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Total variation estimates in the Breuer–Major theorem

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Abstract. This paper provides estimates for the convergence rate of the total variation distance in the framework of the Breuer–Major theorem, assuming some smoothness properties of the underlying function. The results are proved by applying new bounds for the total variation distance between a random variable expressed as a divergence and a standard Gaussian random variable, which are derived by a combination of techniques of Malliavin calculus and Stein's method. The representation of a functional of a Gaussian sequence as a divergence is established by introducing a shift operator on the expansion in Hermite polynomials. Some applications to the asymptotic behavior of power variations of the fractional Brownian motions and to the estimation of the Hurst parameter using power variations are presented.

Résumé. Cet article fournit des estimations pour la vitesse de convergence de la variation totale dans le cadre du théorème de Breuer–Major, en supposant quelques propriétés de régularité de la fonction sous-jacente. Les résultats se démontrent en appliquant des nouvelles bornes pour la distance en variation totale entre une variable aléatoire qui s'exprime comme une divergence et une variable aléatoire gaussienne, qu'on obtient en combinant des techniques du calcul de Malliavin et la méthode de Stein. On établit la représentation d'une fonctionnelle d'une suite gaussienne comme une divergence en introduisant un opérateur de décalage sur le développement en polynômes d'Hermite. Quelques applications au comportement asymptotique des variations puissance pour le mouvement Brownien fractionnaire et à l'estimation du paramètre de Hurst sont aussi présentées.

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1. Introduction

Consider a centered stationary Gaussian family of random variables $X = \{X_n, n \in \mathbb{Z}\}$ with unit variance. For all $k \in \mathbb{Z}$, set $\rho(k) = \mathbb{E}(X_0 X_k)$, so $\rho(0) = 1$ and $\rho(k) = \rho(-k)$. We say that a function $g \in L^2(\mathbb{R}, \gamma)$, where γ is the standard Gaussian measure, has *Hermite rank* $d \geq 1$ if

$$g(x) = \sum_{m=d}^{\infty} c_m H_m(x), \tag{1.1}$$

where $c_d \neq 0$ and H_m is the mth Hermite polynomial. We will make use of the following condition that relates the covariance function ρ to the Hermite rank of a function g:

$$\sum_{j \in \mathbb{Z}} \left|\rho(j)\right|^d < \infty. \tag{1.2}$$

Let us recall the celebrated Breuer–Major theorem for functionals of the stationary Gaussian sequence X (see [7]).

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Theorem 1.1 (Breuer–Major theorem). *Consider a centered stationary Gaussian family of random variables* $X = \{X_n, n \in \mathbb{Z}\}$ *with unit variance and covariance function* ρ. *Let* $g \in L^2(\mathbb{R}, \gamma)$ *be a function with Hermite rank* $d \geq 1$ *and expansion* (1.1). *Suppose that* (1.2) *holds true. Set*

$$\sigma^2 = \sum_{m=d}^{\infty} m! c_m^2 \sum_{k \in \mathbb{Z}} \rho(k)^m. \tag{1.3}$$

Then the sequence

$$Y_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} g(X_j) \tag{1.4}$$

converges in law to the normal distribution $N(0, \sigma^2)$.

The purpose of this paper is to show that, under suitable regularity assumptions on the function g, the sequence Y_n/σ_n, where $\sigma_n^2 = \mathbb{E}(Y_n^2)$, converges in the total variation distance to the standard normal law $N(0, 1)$, and we can estimate the rate of convergence in terms of the covariance function ρ. To show these results we will apply a combination of Stein's method for normal approximations and techniques of Malliavin calculus. The combination of Stein's method with Malliavin calculus to study normal approximations was first developed by Nourdin and Peccati (see the pioneering work [12] and the monograph [13]). For random variables on a fixed Wiener chaos, these techniques provide a quantitative version of the *Fourth Moment Theorem* proved by Nualart and Peccati in [19]. A basic result in this direction is the following proposition. Along the paper Z will denote a $N(0, 1)$ random variable.

Proposition 1.2. *Let* F *be a random variable in the* q*th* ($q \geq 2$) *Wiener chaos with unit variance. Then*

$$d_{\mathrm{TV}}(F, Z) \leq 2\sqrt{\operatorname{Var}\left(\frac{1}{q}\|DF\|_{\mathfrak{H}}^2\right)} \leq 2\sqrt{\frac{q-1}{3q}\left(\mathbb{E}\left(F^4\right) - 3\right)}, \tag{1.5}$$

where D *denotes the derivative in the sense of Malliavin calculus and* d_{TV} *is the total variation distance.*

In the context of the Breuer–Major theorem, this result can be applied to obtain a rate of convergence for the total variation distance $d_{\mathrm{TV}}(Y_n/\sigma_n, Z)$, provided $g = H_d$ and condition (1.2) holds (see [12]). Later on, the rate of convergence was improved in [4] using an approach based on the spectral density.

In the reference [14], with an intensive application of Stein's method combined with Malliavin calculus, Nourdin and Peccati improved the estimate (1.5), obtaining the following matching upper and lower bounds for the total variation distance.

Proposition 1.3. *Let* F *be a random variable in the* q*th* ($q \geq 2$) *Wiener chaos with unit variance. Then, there exist constants* $C_1, C_2 > 0$, *depending on* q, *such that*

$$C_1 \max\left\{\left|\mathbb{E}\left(F^3\right)\right|, \mathbb{E}\left(F^4\right) - 3\right\} \leq d_{\mathrm{TV}}(F, Z) \leq C_2 \max\left\{\left|\mathbb{E}\left(F^3\right)\right|, \mathbb{E}\left(F^4\right) - 3\right\}.$$

In the paper [5], it is proved that that $|\mathbb{E}(F^3)| \leq C\sqrt{\mathbb{E}(F^4) - 3}$, which trivially indicates that the bound in Proposition 1.3 is better than (1.5). Furthermore, using an analytic characterization of cumulants and Edgeworth-type expansions, the authors of [5] proved that, for a normalized sequence F_n which belongs to the qth Wiener chaos and converges to Z in distribution as $n \to \infty$, the rate of convergence of the total variation distance is characterized by the third and fourth cumulants.

The literature on the rate of convergence for normal approximations is focused on random variables on a fixed Wiener chaos. The goal of this paper is to provide an answer to the following question:

Question. To what extent Propositions 1.2 and 1.3 can be generalized to random variables that are not in a fixed chaos and how this approach is applied in the context of the Breuer Major theorem?

We cannot expect that, in this more general framework, the convergence to a normal distribution is characterized by the third and fourth cumulants, and new functionals will appear. In the first part of the paper, we consider random variables that can be written as divergences, that is $F = \delta(u)$, where δ is the adjoint of the derivative operator in the Malliavin calculus. We will use Stein's method and Malliavin calculus to provide three different bounds (see Propositions 3.1, 3.2 and 3.3) for $d_{\mathrm{TV}}(F, Z)$. If F is in some fixed chaos, the bound in Proposition 3.1 should be the same as that of Proposition 1.2 and the bound in Proposition 3.2 should coincide with that of Proposition 1.3. Actually, the proof of Proposition 3.2 has been inspired by the approach used to derive the upper bound in Proposition 1.3.

The second part of the paper is devoted to derive upper bounds for the total variation distance in the context of the Breuer–Major theorem, applying the estimates provided by Propositions 3.1, 3.2 and 3.3. To do this, we need to represent $g(X_j)$ as a divergence $\delta(u)$. A basic ingredient for this representation is the shift operator T_1 (see formula (2.6) below) defined using the expansion of g into a series of Hermite polynomials. It turns out that the representation obtained through T_1 coincides with the classical representation $F = \delta(-DL^{-1}F)$, introduced in [18], that plays a fundamental role in normal approximations by Stein's method and Malliavin calculus. The representation of $g(X_j)$ as a divergence (or an iterated divergence) allows us to apply the integration by parts in the context of Malliavin calculus (or duality between the derivative and divergence operators), which leads to estimates of the expectation of products of random variables of the form $g^{(k)}(X_j)$. For this approach to work, we are going to assume that the function g belongs to the Sobolev space $\mathbb{D}^{k,p}(\mathbb{R}, \gamma)$, for some k and p, of functions that have k weak derivatives with moments of order p with respect to γ.

In this way we have been able to obtain the following results in the framework of Theorem 1.1, for functions of Hermite rank one or two.

(i) For functions g of Hermite rank $d = 1$, assuming $g \in \mathbb{D}^{2,4}(\mathbb{R}, \gamma)$, we have (see Theorem 4.2 below)

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z) \le Cn^{-\frac{1}{2}}.$$

(ii) For functions g of Hermite rank $d = 2$, assuming $g \in \mathbb{D}^{6,8}(\mathbb{R}, \gamma)$, we have (see Theorem 4.3 below)

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z) \le Cn^{-\frac{1}{2}}\left(\sum_{|k|\le n} \left|\rho(k)\right|^{\frac{3}{2}}\right)^2. \tag{1.6}$$

It is worth noticing that the upper bound (1.6) coincides with the optimal rate for the Hermite polynomial $g(x) = x^2 - 1$ obtained in [5]. Furthermore, in Theorem 4.3, rates worse than (1.6) are established under less smoothness on the function g.

For functions g of Hermite rank $d \ge 3$ and assuming $g \in \mathbb{D}^{3d-2,4}(\mathbb{R}, \gamma)$, we have established in Theorem 4.7 an upper bound for the total variation distance $d_{\mathrm{TV}}(Y_n/\sigma_n, Z)$ based on Proposition 3.1, which is a slight modification of the rate derived for the Hermite polynomial H_d. Due to the complexity of the computations, the application of Proposition 3.2 in the case $d \ge 3$ has not been considered in this paper.

A difficult open problem is the derivation of lower bounds for the total variation distance in the case of a general function g. The lower bound given in Proposition 1.3 works for random variables in a fixed Wiener chaos and the approach used to derive this bound does apply to general random variables. For Hermite polynomials, a lower bound is derived in [5, page 491] by applying Stein's equation and Edgeworth-type expansions. The extension of this methodology to the case of general functions is a challenging problem not considered here.

The paper is organized as follows. Section 2 contains some preliminaries on Malliavin calculus and Stein's method, including the definition and properties of the shift operator T_1. In Section 3, we derive the three basic estimates for the total variation distance between a divergence $\delta(u)$ and a $N(0, 1)$ random variable. Section 4 contains the main results of the paper. First we thoroughly analyze the cases $d = 1$ and $d = 2$ and establish bounds for the total variation distance in the framework of the Breuer–Major theorem and later we consider the case $d \ge 3$, applying Proposition 3.1.

As an application, in Section 5 we give the convergence rates for the fractional Gaussian case. We also discuss some applications to the asymptotic behavior of power variations of the fractional Brownian motions and to the consistency of the estimator of the Hurst parameter using power variations. The Appendix contains some technical lemmas used in the proof of the main results and some inequalities, obtained as an application of the rank-one Brascamp–Lieb inequality and Hölder's inequality, which play an important role in the proofs.

2. Preliminaries

In this section, we briefly recall some notions of Malliavin calculus, Stein's method and the Brascamp–Lieb inequality. The shift operator T_1 mentioned above is also introduced here.

2.1. *Gaussian analysis*

Let $\mathfrak{H}$ be a real separable Hilbert space. For any integer $m \geq 1$, we use $\mathfrak{H}^{\otimes m}$ and $\mathfrak{H}^{\odot m}$ to denote the mth tensor product and the mth symmetric tensor product of $\mathfrak{H}$, respectively. Let $\mathbb{X} = \{\mathbb{X}(\phi) : \phi \in \mathfrak{H}\}$ denote an isonormal Gaussian process over the Hilbert space $\mathfrak{H}$. That means, $\mathbb{X}$ is a centered Gaussian family of random variables, defined on some probability space $(\Omega, \mathcal{F}, P)$, with covariance

$$\mathbb{E}\big(\mathbb{X}(\phi)\mathbb{X}(\psi)\big) = \langle \phi, \psi \rangle_{\mathfrak{H}}, \quad \phi, \psi \in \mathfrak{H}.$$

We assume that $\mathcal{F}$ is generated by $\mathbb{X}$.

We denote by $\mathcal{H}_m$ the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_m(\mathbb{X}(\varphi)) : \varphi \in \mathfrak{H}, \|\varphi\|_{\mathfrak{H}} = 1\}$, where H_m is the mth Hermite polynomial defined by

$$H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}, \quad m \geq 1,$$

and $H_0(x) = 1$. The space $\mathcal{H}_m$ is called the Wiener chaos of order m. The mth multiple integral of $\phi^{\otimes m} \in \mathfrak{H}^{\odot m}$ is defined by the identity $I_m(\phi^{\otimes m}) = H_m(\mathbb{X}(\phi))$ for any $\phi \in \mathfrak{H}$. The map I_m provides a linear isometry between $\mathfrak{H}^{\odot m}$ (equipped with the norm $\sqrt{m!}\|\cdot\|_{\mathfrak{H}^{\otimes m}}$) and $\mathcal{H}_m$ (equipped with $L^2(\Omega)$ norm). By convention, $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$.

The space $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_m$. Namely, for any square integrable random variable $F \in L^2(\Omega)$, we have the following expansion,

$$F = \sum_{m=0}^{\infty} I_m(f_m),$$

where $f_0 = \mathbb{E}(F)$, and $f_m \in \mathfrak{H}^{\odot m}$ are uniquely determined by F. This is known as the Wiener chaos expansion. If we use J_m to denote the orthogonal projection of F onto the mth Wiener chaos $\mathcal{H}_m$, we obtain $I_m(f_m) = J_m(F)$ for every $m \geq 0$.

2.2. *Malliavin calculus*

In this subsection we present some background of Malliavin calculus with respect to an isonormal Gaussian process $\mathbb{X}$. We refer the reader to [13,16] for a detailed account on this topic. For a smooth and cylindrical random variable $F = f(\mathbb{X}(\varphi_1), \dots, \mathbb{X}(\varphi_n))$, with $\varphi_i \in \mathfrak{H}$ and $f \in C_b^\infty(\mathbb{R}^n)$ (f and its partial derivatives are bounded), we define its Malliavin derivative as the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big(\mathbb{X}(\varphi_1), \dots, \mathbb{X}(\varphi_n)\big)\varphi_i.$$

By iteration, we can also define the kth derivative $D^k F$ which is an element in the space $L^2(\Omega; \mathfrak{H}^{\otimes k})$. The Sobolev space $\mathbb{D}^{k,p}$ is defined as the closure of the space of smooth and cylindrical random variables with respect to the norm $\|\cdot\|_{k,p}$ defined by

$$\|F\|_{k,p}^p = \mathbb{E}\big(|F|^p\big) + \sum_{i=1}^{k} \mathbb{E}\big(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\big),$$

for any natural number k and any real number $p \geq 1$. We define the divergence operator δ as the adjoint of the derivative operator D. Namely, an element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of δ, denoted by $\operatorname{Dom}\delta$, if there is a constant $c_u > 0$ depending on u and satisfying

$$\big|\mathbb{E}\big(\langle DF, u\rangle_{\mathfrak{H}}\big)\big| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$. If $u \in \operatorname{Dom}\delta$, the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}\big(F\delta(u)\big) = \mathbb{E}\big(\langle DF, u\rangle_{\mathfrak{H}}\big), \tag{2.1}$$

which is valid for all $F \in \mathbb{D}^{1,2}$. In a similar way, for each integer $k \geq 2$, we define the iterated divergence operator δ^k through the duality relationship

$$\mathbb{E}\big(F\delta^k(u)\big) = \mathbb{E}\big(\big\langle D^k F, u\big\rangle_{\mathfrak{H}^{\otimes k}}\big), \tag{2.2}$$

valid for any $F \in \mathbb{D}^{k,2}$, where $u \in \operatorname{Dom}\delta^k \subset L^2(\Omega; \mathfrak{H}^{\otimes k})$.

The Ornstein–Uhlenbeck semigroup $(P_t)_{t\geq 0}$ is the semigroup of operators on $L^2(\Omega)$ defined by

$$P_t F = \sum_{m=0}^{\infty} e^{-mt} I_m(f_m),$$

if F admits the Wiener chaos expansion $F = \sum_{m=0}^{\infty} I_m(f_m)$. Denote by $L = \frac{d}{dt}|_{t=0} P_t$ the infinitesimal generator of $(P_t)_{t\geq 0}$ in $L^2(\Omega)$. Then we have $LF = -\sum_{m=1}^{\infty} m J_m(F)$ for any $F \in \operatorname{Dom} L = \mathbb{D}^{2,2}$. We define the pseudo-inverse of L as $L^{-1}F = -\sum_{m=1}^{\infty} \frac{1}{m} J_m F$. We recall the following formula for any centered and square integrable random variable F,

$$L^{-1}F = -\int_0^{\infty} P_t F\, dt. \tag{2.3}$$

The basic operators D, δ and L satisfy the relation $LF = -\delta DF$, for any random variable $F \in \mathbb{D}^{2,2}$. As a consequence, any centered random variable $F \in L^2(\Omega)$ can be expressed as a divergence:

$$F = \delta\big(-DL^{-1}F\big). \tag{2.4}$$

This representation has intensively been used in normal approximations (see [18,19]).

We denote by γ the standard Gaussian measure on $\mathbb{R}$. The Hermite polynomials $\{H_m(x), m \geq 0\}$ form a complete orthonormal system in $L^2(\mathbb{R}, \gamma)$ and any function $g \in L^2(\mathbb{R}, \gamma)$ admits an orthogonal expansion of the form

$$g(x) = \sum_{m=0}^{\infty} c_m H_m(x). \tag{2.5}$$

If $g \in L^2(\mathbb{R}, \gamma)$ has the expansion (2.5), we define the operator T_1 by

$$T_1(g)(x) = \sum_{m=1}^{\infty} c_m H_{m-1}(x). \tag{2.6}$$

To simplify the notation we will write $T_1(g) = g_1$.

Suppose that F is a random variable in the first Wiener chaos of $\mathbb{X}$ of the form $F = I_1(\varphi)$, where $\varphi \in \mathfrak{H}$ has norm one. In view of the relation between Hermite polynomials and multiple stochastic integrals, it follows that for any $g \in L^2(\mathbb{R}, \gamma)$ of the form (2.5), the random variable $g(F)$ admits the Wiener chaos expansion

$$g(F) = \sum_{m=0}^{\infty} c_m I_m\big(\varphi^{\otimes m}\big). \tag{2.7}$$

For any $k \geq 2$, we can define the iterated operator $T_k = T_1 \circ \overset{k}{\cdots} \circ T_1$ by

$$T_k(g)(x) = \sum_{m=k}^{\infty} c_m H_{m-k}(x). \tag{2.8}$$

We will write $T_k(g) = g_k$.

In the next lemma we establish the connection between the shift operator T_1 defined in (2.6) and the representation of a centered and square integrable random variable as divergence given in (2.4). When the functional g has a general Hermite rank $k \geq 1$, we also provide the representation for the random variable $g(F)$ as an iterated divergence.

Lemma 2.1. *Let F be a random variable in the first Wiener chaos of $\mathbb{X}$ of the form $F = I_1(\varphi)$, where $\|\varphi\|_{\mathfrak{H}} = 1$. Suppose that $g \in L^2(\mathbb{R}, \gamma)$ is centered. Then*

$$g_1(F)\varphi = -DL^{-1}g(F).$$

As a consequence, $g(F) = \delta(g_1(F)\varphi)$. More generally, if $g \in L^2(\mathbb{R}, \gamma)$ has Hermite rank $k \geq 1$, we have the representation

$$g(F) = \delta^k\big(g_k(F)\varphi^{\otimes k}\big). \tag{2.9}$$

Proof. Using the Wiener chaos expansion (2.7), we can write

$$L^{-1}g(F)=-\sum_{m=1}^{\infty}\frac{c_m}{m}H_m(F),$$

which implies, taking into account that $H_m'=mH_{m-1}$, that

$$-DL^{-1}g(F)=\sum_{m=1}^{\infty}c_mH_{m-1}(F)\varphi=g_1(F)\varphi.$$

Property $g(F)=\delta(g_1(F)\varphi)$ is a consequence of (2.4). The proof in the general case $k\geq 2$ follows immediately by an iteration procedure. This completes the proof. □

Lemma 2.2. *Let F be a random variable in the first Wiener chaos of $\mathbb{X}$ of the form $F=I_1(\varphi)$, with $\|\varphi\|_{\mathfrak{H}}=1$. Suppose that $g\in L^2(\mathbb{R},\gamma)$ is centered. Then for any $p>1$,*

$$\|g_1(F)\|_{L^p(\Omega)}\leq c_p\|g(F)\|_{L^p(\Omega)}, \tag{2.10}$$

where $c_p>0$ is a constant depending only on p.

