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MANUSCRIPT

The Flow of Polynomial Roots Under Differentiation

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Abstract

The question about behavior of gaps between zeros of polynomials under differentiation is classical and goes back to Marcel Riesz. Recently, Stefan Steinerberger [42] formally derived a nonlocal nonlinear partial differential equation which models dynamics of roots of polynomials under differentiation. In this paper, we connect rigorously solutions of Steinerberger's PDE and evolution of roots under differentiation for a class of trigonometric polynomials. Namely, we prove that the distribution of the zeros of the derivatives of a polynomial and the corresponding solutions of the PDE remain close for all times. The global in time control follows from the analysis of the propagation of errors equation, which turns out to be a nonlinear fractional heat equation with the main term similar to the modulated discretized fractional Laplacian $(-\Delta)^{1/2}$.

Keywords Zeros of polynomials · Gaps between roots under differentiation · Convergence to equilibrium · Nonlocal transport · Free fractional convolution of measures

Mathematics Subject Classification 26C10 · 35Q70 · 44A15 · 46L54 · 35Q92 · 35Q35 · 60B20

1 Introduction

The analysis of the relation between the zero set of a polynomial or an entire function and the zero set of its derivative has rich history. The Gauss-Lucas theorem [16, 27, 30] says that for a polynomial on complex plane, the zero set of the derivative lies

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in the convex hull of the zero set. This direction remains very active, see e.g. [8, 28, 38, 43, 51] for some recent advances and further references. Classic conjectures by Polya and Wiman [36, 37, 60] dealt with the question of disappearance (or appearance) of complex roots under differentiation for a class of entire functions; see [7, 39] for some resolutions. Closer to our focus in this paper, the question about behavior of gaps between the roots of a real-valued polynomial under differentiation goes back to Marcel Riesz. A result attributed to him [45] shows that the smallest gap between the roots can only increase after differentiation, providing an indication that differentiation tends to "even out" distances between roots (see also [49, 58, 59] for later related works). A rigorous proof of "crystallization" under repeated differentiation - convergence of roots to an ideal lattice - has been established for a class of trigonometric polynomials in [13]. Similar results were also discussed for a class of entire functions in [14], and established for some random entire functions in [35]. A universal nature of oscillations in high order derivatives and their role in asymptotic analysis, as well as connection to quantum theory, has been discussed by Michael Berry in [3]. We also mention a recent related series of papers studying distribution of critical points of a random or deterministic polynomial given the distribution of its roots [4, 20, 24, 25, 32, 34, 35, 48, 50], where further references can be found.

For a trigonometric polynomial, the process of "crystallization" under differentiation is not difficult to understand on an elementary level: repeated differentiation leads to larger factors appearing in front of the leading terms than lower order terms. If the polynomial has order n, after differentiating $\sim An$ times, the leading terms gain at least a constant $\sim e^A$ factor compared to all lower order terms. After sufficient number of differentiations, the leading term will dominate and this will affect the location of roots, enforcing crystallization. One can think of the differentiation as gradually creating a spectral gap, which makes contact with the celebrated Sturm-Hourwitz theorem [23, 46, 47] (see an excellent review [1] for the history, including contributions by Lord Raleigh and Liouville). This theorem provides the estimate on a number of roots for trigonometric polynomials with spectral gap - except that in our case here the limiting polynomial is very simple and so we can say more about the roots. Of course, specific and sufficiently strong bounds on convergence to ideal lattice can be much more subtle to prove. Even harder question is to understand in more detail how the distribution of roots evolves under differentiation.

Recently, Steinerberger [42] proposed a partial differential equation to describe the evolution of roots for polynomials on the real axis. The equation takes form

$$\partial_t u + \frac{1}{\pi}\partial_x \left(\arctan\left(\frac{Hu}{u}\right)\right) = 0, \tag{1.1}$$

where $Hu = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(y)}{x-y}\,dy$ is the Hilbert transform of u. The formal derivation of this PDE in [42] makes certain assumptions - which will be recalled in more detail in Section 2 - that suggest that the PDE should approximate the dynamics of zeroes for polynomials of sufficiently high degree n, provided that their roots are distributed according to a smooth density and maintain this property under repeated differentiation. The unit of time in (1.1) corresponds to n differentiations, so the evolution becomes trivial for $t > 1$. Some interesting explicit solutions, for example corresponding to the semicircle law and to Marchenko-Pastur distribution, are also described in [42], making links to well known asymptotic laws for roots of orthogonal polynomials [11, 12, 52, 53]. Another PDE of a simpler form but related to (1.1) was formally derived by O'Rourke and Steinerberger [33] for the case of complex random polynomials with radial distribution of roots.

Interestingly, the equation (1.1) also appears to be relevant in free probability and random matrices. In a very recent work of Shlyakhtenko and Tao [40] this equation (under a simple change of variables, see [44]) was formally obtained as a PDE for the evolution of free fractional convolution of a probability measure on $\mathbb{R}$.. The free convolution of two probability measures $\mu \boxplus \nu$ is an object in free probability (see e.g. [56]). One can define $\mu \boxplus \nu$ to be the law of $X + Y$, where X, Y are freely independent noncommutative random variables with law μ and ν respectively. One can then define the integer free convolution $\mu^{\boxplus k} = \mu \boxplus \cdots \boxplus \mu$ to be the free convolution of the k copies of μ. This (properly rescaled) object plays a key role in the free analog of the central limit theorem established by Voiculescu [55], where the limiting law is a Wigner semicircle distribution. It turns out that the notion of the integer free convolution $\mu^{\boxplus k}$ can be in a natural way extended to real $k \geq 1$ [2, 31]. Under some additional assumptions, in particular that $d\mu^{\boxplus k} = f_k(x)\,dx$ are sufficiently regular, the equation (1.1) has been formally derived for $f_k(x)$ in [40], Theorem 1.7 and Section 4; the variable $1 - \frac{1}{k}$ plays the role of time. The connection between free fractional convolution and the behavior of roots of polynomials under differentiation can be interpreted through the relation of both these processes to minor process in random matrix theory. Roughly speaking, a minor process consists of a sequence of monotone minors of a random matrix ensemble. The connection between the fractional free convolution $\mu^{\boxplus k}$ and the law of a projection composed with non-commutative random variable X with law μ has been established in [31], see also [40] for a self-contained argument. In this case time is related to the size of the minor. On the other hand, [29,Lemma 1.16] establishes a link between differentiation and minor process, by showing that the expected characteristic polynomial of a random restriction of a matrix is proportional to a derivative of its characteristic polynomial. See also [28] for a direct association between the spectrum of sub-matrices and roots of the derivatives of characteristic polynomials.

In addition to these observations, a very recent preprint [21] by Hoskins and Kabluchko establishes a direct relation between evolution of roots of a polynomial under differentiation and free fractional convolution in the real line setting. The result of [21] applies for each fixed time in the limit of $n \to \infty$, and does not directly involve the PDE (1.1). Rather, it cleverly combines the results of [54] relating the exponential profile of the polynomial coefficients and the Cauchy-Stiltjes transform of the limiting measure, and the dynamics of the exponential profile under differentiation captured in [15], to derive an algorithm for computation of the evolution of the limiting probability measure that involves inversion of the Cauchy-Stiltjes transform. The connection of this process with the PDE (1.1) has not yet been established rigorously. However, in some examples of polynomials in the complex plane with random rotationally invariant coefficients similar to [33], the authors of [21] do verify rigorously that in the limit $n \to \infty$ the limiting distribution satisfies the corresponding PDE.

We point out several more recent papers that are also connected to this circle of ideas, either linking evolution of roots under differentiation and minor process [22], establishing connections between limiting distributions of Bessel and Dunkl processes modeling particle systems and free convolutions [57] or proving a version of "crystallization" for a class of random matrix ensembles [17].

We will consider the equation (1.1) with periodic initial data in $\mathbb{R}$, or, equivalently, set on a circle $\mathbb{S} = (-\pi, \pi]$. Let us rewrite (1.1) as a transport-diffusion equation

$$\partial_t u - \frac{Hu}{\pi(u^2 + Hu^2)} \partial_x u = -\frac{u}{\pi(u^2 + Hu^2)} \Lambda u. \tag{1.2}$$

Here $\Lambda u = (-\Delta)^{1/2} u = \partial_x Hu$ is the fractional Laplacian. This PDE is critical and is similar to some PDEs of fluid mechanics, material science and mathematical biology [5, 6, 9, 10, 19].

