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# Limit theorems for integral functionals of Hermite-driven processes

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Consider a moving average process $X$ of the form $X(t)=\int_{-\infty}^{t}\varphi(t-u)\,dZ_u$, $t\geq 0$, where $Z$ is a (non Gaussian) Hermite process of order $q\geq 2$ and $\varphi:\mathbb{R}_+\to\mathbb{R}$ is sufficiently integrable. This paper investigates the fluctuations, as $T\to\infty$, of integral functionals of the form $t\mapsto\int_0^{Tt}P(X(s))\,ds$, in the case where $P$ is any given polynomial function. It extends a study initiated in (*Stoch. Dyn.* **18** (2018) 1850028, 18), where only the quadratic case $P(x)=x^2$ and the convergence in the sense of finite-dimensional distributions were considered.

*Keywords:* Hermite processes; chaotic decomposition; fractional Brownian motion (fBm); multiple Wiener–Itô integrals

## 1. Introduction

Hermite processes occur naturally when we consider limits of partial sums associated with long-range dependent stationary series. They have become increasingly popular in the recent literature, see for example the book [15] by Pipiras and Taqqu, in particular Section 4.11, which contains bibliographical notes on their history and recent developments. They form a family of stochastic processes, indexed by an integer $q\geq 1$ and a self-similarity index $H\in(\frac{1}{2},1)$, called the Hurst parameter, that contains the fractional Brownian motion ($q=1$) and the Rosenblatt process ($q=2$) as particular cases. We refer the reader to Section 2.2 and the references therein for a precise definition of the Hermite processes. Of primary importance in the sequel is the parameter $H_0$, given in terms of $H$ and $q$ by

$$H_0=1-\frac{1-H}{q}\in\left(1-\frac{1}{2q},1\right). \tag{1}$$

The goal of the present paper is to investigate the fluctuations, as $T\to\infty$, of the family of stochastic processes

$$t\mapsto\int_0^{Tt}P\big(X(s)\big)\,ds,\quad t\in[0,1]\text{ (say)}, \tag{2}$$

in the case where $P(x)$ is a polynomial function and $X$ is a moving average process of the form

$$X(t)=\int_{-\infty}^{t}\varphi(t-u)\,dZ_u,\quad t\geq 0, \tag{3}$$

with $Z$ a Hermite process and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ a sufficiently integrable function. We note that integral functionals such as (2) are often encountered in the context of statistical estimation, see, for example, [7] for a concrete example.

Let us first consider the case where $q = 1$, that is to say the case where $Z$ is the fractional Brownian motion. Note that this is the only case where $Z$ is Gaussian, making the study *a priori* much simpler and more affordable. By linearity and passage to the limit, the process $X$ is also Gaussian. Moreover, it is stationary, since the quantity $\mathbb{E}[X(t)X(s)] =: \rho(t-s)$ only depends on $t-s$. For simplicity and without loss of generality, assume that $\rho(0) = 1$, that is, $X(t)$ has variance 1 for any $t$. As is well known since the eighties (see [4,8,18]), the fluctuations of (2) heavily depends on the centered Hermite rank of $P$, defined as the integer $d \geq 1$ such that $P$ decomposes in the form

$$P = \mathbb{E}\big[P\big(X(0)\big)\big] + \sum_{k=d}^{\infty} a_k H_k, \tag{4}$$

with $H_k$ the $k$th Hermite polynomials and $a_d \neq 0$. (Note that the sum (4) is actually finite, since $P$ is a polynomial, so that $\#\{k : a_k \neq 0\} < \infty$.)

The first result of this paper concerns the fractional Brownian motion. Even if it does not follow directly from the well-known results of Breuer–Major [4], Dobrushin–Major [8] and Taqqu [18], the limits obtained are somehow expected. In particular, the threshold $H = 1 - \frac{1}{2d}$ is well known to specialists. However, the proof of this result is not straightforward, and requires several estimations which are interesting in themselves.

**Theorem 1.** *Let $Z$ be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$, and let $\varphi \in L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Consider the moving average process $X$ defined by* (3) *and assume without loss of generality that* $\operatorname{Var}(X(0)) = 1$ (*if not, it suffices to multiply $\varphi$ by a constant*). *Finally, let $P(x) = \sum_{n=0}^{N} a_n x^n$ be a real-valued polynomial function, and let $d \geq 1$ denotes its centered Hermite rank.*

(1) *If $d \geq 2$ and $H \in (\frac{1}{2}, 1 - \frac{1}{2d})$, then*

$$T^{-\frac{1}{2}}\left\{\int_0^{Tt} \big(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\big)\,ds\right\}_{t\in[0,1]} \tag{5}$$

*converges in distribution in $C([0,1])$ to a standard Brownian motion $W$, up to some multiplicative constant $C_1$ which is explicit and depends only on $\varphi$, $P$ and $H$.*

(2) *If $H \in (1 - \frac{1}{2d}, 1)$ then*

$$T^{d(1-H)-1}\left\{\int_0^{Tt} \big(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\big)\,ds\right\}_{t\in[0,1]} \tag{6}$$

*converges in distribution in $C([0,1])$ to a Hermite process of index $d$ and Hurst parameter $1 - d(1-H)$, up to some multiplicative constant $C_2$ which is explicit and depends only on $\varphi$, $P$ and $H$.*

Now, let us consider the non-Gaussian case, that is, the case where $q \geq 2$. As we will see, the situation is completely different, both in the results obtained (rather unexpected) and in the methods used (very different from the Gaussian case). Let $L > 0$. We define $\mathcal{S}_L$ to be the set of bounded functions $l : \mathbb{R}_+ \to \mathbb{R}$ such that $y^L l(y) \to 0$ as $y \to \infty$. We observe that $\mathcal{S}_L \subset L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$ for any $L > 1$. We can now state the following result.

**Theorem 2.** *Let $Z$ be a Hermite process of order $q \geq 2$ and Hurst parameter $H \in (\frac{1}{2}, 1)$, and let $\varphi \in \mathcal{S}_L$ for some $L > 1$. Recall $H_0$ from (1) and consider the moving average process $X$ defined by (3). Finally, let $P(x) = \sum_{n=0}^{N} a_n x^n$ be a real-valued polynomial function. Then, one and only one of the following two situations takes place at $T \to \infty$:*

(i) *If $q$ is odd and if $a_n \neq 0$ for at least one odd $n \in \{1, \dots, N\}$, then*

$$T^{-H_0}\left\{\int_0^{Tt} \left(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\right) ds\right\}_{t\in[0,1]}$$

*converges in distribution in $C([0,1])$ to a fractional Brownian motion of parameter $H_1 := H_0$, up to some multiplicative constant $K_1$ which is explicit and depends only on $\varphi$, $P$, $q$ and $H$, see Remark 3.*

(ii) *If $q$ is even, or if $q$ is odd and $a_n = 0$ for all odd $n \in \{1, \dots, N\}$, then*

$$T^{1-2H_0}\left\{\int_0^{Tt} \left(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\right) ds\right\}_{t\in[0,1]}$$

*converges in distribution in $C([0,1])$ to a Rosenblatt process of Hurst parameter $H_2 := 2H_0 - 1$, up to some multiplicative constant $K_2$ which is explicit and depends only on $\varphi$, $P$, $q$ and $H$, see Remark 3.*

**Remark 3.** Whether in Theorem 1 or Theorem 2, the multiplicative constants appearing in the limit can be all given explicitly by following the respective proofs. For example, the constant $K_1$ and $K_2$ of Theorem 2 are given by the following intricate expressions:

$$K_1 = \sum_{n=3, n \text{ odd}}^{N} a_n c_{H,q}^n K_{\varphi,n,1}$$

$$K_2 = \sum_{n=2}^{N} a_n c_{H,q}^n K_{\varphi,n,2} + a_1 \mathbb{I}_{\{q=2\}} \int_{\mathbb{R}_+} \varphi(v)\, dv$$

with

$$K_{\varphi,n,i} = \sum_{\alpha \in A_{n,q}, nq-2|\alpha|=i} \frac{C_\alpha K_{\varphi,\alpha,H_0}}{c_{H_i,i}}, \quad i = 1, 2,$$

where the sets and constants in the previous formula are defined in Sections 2, 3 and 4.

**Remark 4.** Note that, unlike the case of a fractional Brownian motion $X$, where the limit depends on the Hermite rank of the polynomial $P$, here the Hermite rank of $P$ plays no role and the limit depends on the parity of the non-vanishing coefficients of $P$. This is not really surprising in our *non-Gaussian* context, since the Hermite rank of $P$ is defined by means of its decomposition into Hermite polynomials, and these latter polynomials only have good probabilistic properties when evaluated in *Gaussian* random variables.

We note that our Theorem 2 contains as a very particular case the main result of [7], which corresponds to the choice $P(x) = x^2$ and thus situation (ii). Moreover, let us emphasize that our Theorem

2 not only studies the convergence of finite-dimensional distributions as in [7], but also provides a *functional* result.

Because the employed method is new, let us sketch the main steps of the proof of Theorem 2, by using the classical notation of the Malliavin calculus (see Section 2 for any unexplained definition or result); in particular we write $I_p^B(h)$ to indicate the $p$th multiple Wiener–Itô integral of kernel $h$ with respect to the standard (two-sided) Brownian motion $B$.

