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Spatial ergodicity of stochastic wave equations in dimensions 1, 2 and 3*

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Abstract

In this note, we study a large class of stochastic wave equations with spatial dimension less than or equal to 3. Via a soft application of Malliavin calculus, we establish that their random field solutions are spatially ergodic.

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1 Introduction

In this article, we fix $d \in \{1, 2, 3\}$ and consider the stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \sigma(u)\dot{W}, \tag{1.1}$$

on $\mathbb{R}_+ \times \mathbb{R}^d$ with initial conditions $u(0, x) = 1$ and $\frac{\partial u}{\partial t}(0, x) = 0$, where Δ is Laplacian in the space variables and $\dot{W}$ is a centered Gaussian noise with covariance

$$\mathbb{E}[\dot{W}(t, x)\dot{W}(s, y)] = \delta_0(t - s)\gamma(x - y). \tag{1.2}$$

Throughout this article, we fix the following conditions:

(C1) $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $L \in (0, \infty)$.

(C2) γ is a tempered nonnegative and nonnegative definite measure, whose Fourier transform μ satisfies Dalang's condition:

$$\int_{\mathbb{R}^d} \frac{\mu(dz)}{1 + |z|^2} < \infty, \tag{1.3}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Conditions **(C1)** and **(C2)** ensure that equation (1.1) has a unique *random field solution*, which is adapted to the filtration generated by W, such that $\sup\big\{\mathbb{E}\big[|u(t, x)|^k\big] : (t, x) \in [0, T] \times \mathbb{R}^d\big\}$ is finite for all $T \in (0, \infty)$ and $k \geq 2$, and

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)\sigma(u(s, y))W(ds, dy), \tag{1.4}$$

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where the stochastic integral is defined in the sense of Dalang-Walsh and $G(t-s,x-y)$ denotes the fundamental solution to the corresponding deterministic wave equation, *i.e.*

$$G(t,\bullet) := \begin{cases} \dfrac{1}{2}\mathbf{1}_{\{|\bullet|<t\}}, & \text{if } d=1 \\ \dfrac{1}{2\pi\sqrt{t^2-|\bullet|^2}}\mathbf{1}_{\{|\bullet|<t\}}, & \text{if } d=2 \\ \dfrac{1}{4\pi t}\sigma_t, & \text{if } d=3, \end{cases} \tag{1.5}$$

with σ_t denoting the surface measure on $\partial\mathbb{B}_t := \{x\in\mathbb{R}^3 : |x|=t\}$; see Example 6 and Theorem 13 in Dalang's paper [3]. The proof of [3, Theorem 13] follows from a standard Picard iteration scheme, from which one can see that $u(t,x)\equiv 1$ if $\sigma(1)=0$.

It is not difficult to see that for each fixed $t>0$, $\{u(t,x) : x\in\mathbb{R}^d\}$ is strictly stationary meaning its law is invariant under spatial shift. Indeed, for each $y\in\mathbb{R}^d$, the random field $\{u(t,x+y) : x\in\mathbb{R}^d\}$ coincides almost surely with the random field u driven by the shifted noise W_y given by

$$W_y(\phi) = \int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\phi(s,x-y)W(ds,dx),\ \phi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d).$$

Here $C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ is the set of real infinitely differentiable functions with compact support defined on $\mathbb{R}_+\times\mathbb{R}^d$. The noise W_y has the same distribution as W, which is enough for us to conclude the stationarity property. We refer readers to Lemma 7.1 in [2] and footnote 1 in [4] for similar arguments.

Then it is natural to define an associated family of shifts $\{\theta_y : y\in\mathbb{R}^d\}$ by setting

$$\theta_y(\{u(t,x), x\in\mathbb{R}^d\}) = \{u(t,x+y), x\in\mathbb{R}^d\},$$

which preserve the law of the process. Then the following question arises:

Are the invariant sets for $\{\theta_y : y\in\mathbb{R}^d\}$ trivial?

That is, for each fixed $t>0$, is $\{u(t,x) : x\in\mathbb{R}^d\}$ ergodic? See the book [11] for more account on ergodic theory. In the following theorem, we provide an affirmative answer to the above question.

Theorem 1.1. *Assume that the spectral measure has no atom at zero,* i.e. $\mu(\{0\})=0$, *then for each $t>0$, $\{u(t,x) : x\in\mathbb{R}^d\}$ is ergodic.*

Condition $\mu(\{0\})=0$ echoes Maruyama's early work [6] on ergodicity of stationary Gaussian processes and it also finds its place in the recent work of Chen, Khoshnevisan, Nualart and Pu [2] on the solution to stochastic heat equations.

Remark 1.2. Under Dalang's condition (1.3), property $\mu(\{0\})=0$ is equivalent to $\gamma(\mathbb{B}_R)=o(R^d)$, as $R\to+\infty$; see [2, Theorem 1.1]. Here and throughout the paper we will make use of the notation $\mathbb{B}_R=\{x\in\mathbb{R}^d : |x|\le R\}$ for any $R>0$. As a consequence, if γ is a function, property $\mu(\{0\})=0$ is equivalent to

$$\lim_{R\to+\infty}\frac{1}{|\mathbb{B}_R|}\int_{\mathbb{B}_R}\gamma(x)dx=0,$$

which means that the asymptotic average of γ is zero.