The global regularity theory for (1.2) has been established in [26]; see also an earlier work of Granero-Belinchon [18]. Moreover, the solution converges to a constant equilibrium $\bar{u} = \frac{1}{2\pi}$ while all derivatives converge to zero exponentially as time approaches infinity, thanks to the dissipation. The relevant result that we will need in this paper is stated in Theorem 3.1 below.

The goal of this paper is to establish a rigorous connection between evolution of roots under differentiation and the PDE (1.1) in the periodic setting. We will consider a class of trigonometric polynomials

$$p_{2n}(x) = \sum_{j=1}^{n} \left(a_j \cos jx + b_j \sin jx\right) = \prod_{j=1}^{2n} \sin \frac{x - x_j}{2}$$

that will be assumed to have exactly $2n$ distinct roots $x_j \in \mathbb{S}$, $j = 1 \ldots, 2n$. Note that by Rolle's theorem and a simple calculation that we outline in Section 2, all derivatives of p_{2n} belong in the same class. Denote $\bar{x}_j = \frac{x_j + x_{j+1}}{2}$ the midpoints of the gaps between the roots (we think here of x_j as of angular coordinates of the roots). We measure closeness between a discrete set of roots $\{x_j\}_{j=1}^{2n}$ and a continuous distribution $u(x)$ in the following way. Define the error

$$E_j = x_{j+1} - x_j - \frac{1}{2nu(\bar{x}_j)}, \quad j = 1, \cdots, 2n. \tag{1.3}$$

We may assume that initially, the roots of the polynomial p_{2n} obey (1.3) for the initial density u_0 with some reasonably small errors. For the subsequent steps, we track

$$\|E^t\|_\infty = \max_j |E^t_j|,$$

with $t = \frac{k}{2n}$ and

$$E^t_j = x^t_{j+1} - x^t_j - \frac{1}{2nu(\bar{x}^t_j, t)}, \quad j = 1, \cdots, 2n.$$

Here x_j^t are the roots of the kth derivative of p_{2n} and $\bar{x}_j^t$ are the corresponding midpoints.

Theorem 1.1 *Let $u_0 \in H^s(\mathbb{S})$, $s > 5/2$. Suppose that $u_0(x) > 0$ for all $x \in \mathbb{S}$, and $\int_{\mathbb{S}} u_0(x)\,dx = 1$. Let $u(x,t)$ be solution of (1.1) with the initial data u_0, and let p_{2n} be any trigonometric polynomial that at the initial time obeys (1.3) with $u = u_0$ and $\|E^0\|_\infty \le Z_0 n^{-1-\epsilon}$ for some $\epsilon > 0$. Then there exist positive constants $C(u_0)$ and $n_0(u_0, Z_0, \epsilon)$ such that if $n \ge n_0(u_0, Z_0, \epsilon)$, the following estimate holds true for all times $t \ge 0$:*

$$\|E^t\|_\infty \le C\left(Z_0 n^{-1-\epsilon} + n^{-3/2} t\right) e^{-\frac{4}{\pi}t(1+O(n^{-\epsilon/2}))}. \tag{1.4}$$

Remark 1.2 1. The largest initial errors E^0 that we can handle have size $\sim n^{-1-\epsilon}$ for arbitrary small $\epsilon > 0$, but the result applies for smaller errors: ϵ can be large or Z_0 can be taken just zero for the initial perfect fit. Hence the theorem applies in the case where root spacings are regular on small scales, but can vary significantly on large scales.

2. In the context of Theorem 1.1, we will sometimes refer to the triple (u_0, Z_0, ϵ) as the initial data, and use short cut notation $\tilde{u}_0$ for this triple, for example in $n_0(\tilde{u}_0) \equiv n_0(u_0, Z_0, \epsilon)$.

3. The error $O(n^{-\epsilon/2})$ in the exponent in (1.4) can be replaced with $O((\log n)^2 n^{-\epsilon})$. We choose the former form for the sake of simplicity.

The surprising aspect of Theorem 1.1 is that we are able to maintain control on the error for all times - and in fact, it is even improving past certain time that only depends on u_0. The philosophical reason for this is that both evolution of roots and the solution $u(x,t)$ tend to uniform distribution, so they have no obligation to diverge for large times. However, there is quite a bit of distance between such observation and the estimate (1.4). On a more detailed level, the estimate (1.4) is enabled by careful analysis of the propagation of the error equation. Amazingly, it turns out to have form

$$\frac{E^{t+\Delta t} - E^t}{\Delta t} = \mathcal{L}^t E^t + lower\ \ order\ \ terms, \tag{1.5}$$

where $\mathcal{L}^t$ is a nonlinear operator of diffusive type that in the main order is similar to a modulated discretized fractional Laplacian $-\Lambda$. In fact, in the limit of large n and large time $\mathcal{L}^t$ converges to exactly the dissipative term

$$-\frac{u}{\pi(u^2 + Hu^2)}\Lambda \sim -\frac{1}{\pi\bar{u}}\Lambda$$

of (1.2), see Theorem 8.2 and remark after it for details. Thus the propagation of the error equation turns out to be essentially a nonlinear fractional heat equation. The dissipative nature of (1.5) is crucial for maintaining control of $\|E^t\|_\infty$ even for a finite time. Global in time bound requires further ingredients, in particular favorable estimates on the leading lower order terms that take advantage of the decay of derivatives (3.2).

Together, (1.5) and results on the evolution of (1.1) stated in Theorem 3.1 describe rigorously specific and delicate mechanisms that modulate the evolution of roots of polynomials under differentiation.

There are many further natural questions. We believe that our scheme of the proof can be useful in establishing the whole line result, as well - but there is an extra issue that one has to handle. The assumption $u_0(x) > 0$ that actually implies $u_0(x) \geq a > 0$ in the compact case is crucial for the proof of global regularity for (1.1). There is no reason to believe that this result holds in the whole line case when $u_0(x)$ has compact support - one would expect the solution to be just Hölder regular near the edges. Thus one has to understand the associated free boundary problem to make progress in such situation. Another reasonable question is whether the current argument can be extended to more general sets of polynomials and some classes of entire functions. Our approach shows how large scale ~ 1 or micro scale $\lesssim n^{-1-\epsilon}$ imbalances in root spacings are evened out, but does not currently apply to intermediate scale irregularities. One can wonder whether repeated differentiation might eventually bring any trigonometric polynomial into the class that we handle here - but new ideas are needed to carefully analyze such conjecture. Potential applications to free probability and random matrices are also an interesting direction.

The paper is organized as follows. In Section 2, we recall the formal derivation of (1.1), recasting it for the periodic case. In Sections 3, 4 we set up and prove estimates on how a single root moves under differentiation, essentially sharpening the estimates in all steps of the formal derivation. In Sections 5, 6 and 7 we set up and prove estimates on how the pairs of neighboring roots move under differentiation, which is used for derivation of the error propagation equation. In Section 8 we further analyze evolution of the error, and prove that it takes form (1.5). In Section 9, we complete the proof of Theorem 1.1.

2 Formal derivation of the PDE

This section follows the original argument from [42], recasting it for trigonometric polynomials. Recall that we consider the following space of periodic functions $\mathbb{P}_{2n}$

$$\mathbb{P}_{2n} = \left\{ p_{2n} \;\middle|\; \begin{array}{l} \exists \, \{a_j, b_j\}_{j=1}^n \in \mathbb{R}, \quad p_{2n}(x) = \sum_{j=1}^n a_j \cos(jx) + b_j \sin(jx), \\ p_{2n} \text{ has } 2n \text{ distinct roots } \{x_j\}_{j=1}^{2n} \text{ in } \mathbb{S} = (-\pi, \pi] \end{array} \right\}.$$

The parameter n is assumed to be large, and the distribution of the roots of p_{2n} is assumed to be close to a smooth function $u(x)$, which is 2π-periodic.