*Step* 1. In Section 3, we represent the moving average process $X$ as a $q$th multiple Wiener–Itô integral with respect to $B$:

$$X(t) = c_{H,q} I_q^B\big(g(t,\cdot)\big),$$

where $c_{H,q}$ is an explicit constant and the kernel $g(t,\cdot)$ is given by

$$g(t,\xi_1,\dots,\xi_q) = \int_{-\infty}^{t} \varphi(t-v) \prod_{j=1}^{q} (v-\xi_j)_+^{H_0-\frac{3}{2}}\, dv, \tag{7}$$

for $\xi_1,\dots,\xi_q \in \mathbb{R}$, $t \ge 0$. Thanks to this representation, we compute in Lemma 5 the chaotic expansion of the $n$th power of $X(t)$ for any $n \ge 2$ and $t > 0$, and obtain an expression of the form

$$X^n(t) = c_{H,q}^n \sum_{\alpha \in A_{n,q}} C_\alpha I_{nq-2|\alpha|}^B\bigg(\bigotimes_\alpha \big(g(t,\cdot),\dots,g(t,\cdot)\big)\bigg),$$

where we have used the novel notation $\bigotimes_\alpha(g(t,\cdot),\dots,g(t,\cdot))$ to indicate iterated contractions whose precise definition is given in Section 2.1, and where $C_\alpha$ are combinatorial constants and the sum runs over a family $A_{n,q}$ of suitable multi-indices $\alpha = (\alpha_{ij}, 1 \le i < j \le n)$. As an immediate consequence, we deduce that our quantity of interest can be decomposed as follows:

$$\begin{aligned}
&\int_0^{Tt} \big(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\big)\, ds \\
&\quad = a_0 \int_0^{Tt} X(s)\, ds \\
&\qquad + \sum_{n=2}^{N} a_n c_{H,q}^n \sum_{\alpha \in A_{n,q}, nq-2|\alpha| \ge 1} C_\alpha \int_0^{Tt} I_{nq-2|\alpha|}^B\bigg(\bigotimes_\alpha \big(g(s,\cdot),\dots,g(s,\cdot)\big)\bigg)\, ds.
\end{aligned} \tag{8}$$

*Step* 2. In Proposition 6, we compute an explicit expression for the iterated contractions $\bigotimes_\alpha(g(t,\cdot), \dots, g(t,\cdot))$ appearing in the right-hand side of (8), by using that $g$ is given by (7).

To ease the description of the remaining steps, let us now set

$$F_{n,q,\alpha,T}(t) = \int_0^{Tt} I_{nq-2|\alpha|}^B\bigg(\bigotimes_\alpha \big(g(s,\cdot),\dots,g(s,\cdot)\big)\bigg)\, ds. \tag{9}$$

*Step* 3. As $T \to \infty$, we show in Proposition 7 that, if $nq - 2|\alpha| < \frac{q}{1-H}$, then $T^{-1+(1-H_0)(nq-2|\alpha|)} \times F_{n,q,\alpha,T}(t)$ converges in distribution to a Hermite process (whose order and Hurst index are specified) up to some multiplicative constant. Similarly, we prove in Proposition 9 that, if $nq - 2|\alpha| \ge 3$, then $T^{\alpha_0} F_{n,q,\alpha,T}(t)$ is tight and converges in $L^2(\Omega)$ to zero, where $\alpha_0$ is given in (28).

*Step* 4. By putting together the results obtained in the previous steps, the two convergences stated in Theorem 2 follow immediately.

To illustrate a possible use of our results, we study in Section 6 an extension of the classical fractional Ornstein–Uhlenbeck process (see, e.g., Cheridito *et al.* [6]) to the case where the driving process is more generally a Hermite process. To the best of our knowledge, there is very little literature devoted to this mathematical object, only [12,16].

The rest of the paper is organized as follows. Section 2 presents some basic results about multiple Wiener–Itô integrals and Hermite processes, as well as some other facts that are used throughout the paper. Section 3 contains preliminary results. The proof of Theorem 1 (resp. Theorem 2) is given in Section 5 (resp. Section 4). In Section 6, we provide a complete asymptotic study of the Hermite–Ornstein–Uhlenbeck process, by means of Theorems 1 and 2 and of an extension of Birkhoff's ergodic Theorem. Finally, Section 6 contains two technical results: a power counting theorem and a version of the Hardy–Littelwood inequality, which both play an important role in the proof of our main theorems.

## 2. Preliminaries on multiple Wiener–Itô integrals and Hermite processes

### 2.1. Multiple Wiener–Itô integrals and a product formula

A function $f:\mathbb{R}^p\to\mathbb{R}$ is said to be *symmetric* if the following relation holds for all permutation $\sigma\in\mathfrak{S}(p)$:

$$f(t_1,\dots,t_p)=f(t_{\sigma(1)},\dots,t_{\sigma(p)}),\quad t_1,\dots,t_p\in\mathbb{R}.$$

The subset of $L^2(\mathbb{R}^p)$ composed of symmetric functions is denoted by $L^2_s(\mathbb{R}^p)$.

Let $B=\{B(t)\}_{t\in\mathbb{R}}$ be a two-sided Brownian motion. For any given $f\in L^2_s(\mathbb{R}^p)$, we consider the *multiple Wiener–Itô integral* of $f$ with respect to $B$, denoted by

$$I_p^B(f)=\int_{\mathbb{R}^p} f(t_1,\dots,t_p)\,dB(t_1)\cdots dB(t_p).$$

This stochastic integral satisfies $\mathbb{E}[I_p^B(f)]=0$ and

$$\mathbb{E}\big[I_p^B(f)I_q^B(g)\big]=\mathbf{1}_{\{p=q\}}p!\langle f,g\rangle_{L^2(\mathbb{R}^p)}$$

for $f\in L^2_s(\mathbb{R}^p)$ and $g\in L^2_s(\mathbb{R}^q)$, see [11] and [13] for precise definitions and further details.

It will be convenient in this paper to deal with multiple Wiener–Itô integrals of possibly nonsymmetric functions. If $f\in L^2(\mathbb{R}^p)$, we put $I_p^B(f)=I_p^B(\tilde{f})$, where $\tilde{f}$ denotes the symmetrization of $f$, that is,

$$\tilde{f}(x_1,\dots,x_p)=\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}(p)} f(x_{\sigma(1)},\dots,x_{\sigma(p)}).$$

We will need the expansion as a sum of multiple Wiener–Itô integrals for a product of the form

$$\prod_{k=1}^n I_q^B(h_k),$$

where $q \geq 2$ is fixed and the functions $h_k$ belong to $L^2_s(\mathbb{R}^q)$ for $k = 1, \dots, n$. In order to present this extension of the product formula and to define the relevant contractions between the functions $h_i$ and $h_j$ that will naturally appear, we introduce some further notation. Let $A_{n,q}$ be the set of multi-indices $\alpha = (\alpha_{ij}, 1 \leq i < j \leq n)$ such that, for each $k = 1, \dots, n$,

$$\sum_{1 \leq i < j \leq n} \alpha_{ij} \mathbf{1}_{k \in \{i,j\}} \leq q.$$

Set $|\alpha| = \sum_{1 \leq i < j \leq n} \alpha_{ij}$,

$$\beta_k^0 = q - \sum_{1 \leq i < j \leq n} \alpha_{ij} \mathbf{1}_{k \in \{i,j\}}, \quad 1 \leq k \leq n$$

and

$$m := m(\alpha) = \sum_{k=1}^n \beta_k^0 = nq - 2|\alpha|. \tag{10}$$

For each $1 \leq i < j \leq n$, the integer $\alpha_{ij}$ will represent the number of variables in $h_i$ which are contracted with $h_j$ whereas, for each $k = 1, \dots, n$, the integer $\beta_k^0$ is the number of variables in $h_k$ which are not contracted. We will also write $\beta_k = \sum_{j=1}^k \beta_j^0$ for $k = 1, \dots, n$ and $\beta_0 = 0$. Finally, we set

$$C_\alpha = \frac{q!^n}{\prod_{k=1}^n \beta_k^0! \prod_{1 \leq i < j \leq n} \alpha_{ij}!}. \tag{11}$$

With these preliminaries, for any element $\alpha \in A_{n,q}$ we can define the contraction $\bigotimes_\alpha (h_1, \dots, h_n)$ as the function of $nq - 2|\alpha|$ variables obtained by contracting $\alpha_{ij}$ variables between $h_i$ and $h_j$ for each couple of indices $1 \leq i < j \leq n$. Define the collection $(u^{i,j})_{1 \leq i,j \leq n, i \neq j}$ in the following way:

$$u^{i,j} = \alpha_{\min(i,j),\max(i,j)}.$$

We then have

$$\begin{aligned}
&\bigotimes_\alpha (h_1, \dots, h_n)(\xi_1, \dots, \xi_{nq-2|\alpha|}) \\
&\quad = \int_{\mathbb{R}^{|\alpha|}} \prod_{k=1}^n h_k \big( s_1^{k,1}, \dots, s_{u^{k,1}}^{k,1}, \dots, s_1^{k,n}, \dots, s_{u^{k,n}}^{k,n}, \xi_{1+\beta_{k-1}}, \dots, \xi_{\beta_k} \big) \\
&\qquad \times \prod_{1 \leq i < j \leq n} ds_1^{i,j} \dots ds_{u^{i,j}}^{i,j}
\end{aligned} \tag{12}$$

When $n = 2$, $\alpha$ has only one component $\alpha_{1,2}$ and $\bigotimes_\alpha (h_1, h_2) = h_1 \bigotimes_{\alpha_{1,2}} h_2$ is the usual contraction of $\alpha_{1,2}$ indices between $h_1$ and $h_2$. Notice that the function $\bigotimes_\alpha (h_1, \dots, h_n)$ is not necessarily symmetric.

Then, we have the following result.

