\mathbf{v} \ne \mathbf{a}$, we have $u(\mathbf{v}) = \sum_{i \in S(\mathbf{v})} d_i \cdot q_i(\mathbf{a})$. For each $\mathbf{v} \in T_1'$ and $\mathbf{v} \ne \mathbf{a}$, letting
>
> $$k = \min(S(\mathbf{v})) \quad \text{and} \quad S'(\mathbf{v}) = S(\mathbf{v}) \setminus \{k\},$$
>
> we have $q_i(\mathbf{v}) = q_i(\mathbf{a})$ for all $i \in S'(\mathbf{v})$, $q_k(\mathbf{v}) = 1 - \sum_{i \in S'(\mathbf{v})} q_i(\mathbf{a})$, and all other entries of $\mathbf{q}(\mathbf{v})$ are 0. Moreover, for each nonessential type-1 vector $\mathbf{v} \in T_1^*$, letting $\mathbf{w} = \textsc{Lower}(\mathbf{v})$, we have $\mathbf{q}(\mathbf{v}) = \mathbf{q}(\mathbf{w})$ and
>
> $$u(\mathbf{v}) = u(\mathbf{w}) + \sum_{j \in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{w}) = \sum_{i \in S(\mathbf{v})} d_i \cdot q_i(\mathbf{a}) + \sum_{j \in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{w}).$$

Note that CONDITION-TYPE-1 does not require $\sum_{i \in [n]} q_i(\mathbf{a}) = 1$. Actually we will only get to impose this condition later in section 5.2.1 after proper choices of a_i's.

We record the following three simple lemmas concerning solutions that satisfy CONDITION-TYPE-1.

LEMMA 5.11. *Assume that* $(u(\cdot), \mathbf{q}(\cdot))$ *satisfies* CONDITION-TYPE-1. *If two type-1 vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *satisfy* $S(\mathbf{w}) \subseteq S(\mathbf{v})$, *then* $q_j(\mathbf{w}) \ge q_j(\mathbf{v})$ *for all* $j \in S(\mathbf{w})$.

LEMMA 5.12. *Assume that* $(u(\cdot), \mathbf{q}(\cdot))$ *satisfies* CONDITION-TYPE-1. *Then we have* $\textsc{Rev}(\mathbf{v}) = \textsc{Rev}(\mathbf{v}')$ *for any two type-1 vectors* $\mathbf{v}$ *and* $\mathbf{v}'$ *in the same block.*

Proof. Let $\mathbf{w} = \textsc{Lower}(\mathbf{v}) = \textsc{Lower}(\mathbf{v}')$. Using Condition-Type-1, $\textsc{Rev}(\mathbf{v})$ is equal to

$$\sum_{i\in[n+2]} v_i \cdot q_i(\mathbf{v}) - u(\mathbf{v}) = \sum_{i\in S(\mathbf{w})} v_i \cdot q_i(\mathbf{w}) - u(\mathbf{w}) - \sum_{i\in S^+(\mathbf{v})} \tau_i \cdot q_i(\mathbf{w})$$
$$= \sum_{i\in S(\mathbf{w})} \ell_i \cdot q_i(\mathbf{w}) - u(\mathbf{w}),$$

which does not depend on $\mathbf{v}$ but only $\mathbf{w} = \textsc{Lower}(\mathbf{v})$. The lemma then follows. ∎

Lemma 5.13. *Let* $\mathbf{q}$ *denote an* $(n+2)$*-dimensional nonnegative vector that sums to at most* 1*. Then there is a unique* $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ *that satisfies* $\mathbf{q}(\mathbf{a}) = \mathbf{q}$ *and* Condition-Type-1*. Moreover,* $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ *satisfies all constraints of* $\text{LP}'(I)$ *in Parts* 0 *and* 1*.*

Proof. Part 0, (5.4), and (5.7) are trivial. For (5.5), given $\mathbf{v} \in T_1$, $i \in S^+(\mathbf{v})$, and $\mathbf{w} = (\mathbf{v}_{-i}, \ell_i)$, we have that

$$u(\mathbf{v}) - u(\mathbf{w}) = \tau_i \cdot q_i\big(\textsc{Lower}(\mathbf{v})\big) = \tau_i \cdot q_i(\mathbf{v})$$

by Condition-Type-1. For (5.6), letting $\mathbf{w} = \textsc{Lower}(\mathbf{v})$, we have $u(\mathbf{v}) - u(\mathbf{w}) = \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{w})$. For (5.8), given $\mathbf{v} \in T_1$, $i \in S(\mathbf{v})$, $\mathbf{w} \in \textsc{Block}(\mathbf{v}_{-i}, a_i)$, $\mathbf{v}' = \textsc{Lower}(\mathbf{v})$, and $\mathbf{w}' = \textsc{Lower}(\mathbf{w})$, we have

$$u(\mathbf{v}) - u(\mathbf{w}) = u(\mathbf{v}) - u(\mathbf{v}') + u(\mathbf{v}') - u(\mathbf{w}') + u(\mathbf{w}') - u(\mathbf{w})$$
$$= \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{v}') + d_i \cdot q_i(\mathbf{a}) - \sum_{j\in S^+(\mathbf{w})} \tau_j \cdot q_j(\mathbf{w}').$$

Applying Lemma 5.11 on $\mathbf{v}$ and $\mathbf{w}'$ (also $\mathbf{q}(\mathbf{v}) = \mathbf{q}(\mathbf{v}')$ and $q_i(\mathbf{a}) \le q_i(\mathbf{v})$ for $i \in S(\mathbf{v})$) we have

$$u(\mathbf{v}) - u(\mathbf{w}) \le \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{v}) + d_i \cdot q_i(\mathbf{v}) - \sum_{j\in S^+(\mathbf{w})} \tau_j \cdot q_j(\mathbf{v}) = \sum_{j\in S(\mathbf{v})} (v_j - w_j) \cdot q_j(\mathbf{v}).$$

This covers all constraints in Parts 0 and 1, and the lemma is proven. ∎

Now we prove Condition-Type-1.

Lemma 5.14. *Given* (5.1)*,* (5.2)*, and* (5.3)*, any optimal solution to* $\text{LP}'(I)$ *satisfies* Condition-Type-1*.*

Proof. Let $(u(\cdot), \mathbf{q}(\cdot))$ be an optimal solution to $\text{LP}'(I)$. Our plan is the following. We first derive a solution $(u^*(\cdot), \mathbf{q}^*(\cdot))$ from $(u(\cdot), \mathbf{q}(\cdot))$, and show that it is feasible to $\text{LP}'(I)$. Then we compare expected revenues from them and show that for $(u(\cdot), \mathbf{q}(\cdot))$ to be optimal as assumed, it must satisfy Condition-Type-1.

Using Lemma 5.13, let $(u'(\mathbf{v}), \mathbf{q}'(\mathbf{v}) : \mathbf{v} \in T_1)$ denote the unique partial solution that satisfies $\mathbf{q}'(\mathbf{a}) = \mathbf{q}(\mathbf{a})$ and Condition-Type-1. Using Lemma 5.13 again $(u'(\mathbf{v}), \mathbf{q}'(\mathbf{v}) : \mathbf{v} \in T_1)$ satisfies all constraints of $\text{LP}'(I)$ in Parts 0 and 1. By Lemma 5.8, $\mathsf{Ext}(u'(\cdot), \mathbf{q}'(\cdot); u(\mathbf{c}_2), u(\mathbf{c}_3), u(\mathbf{c}_4))$ is a well-defined (nonempty) set of feasible solutions to $\text{LP}'(I)$ (here $u(\mathbf{c}_3) \le u(\mathbf{c}_4) \le u(\mathbf{c}_3) + s$ as $(u(\cdot), \mathbf{q}(\cdot))$ is feasible). Now we use $(u^*(\mathbf{v}), \mathbf{q}^*(\mathbf{v}) : \mathbf{v} \in D)$ to denote a full feasible solution to $\text{LP}'(I)$ in $\mathsf{Ext}(u'(\cdot), \mathbf{q}'(\cdot); u(\mathbf{c}_2), u(\mathbf{c}_3), u(\mathbf{c}_4))$. Now we compare expected revenues of $(u^*(\cdot), \mathbf{q}^*(\cdot))$ and $(u(\cdot), \mathbf{q}(\cdot))$.

For this purpose, let Rev_i and Rev_i^* denote expected revenues of $(u(\cdot), \mathbf{q}(\cdot))$ and $(u^*(\cdot), \mathbf{q}^*(\cdot))$ from type-i vectors, and let Rev and Rev^* denote their overall expected revenues. Let $\epsilon = \max_{\mathbf{v}\in T_1} |u(\mathbf{v}) - u^*(\mathbf{v})|$. By Lemmas 5.9 and 5.10 we have

$$\begin{aligned}&\big|(\text{Rev}_2 + \text{Rev}_3 + \text{Rev}_4) - (\text{Rev}_2^* + \text{Rev}_3^* + \text{Rev}_4^*)\big|\\ &\quad\le O\left(\frac{\delta n\epsilon s + \delta^2 n\epsilon t + \delta^3 n\epsilon t}{\beta}\right) = O\left(\frac{\delta ns}{\beta}\right)\cdot \sum_{\mathbf{v}\in T_1}\big|u(\mathbf{v}) - u^*(\mathbf{v})\big|\end{aligned}$$

where we used $s \gg \delta t$ from (5.3) and $\sum_{\mathbf{v}} |u(\mathbf{v}) - u^*(\mathbf{v})|$ as a trivial upper bound for ϵ. By our choice of δ we have $\delta ns/\beta = o(r^{n+1})$. We also have $\Pr[\mathbf{v}] \ge r^n(1-\delta)(1-\delta^2) = \Omega(r^n)$ for all $\mathbf{v} \in T_1$. As a result,

$$\begin{aligned}&\text{Rev} - \text{Rev}^*\\ &\quad\le \text{Rev}_1 - \text{Rev}_1^* + \big|(\text{Rev}_2 + \text{Rev}_3 + \text{Rev}_4) - (\text{Rev}_2^* + \text{Rev}_3^* + \text{Rev}_4^*)\big|\\ &\quad\le \sum_{\mathbf{v}\in T_1} \Pr[\mathbf{v}]\cdot\big(\text{Rev}(\mathbf{v}) - \text{Rev}^*(\mathbf{v})\big) + o(r^{n+1})\cdot\sum_{\mathbf{v}\in T_1}\big|u(\mathbf{v}) - u^*(\mathbf{v})\big|\\ &\quad= \sum_{\mathbf{v}\in T_1}\Pr[\mathbf{v}]\cdot\left(\sum_{i\in[n+2]} v_i\cdot\big(q_i(\mathbf{v}) - q_i^*(\mathbf{v})\big) + (1+\zeta_{\mathbf{v}})\cdot\big(u^*(\mathbf{v}) - u(\mathbf{v})\big)\right)\\ &\quad:= \sum_{\mathbf{v}\in T_1}\Pr[\mathbf{v}]\cdot\text{Diff}(\mathbf{v})\end{aligned}$$

for some $\zeta_{\mathbf{v}}$ with $|\zeta_{\mathbf{v}}| = o(r)$ for all $\mathbf{v} \in T_1$. For convenience we use $\text{Diff}(\mathbf{v})$ to denote each term for $\mathbf{v}$.

We bound $\text{Diff}(\mathbf{v})$ of nonessential type-1 vectors first. Fix a $\mathbf{v} \in T_1^*$. We write $\mathbf{w} = \text{Lower}(\mathbf{v}) \in T_1'$ and $\mathbf{w}_i = \text{Lower}(\mathbf{v}_{-i}, a_i) \in T_1'$ for each $i \in S(\mathbf{v})$. We have for, each $i \in S(\mathbf{v})$,

$$u(\mathbf{v}) - u(\mathbf{w}_i) \le (v_i - a_i)\cdot q_i(\mathbf{v}) + \sum_{i\ne j\in S^+(\mathbf{v})}\tau_j\cdot q_j(\mathbf{v}) = d_i\cdot q_i(\mathbf{v}) + \sum_{j\in S^+(\mathbf{v})}\tau_j\cdot q_j(\mathbf{v}).$$

Applying Condition-Type-1 on $u^*(\cdot)$, we also have

$$\begin{aligned}u(\mathbf{v}) - u(\mathbf{w}_i) &= u(\mathbf{v}) - u^*(\mathbf{v}) + u^*(\mathbf{v}) - u^*(\mathbf{w}) + u^*(\mathbf{w}) - u^*(\mathbf{w}_i) + u^*(\mathbf{w}_i) - u(\mathbf{w}_i)\\ &= (u(\mathbf{v}) - u^*(\mathbf{v})) + \sum_{j\in S^+(\mathbf{v})}\tau_j\cdot q_j^*(\mathbf{w}) + d_i\cdot q_i(\mathbf{a}) + (u^*(\mathbf{w}_i) - u(\mathbf{w}_i)).\end{aligned}$$

Combining these two together (and plugging in $\mathbf{q}^*(\mathbf{w}) = \mathbf{q}^*(\mathbf{v})$), we have

$$d_i\cdot(q_i(\mathbf{a}) - q_i(\mathbf{v})) + \sum_{j\in S^+(\mathbf{v})}\tau_j\cdot(q_j^*(\mathbf{v}) - q_j(\mathbf{v})) \le (u^*(\mathbf{v}) - u(\mathbf{v})) - (u^*(\mathbf{w}_i) - u(\mathbf{w}_i)).$$

Let $k = \min(S(\mathbf{v}))$ ($S(\mathbf{v}) \ne \emptyset$ since $\mathbf{v} \in T_1'$) and $S'(\mathbf{v}) = S(\mathbf{v}) \setminus \{k\}$. We consider the following two cases.

Case 1: $k = \min(S(\mathbf{v})) \notin S^+(\mathbf{v})$. Then $q_j^*(\mathbf{v}) = q_j(\mathbf{a})$ for all $j \in S^+(\mathbf{v})$. Thus,

$$d_i\cdot(q_i(\mathbf{a}) - q_i(\mathbf{v})) + \sum_{j\in S^+(\mathbf{v})}\tau_j\cdot(q_j(\mathbf{a}) - q_j(\mathbf{v})) \le (u^*(\mathbf{v}) - u(\mathbf{v})) - (u^*(\mathbf{w}_i) - u(\mathbf{w}_i)). \tag{5.23}$$

Given $q_k^*(\mathbf{v}) = 1 - \sum_{i\in S'(\mathbf{v})} q_i(\mathbf{a})$ and v_k is the (strictly) largest entry in $\mathbf{v}$, we have

$$\sum_{i\in[n+2]} v_i \cdot q_i^*(\mathbf{v}) = v_k \left(1 - \sum_{i\in S'(\mathbf{v})} q_i(\mathbf{a})\right) + \sum_{i\in S'(\mathbf{v})} v_i \cdot q_i(\mathbf{a})$$
$$= v_k - \sum_{i\in S'(\mathbf{v})} (v_k - v_i)\cdot q_i(\mathbf{a}),$$

$$\sum_{i\in[n+2]} v_i \cdot q_i(\mathbf{v}) \le v_k \left(1 - \sum_{i\in S'(\mathbf{v})} q_i(\mathbf{v})\right) + \sum_{i\in S'(\mathbf{v})} v_i \cdot q_i(\mathbf{v})$$
$$= v_k - \sum_{i\in S'(\mathbf{v})} (v_k - v_i)\cdot q_i(\mathbf{v}).$$

Combining these two we get

$$\sum_{i\in[n+2]} v_i \cdot (q_i(\mathbf{v}) - q_i^*(\mathbf{v})) \le \sum_{i\in S'(\mathbf{v})} (v_k - v_i)\cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})). \tag{5.24}$$

Since $\tau_i = O(\beta) = O(1/2^n) \ll d_i \approx 1$, there exists a unique tuple $(\gamma_i : i \in S'(\mathbf{v})\}$ such that

$$\sum_{i\in S'(\mathbf{v})} (v_k - v_i)\cdot (q_i(\mathbf{a}) - q_i(\mathbf{v}))$$
$$= \sum_{i\in S'(\mathbf{v})} \gamma_i \left(d_i \cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot (q_j(\mathbf{a}) - q_j(\mathbf{v}))\right). \tag{5.25}$$

This is because $(\gamma_i : i \in S'(\mathbf{v}))$ is the unique solution to a linear system with diagonal entries being d_i or $d_i + \tau_i$ and off-diagonal entries being 0 or τ_j for some $j \in S^+(\mathbf{v})$. Furthermore, given $\tau_j = O(\beta)$ and $\beta \le v_k - v_i \le 3n\beta$, we claim that $0 < \gamma_i = O(n\beta)$. To see this, we first prove that $|\gamma_i| \le 6n\beta$ for all i. Assume for contradiction that $|\gamma_i| = \max_j |\gamma_j| > 6n\beta$. Then we have $3n\beta \ge |v_k - v_i| \ge |d_i\gamma_i| - n\cdot O(\beta)\cdot|\gamma_i| > (3/4)\cdot|\gamma_i|$, a contradiction. Next, assume for contradiction that $\gamma_i \le 0$ for some i. Then we have $\beta \le v_k - v_i \le n \cdot O(\beta)\cdot O(n\beta)$, contradicting $\beta = 1/2^n$. It follows from these properties of γ_i's that

$$\sum_{i\in[n+2]} v_i \cdot (q_i(\mathbf{v}) - q_i^*(\mathbf{v})) \le \sum_{i\in S'(\mathbf{v})} \gamma_i \cdot \big((u^*(\mathbf{v}) - u(\mathbf{v})) - (u^*(\mathbf{w}_i) - u(\mathbf{w}_i))\big)$$
$$= \gamma_{\mathbf{v}} \cdot (u^*(\mathbf{v}) - u(\mathbf{v})) + \sum_{i\in S'(\mathbf{v})} \gamma_{\mathbf{v},i}\cdot (u^*(\mathbf{w}_i) - u(\mathbf{w}_i)) \tag{5.26}$$

for some $\gamma_{\mathbf{v}}$ and $\gamma_{\mathbf{v},i}$ that satisfy $|\gamma_{\mathbf{v}}| = O(n^2\beta)$ and $|\gamma_{\mathbf{v},i}| = O(n\beta)$ for all $i \in S'(\mathbf{v})$.