The derivative of a function in $\mathbb{P}_{2n}$ lies in $\mathbb{P}_{2n}$ as well. It also has $2n$ distinct roots, one in each interval (x_j, x_{j+1}). Let us denote

$$\bar{x}_j = \frac{x_j + x_{j+1}}{2}$$

the midpoints of these intervals.

Lemma 2.1 *[An identity on roots and the derivative] Let* $p_{2n} \in \mathbb{P}_{2n}$*, and let* $\{x_j\}_{j=1}^{2n}$ *be the roots of* p_{2n}*. Then, the following identity holds*

$$p'_{2n}(x) = \frac{1}{2}\, p_{2n}(x) \sum_{j=1}^{2n} \cot \frac{x - x_j}{2}. \tag{2.1}$$

Proof It is not hard to show that a trigonometric polynomial $p_{2n} \in \mathbb{P}_{2n}$ that has roots at $2n$ distinct points $x_1, \ldots, x_{2n} \in (-\pi, \pi]$ satisfies

$$p_{2n}(x) = c \prod_{j=1}^{2n} \sin \frac{x - x_j}{2}.$$

Direct differentiation yields (2.1). □

Let $y_m \in (x_m, x_{m+1})$ be the roots of p'_{2n}, $m = 1, \cdots, 2n$. From the identity (2.1), we know

$$\sum_{j=1}^{2n} \cot \frac{y_m - x_j}{2} = 0.$$

Split the sum into two parts:

$$\sum_{j=1}^{2n} \cot \frac{y_m - x_j}{2} = \sum_{|x_j - y_m| \le n^{-1/2}} \cot \frac{y_m - x_j}{2} + \sum_{|x_j - y_m| > n^{-1/2}} \cot \frac{y_m - x_j}{2} = I_m + II_m.$$

For the near field I_m, take the Taylor expansion

$$\cot \frac{y_m - x_j}{2} = \frac{2}{y_m - x_j} + O(|y_m - x_j|).$$

Then,

$$I_m = \sum_{|x_j - y_m| \le n^{-1/2}} \left(\frac{2}{y_m - x_j} + O(|y_m - x_j|) \right) = \sum_{|x_j - y_m| \le n^{-1/2}} \frac{2}{y_m - x_j} + O(1).$$

Recall the cotangent identity

$$\pi \cot \pi x = \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{x + k} + \frac{1}{x - k} \right)$$

for $x \in \mathbb{R} \setminus \mathbb{Z}$. For the first term, since the range of $\{x_j\}$ is small, and the distribution of $\{x_j\}$ is close to $u(x) := u(x, 0)$, we can formally approximate $\{x_j\}$ by equally distributed nodes $\{\tilde{x}_j\}$ centered at x_m, separated by distance $(2nu(\bar{x}_m))^{-1}$. Namely, by

$$\tilde{x}_j = x_m + \frac{j-m}{2nu(\bar{x}_m)}. \tag{2.2}$$

Then, making use of the cotangent identity, we have

$$\sum_{|x_j-y_m|\le n^{-1/2}} \frac{2}{y_m-x_j} \sim \sum_{k=-2u(\bar{x}_m)n^{1/2}}^{2u(\bar{x}_m)n^{1/2}} \frac{2}{y_m-x_m+k(2nu(\bar{x}_m))^{-1}}$$
$$\sim 4\pi nu(\bar{x}_m)\cot(2\pi nu(\bar{x}_m)(y_m-x_m)).$$

For the far field II_m, as the distribution of $\{x_j\}$ is close to $u(x)$, we formally get

$$II_m = \sum_{|x_j-y_m|>n^{-1/2}} \cot\frac{y_m-x_j}{2} \sim 2n\int_{|y-y_m|>n^{-1/2}} u(y)\cot\frac{y_m-y}{2}\,dy \sim 4\pi nHu(y_m).$$

Putting together the two expressions, the leading order $O(n)$ term reads

$$4\pi u(\bar{x}_m)\cot(2\pi nu(\bar{x}_m)(y_m-x_m)) + 4\pi Hu(y_m) = 0.$$

Simplify the equation and get

$$y_m - x_m = -\frac{1}{2n\pi u(\bar{x}_m)}\arctan\left(\frac{u(\bar{x}_m)}{Hu(y_m)}\right). \tag{2.3}$$

To make sure $y_m \in (x_m, x_{m+1})$, we take the branch of arctan x with values in $(-\pi, 0)$. This branch is discontinuous at $x = 0$ but as we will see $u(x,t)/Hu(x,t)$ is bounded away from zero for all times in our setting.

Take the time scale $\Delta t = (2n)^{-1}$, so that $u(x, t = 1)$ represents the distribution of the roots of $p_{2n}^{(2n)}$, the $2n$-th derivative of p_{2n}. Equation (2.3) provides a macroscopic flux

$$v(x_m) = -\frac{1}{\pi u(\bar{x}_m)}\arctan\left(\frac{u(\bar{x}_m)}{Hu(y_m)}\right)$$

of x_m (roots of p_{2n}) to y_m (roots of p'_{2n}). Letting $n \to \infty$, we formally derive

$$0 = \partial_t u + \partial_x(uv) = \partial_t u - \frac{1}{\pi}\partial_x\left(\arctan\left(\frac{u}{Hu}\right)\right) = \partial_t u + \frac{1}{\pi}\partial_x\left(\arctan\left(\frac{Hu}{u}\right)\right). \tag{2.4}$$

Note that here it does not matter which branch of arctan we take as the derivative is the same. This is indeed our main equation.

We would like to comment briefly on the possible path towards establishing rigorous connection of (2.4) with the analog of free fractional convolution on the circle. Suppose u solves (2.4). If we define an analytic function in the unit circle using the complex Poisson kernel

$$F(z) = \int_{\mathbb{S}} u(x) \frac{e^{ix} + z}{e^{ix} - z}\, dx,$$

then it is not hard to check that F satisfies the complex Burgers type equation

$$\partial_t F + 2z\partial_z \log F = 0 \tag{2.5}$$

inside the unit circle. Indeed, applying Plemelj formulas for the circle case (see e.g. [41]) and taking the real part of (2.5) on circle boundary, one obtains (2.4). The equation (2.5) is similar to the real line analogs that were used to formally establish connection between free fractional convolution and (2.4) in [40], and may be useful for establishing the relation rigorously in the periodic case. It may also be provide a lead for deriving the periodic analog of the algorithm due to Hoskins and Kabluchko [21], which is currently missing, and connecting it with the PDE (2.4).

3 The Setup

In this section, we introduce relevant notation and overall set up needed to rigorously connect the main equation (1.1) with the evolution of polynomial roots under differentiation.

3.1 Global regularity for the PDE

The well-posedness theory for (1.1) was first studied by Granero-Belinchon [18]. The results include local regularity for initial data $u_0 \in H^2(\mathbb{S})$, global regularity under a smallness assumption on the initial data, and exponential convergence to equilibrium in L^∞ and in appropriate Wiener spaces.

We state a stronger result [26] on the global regularity and asymptotic behavior of the main equation (1.1).

Theorem 3.1 *([26]) Let $u_0 \in H^s(\mathbb{S})$, $s > 3/2$. Suppose that $u_0(x) > 0$ for all $x \in \mathbb{S}$, and $\int_{\mathbb{S}} u_0(x)\, dx = 1$. The equation (1.1) with initial data u_0 has a unique global smooth solution $u(x,t)$. The H^s norm of the solution is bounded uniformly in time, and all the derivatives of this solution are bounded uniformly in time on any interval $[t_0, \infty)$, $t_0 > 0$.*

Moreover, we have exponential in time convergence to equilibrium

$$\|u(\cdot,t) - \tfrac{1}{2\pi}\|_{L^\infty} \leq C_0 e^{-\frac{4}{\pi}t}, \tag{3.1}$$

and exponential in time decay of all derivatives

$$\|\partial_x^k u(\cdot, t)\|_{L^\infty} \le C_k e^{-\frac{4}{\pi}t} \tag{3.2}$$

for all integer $k \ge 1$, *with constants* C_k, $k = 0, \ldots$ *that may only depend on* u_0. *The estimate (3.2) holds for all t if* u_0 *has necessary regularity, and starting from any fixed* $t_0 > 0$ *otherwise.*

We shall mention that the global regularity result in Theorem 3.1 works for all smooth positive initial data, without any smallness assumptions. Moreover, the exponential decay in higher derivatives (3.2) play an important role in establishing the rigorous connection between the solutions of the PDE and evolution of roots under differentiation.