**Lemma 5.** *Let $n, q \geq 2$ be some integers and let $h_i \in L^2_s(\mathbb{R}^q)$ for $i = 1, \dots, n$. We have*

$$\prod_{k=1}^n I_q^B(h_k) = \sum_{\alpha \in A_{n,q}} C_\alpha I_{nq-2|\alpha|}^B \bigg( \bigotimes_\alpha (h_1, \dots, h_n) \bigg). \tag{13}$$

**Proof.** The product formula for multiple stochastic integrals (see, for instance, [14], Theorem 6.1.1, or formula (2.1) in [2] for $n=2$) says that

$$\prod_{k=1}^{n} I_q^B(h_k) = \sum_{\mathcal{P},\psi} I^B_{\beta_1^0+\cdots+\beta_n^0}\left(\left(\bigotimes_{k=1}^{n} h_k\right)_{\mathcal{P},\psi}\right), \tag{14}$$

where $\mathcal{P}$ denotes the set of all partitions $\{1,\dots,q\} = J_i \cup (\cup_{k=1,\dots,n,k\neq i} I_{ik})$, where for any $i,j = 1,\dots,n$, $I_{ij}$ and $I_{ji}$ have the same cardinality $\alpha_{ij}$, $\psi_{ij}$ is a bijection between $I_{ij}$ and $I_{ji}$ and $\beta_k^0 = |J_k|$. Moreover, $(\bigotimes_{k=1}^n h_k)_{\mathcal{P},\psi}$ denotes the contraction of the indexes $\ell$ and $\psi_{ij}(\ell)$ for any $\ell \in I_{ij}$ and any $i,j = 1\dots,n$. Then, formula (13) follows form (14), by just counting the number of partitions, which is

$$\prod_{k=1}^{n} \frac{q!}{\prod_{i \text{ or } j\neq k} \alpha_{ij}!\beta_k^0!}$$

and multiplying by the number of bijections, which is $\prod_{1\le i<j\le n} \alpha_{ij}!$. □

Notice that when $n=2$, formula (13) reduces to the well-known formula for the product of two multiple integrals. That is, for any two symmetric functions $f \in L_s^2(\mathbb{R}^p)$ and $g \in L_s^2(\mathbb{R}^q)$ we have

$$I_p^B(f) I_q^B(g) = \sum_{r=0}^{\min(p,q)} r! \binom{p}{r}\binom{q}{r} I^B_{p+q-2r}(f \otimes_r g).$$

where, for $0 \le r \le \min(p,q)$, $f \otimes_r g \in L^2(\mathbb{R}^{p+q-2r})$ denotes the contraction of $r$ coordinates between $f$ and $g$.

## 2.2. Hermite processes

Fix $q \ge 1$ and $H \in (\frac{1}{2}, 1)$. The Hermite process of index $q$ and Hurst parameter $H$ can be represented by means of a multiple Wiener–Itô integral with respect to $B$ as follows, see, for example, [9]:

$$Z^{H,q}(t) = c_{H,q} \int_{\mathbb{R}^q} \int_{[0,t]} \prod_{j=1}^{q} (s - x_j)_+^{H_0 - \frac{3}{2}}\, ds\, dB(x_1)\cdots dB(x_q), \quad t \in \mathbb{R}. \tag{15}$$

Here, $x_+ = \max\{x, 0\}$, the constant $c_{H,q}$ is chosen to ensure that $\mathrm{Var}(Z^{H,q}(1)) = 1$, and

$$H_0 = 1 - \frac{1-H}{q} \in \left(1 - \frac{1}{2q}, 1\right).$$

Note that $Z^{H,q}$ is self-similar of index $H$. When $q=1$, the process $Z^{H,1}$ is Gaussian and is nothing but the fractional Brownian motion with Hurst parameter $H$. For $q \ge 2$, the processes $Z^{H,q}$ are no longer Gaussian: they belong to the $q$th Wiener chaos. The process $Z^{H,2}$ is known as the Rosenblatt process.

Let $|\mathcal{H}|$ be the following class of functions:

$$|\mathcal{H}| = \left\{ f : \mathbb{R} \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2}\, du\, dv < \infty \right\}.$$

Maejima and Tudor [9] proved that the stochastic integral $\int_{\mathbb{R}} f(u)\,dZ^{H,q}(u)$ with respect to the Hermite process $Z^{H,q}$ is well defined when $f$ belongs to $|\mathcal{H}|$. Moreover, for any order $q \geq 1$, index $H \in (\frac{1}{2}, 1)$ and function $f \in |\mathcal{H}|$,

$$\int_{\mathbb{R}} f(u)\,dZ^{H,q}(u) = c_{H,q} \int_{\mathbb{R}^q} \left( \int_{\mathbb{R}} f(u) \prod_{j=1}^{q} (u-\xi_j)_+^{H_0-\frac{3}{2}}\,du \right) dB(\xi_1)\cdots dB(\xi_q). \tag{16}$$

As a consequence of the Hardy–Littlewood–Sobolev inequality featured in [1], we observe that $L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R}) \subset |\mathcal{H}|$.

# 3. Chaotic decomposition of $\int_0^{Tt} P(X(s))\,ds$

Assume $\varphi \in |\mathcal{H}|$ and $q \geq 1$. Using (16) and bearing in mind the notation and results from Section 2, it is immediate that $X$ can be written as

$$X(t) = c_{H,q} I_q^B\big(g(t,\cdot)\big), \tag{17}$$

where $g(t,\cdot)$ is given by

$$g(t,\xi_1,\dots,\xi_q) = \int_{-\infty}^{t} \varphi(t-v) \prod_{j=1}^{q} (v-\xi_j)_+^{H_0-3/2}\,dv, \tag{18}$$

and $c_{H,q}$ is defined as in (15).

## 3.1. Computing the chaotic expansion of $X(t)^n$ when $n \geq 2$

Let us denote by $A_{n,q}^0$ the set of elements $\alpha \in A_{n,q}$ such that $nq - 2|\alpha| = 0$ and $A_{n,q}^1$ will be the set of elements $\alpha \in A_{n,q}$ such that $nq - 2|\alpha| \geq 1$. Notice that when $nq$ is odd, $A_{n,q}^0$ is empty. Using (13), we obtain the following formula for the expectation of the $n$th power ($n \geq 2$) of $X$ given by (3):

$$\mathbb{E}\big[X(t)^n\big] = (c_{H,q})^n \sum_{\alpha \in A_{n,q}^0} C_\alpha I_{nq-2|\alpha|}^B\left( \bigotimes_\alpha \big(g(t,\cdot),\dots,g(t,\cdot)\big) \right). \tag{19}$$

We observe in particular that $\mathbb{E}[X(t)^n] = 0$ whenever $nq$ is odd. From (13) and (19), we deduce for $n \geq 2$ that

$$X(t)^n - \mathbb{E}\big[X(t)^n\big] = (c_{H,q})^n \sum_{\alpha \in A_{n,q}^1} C_\alpha I_{nq-2|\alpha|}^B\left( \bigotimes_\alpha \big(g(t,\cdot),\dots,g(t,\cdot)\big) \right). \tag{20}$$

To clarify this formula, let us write down detailed a expression in the cases $n = 2$ and $n = 3$. When $n = 2$, the right-hand side of (20) is

$$(c_{H,q})^2 \sum_{r=0}^{q-1} r! \binom{q}{r}^2 I_{2q-2r}^B\big(g(t,\cdot) \otimes_r g(t,\cdot)\big),$$

because $\alpha$ has just one component $\alpha_{1,2} =: r$ and condition $\alpha \in A^1_{n,q}$ means $0 \le r \le q-1$. For $n=3$, we have

$$A_{3,q} = \{(\alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}) : \alpha_{1,2}+\alpha_{1,3} \le q, \alpha_{1,2}+\alpha_{2,3} \le q, \alpha_{1,3}+\alpha_{2,3} \le q\}$$

and the right-hand side of (20) is

$$(c_{H,q})^3 \sum_{\alpha \in A_{3,q}:3q-2|\alpha|\ge 1} C_\alpha I_{3q-2|\alpha|}\Big(\bigotimes_\alpha \big(g(t,\cdot), g(t,\cdot), g(t,\cdot)\big)\Big),$$

where

$$C_\alpha = \frac{(q!)^3}{\alpha_{1,2}!\alpha_{1,3}!\alpha_{2,2}!(q-\alpha_{1,2}-\alpha_{1,3})!(q-\alpha_{1,2}-\alpha_{2,3})!(q-\alpha_{1,3}-\alpha_{1,3})!}.$$

In this case, the contraction $\bigotimes_\alpha(g(t,\cdot), g(t,\cdot), g(t,\cdot))$ is the function of $3q-2|\alpha|$ variables defined by

$$\int_{\mathbb{R}^{|\alpha|}} g(\bullet, s, u) g(\star, s, v)) g(\circ, u, v)\, ds\, du\, dv,$$

with $s = (s_1, \dots, s_{\alpha_{1,2}})$, $u = (u_1, \dots, u_{\alpha_{1,3}})$ and $v = (v_1, \dots, v_{\alpha_{2,3}})$.

From (20), we obtain

$$\begin{aligned}\int_0^{Tt} &\big(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\big)\, ds \\ &= a_1 \int_0^{Tt} X(s)\, ds + \sum_{n=2}^N a_n (c_{H,q})^n \sum_{\alpha \in A^1_{n,q}} C_\alpha \int_0^{Tt} I^B_{nq-2|\alpha|}\Big(\bigotimes_\alpha \big(g(s,\cdot), \dots, g(s,\cdot)\big)\Big)\, ds. \quad (21)\end{aligned}$$

## 3.2. Expressing the iterated contractions of *g*

We now compute an explicit expression for the iterated contractions appearing in (21).

**Proposition 6.** *Fix $n \ge 2$, $q \ge 1$ and $\alpha \in A_{n,q}$. We have*

$$\begin{aligned}\bigotimes_\alpha \big(g(t,\cdot), \dots, g(t,\cdot)\big)(\xi) &= \beta\Big(H_0 - \frac{1}{2}, 2 - 2H_0\Big)^{|\alpha|} \\ &\quad \times \int_{(-\infty,t]^n} dv_1 \dots dv_n \prod_{k=1}^n \varphi(t - v_k) \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \\ &\quad \times \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k - \xi_\ell)_+^{H_0-\frac{3}{2}},\end{aligned}$$

*with the convention $\beta_0 = 0$.*

**Proof.** The proof is a straightforward consequence of the following identity

$$\int_{\mathbb{R}} (v-\xi)_+^{H_0-3/2}(w-\xi)_+^{H_0-3/2}\,d\xi = \beta\left(H_0-\frac{1}{2}, 2-2H_0\right)|v-w|^{(2H_0-2)}, \tag{22}$$

whose proof is elementary, see, for example, [3]. □

# 4. Proof of Theorem 2

We are now ready to prove Theorem 2. To do so, we will mostly rely on the forthcoming Proposition 7, which might be a result of independent interest by itself, and which studies the asymptotic behavior of $F^B_{n,q,\alpha,T}$ given by (9). We will denote by f.d.d. the convergence in law of the finite-dimensional distributions of a given process. Notice that the hypothesis on $\varphi$ is a bit weaker than the one in the main theorem, the fact that $\varphi \in \mathcal{S}_L$ being required in the forthcoming Proposition 9.