Proof. Observe that, using Lemma 2.1, we can write

$$\|g_1(F)\|_{L^p(\Omega)}=\|-DL^{-1}g(F)\|_{L^p(\Omega;\mathfrak{H})}.$$

Then, using (2.3), Minkowski's inequality and the last inequality in the proof of Proposition 3.2.5 of [17], we can write

$$\begin{aligned}\|-DL^{-1}g(F)\|_{L^p(\Omega;\mathfrak{H})}&\leq\left\|\int_0^{\infty}DP_tg(F)\,dt\right\|_{L^p(\Omega;\mathfrak{H})}\\&\leq\int_0^{\infty}\|DP_tg(F)\|_{L^p(\Omega;\mathfrak{H})}\,dt\\&\leq c_p'\int_0^{\infty}\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\|g(F)\|_{L^p(\Omega)}\,dt,\\&=c_p'\frac{\pi}{2}\|g(F)\|_{L^p(\Omega)}.\end{aligned}$$

This concludes the proof. □

By iteration, we obtain

$$\|g_k(F)\|_{L^p(\Omega)}\leq c_p^k\|g(F)\|_{L^p(\Omega)}, \tag{2.11}$$

for any $k\geq 2$, provided g has Hermite rank k and $F=I_1(\varphi)$, with $\|\varphi\|_{\mathfrak{H}}=1$. If g has Hermite rank strictly less than k, we can write

$$T_kg(x)=T_k\widetilde{g}(x),$$

where $\widetilde{g}(x)=\sum_{m=k}^{\infty}c_mH_m(x)$. Then,

$$\begin{aligned}\|T_kg(F)\|_{L^p(\Omega)}&\leq c_p^k\|g(F)\|_{L^p(\Omega)}+c_p^k\left\|\sum_{m=0}^{k-1}c_mH_m(F)\right\|_{L^p(\Omega)}\\&\leq c_p^k\|g(F)\|_{L^p(\Omega)}+c_p^k\cdot C\left(\sum_{m=0}^{k-1}c_m^2m!\right)^{\frac{1}{2}}\end{aligned}$$

for some constant $C > 0$, where in the second inequality we have used the equivalence of the $L^p(\Omega)$ norms on a finite sum of Wiener chaos, due to the hyprecontractivity property of the Ornstein–Uhlenbeck semigroup (see, for example, Corollary 2.8.14 in [13]).

Consider $\mathfrak{H} = \mathbb{R}$, the probability space $(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$ and the isonormal Gaussian process $\mathbb{X}(h) = h$. For any $k \geq 0$ and $p \geq 1$, denote by $\mathbb{D}^{k,p}(\mathbb{R}, \gamma)$ the corresponding Sobolev spaces of functions. Notice that if $g \in \mathbb{D}^{k,p}(\mathbb{R}, \gamma)$, and $F = I_1(\varphi)$ is an element in the first Wiener chaos of a general isonormal Gaussian process $\mathbb{X}$, then $g(F) \in \mathbb{D}^{k,p}$.

The next lemma provides a regularizing property of the operator T_k.

Lemma 2.3. *Suppose that $g \in \mathbb{D}^{j,p}(\mathbb{R}, \gamma)$ for some $j \geq 0$ and $p > 1$. Then $T_k g \in \mathbb{D}^{j+k,p}(\mathbb{R}, \gamma)$ for all $k \geq 1$.*

Proof. We can assume that g has Hermite rank k, otherwise, we just subtract the first k terms in its expansion. Then, the result is an immediate consequence of the fact that $T_k = (-DL^{-1})^k$ and the equivalence in $L^p(\mathbb{R}, \gamma)$ of the operators D and $(-L)^{1/2}$, which follows from Meyer's inequalities (see, for instance, [16]). □

Notice that T_1 and the derivative operator do not commute. We will write $(g_1)' = g_1'$, which is different from $T_1(g')$. Indeed, for any $g \in L^2(\mathbb{R}, \gamma)$, we have

$$g_1' = T_1(g') - g_2,$$

because if g has the expansion (2.5), we obtain

$$g_1'(x) = \sum_{m=2}^{\infty} c_m (m-1) H_{m-2}(x),$$

$$T_1(g')(x) = \sum_{m=2}^{\infty} c_m m H_{m-2}(x)$$

and

$$g_2(x) = \sum_{m=2}^{\infty} c_m H_{m-2}(x).$$

More generally we can show that for any $k, l \geq 1$,

$$g_k^{(l)} = \sum_{i=0}^{l} \binom{l}{i} \alpha_{k,i} T_{k+i}\big(g^{(l-i)}\big),$$

where $\alpha_{k,i} = (-1)^i k(k+1)\cdots(k+i-1)$, with the convention $\alpha_{k,i} = 1$ if $i = 0$.

2.3. *Brascamp–Lieb inequality*

In this subsection we recall a version of the rank-one Brascamp–Lieb inequality that will be intensively used through this paper (see [1,3,6] and the references therein). This inequality constitutes a generalization of both Hölder's and Young's convolution inequalities.

Proposition 2.4. *Let $2 \leq M \leq N$ be fixed integers. Consider nonnegative measurable functions $f_j : \mathbb{R} \to \mathbb{R}_+$, $1 \leq j \leq N$, and fix nonzero vectors $\mathbf{v}_j \in \mathbb{R}^M$. Fix positive numbers p_j, $1 \leq j \leq N$, verifying the following conditions*:

(i) $\sum_{j=1}^N p_j = M$,
(ii) *For any subset $I \subset \{1, \dots, M\}$, we have*

$$\sum_{j\in I} p_j \leq \dim\big(\mathrm{Span}\{\mathbf{v}_j, j \in I\}\big).$$

Then, there exists a finite constant C, depending on N, M and the p_j's such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^M} \prod_{j=1}^{N} f_j(\mathbf{k}\cdot\mathbf{v}_j) \leq C \prod_{j=1}^{N} \Big(\sum_{k\in\mathbb{Z}} f_j(k)^{1/p_j}\Big)^{p_j}. \tag{2.12}$$

2.4. *Stein's method*

Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel function such that $h \in L^1(\mathbb{R}, \gamma)$. The ordinary differential equation

$$f'(x) - xf(x) = h(x) - \mathbb{E}\big(h(Z)\big) \tag{2.13}$$

is called Stein's equation associated with h. The function

$$f_h(x) := e^{x^2/2} \int_{-\infty}^{x} \big(h(y) - \mathbb{E}\big(h(Z)\big)\big) e^{-y^2/2}\, dy$$

is the unique solution to the Stein's equation satisfying $\lim_{|x|\to\infty} e^{-x^2/2} f_h(x) = 0$. Moreover, if h is bounded, f_h satisfies

$$\|f_h\|_\infty \le \sqrt{\frac{\pi}{2}} \big\| h - \mathbb{E}\big(h(Z)\big)\big\|_\infty \tag{2.14}$$

and

$$\big\|f_h'\big\|_\infty \le 2\big\| h - \mathbb{E}\big(h(Z)\big)\big\|_\infty \tag{2.15}$$

(see [13] and the references therein).

We recall that the total variation distance between the laws of two random variables F, G is defined by

$$d_{\mathrm{TV}}(F, G) = \sup_{B \in \mathcal{B}(\mathbb{R})} \big| P(F \in B) - P(G \in B)\big|,$$

where the supremum runs over all Borel sets $B \subset \mathbb{R}$. Substituting x by F in Stein's equation (2.13) and using the inequalities (2.14) and (2.15) lead to the fundamental estimate

$$d_{\mathrm{TV}}(F, Z) = \sup_{f \in \mathcal{C}^1(\mathbb{R}), \|f\|_\infty \le \sqrt{\pi/2}, \|f'\|_\infty \le 2} \big|\mathbb{E}\big(f'(F) - Ff(F)\big)\big|. \tag{2.16}$$

We also recall here the Poincaré inequality that will be used later to estimate bounds for the total variation distance. Namely, for any $F \in \mathbb{D}^{1,2}$, we have

$$\mathrm{Var}(F) \le \mathbb{E}\big(\|DF\|_{\mathfrak{H}}^2\big). \tag{2.17}$$

3. Basic estimates for the total variation distance

In the framework of an isonormal Gaussian process $\mathbb{X}$, we can use Stein's equation to estimate the total variation distance between a random variable $F = \delta(u)$ and Z. First let us recall the following basic result (see [13,15]), which is an easy consequence of (2.16) and the duality relationship (2.1).

Proposition 3.1. *Assume that* $u \in \operatorname{Dom}\delta$, $F = \delta(u) \in \mathbb{D}^{1,2}$ *and* $\mathbb{E}(F^2) = 1$. *Then,*

$$d_{\mathrm{TV}}(F, Z) \le 2\mathbb{E}\big(\big|1 - \langle DF, u\rangle_{\mathfrak{H}}\big|\big).$$

Notice that, applying the duality relationship (2.1), we can write

$$\mathbb{E}\big(\langle DF, u\rangle_{\mathfrak{H}}\big) = \mathbb{E}\big(F\delta(u)\big) = \mathbb{E}\big(F^2\big) = 1.$$

As a consequence, if $F \in \mathbb{D}^{2,2}$, we apply Cauchy–Schwarz and Poincaré inequalities (2.17) to derive the following estimate

$$d_{\mathrm{TV}}(F, Z) \le 2\sqrt{\mathbb{E}\big(1 - \langle DF, u\rangle_{\mathfrak{H}}\big)^2} = 2\sqrt{\mathrm{Var}(D_u F)} \le 2\sqrt{\mathbb{E}\big(\big\|D(D_u F)\big\|_{\mathfrak{H}}^2\big)}, \tag{3.1}$$

where we have used the notation $D_u F = \langle u, DF\rangle_{\mathfrak{H}}$. We will also write $D_u^{i+1} F = \langle u, D(D_u^i F)\rangle_{\mathfrak{H}}$ for $i \ge 1$, and, by convention, $D_u^1 F = D_u F$.

Furthermore, if the random variable F admits higher order derivatives, iterating the integration by parts argument we can improve the bound (3.1) as follows.

Proposition 3.2. *Assume that* $u \in \operatorname{Dom}\delta$, $F = \delta(u) \in \mathbb{D}^{3,2}$ *and* $\mathbb{E}(F^2) = 1$. *Then*

$$d_{\mathrm{TV}}(F, Z) \leq (8 + \sqrt{32\pi})\mathbb{E}\big(\|D(D_u F)\|_{\mathfrak{H}}^2\big) + \sqrt{2\pi}\big|\mathbb{E}\big(F^3\big)\big| + \sqrt{32\pi}\mathbb{E}\big(\big|D_u^2 F\big|^2\big) + 4\pi\mathbb{E}\big(\big|D_u^3 F\big|\big).$$

Proof. Fix a continuous function $h : \mathbb{R} \to [0, 1]$. Using Stein's equation (2.13) and the estimates (2.14) and (2.15), there exists a function $f_h \in \mathcal{C}^1(\mathbb{R})$ such that $\|f_h\|_\infty \leq \sqrt{\frac{\pi}{2}}$ and $\|f_h'\|_\infty \leq 2$, satisfying

$$I := \big|\mathbb{E}\big(h(F)\big) - \mathbb{E}\big(h(Z)\big)\big| = \big|\mathbb{E}\big(f_h'(F) - F f_h(F)\big)\big|.$$

Applying the duality relationship (2.1), yields

$$I = \big|\mathbb{E}\big(f_h'(F)\big(1 - \langle DF, u\rangle_{\mathfrak{H}}\big)\big)\big|.$$

Taking into account that $\mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}}) = \mathbb{E}(F^2) = 1$, we have

$$I = \big|\mathbb{E}\big(\big(f_h'(F) - \mathbb{E}\big(f_h'(Z)\big)\big)\big(1 - \langle DF, u\rangle_{\mathfrak{H}}\big)\big)\big|.$$

Let f_φ be the solution to Stein's equation (2.13) associated with the function $\varphi = f_h'$. Then, we have

$$I = \big|\mathbb{E}\big(\big(f_\varphi'(F) - F f_\varphi(F)\big)\big(1 - \langle DF, u\rangle_{\mathfrak{H}}\big)\big)\big|,$$

where $\|f_\varphi\|_\infty \leq 4\sqrt{\pi/2}$ and $\|f_\varphi'\|_\infty \leq 8$. Substituting F by $\delta(u)$ and applying again the duality relationship (2.1), yields

$$\begin{aligned} I &= \big|\mathbb{E}\big(f_\varphi'(F)(1 - D_u F) - \big\langle u, D\big(f_\varphi(F)(1 - D_u F)\big)\big\rangle_{\mathfrak{H}}\big)\big| \\ &= \big|\mathbb{E}\big(f_\varphi'(F)(1 - D_u F)^2\big) + \mathbb{E}\big(f_\varphi(F) D_u^2 F\big)\big| \\ &\leq 8\mathbb{E}\big((1 - D_u F)^2\big) + \big|\mathbb{E}\big(\big(f_\varphi(F) - \mathbb{E}\big(f_\varphi(Z)\big)\big) D_u^2 F\big)\big| \\ &\quad + \big|\mathbb{E}\big(f_\varphi(Z)\big)\mathbb{E}\big(D_u^2 F\big)\big| \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.2}$$

For the term I_1, we apply Poincaré inequality to get

$$I_1 \leq 8\mathbb{E}\big(\|D(D_u F)\|_{\mathfrak{H}}^2\big).$$

For the term I_3, taking into account that

$$\mathbb{E}\big(D_u^2 F\big) = \mathbb{E}\big(\langle u, DF\rangle_{\mathfrak{H}}\delta(u)\big) = \frac{1}{2}\mathbb{E}\big(\big\langle u, DF^2\big\rangle_{\mathfrak{H}}\big) = \frac{1}{2}\mathbb{E}\big(F^3\big),$$

we obtain

$$I_3 \leq 2\sqrt{\pi/2}\big|\mathbb{E}\big(F^3\big)\big|.$$

Applying Stein's equation (2.13) to the function $\psi = f_\varphi$ yields $f_\psi'(F) - F f_\psi(F) = \psi(F) - \mathbb{E}(\psi(Z))$. Therefore, for the term I_2 we can write

$$\begin{aligned} I_2 &= \big|\mathbb{E}\big(\big(f_\psi'(F) - F f_\psi(F)\big) D_u^2 F\big)\big| \\ &\leq \big|\mathbb{E}\big(f_\psi'(F)\big(D_u^2 F - D_u F D_u^2 F\big)\big)\big| + \big|\mathbb{E}\big(f_\psi(F) D_u^3 F\big)\big|, \end{aligned}$$

where f_ψ satisfies $\|f_\psi\|_\infty \leq 4\pi$ and $\|f_\psi'\|_\infty \leq 16\sqrt{\pi/2}$. Finally,

$$\begin{aligned} \mathbb{E}\big(\big|D_u^2 F - D_u F D_u^2 F\big|\big) &\leq \frac{1}{2}\big(\mathbb{E}\big(\big|D_u^2 F\big|^2\big) + \mathbb{E}\big(|1 - D_u F|^2\big)\big) \\ &\leq \frac{1}{2}\big(\mathbb{E}\big(\big|D_u^2 F\big|^2\big) + \mathbb{E}\big(\|D D_u F\|_{\mathfrak{H}}^2\big)\big). \end{aligned}$$

This concludes the proof of the proposition. □

If we bound (3.2) in a different way, we would get the following estimate.

Proposition 3.3. *Assume that $u \in \operatorname{Dom}\delta$, $F = \delta(u) \in \mathbb{D}^{2,2}$ and $\mathbb{E}(F^2) = 1$. Then*

$$d_{\mathrm{TV}}(F, Z) \le 8\mathbb{E}\big((1 - D_u F)^2\big) + \sqrt{8\pi}\,\mathbb{E}\big(\big|D_u^2 F\big|\big).$$

4. Main results

Consider a centered stationary Gaussian family of random variables $X = \{X_n, n \in \mathbb{Z}\}$ with unit variance and covariance $\rho(k) = \mathbb{E}(X_0 X_k)$ for $k \in \mathbb{Z}$. Define the Hilbert space $\mathfrak{H}$ as the closure of the linear span of $\mathbb{Z}$ under the inner product $\langle j, k\rangle_{\mathfrak{H}} = \rho(j - k)$. The mapping $k \to X_k$ can be extended to a linear isometry from $\mathfrak{H}$ to the closed linear subspace $L^2(\Omega)$ spanned by X. Then $\{X_\varphi, \varphi \in \mathfrak{H}\}$ is an isonormal Gaussian process.

Consider the sequence $Y_n := \frac{1}{\sqrt{n}} \sum_{j=1}^n g(X_j)$ introduced in (1.4), where $g \in L^2(\mathbb{R}, \gamma)$ has Hermite rank $d \ge 1$ and let $\sigma_n^2 = \mathbb{E}(Y_n^2)$. Under condition (1.2), it is well known that as $n \to \infty$, $\sigma_n^2 \to \sigma^2$, where σ^2 has been defined in (1.3).

Along the paper, we will denote by C a generic constant, whose value can be different from one formula to another one.

Our aim is to establish estimates on the total variation distance between Y_n/σ_n and Z. We will make use of the representation $Y_n = \delta(u_n)$, where

$$u_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n g_1(X_j) j, \tag{4.1}$$

given by Lemma 2.1. Then, if $g \in \mathbb{D}^{2,2}(\mathbb{R}, \gamma)$, by inequality (3.1) and taking into account that $\sigma_n \to \sigma > 0$, we have the estimate

$$\begin{aligned} d_{\mathrm{TV}}(Y_n/\sigma_n, Z) &\le \frac{1}{\sigma_n^2} \sqrt{\mathbb{E}\big(\big|\langle DY_n, u_n\rangle_{\mathfrak{H}} - \sigma_n^2\big|^2\big)} \\ &\le C\sqrt{\mathbb{E}\big(\big\|D\big(\langle DY_n, u_n\rangle_{\mathfrak{H}}\big)\big\|_{\mathfrak{H}}^2\big)} = C\sqrt{A_1}, \end{aligned} \tag{4.2}$$

where $A_1 = \mathbb{E}(\|DD_{u_n} Y_n\|_{\mathfrak{H}}^2)$. Furthermore, using Proposition 3.3, we can write

$$\begin{aligned} d_{\mathrm{TV}}(Y_n/\sigma_n, Z) &\le \frac{8}{\sigma_n^4} \mathbb{E}\big(\|DD_{u_n} Y_n\|_{\mathfrak{H}}^2\big) + \frac{\sqrt{8\pi}}{\sigma_n^3} \sqrt{\mathbb{E}\big(\big|D_{u_n}^2 Y_n\big|^2\big)} \\ &\le C(A_1 + \sqrt{A_2}), \end{aligned} \tag{4.3}$$

where $A_2 = \mathbb{E}(|D_{u_n}^2 Y_n|^2)$ and where we recall that $D_{u_n} Y_n = \langle u_n, DY_n\rangle_{\mathfrak{H}}$ and $D_{u_n}^i Y_n = \langle u_n, DD_{u_n}^{i-1} Y_n\rangle_{\mathfrak{H}}$ for $i \ge 2$. The assumption $g \in \mathbb{D}^{2,2}$ implies that the terms A_1 and A_2 are well defined.