3.2 Measurement of error

Let us recall the way we measure closeness between a discrete set of roots $\{x_j\}_{j=1}^{2n}$ and a continuous distribution $u(x)$.

In the introduction, we defined the error

$$E_j = x_{j+1} - x_j - \frac{1}{2nu(\bar{x}_j)}, \quad j = 1, \cdots, 2n; \quad \|E\|_\infty = \max_{1\le j\le 2n} |E_j| \tag{3.3}$$

We will assume that initially, the roots of the polynomial p_{2n} obey (3.3) with the initial density u_0 and $\|E^0\|_\infty \le Z_0 n^{-1-\epsilon}$. We will assume that $u_0(x) \in H^s(\mathbb{S})$, $s > 5/2$, and that $u_0(x) \ge u_{0,\min} > 0$ for all $x \in \mathbb{S}$, so that the global regularity results apply to solutions of the equation (1.1) with initial data u_0. We will denote the minimum of $u_0(x)$ and $u(x,t)$ by $u_{0,\min}$ and $u_{\min}(t)$ respectively. We aim to control the growth in time of the error E_j^t between the solution $u(x,t)$ of (1.1) and the zeroes of the derivative $p_{2n}^k(x)$, with $t = \frac{k}{2n}$. In order to be able to do so, we will first assume that n is sufficiently large - how large will only depend on u_0. Second, some of our estimates will be done under the assumption that

$$\|E^t\|_\infty = \max_{1\le j\le 2n} |E_j^t| \le Z_{\max}(u_0, Z_0)(n^{-1-\epsilon} + n^{-3/2}), \tag{3.4}$$

where $Z_{\max}$ is a constant that is independent of n and of t, and only depends on u_0 and Z_0. Such constant can certainly be found for a certain initial period of time, and we will eventually show that with a proper choice of $Z_{\max}$, (3.4) indeed continues to hold for all times provided that $n \ge n_0(\tilde{u}_0)$ is sufficiently large.

3.3 The decomposition

Following the heuristic derivation in Section 2, let us make a related but slightly different decomposition

$$\sum_{j=1}^{2n} \cot \frac{y_m - x_j}{2} = \sum_{j \in S_m} \cot \frac{y_m - x_j}{2} + \sum_{j \in S_m^c} \cot \frac{y_m - x_j}{2} = I_m + II_m \tag{3.5}$$

(we are doing such decomposition at any fixed time and omitting time in notation for the sake of simplicity). Recall that y_m is the zero of the next derivative satisfying $x_m < y_m < x_{m+1}$. The set $S_m = \{j_-, \cdots, j_+ - 1\}$ consists of the indices of x_j that lie in the near field of y_m. We define $j_\pm$ so that $\bar{x}_{j_\pm - 1}$ are the closest midpoints to $y_m \pm n^{-1/2}$, respectively. Note that $j_\pm$ depend on m, but to keep notation from getting too heavy we will omit dependence. When doing two roots estimates in the following sections, we will simply use $j_- + 1$ and $j_+ + 1$ as the cutoff indices in the decomposition for $(m+1)$st root. Also, observe that $S_m^c = \{1, \cdots, 2n\} \backslash S_m$.

There is a mismatch between the intervals $(y_m - n^{-1/2}, y_m + n^{-1/2})$ and $(\bar{x}_{j_- - 1}, \bar{x}_{j_+ - 1})$ (shaded area in the figure above). We denote the difference

$$\Omega_m = \Omega_m^- \cup \Omega_m^+, \quad \text{where} \quad \Omega_m^- = [\bar{x}_{j_- - 1}, y_m - n^{-1/2}], \ \Omega_m^+ = [y_m + n^{-1/2}, \bar{x}_{j_+ - 1}].$$

Note that we do not claim that $\bar{x}_{j_- - 1} \le y_m - n^{-1/2}$ or $y_m + n^{-1/2} \le \bar{x}_{j_+ - 1}$. The lengths of the mismatched intervals $\Omega_m^\pm$ are of the order $O(n^{-1})$.

4 Preliminary Estimates at a Single Root

This section follows the outline of the formal derivation, but we make estimates more precise. As we will see, single root estimates are not sufficient to rigorously connect evolution of roots under differentiation and equation (1.1), but these estimates will be one of the ingredients of a more in-depth argument. Let us first estimate the far field part II_m defined in (3.5). All estimates in the next few sections will be carried out at a fixed time, at which we assume (3.4) to hold true. To lighten the presentation, we omit dependence on time in the notation until later sections where evolution of error in time will be considered. All the constants in the estimates, including the O notation, may only depend on u_0. We aim to show that

$$II_m \sim 4\pi n Hu(y_m) = 2n \ P.V. \int_0^{2\pi} u(y) \cot \frac{y_m - y}{2} \, dy$$

with more detailed error estimates than in heuristic derivation. Decompose the integral into two parts and write

$$\begin{aligned} 4\pi n Hu(y_m) - II_m &= 2n \ P.V. \int_{|y - y_m| \le n^{-1/2}} u(y) \cot \frac{y_m - y}{2} \, dy \\ &\quad + \left(2n \int_{|y - y_m| > n^{-1/2}} u(y) \cot \frac{y_m - y}{2} \, dy - \sum_{j \in S_m^c} \cot \frac{y_m - x_j}{2} \right) \\ &=: II_{1,m} + II_{2,m}. \end{aligned} \tag{4.1}$$

The $II_{1,m}$ term can be estimated by symmetry

$$\left|II_{1,m}\right| = 2n \left| \int_{|y-y_m|\le n^{-1/2}} \big(u(y) - u(y_m)\big) \cot \frac{y_m - y}{2} dy \right| \le C\|\partial_x u\|_{L^\infty} n^{1/2} = O(n^{1/2}). \tag{4.2}$$

To estimate $II_{2,m}$, we match each term in the sum of II_m with the part of the integral, and estimate the difference

$$\begin{aligned} II^j_{2,m} &:= 2n \int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y) \cot \frac{y_m - y}{2} dy - \cot \frac{y_m - x_j}{2} \\ &= \cot \frac{y_m - x_j}{2} \left(2n \int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y) dy - 1 \right) + 2n \int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y) \left(\cot \frac{y_m - y}{2} - \cot \frac{y_m - x_j}{2} \right) dy \end{aligned} \tag{4.3}$$

for $j \in S_m^c$.

Let us compute the integral

$$\begin{aligned} \int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y)dy &= \int_{\bar{x}_{j-1}}^{x_j} \big(u(\bar{x}_{j-1}) + O\big((z - \bar{x}_{j-1})\big)\big)\, dz + \int_{x_j}^{\bar{x}_j} \big(u(\bar{x}_j) + O((z - \bar{x}_j))\big)\, dz \\ &= \frac{x_j - x_{j-1}}{2} u(\bar{x}_{j-1}) + \frac{x_{j+1} - x_j}{2} u(\bar{x}_j) + O(n^{-2}) \\ &= \frac{1}{2n} + \frac{u(\bar{x}_{j-1})}{2} E_{j-1} + \frac{u(\bar{x}_j)}{2} E_j + O(n^{-2}). \end{aligned} \tag{4.4}$$

Therefore,

$$2n \int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y)dy - 1 = n \left(u(\bar{x}_{j-1}) E_{j-1} + u(\bar{x}_j) E_j \right) + O(n^{-1}).$$

To estimate $\cot \frac{y_m - x_j}{2}$, we state the following bounds on $|y_m - x_j|$.

Lemma 4.1 *For all n sufficiently large and all $j \neq m, m+1$ we have*

$$\frac{|m-j|}{u_{\min} n} \ge |y_m - x_j| \ge \frac{|m-j|}{8\|u\|_{L^\infty} n}. \tag{4.5}$$

Here $u_{\min} = \min_x u(x)$; we will be using this notation instead of the more compact $m(t)$ that we used in the first part of the paper since the variable m would be too overloaded now.