**Proposition 7.** *Fix $n \geq 2$, $q \geq 1$ and $\alpha \in A_{n,q}$. Assume the function $\varphi$ belongs to $L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$, recall $H_0$ from (1) and let $m$ be defined as in (10). Finally, assume that $2m < \frac{q}{1-H}$ (which is automatically satisfied when $m=1$ or $m=2$). Then, as $T \to \infty$,*

$$\left(T^{-1+(1-H_0)m} F_{n,q,\alpha,T}(t)\right)_{t\in[0,1]} \xrightarrow{f.d.d.} \left(\frac{C_\alpha K_{\varphi,\alpha,H_0}}{c_{H(m),m}} Z^{H(m),m}(t)\right)_{t\in[0,1]}, \tag{23}$$

*where $Z^{H(m),m}$ denotes the $m$th Hermite process of Hurst index $H(m) = 1 - \frac{m}{q}(1-H)$ and the constants $C_\alpha$ and $K_{\varphi,\alpha,H_0}$ are defined in (11) and (24), respectively. Furthermore,*

$$\left\{T^{-1+(1-H_0)m}\left(F_{n,q,\alpha,T}(t)\right)_{t\in[0,1]}, T>0\right\}$$

*is tight in $C([0,1])$.*

**Remark 8.** Note that for $m_1 < m_2$ the chaos of order $m_1$ dominates the chaos of order $m_2$.

**Proof of Proposition 7.** Let $n \geq 2$, $q \geq 1$ and $\alpha \in A_{n,q}$.

Step 1: We will first show the convergence (23). We will make several change of variables in order to transform the expression of $F_{n,q,\alpha,T}(t)$. By means of an application of stochastic's Fubini's theorem, we can write

$$F_{n,q,\alpha,T}(t) = C_\alpha \int_{\mathbb{R}^m} \Psi_T(\xi_1,\dots,\xi_m)\,dB(\xi_1)\cdots dB(\xi_m),$$

where

$$\Psi_T(\xi_1,\dots,\xi_m) := T^{-1+m(1-H_0)} \int_0^{Tt} ds \int_{(-\infty,s]^n} dv_1\cdots dv_n \prod_{k=1}^n \varphi(s-v_k)$$
$$\times \prod_{1\leq i<j\leq n} |v_i-v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k-\xi_\ell)_+^{H_0-\frac{3}{2}}.$$

Using the change of variables $s \to Ts$ and $v_k \to Ts - v_k$, $1 \le k \le n$, we obtain

$$\Psi_T(\xi_1, \ldots, \xi_m) := T^{m(1-H_0)} \int_0^t ds \int_{(-\infty, Ts]^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(Ts - v_k) \times \prod_{1 \le i < j \le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k - \xi_\ell)_+^{H_0 - \frac{3}{2}}$$

$$= T^{-\frac{m}{2}} \int_0^t ds \int_{[0,\infty)^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(v_k) \times \prod_{1 \le i < j \le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} \left(s - \frac{v_k}{T} - \frac{\xi_\ell}{T}\right)_+^{H_0 - \frac{3}{2}}.$$

By the scaling property of the Brownian motion, the processes

$$\big(F_{n,q,\alpha,T}(t)\big)_{\alpha \in A_{n,q}, 2 \le n \le N, t \in [0,1]}$$

and

$$\big(\widehat{F}_{n,q,\alpha,T}(t)\big)_{\alpha \in A_{n,q}, 2 \le n \le N, t \in [0,1]}$$

have the same probability distribution, where

$$\widehat{F}_{n,q,\alpha,T}(t) = C_\alpha \int_{\mathbb{R}^m} \widehat{\Psi}_T(\xi_1, \ldots, \xi_m)\, dB(\xi_1) \cdots dB(\xi_m)$$

$$\widehat{\Psi}_T(\xi_1, \ldots, \xi_m) := \int_0^t ds \int_{(-\infty,0]^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(v_k) \times \prod_{1 \le i < j \le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} \left(s - \frac{v_k}{T} - \xi_\ell\right)_+^{H_0 - \frac{3}{2}}.$$

Set

$$\widehat{\Psi}(\xi_1, \ldots, \xi_m) := K_{\varphi,\alpha,H_0} \int_0^t ds \prod_{\ell=1}^m (s - \xi_\ell)_+^{H_0 - \frac{3}{2}},$$

where

$$K_{\varphi,\alpha,H_0} = \int_{\mathbb{R}_+^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(v_k) \prod_{1 \le i < j \le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}}. \tag{24}$$

Notice that, by Lemma 15, $K_{\varphi,\alpha,H_0}$ is well defined. We claim that

$$\lim_{T \to \infty} \widehat{\Psi}_T = \widehat{\Psi}, \tag{25}$$

where the convergence holds in $L^2(\mathbb{R}^m)$. This will imply the convergence in $L^2(\Omega)$ of $\widehat{F}_{n,q,\alpha,T}(t)$, as $T \to \infty$ to a Hermite process of order $m$, multiplied by the constant $C_\alpha K_{\varphi,\alpha,H_0}$.

*Proof of (25)*: It suffices to show that the inner products $\langle \widehat{\Psi}_T, \widehat{\Psi}_T \rangle_{L^2(\mathbb{R}^m)}$ and $\langle \widehat{\Psi}_T, \widehat{\Psi} \rangle_{L^2(\mathbb{R}^m)}$ converge, as $T \to \infty$, to

$$\|\widehat{\Psi}\|^2_{L^2(\mathbb{R}^m)} = K^2_{\varphi,\alpha,H_0} \beta\left(H_0 - \frac{1}{2}, 2 - 2H_0\right)^m \int_{[0,t]^2} ds\, ds' |s - s'|^{(2H_0-2)m},$$

which is finite because $m < \frac{1}{2(1-H_0)} = \frac{q}{2(1-H)}$. We will show the convergence of $\langle \widehat{\Psi}_T, \widehat{\Psi}_T \rangle_{L^2(\mathbb{R}^m)}$ and the second term can be handled by the same arguments. We have

$$\begin{aligned}
\|\widehat{\Psi}_T\|^2_{L^2(\mathbb{R}^m)} = &\int_{[0,t]^2} ds\, ds' \int_{\mathbb{R}^{2n}_+} dv_1 \cdots dv_n\, dv'_1 \cdots dv'_n \\
&\times \prod_{k=1}^n \varphi(v_k)\varphi(v'_k) \prod_{1 \le i < j \le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}} \\
&\times \prod_{k=1}^n \beta\left(H_0 - \frac{1}{2}, 2 - 2H_0\right)^{\beta_k} \left| s - s' - \frac{v_k - v'_k}{T} \right|^{(2H_0-2)\beta_k}.
\end{aligned}$$

Let us first show that given $w_k \in \mathbb{R}$, $1 \le k \le n$,

$$\lim_{T\to\infty} \int_{[0,t]^2} ds\, ds' \prod_{k=1}^n \left| s - s' - \frac{w_k}{T} \right|^{(2H_0-2)\beta_k} = \int_{[0,t]^2} ds\, ds' |s - s'|^{(2H_0-2)m} \tag{26}$$

and, moreover,

$$\sup_{w_k\in\mathbb{R}, 1\le k\le n} \int_{[0,t]^2} ds\, ds' \prod_{k=1}^n |s - s' - w_k|^{(2H_0-2)\beta_k} < \infty. \tag{27}$$

By the dominated convergence theorem and using Lemma 15, (26) and (27) imply (25).

To show (26), choose $\epsilon$ such that $|w_k|/T < \epsilon$, $1 \le k \le n$, for $T$ large enough (depending on the fixed $w_k$'s). Then, we can write

$$\begin{aligned}
&\int_{[0,t]^2} ds\, ds' \left| \prod_{k=1}^n \left| s - s' - \frac{w_k}{T} \right|^{(2H_0-2)\beta_k} - |s - s'|^{(2H_0-2)m} \right| \\
&\le t \int_{|\xi|>2\epsilon} d\xi \left| \prod_{k=1}^n \left| \xi - \frac{w_k}{T} \right|^{(2H_0-2)\beta_k} - |\xi|^{(2H_0-2)m} \right| \\
&\quad + 2t \sup_{|w_k|<\epsilon} \int_{|\xi|\le 2\epsilon} d\xi \prod_{k=1}^n |\xi - w_k|^{(2H_0-2)\beta_k} \\
&:= B_1 + B_2.
\end{aligned}$$

The term $B_1$ tends to zero a $T \to \infty$, for each $\epsilon > 0$. On the other hand, the term $B_2$ tends to zero as $\epsilon \to 0$. Indeed,

$$B_2 = 2t\epsilon^{(2H_0-2)m+1} \sup_{|w_k|<1} \int_{|\xi|\le 2} d\xi \prod_{k=1}^{n} |\xi - w_k|^{(2H_0-2)\beta_k}.$$

Note that the above supremum is finite because the function $(w_1, \dots, w_k) \to \int_{|\xi|\le 2} d\xi \prod_{k=1}^{n} |\xi - w_k|^{(2H_0-2)\beta_k}$ is continuous.