If $g \in \mathbb{D}^{3,2}(\mathbb{R}, \gamma)$, using Proposition 3.2, we obtain

$$\begin{aligned} d_{\mathrm{TV}}(Y_n/\sigma_n, Z) &\le \frac{8 + \sqrt{32\pi}}{\sigma_n^4} \mathbb{E}\big(\|DD_{u_n} Y_n\|_{\mathfrak{H}}^2\big) + \frac{\sqrt{32\pi}}{\sigma_n^6} \mathbb{E}\big(\big|D_{u_n}^2 Y_n\big|^2\big) \\ &\quad + \frac{\sqrt{2\pi}}{\sigma_n^3} \big|\mathbb{E}\big(Y_n^3\big)\big| + \frac{4\pi}{\sigma_n^4} \sqrt{\mathbb{E}\big(\big|D_{u_n}^3 Y_n\big|^2\big)} \\ &\le C(A_1 + A_2 + A_3 + A_4), \end{aligned} \tag{4.4}$$

where $A_3 = |\mathbb{E}(Y_n^3)|$ and $A_4 = \sqrt{\mathbb{E}(|D_{u_n}^3 Y_n|^2)}$. The assumption $g \in \mathbb{D}^{3,2}$ implies that the term $D_{u_n}^3 Y_n$ is well defined.

In the sequel we will derive estimates on the terms A_i, $i = 1, \dots, 4$ in terms of the covariance function $\rho(k)$. We use the notation $A_i \prec A_j$ if A_i's bound has a better convergence rate to zero than that of A_j or if $A_i \le CA_j$ for some constant $C > 0$. To get the best possible rate, we use the following strategy. If g is just twice differentiable, we can use the estimates (4.2) and (4.3). Then we will compare the rates of the terms A_1 and A_2. If $A_1 \prec A_2$, we just use the bound (4.2). Otherwise, (4.3) would be used. If g has higher order derivatives, we would use the bound (4.4) if $A_2 \prec \sqrt{A_1}$ and the rates of A_3 and A_4 are better than those of $\sqrt{A_2}$ and $\sqrt{A_1}$. Otherwise, if the rate of either A_3 or A_4 is worse than that of $\sqrt{A_1}$ or $\sqrt{A_2}$, we consider the bound (4.3) or (4.2) depending on the comparison between A_2 and A_1.

Before presenting the main results, we will derive some expressions and estimates for the terms A_i, $i = 1, 2, 4$. To simplify the notation, we will write $\rho_{ij} = \rho(l_i - l_j)$ for any $1 \le i, j \le n$.

Lemma 4.1. *Suppose that $g \in \mathbb{D}^{2,4}(\mathbb{R}, \gamma)$. Then,*

$$A_1 \le \frac{2}{n^2} \sum_{i=1}^{2} \sum_{l_1,l_2,l_3,l_4=1}^{n} \left| \mathbb{E}(I_i)\rho(l_1 - l_2)\rho(l_3 - l_4)\rho(l_2 - l_4) \right|,$$

where

$$I_1 = g''(X_{l_2})g''(X_{l_4})g_1(X_{l_1})g_1(X_{l_3}), \tag{4.5}$$

and

$$I_2 = g'(X_{l_1})g'(X_{l_3})g_1'(X_{l_2})g_1'(X_{l_4}). \tag{4.6}$$

Proof. First, we have

$$\mathbb{E}\left(\left\| D\left(\langle DY_n, u_n\rangle_{\mathfrak{H}}\right)\right\|_{\mathfrak{H}}^2\right) \le 2\mathbb{E}\left(\left\| D^2 Y_n \otimes_1 u_n\right\|_{\mathfrak{H}}^2\right) + 2\mathbb{E}\left(\left\|\left\langle D_* Y_n, Du_n(*)\right\rangle_{\mathfrak{H}}\right\|_{\mathfrak{H}}^2\right),$$

where $D^2 Y_n \otimes_1 u_n$ denotes the contraction of one variable between $D^2 Y_n$ and u_n and

$$\left\langle D_* Y_n, Du_n(*)\right\rangle_{\mathfrak{H}} = \sum_{i=1}^{\infty} \langle DY_n, e_i\rangle_{\mathfrak{H}} D\left(\langle u_n, e_i\rangle_{\mathfrak{H}}\right),$$

with $\{e_i, i \ge 1\}$ being a complete orthonormal system in $\mathfrak{H}$. This implies, taking into account (4.1), that

$$D^2 Y_n \otimes_1 u_n = \frac{1}{n} \sum_{j,k=1}^{n} g''(X_k)g_1(X_j)\rho(j-k)k$$

and

$$\left\langle D_* Y_n, Du_n(*)\right\rangle_{\mathfrak{H}} = \frac{1}{n} \sum_{j,k=1}^{n} g'(X_j)g_1'(X_k)\rho(j-k)k.$$

As a consequence,

$$\left\| D^2 Y_n \otimes_1 u_n\right\|_{\mathfrak{H}}^2 = \frac{1}{n^2} \sum_{l_1,l_2,l_3,l_4=1}^{n} I_1 \rho(l_1 - l_2)\rho(l_3 - l_4)\rho(l_2 - l_4),$$

and

$$\left\|\left\langle D_* Y_n, Du_n(*)\right\rangle_{\mathfrak{H}}\right\|_{\mathfrak{H}}^2 = \frac{1}{n^2} \sum_{l_1,l_2,l_3,l_4=1}^{n} I_2 \rho(l_1 - l_2)\rho(l_3 - l_4)\rho(l_2 - l_4),$$

which implies the desired result. □

Next we derive a simple estimate for the term A_2, assuming again that $g \in \mathbb{D}^{2,6}(\mathbb{R}, \gamma)$. Notice that

$$D_{u_n} Y_n = \frac{1}{n} \sum_{l_1,l_2=1}^{n} g_1(X_{l_1})g'(X_{l_2})\rho(l_1 - l_2).$$

Denote

$$f_1(l_1, l_2, l_3) = g_1'(X_{l_1})g'(X_{l_2})g_1(X_{l_3}) \tag{4.7}$$

and

$$f_2(l_1,l_2,l_3)=g_1(X_{l_1})g''(X_{l_2})g_1(X_{l_3}). \tag{4.8}$$

Correspondingly, using the notation $\rho_{ij}=\rho(l_i-l_j)$, we can write

$$D^2_{u_n}Y_n=\frac{1}{\sqrt{n^3}}\sum_{l_1,l_2,l_3=1}^{n}\big(f_1(l_1,l_2,l_3)\rho_{12}\rho_{13}+f_2(l_1,l_2,l_3)\rho_{12}\rho_{23}\big).$$

Thus,

$$\begin{aligned}A_2&=\mathbb{E}\big(\big(D^2_{u_n}Y_n\big)^2\big)\\&\le\frac{2}{n^3}\sum_{l_1,\dots,l_6=1}^{n}\big(\mathbb{E}\big(f_1(l_1,l_2,l_3)f_1(l_4,l_5,l_6)\big)\rho_{12}\rho_{13}\rho_{45}\rho_{46}\\&\quad+\mathbb{E}\big(f_2(l_1,l_2,l_3)f_2(l_4,l_5,l_6)\big)\rho_{12}\rho_{23}\rho_{45}\rho_{56}\big).\end{aligned} \tag{4.9}$$

Finally, let us compute the term A_4, assuming $g\in\mathbb{D}^{3,8}(\mathbb{R},\gamma)$. We have

$$\begin{aligned}D^3_{u_n}Y_n=\frac{1}{n^2}\sum_{l_1,l_2,l_3,l_4=1}^{n}\sum_{i=1}^{3}&\big(f_1^{(i)}(l_1,l_2,l_3)g_1(X_{l_4})\rho_{12}\rho_{13}\rho_{i4}\\&+f_2^{(i)}(l_1,l_2,l_3)g_1(X_{l_4})\rho_{12}\rho_{23}\rho_{i4}\big),\end{aligned}$$

where $f_1^{(i)}=\frac{\partial f_1}{\partial X_{l_i}}$, namely,

$$\begin{aligned}f_1^{(1)}(l_1,l_2,l_3)&=g_1''(X_{l_1})g'(X_{l_2})g_1(X_{l_3}),\\f_1^{(2)}(l_1,l_2,l_3)&=g_1'(X_{l_1})g''(X_{l_2})g_1(X_{l_3}),\\f_1^{(3)}(l_1,l_2,l_3)&=g_1'(X_{l_1})g'(X_{l_2})g_1'(X_{l_3})\end{aligned}$$

and $f_2^{(i)}=\frac{\partial f_2}{\partial X_{l_i}}$, namely,

$$\begin{aligned}f_2^{(1)}(l_1,l_2,l_3)&=g_1'(X_{l_1})g''(X_{l_2})g_1(X_{l_3}),\\f_2^{(2)}(l_1,l_2,l_3)&=g_1(X_{l_1})g'''(X_{l_2})g_1(X_{l_3}),\\f_2^{(3)}(l_1,l_2,l_3)&=g_1(X_{l_1})g''(X_{l_2})g_1'(X_{l_3}).\end{aligned}$$

Therefore,

$$\begin{aligned}A_4^2&=\mathbb{E}\big(\big(D^3_{u_n}Y_n\big)^2\big)\\&\le\frac{6}{n^4}\sum_{i=1}^{3}\sum_{j=1,\dots,8}\sum_{l_j=1}^{n}\mathbb{E}\big(f_1^{(i)}(l_1,l_2,l_3)g_1(X_{l_4})f_1^{(i+4)}(l_5,l_6,l_7)g_1(X_{l_8})\big)\\&\quad\times\rho_{12}\rho_{13}\rho_{i4}\rho_{56}\rho_{57}\rho_{(i+4)8}\\&\quad+\frac{6}{n^4}\sum_{i=1}^{3}\sum_{j=1,\dots,8}\sum_{l_j=1}^{n}\mathbb{E}\big(f_2^{(i)}(l_1,l_2,l_3)g_1(X_{l_4})f_2^{(i+4)}(l_5,l_6,l_7)g_1(X_{l_8})\big)\\&\quad\times\rho_{12}\rho_{23}\rho_{i4}\rho_{56}\rho_{67}\rho_{(i+4)8}.\end{aligned} \tag{4.10}$$

We are now ready to state and prove the main results of this paper. The notation is that of Theorem 1.1.

4.1. *Case $d=1$*

Theorem 4.2. *Let $d=1$ and $g\in\mathbb{D}^{2,4}(\mathbb{R},\gamma)$. Suppose that (1.2) holds true. Then*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le Cn^{-\frac{1}{2}}.$$

Proof. We use the inequality (4.2) and we need to estimate the term A_1. By Lemma 2.3, Hölder's inequality and the fact that $g\in\mathbb{D}^{2,4}(\mathbb{R},\gamma)$, the quantities I_1 and I_2 have finite expectation. Then

$$A_1\le\frac{C}{n^2}\sum_{l_1,l_2,l_3,l_4=1}^{n}\left|\rho(l_1-l_2)\rho(l_3-l_4)\rho(l_2-l_4)\right|.$$

Making the change of variables $k_1=l_1-l_2$, $k_2=l_3-l_4$, $k_3=l_2-l_4$ and using condition (1.2) with $d=1$, we obtain

$$A_1\le\frac{C}{n}\sum_{|k_i|\le n,1\le i\le 3}\left|\rho(k_1)\rho(k_2)\rho(k_3)\right|\le\frac{C}{n},$$

which provides the desired estimate. □

4.2. *Case of $d=2$*

Theorem 4.3. *Let $d=2$ and suppose that (1.2) holds true.*

(i) *If $g\in\mathbb{D}^{2,4}(\mathbb{R},\gamma)$, we have*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le Cn^{-\frac{1}{2}}\Big(\sum_{|k|\le n}\left|\rho(k)\right|\Big)^{\frac{3}{2}}.$$

(ii) *If $g\in\mathbb{D}^{3,4}(\mathbb{R},\gamma)$, we have*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le Cn^{-\frac{1}{2}}\sum_{|k|\le n}\left|\rho(k)\right|.$$

(iii) *If $g\in\mathbb{D}^{4,4}(\mathbb{R},\gamma)$, we have*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le Cn^{-\frac{1}{2}}\Big(\sum_{|k|\le n}\left|\rho(k)\right|\Big)^{\frac{1}{2}}+Cn^{-\frac{1}{2}}\Big(\sum_{|k|\le n}\left|\rho(k)\right|^{\frac{4}{3}}\Big)^{\frac{3}{2}}.$$

(iv) *If $g\in\mathbb{D}^{5,6}(\mathbb{R},\gamma)$, we have*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le Cn^{-\frac{1}{2}}\Big(\sum_{|k|\le n}\left|\rho(k)\right|\Big)^{\frac{1}{2}}+Cn^{-\frac{1}{2}}\Big(\sum_{|k|\le n}\left|\rho(k)\right|^{\frac{3}{2}}\Big)^{2}.$$

(v) *If $g\in\mathbb{D}^{6,8}(\mathbb{R},\gamma)$, we have*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le Cn^{-\frac{1}{2}}\Big(\sum_{|k|\le n}\left|\rho(k)\right|^{\frac{3}{2}}\Big)^{2}.$$

Remark 4.4. The bounds in Theorem 4.3 involve series of the form $S_n:=\sum_{|k|\le n}|\rho(k)|^{\alpha}$ for $\alpha\in[1,2)$. Under condition (1.2), by Hölder's inequality, $S_n\le(1+2n)^{1-\frac{\alpha}{2}}$ and the bounds exhibited in Theorem 4.3 could be divergent. However, we are interested in the cases when the bounds converge to 0 and provide convergence rates in the context of Breuer–Major theorem. See Corollary 4.9 for an example.

Remark 4.5. For $g \in \mathbb{D}^{6,8}(\mathbb{R},\gamma)$ the rate established in point (v) coincides with the rate for the Hermite polynomial $g(x) = x^2 - 1$, obtained by Biermé, Bonami, Nourdin and Peccati in [5] using the optimal bound for the total variation distance in the case of random variables in a fixed Wiener chaos derived by Nourdin and Peccati in [14] (see Proposition 1.3). For $g \in \mathbb{D}^{i,4}(\mathbb{R},\gamma)$, $i = 2, 3, 4$, the estimates in points (i), (ii) and (iii) will be established using Proposition 3.1, whereas, for $g \in \mathbb{D}^{5,6}(\mathbb{R},\gamma)$ we will use Proposition 3.3 to derive the estimate in point (iv) and for $g \in \mathbb{D}^{6,8}(\mathbb{R},\gamma)$ we apply Proposition 3.2.

Remark 4.6. The bound (iii) is better than (ii) based on the inequality (A.19) with $M = 3$ and the fact that $(\sum_{|k|\le n} |\rho(k)|)^{1/2} \le \sum_{|k|\le n} |\rho(k)|$. The bound (iv) is better than (iii) because of the inequality (A.25). Finally, it is straightforward to see (v) is better than (iv).

Proof of Theorem 4.3. The proof will be done in several steps.

Case $g \in \mathbb{D}^{2,4}(\mathbb{R},\gamma)$. We apply Lemma 4.1 to derive the rate of convergence of A_1. Using arguments similar to those in the case $d = 1$ yields

$$A_1 \le \frac{C}{n} \sum_{|k_i|\le n, 1\le i\le 3} \left|\rho(k_1)\rho(k_2)\rho(k_3)\right| = \frac{C}{n}\left(\sum_{|k|\le n} \left|\rho(k)\right|\right)^3, \tag{4.11}$$

which gives the desired estimate in view of (4.2).

We claim that, even if we impose more integrability conditions on the function g, that is, $g \in \mathbb{D}^{2,6}(\mathbb{R},\gamma)$, the estimate (4.3) does not give a rate better than (4.11). In fact, let us estimate the term A_2, which is bounded by the inequality (4.9), where f_1 and f_2 are defined in (4.7) and (4.8). The term $\mathbb{E}(f_2(l_1,l_2,l_3) f_2(l_4,l_5,l_6))$ cannot be integrated by parts because it involves g'' and g is only twice weakly differentiable. Therefore, if $g \in \mathbb{D}^{2,6}(\mathbb{R},\gamma)$, using Lemma 2.3 together with Hölder's inequality, we obtain

$$A_2 \le \frac{C}{n^3}\left(\sum_{l_1,\dots,l_6=1}^{n} |\rho_{12}\rho_{13}\rho_{45}\rho_{46}| + \sum_{l_1,\dots,l_6=1}^{n} |\rho_{12}\rho_{23}\rho_{45}\rho_{56}|\right).$$

Making change of variables, $l_1 - l_2 = k_1$, $l_1 - l_3 = k_2$, $l_4 - l_5 = k_3$, $l_4 - l_6 = k_4$ for the first summand, and $l_1 - l_2 = k_1$, $l_2 - l_3 = k_2$, $l_4 - l_5 = k_3$, $l_5 - l_6 = k_4$ for the second summand, we obtain

$$A_2 \le \frac{C}{n} \sum_{|k_i|\le n, 1\le i\le 4} \prod_{i=1}^{4} \left|\rho(k_i)\right| = \frac{C}{n}\left(\sum_{|k|\le n} \left|\rho(k)\right|\right)^4.$$

Thus, $A_1 \prec A_2$, so we use (4.2) and (4.11) gives the best rate.

Case $g \in \mathbb{D}^{3,4}(\mathbb{R},\gamma)$. Let us first estimate the term A_1. Because g has three derivatives, using Lemma 4.1 and Lemma A.1, we obtain

$$A_1 \le \frac{C}{n^2} \sum_{l_1,l_2,l_3,l_4=1}^{n} \left|\rho(l_1 - l_2)\rho(l_3 - l_4)\rho(l_2 - l_4)\right| \sum_{j\ne 1} \left|\rho(l_1 - l_j)\right|.$$

Making the change of variables $l_1 - l_2 = k_1$, $l_2 - l_4 = k_2$ and $l_3 - l_4 = k_3$, yields

$$A_1 \le \frac{C}{n} \sum_{|k_i|\le n} \left(\left|\rho^2(k_1)\rho(k_2)\rho(k_3)\right| + \left|\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_1 + k_2)\right| \right.$$
$$\left. + \left|\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_1 + k_2 - k_3)\right|\right).$$

Taking into account condition (1.2) and applying (A.20) with $M = 3$, yields

$$A_1 \le \frac{C}{n}\left(\sum_{|k|\le n} \left|\rho(k)\right|\right)^2, \tag{4.12}$$

which gives the desired estimate in view of (4.2).

Again, we claim that imposing more integrability conditions and using either (4.3) or the more refined estimate (4.4) does not improve the above rate. Indeed, let us first estimate the term A_2, assuming $g \in \mathbb{D}^{3,6}(\mathbb{R},\gamma)$. Because g is three times weakly differentiable, we can integrate by parts once in the expectations appearing in (4.9). The two summands in (4.9) are similar, thus it suffices to consider the first one. Recall that $f_1(l_1,l_2,l_3)=g_1'(X_{l_1})g'(X_{l_2})g_1(X_{l_3})$ has been defined in (4.7). Using the representation $g'(X_{l_2})=\delta(T_1(g')(X_{l_2})l_2)$, applying the duality relationship (2.1), and making a change of variables, we obtain

$$A_2 \le \frac{C}{n^3}\sum_{l_1,\dots,l_6=1}^{n}\left(\rho_{12}^2|\rho_{13}\rho_{45}\rho_{46}|+|\rho_{12}\rho_{13}\rho_{45}\rho_{46}\rho_{23}|+|\rho_{12}\rho_{13}\rho_{45}\rho_{46}|\sum_{i=4}^{6}|\rho_{2i}|\right)$$

$$\le \frac{C}{n^2}\sum_{\substack{|k_i|\le n\\1\le i\le 5}}\left(\rho(k_1)^2\prod_{i=2}^{4}\bigl|\rho(k_i)\bigr|+\bigl|\rho(k_1-k_2)\bigr|\prod_{i=1}^{4}\bigl|\rho(k_i)\bigr|+\prod_{i=1}^{5}\bigl|\rho(k_i)\bigr|\right).$$

This implies, using (A.20) with $M=4$ for the second summand, that

$$A_2 \le \frac{C}{n}\left(\sum_{|k|\le n}\bigl|\rho(k)\bigr|\right)^3+\frac{C}{n^2}\left(\sum_{|k|\le n}\bigl|\rho(k)\bigr|\right)^5\le \frac{C}{n}\left(\sum_{|k|\le n}\bigl|\rho(k)\bigr|\right)^3,$$

where we have used the fact that $\sum_{|k|\le n}|\rho(k)|\le C\sqrt{n}$ in the second inequality. Clearly, $A_1 \prec A_2$. So the estimate (4.2) is better than (4.3).