Case 2: $k = \min(\mathbf{v}) \in S^+(\mathbf{v})$. Then we have, for each $i \in S(\mathbf{v})$,

$$d_i \cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})) + \tau_k \cdot (q_k^*(\mathbf{v}) - q_k(\mathbf{v})) + \sum_{j\in S^+(\mathbf{v})\setminus\{k\}} \tau_j \cdot (q_j(\mathbf{a}) - q_j(\mathbf{v}))$$
$$\le (u^*(\mathbf{v}) - u(\mathbf{v})) - (u^*(\mathbf{w}_i) - u(\mathbf{w}_i)). \tag{5.27}$$

For clarity we use LHS_i to denote the LHS of the inequality above for each $i \in S(\mathbf{v})$. Then there exists a unique tuple $(\gamma_i : i \in S(\mathbf{v}))$ such that

$$\sum_{i\in S'(\mathbf{v})} ((v_k - v_i) + \gamma_k)\cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})) + \gamma_k \cdot (q_k^*(\mathbf{v}) - q_k(\mathbf{v})) = \sum_{i\in S'(\mathbf{v})} \gamma_i \cdot \text{LHS}_i.$$

This is because $(\gamma_i : i \in S(\mathbf{v}))$ is the unique solution to a linear system with diagonal entries being either d_i or $d_i + \tau_i$ for $i \neq k$ and -1 for k and off-diagonal entries being either 0 or τ_j in general and -1 for the column that corresponds to k. Similarly we have $0 < \gamma_i \leq O(n\beta)$ for all $i \in S(\mathbf{v})$. This gives us a connection between the LHS above and what we care about since

$$\sum_{i\in S'(\mathbf{v})} ((v_k - v_i) + \gamma_k) \cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})) + \gamma_k \cdot (q_k^*(\mathbf{v}) - q_k(\mathbf{v}))$$

$$= \sum_{i\in S'(\mathbf{v})} (v_k - v_i) \cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})) + \gamma_k - \gamma_k \left(q_k(\mathbf{v}) + \sum_{i\in S'(\mathbf{v})} q_i(\mathbf{v}) \right)$$

$$\geq \sum_{i\in S'(\mathbf{v})} (v_k - v_i) \cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})) \geq \sum_{i\in[n+2]} v_i \cdot (q_i(\mathbf{v}) - q_i^*(\mathbf{v})), \tag{5.28}$$

where the last inequality follows from (5.24). So (5.26) also holds in this case for some $\gamma_{\mathbf{v}}$ and $\gamma_{\mathbf{v},i}$ with absolute values bounded from above by $O(n^2\beta)$ and $O(n\beta)$, respectively.

To summarize our progress so far, we have shown for each nonessential type-1 vector $\mathbf{v} \in T_1^*$, $\textsc{Diff}(\mathbf{v})$ is at most

$$(1+\zeta_{\mathbf{v}}+\gamma_{\mathbf{v}})(u^*(\mathbf{v})-u(\mathbf{v})) + \sum_{i\in S'(\mathbf{v})} \gamma_{\mathbf{v},i} \cdot \big(u^*(\textsc{Lower}(\mathbf{v}_{-i}, a_i)) - u(\textsc{Lower}(\mathbf{v}_{-i}, a_i))\big).$$

Therefore, we have

$$\sum_{\mathbf{v}\in T_1^*} \Pr[\mathbf{v}] \cdot \textsc{Diff}(\mathbf{v})$$

$$\leq \sum_{\mathbf{v}\in T_1^*} \Pr[\mathbf{v}] \cdot (1 + \gamma'_{\mathbf{v}}) \cdot (u^*(\mathbf{v}) - u(\mathbf{v})) + \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \cdot \xi_{\mathbf{v}} \cdot (u^*(\mathbf{v}) - u(\mathbf{v})) \tag{5.29}$$

for some $\gamma'_{\mathbf{v}}$ and $\xi_{\mathbf{v}}$ with $|\gamma'_{\mathbf{v}}| = O(n^2\beta)$ (since $|\zeta_{\mathbf{v}}| = o(r)$) and $|\xi_{\mathbf{v}}| \leq O(n^2p\beta)$. For the latter, we used the fact that for any $\mathbf{v} \in T_1'$ the total probability of all vectors in blocks strictly above $\textsc{Block}(\mathbf{v})$ is at most an $\Omega(np)$-fraction of that of $\mathbf{v}$. We continue to simplify the first part of the RHS above.

Let $\mathbf{w} = \textsc{Lower}(\mathbf{v})$ for some nonessential vector $\mathbf{v} \in T_1^*$. We have

$$u^*(\mathbf{v}) = u^*(\mathbf{w}) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j^*(\mathbf{w}) \quad \text{and} \quad u(\mathbf{v}) \geq u(\mathbf{w}) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{w})$$

by $\textsc{Condition-Type-1}$ and (5.6) in Part 1 of $\text{LP}'(I)$. As a result, we have

$$u^*(\mathbf{v}) - u(\mathbf{v}) \leq u^*(\mathbf{w}) - u(\mathbf{w}) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot (q_j^*(\mathbf{w}) - q_j(\mathbf{w})).$$

Fix an essential vector $\mathbf{w} \in T_1'$ and let $B = \textsc{Block}(\mathbf{w}) \setminus \{\mathbf{w}\}$. Then we have

$$\sum_{\mathbf{v}\in B} \Pr[\mathbf{v}] \cdot (1 + \gamma'_{\mathbf{v}}) \cdot (u^*(\mathbf{v}) - u(\mathbf{v}))$$

$$\leq \sum_{\mathbf{v}\in B} \Pr[\mathbf{v}] \cdot (1 + \gamma'_{\mathbf{v}}) \cdot \left(u^*(\mathbf{w}) - u(\mathbf{w}) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot (q_j^*(\mathbf{w}) - q_j(\mathbf{w})) \right)$$

$$= \Pr[\mathbf{w}] \cdot \alpha_{\mathbf{w}} \cdot (u^*(\mathbf{w}) - u(\mathbf{w})) + \Pr[\mathbf{w}] \sum_{j\in[n+2]} \alpha_{\mathbf{w},j} \cdot (q_j^*(\mathbf{w}) - q_j(\mathbf{w}))$$

for some $\alpha_{\mathbf{w}}$ and $\alpha_{\mathbf{w},j}$ with absolute values bounded by $|\alpha_{\mathbf{w}}| = O(nr/p)$ and $|\alpha_{\mathbf{w},j}| = O(nr\beta/p)$.

Combining all these inequalities together, we have

$$\begin{aligned}
&\sum_{\mathbf{v}\in T_1} \Pr[\mathbf{v}] \cdot \textsc{Diff}(\mathbf{v}) \\
&\quad\le \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \cdot \left(\sum_{j\in[n+2]} v_j \cdot (q_j(\mathbf{v}) - q_j^*(\mathbf{v})) + (1+\zeta_{\mathbf{v}}) \cdot (u^*(\mathbf{v}) - u(\mathbf{v})) \right) \\
&\qquad + \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \cdot \left(\alpha_{\mathbf{v}} \cdot (u^*(\mathbf{v}) - u(\mathbf{v})) + \sum_{j\in[n+2]} \alpha_{\mathbf{v},j} \cdot (q_j^*(\mathbf{v}) - q_j(\mathbf{v})) \right) \\
&\qquad + \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \cdot \xi_{\mathbf{v}} \cdot (u^*(\mathbf{v}) - u(\mathbf{v})) \\
&\quad= \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \cdot \Bigg((1+\zeta_{\mathbf{v}}+\alpha_{\mathbf{v}}+\xi_{\mathbf{v}}) \cdot (u^*(\mathbf{v}) - u(\mathbf{v})) \\
&\qquad + \sum_{j\in[n+2]} (v_j - \alpha_{\mathbf{v},j}) \cdot (q_j(\mathbf{v}) - q_j^*(\mathbf{v})) \Bigg).
\end{aligned}$$

Recall that $|\zeta_{\mathbf{v}}| = o(r)$ and $|\xi_{\mathbf{v}}| \le O(n^2p\beta)$. We have $1+\zeta_{\mathbf{v}}+\alpha_{\mathbf{v}}+\xi_{\mathbf{v}} = 1 \pm o(1)$. Fix an essential $\mathbf{v} \in T_1'$. We have $v_j - \alpha_{\mathbf{v},j} \approx 2$ for $j \in S(\mathbf{v})$, and $k = \min(S(\mathbf{v}))$ still has the (strictly) largest coefficient $v_k - \alpha_{\mathbf{v},k}$ since $|\alpha_{\mathbf{v},j}| = O(nr\beta/p) \ll \beta$. As a result, we have (recall that $S'(\mathbf{v}) = S(\mathbf{v}) \setminus \{k\}$)

$$\begin{aligned}
&\sum_{j\in[n+2]} (v_j - \alpha_{\mathbf{v},j}) \cdot q_j^*(\mathbf{v}) \\
&\qquad = (v_k - \alpha_{\mathbf{v},k}) \cdot \left(1 - \sum_{j\in S'(\mathbf{v})} q_j(\mathbf{a})\right) + \sum_{j\in S'(\mathbf{v})} (v_j - \alpha_{\mathbf{v},j}) \cdot q_j(\mathbf{a}), \\
&\sum_{j\in[n+2]} (v_j - \alpha_{\mathbf{v},j}) \cdot q_j(\mathbf{v}) \\
&\qquad \le (v_k - \alpha_{\mathbf{v},k}) \cdot \left(1 - \sum_{j\in S'(\mathbf{v})} q_j(\mathbf{v})\right) + \sum_{j\in S'(\mathbf{v})} (v_j - \alpha_{\mathbf{v},j}) \cdot q_j(\mathbf{v}).
\end{aligned} \tag{5.30}$$

Let $\phi_{\mathbf{v},j} = v_k - v_j - \alpha_{\mathbf{v},k} + \alpha_{\mathbf{v},j}$ for each $j \in S'(\mathbf{v})$. Then $\Omega(\beta) \le \phi_{\mathbf{v},j} \le O(n\beta)$ and

$$\sum_{j\in[n+2]} (v_j - \alpha_{\mathbf{v},j}) \cdot (q_j(\mathbf{v}) - q_j^*(\mathbf{v})) \le \sum_{j\in S'(\mathbf{v})} \phi_{\mathbf{v},j} \cdot (q_j(\mathbf{a}) - q_j(\mathbf{v})).$$

Plugging this in, we have

$$\begin{aligned}
&\sum_{\mathbf{v}\in T_1} \Pr[\mathbf{v}] \cdot \textsc{Diff}(\mathbf{v}) \\
&\quad\le \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \cdot \left((1 \pm o(1)) \cdot (u^*(\mathbf{v}) - u(\mathbf{v})) + \sum_{j\in S'(\mathbf{v})} \phi_{\mathbf{v},j} \cdot (q_j(\mathbf{a}) - q_j(\mathbf{v})) \right).
\end{aligned}$$

We also have $u^*(\mathbf{v}) - u^*(\mathbf{v}_{-j}, a_j) = d_j \cdot q_j(\mathbf{a})$ for $\mathbf{v} \in T_1'$ and each $j \in S(\mathbf{v})$, and $u(\mathbf{v}) - u(\mathbf{v}_{-j}, a_j) \le d_j \cdot q_j(\mathbf{v})$ by (5.8) of $\text{LP}'(I)$. As a result, we have

$$\sum_{j\in S'(\mathbf{v})} \phi_{\mathbf{v},j} \cdot (q_j(\mathbf{a}) - q_j(\mathbf{v}))$$

$$\le \sum_{j\in S'(\mathbf{v})} \frac{\phi_{\mathbf{v},j}}{d_j} \cdot \big((u^*(\mathbf{v}) - u(\mathbf{v})) - (u^*(\mathbf{v}_{-j}, a_j) - u(\mathbf{v}_{-j}, a_j))\big). \tag{5.31}$$

Plugging it back, we have

$$\sum_{\mathbf{v}\in T_1} \Pr[\mathbf{v}] \cdot \textsc{Diff}(\mathbf{v}) \le \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \cdot (1 \pm o(1)) \cdot (u^*(\mathbf{v}) - u(\mathbf{v}))$$

$$+ \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \sum_{j\in S'(\mathbf{v})} \frac{\phi_{\mathbf{v},j}}{d_j} \cdot \big((u^*(\mathbf{v}) - u(\mathbf{v})) - (u^*(\mathbf{v}_{-j}, a_j) - u(\mathbf{v}_{-j}, a_j))\big)$$

$$\le \sum_{\mathbf{v}\in T_1'} \Pr[\mathbf{v}] \cdot (1 + \delta_{\mathbf{v}}) \cdot (u^*(\mathbf{v}) - u(\mathbf{v}))$$

for some $\delta_{\mathbf{v}}$ with absolute value bounded from above by

$$|\delta_{\mathbf{v}}| \le o(1) + O(n^2\beta) + O(n\beta \cdot np) = o(1).$$

Since $u(\mathbf{v}) \ge u^*(\mathbf{v})$ for all $\mathbf{v} \in T_1'$ (due to (5.4) of $\text{LP}'(I)$), we must have $u(\mathbf{v}) = u^*(\mathbf{v})$ for all $\mathbf{v} \in T_1'$ by the optimality of $(u(\cdot), \mathbf{q}(\cdot))$. This proved the part of Condition-Type-1 on $\mathbf{q}(\mathbf{v})$ of essential vectors.

Combining this with (5.8) of $\text{LP}'(I)$, we have for each $\mathbf{v} \in T_1'$, $i \in S(\mathbf{v})$, and $\mathbf{w} = (\mathbf{v}_{-i}, a_i)$: $d_i \cdot q_i(\mathbf{a}) = u(\mathbf{v}) - u(\mathbf{w}) \le d_i \cdot q_i(\mathbf{v})$ and, thus, $q_i(\mathbf{v}) \ge q_i(\mathbf{a})$. On the other hand, it follows from the optimality of $(u(\cdot), \mathbf{q}(\cdot))$ that (5.31) and (5.30) must be tight. This implies that $\mathbf{q}(\mathbf{v}) = \mathbf{q}^*(\mathbf{v})$ for all essential vectors $\mathbf{v} \in T_1'$.

For a nonessential type-1 vector $\mathbf{v} \in T_1^*$, letting $\mathbf{w} = \textsc{Lower}(\mathbf{v})$, (5.6) in Part 1 of $\text{LP}'(I)$ implies that $u(\mathbf{v}) \ge u(\mathbf{w}) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{w}) = u^*(\mathbf{v})$, as we have proved that $u(\mathbf{w}) = u^*(\mathbf{w})$ and $\mathbf{q}(\mathbf{w}) = \mathbf{q}^*(\mathbf{w})$ (since $\mathbf{w}$ is essential). Then $u(\mathbf{v}) = u^*(\mathbf{v})$ follows from the tightness of (5.29).

Finally, for each nonessential vector $\mathbf{v} \in T_1^*$, we consider the following two cases (letting $k = \min(S(\mathbf{v}))$.

Case 1: $k \notin S^+(\mathbf{v})$. $\mathbf{q}(\mathbf{v}) = \mathbf{q}^*(\mathbf{v})$ follows from the tightness of (5.23) and (5.24). (5.23) yields that

$$d_i \cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot (q_j(\mathbf{a}) - q_j(\mathbf{v})) = 0$$

for all $i \in S'(\mathbf{v})$ (note that we actually do not use $i = k$ in (5.25)). These equations together imply that $q_i(\mathbf{v}) = q_i(\mathbf{a})$ for all $i \in S'(\mathbf{v})$. $q_k(\mathbf{v}) = q_k^*(\mathbf{v})$ follows from the tightness of (5.24).

Case 2: $k \in S^+(\mathbf{v})$. The tightness of (5.28) implies that

$$q_k(\mathbf{v}) = 1 - \sum_{j\in S'(\mathbf{v})} q_j(\mathbf{v}). \tag{5.32}$$

The tightness of (5.27) implies that

$$d_i \cdot (q_i(\mathbf{a}) - q_i(\mathbf{v})) + \tau_k \cdot (q_k^*(\mathbf{v}) - q_k(\mathbf{v})) + \sum_{j\in S^+(\mathbf{v})\setminus\{k\}} \tau_j \cdot (q_j(\mathbf{a}) - q_j(\mathbf{v})) = 0$$

for all $i \in S'(\mathbf{v})$. Plugging in $q_k^*(\mathbf{v}) = 1 - \sum_{j\in S'(\mathbf{v})} q_j(\mathbf{a})$ and (5.32), we must have $q_i(\mathbf{a}) = q_i(\mathbf{v})$ for all $i \in S'(\mathbf{v})$ and thus, $q_k(\mathbf{v}) = q_k^*(\mathbf{v})$ by (5.32). It then follows that $\mathbf{q}(\mathbf{v}) = \mathbf{q}^*(\mathbf{v})$.

This finishes the proof of the lemma. ∎

5.1.7. Characterization of optimal solutions. Let $\mathbf{q}$ be a nonnegative $(n+2)$-dimensional vector that sums to at most 1, and $u_2, u_3, u_4 \geq 0$ that satisfy $u_3 \leq u_4 \leq u_3 + s$. Let $\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ denote the following set of solutions to $\mathrm{LP}'(I)$: Let $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ be the unique partial solution that satisfies both $\mathbf{q}(\mathbf{a}) = \mathbf{q}$ and Condition-Type-1. By Lemma 5.13, $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in T_1)$ satisfies all constraints in Parts 0 and 1 of $\mathrm{LP}'(I)$. Then we set

$$\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4) = \mathsf{Ext}(u(\cdot), \mathbf{q}(\cdot); u_2, u_3, u_4).$$

We record the following lemma.

Lemma 5.15. *Given any nonnegative vector $\mathbf{q}$ that sums to at most 1, and u_2, $u_3, u_4 \geq 0$ with $u_3 \leq u_4 \leq u_3 + s$, $\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ is a nonempty set of feasible solutions to $\mathrm{LP}'(I)$.*

Our characterization of optimal solutions to $\mathrm{LP}'(I)$ is summarized in the theorem below.

Theorem 5.16. *Any optimal solution $(u(\mathbf{v}), \mathbf{q}(\mathbf{v}) : \mathbf{v} \in D)$ to $\mathrm{LP}'(I)$ belongs to $\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$, where $\mathbf{q} = \mathbf{q}(\mathbf{a})$ and $u_i = u(\mathbf{c}_i)$ for each $i = 2, 3, 4$.*

5.2. Choices of parameters and their consequences. Now we pin down the rest of the parameters: a_i, s, h_i, t, and see how they affect optimal solutions of $\mathrm{LP}'(I)$.

5.2.1. Setting a_i's. First, we set a_i's (see (5.34) below) such that they satisfy (5.2), i.e., $|a_i - 1| = O(np)$, and the expected revenue from type-1 vectors in any optimal solution $(u(\cdot), \mathbf{q}(\cdot))$ to $\mathrm{LP}'(I)$ is of the following form,

$$\textsc{const} + c \cdot \sum_{i\in[n]} q_i(\mathbf{a}) \tag{5.33}$$

for some $c \approx 1$. By Theorem 5.16, $(u(\cdot), \mathbf{q}(\cdot))$ lies in $\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ for some nonnegative vector $\mathbf{q}$ that sums to at most 1, and some $u_2, u_3, u_4 \geq 0$ that satisfy $u_3 \leq u_4 \leq u_3 + s$. Given that $(u(\cdot), \mathbf{q}(\cdot))$ satisfies Condition-Type-1, expected revenue from type-1 vectors only depends on $\mathbf{q}(\mathbf{a}) = \mathbf{q}$. We next calculate the expected revenue from type-1 vectors given $\mathbf{q}$, and the choices of a_i's will become clear.