Proof For $j \ge m+2$, estimate

$$|y_m - x_j| \ge x_j - x_{m+1} = \sum_{l=m+1}^{j-1} \left(\frac{1}{2nu(\bar{x}_l)} + E_l \right) \ge \frac{j-m-1}{4n\|u\|_{L^\infty}} \ge \frac{j-m}{8n\|u\|_{L^\infty}}.$$

Note that in the second inequality, we use the assumption (3.4), so that $E_l \ll \frac{1}{4nu(\bar{x}_l)}$ if n is large. A similar argument works for $j \le m-1$:

$$|y_m - x_j| \ge x_m - x_j = \sum_{l=j}^{m-1} \left(\frac{1}{2nu(\bar{x}_l)} + E_l \right) \ge \frac{m-j}{4n\|u\|_{L^\infty}}.$$

The proof of the upper bound is similar. □

Remark 4.2 The lower bound estimates on $|y_m - x_m|$ and $|y_{m+1} - x_{m+1}|$ are more subtle. We will discuss them later in Lemma 4.3. The upper bounds for $j = m, m+1$ are straightforward.

Now, we have

$$\left|\cot \frac{y_m - x_j}{2}\right| < \frac{2}{|y_m - x_j|} \lesssim \frac{n}{|j-m|} \tag{4.6}$$

and, for $y \in (\bar{x}_{j-1}, \bar{x}_j)$,

$$\left|\cot \frac{y_m - x_j}{2} - \cot \frac{y_m - y}{2}\right| \le \frac{1}{\left(\sin \frac{y_m - z}{2}\right)^2} \cdot \frac{|y - x_j|}{2} \lesssim \frac{n}{|j-m|^2}, \tag{4.7}$$

where $z \in (y, x_j)$ and $|y_m - z|$ has a lower bound similar to (4.5).

Collecting all the estimates, we have

$$II_{2,m}^j = n \cot \frac{y_m - x_j}{2} \left(u(\bar{x}_{j-1}) E_{j-1} + u(\bar{x}_j) E_j \right) + O(|m-j|^{-1} + n|m-j|^{-2}).$$

To complete the estimate of $II_{2,m}$, we sum up $II_{2,m}^j$ for all $j \in S_m^c$. Note that there is a mismatch of the integral at the boundary. It can be controlled by

$$\left| 2n \int_{\Omega_m} u(y) \cot \frac{y_m - y}{2} dy \right| \lesssim n \cdot n^{-1} \cdot n^{1/2} = O(n^{1/2}),$$

as the length of Ω_m is $O(n^{-1})$, and $y_m - y = n^{-1/2} + O(n^{-1})$ for any $y \in \Omega_m$.

We can summarize that

$$II_{2,m} = \sum_{j \in S_m^c} II_{2,m}^j + 2n \int_{\Omega_m} u(y) \cot \frac{y_m - y}{2} dy = 4\pi n u(\bar{x}_m) A_{1,m} + O(n^{1/2}), \tag{4.8}$$

where

$$A_{1,m} = \frac{1}{4\pi u(\bar{x}_m)} \sum_{j \in S_m^c} \cot \frac{y_m - x_j}{2} \left(u(\bar{x}_{j-1}) E_{j-1} + u(\bar{x}_j) E_j \right). \tag{4.9}$$

Note that a direct estimate of the term $A_{1,m}$ yields

$$|A_{1,m}| \leq \sum_{|m-j| \gtrsim n^{1/2}} \frac{n}{|m-j|} \cdot \|E\|_\infty = O(n \log n \|E\|_\infty). \tag{4.10}$$

Next, we turn to estimate I_m. The goal is to show that

$$I_m = \sum_{j \in S_m} \cot \frac{y_m - x_j}{2} \sim 4\pi n u(\bar{x}_m) \cot(2\pi n u(\bar{x}_m)(y_m - x_m)).$$

The first step is to replace $\cot \frac{y_m - x_j}{2}$ with $\frac{2}{y_m - x_j}$, with an error

$$I_{1,m} = -\sum_{j \in S_m} \left(\cot \frac{y_m - x_j}{2} - \frac{2}{y_m - x_j} \right). \tag{4.11}$$

Since by (4.5),

$$\left| \cot \frac{y_m - x_j}{2} - \frac{2}{y_m - x_j} \right| = O(|y_m - x_j|) = O\big((|m - j| + 1)n^{-1}\big),$$

we obtain

$$|I_{1,m}| = \sum_{|j-m| \lesssim n^{1/2}} O\big((|m - j| + 1)n^{-1}\big) = O(1). \tag{4.12}$$

In the following lemma, we state a useful simple lower bound on $|y_m - x_j|$, with $j = m, m+1$. Later, we will derive a more precise estimate.

Lemma 4.3 *Suppose (3.4) holds. Then for all sufficiently large n there exists a constant $c = c(u) > 0$, such that*

$$\min\{y_m - x_m, x_{m+1} - y_m\} \geq c n^{-1}. \tag{4.13}$$

As follows from the proof and our earlier bounds, the threshold for n depends only on the constant $Z_{\max}$ in (3.4), on $\|u\|_{C^1}$ and on the minimal value of u - that is, effectively, on u_0. A more precise value for $c(u)$ will be computed below in (4.36).

Proof We start with writing $I_m + II_m = 0$ as

$$\sum_{j \in S_m} \frac{2}{y_m - x_j} - I_{1,m} + (4\pi n H u(y_m) - II_{1,m} - II_{2,m}) = 0. \tag{4.14}$$

Let us first assume y_m is closer to x_m. Split the sum into three pieces

$$\sum_{j=j_-}^{j_+-1} \frac{2}{y_m - x_j} = \frac{2}{y_m - x_m} + \sum_{l=1}^{j_+-m-1} \left(\frac{2}{y_m - x_{m+l}} + \frac{2}{y_m - x_{m-l}} \right) + \sum_{j=j_-}^{2m-j_+} \frac{2}{y_m - x_j}. \tag{4.15}$$

Here, we shall use the convention on summation notation that

$$\sum_{j=a}^{a-1} = 0 \quad \text{and} \quad \sum_{j=a}^{b} = -\sum_{j=b+1}^{a-1} \quad \text{if } \; b \leq a-2. \tag{4.16}$$

The second term on the right hand side of (4.15) matches the $m \pm l$ terms. Compute

$$\frac{2}{y_m - x_{m+l}} + \frac{2}{y_m - x_{m-l}} = \frac{2y_m - x_{m+l} - x_{m-l}}{(y_m - x_{m+l})(y_m - x_{m-l})}.$$

For the numerator, observe that

$$\begin{aligned} x_{m+l} + x_{m-l} &= 2x_m + \sum_{k=1}^{l} \left(\frac{1}{2nu(\bar{x}_{m+k-1})} + E_{m+k-1} - \frac{1}{2nu(\bar{x}_{m-k})} - E_{m-k} \right) \\ &= 2x_m + O(l^2 n^{-2} + l\|E\|_\infty). \end{aligned} \tag{4.17}$$

For the denominator, we use Lemma 4.1 for $|y_m - x_j|$ with $j \neq m+1$. For $|y_m - x_{m+1}|$, as we assume y_m is closer to x_m, we get

$$|y_m - x_{m+1}| \geq \frac{x_{m+1} - x_m}{2} \geq \frac{1}{4nu(\bar{x}_m)} - \|E\|_\infty.$$

Therefore, we can bound the second term on the right hand side of (4.15) by

$$\begin{aligned} \sum_{l=1}^{j_+ - m - 1} \left| \frac{2}{y_m - x_{m+l}} + \frac{2}{y_m - x_{m-l}} \right| &\leq \sum_{l=1}^{\sim n^{1/2}} O\left(\frac{(y_m - x_m) + l^2 n^{-2} + l\|E\|_{L^\infty}}{l^2 n^{-2}} \right) \\ &= O(n + n^2 \log n \|E\|_\infty), \end{aligned} \tag{4.18}$$

which is $O(n)$ if (3.4) is assumed.