The ergodicity gives us the first-order result: With ω_d denoting the volume of $\mathbb{B}_1$,

$$\frac{1}{\omega_d R^d}\int_{\mathbb{B}_R}u(t,x)dx \xrightarrow{R\to\infty} 1$$

in $L^2(\Omega)$. Then it is natural to investigate the corresponding second-order fluctuations. They have been established in several cases briefly recalled below:

- When $d = 1$, the Gaussian noise is white in time and behaves as a fractional noise in space with Hurst parameter $H \in [1/2, 1)$, the authors of [4] prove the Gaussian fluctuations for spatial averages.
- The authors of [1] investigate the case where $d = 2$ and $\gamma(z) = |z|^{-\beta}$ with $\beta \in (0, 2)$.
- In [10], we continued the study of the 2D stochastic wave equation when the covariance kernel γ is integrable.

Our Theorem 1.1 (see also Remark 1.2) establishes the spatial ergodicity for all these cases. The key ingredient in the aforementioned references is a fundamental $L^p(\Omega)$-estimate of the Malliavin derivative of the solution:

$$\big\| D_{s,y} u(t,x) \big\|_p \lesssim G_{t-s}(x-y), \tag{1.6}$$

where D is the Malliavin derivative operator defined over the isonormal Gaussian process $\big\{ W(\phi) : \phi \in \mathfrak{H} \big\}$ that will be defined in Section 2. Such an inequality fails to work when $d = 3$, as the fundamental solution $G(t, \bullet)$ is a measure for $d = 3$ (see (1.5)). The Malliavin derivative $Du(t,x)$, unlike in previous works, is a random measure and it is not clear how to make sense of the left expression in (1.6). We leave this problem for future research that will require some novel ideas in dealing with the Malliavin derivative. In this paper, we do not consider the case of space dimension $d \geq 4$, because the fundamental solution G is a not a nonnegative distribution.

The rest of the article is organized as follows: In Section 2, we briefly collect preliminary facts for our proofs that will be presented in Section 3.

2 Preliminaries

In this section we present some preliminaries on stochastic analysis and Malliavin calculus.

2.1 Basic stochastic analysis

Let $\mathfrak{H}$ be defined as the completion of $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ under the inner product

$$\begin{aligned} \langle f, g \rangle_{\mathfrak{H}} &= \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} f(s,y) g(s,z) \gamma(y-z) dy dz ds \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \mathscr{F} f(s,\xi) \mathscr{F} g(s,-\xi) \mu(d\xi) ds, \end{aligned} \tag{2.1}$$

where $\mathscr{F} f(s,\xi) = \int_{\mathbb{R}^d} e^{-\mathbf{i} x \cdot \xi} f(s,x) dx$. Throughout this article, we may abuse the notation $\gamma(x)$ as γ is assumed to be a measure; for example, the equality in (2.1) should be understood as

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s,y) \big[g(s,\bullet) * \gamma \big](y) dy ds$$

with $*$ denoting the convolution in space. Consider an isonormal Gaussian process associated to the Hilbert space $\mathfrak{H}$, denoted by $W = \big\{ W(\phi) : \phi \in \mathfrak{H} \big\}$. That is, W is a centered Gaussian family of random variables such that $\mathbb{E}\big[W(\phi) W(\psi) \big] = \langle \phi, \psi \rangle_{\mathfrak{H}}$ for any $\phi, \psi \in \mathfrak{H}$. As the noise W is white in time, a martingale structure naturally appears. First we define $\mathcal{F}_t$ to be the σ-algebra generated by the $\mathbb{P}$-negligible sets and the family of random variables $\big\{ W(\phi) : \phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \text{ has compact support contained in } [0,t] \times \mathbb{R}^d \big\}$, so we have a filtration $\mathbb{F} = \{ \mathcal{F}_t : t \in \mathbb{R}_+ \}$. If $\big\{ \Phi(s,y) : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^d \big\}$ is an $\mathbb{F}$-adapted random field such that $\mathbb{E}\big[\|\Phi\|_{\mathfrak{H}}^2 \big] < +\infty$, then

$$M_t = \int_{[0,t] \times \mathbb{R}^d} \Phi(s,y) W(ds, dy),$$

interpreted as the Dalang-Walsh integral ([3, 8, 13]), is a square-integrable $\mathbb{F}$-martingale with quadratic variation

$$\langle M\rangle_t = \int_{[0,t]\times\mathbb{R}^{2d}} \Phi(s,y)\Phi(s,z)\gamma(y-z)dydzds.$$

A suitable version of Burkholder-Davis-Gundy inequality (BDG for short) holds in this setting: If $\{\Phi(s,y) : (s,y)\in\mathbb{R}_+\times\mathbb{R}^d\}$ is an adapted random field with respect to $\mathbb{F}$ such that $\|\Phi\|_{\mathfrak{H}}\in L^p(\Omega)$ for some $p\geq 2$, then

$$\left\|\int_{[0,t]\times\mathbb{R}^d}\Phi(s,y)W(ds,dy)\right\|_p^2 \leq 4p\left\|\int_{[0,t]\times\mathbb{R}^{2d}}\Phi(s,y)\Phi(s,z)\gamma(y-z)dydzds\right\|_{p/2}; \quad (2.2)$$

see *e.g.* [5, Theorem B.1], here $\|\bullet\|_p$ denotes the usual $L^p(\Omega)$-norm.