On the other hand, the estimate (4.4) does not provide a rate better than (4.2), because $\sqrt{A_1}\prec A_3$. Indeed, let us estimate the term A_3. We know that

$$A_3=\bigl|\mathbb{E}\bigl(Y_n^3\bigr)\bigr|=n^{-\frac{3}{2}}\left|\sum_{l_1,l_2,l_3=1}^{n}\mathbb{E}\left(\prod_{i=1}^{3}g(X_{l_i})\right)\right|.$$

Using the representation $g(X_{l_1})=\delta^2(g_2(X_{l_1})l_1^{\otimes 2})$ and applying twice the duality relationship (2.1), we obtain

$$A_3 \le Cn^{-\frac{3}{2}}\sum_{l_1,l_2,l_3=1}^{n}\Bigl(\bigl|\mathbb{E}\bigl(g_2(X_{l_1})g''(X_{l_2})g(X_{l_3})\bigr)\bigr|\rho_{12}^2$$

$$+2\bigl|\mathbb{E}\bigl(g_2(X_{l_1})g'(X_{l_2})g'(X_{l_3})\bigr)\rho_{12}\rho_{13}\bigr|+\bigl|\mathbb{E}\bigl(g_2(X_{l_1})g(X_{l_2})g''(X_{l_3})\bigr)\bigr|\rho_{13}^2\Bigr).$$

Because g is three times differentiable, we can still use the representations $g(X_{l_3})=\delta(g_1(X_{l_3})l_3)$, $g'(X_{l_2})=\delta(T_1(g')(X_{l_2})l_2)$ and $g(X_{l_2})=\delta(g_1(X_{l_2})l_2)$, and apply the duality relationship (2.1) again to produce an additional factor of the form $|\rho_{13}|+|\rho_{23}|$ for the first term and $|\rho_{12}|+|\rho_{23}|$ for the second and third terms. In this way, we obtain

$$A_3 \le Cn^{-\frac{3}{2}}\sum_{l_1,l_2,l_3=1}^{n}\bigl(\bigl|\rho_{12}^2\rho_{13}\bigr|+|\rho_{12}\rho_{13}\rho_{23}|\bigr).$$

We make the change of variables $\rho_{12}=\rho(k_1)$, $\rho_{13}=\rho(k_2)$ and apply (A.18) with $M=2$ to the second summand to obtain

$$A_3 \le Cn^{-\frac{1}{2}}\sum_{|k|\le n}\bigl|\rho(k)\bigr|+Cn^{-\frac{1}{2}}\left(\sum_{|k|\le n}\bigl|\rho(k)\bigr|^{\frac{3}{2}}\right)^2.$$

Clearly, by (A.19), this bound is not better than the bound we have previously obtained for $\sqrt{A_1}$, and (4.12) gives the result in this case.

Case $g\in\mathbb{D}^{4,4}(\mathbb{R},\gamma)$. As before, let us first estimate the term A_1. Taking into account that g has four derivatives, by the results of Lemma 4.1 and Lemma A.1 and using the notation $\rho(l_i-l_j)=\rho_{ij}$, we have

$$A_1 \le \frac{C}{n^2}\sum_{l_1,l_2,l_3,l_4=1}^{n}|\rho_{12}\rho_{34}\rho_{24}|\left(\bigl(|\rho_{12}|+|\rho_{14}|\bigr)\sum_{j\ne 3}|\rho_{j3}|+|\rho_{13}|\right).$$

We further write

$$
\begin{aligned}
A_1 &\le \frac{C}{n^2} \sum_{1\le l_i\le n, 1\le i\le 4} \big(\rho_{12}^2\rho_{34}^2|\rho_{24}| + \rho_{12}^2|\rho_{34}\rho_{24}\rho_{13}| + \rho_{12}^2|\rho_{34}\rho_{24}\rho_{23}| \\
&\quad + \big|\rho_{12}\rho_{34}^2\rho_{24}\rho_{14}\big| + |\rho_{12}\rho_{34}\rho_{24}\rho_{14}\rho_{23}| + |\rho_{12}\rho_{34}\rho_{24}\rho_{14}\rho_{13}| + |\rho_{12}\rho_{34}\rho_{24}\rho_{13}|\big) \\
&\le \frac{C}{n^2} \sum_{1\le l_i\le n, 1\le i\le 4} \big(\rho_{12}^2\rho_{34}^2|\rho_{24}| + |\rho_{12}\rho_{34}\rho_{24}\rho_{13}| + \rho_{12}^2|\rho_{34}\rho_{24}\rho_{23}| \\
&\quad + \big|\rho_{12}\rho_{34}^2\rho_{24}\rho_{14}\big| + |\rho_{12}\rho_{34}\rho_{14}\rho_{23}| + |\rho_{12}\rho_{34}\rho_{24}\rho_{13}| + |\rho_{12}\rho_{34}\rho_{24}\rho_{13}|\big) \\
&\le \frac{C}{n^2} \sum_{1\le l_i\le n, 1\le i\le 4} \rho_{12}^2\rho_{34}^2|\rho_{24}| + \rho_{12}^2|\rho_{34}\rho_{24}\rho_{23}| + |\rho_{12}\rho_{34}\rho_{24}\rho_{13}|. \qquad (4.13)
\end{aligned}
$$

In the second inequality, we have used the fact that $|\rho_{ij}| \le 1$. In the third inequality, we have used that in (4.13) the third and fourth summands are equivalent, and the second, fifth, sixth, and seventh summands are also equivalent. By the change of variables $l_1 - l_2 = k_1$, $l_3 - l_4 = k_2$, $l_2 - l_4 = k_3$, we obtain

$$
\begin{aligned}
A_1 &\le \frac{C}{n} \sum_{|k_i|\le n, 1\le i\le 3} \big(\rho^2(k_1)\rho^2(k_2)\big|\rho(k_3)\big| + \rho^2(k_1)\big|\rho(k_2)\rho(k_3)\rho(k_2-k_3)\big| \\
&\quad + \big|\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_1-k_2+k_3)\big|\big). \qquad (4.14)
\end{aligned}
$$

The first summand is bounded by $\frac{C}{n}\sum_{|k|\le n}|\rho(k)|$ from condition (1.2) with $d = 2$. The second summand is bounded by $\frac{C}{n}\sum_{|k|\le n}|\rho(k)|$ if we apply condition (1.2) and inequality (A.20) with $M = 2$. Finally, we use inequality (A.18) with $M = 3$ for the third summand. In this way, we obtain

$$
A_1 \le \frac{C}{n}\sum_{|k|\le n}\big|\rho(k)\big| + \frac{C}{n}\bigg(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{4}{3}}\bigg)^3. \qquad (4.15)
$$

This gives the desired estimate in view of (4.2).

As in the previous cases, we will show that, even with stronger integrability assumptions, using either (4.3) or (4.4) does not improve the above rate. For this, consider first the term A_2, assuming $g \in \mathbb{D}^{4,6}(\mathbb{R},\gamma)$. Because g has four derivatives, we can apply twice the duality relationship (2.1). Recall that the term A_2 is bounded by (4.9) and it suffices to consider the first summand in the right-hand side of this inequality. We write it here for convenience

$$
A_{21} := \frac{2}{n^3}\sum_{l_1,\dots,l_6=1}^{n} \mathbb{E}\big(f_1(l_1,l_2,l_3)f_1(l_4,l_5,l_6)\big)\rho_{12}\rho_{13}\rho_{45}\rho_{46}, \qquad (4.16)
$$

where $f_1(l_1,l_2,l_3)$ has been defined in (4.7). Notice that the functions g' and g_1 have Hermite rank 1. We first write $g'(X_{l_2}) = \delta(T_1(g')(X_{l_2})l_2)$ and apply duality with respect to this divergence producing factors of the form ρ_{2i}, $i \ne 2$, $1 \le i \le 6$. Next we choose another function that has Hermite rank 1 among the factors $g_1(X_{l_3})$, $g'(X_{l_5})$ and $g_1(X_{l_6})$, write it as a divergence integral and apply duality again to obtain:

$$
\big|\mathbb{E}\big(f_1(l_1,l_2,l_3)f_1(l_4,l_5,l_6)\big)\big| \le C\sum_{\substack{i=1\\ i\ne 2}}^{6}\sum_{\substack{s\in\{3,5,6\}\\ s\ne i}}\sum_{\substack{j=1\\ j\ne s}}^{6}|\rho_{2i}\rho_{sj}|. \qquad (4.17)
$$

Applying inequality (A.5) in Lemma A.3 yields

$$
A_2 \le 2A_{21} \le \frac{C}{n}\bigg(\sum_{|k|\le n}\big|\rho(k)\big|\bigg)^2. \qquad (4.18)
$$

By the inequality (A.19) with $M = 3$, we get that $A_1 \prec A_2$.

Next we will compare this estimate with the bound we can obtain for the term A_3 using the fact that g has four derivatives. We can write

$$A_3 = \left|\mathbb{E}\left(Y_n^3\right)\right| = Cn^{-\frac{3}{2}}\left|\sum_{l_1,l_2,l_3=1}^{n} \mathbb{E}\left(\prod_{i=1}^{3} g(X_{l_i})\right)\right|$$

$$\le Cn^{-\frac{3}{2}} \sum_{l_1,l_2,l_3=1}^{n} \big(\rho_{12}^2\big(|\rho_{13}|+|\rho_{23}|\big)^2$$

$$+ |\rho_{12}\rho_{13}|\big(|\rho_{23}|+|\rho_{12}|\big(|\rho_{13}|+|\rho_{23}|\big)\big)+\rho_{13}^2\big(|\rho_{12}|+|\rho_{23}|\big)^2\big)$$

$$\le Cn^{-\frac{3}{2}} \sum_{l_1,l_2,l_3=1}^{n} \big(\big|\rho_{12}^2\rho_{13}^2\big| + |\rho_{12}\rho_{13}\rho_{23}|\big). \tag{4.19}$$

Note that $n^{-\frac{3}{2}} \sum_{l_1,l_2,l_3=1}^{n} |\rho_{12}^2\rho_{13}^2| = Cn^{-\frac{1}{2}}$. We make the change of variables $\rho_{12} \to \rho(k_1)$, $\rho_{13} \to \rho(k_2)$ and apply (A.18) to the second summand, to obtain

$$A_3 \le Cn^{-\frac{1}{2}}\left(\sum_{|k|\le n}\left|\rho(k)\right|^{\frac{3}{2}}\right)^2. \tag{4.20}$$

By (A.23) with $M=3$ and (A.24), we obtain that $A_1 \prec A_3$. By (A.25), we have $A_3 \prec \sqrt{A_1}$. However, we cannot use the bound (4.4) since the relationship between $\sqrt{A_1}$ and A_2 is not clear, because the sequences $n^{-\frac{1}{2}}(\sum_{|k|\le n}|\rho(k)|)^{\frac{1}{2}}$ and $n^{-1}(\sum_{|k|\le n}|\rho(k)|)^2$ are not comparable. An example could be $\rho(k) \sim k^{-\alpha}$ for $\alpha \in (\frac{1}{2}, \frac{2}{3})$. So, we use the bound (4.2) that is given by (4.15).

Case $g \in \mathbb{D}^{5,6}(\mathbb{R},\gamma)$. For the terms A_1 and A_3 we still have the estimates (4.15) and (4.20). For the term A_2, we continue with the inequalities (4.16) and (4.17), and apply the duality for the third time to $\mathbb{E}(f_1(l_1,l_2,l_3)f_1(l_4,l_5,l_6))$ when there is a factor with Hermite rank 1, to obtain

$$\left|\mathbb{E}\big(f_1(l_1,l_2,l_3)f_1(l_4,l_5,l_6)\big)\right| \le C \sum_{\substack{i\ne s\ne j\\ i,s,j\in\{3,5,6\}}} |\rho_{2i}\rho_{sj}| + C\sum_{(i,s,j,t,h)\in D_3} |\rho_{2i}\rho_{sj}\rho_{th}|,$$

where

$$D_3 = \big\{(i,s,j,t,h): j,h \in \{1,\dots,6\}; s,t\in\{3,5,6\}; i\ne 2, s\notin\{i,j\}; t\notin\{i,s,j,h\}\big\}. \tag{4.21}$$

By inequality (A.6) in Lemma A.3,

$$A_2 \le \frac{C}{n}\sum_{|k|\le n}\left|\rho(k)\right| + \frac{C}{n}\left(\sum_{|k|\le n}\left|\rho(k)\right|^{\frac{3}{2}}\right)^4. \tag{4.22}$$

From (4.15), (4.22) and (A.25) we deduce that $A_2 \prec A_1$ and, therefore, $A_1+\sqrt{A_2} \prec \sqrt{A_1}$. Therefore, (4.3) gives a better rate than (4.2), which is given by

$$A_1+\sqrt{A_2} \le Cn^{-\frac{1}{2}}\left(\sum_{|k|\le n}\left|\rho(k)\right|\right)^{\frac{1}{2}} + Cn^{-\frac{1}{2}}\left(\sum_{|k|\le n}\left|\rho(k)\right|^{\frac{3}{2}}\right)^2. \tag{4.23}$$

Clearly, $A_3 \prec A_1+\sqrt{A_2}$. Whether we choose (4.3) or (4.4) depends on the computation of A_4, where we need to assume $g\in\mathbb{D}^{5,8}(\mathbb{R},\gamma)$. Consider the second summand in the expression (4.10) denoted by

$$(A_{42})^2 := \frac{2}{n^4}\sum_{\substack{l_j=1,\\ j=1,\dots,8}}^{n}\sum_{i=1}^{3}\mathbb{E}\big(f_2^{(i)}(l_1,l_2,l_3)g_1(X_{l_4})f_2^{(i+4)}(l_5,l_6,l_7)g_1(X_{l_8})\big)$$

$$\times \rho_{12}\rho_{23}\rho_{i4}\rho_{56}\rho_{67}\rho_{(i+4)8}. \tag{4.24}$$

Taking into account that g has five derivatives and the terms $f_2^{(2)}$ and $f_2^{(6)}$ involve g''', we can apply duality twice using the factors that have Hermite rank 1. In this way, we get the following item in the bound of A_{42}:

$$\sqrt{\frac{C}{n^4}\sum_{|l_j|=1,j=1,\dots,8}^{n}\rho_{12}^2\rho_{23}\rho_{24}\rho_{56}^2\rho_{67}\rho_{68}},$$

which gives the rate $\frac{1}{n}(\sum_{|k|\le n}|\rho(k)|)^2$. This rate cannot always be better than that of $A_1+\sqrt{A_2}$ bound since the sequences $\frac{1}{n}(\sum_{|k|\le n}|\rho(k)|)^2$ and $n^{-\frac{1}{2}}(\sum_{|k|\le n}|\rho(k)|^{\frac{3}{2}})^2$ are not comparable. An example could be $\rho(k)\sim k^{-\alpha}$ for $\alpha\in(\frac{1}{2},\frac{2}{3})$. This suggests us using the bound (4.3) that is given by (4.23).

Case $g\in\mathbb{D}^{6,8}(\mathbb{R},\gamma)$. For the terms A_1, A_2 and A_3, we still have the estimates (4.15), (4.22) and (4.20). Let us now study the term A_4 given by (4.10). The terms $f_2^{(2)}$ and $f_2^{(6)}$ involve g''' and they can be integrated by parts three times. Therefore, we are going to use only three integration by parts. On the other hand, the terms $f_2^{(2)}$, $f_2^{(6)}$, $f_1^{(1)}$ and $f_1^{(4)}$ have two factors with Hermite rank one that can be represented as divergences, but the other terms have only one. All these terms are similar, with the only difference being the number of factors with Hermite rank one. We will handle only the term $f_1^{(1)}$ that has two factors with Hermite rank one and the term $f_1^{(2)}$ that has only one. The other terms could be treated in a similar way. In this way, for the term $f_1^{(1)}$, we obtain, after integrating by parts three times,

$$\left|\mathbb{E}\left(f_1^{(1)}(l_1,l_2,l_3)g_1(X_{l_4})f_1^{(5)}(l_5,l_6,l_7)g_1(X_{l_8})\right)\right|\le C\sum_{(i,s,j,t,h)\in D_4}|\rho_{2i}\rho_{sj}\rho_{th}|,$$

where

$$D_4=\left\{(i,s,j,t,h):1\le i,j,h\le 8;\ s,t\in\{3,4,6,7,8\};\ i\ne 2;\ s\notin\{i,j\};\ t\notin\{i,s,j,h\}\right\}. \tag{4.25}$$

On the other hand, for the term $f_1^{(2)}$, we obtain, after integrating by parts three times,

$$\begin{aligned}&\left|\mathbb{E}\left(f_1^{(2)}(l_1,l_2,l_3)g_1(X_{l_4})f_1^{(6)}(l_5,l_6,l_7)g_1(X_{l_8})\right)\right|\\&\quad\le C\sum_{\substack{i\ne s\ne j\\ i,s,j\in\{4,7,8\}}}|\rho_{3i}\rho_{sj}|+C\sum_{(i,s,j,t,h)\in D_5}|\rho_{3i}\rho_{sj}\rho_{th}|,\end{aligned}$$

where

$$D_5=\left\{(i,s,j,t,h):1\le i,j,h\le 8;\ s,t\in\{4,7,8\};\ i\ne 3;\ s\notin\{i,j\};\ t\notin\{i,s,j,h\}\right\}. \tag{4.26}$$

By Lemma A.4 and Lemma A.5, we obtain

$$A_4\le\frac{C}{n}\left(\sum_{|k|\le n}\left|\rho(k)\right|\right)^{\frac{3}{2}}+\frac{C}{n}\left(\sum_{|k|\le n}\left|\rho(k)\right|^{\frac{4}{3}}\right)^{3}.$$

Then, from (A.23) with $M=3$ and (A.24), we deduce $A_4\prec A_3$. We already know that $A_2\prec A_1\prec A_3\prec\sqrt{A_2}$. Also using (A.25) it follows that $A_3\prec\sqrt{A_1}$. Thus, we use (4.4) for the bound of $d_{\mathrm{TV}}(Y_n/\sigma_n,Z)$ which is given by the estimate (4.20) of the term A_3. □

4.3. *Case* $d\ge 3$

Theorem 4.7. *Assume* $g\in\mathbb{D}^{3d-2,4}(\mathbb{R},\gamma)$ *has Hermite rank* $d\ge 3$ *and suppose that* (1.2) *holds true. Then we have the following estimate*

$$d_{\mathrm{TV}}(Y_n/\sigma_n,Z)\le Cn^{-\frac{1}{2}}\sum_{|k|\le n}\left|\rho(k)\right|^{d-1}\left(\sum_{|k|\le n}\left|\rho(k)\right|^2\right)^{\frac{1}{2}}+Cn^{-\frac{1}{2}}\left(\sum_{|k|\le n}\left|\rho(k)\right|\right)^{\frac{1}{2}}. \tag{4.27}$$

Proof of Theorem 4.7. Inequality (4.27) will be established using Proposition 3.1 that is specifically expressed as (4.2). The proof will be done in two steps.