First, we have $\textsc{Rev}(\mathbf{a}) = \sum_{i\in[n]} a_i \cdot q_i$ (since $u(\mathbf{a}) = 0$). Given that $(u(\cdot), \mathbf{q}(\cdot))$ satisfies Condition-Type-1, each essential type-1 vector $\mathbf{v} \in T_1'$ and $\mathbf{v} \neq \mathbf{a}$ has revenue (letting $k = \min(S(\mathbf{v}))$)

$$\begin{aligned}\textsc{Rev}(\mathbf{v}) &= \sum_{i\in S'(\mathbf{v})} \ell_i \cdot q_i + \ell_k \cdot \left(1 - \sum_{i\in S'(\mathbf{v})} q_i\right) - \sum_{i\in S(\mathbf{v})} d_i \cdot q_i \\ &= \ell_k - \sum_{i\in S'(\mathbf{v})} (\ell_k - a_i) \cdot q_i - d_k \cdot q_k.\end{aligned}$$

Given Lemma 5.12, the block B that contains $\mathbf{v} \in T_1'$ and $\mathbf{v} \neq \mathbf{a}$ overall contributes

$$\Pr[B] \cdot \textsc{Rev}(\mathbf{v}) = \Pr[B] \cdot \left(\ell_k - \sum_{i \in S'(\mathbf{v})} (\ell_k - a_i) \cdot q_i - (\ell_k - a_k) \cdot q_k \right).$$

It is clear now that expected revenue from type-1 vectors is an affine linear form of q_i's, $i \in [n]$.

Let c_i denote the coefficient of each q_i in the expected revenue from type-1 vectors. Then $\mathbf{a}$ contributes $\Pr[\mathbf{a}] \cdot a_i$ to c_i ($\Pr[\mathbf{a}] \approx 1 - np$ which as we will see is the dominating term). A block B that contains $\mathbf{v} \in T_1'$ and $\mathbf{v} \neq \mathbf{a}$ contributes 0 if $i \notin S(\mathbf{v})$; $-\Pr[B] \cdot (\ell_i - a_i)$ if $i = \min(S(\mathbf{v}))$; and $-\Pr[B] \cdot (\ell_{\min(S(\mathbf{v}))} - a_i)$ if $i \in S'(\mathbf{v})$. More specifically, the total probability of type-1 blocks B and $\mathbf{v} \in B$ with $i = \min(S(\mathbf{v}))$ is

$$(1-\delta) \cdot (1-\delta^2) \cdot (1-p-r)^{i-1} \cdot (p+r);$$

for each $k < i$, the total probability of type-1 blocks B with $i \in S(\mathbf{v})$ and $\min(S(\mathbf{v})) = k$ is

$$(1-\delta) \cdot (1-\delta^2) \cdot (1-p-r)^{k-1} \cdot (p+r) \cdot (p+r).$$

As a result, c_i becomes

$$(1-\delta)(1-\delta^2)\Bigg((1-p-r)^n a_i - \sum_{k<i} (1-p-r)^{k-1} (p+r)^2 (\ell_k - a_i) \\ - (1-p-r)^{i-1}(p+r)(\ell_i - a_i) \Bigg).$$

To meet both goals, i.e., $c_1 = \cdots = c_n \approx 1$ and $|a_i - 1| \le O(np)$, we set

$$a_i = \frac{1 + \sum_{k<i} (1-p-r)^{k-1} \cdot (p+r)^2 \cdot \ell_k + (1-p-r)^{i-1} \cdot (p+r) \cdot \ell_i}{(1-p-r)^n + \sum_{k<i} (1-p-r)^{k-1} \cdot (p+r)^2 + (1-p-r)^{i-1} \cdot (p+r)}. \tag{5.34}$$

It is easy to verify that the a_i's satisfy $1 < a_i \le 1 + O(np)$. The length of binary representations of each a_i is polynomial in n and a_i's can be computed efficiently, given p, r, and ℓ_i's as in (5.1) and (5.2).

We summarize the consequence of our choices of a_i's in the following lemma.

Lemma 5.17. *Given choices of a_i's in* (5.34)*, revenue from type*-1 *vectors in any feasible solution to* $\mathrm{LP}'(I)$ *that satisfies* Condition-Type-1 *is of the form in* (5.33) *with* $c = (1-\delta)(1-\delta^2) \approx 1$.

It is now time to prove that $\mathbf{q}(\mathbf{a})$ sums to 1 in any optimal solution to $\mathrm{LP}'(I)$.

Lemma 5.18. *Given our choices of a_i's in* (5.34)*, any optimal solution to* $\mathrm{LP}'(I)$ *satisfies* $\sum_{i\in[n]} q_i(\mathbf{a}) = 1$.

Proof. Assume for contradiction that $(u(\cdot), \mathbf{q}(\cdot))$ is optimal but $\sum_{i\in[n]} q_i(\mathbf{a}) = 1$ does not hold. Let $\mathbf{q}'$ be the vector obtained from $\mathbf{q}(\mathbf{a})$ as follows: if $\sum_{i\in[n+2]} q_i(\mathbf{a}) < 1$, we replace its first entry by $q_1' = q_1(\mathbf{a}) + \epsilon$, where $\epsilon = 1 - \sum_{i\in[n+2]} q_i(\mathbf{a}) > 0$; otherwise, letting $\epsilon = q_{n+1}(\mathbf{a}) + q_{n+2}(\mathbf{a}) > 0$, we set $q_1' = q_1(\mathbf{a}) + \epsilon$ and $q_{n+1}' = q_{n+2}' = 0$. Let $(u'(\cdot), \mathbf{q}'(\cdot))$ be a feasible solution from $\mathsf{Ext}(\mathbf{q}', u(\mathbf{c}_2), u(\mathbf{c}_3))$. It follows from Lemma 5.17 that the expected revenue from type-1 vectors goes up by $\Omega(\epsilon)$ in $(u'(\cdot), \mathbf{q}'(\cdot))$. However, by Condition-Type-1, we have $|u(\mathbf{v}) - u'(\mathbf{v})| \le O(\epsilon)$ for all $\mathbf{v} \in T_1$.

By Lemma 5.10, expected revenue from types-2, -3 and -4 vectors goes down in $(u'(\cdot), \mathbf{q}'(\cdot))$ by at most $O(\delta n\epsilon s/\beta) + O(\delta^2 n\epsilon t/\beta) \ll \epsilon$. This contradicts the assumption that $(u(\cdot), \mathbf{q}(\cdot))$ is optimal. ∎

Given Lemma 5.18, from now on we restrict $\mathbf{q}$ to be a nonnegative n-dimensional vector that sums to exactly 1 in $\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$. We also use $\textsc{Rev}(\mathbf{q}, u_2, u_3, u_4)$ to denote the expected revenue of solutions in $\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$. All parameters of I have been chosen except s, h_i's, and t.

5.2.2. Setting s. Our goal in this section is to show that, by setting

$$s = 2 + \frac{1}{(n - 0.5)p} = \Theta\left(\frac{1}{np}\right),$$

any optimal solution to $\mathrm{LP}'(I)$ from $\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ must satisfy

$$d_1 \cdot q_1(\mathbf{a}) = d_2 \cdot q_2(\mathbf{a}) = \cdots = d_n \cdot q_n(\mathbf{a}). \tag{5.35}$$

Note that we have chosen both a_i and ℓ_i, and so $d_i = \ell_i - a_i$. (5.35) then uniquely determines $\mathbf{q}(\mathbf{a})$ in any optimal solution, as by Lemma 5.18, $\mathbf{q}(\mathbf{a})$ must sum to 1. (5.35) also implies that $\mathbf{q}(\mathbf{a})$ is indeed very close to the uniform distribution over $[n]$ since $d_i \approx 1$ (more exactly, $|d_i - 1| = O(np + n\beta)$).

In the rest of section 5.2.2 we use $\mathbf{v}_i$ for each $i \in [n]$ to denote the type-2 vector with $v_{i,i} = \ell_i$, $v_{i,j} = a_j$ for other $j \in [n]$, $v_{i,n+1} = s$ and $v_{i,n+2} = 0$. To prove (5.35), we start with the following lemma.

Lemma 5.19. *Let $\mathbf{q}$ be any nonnegative n-dimensional vector that sums to* 1*, and $u_2' = \min_{i\in[n]} d_i \cdot q_i$. If $u_2, u_3, u_4 \geq 0$ satisfy $u_3 \leq u_4 \leq u_3 + s$ and $u_2 \neq u_2'$, then* $\textsc{Rev}(\mathbf{q}, u_2, u_3, u_4) < \textsc{Rev}(\mathbf{q}, u_2', u_3, u_4)$.

Proof. Let $(u(\cdot), \mathbf{q}(\cdot))$ be a feasible solution in $\mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ and $(u'(\cdot), \mathbf{q}'(\cdot))$ be a feasible solution in $\mathsf{Ext}(\mathbf{q}, u_2', u_3, u_4)$. Below we compare their revenues $\textsc{Rev}_2$ and $\textsc{Rev}_2'$ from type-2 vectors since it is clear that $\textsc{Rev}_i = \textsc{Rev}_i'$ for all $i \in \{1, 3, 4\}$. We consider two cases: $u_2 < u_2'$ or $u_2 > u_2'$. In both cases we have from Lemma 5.4 that $|\textsc{Rev}'(\mathbf{v}) - \textsc{Rev}(\mathbf{v})| = O(n\epsilon s/\beta)$, given that $|u(\mathbf{w}) - u(\mathbf{w}')| \leq \epsilon$. We focus on comparing $\textsc{Rev}'(\mathbf{v}_i)$ and $\textsc{Rev}(\mathbf{v}_i)$ in both cases.

Case 1: $u_2 < u_2'$. Let $\epsilon = u_2' - u_2 > 0$. We have $\textsc{Rev}'(\mathbf{c}_2) = \textsc{Rev}(\mathbf{c}_2) - \epsilon$ for $\mathbf{c}_2$. We compare $\textsc{Rev}'(\mathbf{v}_i)$ and $\textsc{Rev}(\mathbf{v}_i)$ for each i. Since $u_2 < u_2'$, by Condition-Type-2 and the definition of u_2', we have $u(\mathbf{v}_i) = u'(\mathbf{v}_i)$. Constraints of $\mathrm{LP}(\mathbf{v}_i : u)$ are

$$q_j \geq 0 \text{ for all } j \in [n+2], \quad \sum_{j\in[n+2]} q_j \leq 1, \quad \text{and} \quad d_i \cdot q_i \geq u(\mathbf{v}_i) - u_2;$$

constraints in $\mathrm{LP}(\mathbf{v}_i : u')$ are

$$q_i \geq 0 \text{ for all } j \in [n+2], \quad \sum_{j\in[n+2]} q_j \leq 1, \quad \text{and} \quad d_i \cdot q_i \geq u'(\mathbf{v}_i) - u_2'.$$

It follows that $\mathbf{q}(\mathbf{v}_i)$ satisfies $q_i(\mathbf{v}_i) = (u(\mathbf{v}_i) - u_2)/d_i$ and puts the rest of the probability onto $q_{n+1}(\mathbf{v}_i)$, while $\mathbf{q}'(\mathbf{v}_i)$ satisfies $q_i'(\mathbf{v}_i) = (u'(\mathbf{v}_i) - u_2')/d_i = q_i(\mathbf{v}_i) - (\epsilon/d_i)$ and puts the rest onto $q_{n+1}'(\mathbf{v}_i)$. As a result we have $\textsc{Rev}'(\mathbf{v}_i) = \textsc{Rev}(\mathbf{v}_i) + (\epsilon/d_i) \cdot (s - \ell_i)$ for each $i \in [n]$. To summarize, we have

$$\textsc{Rev}_2' - \textsc{Rev}_2 \geq \sum_{i\in[n]} \Pr[\mathbf{v}_i] \cdot \left(\frac{\epsilon}{d_i} \cdot (s - \ell_i)\right) - \Pr[\mathbf{c}_2] \cdot \epsilon - O(nr\delta) \cdot O\left(\frac{n\epsilon s}{\beta}\right).$$

Plugging in that $\Pr[\mathbf{c}_2] \le \delta$, $1/d_i \ge 1 - O(n\beta)$, $s - \ell_i \ge s - 2 - O(n\beta)$, and

$$\Pr[\mathbf{v}_i] = p \cdot (1-p-r)^{n-1} \cdot \delta \cdot (1-\delta^2) \ge p\delta \cdot (1 - O(np)),$$

we have

$$\begin{aligned}\sum_{i\in[n]} \Pr[\mathbf{v}_i] \cdot \frac{\epsilon}{d_i} \cdot (s-\ell_i) \\ &\ge n \cdot p\delta \cdot (1-O(np)) \cdot \epsilon \cdot (1-O(n\beta)) \cdot \left(\frac{1}{(n-0.5)p} - O(n\beta)\right) \\ &\ge \frac{n\delta\epsilon}{n-0.5} \cdot (1-O(n\beta)).\end{aligned}$$

As a result, we have $\text{Rev}'_2 - \text{Rev}_2 > 0$ given our choices of p, r, and β.

Case 2: $u_2 > u'_2$. Let $\epsilon = u_2 - u'_2 > 0$. In this case, $\text{Rev}'(\mathbf{c}_2) = \text{Rev}(\mathbf{c}_2) + \epsilon$. For each $i \in [n]$, a similar analysis of $\text{LP}(\mathbf{v}_i : u)$ and $\text{LP}(\mathbf{v}_i : u')$ as in Case 1 implies that $q_i(\mathbf{v}_i) = (u(\mathbf{v}_i) - u_2)/d_i$ and $q'_i(\mathbf{v}_i) = (u'(\mathbf{v}_i) - u'_2)/d_i$, and both vectors $\mathbf{q}(\mathbf{v}_i)$ and $\mathbf{q}'(\mathbf{v}_i)$ have the rest of the probability allocated onto their $(n+1)$th entries.

Let I denote the nonempty set of i that has the minimum $d_i q_i$ among all indices in $[n]$. It then follows from the definition of u'_2 and the assumption of $u_2 > u'_2$ that $u(\mathbf{v}_i) = u_2$ and $u'(\mathbf{v}_i) = u'_2$ for each $i \in I$ and, thus, $\text{Rev}'(\mathbf{v}_i) = \text{Rev}(\mathbf{v}_i) + \epsilon$ for each $i \in I$. For each $i \notin I$, we have $u'(\mathbf{v}_i) - u'_2 \le u(\mathbf{v}_i) - u_2 + \epsilon$. This follows by considering the two cases of $u(\rho(\mathbf{v}_i)) \le u_2$ or $u(\rho(\mathbf{v}_i)) > u_2$:

- If $u(\rho(\mathbf{v}_i)) \le u_2$, we have $u(\mathbf{v}_i) = u_2$ and $u'(\mathbf{v}_i) = \max(u(\rho(\mathbf{v}_i)), u'_2) \le u_2$ and, thus, $u'(\mathbf{v}_i) - u'_2 \le u_2 - u'_2 = \epsilon \le u(\mathbf{v}_i) - u_2 + \epsilon$.
- If $u(\rho(\mathbf{v}_i)) > u_2$, we have $u(\mathbf{v}_i) = u(\rho(\mathbf{v}_i))$ and $u'(\mathbf{v}_i) = \max(u(\rho(\mathbf{v}_i)), u'_2) = u(\rho(\mathbf{v}_i))$ and thus, $u'(\mathbf{v}_i) - u'_2 = u(\mathbf{v}_i) - u_2 + \epsilon$.

So for each $i \notin I$ we have $\text{Rev}'(\mathbf{v}_i) \ge \text{Rev}(\mathbf{v}_i) - (\epsilon/d_i)\cdot(s-\ell_i)$. Combining everything we have $\text{Rev}'_2 - \text{Rev}_2$ is at least

$$\Pr[\mathbf{c}_2]\cdot\epsilon + \sum_{i\in I}\Pr[\mathbf{v}_i]\cdot\epsilon - \sum_{i\notin I}\Pr[\mathbf{v}_i]\cdot\left(\frac{\epsilon}{d_i}\cdot(s-\ell_i)\right) - O(nr\delta)\cdot O\left(\frac{n\epsilon s}{\beta}\right).$$

Plugging in $\Pr[\mathbf{c}_2] \ge \delta(1-O(np))$ and

$$\begin{aligned}\sum_{i\notin I}\Pr[\mathbf{v}_i]\cdot\frac{\epsilon}{d_i}\cdot(s-\ell_i) &\le (n-1)\cdot p\delta\cdot\epsilon\cdot(1+O(n\beta))\cdot\frac{1}{(n-0.5)p} \\ &= \delta\epsilon\cdot\frac{n-1}{n-0.5}\cdot(1+O(n\beta)),\end{aligned}$$

we have $\text{Rev}'_2 - \text{Rev}_2 > 0$. This finishes the proof of the lemma. ∎

We are now ready to prove the main lemma of this section.

Lemma 5.20. *Any optimal solution* $(u(\cdot), \mathbf{q}(\cdot))$ *to* $\text{LP}'(I)$ *satisfies*

$$d_1 \cdot q_1(\mathbf{a}) = \cdots = d_n \cdot q_n(\mathbf{a}).$$

Proof. Let $(u(\cdot), \mathbf{q}(\cdot)) \in \mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ be an optimal solution to $\text{LP}'(I)$, where $\mathbf{q}$ is an n-dimensional nonnegative vector that sums to 1 and $u_2, u_3, u_4 \ge 0$ with $u_3 \le u_4 \le u_3 + s$. By Lemma 5.19, we have $u_2 = \min_{i\in[n]} d_i \cdot q_i$. Assume for contradiction that $\mathbf{q}$ does not satisfy $d_1 \cdot q_1 = \cdots = d_n \cdot q_n$. We use $K \subset [n]$ to denote the set of indices k with $d_k \cdot q_k = \min_i d_i \cdot q_i$, and $t \in [n]$ denotes an index with $d_t \cdot q_t > \min_i d_i \cdot q_i$. Then we replace $\mathbf{q}$ by $\mathbf{q}'$, where $q'_k = q_k + (\epsilon/d_k)$ for each $k \in K$ and $q'_t = q_t - \sum_{k\in K}(\epsilon/d_k)$, for a sufficiently small $\epsilon > 0$ such that $\mathbf{q}'$ remains nonnegative and indices $k \in K$ still have the smallest $d_k \cdot q'_k = d_k \cdot q_k + \epsilon$ in $\mathbf{q}'$. We also replace u_2 by $u'_2 = u_2 + \epsilon$. Let $(u'(\cdot), \mathbf{q}'(\cdot)) \in \mathsf{Ext}(\mathbf{q}', u'_2, u_3, u_4)$ be a feasible solution. Then we reach a contradiction by showing that the revenue of $(u'(\cdot), \mathbf{q}'(\cdot))$ is strictly higher than that of $(u(\cdot), \mathbf{q}(\cdot))$.