2.2 Malliavin calculus

Now let us recall some basic facts on the Malliavin calculus associated with W. For any unexplained notation and result, we refer to the book [7]. We denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let $\mathcal{S}$ be the space of simple functionals of the form $F = f(W(h_1),\dots,W(h_n))$ for $f\in C_p^\infty(\mathbb{R}^n)$ and $h_i\in\mathfrak{H}$, $1\leq i\leq n$. Then, the Malliavin derivative DF is the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1),\dots,W(h_n))h_i\,.$$

The derivative operator D is closable from $L^p(\Omega)$ into $L^p(\Omega;\mathfrak{H})$ for any $p\geq 1$ and we define $\mathbb{D}^{1,p}$ to be the completion of $\mathcal{S}$ under the norm

$$\|F\|_{1,p} = \left(\mathbb{E}\big[|F|^p\big] + \mathbb{E}\big[\|DF\|_{\mathfrak{H}}^p\big]\right)^{1/p}.$$

The *chain rule* for D asserts that if $F\in\mathbb{D}^{1,2}$ and $h:\mathbb{R}\to\mathbb{R}$ is Lipschitz, then $h(F)\in\mathbb{D}^{1,2}$ with

$$D[h(F)] = h'(F)DF, \quad (2.3)$$

where h' denotes any version of the almost everywhere derivative (in view of Rademacher's theorem) satisfying

$$h(x) = h(0) + \int_0^x h'(t)dt \quad \text{for } x\geq 0, \quad h(0) = h(x) + \int_x^0 h'(t)dt \quad \text{for } x<0$$

and $\|h'\|_\infty$ is bounded by the Lipschitz constant of h.

We denote by δ the adjoint of D given by the duality formula

$$\mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF\rangle_{\mathfrak{H}}] \quad (2.4)$$

for any $F\in\mathbb{D}^{1,2}$ and $u\in\operatorname{Dom}\delta\subset L^2(\Omega;\mathfrak{H})$, the domain of δ. The operator δ is also called the Skorohod integral and in our context, the Dalang-Walsh integral coincides with the Skorohod integral: Any adapted random field Φ that satisfies $\mathbb{E}\big[\|\Phi\|_{\mathfrak{H}}^2\big]<\infty$ belongs to the domain of δ and

$$\delta(\Phi) = \int_0^\infty\int_{\mathbb{R}^d}\Phi(s,y)W(ds,dy). \quad (2.5)$$

The operators D, δ satisfy the Heisenberg's commutation relation:

$$(D\delta - \delta D)(V) = V.$$

From this relation, we have for any adapted random field Φ belonging to $\mathbb{D}^{1,2}(\mathfrak{H})$ given as in (2.5),

$$D_{s,y} \int_0^\infty \int_{\mathbb{R}^d} \Phi(r,z) W(dr,dz) = \Phi(s,y) + \int_0^\infty \int_{\mathbb{R}^d} D_{s,y}\Phi(r,z) W(dr,dz). \tag{2.6}$$

It is known that for a random variable $F \in \mathbb{D}^{1,2}$, one can represent it as a stochastic integral:

$$F = \mathbb{E}[F] + \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mathbb{E}\big[D_{s,y}F | \mathcal{F}_s\big] W(ds,dy)$$

(see *e.g.* [2, Proposition 6.3]). This is known as Clark-Ocone formula and it leads to the following Poincaré inequality: For any such two random variables $F, G \in \mathbb{D}^{1,2}$, we have

$$|\mathrm{Cov}(F,G)| \leq \int_0^\infty \int_{\mathbb{R}^{2d}} \|D_{s,y}F\|_2 \|D_{s,z}G\|_2 \gamma(y-z) dydzds. \tag{2.7}$$

Throughout this note, we write $A \lesssim B$ to mean that $A \leq KB$ for some immaterial constant which may vary from line to line.

3 Proof of Theorem 1.1

We first introduce the following regularization of the kernel G: Given a nonnegative function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(z)dz = 1$, we define $\psi_n(z) = n^d \psi(nz)$ for all $z \in \mathbb{R}^d$ and

$$G_n(t,x) = \int_{\mathbb{R}^d} G(t,dy)\psi_n(x-y). \tag{3.1}$$

Here $G(t,dy)$ denotes $G(t,y)dy$, when $d = 1,2$. Consider the approximating sequence of random fields $\{u_n\}_{n\geq 1}$ defined by

$$u_n(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_n(t-s,x-y)\sigma\big(u_n(s,y)\big) W(ds,dy). \tag{3.2}$$

It holds that, for any $p \geq 1$

$$\lim_{n\to+\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \big\|u_n(t,x) - u(t,x)\big\|_p = 0 \tag{3.3}$$

for any $T \in (0,\infty)$, see [12, Proposition 1]. Fix $n \geq 1$ and consider the Picard iteration scheme for u_n: We put $u_{n,0}(t,x) = 1$ and for $k \geq 0$,

$$u_{n,k+1}(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_n(t-s,x-y)\sigma\big(u_{n,k}(s,y)\big) W(ds,dy). \tag{3.4}$$

It is known that for any $T > 0$ and any $p \in [1,\infty)$,

$$\lim_{k\to+\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \big\|u_{n,k}(t,x) - u_n(t,x)\big\|_p = 0; \tag{3.5}$$

the proof can be done following the same arguments as in the proof of [3, Theorem 13].