Step 1: First, we consider the case when g is the Hermite polynomial H_d. By Lemma 4.1 and Lemma A.2, we have

$$A_1 \le \frac{C}{n^2} \sum_{l_1,l_2,l_3,l_4=1}^{n} \big|\rho(l_1-l_2)^{\beta_1}\rho(l_3-l_4)^{\beta_2}\rho(l_2-l_4)^{\beta_3} \times \rho(l_1-l_3)^{\beta_4}\rho(l_1-l_4)^{\beta_5}\rho(l_2-l_3)^{\beta_6}\big|,$$

where the β_i's satisfy $\sum_{i=1}^{6}\beta_i = 2d$, $\beta_2+\beta_3+\beta_5 = d$, $\beta_1+\beta_3+\beta_6 = d$, $\beta_1+\beta_4+\beta_5 = d$, $\beta_2+\beta_4+\beta_6 = d$ and $\beta_j \ge 1$ for $j = 1,2,3$. Making the change of variables, $l_i - l_4 \to k_i$, $i = 1,2,3$ yields

$$A_1 \le \frac{C}{n} \sum_{k_1,k_2,k_3=1}^{n} \big|\rho(k_1-k_2)^{\beta_1}\rho(k_3)^{\beta_2}\rho(k_2)^{\beta_3}\rho(k_1-k_3)^{\beta_4}\rho(k_1)^{\beta_5}\rho(k_2-k_3)^{\beta_6}\big|.$$

Applying the Brascamp–Lieb inequality (2.12), we can write

$$A_1 \le \frac{C}{n}\prod_{i=1}^{6}\bigg(\sum_{|k_i|\le n}\big|\rho(k_i)\big|^{\frac{\beta_i}{p_i}}\bigg)^{p_i},$$

where the p_i's satisfy $\sum_{i=1}^{6} p_i = 3$, $p_i \le 1$, $p_1+p_3+p_5 \le 2$, $p_2+p_3+p_6 \le 2$, $p_2+p_4+p_5 \le 2$ and $p_1+p_4+p_6 \le 2$. The restriction of β_i could be further simplified as

$$\beta_1 = \beta_2, \qquad \beta_3 = \beta_4, \qquad \beta_5 = \beta_6, \qquad \beta_1+\beta_3+\beta_5 = d, \quad \text{and} \quad \beta_1,\beta_3 \ge 1.$$

Then we choose $p_1 = p_2$, $p_3 = p_4$, $p_5 = p_6$ to obtain

$$A_1 \le \frac{C}{n}\bigg(\prod_{i=1,3,5}\bigg(\sum_{|k_i|\le n}\big|\rho(k_i)\big|^{\frac{\beta_i}{p_i}}\bigg)^{p_i}\bigg)^2. \tag{4.28}$$

We are going to choose $p_i = \frac{\beta_i}{d-1} + \epsilon_i$ for $i = 1,3,5$, where the ϵ_i's satisfy $\epsilon_i \ge 0$ and $\frac{d}{d-1} + \sum_{i=1,3,5}\epsilon_i = \frac{3}{2}$. To choose the values of the ϵ_i's we consider two cases. Set $\delta = \frac{1}{2} - \frac{1}{d-1}$.

(i) Suppose that $\delta \le 1 - \frac{\beta_1}{d-1}$. Then, we take $\epsilon_1 = \delta$ and $\epsilon_3 = \epsilon_5 = 0$ and we obtain $p_1 = \frac{\beta_1}{d-1} + \frac{1}{2} - \frac{1}{d-1}$, $p_3 = \frac{\beta_3}{d-1}$ and $p_5 = \frac{\beta_5}{d-1}$.

(ii) Suppose that $\delta \ge 1 - \frac{\beta_1}{d-1}$. Then, we take $\epsilon_1 = 1 - \frac{\beta_1}{d-1}$ and $\epsilon_3 = \delta - \epsilon_1$ and $\epsilon_5 = 0$ and we obtain $p_1 = 1$, $p_3 = \frac{\beta_3}{d-1} + \frac{\beta_1}{d-1} - \frac{1}{2} - \frac{1}{d-1}$ and $p_5 = \frac{\beta_5}{d-1}$.

It is easy to show that these p_i's satisfy the desired conditions and, furthermore, $\beta_i \ge 2p_i$ for $i = 1,3,5$. This allows us to choose the pair (α_i,γ_i) that satisfies the following equations

$$\frac{\alpha_i}{2} + \frac{\gamma_i}{d-1} = 1, \quad \text{and} \quad \alpha_i + \gamma_i = \frac{\beta_i}{p_i}. \tag{4.29}$$

Then Hölder inequality implies

$$\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{\beta_i}{p_i}} \le \bigg(\sum_{|k|\le n}\big|\rho(k)\big|^2\bigg)^{\frac{\alpha_i}{2}}\bigg(\sum_{|k|\le n}\big|\rho(k)\big|^{d-1}\bigg)^{\frac{\gamma_i}{d-1}}.$$

We plug this inequality into (4.28) and solve α_i, γ_i from (4.29). In this way, we obtain the inequality

$$A_1 \le \frac{C}{n}\bigg(\sum_{|k|\le n}\big|\rho(k)\big|^{d-1}\bigg)^2\sum_{|k|\le n}\big|\rho(k)\big|^2. \tag{4.30}$$

Step 2: We consider the case $g \in \mathbb{D}^{3d-2}(\mathbb{R}, \gamma)$. By Lemma 4.1 and Lemma A.2, we have

$$A_1 \le \frac{C}{n^2} \sum_{l_1,l_2,l_3,l_4=1}^{n} \big|\rho(l_1-l_2)^{\beta_1}\rho(l_3-l_4)^{\beta_2}\rho(l_2-l_4)^{\beta_3} \times \rho(l_1-l_3)^{\beta_4}\rho(l_1-l_4)^{\beta_5}\rho(l_2-l_3)^{\beta_6}\big|, \tag{4.31}$$

where the β_i's satisfy $\beta_i \le d$, $\beta_j \ge 1$ for $j = 1, 2, 3$, $\sum_{i=1}^{6} \beta_i \le 3d-1$ and the lower bounds

$$\beta_2+\beta_3+\beta_5 \ge d,$$
$$\beta_1+\beta_3+\beta_6 \ge d,$$
$$\beta_1+\beta_4+\beta_5 \ge d,$$
$$\beta_2+\beta_4+\beta_6 \ge d.$$

When all the above β_i's inequalities attain the lower bound d, the right hand-side of (4.31) coincides with the case when g is the Hermite polynomial H_d. This case has been discussed in Step 1. On the other hand, if $\beta_1\wedge\beta_2+\beta_3\wedge\beta_4+\beta_5\wedge\beta_6 \ge d$ and $\beta_3\wedge\beta_4 \ge 1$, taking into account that $|\rho| \le 1$, the right-hand side of (4.31) is actually dominated by the case where all the β_i's inequalities attain the lower bound d.

Now we need to consider all the other possible cases. In each case, we make the change of variables $l_1-l_2 = k_1$, $l_3-l_4 = k_2$, $l_2-l_4 = k_3$.

(i) *Case* $\beta_4=\beta_5=\beta_6=0$. Then $\beta_1=\beta_2=d$, $\beta_3=1$. For these values of the β_i's we can write the right hand-side of (4.31) as

$$\begin{aligned}
&\frac{1}{n^2}\sum_{l_1,l_2,l_3,l_4=1}^{n} \big|\rho(l_1-l_2)^d\rho(l_3-l_4)^d\rho(l_2-l_4)\big| \\
&\quad= \frac{1}{n}\sum_{|k_i|\le n, 1\le i\le 3} \big|\rho(k_1)\big|^d\big|\rho(k_2)\big|^d\big|\rho(k_3)\big| \\
&\quad\le \frac{C}{n}\sum_{|k|\le n}\big|\rho(k)\big|.
\end{aligned}$$

(ii) *Case* $\beta_4=\beta_5=0$, $\beta_6>0$. Then $\beta_1=d$, $\beta_2<d$, $\beta_2+\beta_3\ge d$ and $\beta_2+\beta_6\ge d$. Using (1.2), we can write

$$\begin{aligned}
A_1 &\le \frac{C}{n}\sum_{|k_i|\le n, i=2,3} \big|\rho(k_2)\big|^{\beta_2}\big|\rho(k_3)\big|^{\beta_3}\big|\rho(k_3-k_2)\big|^{\beta_6} \\
&\le \frac{C}{n}\sum_{|k_i|\le n, i=2,3} \big|\rho(k_2)\big|^{\beta_2}\big|\rho(k_3)\big|^{d-\beta_2}\big|\rho(k_3-k_2)\big|^{d-\beta_2} \\
&\le \frac{C}{n}\sum_{|k|\le n}\big|\rho(k)\big|^{d-\beta_2} \le \frac{C}{n}\sum_{|k|\le n}\big|\rho(k)\big|,
\end{aligned}$$

where in the third inequality we have used (2.12) with $p_1=\frac{\beta_2}{d}$, $p_2=1$ and $p_3=\frac{d-\beta_2}{d}$.

(iii) *Case* $\beta_4=\beta_6=0$, $\beta_5>0$. This case is similar to (ii).

(iv) *Case* $\beta_5=\beta_6=0$, $\beta_4>0$. Then $\beta_2+\beta_3\ge d$, $\beta_1+\beta_3\ge d$, $\beta_1+\beta_4\ge d$, $\beta_2+\beta_4\ge d$. It is easy to see $\beta_1\wedge\beta_2+\beta_3\wedge\beta_4+\beta_5\wedge\beta_6\ge d$ and, furthermore, $\beta_3\wedge\beta_4\ge 1$. This situation has been discussed before and A_1 is dominated by the bound in the case where g is the Hermite polynomial.

(v) $\beta_4 = 0,\ \beta_5 > 0,\ \beta_6 > 0$. Then $\beta_1 < d$, $\beta_2 < d$, $\beta_1 + \beta_5 \geq d$, $\beta_2 + \beta_6 \geq d$. As a consequence, we obtain

$$A_1 \leq \frac{C}{n} \sum_{|k_i| \leq n, 1 \leq i \leq 3} \left|\rho(k_1)\right|^{\beta_1} \left|\rho(k_2)\right|^{\beta_2} \left|\rho(k_3)\right|^{\beta_3} \times \left|\rho(k_1 + k_3)\right|^{d-\beta_1} \left|\rho(k_3 - k_2)\right|^{d-\beta_2} \leq \frac{C}{n} \sum_{|k| \leq n} \left|\rho(k)\right|,$$

where have used (2.12) for $p_i = \frac{\beta_i}{d}$ for $i = 1, 2$, $p_3 = 1$ and $p_{i+3} = \frac{d-\beta_i}{d}$ for $i = 1, 2$.

(vi) $\beta_5 = 0,\ \beta_4 > 0,\ \beta_6 > 0$. Then $\beta_2 + \beta_3 \geq d$, $\beta_1 + \beta_4 \geq d$. This case is similar to (v).

(vii) $\beta_6 = 0,\ \beta_4 > 0,\ \beta_5 > 0$. This case is similar to (v) and (vi).

(viii) $\beta_i > 0$ *for all* $1 \leq i \leq 6$, *and* $\beta_1 \wedge \beta_2 + \beta_3 \wedge \beta_4 + \beta_5 \wedge \beta_6 < d$. Without loss of generality, we may assume that $\beta_1 \leq \beta_2$. We take into account of $\beta_1 + \beta_4 + \beta_5 \geq d$ and $\beta_1 + \beta_3 + \beta_6 \geq d$, so there are two cases: $\beta_3 \leq \beta_4$, $\beta_5 \leq \beta_6$; and $\beta_4 \leq \beta_3$, $\beta_6 \leq \beta_5$. These two cases are actually equivalent, because in the second case, we can make the change of variable $l_3 - l_1 \to k_3$, instead of $l_2 - l_4 \to k_3$ for the first case. Thus it sufficies to consider the first case, i.e.,

$$A_1 \leq \frac{C}{n} \sum_{\substack{|k_i| \leq n, \\ 1 \leq i \leq 3}} \left|\rho(k_1)\right|^{\beta_1} \left|\rho(k_2)\right|^{\beta_2} \left|\rho(k_3)\right|^{\beta_3} \times \left|\rho(k_1 - k_2 + k_3)\right|^{\beta_4} \left|\rho(k_1 + k_3)\right|^{\beta_5} \left|\rho(k_3 - k_2)\right|^{\beta_6},$$

where $\beta_1 + \beta_3 + \beta_5 < d$, $\beta_2 + \beta_4 + \beta_6 > d$ since $\sum_{i=1}^{6} \beta_i > 2d$.

Next we will apply Brascamp–Lieb inequality (2.12) according to several different subcases.

(1) Suppose $\beta_1 \wedge \beta_3 \wedge \beta_5 = \beta_1$. Then if $\sum_{i=2}^{6} \beta_i \geq 2d$, the right-hand side of the above inequality is bounded by the case $\sum_{i=2}^{6} \beta_i = 2d$ when we decrease β_i's, $i = 2, 4, 6$ appropriately. We use (2.12) with $p_1 = 1$, $p_i = \frac{\beta_i}{d}$ for $i \geq 2$, taking into account that $|\rho| \leq 1$, to obtain

$$A_1 \leq \frac{C}{n} \sum_{|k| \leq n} \left|\rho(k)\right|^{\beta_1} \leq \frac{C}{n} \sum_{|k| \leq n} \left|\rho(k)\right|.$$

If $\sum_{i=2}^{6} \beta_i < 2d$, for which an example could be $\beta_1 = 2$, $\beta_3 = 2$, $\beta_5 = d - 5$, $\beta_2 = 3$, $\beta_4 = 3$, $\beta_6 = d - 4$, we can see that $\beta_1 \geq 2$ because of $\sum_{i=1}^{d} \beta_i > 2d$ and $\sum_{i=2}^{6} \beta_i < 2d$, and correspondingly $d > \sum_{i=1,3,5} \beta_i \geq 6$. Furthermore, $\beta_1 < \frac{d}{3}$. In order to apply (2.12), we choose with $p_1 = 1$ and $p_i = \frac{2\beta_i}{\sum_{i=2}^{6} \beta_i}$ for $i = 2, \dots, 6$. One can easily check that $p_i \leq 1$, $\sum_{i=1}^{6} p_i = 3$, $\sum_{i=1,3,5} p_i < 2$ and $\sum_{i=2,3,6} p_i < 2$. In this way, we obtain

$$\begin{aligned} A_1 &\leq \frac{C}{n} \sum_{|k| \leq n} \left|\rho(k)\right|^{\beta_1} \left(\sum_{|k| \leq n} \left|\rho(k)\right|^{\frac{\beta_2 + \dots + \beta_6}{2}} \right)^2 \\ &\leq \frac{C}{n} \sum_{|k| \leq n} \left|\rho(k)\right|^{\beta_1} \left(\sum_{|k| \leq n} \left|\rho(k)\right|^{d - \frac{\beta_1}{2}} \right)^2 \\ &\leq \sup_{2 \leq a < \frac{d}{3}} \frac{C}{n} \sum_{|k| \leq n} \left|\rho(k)\right|^{a} \left(\sum_{|k| \leq n} \left|\rho(k)\right|^{d - \frac{a}{2}} \right)^2. \end{aligned}$$

In view of Lemma A.10 the supremum in the above display is attained at $a = 2$ and it coincides with the first term in the estimate (4.27).

(2) $\beta_1 \wedge \beta_3 \wedge \beta_5 = \beta_5$. We use the same approach as for the subcase (1).

(3) $\beta_1 \wedge \beta_3 \wedge \beta_5 = \beta_3$. We follow the same methodology. When $\sum_{i \neq 3} \beta_i < 2d$, the arguments are the same. When $\sum_{i \neq 3} \beta_i \geq 2d$, since $d \leq \beta_1 + \beta_4 + \beta_5 < 2d$, we can decrease β_2, β_6 appropriately such that $\sum_{i \neq 3} \beta_i = 2d$ and at

the same time this implies $\beta_2+\beta_6\le d$. Then we use (2.12) with $p_3=1$, $p_i=\frac{\beta_i}{d}$ for $i\neq 3$ to obtain

$$A_1\le \frac{C}{n}\sum_{|k|\le n}\left|\rho(k)\right|^{\beta_3}\le \frac{C}{n}\sum_{|k|\le n}\left|\rho(k)\right|.$$

This completes the proof of the theorem. □

Remark 4.8. In the case of the Hermite polynomial $g=H_d$, $d\ge 3$, the proof of Theorem 4.7, based on Proposition 3.1, yields

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le Cn^{-\frac{1}{2}}\sum_{|k|\le n}\left|\rho(k)\right|^{d-1}\left(\sum_{|k|\le n}\left|\rho(k)\right|^2\right)^{\frac{1}{2}}. \tag{4.32}$$

In this case Proposition 3.2 reduces to the computation of the third and fourth cumulants and one can derive the following bound (see [5]), which is better than (4.32):

$$\begin{aligned}&d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\\&\quad\le \frac{C}{n}\left(\sum_{|k|\le n}\left|\rho(k)\right|^{d-1}\right)^2\sum_{|k|\le n}\left|\rho(k)\right|^2+\frac{C}{\sqrt{n}}\left(\sum_{|k|\le n}\left|\rho(k)\right|^{\frac{3d}{4}}\right)^2\mathbf{1}_{\{d\text{ even}\}}.\end{aligned}$$

However, applying Proposition 3.2 to the case of a general function g is a much harder problem and it will not be dealt in this paper.

Consider the particular case where $\rho(k)\sim k^{-\alpha}$, as k tends to infinity, for some $\alpha>0$. Then, condition (1.2) is satisfied provided $\alpha d>1$. In this case, Theorems 4.2, 4.3 and 4.7 imply the following results.

Corollary 4.9. *Suppose that $\rho(k)\sim k^{-\alpha}$, as k tends to infinity, where $\alpha>0$ is such that $\alpha d>1$. Then, the following estimates hold true in the context of Theorem 1.1*:

(i) *If $g\in\mathbb{D}^{2,4}(\mathbb{R},\gamma)$ has Hermite rank 1 and $\alpha>1$,*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le Cn^{-\frac{1}{2}}.$$

(ii) *If $g\in\mathbb{D}^{2,4}(\mathbb{R},\gamma)$ has Hermite rank 2 and $\alpha>\frac{2}{3}$,*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le\begin{cases}Cn^{-\frac{1}{2}} & \text{if } \alpha>1,\\ Cn^{-\frac{1}{2}}(\log n)^{\frac{3}{2}} & \text{if } \alpha=1,\\ Cn^{1-\frac{3}{2}\alpha} & \text{if } \alpha\in(\frac{2}{3},1).\end{cases}$$

(iii) *If $g\in\mathbb{D}^{3,4}(\mathbb{R},\gamma)$ has Hermite rank 2,*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le\begin{cases}Cn^{-\frac{1}{2}} & \text{if } \alpha>1,\\ Cn^{-\frac{1}{2}}\log n & \text{if } \alpha=1,\\ Cn^{\frac{1}{2}-\alpha} & \text{if } \alpha\in(\frac{1}{2},1).\end{cases}$$

(iv) *If $g\in\mathbb{D}^{4,4}(\mathbb{R},\gamma)$ has Hermite rank 2,*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z)\le\begin{cases}Cn^{-\frac{1}{2}} & \text{if } \alpha>1,\\ Cn^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} & \text{if } \alpha=1,\\ Cn^{-\frac{\alpha}{2}} & \text{if } \alpha\in(\frac{2}{3},1),\\ Cn^{1-2\alpha} & \text{if } \alpha\in(\frac{1}{2},\frac{2}{3}].\end{cases}$$

(v) *If* $g \in \mathbb{D}^{5,6}(\mathbb{R}, \gamma)$ *has Hermite rank* 2,

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z) \le \begin{cases} Cn^{-\frac{1}{2}} & \text{if } \alpha > 1, \\ Cn^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} & \text{if } \alpha = 1, \\ Cn^{-\frac{\alpha}{2}} & \text{if } \alpha \in (\frac{3}{5}, 1), \\ Cn^{\frac{3}{2}-3\alpha} & \text{if } \alpha \in (\frac{1}{2}, \frac{3}{5}]. \end{cases}$$

(vi) *If* $g \in \mathbb{D}^{6,8}(\mathbb{R}, \gamma)$ *has Hermite rank* 2,

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z) \le \begin{cases} Cn^{-\frac{1}{2}} & \text{if } \alpha > \frac{2}{3}, \\ Cn^{-\frac{1}{2}}(\log n)^{2} & \text{if } \alpha = \frac{2}{3}, \\ Cn^{\frac{3}{2}-3\alpha} & \text{if } \alpha \in (\frac{1}{2}, \frac{2}{3}). \end{cases}$$

(vii) *If* $g \in \mathbb{D}^{3d-2,4}(\mathbb{R}, \gamma)$ *has Hermite rank* $d \ge 3$,

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z) \le \begin{cases} Cn^{-\frac{1}{2}} & \text{if } \alpha > 1, \\ Cn^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} & \text{if } \alpha = 1, \\ Cn^{-\frac{\alpha}{2}} & \text{if } \alpha \in (\frac{1}{2}, 1), \\ Cn^{-\frac{\alpha}{2}}\sqrt{\log n} & \text{if } \alpha = \frac{1}{2}, \\ Cn^{-\frac{\alpha}{2}} & \text{if } \alpha \in (\frac{2}{2d-1}, \frac{1}{2}), \\ Cn^{1-\alpha d} & \text{if } \alpha \in (\frac{1}{d}, \frac{2}{2d-1}]. \end{cases}$$

(viii) *When* $g = H_d$, $d \ge 3$, *the bound* (4.2) *combined with the estimate* (4.30) *yields*

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z) \le \begin{cases} Cn^{-\frac{1}{2}} & \text{if } \alpha > \frac{1}{2}, \\ Cn^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} & \text{if } \alpha = \frac{1}{2}, \\ Cn^{-\alpha} & \text{if } \alpha \in (\frac{1}{d-1}, \frac{1}{2}), \\ Cn^{-\alpha}\log n & \text{if } \alpha = \frac{1}{d-1}, \\ Cn^{1-\alpha d} & \text{if } \alpha \in (\frac{1}{d}, \frac{1}{d-1}). \end{cases}$$

We remark that the bounds derived in point (viii) coincide with the estimates obtained by Biermé, Bonami and León in [4] using techniques of Fourier analysis. Corollary 4.9 can be applied to any function g with an expansion $g(x) = \sum_{m=d}^{d+k} c_m H_m(x)$ for any $k \ge 0$.