Property (27) follows immediately from the fact that the function

$$(w_1, \dots, w_k) \to \int_{[0,t]^2} ds\, ds' \prod_{k=1}^{n} \big|s - s' - w_k\big|^{(2H_0-2)\beta_k}$$

is continuous and vanishes as $|(w_1, \dots, w_k)|$ tends to infinity.

We have $H_0 = 1 - \frac{1-H}{q} = 1 - \frac{1-H(m)}{m}$ with $H(m)$ as above. As a result, we obtain the convergence of the finite-dimensional distributions of $T^{-1+(1-H_0)m} F_{n,q,\alpha,T}(t)$ to those the $m$th Hermite process $Z^{H(m),m}$ multiplied by the constant $\frac{C_\alpha K_{\varphi,\alpha,H_0}}{c_{H(m),m}}$.

*Step 2*: *Tightness*. Fix $0 \le s < t \le 1$. To check that tightness holds in $C([0, 1])$, let us compute the squared $L^2(\Omega)$-norm

$$\Phi_T := T^{-1+(1-H_0)m} \mathbb{E}\big(\big|F_{n,q,\alpha,T}(t) - F_{n,q,\alpha,T}(s)\big|^2\big).$$

Proceeding as in the first step of the proof, we obtain

$$\begin{aligned}\Psi_T &= \mathbb{E}\Bigg(\Bigg|\int_{\mathbb{R}^m} dB(\xi_1)\cdots dB(\xi_m) \int_s^t du \int_{\mathbb{R}_+^n} dv_1 \cdots dv_n \prod_{k=1}^{n} \varphi(v_k) \\ &\quad \times \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^{n} \prod_{\ell=1+\beta_{k-1}}^{\beta_k} \left(u - \frac{v_k}{T} - \xi_\ell\right)_+^{H_0-\frac{3}{2}}\Bigg|^2\Bigg) \\ &\le m! \int_{\mathbb{R}^m} d\xi_1 \cdots d\xi_m \Bigg|\int_s^t du \int_{\mathbb{R}_+^n} dv_1 \cdots dv_n \prod_{k=1}^{n} \varphi(v_k) \\ &\quad \times \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^{n} \prod_{\ell=1+\beta_{k-1}}^{\beta_k} \left(u - \frac{v_k}{T} - \xi_\ell\right)_+^{H_0-\frac{3}{2}}\Bigg|^2.\end{aligned}$$

Using (22) yields

$$\begin{aligned}\Psi_T &\le m! \int_{[s,t]^2} du\, du' \int_{\mathbb{R}_+^{2n}} dv_1 \cdots dv_n\, dv_1' \cdots dv_n' \\ &\quad \times \prod_{k=1}^{n} \varphi(v_k)\varphi(v_k') \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \big|v_i' - v_j'\big|^{(2H_0-2)\alpha_{ij}} \\ &\quad \times \prod_{k=1}^{n} \beta\left(H_0 - \frac{1}{2}, 2 - 2H_0\right)^{\beta_k} \left|u - u' - \frac{v_k - v_k'}{T}\right|^{(2H_0-2)\beta_k}\end{aligned}$$

$$\begin{aligned}
&\le m!(t-s)\int_{-1}^{1} d\xi \int_{\mathbb{R}_+^{2n}} dv_1\cdots dv_n\, dv_1'\cdots dv_n' \\
&\quad \times \prod_{k=1}^{n} \varphi(v_k)\varphi\big(v_k'\big) \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \big|v_i' - v_j'\big|^{(2H_0-2)\alpha_{ij}} \\
&\quad \times \prod_{k=1}^{n} \beta\bigg(H_0 - \frac{1}{2}, 2-2H_0\bigg)^{\beta_k} \bigg|\xi - \frac{v_k - v_k'}{T}\bigg|^{(2H_0-2)\beta_k} \\
&\le C(t-s).
\end{aligned}$$

Then the equivalence of all $L^p(\Omega)$-norms, $p \ge 2$, on a fixed Wiener chaos, also known as the hypercontractivity property, allows us to conclude the proof of the tightness. □

We will make use of the notation

$$\alpha_0 = (1-2H_0)\mathbf{1}_{\{nq \text{ is even}\}} - H_0 \mathbf{1}_{\{nq \text{ is odd}\}}. \tag{28}$$

**Proposition 9.** *Fix $n, q \ge 2$ and $\alpha \in A_{n,q}$, assume that the function $\varphi$ belongs to $\mathcal{S}_L$ for some $L > 1$ and that $m \ge 3$. Then for any $t \in [0,1]$, $T^{\alpha_0} F_{n,q,\alpha,T}(t)$ converge in $L^2(\Omega)$ to zero as $T \to \infty$; furthermore, the family $\{(F_{n,q,\alpha,T}(t))_{t\in[0,1]}, T > 0\}$ is tight in $C([0,1])$.*

**Proof.** If $(2H_0 - 2)m > -1$, by Proposition 7, we know that $T^{-1+m(1-H_0)} F_{n,q,\alpha,T}(t)$ converges to zero in $L^2(\Omega)$ as $T \to \infty$. This implies the convergence to zero in $L^2(\Omega)$ as $T \to \infty$ of $T^{-\alpha_0} F_{n,q,\alpha,T}(t)$ because $-1 + m(1-H_0) > \alpha_0$. We should then concentrate on the case $(2H_0 - 2)m \le -1$. Once again, we shall divide the proof in two steps:

Step 1: Let us first prove the convergence in $L^2(\Omega)$. Fix $\alpha \in A_{n,q}$. We are going to show that

$$\lim_{T\to\infty} T^{2\alpha_0} \mathbb{E}\big(\big|F_{n,q,\alpha,T}(t)\big|^2\big) = 0.$$

We know that

$$T^{2\alpha_0} \mathbb{E}\big(\big|F_{n,q,\alpha,T}(t)\big|^2\big) = T^{2\alpha_0} m! \times \bigg\| \int_{[0,Tt]} ds \bigotimes_{\alpha} \big(g(s,\cdot),\ldots,g(s,\cdot)\big) \bigg\|_{L^2(\mathbb{R}^m)}^2.$$

In view of the expression for the contractions obtained in Proposition 7, it suffices to show that

$$\begin{aligned}
&\lim_{T\to\infty} T^{2\alpha_0} \int_{[0,Tt]^2} ds\, ds' \int_{\mathbb{R}^m} \int_{(-\infty,s]^n} \int_{(-\infty,s']^n} dv_1\cdots dv_n\, dv_1'\cdots dv_n'\, d\xi_1\cdots d\xi_m \\
&\quad \times \prod_{k=1}^{n} \varphi(s - v_k)\varphi\big(s' - v_k'\big) \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \big|v_i' - v_j'\big|^{(2H_0-2)\alpha_{ij}} \\
&\quad \times \prod_{k=1}^{n} \prod_{\ell=1+\beta_{j-1}}^{\beta_j} (v_k - \xi_\ell)_+^{H_0-\frac{3}{2}} \prod_{\ell=1+\beta_{j-1}}^{\beta_j} \big(v_k' - \xi_\ell\big)_+^{H_0-\frac{3}{2}} = 0.
\end{aligned}$$

Integrating in the variables $\xi$'s and using (22), it remains to show that

$$\lim_{T\to\infty} T^{2\alpha_0} \int_{[0,Tt]^2} ds\, ds' \int_{(-\infty,s]^n} \int_{(-\infty,s']^n} dv_1 \cdots dv_n\, dv_1' \cdots dv_n'$$
$$\times \prod_{k=1}^n \varphi(s-v_k)\varphi\big(s'-v_k'\big) \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \big|v_i' - v_j'\big|^{(2H_0-2)\alpha_{ij}}$$
$$\times \prod_{k=1}^n \big|v_k - v_k'\big|^{(2H_0-2)\beta_k} = 0.$$

Set

$$\Phi_T := T^{2\alpha_0} \int_{[0,Tt]^2} ds\, ds' \int_{(-\infty,s]^n} \int_{(-\infty,s']^n} dv_1 \cdots dv_n\, dv_1' \cdots dv_n'$$
$$\times \prod_{k=1}^n \varphi(s-v_k)\varphi\big(s'-v_k'\big) \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \big|v_i' - v_j'\big|^{(2H_0-2)\alpha_{ij}}$$
$$\times \prod_{k=1}^n \big|v_k - v_k'\big|^{(2H_0-2)\beta_k}.$$

Making the change of variables $w_k = s - v_k$, $w_k' = s' - v_k'$ for $k = 1, \dots, n$, yields

$$\Phi_T = T^{2\alpha_0} \int_{[0,Tt]^2} ds\, ds' \int_{\mathbb{R}_+^n} \int_{[0,\infty)^n} dw_1 \cdots dw_n\, dw_1' \cdots dw_n'$$
$$\times \prod_{k=1}^n \varphi(w_k)\varphi\big(w_k'\big) \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} \big|w_i' - w_j'\big|^{(2H_0-2)\alpha_{ij}}$$
$$\times \prod_{k=1}^n \big|s - s' - w_k + w_k'\big|^{(2H_0-2)\beta_k}.$$

Now we use Fubini's theorem and make the change of variables $s - s' = \xi$ to obtain

$$\Phi_T = tT^{2\alpha_0+1} \int_{\mathbb{R}_+^{2n}} dw_1 \cdots dw_n\, dw_1' \cdots dw_n'$$
$$\times \prod_{k=1}^n \big|\varphi(w_k)\varphi\big(w_k'\big)\big| \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} \big|w_i' - w_j'\big|^{(2H_0-2)\alpha_{ij}}$$
$$\times \int_{-tT}^{tT} d\xi \prod_{k=1}^n \big|\xi - w_k + w_k'\big|^{(2H_0-2)\beta_k}.$$

We shall distinguish again two subcases:
*Case* $(2H_0-2)m < -1$: Notice that the exponent $2\alpha_0 + 1$ is negative:

(i) If $nq$ is even, then $\alpha_0 = 1 - 2H_0$ and

$$2\alpha_0 + 1 = 3 - 4H_0 < 0$$

because $H_0 > \frac{3}{4}$.