In the following, we present the key ingredient to prove our main result.

Proposition 3.1. *Let $u_{n,k}$ be given as in* (3.4) *and fix $T\in(0,\infty)$. Then for any $p\geq 1$, the following estimate holds for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and for almost every $(s,y)\in[0,t]\times\mathbb{R}^d$*

$$\big\|D_{s,y}u_{n,k}(t,x)\big\|_p \lesssim \mathbf{1}_{\mathbb{B}_{\frac{a(k+1)}{n}+T}}(x-y),$$

where $\mathbb{B}_a=\{x\in\mathbb{R}^d:|x|\leq a\}$ contains the support of ψ for some $a>0$ and the implicit constant only depends on (p,T,L,γ,n,k).

Before we proceed with the proof of Proposition 3.1 we show two technical lemmas.

Lemma 3.2. *Suppose the Dalang's condition* (1.3) *is satisfied. For any $T\in(0,\infty)$, we have*

$$\mathcal{U}_T:=\sup_{b\in[0,T]}\int_{\mathbb{R}^d}\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi)<\infty. \tag{3.6}$$

Proof. Let us recall from [9, Lemma 2.1] that

$$\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2=\left|\int_{\mathbb{B}_b}e^{-\mathbf{i}x\xi}dx\right|^2=(2\pi b)^d|\xi|^{-d}J_{\frac{d}{2}}(b|\xi|)^2,$$

where, for $p>0$,

$$J_p(x):=\frac{(x/2)^p}{\sqrt{\pi}\Gamma(p+\frac12)}\int_0^\pi(\sin\theta)^{2p}\cos(x\cos\theta)d\theta$$

is the Bessel function of first kind with order p, which satisfies

(i) $\sup\big\{|J_p(x)|:x\in\mathbb{R}_+\big\}<\infty$,

(ii) $|J_p(x)|\leq C|x|^{-1/2}$ for any $x\in\mathbb{R}$ and for some absolute constant $C>0$.

It is also clear that $|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}|\lesssim b^d$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d}\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi)&=\int_{|\xi|\leq1}\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi)+\int_{|\xi|>1}\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi)\\&\lesssim b^{2d}\mu\big(\{\xi\in\mathbb{R}^d:|\xi|\leq1\}\big)+b^d\int_{|\xi|>1}|\xi|^{-d}J_{d/2}(b|\xi|)^2\mu(d\xi)\\&\lesssim b^{2d}\mu\big(\{\xi\in\mathbb{R}^d:|\xi|\leq1\}\big)+b^{d-1}\int_{|\xi|>1}|\xi|^{-d-1}\mu(d\xi)\end{aligned}$$

using point (ii) in the last step. Because of (1.3) and $d\geq1$, the two integrals in the last display are both finite. Hence the result (3.6) follows. □

Lemma 3.3. *For each $n\geq1$ and $T\in(0,\infty)$*

$$\Theta(T,n):=\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\big|G_n(t,x)\big|<\infty.$$

Proof. By definition,

$$\big|G_n(t,x)\big|=\int_{\mathbb{R}^d}\psi_n(x-y)G(t,dy)\leq\|\psi_n\|_\infty G(t,\mathbb{R}^d).$$

It is known that $\sup_{t\leq T}G(t,\mathbb{R}^d)$ is finite for any $T\in(0,\infty)$, so that $\Theta(T,n)<\infty$. □

Proof of Proposition 3.1. Recall the Picard iterations from (3.4). Now let us fix $p\in[2,\infty)$, $T\in(0,\infty)$ and the integers n,k. Then, by standard arguments one can show that for any

$(t,x)\in[0,T]\times\mathbb{R}^d$, $u_{n,k+1}(t,x)$ belongs to the space $\mathbb{D}^{1,p}$ and, in view of (2.3) and (2.2), we can write for almost all $(s,y)\in[0,t]\times\mathbb{R}^d$,

$$D_{s,y}u_{n,k+1}(t,x)=G_n(t-s,x-y)\sigma\big(u_{n,k}(s,y)\big)\\ +\int_s^t\int_{\mathbb{R}^d}G_n(t-r,x-z)\sigma'\big(u_{n,k}(r,z)\big)D_{s,y}u_{n,k}(r,z)W(dr,dz).$$