First it is clear that $\textsc{Rev}'_1 = \textsc{Rev}_1$ since both $\mathbf{q}$ and $\mathbf{q}'$ sum to 1. By Lemma 5.10, we have

$$\big|(\textsc{Rev}_3 + \textsc{Rev}_4) - (\textsc{Rev}'_3 + \textsc{Rev}'_4)\big| \le O\left(\frac{\delta^2 n^2 \epsilon t}{\beta}\right) + O\left(\frac{\delta^3 n^2 \epsilon t}{\beta}\right) = O\left(\frac{\delta^2 n^2 \epsilon t}{\beta}\right),$$

where we used the loose bound of $|u(\mathbf{w}) - u(\mathbf{w}')| \le O(n\epsilon)$ for all $\mathbf{w} \in T_1$. The RHS above is negligible as we will see due to δ^2. Recall that $\delta = 1/2^{n^6}$ and $t = 2^{\Theta(n^5)}$. It remains to compare $\textsc{Rev}_2$ and $\textsc{Rev}'_2$. For all $\mathbf{v} \in T_2$ other than $\mathbf{c}_2$ and $\mathbf{v}_i$, $i \in [n]$, by Lemma 5.4, $|\textsc{Rev}'(\mathbf{v}) - \textsc{Rev}(\mathbf{v})| = O(n \cdot n\epsilon \cdot s/\beta) = O(n^2\epsilon s/\beta)$. On the other hand, we have $\textsc{Rev}'(\mathbf{c}_2) = \textsc{Rev}(\mathbf{c}_2) - \epsilon$. For each $i \in [n]$, it follows from $\text{LP}(\mathbf{v}_i : u)$ that

$$\textsc{Rev}(\mathbf{v}_i) = \ell_i \cdot q_i(\mathbf{v}) + s \cdot (1 - q_i(\mathbf{v})) - d_i \cdot q_i(\mathbf{a}) = s - (s - \ell_i) \cdot \frac{d_i \cdot q_i - u_2}{d_i} - d_i \cdot q_i.$$

A similar expression holds for $\textsc{Rev}'(\mathbf{v}_i)$ (replacing q_i by q'_i and u_2 by u'_2). As a result,

$$\begin{aligned}
&\sum_{i\in[n]} \Pr[\mathbf{v}_i] \cdot \big(\textsc{Rev}'(\mathbf{v}_i) - \textsc{Rev}(\mathbf{v}_i)\big) \\
&\quad = \delta(1-\delta^2)p(1-p-r)^{n-1} \cdot \left(\sum_{i\in[n]} \frac{s-\ell_i}{d_i} \cdot \epsilon + (s - a_t)\sum_{k\in K} \frac{\epsilon}{d_k} - \sum_{k\in K}(s - a_k) \cdot \frac{\epsilon}{d_k}\right) \\
&\quad \ge \delta(1-\delta^2)p(1-p-r)^{n-1} \cdot \left(\frac{n}{(n-0.5)p} \cdot (1 - O(n\beta)) \cdot \epsilon - O\left(np \cdot n\epsilon\right)\right) \\
&\quad = \frac{n\delta\epsilon}{n-0.5} \cdot (1 - O(n\beta)) - O\big(n^2p^2\delta\epsilon\big),
\end{aligned}$$

where we used $|a_i - 1| = O(np)$. Combining all these bounds together, we have

$$\begin{aligned}
\textsc{Rev}'_2 - \textsc{Rev}_2 &\ge \frac{n\delta\epsilon}{n-0.5} \cdot (1 - O(n\beta)) - O\big(n^2p^2\delta\epsilon\big) - \epsilon \cdot \delta - O(nr\delta) \cdot O(n^2\epsilon s/\beta) \\
&\gg O\left(\frac{\delta^2 n^2 \epsilon t}{\beta}\right),
\end{aligned}$$

given our choices of parameters. This contradicts the optimality of $(u(\cdot), \mathbf{q}(\cdot))$. ∎

Given that $\mathbf{q}(\mathbf{a})$ is close to a uniform distribution, we record a lemma that will be useful later.

LEMMA 5.21. *Let $\mathbf{v}, \mathbf{v}' \in D$ denote two valuation vectors that differ at the ith entry only, for some $i \in [n]$, and $v'_i > v_i$. Then we have $u(\mathbf{v}') \ge u(\mathbf{v})$ in any optimal solution to* $\text{LP}'(I)$.

Proof. It suffices to prove the lemma for two type-1 vectors $\mathbf{v}, \mathbf{v}' \in T_1$ (due to CONDITION-TYPE-2, -3, and -4). The case when $v_i = \ell_i$ and $v'_i = h_i$ follows directly from CONDITION-TYPE-1. The case when $v_i = a_i$ and $v_i = \ell_i$ follows from Lemma 5.20, that $\mathbf{q}(\mathbf{a})$ is close to a uniform distribution. In particular, we have

$$u(\text{LOWER}(\mathbf{v}')) = u(\text{LOWER}(\mathbf{v})) + d_i \cdot q_i(\mathbf{a}) \approx u(\text{LOWER}(\mathbf{v})) + (1/n),$$

while both $u(\mathbf{v}') - u(\text{LOWER}(\mathbf{v}'))$ and $u(\mathbf{v}) - u(\text{LOWER}(\mathbf{v}))$ are much smaller than $1/n$ since $\tau_i = O(n\beta)$ for all i and $\beta = 1/2^n$. ∎

5.2.3. Setting h_i's and t. Before giving our choices of h_i's and t, we introduce the problem COMP, and show that it is #P-hard. Here an input $(\mathcal{G}, H, M)$ of COMP consists of a tuple $\mathcal{G} = (g_2, \ldots, g_n)$ of $n-1$ integers between 1 and $N = 2^n$, a subset $H \subset [2:n]$ of size $|H| = m = \lceil n/2 \rceil$, and an integer M between 1 and $\binom{n-1}{m}$. For convenience, we write $\mathsf{Sum}(T) = \sum_{i\in T} g_i$ for $T \subseteq [2:n]$. We use t^* to denote the Mth largest integer in the multiset

$$\big\{\mathsf{Sum}(T) : T \subset [2:n] \text{ and } |T| = m\big\}. \tag{5.36}$$

The problem is then to decide whether $\mathsf{Sum}(H) > t^*$ or $\mathsf{Sum}(H) \le t^*$, i.e., compare $\mathsf{Sum}(H)$ to the Mth largest integer in (5.36). We first show that COMP is #P-hard.

LEMMA 5.22. COMP *is* #P-*hard.*

Proof. We reduce from a related problem called LEX-RANK, which was shown to be #P-hard in [DDT14a]. In LEX-RANK, the input consists of a collection $C = \{c_1, \ldots, c_n\}$ of positive integers, a subset $S \subseteq [n]$, and a positive integer k. Order the subsets T of $[n]$ of cardinality $|S|$ according to their sums, $\mathsf{Sum}_C(T) = \sum_{i\in T} c_i$, from smallest to largest, with subsets that have equal sums ordered lexicographically; that is, we have $T <_C T'$ if and only if $\mathsf{Sum}_C(T) < \mathsf{Sum}_C(T')$, or $\mathsf{Sum}_C(T) = \mathsf{Sum}_C(T')$ and the largest element in the symmetric difference $T \Delta T'$ belongs to T'. The LEX-RANK problem is to determine for a given input (C, S, k) whether the rank of S in this ordering (among subsets of cardinality $|S|$) is at most k.

Let (C, S, k) be an instance of LEX-RANK. Let $c_i' = 2^{2n} \cdot c_i + 2^i$ for all i, and let $C' = \{c_1', \ldots, c_n'\}$. Clearly, any two subsets of C' have unequal sums and, furthermore, $T <_C T'$ if and only if $\mathsf{Sum}_{C'}(T) < \mathsf{Sum}_{C'}(T')$ for all $T, T' \subseteq [n]$. In the new instance (C', S, k), the rank of a set S (among sets of the same cardinality) is the same as its rank in the old (C, S, k), and the rank of S is at most k if and only if $\mathsf{Sum}_{C'}(S)$ is at most the Mth largest sum, where $M = \binom{n}{|S|} - k + 1$. Thus, the LEX-RANK problem in the new instance is equivalent to the COMP problem, except that in the latter problem we also require that $|S| = \lceil n/2 \rceil$ and that all input integers are at most 2^n.

Let B be the maximum number of bits of the integers in C'; note, $B \ge 2n$. Add $2B - n - 1$ new elements to the set C' to form the new set $\mathcal{G}$; $B - |S|$ of the new elements have value $n2^{B+1}$, and the rest have value 1. Let H be the set that consists of S and the new elements with value $n2^{B+1}$. Thus, $\mathcal{G}$ has $2B - 1 = n' - 1$ elements, S has size $B = n'/2$, and all the integers are between 1 and $2^{2B} = 2^{n'}$. Let $M = \binom{n}{|S|} - k + 1$, as above. The instance $(\mathcal{G}, H, M)$ of COMP now satisfies the required constraints. If we order the subsets of cardinality $B = |H|$ from largest sum to smallest, the first $\binom{n}{|S|}$ subsets will each consist of the $B - |S|$ new elements with the large value of $n2^{B+1}$ and then a subset of cardinality $|S|$ of the original elements, ordered according to their sum. Therefore, $\mathsf{Sum}_{\mathcal{G}}(H)$ is at most the Mth largest sum in the instance $(\mathcal{G}, H, M)$ of COMP if and only if $\mathsf{Sum}_{C'}(S)$ is at most the Mth largest sum in (C', S, k), i.e., if and only if the rank of S is at most k in the original instance (C, S, k) of LEX-RANK. ∎

We embed COMP in I. Let $(\mathcal{G}, H, M)$ be an instance of COMP, where $\mathcal{G} = (g_2, \ldots, g_n)$ is a sequence of $n-1$ integers between 1 and $N = 2^n$, $H \subset [2:n]$ with $|H| = m = \lceil n/2 \rceil$, and M is an integer between 1 and $\binom{n-1}{m}$. Here are our choices of τ_i's and then $h_i = \ell_i + \tau_i$. Recall that we promised in (5.2), (5.3) that

$$\beta \le \tau_i \le \left(1 + \frac{1}{N^2}\right)\beta, \quad t = O\left(\frac{\beta}{r^{m+1}m}\right), \quad \text{and} \quad t = \Omega\left(\frac{\beta}{r^{m+1}m2^n}\right). \tag{5.37}$$

By our choices and a_i's and ℓ_i's, $d_1 = \max_{j\in[n]} d_j$. Set $\tau_i = \tau_i' + \beta$ for each $i \in [n]$ with $\tau_1' = \beta/N^2$ and

$$\tau_i' = \frac{\beta}{N^2} \cdot \frac{d_1 - d_i}{d_1} + g_i \cdot \frac{d_i\beta}{N^4} = O\left(\frac{n\beta^2}{N^2}\right) \quad \text{for each } i > 1.$$

Recall g_i is from $\mathcal{G}$. As $\beta = 1/N$, (5.37) on τ_i is satisfied. The choice of t needs to be done more carefully.

Let R denote the set of $\mathbf{v} \in T_3$ satisfying $|S(\mathbf{v})| = m+1$ and $|S^+(\mathbf{v})| = m$, and let R' denote the set of $\mathbf{v} \in T_3$ with $|S(\mathbf{v})| = |S^+(\mathbf{v})| = m+1$. Let R^* denote the set of $\mathbf{v} \in R'$ with $1 \in S^+(\mathbf{v})$. Let h denote the probability $\Pr[\mathbf{v}]$ of each vector $\mathbf{v} \in R'$ (note that they all share the same probability $\Pr[\mathbf{v}]$):

$$h = (1-\delta) \cdot \delta^2 \cdot r^{m+1} \cdot (1-p-r)^{n-m} \approx \delta^2 r^{m+1}.$$

We are now ready to set t using M as follows,

$$t = 2 + \frac{\beta\delta^2}{h(m+1)(M-(1/2))},$$

which clearly satisfies the promise on t in (5.37).

Fix a type-3 vector $\mathbf{v} \in R^*$, and let $\mathbf{w} = \rho(\mathbf{v}) \in T_1$. Let $(u(\cdot), \mathbf{q}(\cdot)) \in \mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ be a feasible solution to $\mathrm{LP}'(I)$ for some nonnegative $\mathbf{q}$ that sums to 1, and $u_2, u_3, u_4 \ge 0$ that satisfy $u_2 = d_1 \cdot q_1 = \cdots = d_n \cdot q_n = \Theta(1/n)$ and $u_3 \le u_4 \le u_3 + s$. To see the connection between these two problems, we calculate $u(\mathbf{w})$. Given $\min(S(\mathbf{w})) = \min(S(\mathbf{v})) = 1$, we have

$$\begin{aligned}
u(\mathbf{w}) &= \sum_{i\in S(\mathbf{v})} d_i \cdot q_i + \tau_1 \cdot \left(1 - \sum_{i\in S'(\mathbf{v})} q_i\right) + \sum_{i\in S'(\mathbf{v})} \tau_i \cdot q_i \\
&= (m+1)\cdot u_2 + \tau_1 - \sum_{i\in S'(\mathbf{v})} (\tau_1' - \tau_i') \cdot \frac{u_2}{d_i} \\
&= (m+1)\cdot u_2 + \tau_1 - \sum_{i\in S'(\mathbf{v})} \left(\frac{\beta d_i}{N^2 d_1} - g_i \cdot \frac{d_i\beta}{N^4}\right) \cdot \frac{u_2}{d_i} \\
&= C + \frac{\beta u_2}{N^4} \sum_{i\in S'(\mathbf{v})} g_i,
\end{aligned}$$

where we write the constant C (independent of the choice of $\mathbf{v} \in R^*$) as

$$C = (m+1)\cdot u_2 + \tau_1 - \frac{m\beta u_2}{N^2 d_1}.$$

This suggests a natural one-to-one correspondence: $T \mapsto \mathbf{v} \in R^*$ with $S(\mathbf{v}) = \{1\} \cup T$, between

$$\big\{T : T \subset [2:n] \text{ and } |T| = m\big\}$$

and R^* with respect to which the order over $\mathsf{Sum}(T)$ is the same as that over $u(\rho(\mathbf{v}))$. Moreover, since τ_1' is much larger than τ_i' with $i > 1$, other $\mathbf{v} \in R'$ have strictly smaller utility $u(\rho(\mathbf{v}))$ than those in R^*.

To see this, note that for each $\mathbf{v} \in R^*$, we have

$$u(\rho(\mathbf{v})) \geq (m+1) \cdot u_2 + \beta + \tau_1' - \sum_{i \in S'(\mathbf{v})} \tau_1' \cdot q_i = (m+1) \cdot u_2 + \beta + \Omega(\tau_1').$$

On the other hand, let $k = \min(S(\mathbf{v}')) > 1$ for some $\mathbf{v}' \in R' \setminus R^*$. We have

$$u(\rho(\mathbf{v}')) \leq (m+1) \cdot u_2 + \beta + \tau_k' + \sum_{i \in S'(\mathbf{v})} \tau_i' \cdot q_i = (m+1) \cdot u_2 + \beta + O\left(\max_{i \geq 2} \tau_i'\right).$$

It is also easier to verify that $u(\rho(\mathbf{v}))$ with $\mathbf{v} \in R^*$ are strictly higher than $u(\rho(\mathbf{v}'))$ of $\mathbf{v}' \in R$.

We write u^* to denote the Mth largest element of multiset $\{u(\rho(\mathbf{v})) : \mathbf{v} \in R^*\}$. Then the next two lemmas together show that $u(\mathbf{c}_3) = u_3$ must be exactly u^* in any optimal solution to $\mathrm{LP}'(I)$.

LEMMA 5.23. *Any optimal solution* $(u(\cdot), \mathbf{q}(\cdot))$ *to* $\mathrm{LP}'(I)$ *must satisfy* $u(\mathbf{c}_3) \leq u^*$.

Proof. This is the easier direction. Assume for contradiction that $(u(\cdot), \mathbf{q}(\cdot)) \in \mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ is optimal but $u_3 > u^*$. Let $\epsilon > 0$ be sufficiently small such that $u(\mathbf{v}) < u_3$ implies that $u(\mathbf{v}) < u_3 - \epsilon$ for all $\mathbf{v} \in D$.

We show that $(u'(\cdot), \mathbf{q}'(\cdot)) \in \mathsf{Ext}(\mathbf{q}, u_2, u_3', u_4')$, where $u_3' = u_3 - \epsilon$ and $u_4' = u_4 - \epsilon$ (note that we still have $u_3' \leq u_4' \leq u_3' + s$), results in strictly higher expected revenue from type-3 and type-4 vectors, which contradicts the optimality of $(u(\cdot), \mathbf{q}(\cdot))$. By Lemma 5.10, we have $|\textsc{Rev}_4' - \textsc{Rev}_4| = O(\delta^3 n\epsilon t/\beta)$.

We now bound $\textsc{Rev}_3' - \textsc{Rev}_3$. To this end, let A denote the set of $\mathbf{v} \in T_3$ with $u(\rho(\mathbf{v})) \geq u_3$ and let B denote the rest of the type-3 vectors with $u(\rho(\mathbf{v})) < u_3$ (so $\mathbf{c}_3 \in B$). For each $\mathbf{v} \in B$, we have $u(\mathbf{v}) = u_3$ and $u'(\mathbf{v}) = u_3'$ (by our choice of ϵ). By $\mathrm{LP}(\mathbf{v} : u)$ and $\mathrm{LP}(\mathbf{v} : u')$, we have both $\mathbf{q}(\mathbf{v})$ and $\mathbf{q}'(\mathbf{v})$ putting probability 1 on item $n+2$. As a result, we have $\textsc{Rev}'(\mathbf{v}) = \textsc{Rev}(\mathbf{v}) + \epsilon$ for each $\mathbf{v} \in B$. On the other hand, for each $\mathbf{v} \in A$, by Lemma 5.4 we have $\textsc{Rev}'(\mathbf{v}) \geq \textsc{Rev}(\mathbf{v}) - O(n\epsilon t/\beta)$.