(ii) If $nq$ is odd, then $\alpha_0 = -H_0$ and

$$2\alpha_0 + 1 = 1 - 2H_0 < 0.$$

Therefore, in order to show that $\lim_{T\to\infty} \Phi_T = 0$, it suffices to check that

$$\begin{aligned} J := \int_{\mathbb{R}^{2n}} dw_1 \cdots dw_n \, dw'_1 \cdots dw'_n \prod_{k=1}^{n} \bigl|\varphi(w_k)\varphi\bigl(w'_k\bigr)\bigr| \\ \times \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} \bigl|w'_i - w'_j\bigr|^{(2H_0-2)\alpha_{ij}} \\ \times \int_{\mathbb{R}} d\xi \prod_{k=1}^{n} \bigl|\xi - w_k + w'_k\bigr|^{(2H_0-2)\beta_k} < \infty, \end{aligned} \tag{29}$$

where, by convention $\varphi(w) = 0$ if $w < 0$. We will apply the Power Counting Theorem 14 to prove that this integral is finite. We consider functions on $\mathbb{R}^{2n+1}$ with variables $\{(w_k)_{k\le n}, (w'_k)_{k\le n}, \xi\}$. The set of linear functions is

$$\begin{aligned} T = \bigl\{\omega_k, \omega'_k, 1 \le k \le n\bigr\} \cup \bigl\{w_i - w_j, w'_i - w'_j, 1 \le i < j \le n\bigr\} \\ \cup \bigl\{\xi - w_k + w'_k, 1 \le k \le n\bigr\}. \end{aligned}$$

The corresponding exponents $(\mu_M, \nu_M)$ for each $M \in T$ are $(0, -L)$ for the linear functions $w_k$ and $w'_k$ (taking into account that $\varphi \in \mathcal{S}_L$), $(2H_0 - 2)\alpha_{ij}$ for each function of the form $w_i - w_j$ or $w'_i - w'_j$ and $(2H_0 - 2)\beta_k$ for each function of the form $\xi - w_k + w'_k$.

Then $J < \infty$, provided conditions (a) and (b) are satisfied.

• *Verification of (b)*: Let $W \subset T$ be a linearly independent proper subset of $T$, and

$$d_\infty = 2n + 1 - \dim\bigl(\mathrm{Span}(W)\bigr) + \sum_{M\in T\setminus(\mathrm{Span}(W)\cap T)} \nu_M.$$

Let $S$ be the following subset of $T$: $S = \{w_k, w'_k, 1 \le k \le n\}$. Let $e = \mathrm{Card}(S \cap \mathrm{Span}(W))$. Consider the following two cases:

(i) There exists $k \le n$ such that $\xi - w_k + w'_k \in \mathrm{Span}(W) \cap T$. Then $\dim(\mathrm{Span}(W)) \ge e + 1$. As a consequence,

$$d_\infty \le 2n + 1 - (e + 1) - (2n - e)L < 0,$$

because $L > 1$ and in this case, we should have $e < 2n$ because $W$ is a proper subset of $T$.

(ii) Otherwise,

$$d_\infty \le 2n + 1 - e - (2n - e)L + (2H_0 - 2)m < 0,$$

because $L > 1$ and $(2H_0 - 2)m < -1$.

• *Verification of (a)*: A direct verification would require to solve a seemingly difficult combinatorial problem. We can simply remark that

$$\begin{aligned}
&\int_{[-1,1]^{2n}} dw_1 \cdots dw_n \, dw_1' \cdots dw_n' \\
&\quad \times \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} \left|w_i' - w_j'\right|^{(2H_0-2)\alpha_{ij}} \\
&\quad \times \int_{-1}^{1} d\xi \prod_{k=1}^{n} \left|\xi - w_k + w_k'\right|^{(2H_0-2)\beta_k} \\
=& m! \frac{1}{\beta(H_0-\frac{1}{2}, 2-2H_0)^{|\alpha|}} \mathbb{E}\left[\left(\int_0^1 I^B_{nq-2|\alpha|}\big(f(s,\cdot),\dots,f(s,\cdot)\big)\right)^2\right] < \infty
\end{aligned}$$

where $f(s,\xi_1,\dots\xi_q) = \int_{-\infty}^{+\infty} \mathbb{I}_{[-1,1]}(s-v) \prod_{j=1}^{q} (v-\xi_j)_+^{H_0-3/2} \, dv$. Since $\varphi \in \mathcal{S}_L$, $\varphi$ is bounded on $[-1,1]$. This implies that (a) is verified by the converse side of the Power Counting theorem.

*Case* $(2H_0-2)m = -1$: In this case, we can apply Hölder and Jensen inequalities to $\Phi_T$ in order to get

$$\Phi_T \le T^{2\alpha_0+1} A^{\frac{\epsilon}{1+\epsilon}} B^{\frac{1}{1+\epsilon}},$$

with $2\alpha_0 + 1 < 0$, $A = (\int_{\mathbb{R}} |\varphi(w)|\, dw)^{2n}$ and

$$\begin{aligned}
B = &\int_{\mathbb{R}^{2n}} dw_1 \cdots dw_n \, dw_1' \cdots dw_n' \prod_{k=1}^{n} \left|\varphi(w_k)\varphi\big(w_k'\big)\right| \\
&\times \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0'-2)\alpha_{ij}} \left|w_i' - w_j'\right|^{(2H_0'-2)\alpha_{ij}} \\
&\times \int_{\mathbb{R}} d\xi \prod_{k=1}^{n} \left|\xi - w_k + w_k'\right|^{(2H_0'-2)\beta_k},
\end{aligned}$$

where $H_0' = H_0(1+\epsilon) - \epsilon$. If $\epsilon$ is small enough, $H_0'$ can still be expressed as $1 - \frac{1-H'}{q}$ for some $\frac{1}{2} < H' < H$. Moreover, in this case $(2H_0'-2)m < -1$ so we are exactly in the situation of the previous case, and the integral $B$ is finite.

Step 2: Using the same arguments as previously and the hypercontractivity property, we deduce that there exists a constant $K > 0$ such that for all $0 \le s < t \le 1$,

$$\mathbb{E}\left(\left|F_{n,q,\alpha,T}(t) - F_{n,q,\alpha,T}(s)\right|^4\right) \le K|t-s|^2,$$

which proves the tightness in $C([0,1])$. □

It remains to study what happens when $n = 1$. The proof of Proposition 10 is very similar to that of Proposition 7 (although much simpler) and details are left to the reader.

**Proposition 10.** *Fix $q \geq 1$ and assume the function $\varphi$ belongs to $L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Then the finite-dimensional distributions of the process*

$$G_T(t) := T^{q(1-H_0)-1} \int_0^{Tt} ds I_q^B\big(g(s,\cdot)\big), \quad t \in [0,1], \tag{30}$$

*where $g(s,\cdot)$ is defined in (18), converge in law to those of a $q$th Hermite process of Hurst parameter $1-q(1-H_0)$ multiplied by the constant $c_{H_0,q}^{-1} \int_0^\infty \varphi(w)\,dw$, and the family $\{(G_T(t))_{t\in[0,1]}, T>0\}$ is tight in $C([0,1])$.*

We are now ready to make the proof of Theorem 2.

*Proof of Theorem 2.* It suffices to consider the decomposition (21) and to apply the results shown in Propositions 7 and 9. □

# 5. Proof of Theorem 1

Let $Z$ be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$, and let $\varphi \in L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Consider the moving average process $X$ defined by

$$X(t) = \int_{-\infty}^t \varphi(t-u)\,dZ_u, \quad t \geq 0,$$

which is easily checked to be a stationary centered Gaussian process. Denote by $\rho : \mathbb{R} \to \mathbb{R}$ the correlation function of $X$, that is, $\rho(t-s) = \mathbb{E}[X(t)X(s)]$, $s,t \geq 0$. By multiplying the function $\varphi$ by a constant if necessary, we can assume without loss of generality that $\rho(0) = 1$ $(= \mathrm{Var}(X(t))$ for all $t)$. Let $P(x) = \sum_{n=0}^N a_n x^n$ be a real-valued polynomial function, and let $d$ denotes its centered Hermite rank.

## 5.1. Proof of (6)

In this section, we assume that $d \geq 1$ and that $H \in (1-\frac{1}{2d}, 1)$, and our goal is to show that

$$T^{d(1-H)-1}\left\{\int_0^{Tt} \big(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\big)\,ds\right\}_{t\in[0,1]}$$

converges in distribution in $C([0,1])$ to a Hermite process of index $d$ and Hurst parameter $1-d(1-H)$, up to some multiplicative constant $C_2$. Since $P$ has centered Hermite rank $d$, it can be rewritten as

$$P(x) = \mathbb{E}\big[P(X(s)\big] + \sum_{l=d}^N b_l H_l(x),$$

for some $b_d, \ldots, b_N \in \mathbb{R}$, with $b_d \neq 0$ and $H_l$ the $l$th Hermite polynomial. As a result, we have

$$\int_0^{Tt} \big(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\big)\,ds = \sum_{l=d}^N b_l (c_{H,1})^l \int_0^{Tt} I_l^B\big(g(s,\cdot)^{\otimes l}\big)\,ds,$$

and the desired conclusion follows thanks to Propositions 7 and 10.

### 5.2. Proof of (5)

In this section, we assume that $d \geq 2$ and that $H \in (\frac{1}{2}, 1 - \frac{1}{2d})$, and our goal is to show that

$$T^{-\frac{1}{2}} \left\{ \int_0^{Tt} \big(P\big(X(s)\big) - \mathbb{E}\big[P\big(X(s)\big)\big]\big)\, ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0, 1])$ to a standard Brownian motion $W$, up to some multiplicative constant $C_1$. To do so, we will rely on the Breuer–Major theorem, which asserts that the desired conclusion holds as soon as

$$\int_{\mathbb{R}} \big|\rho(s)\big|^d\, ds < \infty, \tag{31}$$

where $\rho(s) = \mathbb{E}[X(s)X(0)]$ (see, e.g., [5] for a continuous version of the Breuer–Major theorem).