Iterating this equation yields, with $r_0=t, z_0=x$,

$$\begin{aligned}D_{s,y}u_{n,k+1}(t,x)&=G_n(t-s,x-y)\sigma\big(u_{n,k}(s,y)\big)\\&+\int_s^t\int_{\mathbb{R}^d}G_n(t-r_1,x-z_1)\sigma'\big(u_{n,k}(r_1,z_1)\big)G_n(r_1-s,z_1-y)\sigma\big(u_{n,k-1}(s,y)\big)W(dr_1,dz_1)\\&+\sum_{\ell=2}^k\sigma\big(u_{n,k-\ell}(s,y)\big)\int_s^t\cdots\int_s^{r_{\ell-1}}\int_{\mathbb{R}^{d\ell}}G_n(r_\ell-s,z_\ell-y)\\&\quad\times\prod_{j=1}^{\ell}G_n(r_{j-1}-r_j,z_{j-1}-z_j)\sigma'\big(u_{n,k+1-j}(r_j,z_j)\big)W(dr_j,dz_j)=:\sum_{\ell=0}^kT_\ell.\end{aligned}\tag{3.7}$$

Note that by the uniform L^p-convergence of $u_{n,k}(t,x)$ as $k\to\infty$ and $n\to\infty$, we have

$$\Lambda(T,p):=\sup_{n,k\geq1}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\Big\|\sigma\big(u_{n,k}(t,x)\big)\Big\|_p<\infty.$$

Now let us estimate $\|T_\ell\|_p$ for each $\ell\in\{0,1,\dots,k\}$.

Case $\ell=0$: It is clear that $\|T_0\|_p\leq\Lambda(t,p)G_n(t-s,x-y)$. From now on, let us assume that

the support of ψ is contained in $\mathbb{B}_a$ for some $a>0$.

Then the function $x\in\mathbb{R}^d\longmapsto G_n(t,x)$ has a compact support that is contained in $\mathbb{B}_{\frac{a}{n}+t}$. So that

$$G_n(t-s,x-y)\leq\Theta(t-s,n)\mathbf{1}_{\mathbb{B}_{\frac{a}{n}+t}}(x-y).\tag{3.8}$$

It follows that

$$\big\|T_0\big\|_p\leq\Lambda(T,p)\Theta(T,n)\mathbf{1}_{\mathbb{B}_{\frac{a}{n}+T}}(x-y).\tag{3.9}$$

Case $\ell=1$: By the BDG inequality (2.2),

$$\begin{aligned}\big\|T_1\big\|_p^2&\leq4p\bigg\|\int_s^tdr_1\int_{\mathbb{R}^{2d}}dz_1dz_1'G_n(t-r_1,x-z_1)\sigma'\big(u_{n,k}(r_1,z_1)\big)G_n(r_1-s,z_1-y)\\&\quad\times G_n(t-r_1,x-z_1')\sigma'\big(u_{n,k}(r_1,z_1')\big)G_n(r_1-s,z_1'-y)\sigma^2\big(u_{n,k-1}(s,y)\big)\gamma(z_1-z_1')\bigg\|_{p/2}\\&\leq4pL^2\Lambda(T,p)^2\int_s^tdr_1\int_{\mathbb{R}^{2d}}dz_1dz_1'G_n(t-r_1,x-z_1)G_n(r_1-s,z_1-y)\\&\quad\times G_n(t-r_1,x-z_1')G_n(r_1-s,z_1'-y)\gamma(z_1-z_1').\end{aligned}$$

Note that a necessary condition for $G_n(t-r_1,x-z_1)G_n(r_1-s,z_1-y)\neq0$ is

$$x-z_1\in\mathbb{B}_{\frac{a}{n}+t-r_1}\text{ and }z_1-y\in\mathbb{B}_{\frac{a}{n}+r_1-s}$$

which implies $x-y\in\mathbb{B}_{\frac{2a}{n}+t-s}$. This fact, together with Lemma 3.3 and (3.8), leads to

$$\begin{aligned}\big\|T_1\big\|_p^2 &\le 4pL^2\Lambda(T,p)^2\Theta(T,n)^2\mathbf{1}_{\mathbb{B}_{\frac{2a}{n}+T}}(x-y)\\ &\quad\times\int_s^t dr_1\int_{\mathbb{R}^{2d}}dz_1dz_1'G_n(t-r_1,x-z_1)G_n(t-r_1,x-z_1')\gamma(z_1-z_1')\\ &\le 4pL^2(t-s)\Lambda(T,p)^2\Theta(T,n)^4\mathbf{1}_{\mathbb{B}_{\frac{2a}{n}+T}}(x-y)\\ &\qquad\qquad\times\int_{\mathbb{R}^{2d}}dz_1dz_1'\mathbf{1}_{\mathbb{B}_{\frac{a}{n}+t}}(x-z_1)\mathbf{1}_{\mathbb{B}_{\frac{a}{n}+t}}(x-z_1')\gamma(z_1-z_1')\\ &\le 4pL^2(t-s)\Lambda(T,p)^2\Theta(T,n)^4\mathcal{U}_{\frac{a}{n}+T}\mathbf{1}_{\mathbb{B}_{\frac{2a}{n}+T}}(x-y),\end{aligned}$$

by Lemma 3.2. It follows that

$$\big\|T_1\big\|_p\le 2\sqrt{p\,\mathcal{U}_{\frac{a}{n}+T}(t-s)}\,\Lambda(T,p)\Theta(T,n)^2\mathbf{1}_{\mathbb{B}_{\frac{2a}{t}+T}}(x-y).\tag{3.10}$$