The third part on the right hand side of (4.15) represents the mismatched terms, as there might not be precisely the same number of points in S_m on each side of m, namely $m - (j_- - 1) \neq j_+ - m$. However, we can estimate the number of mismatched indices. Indeed, from the definition of S_m, we have

$$x_{j_+} - x_{m+1} = n^{-1/2} + O(n^{-1}), \quad x_m - x_{j_- - 1} = n^{-1/2} + O(n^{-1}). \tag{4.19}$$

On the other hand,

$$x_{j_+} - x_{m+1} = \sum_{l=m+1}^{j_+ - 1} \left(\frac{1}{2nu(\bar{x}_l)} + E_l \right) = \frac{j_+ - 1 - m}{2nu(\bar{x}_m)} + O(n^{1/2}\|E\|_\infty + n^{-1}),$$

$$x_m - x_{j_- - 1} = \sum_{l=j_- - 1}^{m-1} \left(\frac{1}{2nu(\bar{x}_l)} + E_l \right) = \frac{m - j_- + 1}{2nu(\bar{x}_m)} + O(n^{1/2}\|E\|_\infty + n^{-1}).$$

Taking the difference and using (4.19), we get

$$2m - j_+ - j_- = O(1 + n^{3/2}\|E\|_\infty). \tag{4.20}$$

So, the third term can be controlled by

$$\left|\sum_{j=j_-}^{2m-j_+} \frac{2}{y_m - x_j}\right| \le O(1 + n^{3/2}\|E\|_\infty) \cdot \frac{2}{O(n^{-1/2})} = O(n^{1/2} + n^2\|E\|_\infty). \tag{4.21}$$

Putting together all the estimates (4.2), (4.8), (4.10), (4.12), (4.18), (4.21) into (4.14), we obtain

$$\frac{2}{y_m - x_m} = -4\pi n H u(y_m) + O(n + n^2 \log n \|E\|_\infty) = O(n).$$

This implies a lower bound on $|y_m - x_m| \gtrsim n^{-1}$.

In the case when y_m is closer to x_{m+1}, we split the sum differently, singling out the $(m+1)$st term. Then, a similar argument yields the same lower bound. □

Next, we approximate $\frac{2}{y_m - x_j}$ by $\frac{2}{y_m - \tilde{x}_j}$, where $\tilde{x}_j = x_m + \frac{j-m}{2nu(\bar{x}_m)}$ are equally distributed points defined in (2.2). Recall the cotangent identity

$$\sum_{j\in\mathbb{Z}} \frac{2}{y_m - \tilde{x}_j} = \sum_{j\in\mathbb{Z}} \frac{2}{y_m - x_m - \frac{j-m}{2nu(\bar{x}_m)}} = 4\pi n u(\bar{x}_m) \cot(2\pi n u(\bar{x}_m)(y_m - x_m)).$$

Split the sum into two parts

$$\sum_{j\in\mathbb{Z}} \frac{2}{y_m - \tilde{x}_j} = \sum_{j\in\mathbb{Z}\setminus S_m} \frac{2}{y_m - \tilde{x}_j} + \sum_{j\in S_m} \frac{2}{y_m - \tilde{x}_j},$$

and group the second term with $\sum_{j\in S_m} \frac{2}{y_m - x_j}$. Define

$$I_{2,m} = \sum_{j\in\mathbb{Z}\setminus S_m} \frac{2}{y_m - \tilde{x}_j} \tag{4.22}$$

and

$$I_{3,m} = \sum_{j\in S_m} \left(\frac{2}{y_m - \tilde{x}_j} - \frac{2}{y_m - x_j}\right) = \sum_{j\in S_m\setminus\{m\}} \frac{-2(x_j - \tilde{x}_j)}{(y_m - \tilde{x}_j)(y_m - x_j)}. \tag{4.23}$$

Note that we can exclude the term $j = m$ as $\tilde{x}_m = x_m$. This yields

$$I_m = 4\pi n u(\bar{x}_m) \cot(2\pi n u(\bar{x}_m)(y_m - x_m)) - I_{1,m} - I_{2,m} - I_{3,m}. \tag{4.24}$$

It remains to estimate $I_{2,m}$ and $I_{3,m}$.

For $I_{2,m}$, we match the terms $m \pm k$ over the summation range

$$\left| \frac{2}{y_m - x_m - \frac{k}{2nu(\bar{x}_m)}} + \frac{2}{y_m - x_m + \frac{k}{2nu(\bar{x}_m)}} \right| \lesssim \frac{|y_m - x_m|}{k^2 n^{-2}}. \tag{4.25}$$

As before, the number of mismatched indices is at most $O(1+n^{3/2}\|E\|_\infty)$ (see (4.20)). Thus we obtain

$$\begin{aligned} I_{2,m} &= \sum_{k=j_+-m}^{\infty} \left(\frac{2}{y_m - \tilde{x}_{m+k}} + \frac{2}{y_m - \tilde{x}_{m-k}} \right) + \sum_{j=2m-j_++1}^{j_--1} \frac{2}{y_m - \tilde{x}_j} \\ &= \sum_{k \gtrsim n^{1/2}} O\left(\frac{n^{-1}}{k^2 n^{-2}} \right) + O(1 + n^{3/2}\|E\|_\infty)) \cdot \frac{2}{O(n^{-1/2})} = O(n^{1/2} + n^2\|E\|_\infty). \end{aligned} \tag{4.26}$$

Finally, let us estimate $I_{3,m}$. Compute (using convention (4.16) for $j < m$ case)

$$\begin{aligned} x_j - \tilde{x}_j &= \sum_{l=m}^{j-1} \left(\frac{1}{2nu(\bar{x}_l)} + E_l \right) - \frac{j-m}{2nu(\bar{x}_m)} = \sum_{l=m}^{j-1} E_l + \frac{1}{2n} \sum_{l=m}^{j-1} \left(\frac{1}{u(\bar{x}_l)} - \frac{1}{u(\bar{x}_m)} \right) \\ &= \sum_{l=m}^{j-1} E_l + O(n^{-2}|m-j|^2). \end{aligned} \tag{4.27}$$

For the denominators, we make use of the lower bound estimates (4.5) and (4.13).

$$|y_m - x_j| > c|m-j|n^{-1}, \quad \forall\, j \neq m.$$

Similar estimates are also valid for $|y_m - \tilde{x}_j|$. Indeed, for $j \leq m-1$,

$$y_m - \tilde{x}_j > x_m - \tilde{x}_j = \frac{m-j}{2nu(\bar{x}_m)},$$

for $j \geq m+2$,

$$\tilde{x}_j - y_m > \frac{j-m}{2nu(\bar{x}_m)} - (x_{m+1} - x_m) > \frac{j-m-1}{2nu(\bar{x}_m)} - \|E\|_\infty > \frac{j-m}{6nu(\bar{x}_m)},$$

and for $j = m+1$,

$$\tilde{x}_{m+1} - y_m = x_{m+1} - y_m - E_m > cn^{-1} - \|E\|_\infty > \frac{c}{2}n^{-1}$$

for all sufficiently large n.

Collecting all the estimates and using the convention (4.16), we arrive at

$$I_{3,m} = \sum_{j\in S_m\setminus\{m\}} \left(\frac{-2\sum_{l=m}^{j-1} E_l}{(y_m-\tilde{x}_j)(y_m-x_j)} + O\left(\frac{n^{-2}|m-j|^2}{n^{-2}|m-j|^2}\right)\right)$$
$$= 4\pi nu(\bar{x}_m)A_{2,m} + O(n^{1/2}), \tag{4.28}$$

where we specify the expression $A_{2,m}$

$$A_{2,m} = \frac{1}{2\pi nu(\bar{x}_m)}\left(\sum_{j=j_-}^{m-1}\sum_{l=j}^{m-1}\frac{E_l}{(y_m-\tilde{x}_j)(y_m-x_j)} - \sum_{j=m+1}^{j_+-1}\sum_{l=m}^{j-1}\frac{E_l}{(y_m-\tilde{x}_j)(y_m-x_j)}\right). \tag{4.29}$$

A direct estimate of this term leads to the bound

$$A_{2,m} = \sum_{1\le|m-j|\lesssim n^{1/2}} O\left(n^{-1}\cdot\frac{|m-j|\|E\|_\infty}{n^{-2}|m-j|^2}\right) = O(n\log n\|E\|_\infty). \tag{4.30}$$

To summarize our computations, from (4.1) and (4.24), we have

$$4\pi nu(\bar{x}_m)\cot(2\pi nu(\bar{x}_m)(y_m-x_m) = -4\pi nHu(y_m) + I_{1,m} + I_{2,m} + I_{3,m} + II_{1,m} + II_{2,m}. \tag{4.31}$$

Applying estimates (4.2), (4.8), (4.12), (4.26) and (4.28) we get

$$\cot(2\pi nu(\bar{x}_m)(y_m-x_m)) = -\frac{Hu(y_m)}{u(\bar{x}_m)} + A_{1,m} + A_{2,m} + O(n^{-1/2} + n\|E\|_\infty), \tag{4.32}$$

where $A_{1,m}$ and $A_{2,m}$ are given by (4.9) and (4.29) and are of order $O(n\log n\|E\|_\infty)$.