5. Application to fractional Brownian motion

Recall that the fractional Brownian motion (fBm) $B = \{B_t, t \in \mathbb{R}\}$ with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with the covariance function

$$\mathbb{E}(B_s B_t) = \frac{1}{2}\big(|s|^{2H} + |t|^{2H} - |s - t|^{2H}\big).$$

The fractional noise defined by $X_j = B_{j+1} - B_j$, $j \in \mathbb{Z}$ is an example of a Gaussian stationary sequence with unit variance. The covariance function is given by

$$\rho_H(j) = \frac{1}{2}\big(|j + 1|^{2H} + |j - 1|^{2H} - 2|j|^{2H}\big).$$

Notice that $\rho_H(j)$ behaves as $H(2H - 1)j^{2H-2}$ as $j \to \infty$. Thus, this covariance function has a power decay at infinity with $\alpha = 2 - 2H$. Consider the sequence Y_n defined by

$$Y_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} g(B_{j+1} - B_j),$$

where $g \in L^2(\mathbb{R}, \gamma)$ has Hermite rank $d \geq 1$. As a consequence, the estimates obtained in Corollary 4.9 hold with $\alpha = 2 - 2H$. Here are some examples where these results can be applied.

Example 1. Consider the function $g(x) = e^x - \sqrt{e}$. It is easy to check that this function has Hermite rank $d = 1$ and $g \in \mathbb{D}^{\infty,p}$ for any $p \geq 1$. Then g admits the expansion $g(x) = \sum_{m\geq 1} c_m H_m(x)$. By Corollary 4.9(i), we have for $H \in (0, \frac{1}{2})$,

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z) \leq Cn^{-\frac{1}{2}}.$$

Example 2. Consider the function $g(x) = \sin(x) - x\mathbb{E}(Z\sin(Z))$. We can check that this function has Hermite rank $d = 3$ and $g \in \mathbb{D}^{\infty,p}$ for any $p > 1$. Suppose g admits the expansion $g(x) = \sum_{m\geq 3} c_m H_m(x)$. By Corollary 4.9(vii), we have

$$d_{\mathrm{TV}}(Y_n/\sigma_n, Z) \leq \begin{cases} Cn^{-\frac{1}{2}} & \text{if } H \in (0, \frac{1}{2}), \\ Cn^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} & \text{if } H = \frac{1}{2}, \\ Cn^{H-1} & \text{if } H \in (\frac{1}{2}, \frac{3}{4}), \\ Cn^{H-1}\sqrt{\log n} & \text{if } H = \frac{3}{4}, \\ Cn^{H-1} & \text{if } H \in (\frac{3}{4}, \frac{4}{5}), \\ Cn^{6H-5} & \text{if } H \in [\frac{4}{5}, \frac{5}{6}). \end{cases}$$

In the next subsections, we will review some applications to the power variations in which functionals with Hermite rank 2 will be considered.

5.1. *Application to the asymptotic behavior of power variations*

For any $p \geq 1$, the power variation of the fBm on the time interval [0, 1] is given by

$$V_n^p(B) = \sum_{j=0}^{n-1} |B_{\frac{j+1}{n}} - B_{\frac{j}{n}}|^p.$$

By the self-similarity property of fBm, the sequence $\{n^H(B_{\frac{j+1}{n}} - B_{\frac{j}{n}}), j \geq 0\}$ has the same distribution as $\{B_{j+1} - B_j, j \geq 0\}$, which is stationary and ergodic. By the Ergodic Theorem, we have, as $n \to \infty$,

$$n^{pH-1}V_n^p(B) \to c_p$$

almost surely and in $L^q(\Omega)$ for any $q \geq 1$, where $c_p = \mathbb{E}(|Z|^p)$. Moreover, when $H \in (0, \frac{3}{4})$, using the fact that the function $g(x) = |x|^p - c_p$ has Hermite rank 2, the Breuer–Major theorem leads to the following central limit theorem

$$S_n := \sqrt{n}\big(n^{pH-1}V_n^p(B) - c_p\big) \to N\big(0, \sigma_{H,p}^2\big), \tag{5.1}$$

where $\sigma_{H,p}^2 = \sum_{m=2}^{\infty} c_m^2 m! \sum_{k\in\mathbb{Z}} \rho_H(k)^m$, with $|x|^p - c_p = \sum_{m=2}^{\infty} c_m H_m(x)$. A functional version of this central limit theorem can also be proved (see [9]).

We can apply the results obtained in Section 3 to derive the rate of convergence for the total variation distance in (5.1). Indeed, the sequence S_n has the same distribution as

$$Y_n = \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n} |B_{j+1} - B_j|^p - c_p\right),$$

and it suffices to consider the case of the fractional noise $X_j = B_{j+1} - B_j$ and the function $g(x) = |x|^p - c_p$ that has Hermite rank 2. More precisely, if $N \leq p < N + 1$ where $N \geq 2$ is an integer, then the function g belongs to $\mathcal{D}^N := \bigcap_{q\geq 1} \mathbb{D}^{N,q}(\mathbb{R}, \gamma)$ and Corrollary 4.9 gives the convergence rate to zero of $d_{\mathrm{TV}}(S_n/\sigma_n, Z)$ with $\alpha = 2 - 2H$. Here are some examples.

Example 3. Let $p = 2.5$ and $\sigma_n^2 = \mathbb{E}(S_n^2) = \mathbb{E}(Y_n^2)$. Then $g \in \mathcal{D}^2$ and

$$d_{\mathrm{TV}}(S_n/\sigma_n, Z) \le \begin{cases} Cn^{-\frac{1}{2}} & \text{if } H \in (0, \frac{1}{2}), \\ Cn^{-\frac{1}{2}}(\log n)^{\frac{3}{2}} & \text{if } H = \frac{1}{2}, \\ Cn^{3H-2} & \text{if } H \in (\frac{1}{2}, \frac{2}{3}). \end{cases}$$

Example 4. Let $p = 3$ and $\sigma_n^2 = \mathbb{E}(S_n^2) = \mathbb{E}(Y_n^2)$. Then $g \in \mathcal{D}^3$ and

$$d_{\mathrm{TV}}(S_n/\sigma_n, Z) \le \begin{cases} Cn^{-\frac{1}{2}} & \text{if } H \in (0, \frac{1}{2}), \\ Cn^{-\frac{1}{2}}\log n & \text{if } H = \frac{1}{2}, \\ Cn^{2H-\frac{3}{2}} & \text{if } H \in (\frac{1}{2}, \frac{3}{4}). \end{cases}$$

Example 5. Let $p = 4$ and $\sigma_n^2 = \mathbb{E}(S_n^2) = \mathbb{E}(Y_n^2)$. Then $g \in \mathcal{D}^4$ and

$$d_{\mathrm{TV}}(S_n/\sigma_n, Z) \le \begin{cases} Cn^{-\frac{1}{2}} & \text{if } H \in (0, \frac{1}{2}), \\ Cn^{-\frac{1}{2}}\sqrt{\log n} & \text{if } H = \frac{1}{2}, \\ Cn^{H-1} & \text{if } H \in (\frac{1}{2}, \frac{2}{3}], \\ Cn^{4H-3} & \text{if } H \in (\frac{2}{3}, \frac{3}{4}). \end{cases}$$

Example 6. Let $p = 6$ and $\sigma_n^2 = \mathbb{E}(S_n^2) = \mathbb{E}(Y_n^2)$. Then $g \in \mathcal{D}^6$ and

$$d_{\mathrm{TV}}(S_n/\sigma_n, Z) \le \begin{cases} Cn^{-\frac{1}{2}} & \text{if } H \in (0, \frac{2}{3}), \\ Cn^{-\frac{1}{2}}(\log n)^2 & \text{if } H = \frac{2}{3}, \\ Cn^{6H-\frac{9}{2}} & \text{if } H \in (\frac{2}{3}, \frac{3}{4}). \end{cases}$$

5.2. *Application to the estimation of the Hurst parameter*

As an application of the convergence rates of power variations, we establish the consistency of the estimatior of the Hurst parameter H for the fBm, defined by means of p-power variations. This problem has been studied for $H > \frac{1}{2}$ using quadratic variations in the papers [2,10,11,20] and the references therein. In the paper [8], a consistent estimator based on the p-power variation is adopted, defined as

$$\tilde{H} = \frac{\log C_p - \log(n^{-1}V_n^p(B))}{p \log n},$$

where the specific constant C_p depends on p. In the paper [8], the author also discusses other filters to define the power variation and obtains a normalizing factor for the central limit theorem equal to $1/\sqrt{n}\log n$. Here we construct another estimator based on the p-power variation, which is motivated by the papers [2,11], where the quadratic variation is used.

Let $\lambda > 1$, $\lambda \in \mathbb{N}$ be a scaling parameter. Fix $p \ge 2$, and consider the statistics $T_{\lambda,n}$ defined by

$$T_{\lambda,n} := \frac{V_{\lambda n}^p(B)}{V_n^p(B)} = \frac{\sum_{j=0}^{\lambda n-1}|B_{\frac{j+1}{\lambda n}} - B_{\frac{j}{\lambda n}}|^p}{\sum_{j=0}^{n-1}|B_{\frac{j+1}{n}} - B_{\frac{j}{n}}|^p}.$$

Then we propose the following estimator for the Hurst parameter H:

$$\hat{H}_{\lambda,n} = \frac{1}{p}\left(1 - \frac{\log T_{\lambda,n}}{\log \lambda}\right). \tag{5.2}$$

In the next proposition we show the consistency of this estimator. Though the consistency could be clearly obtained from the ergodic theorem, we will apply the main results obtained in this paper to prove the consistency as well as the convergence rate.

Proposition 5.1. *When* $H \in (0, \frac{3}{4})$, *for* $p \in \{2\} \cup [3, \infty)$,

$$\lim_{n\to\infty} \sqrt{\frac{n}{\log n}} (\hat{H}_{\lambda,n} - H) = 0,$$

in probability.

Proof. Denote $\alpha_n = n^{-1+pH} V_n^p(B)$. Then

$$\log \alpha_{\lambda n} - \log \alpha_n = (-1 + pH) \log \lambda + \log T_{\lambda,n}.$$

Thus

$$\hat{H}_{\lambda,n} - H = -\frac{\log \alpha_{\lambda n} - \log \alpha_n}{p \log \lambda}. \tag{5.3}$$

Let $\sigma_n^2 = \mathbb{E}[(\sqrt{n}(\alpha_n - c_p))^2]$. By previous results, we know that $\sqrt{n}(\alpha_n - c_p) \to \sigma_{H,p} Z$ where $\sigma_n^2 \to \sigma_{H,p}^2$, and

$$d_{\mathrm{TV}}\left(\frac{\sqrt{n}(\alpha_n - c_p)}{\sigma_n}, Z\right) < n^{-a}$$

for some $a > 0$. Then for any $\epsilon > 0$,

$$\begin{aligned} P\left(\left|\frac{\sqrt{n}(\alpha_n - c_p)}{\sigma_n}\right| > \epsilon\sqrt{\log n}\right) &\le P\big(|Z| > \epsilon\sqrt{\log n}\big) + n^{-a} \\ &\le \frac{C_\epsilon}{n^{\frac{\epsilon^2}{2}}\sqrt{\log n}} + n^{-a}, \end{aligned}$$

where we have used the estimate for the tail of a standard Gaussian random variable, i.e., $P(Z > x) \le \frac{e^{-x^2/2}}{x\sqrt{2\pi}}$. This implies that $\frac{\sqrt{n}(\alpha_n - c_p)}{\sqrt{\log n}} \to 0$ in probability as $n \to \infty$. Back to equation (5.3), note that $\log \alpha_n - \log c_p = \frac{1}{\alpha_n^*}(\alpha_n - c_p)$ for some α_n^* between α_n and c_p. These results are true for $\alpha_{\lambda n}$ as well, so we conclude that $\sqrt{\frac{n}{\log n}}(\hat{H}_{\lambda,n} - H) \to 0$ in probability. □

Appendix

In this section we show some technical lemmas that play a crucial role in the proof of our main results.

Lemma A.1. *Under the notation and assumptions of Theorem 1.1, let* I_1 *and* I_2 *be the random variables defined in* (4.5) *and* (4.6), *respectively. Suppose* $d = 2$. *Then we have the following estimates.*

(1) *If* $g \in \mathbb{D}^{3,4}(\mathbb{R}, \gamma)$, *then for* $i = 1, 2$, *we have*

$$\big|\mathbb{E}(I_i)\big| \le C \sum_{i \ne 1} \big|\rho(l_1 - l_i)\big|.$$

(2) *If* $g \in \mathbb{D}^{4,4}(\mathbb{R}, \gamma)$, *then for* $i = 1, 2$, *we have*

$$\big|\mathbb{E}(I_i)\big| \le C \left|\big(\rho(l_1 - l_2) + \rho(l_1 - l_4)\big) \sum_{j \ne 3} \rho(l_j - l_3) + \rho(l_1 - l_3)\right|. \tag{A.1}$$

(3) *If* g *is the Hermite polynomial* $x^2 - 1$, *then*

$$\big|\mathbb{E}(I_i)\big| \le C \big|\rho(l_1 - l_3)\big|.$$

Proof. We first consider the term I_1. Observe that

$$g_1(X_{l_1}) = \delta\big(g_2(X_{l_1}) l_1\big).$$

Applying the duality relationship (2.1), we obtain

$$\mathbb{E}(I_1) = \sum_{a+b+c=1} \mathbb{E}\big(g^{(a+2)}(X_{l_2})g^{(b+2)}(X_{l_4})(g_1)^{(c)}(X_{l_3})g_2(X_{l_1})\big) \times \langle l_1, l_2^{\otimes a} \otimes l_4^{\otimes b} \otimes l_3^{\otimes c}\rangle_{\mathfrak{H}}.$$

When g is the Hermite polynomial x^2-1, we just need to consider the case $a=0$, $b=0$ and $c=1$. In this way we get

$$\big|\mathbb{E}(I_1)\big| \le C\big|\rho(l_1-l_3)\big|.$$

When $g \in \mathbb{D}^{3,4}(\mathbb{R},\gamma)$, we obtain

$$\big|\mathbb{E}(I_1)\big| \le C\sum_{i\neq 1}\big|\rho(l_1-l_i)\big|.$$

When $g \in \mathbb{D}^{4,4}(\mathbb{R},\gamma)$, in the case of $c=0$, we apply duality again to obtain

$$\begin{aligned}\mathbb{E}(I_1) = &\sum_{a+b=1}\sum_{a'+b'+c'=1} \mathbb{E}\big(g^{(a+a'+2)}(X_{l_2})g^{(b+b'+2)}(X_{l_4})g_2(X_{l_3})g_2^{(c')}(X_{l_1})\big)\\ &\times \langle l_1, l_2^{\otimes a} \otimes l_4^{\otimes b}\rangle_{\mathfrak{H}}\langle l_3, l_2^{\otimes a'} \otimes l_4^{\otimes b'} \otimes l_1^{\otimes c'}\rangle_{\mathfrak{H}}\\ &+ \mathbb{E}\big(g''(X_{l_2})g''(X_{l_4})g_1'(X_{l_3})g_2(X_{l_1})\rho(l_1-l_3)\big).\end{aligned}$$

Then the inequality (A.1) for $i=1$ is derived from expanding the above identities.

Similarly, for the term I_2, since $g'(X)$ has the Hermite rank 1, we can write

$$g'(X_{l_i}) = \delta\big((g')_1(X_{l_i})l_i\big).$$

Using this representation, we have

$$\mathbb{E}(I_2) = \mathbb{E}\big(\delta\big((g')_1(X_{l_1})l_1\big)\delta\big((g')_1(X_{l_3})l_3\big)g_1'(X_{l_2})g_1'(X_{l_4})\big).$$

We use the similar arguments as the term I_1 to obtain the inequality (A.1) for $i=2$. □

Lemma A.2. *Under the notation and assumptions of Theorem 1.1, let I_1 and I_2 be the random variables defined in (4.5) and (4.6), respectively. Suppose $d\ge 3$. Then for $i=1,2$,*

$$\begin{aligned}\big|\mathbb{E}(I_i)\big| \le C\sum_{\beta\in\mathcal{I}_1}\big|&\rho(l_1-l_2)^{\beta_1}\rho(l_1-l_3)^{\beta_2}\rho(l_1-l_4)^{\beta_3}\\ &\times\rho(l_3-l_2)^{\beta_4}\rho(l_2-l_4)^{\beta_5}\rho(l_3-l_4)^{\beta_6}\big|,\end{aligned}$$

where $\beta=(\beta_1,\dots,\beta_6)$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and

$$\mathcal{I}_1 = \Bigg\{\beta\in\mathbb{N}_0^6 : d-1\le\beta_1+\beta_2+\beta_3, d-1\le\beta_2+\beta_4+\beta_6, d-2\le\beta_1+\beta_4+\beta_5, d-2\le\beta_3+\beta_5+\beta_6, \sum_{i=1}^6\beta_i\le 3d-4\Bigg\}. \tag{A.2}$$

Moreover, if g is the Hermite polynomial H_d, we obtain

$$\begin{aligned}\big|\mathbb{E}(I_i)\big| \le C\sum_{\beta\in\mathcal{I}_3}\big|&\rho(l_1-l_2)^{\beta_1}\rho(l_1-l_3)^{\beta_2}\rho(l_1-l_4)^{\beta_3}\\ &\times\rho(l_3-l_2)^{\beta_3}\rho(l_2-l_4)^{\beta_2-1}\rho(l_3-l_4)^{\beta_1}\big|,\end{aligned} \tag{A.3}$$

where

$$\mathcal{I}_3 = \{\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3 : \beta_1 + \beta_2 + \beta_3 = d - 1\}.$$

Proof. We can represent the factor $g_1(X_{l_1})$ appearing in I_1 as

$$g_1(X_{l_1}) = \delta^{d-1}\big(g_d(X_{l_1})l_1^{\otimes(d-1)}\big).$$

Then applying the duality relationship (2.2) and Leibniz's rule yields

$$\mathbb{E}(I_1) = \sum_{a+b+c=d-1} \mathbb{E}\big(g^{(a+2)}(X_{l_2})g^{(b+2)}(X_{l_4})g_d(X_{l_1})g_1^{(c)}(X_{l_3})\big) \times \rho(l_1-l_2)^a\rho(l_1-l_4)^b\rho(l_1-l_3)^c.$$

We write

$$g_1^{(c)}(X_{l_3}) = \delta^{d-1-c}\big(T_{d-1-c}\big(g_1^{(c)}\big)(X_{l_3})l_3^{\otimes(d-1-c)}\big).$$

Then, applying again the duality relationship (2.2) and Leibniz's rule, we obtain

$$\begin{aligned}\mathbb{E}(I_1) = \sum_{a+b+c=d-1}\ \sum_{a'+b'+c'=d-1-c} &\mathbb{E}\big(g^{(a+a'+2)}(X_{l_2})g^{(b+b'+2)}(X_{l_4}) \\ &\times g_d^{(c')}(X_{l_1})T_{d-1-c}\big(g_1^{(c)}\big)(X_{l_3})\big) \\ &\times \rho(l_1-l_2)^a\rho(l_1-l_4)^b\rho(l_1-l_3)^{c+c'}\rho(l_3-l_2)^{a'}\rho(l_3-l_4)^{b'}.\end{aligned}$$

We can still represent the factors $g^{(a+a'+2)}(X_{l_2})$ and $g^{(b+b'+2)}(X_{l_4})$ as divergences:

$$g^{(a+a'+2)}(X_{l_2}) = \delta^{d-(a+a'+2)}\big(T_{d-(a+a'+2)}\big(g^{(a+a'+2)}\big)(X_{l_2})l_2^{\otimes(d-(a+a'+2))}\big)$$

and

$$g^{(b+b'+2)}(X_{l_4}) = \delta^{d-(b+b'+2)}\big(T_{d-(b+b'+2)}\big(g^{(b+b'+2)}\big)(X_{l_4})l_4^{\otimes(d-(b+b'+2))}\big).$$

Then, we repeat the above process to obtain, using the fact that $g \in \mathcal{D}^{3d-2}$,

$$\begin{aligned}\big|\mathbb{E}(I_1)\big| \le C\sum\big|&\rho(l_1-l_2)^{a+b''}\rho(l_1-l_4)^{b+b'''}\rho(l_1-l_3)^{c+c'} \\ &\times \rho(l_3-l_2)^{a'+c''}\rho(l_3-l_4)^{b'+c'''}\rho(l_2-l_4)^{a''+a'''}\big|,\end{aligned} \tag{A.4}$$

where the sum runs over all nonnegative integers $a, b, c, a', b', c', a'', b'', c'', a''', b''', c'''$ satisfying

$$\begin{aligned}&a+b+c = d-1, \\ &a'+b'+c' = d-1-c, \\ &a''+b''+c'' = \big(d-a'-a-2\big)\vee 0, \\ &a'''+b'''+c''' = \big(d-b-b'-a''-2\big)\vee 0.\end{aligned}$$

Inequality (A.4) can be equivalently written as

$$\begin{aligned}\big|\mathbb{E}(I_1)\big| \le C\sum_{\beta\in\mathcal{I}_1}\big|&\rho(l_1-l_2)^{\beta_1}\rho(l_1-l_3)^{\beta_2}\rho(l_1-l_4)^{\beta_3} \\ &\times \rho(l_3-l_2)^{\beta_4}\rho(l_2-l_4)^{\beta_5}\rho(l_3-l_4)^{\beta_6}\big|,\end{aligned}$$

where $\beta = (\beta_1, \dots, \beta_6)$ and $\mathcal{I}_1$ is the set defined in (A.2). Notice that we have the lower bound $\sum_{i=1}^6 \beta_i \ge 2d-3$. On the other hand, the upper bound $\sum_{i=1}^6 \beta_i \le 3d-4$ is attained when $a = d-1$, $a' = d-1$, $a''' = d-2$ and the other numbers vanish. Taking into account that in this case the function g'' might be differentiated $3d-4$ times, we need $g \in \mathcal{D}^{3d-2}$.