Case $\ell\in\{2,\dots,k\}$: We can first represent T_ℓ as

$$T_\ell=\int_s^t\int_{\mathbb{R}^d}G_n(t-r_1,x-z_1)\sigma'\big(u_{n,k}(r_1,z_1)\big)\mathcal{J}(r_1,z_1)W(dr_1,dz_1),$$

with $\mathcal{J}(r_1,z_1)$ defined by

$$\begin{aligned}\mathcal{J}(r_1,z_1)&=\int_s^{r_1}\cdots\int_s^{r_{\ell-1}}\int_{\mathbb{R}^{d\ell-d}}\sigma\big(u_{n,k-\ell}(s,y)\big)G_n(r_\ell-s,z_\ell-y)\\ &\qquad\times\prod_{j=2}^{\ell}G_n(r_{j-1}-r_j,z_{j-1}-z_j)\sigma'\big(u_{n,k+1-j}(r_j,z_j)\big)W(dr_j,dz_j).\end{aligned}$$

In this way, we get $\big\|T_\ell\big\|_p^2$ bounded by

$$\begin{aligned}&4pL^2\int_s^t dr_1\int_{\mathbb{R}^{2d}}dz_1dz_1'\gamma(z_1-z_1')G_n(t-r_1,x-z_1)G_n(t-r_1,x-z_1')\|\mathcal{J}(r_1,z_1')\mathcal{J}(r_1,z_1)\|_{\frac{p}{2}}\\ &\le 4pL^2\int_s^t dr_1\int_{\mathbb{R}^{2d}}dz_1dz_1'\gamma(z_1-z_1')G_n(t-r_1,x-z_1)G_n(t-r_1,x-z_1')\|\mathcal{J}(r_1,z_1)\|_p^2,\end{aligned}$$

using symmetry and the fact that $\|XY\|_{p/2}\le\|X\|_p\|Y\|_p\le\frac{\|X\|_p^2+\|Y\|_p^2}{2}$. Iterating the above procedure for finite times yields

$$\begin{aligned}\big\|T_\ell\big\|_p^2&\le\big(4pL^2\big)^{\ell-1}\int_s^t dr_1\cdots\int_s^{r_{\ell-2}}dr_{\ell-1}\int_{\mathbb{R}^{2d\ell-2d}}\|\widehat{\mathcal{J}}(r_{\ell-1},z_{\ell-1})\|_p^2\\ &\qquad\times\prod_{j=1}^{\ell-1}\gamma(z_j-z_j')G_n(r_{j-1}-r_j,z_{j-1}-z_j)G_n(r_{j-1}-r_j,z_{j-1}-z_j')dz_jdz_j',\end{aligned}$$

with $\widehat{\mathcal{J}}(r_{\ell-1},z_{\ell-1})$ given by

$$\int_s^{r_{\ell-1}}\int_{\mathbb{R}^d}\sigma(u_{n,k-\ell}(s,y))G_n(r-s,z-y)G_n(r_{\ell-1}-r,z_{\ell-1}-z)\sigma'(u_{n,k+1-\ell}(r,z))W(dr,dz).$$

Similarly to how we estimate $\|T_1\|_p$, we get

$$\big\|\widehat{\mathcal{J}}(r_{\ell-1},z_{\ell-1})\big\|_p^2\le 4pL^2T\Lambda(T,p)^2\Theta(T,n)^4\mathcal{U}_{\frac{a}{n}+T}\mathbf{1}_{\mathbb{B}_{\frac{2a}{n}+r_{\ell-1}-s}}(z_{\ell-1}-y).$$

As in Case $\ell = 1$, we have the following implication:

$$\mathbf{1}_{\mathbb{B}_{\frac{2a}{n}+r_{\ell-1}-s}}(z_{\ell-1}-y)\prod_{j=1}^{\ell-1}G_n(r_{j-1}-r_j, z_{j-1}-z_j)G_n(r_{j-1}-r_j, z_{j-1}-z_j')\neq 0$$
$$\Longrightarrow x-y\in\mathbb{B}_{\frac{a(\ell+1)}{n}+t-s}.$$