The rest of this section is devoted to checking that (31) holds true. Let us first compute $\rho$:

$$\begin{aligned}
\rho(t-s) &= \mathbb{E}\big[X(t)X(s)\big] \\
&= H(2H-1) \iint_{\mathbb{R}^2} \varphi(t-v)\mathbf{1}_{(-\infty,t]}(v)\varphi(s-u)\mathbf{1}_{(-\infty,s]}(u)|v-u|^{2H-2}\, du\, dv \\
&= H(2H-1) \iint_{\mathbb{R}^2} \varphi(u)\varphi(v)|t-s-v+u|^{2H-2}\, du\, dv,
\end{aligned}$$

with the convention that $\varphi(u) = 0$ if $u < 0$. This allows us to write

$$\rho(s) = c_H\big[\tilde{\varphi} * \big(I^{2H-1}\varphi\big)\big](s),$$

where $\tilde{\varphi}(u) = \varphi(-u)$, $I^{2H-1}$ is the fractional integral operator of order $2H - 1$ and $c_H$ is a constant depending on $H$. As a consequence, applying Young's inequality and Hardy-Littlewood's inequality (see [17], Theorem 1) yields

$$\|\rho\|_{L^d(\mathbb{R})} \leq c_H \|\varphi\|_{L^p(\mathbb{R})} \big\|I^{2H-1}\varphi\big\|_{L^q(\mathbb{R})} \leq c_{H,p}\|\varphi\|^2_{L^p(\mathbb{R})},$$

where $\frac{1}{d} = \frac{1}{p} + \frac{1}{q} - 1$ and $\frac{1}{q} = \frac{1}{p} - (2H-1)$. This implies $p = (H + \frac{1}{2d})^{-1}$ and we have $\|\varphi\|_{L^p(\mathbb{R})} < \infty$, because $p \in (1, \frac{1}{H})$ and $\varphi \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$. The proof of (5) is complete.

## 6. The stationary Hermite–Ornstein–Uhlenbeck process

We dedicate this section to the study of the extension of the Ornstein–Uhlenbeck process to the case where the driving process is a Hermite process. To our knowledge, there is not much literature about this object. Among the few existing references, we mention [16] and [12]. The special case in which the driving process is a fractional Brownian motion has been, in contrast, well studied, see, for instance, [6]. In what follows, we will prove a first-order ergodic theorem for the stationary Hermite–Ornstein–Uhlenbeck process. Then, we will use Theorem 2 to study its second order fluctuations.

Let $\alpha > 0$. Consider the function $\varphi(s) = e^{-\alpha s}\mathbb{I}_{s>0}$ and let $Z^{H,q}$ be a Hermite process of order $q \geq 1$ and Hurst index $H > \frac{1}{2}$. Then $\varphi \in S_L$ for all $L > 0$, and we can define the stationary Hermite–Ornstein–Uhlenbeck process as:

$$(U_t)_{t\geq 0} = \int_{-\infty}^t \varphi(t-s)\, dZ_s^{H,q}. \tag{32}$$

As its name suggests, this process is strongly stationary, that is, for any $h > 0$ the processes $(U_t)_{t\geq 0}$ and $(U_{t+h})_{t\geq 0}$ have the same finite-dimensional distributions. We then state the following general ergodic type result.

**Proposition 11.** *Let $(u_t)_{t\geq 0}$ be a real valued process of the form $u_t = I_q^B(f_t)$, where $f_t \in L_s^2(\mathbb{R}^q)$ for each $t \geq 0$. Assume that $u$ is strongly stationary, has integrable sample paths and satisfies, for each $1 \leq r \leq q$,*

$$\|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})} \underset{s\to\infty}{\longrightarrow} 0.$$

*Then, for all measurable function such that $\mathbb{E}[|f(u_0)|] < +\infty$,*

$$\frac{1}{T}\int_0^T f(u_s)\,ds \underset{T\to\infty}{\overset{a.s}{\longrightarrow}} \mathbb{E}\big[f(u_0)\big].$$

**Proof.** According to Theorem 1.3 in [10], the process $u$ is strongly mixing if for all $t > 0$ and $1 \leq r \leq q$, the following convergence holds

$$\|f_t \otimes_r f_{t+s}\|_{L^2(\mathbb{R}^{2q-2r})} \underset{s\to\infty}{\longrightarrow} 0.$$

Taking into account that $u$ is strongly stationary, we can write

$$\|f_t \otimes_r f_{t+s}\|_{L^2(\mathbb{R}^{2q-2r})} = \|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})},$$

and the conclusion follows immediately from Birkhoff's continuous ergodic theorem. □

We can now particularize to the Hermite–Ornstein–Uhlenbeck process.

**Theorem 12.** *Let $U$ be the Hermite–Ornstein–Uhlenbeck process defined by (32). Let $f$ be a measurable function such that $|f(x)| \leq \exp(|x|^\gamma)$ for some $\gamma < \frac{2}{q}$. Then,*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(U_s)\,ds = \mathbb{E}\big[f(U_0)\big] \quad a.s.$$

**Proof.** We shall prove that the process $U$ verifies the conditions of Proposition 11. We have $U_t = I_q^B(f_t)$ with

$$f_t(x_1,\dots,x_q) = c_{H,q}\mathbb{I}_{[-\infty,t]^q}(x_1,\dots,x_q)\int_{x_1\vee\cdots\vee x_q}^t e^{-\alpha(t-u)}\prod_{i=1}^q (u-x_i)^{H_0-\frac{3}{2}}\,du.$$

*Step 1.* Let us first show the mixing condition, that is

$$\lim_{s\to\infty}\|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})} = 0$$

for all $r \in \{1,\dots,q\}$. We can write

$$\begin{aligned} &f_0 \otimes_r f_s(y_1,\dots,y_{2q-2r}) \\ &\quad= c_{H,q}^2\int_{(-\infty,0]^r}\left(\int_{x_1\vee\cdots\vee x_r\vee y_1\cdots\vee y_{q-r}}^0 e^{\alpha u}\prod_{i=1}^r\prod_{j=1}^{q-r}(u-x_i)^{H_0-\frac{3}{2}}(u-y_j)^{H_0-\frac{3}{2}}\,du\right)\end{aligned}$$

$$
\begin{aligned}
&\times \left( \int_{x_1 \vee \cdots \vee x_r \vee y_{q-r+1} \cdots \vee y_{2q-2r}}^{s} e^{-\alpha(s-u)} \right. \\
&\left. \times \prod_{i=1}^{r} \prod_{j=q-r+1}^{2q-2r} (u-x_i)^{H_0-\frac{3}{2}} (u-y_j)^{H_0-\frac{3}{2}} \, du \right) dx_1 \cdots dx_r \\
&= c_{H,q}^2 \int_{y_1 \vee \cdots \vee y_{q-r}}^{0} e^{\alpha u} \int_{y_{q-r+1} \vee \cdots \vee y_{2q-2r}}^{s} e^{-\alpha(s-v)} \\
&\times \left( \int_{(-\infty, u \wedge v]} (u-x)^{H_0-\frac{3}{2}} (v-x)^{H_0-\frac{3}{2}} \, dx \right)^r \\
&\times \prod_{j=1}^{q-r} \prod_{l=q-r+1}^{2q-2r} (u-y_j)^{H_0-\frac{3}{2}} (v-y_l)^{H_0-\frac{3}{2}} \, dv \, du \\
&= c_{H,q}^2 \beta\left(H_0 - \frac{1}{2}, 2-2H_0\right)^r \int_{y_1 \vee \cdots \vee y_{q-r}}^{0} e^{\alpha u} \int_{y_{q-r+1} \vee \cdots \vee y_{2q-2r}}^{s} e^{-\alpha(s-v)} |u-v|^{r(2H_0-2)} \\
&\times \prod_{j=1}^{q-r} \prod_{l=q-r+1}^{2q-2r} (u-y_j)^{H_0-\frac{3}{2}} (v-y_l)^{H_0-\frac{3}{2}} \, dv \, du,
\end{aligned}
$$

where we used again the identity (22). We then have

$$
\begin{aligned}
&\| f_0 \otimes_r f_s \|_{L^2(\mathbb{R}^{2q-2r})}^2 \\
&\quad = c_{H,q}^4 \beta\left(H_0 - \frac{1}{2}, 2-2H_0\right)^{2q} \\
&\quad \times \int_{(-\infty,0]^2} \int_{(-\infty,s]^2} e^{\alpha(u+u_1)} e^{-\alpha(2s-(v+v_1))} |u-u_1|^{(q-r)(2H_0-2)} \\
&\quad \times |v-v_1|^{(q-r)(2H_0-2)} |u-v|^{r(2H_0-2)} |u_1-v_1|^{r(2H_0-2)} \, dv_1 \, dv \, du_1 \, du \\
&\quad \leq c_{H,q}^4 \beta\left(H_0 - \frac{1}{2}, 2-2H_0\right)^{2q} A_0 A_s R_s^2,
\end{aligned}
$$

with

$$
A_x = \left( \int_{(-\infty,x]^2} e^{-q\alpha(2x-(u+u_1))} |u-u_1|^{q(2H_0-2)} \, du \, du_1 \right)^{\frac{1}{a}}
$$

and

$$
R_s = \left( \int_{-\infty}^{0} \int_{-\infty}^{s} e^{-q\alpha(s-(u+v))} |u-v|^{q(2H_0-2)} \, dv \, du \right)^{\frac{1}{b}},
$$

where we used the Hölder inequality with $a = \frac{q}{q-r}$, $b = \frac{q}{r}$. Making the change of variable $x - u = v$, $x - u_1 = v_1$, we obtain

$$\int_{(-\infty,x]^2} e^{-q\alpha(2x-(u+u_1))}|u - u_1|^{q(2H_0-2)}\, du\, du_1 = \int_{[0,\infty)^2} e^{-q\alpha(v+v_1)}|v - v_1|^{q(2H_0-2)}\, du\, du_1 < \infty.$$

On the other hand, we have $q(2H_0 - 2) = 2H - 2$, so

$$R_s^b = \operatorname{Cov}\big(U_0^H, U_s^H\big)$$

where $U^H$ is a stationary Ornstein Uhlenbeck process driven by a fractional Brownian motion of index $H$ (and with $\alpha^H = q\alpha$). According to [6], Lemma 2.2, one has $R_s^b = O_{s\to\infty}(s^{-H})$, implying in turn that $\lim_{s\to\infty} R_s = 0$ and concluding the proof of the mixing condition.