When g is the Hermite polynomial H_d, $g_d = 1$ and $g_1 = H_{d-1}$, so we have $T_{d-1-c}(g_1^{(c)}) = (d-1)(d-2)\cdots(d-c)$. In this case, taking into account of the orthogonality of Hermite polynomials of different order, we obtain

$$\big|\mathbb{E}(I_1)\big| \le C \sum_{\substack{a+b+c=d-1,\\ a'+b'=d-1-c,\\ a+a'=b+b'=\tilde{c}}} \big|\rho(l_1-l_2)^a \rho(l_1-l_4)^b \rho(l_1-l_3)^c \times \rho(l_3-l_2)^{a'} \rho(l_3-l_4)^{b'} \rho(l_2-l_4)^{d-2-\tilde{c}}\big|.$$

Again this can be written as

$$\big|\mathbb{E}(I_1)\big| \le C \sum_{\beta\in\mathcal{I}_2} \big|\rho(l_1-l_2)^{\beta_1} \rho(l_1-l_3)^{\beta_2} \rho(l_1-l_4)^{\beta_3} \times \rho(l_3-l_2)^{\beta_4} \rho(l_2-l_4)^{\beta_5} \rho(l_3-l_4)^{\beta_6}\big|,$$

where $\mathcal{I}_2$ is the set of $\beta \in \mathbb{N}_0^6$ such that $\beta_1+\beta_2+\beta_3 = d-1$, $\beta_4+\beta_6+\beta_2 = d-1$ and $\beta_1+\beta_4 = \beta_3+\beta_6 = d-2-\beta_5$. This implies $\beta_1 = \beta_6$, $\beta_3 = \beta_4$, $\beta_5 = \beta_2 - 1$ and $\beta_1+\beta_2+\beta_3 = d-1$, and this completes the proof of (A.3).

Similar arguments could be applied to handle the term I_2. □

Lemma A.3. *Assume condition* (1.2) *with $d = 2$. Define*

$$J_1 = \frac{1}{n^3} \sum_{l_1,\dots,l_6=1}^{n} \sum_{\substack{i=1\\ i\neq 2}}^{6} \sum_{\substack{s\in\{3,5,6\}\\ s\neq i}} \sum_{\substack{j=1\\ j\neq s}}^{6} |\rho_{2i}\rho_{sj}\rho_{12}\rho_{13}\rho_{45}\rho_{46}|$$

and

$$J_2 := \frac{1}{n^3} \sum_{l_1,\dots,l_6=1}^{n} \Bigg(\sum_{\substack{i\neq s\neq j\\ i,s,j\in\{3,5,6\}}} |\rho_{2i}\rho_{sj}\rho_{12}\rho_{13}\rho_{45}\rho_{46}| + \sum_{(i,s,j,t,h)\in D_3} |\rho_{2i}\rho_{sj}\rho_{th}\rho_{12}\rho_{13}\rho_{45}\rho_{46}| \Bigg),$$

where the set D_3 has been defined in (4.21) *and we recall that $\rho_{ij} = \rho(l_i - l_j)$. Then,*

$$J_1 \le \frac{C}{n} \bigg(\sum_{|k|\le n} \big|\rho(k)\big| \bigg)^2 \tag{A.5}$$

and

$$J_2 \le \frac{C}{n} \sum_{|k|\le n} \big|\rho(k)\big| + \frac{C}{n} \bigg(\sum_{|k|\le n} \big|\rho(k)\big|^{\frac{3}{2}} \bigg)^4. \tag{A.6}$$

Proof. *Step* 1: We show first the inequality (A.5). We make change of variables $l_1 - l_2 = k_1$, $l_1 - l_3 = k_2$, $l_4 - l_5 = k_3$, $l_4 - l_6 = k_4$. We first consider the term ρ_{2i} that has three possibilities: $\rho(k_1)$, $\rho(k_1 - k_2)$, or a new factor $\rho(k_5)$ where $k_5 = l_2 - l_i$ is linearly independent of k_t, $t = 1,\dots,4$. If ρ_{2i} is one of the first two cases, ρ_{sj} have three possibilities: $\rho(k_i)$ for $i = 2,3,4$; $\rho(k_1 - k_2)$ or $\rho(k_3 - k_4)$; a new factor $\rho(k_5)$ where $k_5 = l_j - l_s$ is independent of k_t, $1 \le t \le 4$. If ρ_{2i} is in the third case, i.e. a new factor, then ρ_{sj} have several possibilities: $\rho(k_i)$ for $i = 2,3,4$; $\rho(\mathbf{k}\cdot\mathbf{v})$ where $\mathbf{k}\cdot\mathbf{v}$ is a linear combination of two, three or four or five k_t's, $1 \le t \le 5$. Through this analysis, by taking advantage of the symmetry, we obtain

$$J_1 \le \frac{C}{n^2} \sum_{i=1}^{9} \sum_{|k_j|\le n, 1\le j\le 5} |J_{1i}|,$$

where

$$J_{11} = \rho(k_1)^2\rho(k_2)^2\rho(k_3)\rho(k_4),$$
$$J_{12} = \rho(k_1)^2\rho(k_2)\rho(k_1-k_2)\rho(k_3)\rho(k_4),$$
$$J_{13} = \rho(k_1)^2\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_3-k_4),$$
$$J_{14} = \rho(k_1)^2\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_5),$$
$$J_{15} = \rho(k_1)\rho(k_2)\rho(k_1-k_2)\rho(k_3)\rho(k_4)\rho(k_3-k_4),$$
$$J_{16} = \rho(k_1)\rho(k_2)\rho(k_1-k_2)\rho(k_3)\rho(k_4)\rho(k_5),$$
$$J_{17} = \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_1-k_5-k_2),$$
$$J_{18} = \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_1-k_2+k_3-k_4),$$
$$J_{19} = \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_1-k_2+k_3-k_4+k_5).$$

We claim that for $i = 1, \dots, 9$, the following estimate holds true

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5} |J_{1i}| \le \frac{C}{n}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^2. \tag{A.7}$$

The estimate (A.7) holds clearly for $i = 1$ and $i = 4$ due to condition (1.2) with $d = 2$. By the Cauchy–Schwartz inequality we have

$$\sum_{|k_1|,|k_2|\le n} \rho(k_1)^2\big|\rho(k_2)\rho(k_1-k_2)\big| < \infty$$

and (A.7) is true for $i = 2$. For $i = 3, 5, 6$, the estimate (A.7) follows from (A.18) and (A.19) with $M = 2$ and for $i = 7, 8, 9$ we use these inequalities with $M = 3, 4, 5$, respectively.

Step 2: We proceed to prove the inequality (A.6). Note that for the first summand in J_2, the product $\rho_{2i}\rho_{sj}$ can be only one of the following terms: $\rho_{23}\rho_{56}$, $\rho_{26}\rho_{35}$, or $\rho_{25}\rho_{36}$. In the first case, we obtain the term J_{15}, for which we have, by (A.18) with $M = 2$,

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5} |J_{15}| \le \frac{C}{n}\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{3}{2}}\Big)^4.$$

In the second and third case, we obtain the term J_{19}, for which we have, by (A.18) with $M = 5$,

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5} |J_{19}| \le \frac{C}{n^2}\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{6}{5}}\Big)^5. \tag{A.8}$$

By Hölder's inequality,

$$\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{6}{5}}\Big)^5 \le n\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{3}{2}}\Big)^4, \tag{A.9}$$

and we obtain the desired bound.

Let us now consider the second summand in the expression of J_2. This summand will consists of terms of the form $J_{1i}\rho_{th}$ for $i = 1, \dots, 4, 6, \dots, 9$, where ρ_{th} can be written as a linear combination of $k_1, \dots, k_5$. For $i = 6, \dots, 8$, we estimate the factor $|\rho_{th}|$ by one and apply the estimate (A.18) with $M = 2, 3, 4$ to obtain

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5} |J_{16}| \le \frac{C}{n^2}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^3\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{3}{2}}\Big)^2, \tag{A.10}$$

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5} |J_{17}| \le \frac{C}{n^2}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^2\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{4}{3}}\Big)^3, \tag{A.11}$$

and

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5}|J_{18}| \le \frac{C}{n^2}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{5}{4}}\Big)^4. \tag{A.12}$$

Then, from (A.10) and (A.24), we get

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5}|J_{16}| \le \frac{C}{n}\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{3}{2}}\Big)^4.$$

From (A.11), (A.23) with $M=3$ and (A.24),

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5}|J_{17}| \le \frac{C}{n}\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{3}{2}}\Big)^4.$$

Finally, from (A.12), (A.23) with $M=4$ and the above inequality of J_{17},

$$\frac{1}{n^2}\sum_{|k_j|\le n, 1\le j\le 5}|J_{18}| \le \frac{C}{n}\Big(\sum_{|k|\le n}\big|\rho(k)\big|^{\frac{3}{2}}\Big)^4.$$

The term J_{19} can be handled applying (A.8) and (A.9).

For J_{11}, J_{12}, t can be just chosen from the set $\{5,6\}$ and the possible values of the factor ρ_{th} (after a change of variable) can be $\rho(k_3)$, $\rho(k_4)$, $\rho(k_3-k_4)$ or $\rho(k_5)$ where k_5 is linearly independent of $k_1,\dots,k_4$. Then we first sum up the variables k_1 and k_2 and this part produces a constant. The sum with respect to k_3, k_4, k_5 is as follows.

$$\sum_{|k_j|\le n}\big|\rho(k_3)^2\rho(k_4)\big| \le C\sum_{|k|\le n}\big|\rho(k)\big|,$$

$$\sum_{|k_j|\le n}\big|\rho(k_3)\rho(k_4)\rho(k_5)\big| = \Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^3 \le n\sum_{|k|\le n}\big|\rho(k)\big|,$$

and

$$\sum_{|k_j|\le n}\big|\rho(k_3)\rho(k_4)\rho(k_3-k_4)\big| \le C\sum_{|k|\le n}\big|\rho(k)\big|,$$

where we have used (A.18) and (A.19) with $M=2$. Therefore,

$$\frac{1}{n^2}\sum_{j=1}^5\sum_{|k_j|\le n}|J_{1i}\rho_{th}| \le \frac{C}{n}\sum_{|k|\le n}\big|\rho(k)\big|, \quad i=1,2.$$

For J_{13}, $t=3$ and possible values of ρ_{th} can be $\rho(k_2)$, $\rho(k_2-k_1)$ or $\rho(k_5)$ where k_5 is linearly independent of $k_1,\dots,k_4$. The first two cases have been considered above in the discussion of the terms $J_{11}\rho_{th}$ and $J_{12}\rho_{th}$. For the third case, observe that

$$\frac{1}{n^2}\sum_{j=1}^5\sum_{|k_j|\le n}\big|\rho(k_1)^2\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_3-k_4)\rho(k_5)\big|$$

$$\le \frac{C}{n^2}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^3 \le \frac{C}{n}\sum_{|k|\le n}\big|\rho(k)\big|,$$

where we have used (A.18) and (A.19) with $M=2$. Thus,

$$\frac{1}{n^2}\sum_{j=1}^5\sum_{|k_j|\le n}|J_{13}\rho_{th}| \le \frac{C}{n}\sum_{|k|\le n}\big|\rho(k)\big|.$$

Finally, for J_{14}, the term ρ_{th} could be $\rho(k_i)$, $i = 2, \dots, 4$ or $\rho(\star)$ where $\star$ is a linear combination of k_i's which at least involves two different terms k_{h_1} and k_{h_2} where $h_1, h_2 \in \{2, 3, 4, 5\}$. The first case has been considered above in the discussion of the terms $J_{1i}\rho_{th}$, $i = 1, 2, 3$. For the second case, we apply inequalities (A.18) and (A.19) with $M = 2, 3, 4$ and we get

$$\frac{1}{n^2}\sum_{j=1}^{5}\sum_{|k_j|\le n}\left|\rho(k_1)^2\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(\star)\right|$$

$$\le \frac{C}{n^2}\left(\sum_{|k|\le n}\left|\rho(k)\right|\right)^3 \le \frac{C}{n}\sum_{|k|\le n}\left|\rho(k)\right|.$$

Therefore, $\frac{1}{n^2}\sum_{j=1}^{5}\sum_{|k_j|\le n}|J_{14}\rho_{th}| \le \frac{C}{n}\sum_{|k|\le n}|\rho(k)|$ and this finishes the proof. □

Lemma A.4. *Assume condition* (1.2) *with* $d = 2$. *Define*

$$\mathcal{L}_1 := n^{-4}\sum_{l_1,\dots,l_8=1}^{n}\sum_{(i,s,j,t,h)\in D_4}|\rho_{12}\rho_{13}\rho_{14}\rho_{56}\rho_{57}\rho_{58}\rho_{2i}\rho_{sj}\rho_{th}|,$$

where the set D_4 *has been defined in* (4.25). *Then*

$$\mathcal{L}_1 \le \frac{C}{n^2}\left(\sum_{|k|\le n}\left|\rho(k)\right|\right)^3. \tag{A.13}$$

Proof. We make the change of variables $l_1 - l_2 = k_1$, $l_1 - l_3 = k_2$, $l_1 - l_4 = k_3$, $l_5 - l_6 = k_4$, $l_5 - l_7 = k_5$, $l_5 - l_8 = k_6$. The factors ρ_{2i}, ρ_{sj} and ρ_{th} can be one of the two forms:

(i) $\rho_{\alpha\beta}$, where $\alpha, \beta \in \{1, 2, 3, 4\}$ or $\alpha, \beta \in \{5, 6, 7, 8\}$.
(ii) $\rho_{\alpha\beta}$, where $\alpha \in \{1, 2, 3, 4\}$ and $\beta \in \{5, 6, 7, 8\}$ or $\beta \in \{1, 2, 3, 4\}$ and $\alpha \in \{5, 6, 7, 8\}$.

For factors of the form (i), we have $\rho_{\alpha\beta} = \rho(\mathbf{k}\cdot\mathbf{v})$, where $\mathbf{k}$ is one of the vectors (k_1, k_2, k_3) or (k_4, k_5, k_6) and $\mathbf{v}$ is a vector in $\mathbb{R}^3$ whose components are 0, 1 or -1. For the first factor of the form (ii), we write $\rho_{\alpha\beta} = \rho(k_7)$, where k_7 is a new variable independent of the k_i's, $1 \le i \le 6$. If there are more than one factor of the form (ii), then these extra factor(s) can be written as $\rho(\mathbf{k}\cdot\mathbf{v})$, where $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5, k_6, k_7)$ and $\mathbf{v}$ is a vector in $\mathbb{R}^7$ whose components are 0, 1 or -1.

Then we decompose $\mathcal{L}_1$ as the sum of several terms $\mathcal{L}_{1j}$, according to the following cases:

Case 1: There are three factors that have power 2. We denote the corresponding term by $\mathcal{L}_{11}$. For this term we have

$$\mathcal{L}_{11} = \frac{1}{n^2}\sum_{\substack{|k_i|\le n\\ i=1,\dots,6}}\rho(k_1)^2\rho(k_2)^2\rho(k_3)^2\left|\rho(k_4)\rho(k_5)\rho(k_6)\right| \le \frac{C}{n^2}\left(\sum_{|k|\le n}\left|\rho(k)\right|\right)^3.$$

Case 2: Two factors have power 2. Then we have the following possibilities by taking into account of the symmetry.

$$\mathcal{L}_{12} := \frac{1}{n^3}\sum_{\substack{|k_i|\le n\\ i=1,\dots,7}}\left|\rho^2(k_1)\rho^2(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_6)\rho(k_7)\right|$$

and

$$\mathcal{L}_{13} := \frac{1}{n^2}\sum_{\substack{|k_i|\le n\\ i=1,\dots,6}}\left|\rho^2(k_1)\rho^2(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_6)\rho(\mathbf{k}\cdot\mathbf{v})\right|,$$

where $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5, k_6)$ and $\mathbf{v}$ is a vector in $\mathbb{R}^6$ whose components are 0, 1 or -1. Clearly,

$$\mathcal{L}_{12} \le \frac{C}{n^3}\left(\sum_{|k|\le n}\left|\rho(k)\right|\right)^5 \le \frac{C}{n^2}\left(\sum_{|k|\le n}\left|\rho(k)\right|\right)^3.$$

For $\mathcal{L}_{13}$, $\mathbf{k}\cdot\mathbf{v}$ involves at least two factors k_j, $k_{j'}$ but $\mathbf{k}\cdot\mathbf{v}$ cannot be a linear combination of only k_1 and k_2. Applying inequality (A.21) with $M=5$, yields

$$\mathcal{L}_{13} \le n^{-2}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^3.$$

Case 3: Only one factor has power 2. Then we have the following two possibilities, taking into account the symmetry. The first one is

$$\mathcal{L}_{14} = \frac{1}{n^3}\sum_{\substack{|k_i|\le n\\ i=1,\dots,7}}\big|\rho^2(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_6)\rho(k_7)\rho(\mathbf{k}\cdot\mathbf{v})\big|,$$

where $\mathbf{k}=(k_1,k_2,k_3,k_4,k_5,k_6,k_7)$ and $\mathbf{v}$ is a vector in $\mathbb{R}^7$ whose components are 0, 1 or -1 and it has at least two nonzero components. By (A.21) with $M=7$, we can write

$$\mathcal{L}_{14} \le \frac{C}{n^3}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^5 \le \frac{C}{n^2}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^3.$$

The second possibility is

$$\mathcal{L}_{15} := \frac{1}{n^2}\sum_{\substack{|k_i|\le n\\ i=1,\dots,6}}\big|\rho^2(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_6)\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})\big|,$$

where $\mathbf{k}=(k_1,k_2,k_3,k_4,k_5,k_6)$ and $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^6$ in such a way that $\mathbf{k}\cdot\mathbf{v}$ and $\mathbf{k}\cdot\mathbf{w}$ are linear combinations of k_1, k_2, k_3 or k_4, k_5, k_6 with exactly two nonzero components equal to 1 and -1 and satisfying some additional restrictions, due to the definition of the set D_4. There are several combinations:

(i) $\mathbf{k}\cdot\mathbf{v}$ and $\mathbf{k}\cdot\mathbf{w}$ are chosen differently from $\{k_1-k_2, k_2-k_3, k_1-k_3\}$. In this case, by Proposition 2.4 with $p_1=1$, $p_i=\frac{1}{2}$ for $2\le i\le 5$, we have

$$\sum_{|k_i|\le n, 1\le i\le 3}\big|\rho^2(k_1)\rho(k_2)\rho(k_3)\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})\big| \le C,$$

and we obtain

$$\mathcal{L}_{15} \le \frac{C}{n^2}\Big(\sum_{|k|\le n}\big|\rho(k)\big|\Big)^3. \tag{A.14}$$

(ii) $\mathbf{k}\cdot\mathbf{v}$ and $\mathbf{k}\cdot\mathbf{w}$ are two different linear combinations chosen among $\{k_4-k_5, k_4-k_6, k_5-k_6\}$. Then, the inequality (A.22) with $M=3$ yields

$$\sum_{|k_i|\le n, i=4,5,6}\big|\rho(k_4)\rho(k_5)\rho(k_6)\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})\big| \le \sum_{|k|\le n}\big|\rho(k)\big|,$$

which implies (A.14).