We need to take a closer look at vectors $\mathbf{v} \in R^* \cap A$ (which can be empty but by $u_3 > u^*$, $|R^* \cap A|$ is at most $M-1$). To understand $\mathbf{q}(\mathbf{v})$ and $\mathbf{q}'(\mathbf{v})$, we note that all $u(\mathbf{w})$ in $\mathrm{LP}(\mathbf{v} : u)$ are u_3 and all $u'(\mathbf{w})$ in $\mathrm{LP}(\mathbf{v} : u')$ are u_3'. As a result, we only need to consider the following constraints in $\mathrm{LP}(\mathbf{v} : u)$: $q_i \geq 0$, $\sum_{j \in [n+2]} q_j \leq 1$, and $\tau_i \cdot q_i \geq u(\mathbf{v}) - u_3$, for $i \in S^+(\mathbf{v}) = S(\mathbf{v})$, as all other constraints would be implied. Thus, $q_i(\mathbf{v}) = (u(\mathbf{v}) - u_3)/\tau_i$ for $i \in S^+(\mathbf{v})$, and $\mathbf{q}(\mathbf{v})$ puts the rest of the probability onto $q_{n+2}(\mathbf{v})$. Similarly, $q_i'(\mathbf{v}) = (u(\mathbf{v}) - u_3')/\tau_i$ for $i \in S^+(\mathbf{v})$ and $\mathbf{q}'(\mathbf{v})$ puts the rest of the probability onto $q_{n+2}'(\mathbf{v})$. This implies that $\textsc{Rev}'(\mathbf{v}) > \textsc{Rev}(\mathbf{v}) - (m+1) \cdot (\epsilon/\beta) \cdot t$ for each $\mathbf{v} \in R^* \cap A$.

Combining all these inequalities, we have

$$\textsc{Rev}_3' - \textsc{Rev}_3 \geq \Pr[B] \cdot \epsilon - \Pr[A \setminus R^*] \cdot O(n\epsilon t/\beta) - (M-1) \cdot h \cdot (m+1) \cdot \frac{\epsilon}{\beta} \cdot t.$$

Plugging in $\Pr[B] \geq \Pr[\mathbf{c}_3] \geq \delta^2(1 - O(np))$ (since $\mathbf{c}_3 \in B$) and $\Pr[A \setminus R^*] < 3^n \cdot \delta^2 \cdot p^{m+2}$ since $A \setminus R^*$ only has vectors $\mathbf{v} \in T_3$ with $|S(\mathbf{v})| \geq m+2$, we have

$$\begin{aligned}\textsc{Rev}'_3 - \textsc{Rev}_3 &\geq \delta^2\epsilon(1 - O(np)) - O\left(\frac{3^n\delta^2 p^{m+2} n\epsilon t}{\beta}\right)\\ &\quad + (M-1)h(m+1)\frac{\epsilon}{\beta}\cdot\frac{\beta\delta^2(1+o(r^m))}{h(m+1)(M-(1/2))}\\ &= \delta^2\epsilon\cdot\left(\frac{1}{2M-1} - O(np) - o(r^m) - O\left(\frac{3^n p^{m+2} n}{r^{m+1} m}\right)\right)\\ &\gg O(\delta^3 n\epsilon t/\beta),\end{aligned}$$

which follow from choices of p, r, and δ in (5.1). This finishes the proof. ∎

Lemma 5.24. *Any optimal solution* $(u(\cdot), \mathbf{q}(\cdot))$ *to* $\mathrm{LP}'(I)$ *must satisfy* $u(\mathbf{c}_3) \geq u^*$.

Proof. This direction is more difficult. Assume for contradiction that $(u(\cdot), \mathbf{q}(\cdot)) \in \mathsf{Ext}(\mathbf{q}, u_2, u_3, u_4)$ is an optimal solution but $u_3 < u^*$. Let $\epsilon > 0$ be a sufficiently small positive number such that $u(\mathbf{v}) > u_3$ implies $u(\mathbf{v}) > u_3 + \epsilon$ for all $\mathbf{v} \in D$. Our plan is to show that $(u'(\cdot), \mathbf{q}'(\cdot)) \in \mathsf{Ext}(\mathbf{q}, u_2, u'_3, u'_4)$, where $u'_3 = u_3 + \epsilon$ and $u'_4 = u_4 + \epsilon$ result in strictly higher expected revenue, a contradiction.

By Lemma 5.10, we have $|\textsc{Rev}'_4 - \textsc{Rev}_4| = O(\delta^3 n\epsilon t/\beta)$. Next we compare $\textsc{Rev}'_3$ and $\textsc{Rev}_3$. For this purpose we define A as the set of $\mathbf{v} \in T_3$ with $u(\rho(\mathbf{v})) > u_3$ and B as the rest of $\mathbf{v} \in T_3$ with $u(\rho(\mathbf{v})) \leq u_3$ (so $\mathbf{c}_3 \in B$). By an argument similar to the previous lemma, we have $\textsc{Rev}'(\mathbf{v}) = \textsc{Rev}(\mathbf{v}) - \epsilon$ for all $\mathbf{v} \in B$. For $\mathbf{v} \in A$, we have $u(\mathbf{v}) = u'(\mathbf{v})$ by our choice of ϵ. Since $u'_3 = u_3 + \epsilon$, we have $u'(\mathbf{w}) \geq u(\mathbf{w})$ for each $\mathbf{w}$ in $\mathrm{LP}(\mathbf{v} : u)$ and, thus, constraints in $\mathrm{LP}(\mathbf{v} : u)$ are at least as strong as those in $\mathrm{LP}(\mathbf{v} : u')$. As a result, we have $\textsc{Rev}'(\mathbf{v}) \geq \textsc{Rev}(\mathbf{v}) - \epsilon$ for every $\mathbf{v} \in A$.

Let C denote the subset of $\mathbf{v} \in A$ that satisfies (1) $\mathbf{v}$ is below a vector in R^* and (2) every type-3 vector below $\mathbf{v}$ has $u(\mathbf{w}) = u_3$ (so $\mathbf{w} \in B$). Note that C can be empty. Fix a $\mathbf{v} \in C$ when it is nonempty. We have $u'(\mathbf{v}) = u(\mathbf{v}) > u'_3 = u_3 + \epsilon$ and $u'(\mathbf{w}) = u'_3 = u(\mathbf{w}) + \epsilon$, for all type-3 vectors $\mathbf{w}$ below $\mathbf{v}$. As a result, every constraint (other than those on $\mathbf{q}$ only) in $\mathrm{LP}(\mathbf{v} : u)$ has its RHS larger than the corresponding RHS of $\mathrm{LP}(\mathbf{v} : u')$ by ϵ. We claim that $\textsc{Rev}'(\mathbf{v}) \geq \textsc{Rev}(\mathbf{v}) + \Omega(t\epsilon)$. To see this, let $\mathbf{q}^*$ denote the vector derived from $\mathbf{q}(\mathbf{v})$ as follows: $q^*_i = q_i(\mathbf{v}) - (\epsilon/2)$ for some $i \in S(\mathbf{v})$ and $q^*_{n+2} = q_{n+2}(\mathbf{v}) + (\epsilon/2)$; all other entries remain the same. It is clear that $\mathbf{q}^*$ is nonnegative (since $d_i \cdot q_i(\mathbf{v}) \geq u(\mathbf{v}) - u_3 > \epsilon$) and is also a feasible solution to $\mathrm{LP}(\mathbf{v} : u')$. It follows that $\textsc{Rev}'(\mathbf{v}) \geq \textsc{Rev}(\mathbf{v}) + \Omega(t\epsilon)$.

To finish the proof, we consider the following two cases.

Case 1: $C \neq \emptyset$. Then (taking the worst case that $|C| = 1$ and the vector is in R) we have

$$\begin{aligned}\textsc{Rev}'_3 - \textsc{Rev}_3 &\geq \delta^2 \cdot r^m \cdot p \cdot \Omega(t\epsilon) - \delta^2 \cdot \epsilon\\ &> \delta^2\epsilon\cdot\left(\frac{r^m p\beta\delta^2}{h(m+1)(M-0.5)} - 1\right) \gg O(\delta^3 n\epsilon t/\beta),\end{aligned}$$

where the second to the last inequality follows from $p/r = 2^{n^2} \gg mM/\beta$.

Case 2: $C = \emptyset$. Then every $\mathbf{v} \in R^* \cap A$ satisfies that all vectors $\mathbf{w}$ below $\mathbf{v}$ have $u(\mathbf{w}) = u_3$, and for each $\mathbf{v} \in R^* \cap A$, $\mathrm{LP}(\mathbf{v} : u)$ boils down to the following constraints: $q_i \geq 0$, $\sum_{i\in[n+2]} q_i \leq 1$, and $\tau_i \cdot q_i \geq u(\mathbf{v}) - u_3$ for all $i \in S^+(\mathbf{v}) = S(\mathbf{v})$, since all other constraints would be trivially implied. As a result, $q_i(\mathbf{v}) = (u(\mathbf{v}) - u_3)/\tau_i$ for each

$i \in S(\mathbf{v})$ and $q_{n+2}(\mathbf{v})$ takes the rest of the probability; $q_i'(\mathbf{v}) = (u(\mathbf{v}) - u_3')/\tau_i$ for $i \in S(\mathbf{v})$ and $q_{n+2}'(\mathbf{v})$ takes the rest of the probability. Plugging in $u_3' = u_3 + \epsilon$,

$$\textsc{Rev}'(\mathbf{v}) \geq \textsc{Rev}(\mathbf{v}) + (m+1) \cdot \frac{\epsilon}{\max_{i \in [n]} \tau_i} \cdot \left(t - \max_{i \in [n]} h_i \right) - \epsilon.$$

Given that $u_3 < u^*$, we have $|A \cap R^*| \geq M$. Combining all bounds together, we have

$$\begin{aligned} \textsc{Rev}_3' - \textsc{Rev}_3 &\geq Mh \cdot \frac{(m+1)\epsilon}{\beta(1 + O(1/N^2))} \cdot \big(t - (2 + 3n\beta)\big) - \delta^2 \epsilon \\ &= \delta^2 \epsilon \left(\frac{M(1 - O(1/N^2))}{M - 0.5} - 1 \right) > 0 \end{aligned}$$

and is $\gg O(\delta^3 n \epsilon t/\beta)$.

This finishes the proof of the lemma. ∎

Before we pin down $u(\mathbf{c}_4)$, recall that the second part of the input $(\mathcal{G}, H, M)$ is a set $H \subset [2:n]$ of size m. Let $\mathbf{v}_H$ denote the vector in R^* with $S^+(\mathbf{v}_H) = S(\mathbf{v}_H) = \{1\} \cup H$. Given that $u_3 = u^*$, we have (1) if $\mathsf{Sum}(H) > t^*$, then $u(\mathbf{v}_H) > u(\mathbf{c}_3)$, and (2) if $\mathsf{Sum}(H) \leq t^*$ then $u(\mathbf{v}_H) = u(\mathbf{c}_3)$ in any optimal solution $(u(\cdot), \mathbf{q}(\cdot))$ to $\mathrm{LP}'(I)$. It also follows from $\mathrm{LP}(\mathbf{v}:u)$ that (1) if $\mathsf{Sum}(H) > t^*$, then $q_{n+2}(\mathbf{v}_H) < 1$, and (2) if $\mathsf{Sum}(H) \leq t^*$, then $q_{n+2}(\mathbf{v}_H) = 1$ in any optimal solution. We summarize it below.

COROLLARY 5.25. *If* $\mathsf{Sum}(H) > t^*$*, then* $q_{n+2}(\mathbf{v}_H) < 1$ *in every optimal solution to* $\mathrm{LP}'(I)$*. If* $\mathsf{Sum}(H) \leq t^*$*, then* $q_{n+2}(\mathbf{v}_H) = 1$ *in every optimal solution to* $\mathrm{LP}'(I)$*.*

Finally we show that $u(\mathbf{c}_4) = u(\mathbf{c}_3) = u^*$ in any optimal solution to $\mathrm{LP}'(I)$.

LEMMA 5.26. *Any optimal solution to* $\mathrm{LP}'(I)$ *must satisfy* $u(\mathbf{c}_4) = u(\mathbf{c}_3) = u^*$*.*

Proof. The proof is similar to that of Lemma 5.23. Suppose that $(u(\cdot), \mathbf{q}(\cdot)) \in \mathsf{Ext}(\mathbf{q}, u_2, u^*, u_4)$ is optimal but $u_4 > u^*$ (and $u_4 \leq u^* + s$ for it to be feasible). Let $\epsilon > 0$ be a sufficiently small positive number, such that $u_4 - \epsilon \geq u^*$ and $u(\mathbf{v}) < u_4$ implies that $u(\mathbf{v}) < u_4 - \epsilon$ for every $\mathbf{v} \in D$. Our goal is then to show that $(u'(\cdot), \mathbf{q}'(\cdot)) \in \mathsf{Ext}(\mathbf{q}, u_2, u^*, u_4')$ is strictly better, where $u_4' = u_4 - \epsilon$, a contradiction.

It suffices to compare $\textsc{Rev}_4'$ and $\textsc{Rev}_4$ since $\textsc{Rev}_i' = \textsc{Rev}_i$ for $i = 1, 2, 3$.

For $\mathbf{c}_4$ we have $\textsc{Rev}'(\mathbf{c}_4) \geq \textsc{Rev}(\mathbf{c}_4) + \Omega(\epsilon t/s)$. Let A be the set of $\mathbf{v} \in T_4 \setminus \{\mathbf{c}_4\}$ with $u(\rho(\mathbf{v})) \geq u_4$ and B be the rest of $\mathbf{v} \in T_4 \setminus \{\mathbf{c}_4\}$ with $u(\rho(\mathbf{v})) < u_4$. Following the same argument used in Lemma 5.23, we have $\textsc{Rev}'(\mathbf{v}) = \textsc{Rev}(\mathbf{v}) + \epsilon$ for each $\mathbf{v} \in B$, and $\textsc{Rev}'(\mathbf{v}) \geq \textsc{Rev}(\mathbf{v}) - O(n\epsilon t/\beta)$ for each $\mathbf{v} \in A$.

These bounds are strong enough for the current lemma. Given u^* we have $\Pr[A] \leq 3^n \cdot \delta^3 \cdot r^{m+1}$. Thus $\textsc{Rev}_4' - \textsc{Rev}_4 \geq \Pr[\mathbf{c}_4] \cdot \Omega(\epsilon t/s) - \Pr[A] \cdot O(n\epsilon t/\beta) = \Omega(\delta^3 \epsilon t/s) - O(3^n \delta^3 r^{m+1} n \epsilon t/\beta) > 0$. ∎

5.3. Returning to the standard linear program. Let $(\mathcal{G}, H, M)$ be an input instance of COMP and I be the input instance of the optimal mechanism design problem (or the lottery problem) constructed from $(\mathcal{G}, H, M)$ in sections 5.1 and 5.2. We show that any optimal solution to $\mathrm{LP}'(I)$ is a feasible solution to the standard $\mathrm{LP}(I)$.

LEMMA 5.27. *Any optimal solution* $(u(\cdot), \mathbf{q}(\cdot))$ *to* $\mathrm{LP}'(I)$ *is a feasible solution to* $\mathrm{LP}(I)$*.*

Before proving Lemma 5.27, we use it to prove Theorem 1.4.

Proof of Theorem 1.4 *assuming Lemma* 5.27. Using Lemma 5.27, we claim that $(u(\cdot), \mathbf{q}(\cdot))$ is an optimal solution to $\mathrm{LP}(I)$ if and only if it is an optimal solution to $\mathrm{LP}'(I)$. To see this, let $\textsc{Opt}$ and $\textsc{Opt}'$ denote the optimal values of $\mathrm{LP}(I)$ and $\mathrm{LP}'(I)$, respectively. As $\mathrm{LP}'(I)$ is a relaxation of $\mathrm{LP}(I)$, we have $\textsc{Opt}' \geq \textsc{Opt}$. Lemma 5.27, on the other hand, implies $\textsc{Opt} \geq \textsc{Opt}'$. So $\textsc{Opt} = \textsc{Opt}'$, and from here the claim follows easily.

Suppose that $\mathcal{A}(\cdot, \cdot)$ satisfies both properties stated in Theorem 1.4. Then it follows from the connection between the optimal mechanism design problem and $\mathrm{LP}(I)$ (see section 2.1) that $(\overline{\mathcal{A}}(I, \mathbf{v}) : \mathbf{v} \in D)$ is an optimal solution to $\mathrm{LP}(I)$ and thus, an optimal solution to $\mathrm{LP}'(I)$.

It follows from Corollary 5.25 that (1) when $\mathsf{Sum}(H) > t^*$, $\mathcal{A}(I, \mathbf{v}_H)$ assigns an item other than $n+2$ or no item to the buyer with a positive probability; (2) when $\mathsf{Sum}(H) \leq t^*$, $\mathcal{A}(I, \mathbf{v}_H)$ always assigns item $n+2$ with probability 1. Given I and $\mathbf{v}_H$, the problem of deciding which case it is belongs to NP, because $\mathcal{A}$ is a randomized algorithm that always terminates in polynomial time by assumption.

The theorem follows from the #P-hardness of COMP proved in Lemma 5.22. ∎

We prove Lemma 5.27 in the rest of the section. It suffices to show that any optimal solution $(u(\cdot), \mathbf{q}(\cdot))$ to $\mathrm{LP}'(I)$ satisfies (2.1) for all ordered pairs $(\mathbf{v}, \mathbf{w})$ in D.

5.3.1. Reducing to $(\mathbf{v}, \mathbf{w})$ with $S(\mathbf{w}) \subseteq S(\mathbf{v})$. First we handle the special case when $\mathbf{v} = \mathbf{a}$. Let $(u(\cdot), \mathbf{q}(\cdot))$ be an optimal solution to $\mathrm{LP}'(I)$ and $\mathbf{w} \in T_1$. By Condition-Type-1, we have $u(\mathbf{w}) \geq \sum_{i \in S(\mathbf{w})}(w_i - a_i) \cdot q_i(\mathbf{a})$. We extend ρ by setting $\rho(\mathbf{w}) = \mathbf{w}$ if $\mathbf{w} \in T_1$. Then $u(\mathbf{w}) \geq u(\rho(\mathbf{w}))$ for all $\mathbf{w} \in D$. Thus, $u(\mathbf{w}) \geq u(\rho(\mathbf{w})) \geq \sum_{i \in S(\rho(\mathbf{w}))}(w_i - a_i) \cdot q_i(\mathbf{a}) = \sum_{i \in S(\mathbf{w})}(w_i - a_i) \cdot q_i(\mathbf{a})$. Since $u(\mathbf{a}) = 0$, this implies (2.1) on $(\mathbf{a}, \mathbf{w})$ for all $\mathbf{w} \in D$. We assume $\mathbf{v} \neq \mathbf{a}$ in $(\mathbf{v}, \mathbf{w})$ from now on.