Without the error terms, the equation becomes (2.3).

We can now return to the question addressed in Lemma 4.3 and derive a more precise estimate on $y_m - x_m$ and $x_{m+1} - y_m$ that we will need later.

Lemma 4.4 *With the choice of the branch of* arccot x *with values in* $(0,\pi)$, *we have*

$$y_m - x_m = \frac{1}{2\pi nu(\bar{x}_m)}\operatorname{arccot}\left(-\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right) + O(n^{-3/2} + \log n\|E\|_\infty) \tag{4.33}$$

and

$$x_{m+1} - y_m = \frac{1}{2\pi nu(\bar{x}_m)}\operatorname{arccot}\left(\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right) + O(n^{-3/2} + \log n\|E\|_\infty). \tag{4.34}$$

Proof It follows directly from (4.32) and estimates for $A_{1,m}$ and $A_{2,m}$ that

$$y_m - x_m = \frac{1}{2\pi n u(\bar{x}_m)} \operatorname{arccot}\left(-\frac{Hu(y_m)}{u(\bar{x}_m)} + O(n^{-1/2} + n\log n\|E\|_\infty)\right). \quad (4.35)$$

Since the derivative of arccot is globally bounded, the estimate (4.33) follows. Note that $Hu(y_m)$ can be replaced in (4.35) by $Hu(\bar{x}_m)$ creating a difference of order $O(n^{-1})$ that can be absorbed in the error.

The second estimate could be derived in a similar way by repeating the computations taking x_{m+1} as the center of approximation. But it is much simpler to observe that

$$x_{m+1} - x_m = \frac{1}{2nu(\bar{x}_m)} + E_m,$$

while $\operatorname{arccot}(\theta) + \operatorname{arccot}(-\theta) = \pi$ for all θ. □

Note that Lemma 4.4 yields a sharp bound for the constant $c(u)$ in Lemma 4.3:

$$c(u) = \frac{1}{2\pi u(\bar{x}_m)} \min\left(\operatorname{arccot}\left(\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right), \pi - \operatorname{arccot}\left(\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right)\right) + O(n^{-3/2} + \log n\|E\|_\infty). \quad (4.36)$$

To determine how the errors change over a single time step, we can subtract $y_m - x_m$ from $y_{m+1} - x_{m+1}$, obtaining

$$\begin{aligned} E_m^{t+\Delta t} - E_m^t = &-\frac{1}{2nu(\bar{y}_m, t+\Delta t)} + \frac{1}{2nu(\bar{x}_m, t)} + \frac{1}{2\pi n u(\bar{x}_{m+1})} \operatorname{arccot}\left(-\frac{Hu(\bar{x}_{m+1})}{u(\bar{x}_{m+1})}\right) \\ &- \frac{1}{2\pi n u(\bar{x}_m)} \operatorname{arccot}\left(-\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right) + O(n^{-3/2} + \log n\|E\|_\infty). \end{aligned} \quad (4.37)$$

We could now try to use the evolution equation on u to absorb the main terms - but the error estimates in (4.37) are too crude to yield anything useful. In fact, the difference of the arccot terms, which are the dominant ones in (4.33), is already of the order $O(n^{-2})$ in (4.37), smaller than the error. So, in a sense, from the one point estimates it is not even clear for sure that (1.1) is the right equation. If we tried to propagate $O(n^{-3/2})$ errors for $\sim n$ steps, a unit time, the accumulation of error could be $\sim n^{-1/2}$. This is much larger than the spacing between roots and would definitely destroy the approximation argument using cotangent identity in the near field. To overcome these obstacles, we need to go deeper and consider two point estimates, where additional cancellations will yield much more favorable bounds on errors.

5 Estimates for Pairs of Roots: General Setup

Before going into details, let us recap from the introduction what form of the error propagation equation we are aiming at, as well as outline informally the general plan of the argument. We are going to show that

$$E_m^{t+\Delta t} - E_m^t = \sum_{j \neq m} \kappa(j, m)(E_j^t - E_m^t) + \text{errors}, \tag{5.1}$$

where $\kappa(j, m) \sim |m-j|^{-2}$. Essentially, the evolution is a discretization of a nonlinear fractional heat equation $\partial_t E^t = \mathcal{L}^t E^t + n \cdot$ errors at a scale $\sim n^{-1}$, where $\mathcal{L}^t$ is a dissipative operator that in the main order is similar to $-(-\Delta)^{1/2}$. In order to follow the estimates below, it is useful to understand how one can classify the error terms in (5.1). The most senior inhomogeneous errors in (5.1) will have the order $O(n^{-5/2})$, but in fact any forcing term containing no E^t in (5.1) would prevent us from proving convergence of the error to zero as time goes to infinity as stated in Theorem 1.1. A crucial observation will be that all such error terms have factors involving derivatives of u in front of them. By (3.2) we know that all derivatives of u decay exponentially in time, allowing for much better estimates.

The linear errors of the order $O(n^{-1}\|E^t\|_\infty)$ can be thought of as critical. These errors can lead to growth by a constant factor over $\sim n$ iterations, that is time ~ 1. Any larger linear in E^t errors are supercritical: a brutal absolute value estimate on them will lead to only exploding upper bound on the error. The expressions $A_{1,m}$ and $A_{2,m}$ that we saw above in (4.9), (4.29) are examples of supercritical terms (or, rather, they lead to supercritical terms of order $O(n^{-1}\log n\|E^t\|_\infty)$ in the final error propagation equation). We will have to use the detailed structure of the supercritical terms in order to be able to handle them; it will turn out that any such terms can be absorbed into the dissipative sum in (5.1). But even critical terms could lead to exponential growth in time - all such errors, however, turn out to also have factors involving derivatives of u that decay exponentially in time. Thus, given a threshold, in finite time that only depends on u_0 and this threshold, the coefficients in front of the linear critical terms will become smaller than the threshold. This will allow us to use a sort of spectral gap estimate on dissipation to dominate these errors. To deploy the variant of a spectral gap bound, we will need to control the mean, the zeroth mode, of E^t. This will be done by using the definition of the error and conservation of $\int_{\mathbb{S}} u(x,t)\,dx$.

Among the nonlinear in E errors, the most dangerous term will have order $O(n^{1/2}\|E^t\|_\infty^2)$. The criticality of nonlinear terms is determined by how large $\|E^t\|_\infty$ can become. At the very least, even when the initial errors are small or zero, the inhomogeneous error terms will only allow an upper bound of $\|E^t\|_\infty \lesssim n^{-3/2}$ for $t \sim 1$. At this level, the $O(n^{1/2}\|E\|_\infty^2)$ term is critical; but if linear critical terms allow to maintain $\|E^t\|_\infty \lesssim n^{-3/2}$ bound for arbitrary long finite time with worsening constant, the nonlinear one would only allow a fixed finite time ~ 1. However, we will be able to control the error term of $O(n^{1/2}\|E^t\|_\infty^2)$ type by absorbing it into dissipation. There will also be subcritical errors, that will be easier to handle by making sure that n is sufficiently large.

There will be many places in the argument that will generate inhomogeneous and critical errors, and to keep track of the decaying factors in front of them, we define

$$\delta(t) = \max\left\{\|\partial_x u(\cdot,t)\|_{L^\infty},\ \|\partial_x^2 u(\cdot,t)\|_{L^\infty},\ \|\partial_x^2 u(\cdot,t)\|_{C^\gamma}\right\}, \quad \gamma \in (0, s-\tfrac{5}{2}). \tag{5.2}$$

Using Sobolev embedding, we get $\delta(t) \leq C\|u(\cdot, t)\|_{H^s}$, which is bounded for all times. Moreover, from (3.2), we can easily verify that $\delta(t)$ decays exponentially in time.