(iii) $\mathbf{k}\cdot\mathbf{v}$ is chosen from $\{k_1-k_2, k_2-k_3, k_1-k_3\}$, and $\mathbf{k}\cdot\mathbf{w}$ is chosen from $\{k_4-k_5, k_4-k_6, k_5-k_6\}$. Take $\mathbf{k}\cdot\mathbf{v}=k_1-k_2$ for example, then (A.14) follows from

$$\sum_{|k_1|\le n, |k_2|\le n}\big|\rho^2(k_1)\rho(k_2)\rho(k_1-k_2)\big| \le C$$

and (A.20) with $M=3$. Similar arguments apply if $\mathbf{k}\cdot\mathbf{v}=k_1-k_3$. If $\mathbf{k}\cdot\mathbf{v}=k_2-k_3$, we use (A.20) and (A.21) with $M=3$.

Case 4: All factors have power 1, $i \in \{3, 4\}$, and $\rho_{sj} = \rho(\mathbf{k} \cdot \mathbf{v})$, $\rho_{th} = \rho(\mathbf{k} \cdot \mathbf{w})$ where $\mathbf{k} \cdot \mathbf{v}$ is a linear combination of k_1, k_2, k_3 and $\mathbf{k} \cdot \mathbf{w}$ is a linear combination of k_4, k_5, k_6, or vice versa. We denote the corresponding term by $\mathcal{L}_{16}$. Then the estimate

$$\mathcal{L}_{16} \le n^{-2}\left(\sum_{|k|\le n} \left|\rho(k)\right|\right)^3$$

follows from (A.22) with $M = 3$ and (A.20) with $M = 3$.

Case 5: All factors have power 1, and there is one of the differences $l_i - l_2$, $l_j - l_s$ or $l_h - l_t$ linearly independent of $k_1, \dots, k_6$. We denote this difference by k_7. The other two factors are of the form $\rho(\mathbf{k} \cdot \mathbf{v})$ and $\rho(\mathbf{k} \cdot \mathbf{w})$, where $\mathbf{k} \cdot \mathbf{v}$ and $\mathbf{k} \cdot \mathbf{w}$ are linear combinations of $k_1, \dots, k_6, k_7$. In this case, the desired estimate follows from the inequality (A.22), with $M = 7$. In fact, if we denote the corresponding term by $\mathcal{L}_{17}$, we obtain

$$\mathcal{L}_{17} \le \frac{C}{n^3}\left(\sum_{|k|\le n} \left|\rho(k)\right|\right)^5 \le \frac{C}{n^2}\left(\sum_{|k|\le n} \left|\rho(k)\right|\right)^3.$$

This finishes the lemma. □

Lemma A.5. *Define*

$$\mathcal{L}_2 := n^{-4} \sum_{l_1,\dots,l_8=1}^{n} \sum_{\substack{i\ne s\ne j \\ i,s,j\in\{4,7,8\}}} |\rho_{12}\rho_{13}\rho_{24}\rho_{56}\rho_{57}\rho_{68}\rho_{3i}\rho_{sj}|$$

and

$$\mathcal{L}_3 := n^{-4} \sum_{l_1,\dots,l_8=1}^{n} \sum_{(i,s,j,t,h)\in D_5} |\rho_{12}\rho_{13}\rho_{24}\rho_{56}\rho_{57}\rho_{68}\rho_{3i}\rho_{sj}\rho_{th}|,$$

where the set D_5 has been defined in (4.26). Then

$$\mathcal{L}_2 \le \frac{C}{n^2}\left(\sum_{|k|\le n} \left|\rho(k)\right|^{\frac{4}{3}}\right)^6, \tag{A.15}$$

and

$$\mathcal{L}_3 \le \frac{C}{n^2}\left(\sum_{|k|\le n} \left|\rho(k)\right|\right)^3. \tag{A.16}$$

Proof. Let us first show (A.15). We make the change of variables $l_1 - l_2 = k_1$, $l_1 - l_3 = k_2$, $l_2 - l_4 = k_3$, $l_5 - l_6 = k_4$, $l_5 - l_7 = k_5$, $l_6 - l_8 = k_6$. By symmetry, it suffices to analyze the cases $i = 4$ and $i = 7$. If $i = 4$, then $\rho_{34} = \rho(k_1 - k_2 + k_3)$ and $s = 8$, $j = 7$ or $s = 7$, $j = 8$, which gives $\rho_{sj} = \rho(k_4 - k_5 + k_6)$. In this case, we obtain a term of the form

$$\mathcal{L}_{21} := n^{-2} \sum_{\substack{|k_i|\le n, \\ i=1,\dots,6}} \left|\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_1 - k_2 + k_3)\rho(k_4)\rho(k_5)\rho(k_6)\rho(k_4 - k_5 + k_6)\right|.$$

Applying inequality (A.18) with $M = 3$ yields

$$\mathcal{L}_{21} \le \frac{C}{n^2}\left(\sum_{|k|\le n} \left|\rho(k)\right|^{\frac{4}{3}}\right)^6.$$

In the case $i = 7$, we set $\rho_{37} = \rho(k_7)$ and have two possibilities for sj: 48 and 84, which produce the following term

$$\mathcal{L}_{23} := n^{-3} \sum_{\substack{|k_i|\le n \\ i=1,\dots,7}} \left|\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_6)\rho(k_7) \times \rho(k_2 + k_7 - k_3 - k_1 - k_5 + k_4 + k_6)\right|.$$

Applying the inequality (A.18) with $M=7$ and Hölder's inequality, we obtain

$$\mathcal{L}_{23} \le \frac{C}{n^3}\Big(\sum_{|k|\le n}|\rho(k)|^{\frac{8}{7}}\Big)^7 \le \frac{C}{n^2}\Big(\sum_{|k|\le n}|\rho(k)|^{\frac{4}{3}}\Big)^6.$$

This finishes the proof of (A.15). The proof of (A.16) is analogous to that of (A.13). Namely, we can make the change of variables $l_1-l_2=k_1$, $l_1-l_3=k_2$, $l_2-l_4=k_3$, $l_5-l_6=k_4$, $l_5-l_7=k_5$, $l_6-l_8=k_6$, and follow the arguments of (A.13). A subtle difference might be the verification of (A.14). That is, the estimation of

$$\mathcal{L}_{15} := \frac{1}{n^2}\sum_{\substack{|k_i|\le n\\ i=1,\dots,6}} \big|\rho(k_1)\rho^2(k_2)\rho(k_3)\rho(k_4)\rho(k_5)\rho(k_6)\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})\big|,$$

where $\mathbf{k}\cdot\mathbf{v}$, $\mathbf{k}\cdot\mathbf{w}$ have the following two cases:

(i) They are linear combinations of k_4, k_5, k_6.
(ii) $\mathbf{k}\cdot\mathbf{v}$ is a linear combination of k_1, k_2, k_3 (k_1-k_2 with respect to $i=2$ or $k_2-k_1-k_3$ with respect to $i=4$), and $\mathbf{k}\cdot\mathbf{w}$ is a linear combination of k_4, k_5, k_6.

In the case (i), we apply the inequality (A.22) with $M=3$ to obtain

$$\mathcal{L}_{15} \le \frac{C}{n^2}\Big(\sum_{|k|\le n}|\rho(k)|\Big)^3. \tag{A.17}$$

In the case (ii), we apply (A.21) with $M=3$ and (A.20) with $M=3$ to obtain the desired the inequality (A.17). □

The next lemma contains several inequalities that are used along the paper.

Lemma A.6. *Fix an integer $M\ge 2$. We have*

$$\sum_{\substack{|k_j|\le n\\ 1\le j\le M}} |\rho(\mathbf{k}\cdot\mathbf{v})|\prod_{j=1}^M|\rho(k_j)| \le C\Big(\sum_{|k|\le n}|\rho(k)|^{1+\frac{1}{M}}\Big)^M, \tag{A.18}$$

where $\mathbf{k}=(k_1,\dots,k_M)$ and $\mathbf{v}\in\mathbb{R}^M$ is a fixed vector whose components are 1 or -1. Furthermore, if $\sum_{k\in\mathbb{Z}}\rho(k)^2<\infty$, then

$$\Big(\sum_{|k|\le n}|\rho(k)|^{1+\frac{1}{M}}\Big)^M \le C\Big(\sum_{|k|\le n}|\rho(k)|\Big)^{M-1} \tag{A.19}$$

and if $\mathbf{v}\in\mathbb{R}^M$ is a nonzero vector whose components are 0, 1 or -1,

$$\sum_{\substack{|k_j|\le n\\ 1\le j\le M}} |\rho(\mathbf{k}\cdot\mathbf{v})|\prod_{j=1}^M|\rho(k_j)| \le C\Big(\sum_{|k|\le n}|\rho(k)|\Big)^{M-1}. \tag{A.20}$$

Proof. Applying the Brascamp–Lieb inequality (2.12), we have

$$\sum_{\substack{|k_j|\le n\\ 1\le j\le M}} \prod_{j=1}^M|\rho(k_j)||\rho(\mathbf{k}\cdot\mathbf{v})| \le C\prod_{i=1}^{M+1}\Big(\sum_{|k|\le n}|\rho(k)|^{\frac{1}{p_i}}\Big)^{p_i},$$

where $p_i\le 1$ and $\sum_{i=1}^{M+1}p_i=M$. Choosing $p_i=M/(M+1)$ for $i=1,\dots,M+1$, we get inequality (A.18). To show (A.19), we make the decomposition $|\rho(k)|^{1+\frac{1}{M}}=|\rho(k)|^{1-\frac{1}{M}}|\rho(k)|^{\frac{2}{M}}$ and apply Hölder's inequality with exponents $p=\frac{M}{M-1}$ and $q=M$. Finally, to show (A.20), we decompose the sum into the product of the sum with respect to the k_i's that appear in $\mathbf{k}\cdot\mathbf{v}$ and the sum of the remaining terms. □

Lemma A.7. *Fix an integer $M \geq 3$ and assume $\sum_{k\in\mathbb{Z}} \rho(k)^2 < \infty$. We have*

$$\sum_{\substack{|k_j|\leq n\\1\leq j\leq M}} \rho(k_1)^2 \big|\rho(\mathbf{k}\cdot\mathbf{v})\big| \prod_{j=2}^{M} \big|\rho(k_j)\big| \leq C\Big(\sum_{|k|\leq n} \big|\rho(k)\big|\Big)^{M-2}, \tag{A.21}$$

where $\mathbf{k} = (k_1, \dots, k_M)$ and $\mathbf{v} \in \mathbb{R}^M$ is a fixed vector whose components are 0, 1 *or* −1 *and it has at least two nonzero components.*

Proof. It suffices to assume that all the components of $\mathbf{v}$ are nonzero. In this case, we apply the Brascamp–Lieb inequality (2.12) with exponents $p_1 = 1$ and $p_2 = \cdots = p_{M+1} = \frac{M-1}{M}$ and inequality (A.19) with M replaced by $M - 1$. □

Lemma A.8. *Fix an integer $M \geq 3$ and assume $\sum_{k\in\mathbb{Z}} \rho(k)^2 < \infty$. We have*

$$\sum_{\substack{|k_j|\leq n\\1\leq j\leq M}} \big|\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})\big| \prod_{j=1}^{M} \big|\rho(k_j)\big| \leq C\Big(\sum_{|k|\leq n} \big|\rho(k)\big|\Big)^{M-2}, \tag{A.22}$$

where $\mathbf{k} = (k_1, \dots, k_M)$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^M$ are linearly independent vectors, whose components are 0, 1 *or* −1 *and they have at least two nonzero components.*

Proof. Suppose first that $\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})$ involves only three k_i's, for instance, k_1, k_2, k_3. In this case, applying the Brascamp–Lieb inequality (2.12) with exponents $p_i = 3/5$, $1 \leq i \leq 5$, yields,

$$\sum_{\substack{|k_i|\leq n\\1\leq i\leq 3}} \big|\rho(k_1)\rho(k_2)\rho(k_3)\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})\big| \leq \Big(\sum_{|k|\leq n} \big|\rho(k)\big|^{\frac{5}{3}}\Big)^3.$$

Notice that assumption (ii) in Proposition 2.4 is satisfied because three of the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $\mathbf{v}$, $\mathbf{w}$ may span a subspace of dimension 2, and we have $3 \times 3/5 = 9/5 \leq 2$. Then, making the decomposition $|\rho(k)|^{\frac{5}{3}} = |\rho(k)|^{\frac{1}{3}}|\rho(k)|^{\frac{4}{3}}$ and using Hölder's inequality with exponents $p = 3$ and $q = \frac{3}{2}$, yields

$$\Big(\sum_{|k|\leq n} \big|\rho(k)\big|^{\frac{5}{3}}\Big)^3 \leq C\sum_{|k|\leq n} \big|\rho(k)\big|,$$

which gives the desired estimate.

If $\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})$ involves four k_i's, for instance, k_1, k_2, k_3, k_4, we apply the Brascamp–Lieb inequality (2.12) with exponents $p_i = 2/3$, $1 \leq i \leq 6$, and we obtain

$$\sum_{\substack{|k_i|\leq n\\1\leq i\leq 3}} \big|\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})\big| \leq \Big(\sum_{|k|\leq n} \big|\rho(k)\big|^{\frac{3}{2}}\Big)^4.$$

Then, using (A.19) with $M = 2$, yields

$$\Big(\sum_{|k|\leq n} \big|\rho(k)\big|^{\frac{3}{2}}\Big)^4 \leq C\Big(\sum_{|k|\leq n} \big|\rho(k)\big|\Big)^2,$$

which gives the desired estimate. Finally, if $\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})$ involves more than four k_i's, the result follows again from the Brascamp–Lieb inequality (2.12), where we choose $p_i = 2/3$ for the factors $\rho(\mathbf{k}\cdot\mathbf{v})$, $\rho(\mathbf{k}\cdot\mathbf{w})$ and for the four factors $\rho(k_i)$ such that k_i appears in the linear combination with less factors, and we choose $p_i = 1$ for all the remaining factors $\rho(k_i)$ appearing in the linear combinations $\rho(\mathbf{k}\cdot\mathbf{v})$ or $\rho(\mathbf{k}\cdot\mathbf{w})$. □

The last lemma summarizes some inequalities derived from the application of Hölder's inequality.

Lemma A.9. *For any $M \geq 2$, we have*

$$\Big(\sum_{|k|\leq n} |\rho(k)|^{1+\frac{1}{M}}\Big)^M \leq \Big(\sum_{|k|\leq n} |\rho(k)|\Big)\Big(\sum_{|k|\leq n} |\rho(k)|^{\frac{M}{M-1}}\Big)^{M-1} \tag{A.23}$$

and

$$\Big(\sum_{|k|\leq n} |\rho(k)|\Big)^3 \leq n\Big(\sum_{|k|\leq n} |\rho(k)|^{\frac{3}{2}}\Big)^2. \tag{A.24}$$

Furthermore, if $\sum_{|k|\leq n} |\rho(k)|^2 < \infty$, *then*

$$\sum_{|k|\leq n} |\rho(k)|^{\frac{3}{2}} \leq C\Big(\sum_{|k|\leq n} |\rho(k)|^{\frac{4}{3}}\Big)^{\frac{3}{4}}. \tag{A.25}$$

Proof. To show (A.23) we make use of the decomposition $|\rho(k)|^{1+\frac{1}{M}} = |\rho(k)||\rho(k)|^{\frac{1}{M}}$ and apply Hölder's inequality with exponents $p = \frac{M}{M-1}$ and $q = M$. For (A.25) we use the decomposition $|\rho(k)|^{\frac{3}{2}} = |\rho(k)||\rho(k)|^{\frac{1}{2}}$ and apply Hölder's inequality with exponents $p = \frac{4}{3}$ and $q = 4$. Finally, for (A.24) we use again Hölder's inequality. □

The following lemma has been used in the proof of Theorem 4.7.

Lemma A.10. *The function*

$$\Psi(a) := \Big(\sum_{|k|\leq n} |\rho(k)|^{d-\frac{a}{2}}\Big)^2 \sum_{|k|\leq n} |\rho(k)|^a,$$

defined on the interval $[2, \frac{2d}{3}]$ *is nonincreasing*.

Proof. Without any loss of generality, we can assume that $\rho(k) > 0$ for all $|k| \leq n$. It suffices to check that the derivative of Ψ is nonpositive. We have

$$\begin{aligned}
\Psi'(a) &= -\sum_{|k|\leq n} |\rho(k)|^{d-\frac{a}{2}} \Big(\sum_{|k|\leq n} |\rho(k)|^{d-\frac{a}{2}} \log(|\rho(k)|)\Big) \sum_{|k|\leq n} |\rho(k)|^a \\
&\quad + \Big(\sum_{|k|\leq n} |\rho(k)|^{d-\frac{a}{2}}\Big)^2 \sum_{|k|\leq n} |\rho(k)|^a \log(|\rho(k)|) \\
&= \Big(\sum_{|k|\leq n} |\rho(k)|^{d-\frac{a}{2}}\Big) \\
&\quad \times \sum_{|k_1|,|k_2|\leq n} \big(|\rho(k_1)|^{d-\frac{a}{2}} |\rho(k_2)|^a - |\rho(k_2)|^{d-\frac{a}{2}} |\rho(k_1)|^a\big) \log(|\rho(k_2)|) \\
&= \Big(\sum_{|k|\leq n} |\rho(k)|^{d-\frac{a}{2}}\Big) \\
&\quad \times \sum_{|k_1|,|k_2|\leq n} |\rho(k_1)|^a |\rho(k_2)|^a \big[|\rho(k_1)|^{d-\frac{3a}{2}} - |\rho(k_2)|^{d-\frac{3a}{2}}\big] \log(|\rho(k_2)|).
\end{aligned}$$

By symmetry, we obtain

$$\begin{aligned}
\Psi'(a) &= \frac{1}{2}\Big(\sum_{|k|\leq n} |\rho(k)|^{d-\frac{a}{2}}\Big) \sum_{|k_1|,|k_2|\leq n} |\rho(k_1)|^a |\rho(k_2)|^a \big[|\rho(k_1)|^{d-\frac{3a}{2}} - |\rho(k_2)|^{d-\frac{3a}{2}}\big] \\
&\quad \times \big(\log(|\rho(k_2)|) - \log(|\rho(k_1)|)\big) \leq 0.
\end{aligned}$$

The proof of the lemma is complete. □

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