Now we claim that it suffices to prove (2.1) for $(\mathbf{v}, \mathbf{w})$ that satisfies $S(\mathbf{w}) \subseteq S(\mathbf{v})$ (though $\mathbf{v}$ and $\mathbf{w}$ here may belong to different blocks). Suppose that we have proved (2.1) over $(\mathbf{v}, \mathbf{w})$ with $S(\mathbf{w}) \subseteq S(\mathbf{v})$. Given any general pair $(\mathbf{v}, \mathbf{w})$ with $\mathbf{v} \neq \mathbf{a}$ (otherwise it is done), we use $\mathbf{w}'$ to denote the vector obtained from $\mathbf{w}$ by replacing every w_i, $i \in S(\mathbf{w}) \setminus S(\mathbf{v})$, by a_i. Then clearly we have $S(\mathbf{w}') \subseteq S(\mathbf{v})$. Because (2.1) holds for $(\mathbf{v}, \mathbf{w}')$, by monotonicity of $u(\cdot)$ (Lemma 5.21), we have

$$u(\mathbf{v}) - u(\mathbf{w}) \leq u(\mathbf{v}) - u(\mathbf{w}') \leq \sum_{i \in [n]} (v_i - w'_i) \cdot q_i(\mathbf{v}) + \sum_{i \in \{n+1, n+2\}} (v_i - w'_i) \cdot q_i(\mathbf{v}),$$

where the latter is equal to

$$\sum_{i \in [n]} (v_i - w_i) \cdot q_i(\mathbf{v}) + \sum_{i \in \{n+1, n+2\}} (v_i - w_i) \cdot q_i(\mathbf{v})$$

using $w_{n+1} = w'_{n+1}$, $w_{n+2}, = w'_{n+2}$ and for every $i \in [n]$ but $i \notin S(\mathbf{v})$, $q_i(\mathbf{v}) = 0$ (Condition-Type-1, Lemmas 5.1 and 5.3). From now on we consider pairs $(\mathbf{v}, \mathbf{w})$ that satisfy $S(\mathbf{w}) \subseteq S(\mathbf{v})$.

5.3.2. Both v and w are type 1. We start with the case when $\mathbf{v}$ and $\mathbf{w}$ are both type-1 vectors (and satisfy $S(\mathbf{w}) \subseteq S(\mathbf{v})$).

Note that (2.1) means that $\mathbf{w}$ does not envy the lottery of $\mathbf{v}$. As $\mathbf{v}$ buys the same lottery as $\textsc{Lower}(\mathbf{v})$ (by Condition-Type-1), we may assume without loss of generality that $\mathbf{v} \in T'_1$ and $S(\mathbf{w}) \subset S(\mathbf{v})$. Then

$$u(\mathbf{v}) - u(\mathbf{w}) = \sum_{i \in S(\mathbf{v}) \setminus S(\mathbf{w})} d_i \cdot q_i(\mathbf{a}) - \sum_{i \in S^+(\mathbf{w})} \tau_i \cdot q_i\big(\textsc{Lower}(\mathbf{w})\big)$$
$$\leq \sum_{i \in S(\mathbf{v}) \setminus S(\mathbf{w})} d_i \cdot q_i(\mathbf{v}) - \sum_{i \in S^+(\mathbf{w})} \tau_i \cdot q_i(\mathbf{v}),$$

where the last inequality follows from $q_i(\mathbf{v}) \geq q_i(\mathbf{a})$ for every $i \in S(\mathbf{v})$, and that $q_i(\textsc{Lower}(\mathbf{w})) \geq q_i(\mathbf{v})$ for every $i \in S(\mathbf{w})$ by Lemma 5.11.

5.3.3. Both v and w are type 2. Next we prove (2.1) for pairs $(\mathbf{v}, \mathbf{w})$ of type-2 vectors that satisfy $S(\mathbf{w}) \subseteq S(\mathbf{v})$.

The special case of $|S(\mathbf{v})| \leq 1$ is easy to check. Let $\mathbf{v}_i$ be the type-2 vector with $S(\mathbf{v}_i) = \{i\}$ and its ith entry being ℓ_i and let $\mathbf{v}'_i$ denote the type-2 vector with $S(\mathbf{v}'_i) = \{i\}$ and its ith entry being h_i. The constraint (2.1) over $(\mathbf{v}, \mathbf{w}) = (\mathbf{v}_i, \mathbf{c}_2)$, $(\mathbf{v}'_i, \mathbf{c}_2)$, or $(\mathbf{v}'_i, \mathbf{v}_i)$ is part of $\text{LP}'(I)$; for $(\mathbf{v}, \mathbf{w}) = (\mathbf{v}_i, \mathbf{v}'_i)$, (2.1) follows trivially from the fact that $u(\mathbf{v}'_i) \geq u(\mathbf{v}_i)$ (by Condition-Type-2), and $q_i(\mathbf{v}_i) = 0$. To see the latter, note that by Lemma 5.20 we have $u(\mathbf{v}_i) = u(\mathbf{c}_2)$ and, thus, an optimal solution to $\text{LP}(\mathbf{v}_i : u)$ must have $q_{n+1} = 1$.

For type-2 $(\mathbf{v}, \mathbf{w})$ with $|S(\mathbf{v})| \geq 2$, we need to understand $\mathbf{q}(\mathbf{v})$ better. We prove the following lemma regarding $\mathbf{v} \in T_2 \cup T_3 \cup T_4 \setminus \{\mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$ that satisfies certain conditions.

Lemma 5.28. *Let $(u(\cdot), \mathbf{q}(\cdot))$ be an optimal solution to $\text{LP}'(I)$ and $\mathbf{v} \in T_2 \cup T_3 \cup T_4 \setminus \{\mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$. Assume that $u(\mathbf{v}) = u(\rho(\mathbf{v}))$ and $u(\mathbf{w}) = u(\rho(\mathbf{w}))$ for every $\mathbf{w}$ that appears in $\text{LP}(\mathbf{v} : u)$. Then $\text{LP}(\mathbf{v} : u)$ has the following unique optimal solution $\mathbf{q}$ (letting $k = \min(S(\mathbf{v}))$ and $S'(\mathbf{v}) = S(\mathbf{v}) \setminus \{k\}$):*

- *If $k \notin S^+(\mathbf{v})$, $q_i = q_i(\mathbf{a})$ for all $i \in S(\mathbf{v})$, and $\mathbf{q}$ puts the rest of the probability $1 - \sum_{i \in S(\mathbf{v})} q_i(\mathbf{a})$ (if any) on q_{n+1} if $\mathbf{v} \in T_2$ or q_{n+2} if $\mathbf{v} \in T_3 \cup T_4$; all other entries of $\mathbf{q}$ are 0.*
- *If $k \in S^+(\mathbf{v})$, $q_i(\mathbf{v}) = q_i(\mathbf{a})$ for all $i \in S'(\mathbf{v})$ and $q_k(\mathbf{v}) = 1 - \sum_{i \in S'(\mathbf{v})} q_i(\mathbf{a})$; all other entries of $\mathbf{q}$ are 0. In this case we have $\mathbf{q}(\mathbf{v}) = \mathbf{q}(\rho(\mathbf{v}))$.*

Proof. We relax $\text{LP}(\mathbf{v} : u)$: its second batch of constraints is now over $i \in S(\mathbf{v})$ and $\mathbf{w}_i = \textsc{Lower}(\mathbf{v}_{-i}, a_i)$ only. Denote this linear program by $\text{LP}^*(\mathbf{v} : u)$:

$$\text{maximize} \quad \sum_{j \in [n+2]} v_j \cdot q_j - u(\mathbf{v}) \quad \text{subject to}$$
$$q_i \geq 0 \quad \text{and} \quad \sum_{j \in [n+2]} q_j \leq 1 \quad \text{for } i \in [n+2],$$
$$\tau_i \cdot q_i \geq u(\mathbf{v}) - u(\mathbf{w}) \quad \text{for } i \in S^+(\mathbf{v}) \text{ and } \mathbf{w} = (\mathbf{v}_{-i}, \ell_i),$$
$$\sum_{j \in [n]} (v_j - w_j) \cdot q_j \geq u(\mathbf{v}) - u(\mathbf{w}) \quad \text{for } i \in S(\mathbf{v}) \text{ and } \mathbf{w} = \textsc{Lower}(\mathbf{v}_{-i}, a_i).$$

We start with the case when $k = \min(S(\mathbf{v})) \notin S^+(\mathbf{v})$. The first batch of constraints yields $q_i \geq q_i(\mathbf{a})$ for all $i \in S^+(\mathbf{v})$, where we used $u(\mathbf{v}) = u(\rho(\mathbf{v}))$, $u(\mathbf{w}) = u(\rho(\mathbf{w}))$, and Condition-Type-1. For each $i \in S(\mathbf{v})$, the second batch requires

$$(v_i - a_i) \cdot q_i + \sum_{j \in S(\mathbf{w})} (v_j - w_j) \cdot q_j \geq d_i \cdot q_i(\mathbf{a}) + \sum_{j \in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{a}).$$

Rearranging terms results in

$$d_i \cdot (q_i - q_i(\mathbf{a})) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot (q_j - q_j(\mathbf{a})) \geq 0$$

for each $i \in S(\mathbf{v})$. These are the only constraints in $\mathrm{LP}^*(\mathbf{v} : u)$ other than those on $\mathbf{q}$ itself. We now show that $\mathrm{LP}^*(\mathbf{v} : u)$ has a unique optimal solution $\mathbf{q}$ with $q_i = q_i(\mathbf{a})$ for all $i \in S(\mathbf{v})$, and $\mathbf{q}$ allocates all the rest of probability onto q_{n+1} or q_{n+2}, depending on whether $\mathbf{v} \in T_2$ or $\mathbf{v} \in T_3 \cup T_4$.

Assume for contradiction that $q_\ell < q_\ell(\mathbf{a})$ for some $\ell \in S(\mathbf{v})$ (this is actually without loss of generality since if $q_i \geq q_i(\mathbf{a})$ for all $i \in S(\mathbf{v})$, then to be optimal $\mathbf{q}$ must be the vector described above). Take ℓ to be an index in $S(\mathbf{v})$ that maximizes $q_\ell(\mathbf{a}) - q_\ell$, denoted by $\epsilon > 0$. For the second constraint on ℓ we must have $q_t - q_t(\mathbf{a}) \geq \Omega(\epsilon/(n\beta))$ for some $t \in S(\mathbf{v})$. Let $\mathbf{q}'$ denote the following vector derived from $\mathbf{q}$: $q'_i = q_i + \epsilon$ for all $i \in S(\mathbf{v})$ and $i \neq t$; $q'_t = q_t - 2n\epsilon$; q'_{n+1} or q'_{n+2} takes the rest of the probability. Then $\mathbf{q}'$ is feasible and strictly better than $\mathbf{q}$. For feasibility, the only nontrivial constraint to check is the second one on t:

$$d_t \cdot (q'_t - q_t(\mathbf{a})) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot (q_j - q_j(\mathbf{a})) \geq \Omega(\epsilon/(n\beta)) - n \cdot O(\beta) \cdot \epsilon > 0.$$

Given that $\mathbf{q}$ described above is the unique optimal solution to $\mathrm{LP}^*(\mathbf{v} : u)$, it is easy to verify that $\mathbf{q}$ is indeed a feasible solution to $\mathrm{LP}(\mathbf{v} : u)$. Taking a $\mathbf{w} \in \textsc{Block}(\mathbf{v}_{-i}, a_i)$ for some $i \in S(\mathbf{v})$, we have

$$\begin{aligned} u(\mathbf{v}) - u(\mathbf{w}) &= d_i \cdot q_i(\mathbf{a}) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j\big(\rho(\mathbf{v})\big) - \sum_{j\in S^+(\mathbf{w})} \tau_j \cdot q_j\big(\rho(\mathbf{w})\big) \\ &\leq d_i \cdot q_i(\mathbf{a}) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{a}) - \sum_{j\in S^+(\mathbf{w})} \tau_j \cdot q_j(\mathbf{a}) \\ &= \sum_{j\in S(\mathbf{v})} (v_j - w_j) \cdot q_j(\mathbf{a}) = \sum_{j\in S(\mathbf{v})} (v_j - w_j) \cdot q_j. \end{aligned}$$

This finishes the proof of the case when $k = \min(S(\mathbf{v})) \notin S^+(\mathbf{v})$.

We consider the case when $k = \min(\mathbf{v}) \in S^+(\mathbf{v})$. Let $\mathbf{v}' = \rho(\mathbf{v})$. The first batch requires $q_i \geq q_i(\mathbf{v}')$ for all $i \in S^+(\mathbf{v})$ (including k). For each $i \in S^-(\mathbf{v})$, the second batch of $\mathrm{LP}^*(\mathbf{v} : u)$ requires

$$(v_i - a_i) \cdot q_i + \sum_{j\in S(\mathbf{w})} (v_j - w_j) \cdot q_j \geq d_i \cdot q_i(\mathbf{a}) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot q_j(\mathbf{v}').$$

As $i \in S^-(\mathbf{v})$ and $i \neq k$, we have $q_i(\mathbf{a}) = q_i(\mathbf{v}')$ and thus $d_i \cdot (q_i - q_i(\mathbf{v}')) + \sum_{j\in S^+(\mathbf{v})} \tau_j \cdot (q_j - q_j(\mathbf{v}')) \geq 0$ for each $i \in S^-(\mathbf{v})$. It turns out that $\mathbf{q} = \mathbf{q}(\mathbf{v}')$ is the unique *feasible* solution to these constraints (as $\mathbf{q}(\mathbf{v}')$ sums to 1, $\mathbf{q}$ sums to at most 1, and $d_i \gg \tau_j$). Hence, we have $\mathbf{q}(\mathbf{v}) = \mathbf{q}(\mathbf{v}')$ (as $\mathrm{LP}(\mathbf{v} : u)$ is feasible and $\mathbf{q} = \mathbf{q}(\mathbf{v}')$ is the only feasible solution to $\mathrm{LP}^*(\mathbf{v} : u)$). This finishes the proof when $k \in S^+(\mathbf{v})$. ∎

We summarize below the following property of $\mathbf{q}(\mathbf{v})$ for all $\mathbf{v} \in T_2$ that will be useful later.

LEMMA 5.29. *For all $\mathbf{v} \in T_2$ and $i \in S(\mathbf{v})$, $q_i(\mathbf{v}) \leq q_i(\rho(\mathbf{v}))$. Moreover,*

$$q_{n+1}(\mathbf{v}) = 1 - \sum_{i\in S(\mathbf{v})} q_i(\mathbf{v}).$$

Proof. Recall $\mathbf{v}_i$ and $\mathbf{v}'_i$ at the beginning of section 5.3.3. For $\mathbf{c}_2$ and $\mathbf{v}_i$ we have $q_{n+1}(\mathbf{c}_2) = q_{n+1}(\mathbf{v}_i) = 1$; for $\mathbf{v}'_i$ we have $q_i(\mathbf{v}'_i) = 1 = q_i(\rho(\mathbf{v}'_i))$. The rest of $\mathbf{v} \in T_2$ follows from Lemma 5.28. ◻

Now let $(\mathbf{v}, \mathbf{w})$ be a pair of type-2 vectors with $S(\mathbf{w}) \subseteq S(\mathbf{v})$ and $|S(\mathbf{v})| \geq 2$ (so Lemma 5.28 applies to $\mathbf{v}$ and we know exactly what $\mathbf{q}(\mathbf{v})$ is). The rest of the proof is similar to that for type-1 vectors.

Using $u(\mathbf{v}) = u(\rho(\mathbf{v}))$ and $u(\mathbf{w}) \geq u(\rho(\mathbf{w}))$, we have

$$u(\mathbf{v}) - u(\mathbf{w}) \leq \sum_{i \in S(\mathbf{v}) \setminus S(\mathbf{w})} d_i \cdot q_i(\mathbf{a}) + \sum_{i \in S^+(\mathbf{v})} \tau_i \cdot q_i(\rho(\mathbf{v})) - \sum_{i \in S^+(\mathbf{w})} \tau_i \cdot q_i(\rho(\mathbf{w})).$$

When $k = \min(\mathbf{v}) \notin S^+(\mathbf{v})$, we have $q_i(\mathbf{v}) = q_i(\mathbf{a})$ for all $i \in S(\mathbf{v})$. We have

$$\begin{aligned} u(\mathbf{v}) - u(\mathbf{w}) &\leq \sum_{i \in S(\mathbf{v}) \setminus S(\mathbf{w})} d_i \cdot q_i(\mathbf{a}) + \sum_{i \in S^+(\mathbf{v})} \tau_i \cdot q_i(\mathbf{a}) - \sum_{i \in S^+(\mathbf{w})} \tau_i \cdot q_i(\mathbf{a}) \\ &= \sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\mathbf{v}), \end{aligned}$$

where we used $q_i(\rho(\mathbf{v})) = q_i(\mathbf{a})$ for $i \neq \min(S(\mathbf{v}))$ and $q_i(\rho(\mathbf{w})) \geq q_i(\mathbf{a})$ for $i \in S(\mathbf{w})$.

For the case when $k = \min(\mathbf{v}) \in S^+(\mathbf{v})$, we have $\mathbf{q}(\mathbf{v}) = \mathbf{q}(\rho(\mathbf{v}))$. Then

$$\begin{aligned} u(\mathbf{v}) - u(\mathbf{w}) &\leq \sum_{i \in S(\mathbf{v}) \setminus S(\mathbf{w})} d_i \cdot q_i(\mathbf{v}) + \sum_{i \in S^+(\mathbf{v})} \tau_i \cdot q_i(\mathbf{v}) - \sum_{i \in S^+(\mathbf{w})} \tau_i \cdot q_i(\mathbf{v}) \\ &= \sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\mathbf{v}), \end{aligned}$$

where we used $q_i(\rho(\mathbf{w})) \geq q_i(\rho(\mathbf{v})) = q_i(\mathbf{v})$ for $i \in S(\rho(\mathbf{w})) = S(\mathbf{w})$ by Lemma 5.11.

This finishes the proof of (2.1) over all pairs $(\mathbf{v}, \mathbf{w})$ of type-2 vectors.

5.3.4. Both v and w are type 3. Now we turn to pairs $(\mathbf{v}, \mathbf{w})$ of type-3 vectors that satisfy $S(\mathbf{w}) \subseteq S(\mathbf{v})$.

When $|S(\mathbf{v})| \geq m + 3$, we note that by Lemma 5.23 and 5.24, $\mathbf{v}$ satisfies the condition of Lemma 5.28 which completely characterizes $\mathbf{q}(\mathbf{v})$. The same argument above for type-2 vectors with $|S(\mathbf{v})| \geq 2$ can be used to prove (2.1) for type-3 $(\mathbf{v}, \mathbf{w})$ with $S(\mathbf{w}) \subseteq S(\mathbf{v})$ and $|S(\mathbf{v})| \geq m + 3$.