*Step 2.* We now show the integrability condition $\mathbb{E}[|f(U_0)|] < \infty$. From the results of [6], we have $\mathbb{E}[U_0^2] = \frac{1}{\alpha^{2H}}\Gamma(2H)$. A power series development yields

$$\mathbb{E}\big[\big|f(U_0)\big|\big] \le \sum_{k=0}^{\infty} \frac{1}{k!}\mathbb{E}\big[|U_0|^{\gamma k}\big],$$

where $U_0$ is an element of the $q$th Wiener chaos. By the hypercontractivity property, for all $k \ge \frac{2}{\gamma}$,

$$\mathbb{E}\big[|U_0|^{\gamma k}\big] \le g(k) := (k-1)^{\frac{\gamma q k}{2}}\left(\frac{1}{\alpha^{2H}}\Gamma(2H)\right)^{\frac{\gamma k}{2}}.$$

Stirling formula allows us to write

$$\frac{g(k)}{k!} \sim_{k\to\infty} \frac{(k-1)^{\frac{\gamma q k}{2}}}{k^k}\frac{(\frac{1}{\alpha^{2H}}\Gamma(2H))^{\frac{\gamma k}{2}} e^k}{\sqrt{2\pi k}}, \tag{33}$$

and the associated series converges if $\gamma q < 2$. □

The next result analyzes the fluctuations in the ergodic theorem proved in Theorem 12.

**Theorem 13.** *(A)* [Case $q = 1$] *Let $f$ be in $L^2(\mathbb{R}, \gamma)$ for $\gamma = \mathcal{N}(0, \frac{\Gamma(2H)}{\alpha^{2H}})$. We denote by $(a_i)_{i\ge 0}$ the coefficients of $f$ in its Hermite expansion, and we let $d$ be the centered Hermite rank of $f$. Then,*

- *if* $\frac{1}{2} < H < 1 - \frac{1}{2d}$,

$$\frac{1}{\sqrt{T}}\int_0^{Tt}\big(f(U_s) - \mathbb{E}\big[f(U_0)\big]\big)\, ds \xrightarrow[T\to\infty]{f.d.d} c_{f,H} W_t,$$

- *if* $H = 1 - \frac{1}{2d}$,

$$\frac{1}{\sqrt{T\log T}}\int_0^{Tt}\big(f(U_s) - \mathbb{E}\big[f(U_0)\big]\big)\, ds \xrightarrow[T\to\infty]{f.d.d} c_{f,H} W_t,$$

- *if* $H > 1 - \frac{1}{2d}$,

$$T^{q(1-H)-1} \int_0^{Tt} \big(f(U_s) - \mathbb{E}\big[f(U_0)\big]\big)\, ds \xrightarrow[T\to\infty]{f.d.d} c_{f,H} Z_t^{d,H},$$

*where* $Z^{d,H}$ *is a Hermite process of order d and index* $d(H-1)+1$, $W$ *is a Brownian motion and*

$$c_{f,H} = \begin{cases} \sqrt{\sum_{k\geq d} k! a_k^2 \int_{\mathbb{R}_+} \big|\rho(s)\big|^k} & \text{if } H < 1 - \dfrac{1}{2d} \\ a_d \sqrt{d! \dfrac{3}{16\alpha^2}} & \text{if } H = 1 - \dfrac{1}{2d} \\ a_d \sqrt{d! \dfrac{H^d \Gamma(2H)^d}{\alpha^{2Hd}}} & \text{if } H > 1 - \dfrac{1}{2d} \end{cases} \tag{34}$$

*with*

$$\rho(s) = \mathbb{E}[U_s U_0] = \int_{-\infty}^0 \int_{-\infty}^s e^{-\alpha(s-(u+v))} |u - v|^{2H-2}\, du\, dv.$$

*Moreover, if* $f \in L^p(\mathbb{R}, \gamma)$ *for some* $p > 2$, *the previous convergences holds true in the Banach space* $C([0, 1])$.

*(B)* [Case $q > 1$] *Let P be a real valued polynomial. Then, the conclusions of Theorem 2 apply to* $U$.

**Proof.** Except for $H = 1 - \frac{1}{2d}$ in Part A, this is a direct consequence of Theorems 1 and 2. The convergence in the critical case can be checked through easy but tedious computations, by reducing to the case where $f$ is the $d$th Hermite polynomial. Details are left to the reader. □

# Appendix

In this section, we present two technical lemmas that play an important role along the paper. First, we shall reproduce a very useful result from [19].

**Theorem 14 (Power Counting t Theorem).** *Let* $T = \{M_1, \dots, M_K\}$ *a set of linear functionals on* $\mathbb{R}^n$, $\{f_1, \dots, f_K\}$ *a set of real measurable functions on* $\mathbb{R}^n$ *such that there exist real numbers* $(a_i, b_i, \mu_i, \nu_i)_{1\leq i\leq K}$, *satisfying for each* $i = 1, \dots, K$,

$$\begin{aligned} &0 < a_i \leq b_i, \\ &\big|f_i(x)\big| \leq |x|^{\mu_i} \quad \text{if } |x| \leq a_i, \\ &\big|f_i(x)\big| \leq |x|^{\nu_i} \quad \text{if } |x| \geq b_i, \\ &f_i \text{ is bounded over } [a_i, b_i]. \end{aligned}$$

*For a linearly independent subset of W of T, we write* $S_T(W) = \mathrm{Span}(M) \cap T$. *We also define*

$$d_0(W) = \dim\big(\mathrm{Span}(W)\big) + \sum_{i:M_i\in S_T(W)} \mu_i,$$

$$d_\infty(W) = n - \dim\big(\mathrm{Span}(W)\big) + \sum_{i:M_i\in T\setminus S_T(W)} \nu_i.$$

*Assume* $\dim(\mathrm{Span}(T)) = n$. *Then, the two conditions* $(a) : d_0(W) > 0$ *for all linearly independent subsets* $W \subset T$, $(b) : d_\infty(W') < 0$ *for all linearly independent proper subsets* $W' \subset T$, *imply*

$$\int_{\mathbb{R}^n} \prod_{i=1}^{K} \big| f_i\big(M_i(x)\big)\big| \, dx < \infty \tag{35}$$

*Moreover, assume that* $|f_i(x)| = |x|^{\mu_i}$ *if* $|x| \le a_i$, *Then* $\int_{[-1,1]^n} \prod_{i=1}^{K} |f_i(M_i(x))| \, dx < \infty$, *if an only if for any linearly independent subset* $W \subset T$ *condition* $(a)$ *holds.*

The next lemma is an application of the Hardy–Littlewood–Sobolev inequality,

**Lemma 15.** *Fix* $n, q \ge 2$ *and* $\alpha \in A_{n,q}$. *Recall* $H$ *and* $H_0$ *from* (1). *Assume* $\varphi \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$ *Then*

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} \big|\varphi(\eta_k)\big| \prod_{1 \le i < j \le n} |\eta_i - \eta_j|^{(2H_0-2)\alpha_{ij}} \, d\eta_1 \dots d\eta_n < \infty.$$

**Proof.** We are going to use the multilinear Hardy–Littlewood–Sobolev inequality, that we recall here for the convenience of the reader (see [1], Theorem 6): if $f : \mathbb{R} \to \mathbb{R}$ is a measurable function, if $p \in (1, n)$ and if the $\gamma_{ij} \in (0, 1)$ are such that $\sum_{1 \le i < j \le n} \gamma_{ij} = 1 - \frac{1}{p}$, then there exists $c_{p,\gamma} > 0$ such that

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} \big| f(u_k)\big| \prod_{1 \le i < j \le n} |u_i - u_j|^{-\gamma_{ij}} \, du_1 \dots du_n \le c_{p,\gamma} \left( \int_{\mathbb{R}} \big| f(u)\big|^p \, du \right)^{\frac{n}{p}}. \tag{36}$$

Set $p = 1/(1 - (1 - H)\frac{2|\alpha|}{nq})$. Since $2|\alpha| \le nq$, we have that $p > 1$. On the other hand, since $H > \frac{1}{2}$, one has $nH > \frac{n}{2} \ge 1$; this implies that $(1 - H)\frac{2|\alpha|}{q} < (1 - H)n < n - 1$, that is, $p < n$. Moreover, set $\gamma_{ij} = (2 - 2H_0)\alpha_{ij} = (1 - H)\frac{2\alpha_{ij}}{q} \in (0, 1)$; we have $\sum_{1 \le i < j \le n} \gamma_{ij} = 2(1 - H)\frac{|\alpha|}{q} \le (1 - H)n < n - 1$. We deduce from (36) that

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} \big|\varphi(\eta_k)\big| \prod_{1 \le i < j \le n} |\eta_i - \eta_j|^{(2H_0-2)\alpha_{ij}} \, d\eta_1 \dots d\eta_n \le c_{p,\gamma} \left( \int_{-\infty}^{\infty} \big|x(u)\big|^p \, du \right)^{\frac{n}{p}}.$$

But $p \in (1, \frac{1}{H})$ and $x \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$, so the claim follows. □

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