Note also that using (3.8) and integrating out $dz_{\ell-1}dz_{\ell-1}'$, ..., dz_2dz_2' and dz_1dz_1' yields

$$\begin{aligned}
&\int_{\mathbb{R}^{2d\ell-2d}}\prod_{j=1}^{\ell-1}\gamma(z_j-z_j')G_n(r_{j-1}-r_j, z_{j-1}-z_j)G_n(r_{j-1}-r_j, z_{j-1}-z_j')dz_jdz_j'\\
&\leq \Theta(T,n)^{2\ell-2}\int_{\mathbb{R}^{2d\ell-2d}}\prod_{j=1}^{\ell-1}\gamma(z_j-z_j')\mathbf{1}_{\mathbb{B}_{\frac{a}{n}+T}}(z_{j-1}-z_j)\mathbf{1}_{\mathbb{B}_{\frac{a}{n}+T}}(z_{j-1}-z_j')dz_jdz_j'\\
&=\Theta(T,n)^{2\ell-2}\left(\int_{\mathbb{R}^{2d}}\gamma(z-z')\mathbf{1}_{\mathbb{B}_{\frac{a}{n}+T}}(z)\mathbf{1}_{\mathbb{B}_{\frac{a}{n}+T}}(z')dzdz'\right)^{\ell-1}\leq \Theta(T,n)^{2\ell-2}\mathcal{U}_{\frac{a}{n}+T}^{\ell-1},
\end{aligned}$$

where $\mathcal{U}_{\frac{a}{n}+T}$ is defined in Lemma 3.2. This leads to

$$\big\|T_\ell\big\|_p^2\leq\frac{\Lambda(T,p)^2\Theta(T,n)^2}{(\ell-1)!}\big(4pL^2T\mathcal{U}_{\frac{a}{n}+T}\Theta(T,n)^2\big)^\ell\mathbf{1}_{\mathbb{B}_{\frac{a(\ell+1)}{n}+T}}(x-y).$$

Combining the above cases, we obtain

$$\big\|D_{s,y}v_{k+1}(t,x)\big\|_p\leq\sum_{\ell=0}^k\|T_\ell\|_p\lesssim\mathbf{1}_{\mathbb{B}_{\frac{a(k+1)}{n}+T}}(x-y).$$

That is, Proposition 3.1 is proved. □

We finally proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. In view of [2, Lemma 7.2], it suffices to prove

$$V(R):=\mathrm{Var}\left(R^{-d}\int_{\mathbb{B}_R}\prod_{j=1}^m g_j\big(u(t,x+\zeta^j)\big)dx\right)\xrightarrow{R\to\infty}0,$$

for any fixed $\zeta^1,\dots,\zeta^m\in\mathbb{R}^d$ and $g_1,\dots,g_m\in C_b(\mathbb{R})$ such that each g_j vanishes at zero and has Lipschitz constant bounded by 1.

Using the elementary fact $\mathrm{Var}(X+Y)\leq 2\mathrm{Var}(X)+2\mathrm{Var}(Y)$ for any two square-integrable random variables X and Y, we write

$$\begin{aligned}
V(R)&\leq 2\mathrm{Var}\left(R^{-d}\int_{\mathbb{B}_R}\mathcal{R}_{n,k}(x)dx\right)+4\mathrm{Var}\left(R^{-d}\int_{\mathbb{B}_R}\Big[\mathcal{R}_n(x)-\mathcal{R}_{n,k}(x)\Big]dx\right)\\
&\quad+4\mathrm{Var}\left(R^{-d}\int_{\mathbb{B}_R}\Big[\mathcal{R}(x)-\mathcal{R}_n(x)\Big]dx\right):=2V_{1,n,k}(R)+4V_{2,n,k}(R)+4V_{3,n}(R),
\end{aligned}$$

where

$$\mathcal{R}(x)=\prod_{j=1}^m g_j\big(u(t,x+\zeta^j)\big),\ \mathcal{R}_n(x)=\prod_{j=1}^m g_j\big(u_n(t,x+\zeta^j)\big)\text{ and }\mathcal{R}_{n,k}(x)=\prod_{j=1}^m g_j\big(u_{n,k}(t,x+\zeta^j)\big).$$

Using the stationarity and Minkowski's inequality,

$$\begin{aligned}
V_{3,n}(R)&\leq\left\|R^{-d}\int_{\mathbb{B}_R}\big[\mathcal{R}_n(x)-\mathcal{R}(x)\big]dx\right\|_2^2\leq\left(R^{-d}\int_{\mathbb{B}_R}\big\|\mathcal{R}_n(x)-\mathcal{R}(x)\big\|_2dx\right)^2\\
&=\omega_d^2\Big\|\mathcal{R}_n(0)-\mathcal{R}(0)\Big\|_2^2\xrightarrow{n\to+\infty}0,\quad\text{by (3.3)}.
\end{aligned}$$

The above limit takes place uniformly in $R>0$. Therefore, for any given $\varepsilon>0$, we can find $n\geq N_\varepsilon$ big enough such that $V_{3,n}(R)\leq\varepsilon$, $\forall R>0$. *From now on, let us fix such an integer* n. And we can estimate $V_{2,n,k}(R)$ similarly: Using Minkowski's inequality,

$$V_{2,n,k}(R)\leq\left\|R^{-d}\int_{\mathbb{B}_R}\Big[\mathcal{R}_n(x)-\mathcal{R}_{n,k}(x)\Big]dx\right\|_2^2\leq\omega_d^2\sup_{x\in\mathbb{R}^d}\Big\|\mathcal{R}_n(x)-\mathcal{R}_{n,k}(x)\Big\|_2^2\xrightarrow{k\to\infty}0,$$

as a consequence of (3.5). So we can find some big $k\geq K_{\varepsilon,n}$ such that $V_{2,n,k}(R)\leq\varepsilon$, $\forall R>0$. *From now on, let us fix such an integer* k.