Next we check the case when $|S(\mathbf{v})| \leq m+1$. The case when $u(\mathbf{v}) = u(\mathbf{c}_3)$ is simple as $q_{n+2}(\mathbf{v}) = 1$ (note that this includes $\mathbf{v} = \mathbf{c}_3$). As a result, we have (using $u(\mathbf{w}) \geq u(\mathbf{c}_3)$ by CONDITION-TYPE-3) that $u(\mathbf{v}) - u(\mathbf{w}) \leq 0 = \sum_{i \in [n+2]} (v_i - w_i) \cdot q_i(\mathbf{v})$.

For the case when $u(\mathbf{v}) > u(\mathbf{c}_3)$ and $|S(\mathbf{v})| \leq m+1$, by Lemmas 5.23 and 5.24 we must have $\mathbf{v} \in R^*$. $\mathbf{q}(\mathbf{v})$ is an optimal solution to the following (relaxed) LP (from LP$(\mathbf{v} : u)$): $q_i \geq 0$, $\sum_{i \in [n+2]} q_i \leq 1$, $\tau_i \cdot q_i \geq u(\mathbf{v}) - u(\mathbf{c}_3)$ for $i \in S^+(\mathbf{v})$, since all other constraints in LP$(\mathbf{v} : u)$ would be implied. This implies that $q_i(\mathbf{v}) = (u(\mathbf{v}) - u(\mathbf{c}_3))/\tau_i$ for all $i \in S(\mathbf{v})$ and $q_{n+2}(\mathbf{v})$ takes the rest of the probability. We now prove (2.1) on $(\mathbf{v}, \mathbf{w})$. Using $S(\mathbf{w}) \subseteq S(\mathbf{v})$ and $\mathbf{w} \neq \mathbf{v}$ (so $u(\mathbf{w}) = u(\mathbf{c}_3)$), there must be an index $t \in S(\mathbf{v})$ such that $w_t < v_t$. As a result we have

$$\begin{aligned} \sum_{i \in [n+2]} (v_i - w_i) \cdot q_i(\mathbf{v}) &= \sum_{i \in S(\mathbf{v})} (h_i - w_i) \cdot q_i(\mathbf{v}) \\ &\geq \tau_t \cdot q_t(\mathbf{v}) = u(\mathbf{v}) - u(\mathbf{c}_3) = u(\mathbf{v}) - u(\mathbf{w}). \end{aligned}$$

The only case left for type 3 $(\mathbf{v}, \mathbf{w})$ is when $|S(\mathbf{v})| = m + 2$. We need the next lemma about its $\mathbf{q}(\mathbf{v})$.

LEMMA 5.30. *For each* $\mathbf{v} \in T_3$ *with* $|S(\mathbf{v})| = m+2$, $\mathbf{q}(\mathbf{v})$ *satisfies* $q_i(\mathbf{v}) = q_i(\rho(\mathbf{v}))$ *for each* $i \in S^+(\mathbf{v})$, $q_i(\mathbf{v}) \le q_i(\mathbf{a})$ *for each* $i \in S^-(\mathbf{v})$, *and* $\mathbf{q}(\mathbf{v})$ *puts the rest of probability onto* $q_{n+2}(\mathbf{v})$.

Proof. By $\text{LP}(\mathbf{v} : u)$, $q_i(\mathbf{v})$ for each $i \in S^+(\mathbf{v})$ must satisfy

$$\tau_i \cdot q_i(\mathbf{v}) \ge u(\mathbf{v}) - u(\mathbf{v}_{-i}, \ell_i) = u(\rho(\mathbf{v})) - u(\rho(\mathbf{v}_{-i}, \ell_i)) = \tau_i \cdot q_i(\rho(\mathbf{v})), \tag{5.38}$$

since we have $u(\mathbf{w}) = u(\rho(\mathbf{w}))$ for $\mathbf{w} \in T_3$ with $|S(\mathbf{w})| \ge m + 2$. Let $\mathbf{q}$ be the vector with $q_i = q_i(\rho(\mathbf{v}))$ for all $i \in S^+(\mathbf{v})$ and $q_i = q_i(\mathbf{a})$ for all $i \in S^-(\mathbf{v})$. Let $c = \max_i(q_i(\mathbf{v}) - q_i)$, and assume for contradiction that $c > 0$. Let $t \in S(\mathbf{v})$ denote an index with $q_t(\mathbf{v}) = q_t + c$. We consider two cases below.

Case 1: One of the constraints in the second batch of $\text{LP}(\mathbf{v} : u)$ with $i = t$ is tight, i.e., there is a type-3 vector $\mathbf{w} \in \textsc{Block}(\mathbf{v}_{-t}, a_t)$ such that $\sum_{i \in S(\mathbf{v})}(v_i - w_i) \cdot q_i(\mathbf{v}) = u(\mathbf{v}) - u(\mathbf{w})$. Since $u(\mathbf{v}) = u(\rho(\mathbf{v}))$ and $u(\mathbf{w}) \ge u(\rho(\mathbf{w}))$, we have

$$\sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\mathbf{v}) \le u(\rho(\mathbf{v})) - u(\rho(\mathbf{w}))$$
$$= d_t \cdot q_t(\mathbf{a}) + \sum_{i \in S^+(\mathbf{v})} \tau_i \cdot q_i(\rho(\mathbf{v})) - \sum_{i \in S^+(\mathbf{w})} \tau_i \cdot q_i(\rho(\mathbf{w})).$$

Plugging in $q_t(\mathbf{a}) \le q_t$, $q_i(\rho(\mathbf{v})) = q_i$ for $i \in S^+(\mathbf{v})$, and $q_i(\rho(\mathbf{w})) \ge q_i(\rho(\mathbf{v})) \ge q_i$ for $i \in S(\mathbf{w})$, we have

$$d_t \cdot (q_t(\mathbf{v}) - q_t) + \sum_{i \in S^+(\mathbf{v})} \tau_i \cdot (q_i(\mathbf{v}) - q_i) \le \sum_{i \in S^+(\mathbf{w})} \tau_i \cdot (q_i(\mathbf{v}) - q_i) \le n \cdot O(\beta) \cdot c.$$

Given that $d_t \approx 1 \gg O(n\beta)$, there must exist an $i \in S^+(\mathbf{v})$ such that $q_i(\mathbf{v}) - q_i < 0$, contradicting (5.38).

Case 2: All constraints in the second batch with $i = t$ are loose. For this case we lower $q_t(\mathbf{v})$ by ϵ for some sufficiently small $\epsilon > 0$, increase $q_i(\mathbf{v})$ by $\epsilon/(2n)$ for other $i \in S(\mathbf{v})$, and move the rest of (at least $\epsilon/2$) probability to $q_{n+2}(\mathbf{v})$. This gives a feasible solution that is strictly better than $\mathbf{q}(\mathbf{v})$, a contradiction. ∎

We summarize below the following property of $\mathbf{q}(\mathbf{v})$ for all $\mathbf{v} \in T_3$ that will be useful later:

LEMMA 5.31. *For all* $\mathbf{v} \in T_3$ *and* $i \in S(\mathbf{v})$, $q_i(\mathbf{v}) \le q_i(\rho(\mathbf{v}))$. *Moreover,*

$$q_{n+1}(\mathbf{v}) = 1 - \sum_{i \in S(\mathbf{v})} q_i(\mathbf{v}).$$

Proof. The case of $u(\mathbf{v}) = u(\mathbf{c}_3)$ is trivial. The case of $u(\mathbf{v}) > u(\mathbf{c}_3)$ and $|S(\mathbf{v})| \le m+1$ follows from $q_i(\mathbf{v}) = (u(\mathbf{v}) - u(\mathbf{c}_3))/\tau_i \le \big(u(\rho(\mathbf{v})) - u(\rho(\mathbf{v}_{-i}, \ell_i))\big)/\tau_i = q_i(\rho(\mathbf{v}))$. The rest of $\mathbf{v} \in T_4$ follows from either Lemmas 5.28 or 5.30. ∎

We now return to prove (2.1) for pairs $(\mathbf{v}, \mathbf{w})$ of type-3 vectors with $S(\mathbf{w}) \subseteq S(\mathbf{v})$ and $|S(\mathbf{v})| = m+2$. The only nontrivial case here is when $\mathbf{w}$ also has $|S(\mathbf{w})| = m+2$. For other cases, we have the following:

1. $|S(\mathbf{w})| = m + 1$: Trivial since the constraint is indeed part of $\text{LP}'(I)$.
2. $|S(\mathbf{w})| < m + 1$: Let $\mathbf{w}^*$ denote a type-3 vector in R such that $\mathbf{w} \prec \mathbf{w}^* \prec \mathbf{v}$ and $w^*_i = h_i$ for all $i \in S(\mathbf{w})$. Then we have $u(\mathbf{w}^*) = u(\mathbf{w}) = u(\mathbf{c}_3)$. It follows from (2.1) over $(\mathbf{v}, \mathbf{w}^*)$ that

$$u(\mathbf{v}) - u(\mathbf{w}) = u(\mathbf{v}) - u(\mathbf{w}^*) \le \sum_{i \in S(\mathbf{v})} (v_i - w_i^*) \cdot q_i(\mathbf{v}) \le \sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\mathbf{v}),$$

where the last inequality follows from $w_i \le w_i^*$ for all i.

When $|S(\mathbf{w})| = m+2$, we have $S(\mathbf{v}) = S(\mathbf{w})$. Then $u(\mathbf{v}) - u(\mathbf{w}) = u(\rho(\mathbf{v})) - u(\rho(\mathbf{w}))$ and

$$\begin{aligned} u(\rho(\mathbf{v})) - u(\rho(\mathbf{w})) &= \sum_{i \in S^+(\mathbf{v}) \setminus S^+(\mathbf{w})} \tau_i \cdot q_i(\rho(\mathbf{v})) - \sum_{i \in S^+(\mathbf{w}) \setminus S^+(\mathbf{v})} \tau_i \cdot q_i(\rho(\mathbf{v})) \\ &\le \sum_{i \in S^+(\mathbf{v}) \setminus S^+(\mathbf{w})} \tau_i \cdot q_i(\mathbf{v}) - \sum_{i \in S^+(\mathbf{w}) \setminus S^+(\mathbf{v})} \tau_i \cdot q_i(\mathbf{v}) \\ &= \sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\mathbf{v}), \end{aligned}$$

since $q_i(\mathbf{v}) = q_i(\rho(\mathbf{v}))$ for all $i \in S^+(\mathbf{v})$ and $q_i(\mathbf{v}) \le q_i(\mathbf{a}) \le q_i(\rho(\mathbf{v}))$ for all $i \in S^-(\mathbf{v})$.

This finishes the proof of (2.1) over pairs of type-3 vectors.

5.3.5. Both v and w are type 4. For each $\mathbf{v} \in T_4$, let $\Phi(\mathbf{v}) = (\mathbf{v}_{-(n+1)}, 0)$. So Φ is a one-to-one correspondence between type-4 and type-3 vectors. As $u(\mathbf{c}_4) = u(\mathbf{c}_3)$, we have $u(\mathbf{v}) = u(\Phi(\mathbf{v}))$ for all $\mathbf{v} \in T_4$. This suggests the following lemma.

LEMMA 5.32. *Let $(u(\cdot), \mathbf{q}(\cdot))$ be an optimal solution to $\text{LP}'(I)$ and $\mathbf{v} \in T_4$. Then $(u(\cdot), \mathbf{q}(\cdot))$ remains to be an optimal solution to $\text{LP}'(I)$ after replacing $\mathbf{q}(\Phi(\mathbf{v}))$ by $\mathbf{q}(\mathbf{v})$.*

Proof. The statement is trivial for $\mathbf{v} = \mathbf{c}_4$ since $q_{n+2}(\mathbf{c}_3) = q_{n+2}(\mathbf{c}_4) = 1$.

For $\mathbf{v} \ne \mathbf{c}_4$, note that $\text{LP}(\mathbf{v} : u)$ is essentially the same as $\text{LP}(\Phi(\mathbf{v}) : u)$, with the only subtle difference being that the coefficient of q_{n+1} is s in $\text{LP}(\mathbf{v} : u)$ but 0 in $\text{LP}(\Phi(\mathbf{v}) : u)$. However, neither $\mathbf{q}(\Phi(\mathbf{v}))$ nor $\mathbf{q}(\mathbf{v})$ can put any probability onto q_{n+1}. The lemma then follows. ∎

To prove (2.1) on a pair $(\mathbf{v}, \mathbf{w})$ of type-4 vectors we simply replace $\mathbf{q}(\Phi(\mathbf{v}))$ by $\mathbf{q}(\mathbf{v})$ to get a new optimal solution by Lemma 5.32, and (2.1) must hold on $(\Phi(\mathbf{v}), \Phi(\mathbf{w}))$ in the new solution (since we have proved (2.1) between type-3 vectors in any optimal solution). This then implies (2.1) on $(\mathbf{v}, \mathbf{w})$ in the original solution.

5.3.6. Pairs with different types. Finally we prove (2.1) for pairs $(\mathbf{v}, \mathbf{w})$ of vectors with $S(\mathbf{w}) \subseteq S(\mathbf{v})$ and of different types.

The following lemma helps us further reduce cases that need to be considered.

LEMMA 5.33. *Assume that $\mathbf{v}, \mathbf{v}' \in D$ differ at the ith entry only, for some $i \in \{n+1, n+2\}$, and $v_i' > v_i$. Then we have $u(\mathbf{v}') \ge u(\mathbf{v})$ in any optimal solution to $\text{LP}'(I)$.*

Proof. The case when $\mathbf{v} \in T_1$ follows directly from CONDITION-TYPE-2 and CONDITION-TYPE-3.

The case when $\mathbf{v} \in T_3$ and $i = n+1$ follows from $u(\mathbf{c}_3) = u(\mathbf{c}_4)$.

The case when $\mathbf{v} \in T_2$ and $i = n+2$ follows from the fact that $u(\mathbf{c}_3) > u(\mathbf{c}_2)$. ∎

It suffices to prove (2.1) for $(\mathbf{v}, \mathbf{w})$ that satisfies $v_{n+1} \ge w_{n+2}$ and $v_{n+2} \ge w_{n+2}$. To see this, we let $\mathbf{w}'$ denote the vector obtained from $\mathbf{w}$ by replacing w_i by $\min(w_i, v_i)$, $i \in \{n+1, n+2\}$. Then $u(\mathbf{w}) \ge u(\mathbf{w}')$ by Lemma 5.33 and $(\mathbf{v}, \mathbf{w}')$ satisfies $v_{n+1} \ge w'_{n+1}$ and $v_{n+2} \ge w'_{n+2}$. Assuming that (2.1) holds for $(\mathbf{v}, \mathbf{w}')$,

$$u(\mathbf{v}) - u(\mathbf{w}) \le u(\mathbf{v}) - u(\mathbf{w}') \le \sum_{i \in [n+2]} (v_i - w_i') q_i(\mathbf{v})$$
$$= \sum_{i \in [n]} (v_i - w_i) q_i(\mathbf{v}) + \sum_{i \in \{n+1, n+2\}} (v_i - w_i') q_i(\mathbf{v}).$$

The RHS is indeed the same as $\sum_i (v_i - w_i) \cdot q_i(\mathbf{v})$. This is because for either $i \in \{n+1, n+2\}$, $w_i \neq w_i'$ would imply that $v_i = 0$ and, thus, $q_i(\mathbf{v}) = 0$.

Now we need to consider the following cases of types of $(\mathbf{v}, \mathbf{w})$: $(2,1)$, $(3,1)$, $(4,1)$, $(4,2)$, and $(4,3)$. We start with the case when $\mathbf{v}$ is type 2 and $\mathbf{w}$ is type 1.

We consider the following two cases: $u(\mathbf{v}) = u(\mathbf{c}_2)$ or $u(\mathbf{v}) > u(\mathbf{c}_2)$. For the former, $q_{n+1}(\mathbf{v}) = 1$ and, thus, $u(\mathbf{v}) - u(\mathbf{w}) \le u(\mathbf{c}_2) \ll s = \sum_{i \in [n+2]} (v_i - w_i) \cdot q_i(\mathbf{v})$. For the latter, $u(\mathbf{v}) = u(\rho(\mathbf{v}))$. By Lemma 5.29, let $\gamma_i = q_i(\rho(\mathbf{v})) - q_i(\mathbf{v}) \ge 0$ for each $i \in S(\mathbf{v})$. Then

$$\begin{aligned} u(\mathbf{v}) - u(\mathbf{w}) = u(\rho(\mathbf{v})) - u(\mathbf{w}) &\le \sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\rho(\mathbf{v})) \\ &= \sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\mathbf{v}) + \sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot \gamma_i \\ &\le \sum_{i \in S(\mathbf{v})} (v_i - w_i) \cdot q_i(\mathbf{v}) + s \cdot \sum_{i \in S(\mathbf{v})} \gamma_i \\ &= \sum_{i \in [n+2]} (v_i - w_i) \cdot q_i(\mathbf{v}), \end{aligned}$$

where the last equation used $q_{n+1}(\mathbf{v}) = 1 - \sum_{i \in S(\mathbf{v})} q_i(\mathbf{v})$ from Lemma 5.29.

The case when $\mathbf{v}$ is type 3 and $\mathbf{w}$ is type 1 can be proved similarly using Lemma 5.31. From this case, the case when $\mathbf{v}$ is type 3 and $\mathbf{w}$ is type 2 follows from Lemma 5.33 (we mention it since it is used below).

For the case when $\mathbf{v}$ is type 4 and $\mathbf{w}$ is type 3, we simply replace $\mathbf{q}(\Phi(\mathbf{v}))$ by $\mathbf{q}(\mathbf{v})$ to get a new optimal solution by Lemma 5.32. (2.1) on $(\mathbf{v}, \mathbf{w})$ in the original solution then follows from that on $(\Phi(\mathbf{v}), \mathbf{w})$ in the new solution (note that this is the $(3,2)$ case we already handled), given that $q_{n+1}(\mathbf{v}) = 0$.

For the case when $\mathbf{v}$ is type 4 and $\mathbf{w}$ is type 2, we again replace $\mathbf{q}(\Phi(\mathbf{v}))$ by $\mathbf{q}(\mathbf{v})$ to get a new optimal solution by Lemma 5.32. (2.1) on $(\mathbf{v}, \mathbf{w})$ in the original solution then follows from that on $(\Phi(\mathbf{v}), \mathbf{w})$ in the new solution, given $q_{n+1}(\mathbf{v}) = 0$. The same argument works for the case when $\mathbf{v}$ is type 4 and $\mathbf{w}$ is type 1.

This finishes the proof of Lemma 5.27.

6. Conclusions.

In this paper we studied the complexity of optimal lottery pricing and randomized mechanisms for a unit-demand buyer with a product distribution. We showed that the menu size complexity of the problem is exponential even when the distribution of each item has support size 2. For the computational complexity, we showed that the problem is unlikely to have a randomized polynomial-time algorithm unless $\mathrm{P}^{\mathrm{NP}} = \mathrm{P}^{\#\mathrm{P}}$, and this holds even when the distribution of each item has support size 3.

Appendix A. Two items with support size 2.

In this section we show Theorem 1.3, i.e., that offering lotteries does not improve the expected revenue when there are two items and both distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are of support size 2.

Let $\{a_i, b_i\}$ be the support of $\mathcal{D}_i$ for $i \in \{1,2\}$, where $0 \le a_i < b_i$. Let q_i be the probability that item i has value a_i (and $1-q_i$ that it has value b_i). Without loss of generality, we assume that $b_2 \ge b_1$ and write $t = b_1 - a_1$. We consider the following four item pricings: $(a_1, b_2), (b_1, a_2), (b_1, b_2), (a_1, b_2 - t)$ (according to the algorithm for the optimal item pricing in the support-2 case [CDP+18], one of them is optimal).

In Table A.1 below, we list the revenue for each of the four item pricings (the rows of the table) at each of the four possible valuations (the columns). The bottom left entry δ of the table is equal to a_1 if $a_2 < b_2 - t$ (i.e., if $t < b_2 - a_2$), and is equal to $b_2 - t$ if $a_2 \ge b_2 - t$.

Table A.1

Revenue for each potentially optimal pricing (rows) and each possible valuation vector (columns).

	(a_1, a_2)	(b_1, a_2)	(a_1, b_2)	(b_1, b_2)
(a_1, b_2)	a_1	a_1	b_2	a_1
(b_1, a_2)	a_2	$\max\{b_1, a_2\}$	a_2	a_2
(b_1, b_2)	0	b_1	b_2	b_2
$(a_1, b_2 - t)$	δ	a_1	$b_2 - t$	$b_2 - t$

Consider now an optimal menu L^* of lotteries. By Lemma 2.4, all the lotteries, except for the one bought for valuation (a_1, a_2), are complete. In Table A.2 we list the allocation and price of each lottery bought.

Table A.2

An optimal menu.

		Allocation		
		Item 1	Item 2	Price
Valuation	(a_1, a_2)	w_1	w_2	p_1
	(b_1, a_2)	$1 - x$	x	p_2
	(a_1, b_2)	y	$1 - y$	p_3
	(b_1, b_2)	z	$1 - z$	p_4

Our plan is to show that the revenue of L^* is upperbounded by a convex combination of revenues from the four item pricings. We use the following strategy.

Let $\alpha = (1-q_1)/q_1$. Note that this is the ratio between probabilities of valuations (b_1, a_2) and (a_1, a_2), and also those of (b_1, b_2) and (a_1, b_2). The expected revenue of L^* then can be written as

$$q_1 q_2 \cdot (p_1 + \alpha p_2) + q_1(1 - q_2) \cdot (p_3 + \alpha p_4).$$

Denote by $\mathbf{C}_i$ the ith column vector of Table A.1. Our goal is to find a nonnegative vector $\mathbf{s} = (s_1, s_2, s_3, s_4)$ of weights (view s_i as the weight of the item pricing on the ith row of Table A.1) with $\sum_{i=1}^{4} s_i = 1$ such that

$$\text{(A.1)} \qquad \mathbf{s} \cdot (\mathbf{C}_1 + \alpha \mathbf{C}_2) \ge p_1 + \alpha p_2 \quad \text{and} \quad \mathbf{s} \cdot (\mathbf{C}_3 + \alpha \mathbf{C}_4) \ge p_3 + \alpha p_4.$$

Let R^* be the revenue of L^* and R_i be the revenue of the item pricing on the ith column of Table A.1. Such a weight vector $\mathbf{s}$ then implies that $R^* \le \sum_{i=1}^{4} s_i \cdot R_i$, and Theorem 1.3 follows.

Here is the plan of the rest of the section. In section A.1 we bound the prices p_i of L^*, and then bound $p_1 + \alpha p_2$ and $p_3 + \alpha p_4$. We then choose an appropriate $\mathbf{s}$ and use these bounds to prove (A.1) in section A.2.

A.1. Upper bounds for the prices in L^*. We start with upper bounds for p_i, $i \in \{1, 2, 3, 4\}$.

Bounding p_1*:* For valuation (a_1, a_2), the buyer buys (w_1, w_2, p_1). Since it has nonnegative utility,

$$p_1 \leq a_1 w_1 + a_2 w_2. \tag{A.2}$$

Bounding p_2*:* For valuation (b_1, a_2), the buyer prefers lottery $(1 - x, x, p_2)$ over (w_1, w_2, p_1). Thus,

$$b_1(1-x) + a_2 x - p_2 \geq b_1 w_1 + a_2 w_2 - p_1 \xrightarrow{\text{(A.2)}} p_2 \leq b_1 - x(b_1 - a_2) - w_1(b_1 - a_1).$$

Bounding p_4*:* For valuation (b_1, b_2), the buyer prefers lottery $(z, 1 - z, p_4)$ over (w_1, w_2, p_1), so

$$b_1 z + b_2(1-z) - p_4 \geq b_1 w_1 + b_2 w_2 - p_1 \xrightarrow{\text{(A.2)}}$$
$$p_4 \leq b_1 z + b_2(1-z) - w_1(b_1 - a_1) - w_2(b_2 - a_2). \tag{A.3}$$

For valuation (b_1, b_2), lottery $(z, 1 - z, p_2)$ is also preferred over $(1 - x, x, p_2)$, so we have

$$b_1 z + b_2(1-z) - p_4 \geq b_1(1-x) + b_2 x - p_2 \xrightarrow{\text{(A.3)}}$$
$$p_4 \leq b_1 z + b_2(1-z) - w_1(b_1 - a_1) - x(b_2 - a_2). \tag{A.4}$$

Hence, from (A.3) and (A.4) it follows that

$$p_4 \leq b_2 - z(b_2 - b_1) - w_1(b_1 - a_1) - \max\{w_2, x\}(b_2 - a_2). \tag{A.5}$$

Bounding p_3*:* For valuation (a_1, b_2), lottery $(y, 1 - y, p_3)$ is preferred over $(z, 1 - z, p_4)$, so we have

$$a_1 y + b_2(1-y) - p_3 \geq a_1 z + b_2(1-z) - p_4 \xrightarrow{\text{(A.5)}}$$
$$p_3 \leq b_2 - (b_2 - a_1)y + z(b_1 - a_1) - w_1(b_1 - a_1) - \max\{w_2, x\}(b_2 - a_2). \tag{A.6}$$

Similarly, for valuation (a_1, b_2), lottery $(y, 1 - y, p_3)$ is preferred over (w_1, w_2, p_1), so we have

$$a_1 y + b_2(1-y) - p_3 \geq a_1 w_1 + b_2 w_2 - p_1 \xrightarrow{\text{(A.2)}}$$
$$p_3 \leq a_1 y + b_2(1-y) - w_2(b_2 - a_2). \tag{A.7}$$

Plugging in $b_2 \geq a_1$ and $y \geq 0$, we have from (A.6) and (A.7) that

$$p_3 \leq b_2 + z(b_1 - a_1) - w_1(b_1 - a_1) - \max\{w_2, x\}(b_2 - a_2) \quad \text{and}$$
$$p_3 \leq b_2 - w_2(b_2 - a_2). \tag{A.8}$$

Bounding $p_1 + \alpha p_2$*:* From (A.2) and (A.3) we get

$$p_1 + \alpha p_2 \leq \alpha b_1 - w_1(\alpha b_1 - (1+\alpha)a_1) + w_2 a_2 - x\alpha(b_1 - a_2). \tag{A.9}$$

Bounding $p_3 + \alpha p_4$: Combining the first part of (A.8) and (A.5) we get

$$p_3 + \alpha p_4 \leq (1+\alpha)(b_2 - w_1(b_1 - a_1) - \max\{w_2, x\}(b_2 - a_2)) \\ - z(\alpha(b_2 - b_1) - (b_1 - a_1)). \tag{A.10}$$

Similarly, from the second part of (A.8) and (A.5) we get

$$p_3 + \alpha p_4 \leq b_2(1+\alpha) - z\alpha(b_2 - b_1) - w_1\alpha(b_1 - a_1) \\ - w_2(b_2 - a_2) - \max\{w_2, x\}\alpha(b_2 - a_2). \tag{A.11}$$

Next we will prove that there are nonnegative weights s_1, s_2, s_3, and s_4 that sum to 1 and satisfy (A.1).

A.2. Upper bounds for the expected revenue. First we note the following useful inequality:

$$w_2 a_2 - x\alpha(b_1 - a_2) \leq \max\{w_2, x\} \cdot \big(a_2 + \alpha(\max\{b_1, a_2\} - b_1)\big),$$

which can be verified by checking the two cases $b_1 \geq a_2$ and $b_1 < a_2$.

We start with a sufficient condition on $\mathbf{s} = (s_1, \ldots, s_4)$ to satisfy the first part of (A.1).

LEMMA A.1. *Suppose that $s_1, s_4 \geq 0$ satisfy $s_1 + s_4 = w_1$, s_2 satisfies $0 \leq s_2 \leq \max\{w_2, x\}$,*

$$w_2 a_2 - x\alpha(b_1 - a_2) \leq s_2 \cdot \big(a_2 + \alpha(\max\{b_1, a_2\} - b_1)\big), \tag{A.12}$$

and $s_3 = 1 - w_1 - s_2$. Then $s_i \geq 0$ for all i, $\sum_{i=1}^4 s_i = 1$, and $\mathbf{s}$ satisfies

$$\mathbf{s} \cdot (\mathbf{C}_1 + \alpha \mathbf{C}_2) \geq p_1 + \alpha p_2.$$

Proof. We have $s_1, s_2, s_4 \geq 0$ by the assumption of the lemma. To see that $s_3 \geq 0$ note that by Lemma 2.3 $w_1 \leq 1 - x$. As $w_1 + w_2 \leq 1$, we have $1 - w_1 \geq \max\{w_2, x\}$ and, thus, $s_3 \geq 0$. $\sum_{i=1}^4 s_i = 1$ is obvious.

Recall that δ in Table A.1 is a_1 or $b_2 - t = b_2 - b_1 + a_1 \geq a_1$. Letting $A = \mathbf{s} \cdot (\mathbf{C}_1 + \alpha \mathbf{C}_2)$, we have

$$\begin{aligned} A &= s_1(1+\alpha)a_1 + s_2 a_2 + s_2\alpha \max\{b_1, a_2\} + s_3\alpha b_1 + s_4\delta + s_4\alpha a_1 \\ &\geq s_1(1+\alpha)a_1 + s_2 a_2 + s_2\alpha \max\{b_1, a_2\} + s_3\alpha b_1 + s_4 a_1 + s_4\alpha a_1 \\ &= (s_1 + s_4)(1+\alpha)a_1 + s_2 a_2 + s_2\alpha \max\{b_1, a_2\} + s_3\alpha b_1. \end{aligned}$$

From the choice of the s_i's: $s_1 + s_4 = w_1$ and $s_4 = 1 - w_1 - s_2$, the above inequality becomes

$$\begin{aligned} A &\geq w_1(1+\alpha)a_1 + s_2 a_2 + s_2\alpha \max\{b_1, a_2\} + (1 - w_1 - s_2)\alpha b_1 \\ &= \alpha b_1 - w_1(\alpha b_1 - (1+\alpha)a_1) + s_2 a_2 + s_2\alpha(\max\{b_1, a_2\} - b_1). \end{aligned}$$

The lemma then follows directly from (A.9) and the assumption (A.12). ∎

We next show that there is an $\mathbf{s}$ that satisfies the second part of (A.1) as well as conditions of Lemma A.1.

LEMMA A.2. *There exists an $\mathbf{s}$ that satisfies conditions of Lemma A.1 and $\mathbf{s} \cdot (\mathbf{C}_3 + \alpha \mathbf{C}_4) \geq p_3 + \alpha p_4$.*

Proof. Let $B = \mathbf{s} \cdot (\mathbf{C}_3 + \alpha \mathbf{C}_4)$. It follows from Table A.1 that we have

$$B = s_1 b_2 + s_1 \alpha a_1 + s_2(1+\alpha)a_2 + s_3(1+\alpha)b_2 + s_4(1+\alpha)(b_2 - t). \tag{A.13}$$

We will distinguish two cases.

Case 1*:* $(b_2 - b_1)\alpha \geq b_1 - a_1$. Set $s_2 = \max\{w_2, x\}$, $s_3 = 1 - w_1 - \max\{w_2, x\}$, $s_1 = 0$, and $s_4 = w_1$. Clearly, this assignment satisfies the conditions of Lemma A.1. Equation (A.13) gives

$$B = (1+\alpha)\big(b_2 - w_1(b_1 - a_1) - \max\{w_2, x\}(b_2 - a_2)\big). \tag{A.14}$$

Furthermore, in this case $-z((b_2 - b_1)\alpha - (b_1 - a_1)) \leq 0$, therefore (A.10) and (A.14) give $p_3 + \alpha p_4 \leq B$.

Case 2*:* $(b_2 - b_1)\alpha < (b_1 - a_1)$. For this case we distinguish 3 subcases.

Case 2.1*:* $z \leq w_1$. Set $s_1 = w_1$, $s_2 = \max\{w_2, x\}$, $s_3 = 1 - w_1 - \max\{w_2, x\}$, and $s_4 = 0$. Then

$$\begin{aligned} B &= w_1 b_2 + w_1 \alpha a_1 + \max\{w_2, x\}(1+\alpha)a_2 + (1 - w_1 - \max\{w_2, x\})(1+\alpha)b_2 \\ &= b_2(1+\alpha) - w_1\alpha(b_2 - a_1) - \max\{w_2, x\}(1+\alpha)(b_2 - a_2). \end{aligned} \tag{A.15}$$

Using $(b_2 - b_1)\alpha < (b_1 - a_1)$ and $z \leq w_1$ in (A.10), we have

$$\begin{aligned} p_3 + \alpha p_4 &\leq (1+\alpha)(b_2 - w_1(b_1 - a_1) - \max\{w_2, x\}(b_2 - a_2)) \\ &\qquad - w_1(\alpha(b_2 - b_1) - (b_1 - a_1)) \\ &= b_2(1+\alpha) - w_1\alpha(b_2 - a_1) - \max\{w_2, x\}(1+\alpha)(b_2 - a_2) = B. \end{aligned}$$

Case 2.2*:* $z > w_1$ *and* $x \leq w_2$. Using the same assignment of $\mathbf{s}$ as in Case 2.1, by $z > w_1$, (A.11) gives

$$\begin{aligned} p_3 + \alpha p_4 \leq\ & b_2(1+\alpha) - w_1\alpha(b_2 - a_1) \\ & - w_2(b_2 - a_2) - \max\{w_2, x\}\alpha(b_2 - a_2). \end{aligned}$$

Furthermore, $x \leq w_2$ implies that $w_2 = \max\{w_2, x\}$. It follows from (38) and (A.15) that $p_3 + \alpha p_4 \leq B$.

Case 2.3*:* $z > w_1$ *and* $x > w_2$. Set $s_1 = w_1$, $s_3 = 1 - w_1 - s_2$, $s_4 = 0$ with

$$s_2 = (w_2 + x\alpha)/(1+\alpha).$$

Clearly $s_2 \leq \max\{w_2, x\}$. We verify (A.12) at the end but first compare B and $p_3 + \alpha p_4$. We have

$$\begin{aligned} B &= w_1 b_2 + w_1 \alpha a_1 + s_2(1+\alpha)a_2 + (1 - w_1 - s_2)(1+\alpha)b_2 \\ &= b_2(1+\alpha) - w_1\alpha(b_2 - a_1) - s_2(1+\alpha)(b_2 - a_2). \end{aligned} \tag{A.16}$$

Since $x > w_2$, (38) gives

$$p_3 + \alpha p_4 \leq b_2(1+\alpha) - w_1\alpha(b_2 - a_1) - w_2(b_2 - a_2) - x\alpha(b_2 - a_2). \tag{A.17}$$

It follows from our choice of s_2 and $b_2 - a_2 \geq 0$ that $p_3 + \alpha p_4 \leq B$.

Finally we verify that our choice of s_2 satisfies (A.12) in this case. To see this,

$$\begin{aligned} &(1+\alpha)\big(w_2 a_2 - x\alpha(b_1 - a_2)\big) - (w_2 + x\alpha)\big(a_2 + \alpha(\max\{b_1, a_2\} - b_1)\big) \\ &\quad = w_2\alpha a_2 - x\alpha b_1 + x\alpha^2 a_2 - w_2\alpha \max\{b_1, a_2\} + w_2\alpha b_1 - x\alpha^2 \max\{b_1, a_2\} \leq 0. \end{aligned}$$

The last inequality used $x > w_2$. The lemma follows by combining all the cases. ∎

Theorem 1.3 follows from Lemmas A.1 and A.2.

Appendix B. Small instances where lotteries help.

In this section, we give examples where lotteries can extract a strictly higher revenue than the optimal item pricing. In the first example, there are three items and each $\mathcal{D}_i$ has support size 2; in the second example, there are two items and each $\mathcal{D}_i$ has support size 3.

Three items, support size 2: We consider the following instance I with three items. The three items have distributions with support $\{5, b_i\}$ for $i \in [3]$, where $b_1 = 10$ and $b_2 = b_3 = 6$. Let p_i be the probability that item i has value 5. Then set $p_1 = 0.6$, $p_2 = 0.7$, and $p_3 = 0.8$.

There are two optimal item pricings: $(10, 6, 5)$ and $(9, 6, 5)$, with expected revenue 6.744. The optimal menu for I consists of four lotteries: $\mathbf{x}_1 = (1, 0, 0)$ at price 9.5, $\mathbf{x}_2 = (0, 1, 0)$ at price 5.5, $\mathbf{x}_3 = (0, 0, 1)$ at price 5.5, and $\mathbf{x}_4 = (0, 0.5, 0.5)$ at price 5. The expected revenue of this menu is 6.806.

Two items, support size 3: Consider the following instance J with two items and identical distributions. Each item has value 4 with probability 0.5, value 6 with probability 0.2, and value 7 with probability 0.3.

There are also two optimal item pricings: $(6, 4)$ and $(6, 6)$, with expected revenue 4.5. The optimal menu for instance J consists of three lotteries: $\mathbf{x}_1 = (1, 0)$ at price 6, $\mathbf{x}_2 = (0, 1)$ at price 5, and $\mathbf{x}_3 = (0, 0.5)$ at price 2. The expected revenue of this menu is 4.56.

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