Finally, let us estimate the term $V_{1,n,k}(R)$: First by using the Poincaré inequality (2.7), we obtain

$$\begin{aligned}&V_{1,n,k}(R)\leq R^{-2d}\int_{\mathbb{B}_R^2}dxdy\big|\mathrm{Cov}(\mathcal{R}_{n,k}(x),\mathcal{R}_{n,k}(y))\big|\\&\leq R^{-2d}\int_{\mathbb{B}_R^2}\int_0^t\int_{\mathbb{R}^{2d}}\|D_{s,z}\mathcal{R}_{n,k}(x)\|_2\|D_{s,z'}\mathcal{R}_{n,k}(y)\|_2\,\gamma(z-z')dzdz'dsdxdy.\end{aligned}\tag{3.11}$$

By the chain rule (2.3),

$$\big|D_{s,z}\mathcal{R}_{n,k}(x)\big|\leq\mathbf{1}_{(0,t)}(s)\sum_{j_0=1}^m\left|\prod_{j=1,j\neq j_0}^m g_j(u_{n,k}(t,x+\zeta^j))\right|\big|D_{s,y}u_{n,k}(t,x+\zeta^{j_0})\big|,$$

which implies, for any $s\in[0,t]$,

$$\|D_{s,z}\mathcal{R}_{n,k}(x)\|_2\leq\max_{j=1}^m\|g_j\|_\infty^{m-1}\sum_{j_0=1}^m\big\|D_{s,y}u_{n,k}(t,x+\zeta^{j_0})\big\|_2\lesssim\sum_{j_0=1}^m\mathbf{1}_{\mathbb{B}_b}\big(x-y+\zeta^{j_0}\big),\tag{3.12}$$

where $b=a(k+1)/n+T$, as a consequence of Proposition 3.1. Plugging (3.12) into (3.11), yields

$$\begin{aligned}V_{1,n,k}(R)&\lesssim R^{-2d}\sum_{j,\ell=1}^m\int_{\mathbb{B}_R^2}\int_0^t\int_{\mathbb{R}^{2d}}\mathbf{1}_{\mathbb{B}_b}(x-z+\zeta^j)\mathbf{1}_{\mathbb{B}_b}(y-z'+\zeta^\ell)\gamma(z-z')dzdz'dsdxdy\\&\lesssim R^{-2d}\sum_{j,\ell=1}^m\int_{\mathbb{B}_R^2}\int_{\mathbb{R}^{2d}}\mathbf{1}_{\mathbb{B}_b}(x-z+\zeta^j)\mathbf{1}_{\mathbb{B}_b}(y-z'+\zeta^\ell)\gamma(z-z')dzdz'dxdy.\end{aligned}$$

Therefore using Fourier transform, we write

$$\begin{aligned}\widehat{V}&:=\int_{\mathbb{B}_R^2}\int_{\mathbb{R}^{2d}}\mathbf{1}_{\mathbb{B}_b}(x-z+\zeta^j)\mathbf{1}_{\mathbb{B}_b}(y-z'+\zeta^\ell)\gamma(z-z')dzdz'dxdy\\&=\int_{\mathbb{B}_R^2}\int_{\mathbb{R}^d}e^{-\mathbf{i}(x-y+\zeta^j-\zeta^\ell)\xi}\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi)dxdy.\end{aligned}$$

Put $\ell_R(\xi)=\int_{\mathbb{B}_R^2}e^{-\mathbf{i}(x-y)\xi}dxdy$, which is a nonnegative function. So we get

$$\begin{aligned}\widehat{V}&\leq\int_{\mathbb{R}^d}\ell_R(\xi)\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi)dxdy=\int_{\mathbb{B}_R^2}\int_{\mathbb{R}^d}e^{-\mathbf{i}(x-y)\xi}\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi)dxdy\\&=R^{2d}\int_{\mathbb{B}_1^2}\int_{\mathbb{R}^d}e^{-\mathbf{i}R(x-y)\xi}\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi)dxdy.\end{aligned}$$

That is,

$$R^{-2d}\widehat{V}\leq\int_{\mathbb{R}^d}\left(\int_{\mathbb{B}_1^2}e^{-\mathbf{i}R(x-y)\xi}dxdy\right)\big|\mathscr{F}\mathbf{1}_{\mathbb{B}_b}(\xi)\big|^2\mu(d\xi).$$

Since $\mu(\{0\})=0$, for μ-almost every ξ, $\int_{\mathbb{B}_1^2} e^{-\mathbf{i}R(x-y)\xi}dxdy$ converges to zero as $R\to\infty$, by Riemann-Lebesgue's lemma. Thus, by dominated convergence theorem with the dominance condition (3.6), we deduce that $R^{-2d}\widehat{V}$ converges to zero as $R\to+\infty$. This leads to $V_{1,n,k}(R)\to 0$, as $R\to+\infty$. It follows that $\limsup_{R\to+\infty} V(R)\le 8\varepsilon$, where $\varepsilon>0$ is arbitrary. Hence we can conclude our proof. □

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