In the argument, we will usually omit the dependence of δ on t to save space.

Now we begin with two point estimates. Consider x_m, x_{m+1} and x_{m+2} at some time t. Let $y_m \in (x_m, x_{m+1})$ and $y_{m+1} \in (x_{m+1}, x_{m+2})$ be roots after one more differentiation. To estimate $y_{m+1} - y_m - x_{m+1} + x_m$, we start with (4.31).

Recall that

$$\cot(2\pi nu(\bar{x}_m)n(y_m - x_m)) = -\frac{Hu(y_m)}{u(\bar{x}_m)} + \frac{1}{4\pi nu(\bar{x}_m)}G_m,$$

where we define

$$G_m = I_{1,m} + I_{2,m} + I_{3,m} + II_{1,m} + II_{2,m}, \tag{5.3}$$

with $I_{1,m}$, $I_{2,m}$, $I_{3,m}$, $II_{1,m}$ and $II_{2,m}$ given by (4.11), (4.22), (4.23) and (4.1) respectively. A completely analogous equality holds for $m+1$; here we naturally choose $S_{m+1} = \{j_- + 1, \ldots, j_+\}$ in the splitting of near field and far field terms. All estimates that we derived for $I_{i,m}$ and $II_{i,m}$ extend to $I_{i,m+1}$ and $II_{i,m+1}$.

Applying the mean value theorem, we get

$$\begin{aligned}
&\cot(2\pi nu(\bar{x}_{m+1})(y_{m+1} - x_{m+1})) - \cot(2\pi nu(\bar{x}_m)(y_m - x_m)) \\
&= -\frac{1}{\sin^2 z}\big(2\pi nu(\bar{x}_{m+1})(y_{m+1} - x_{m+1}) - 2\pi nu(\bar{x}_m)(y_m - x_m)\big),
\end{aligned}$$

where

$$z \in [2\pi nu(\bar{x}_m)(y_m - x_m), 2\pi nu(\bar{x}_{m+1})(y_{m+1} - x_{m+1})]. \tag{5.4}$$

Therefore,

$$\begin{aligned}
y_{m+1} - y_m - x_{m+1} + x_m &= -\frac{u(\bar{x}_{m+1}) - u(\bar{x}_m)}{u(\bar{x}_m)}(y_{m+1} - x_{m+1}) \\
+ \frac{\sin^2 z}{2\pi nu(\bar{x}_m)}\left(\frac{Hu(y_{m+1})}{u(\bar{x}_{m+1})} - \frac{Hu(y_m)}{u(\bar{x}_m)}\right) &- \frac{\sin^2 z}{8\pi^2 n^2 u(\bar{x}_m)^2}\big(G_{m+1} - G_m\big)
\end{aligned} \tag{5.5}$$

The first term can be replaced by

$$\begin{aligned}
&-\frac{u(\bar{x}_{m+1}) - u(\bar{x}_m)}{u(\bar{x}_m)}(y_{m+1} - x_{m+1}) = -\frac{\partial_x u(\bar{x}_m)}{u(\bar{x}_m)}(\bar{x}_{m+1} - \bar{x}_m)(y_{m+1} - x_{m+1}) + O(\delta n^{-3}) \\
&= -\frac{\partial_x u(\bar{x}_m)}{2nu^2(\bar{x}_m)}(y_{m+1} - x_{m+1}) + O(\delta n^{-3} + \delta n^{-1}\|E\|_\infty).
\end{aligned} \tag{5.6}$$

Next, we observe that $\sin^2 z = \frac{1}{1+\cot^2 z}$. Since z lies in the interval appearing in (5.4), we focus on the two endpoints. The formula (4.32) provides a useful expression for $\cot(2\pi nu(\bar{x}_m)n(y_m - x_m))$, while the analogous formula for

$\cot(2\pi nu(\bar{x}_{m+1})n(y_{m+1}-x_{m+1}))$ is

$$\begin{aligned}
&\cot(2\pi nu(\bar{x}_{m+1})n(y_{m+1}-x_{m+1}))\\
&=-\frac{Hu(y_{m+1})}{u(\bar{x}_{m+1})}+A_{1,m+1}+A_{2,m+1}+O(n^{-1/2}+n\|E\|_\infty)\\
&=-\frac{Hu(y_m)}{u(\bar{x}_m)}+A_{1,m}+A_{2,m}+O(n^{-1/2}+n\|E\|_\infty)\\
&\quad-\left(\frac{Hu(y_{m+1})}{u(\bar{x}_{m+1})}-\frac{Hu(y_m)}{u(\bar{x}_m)}\right)+(A_{1,m+1}-A_{1,m})+(A_{2,m+1}-A_{2,m}). \qquad (5.7)
\end{aligned}$$

Let us verify that the terms in the last line of the estimate above are small. For the first term, let us replace $Hu(y_m)$ and $Hu(y_{m+1})$ by $Hu(\bar{x}_m)$ and $Hu(\bar{x}_{m+1})$ respectively. The difference is

$$\begin{aligned}
&\frac{Hu(y_{m+1})-Hu(\bar{x}_{m+1})}{u(\bar{x}_{m+1})}-\frac{Hu(y_m)-Hu(\bar{x}_m)}{u(\bar{x}_m)}\\
&=\frac{1}{u(\bar{x}_m)}\big(Hu(y_{m+1})-Hu(\bar{x}_{m+1})-Hu(y_m)+Hu(\bar{x}_m)\big)+O(\delta n^{-2})\\
&=\frac{\partial_x Hu(\bar{x}_m)}{u(\bar{x}_m)}\big(y_{m+1}-y_m-\bar{x}_{m+1}+\bar{x}_m\big)+O(\delta n^{-2})\\
&=\frac{\partial_x Hu(\bar{x}_m)}{u(\bar{x}_m)}\left(y_{m+1}-y_m-x_{m+1}+x_m+\frac{1}{4nu(\bar{x}_m)}-\frac{1}{4nu(\bar{x}_{m+1})}+\frac{E_m-E_{m+1}}{2}\right)+O(\delta n^{-2})\\
&=O(\delta|y_{m+1}-y_m-x_{m+1}+x_m|)+O(\delta n^{-2}+\delta\|E\|_\infty). \qquad (5.8)
\end{aligned}$$

On the other hand,

$$\left|\frac{Hu(\bar{x}_{m+1})}{u(\bar{x}_{m+1})}-\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right|\le\left\|\partial_x\left(\frac{Hu}{u}\right)\right\|_\infty|\bar{x}_{m+1}-\bar{x}_m|\le C(\|\partial_x u\|_\infty+\|\partial_x^2 u\|_\infty)n^{-1}=O(\delta n^{-1}). \qquad (5.9)$$

due to global regularity.

Next, for the term $A_{1,m+1}-A_{1,m}$, we start with a simplification of $A_{1,m}$ by replacing $u(\bar{x}_{j-1})$ and $u(\bar{x}_j)$ in (4.9) with $u(\bar{x}_m)$. The difference is bounded by $O(\delta n^{-1}|m-j|)$. We obtain after summation

$$A_{1,m}=\frac{1}{4\pi}\sum_{j\in S_m^c}\cot\frac{y_m-x_j}{2}(E_{j-1}+E_j)+O(\delta n\|E\|_\infty). \qquad (5.10)$$

Then, we can telescope the difference $A_{1,m+1}-A_{1,m}$ as follows

$$\begin{aligned}
&A_{1,m+1}-A_{1,m}\\
&=\frac{1}{4\pi}\sum_{j\in S_m^c}\left(\cot\frac{y_{m+1}-x_{j+1}}{2}(E_j+E_{j+1})-\cot\frac{y_m-x_j}{2}(E_{j-1}+E_j)\right)+O(\delta n\|E\|_\infty)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi} \sum_{j\in S_m^c} \left(\cot \frac{y_{m+1}-x_{j+1}}{2} + \cot \frac{y_{m+1}-x_j}{2} - \cot \frac{y_m - x_{j+1}}{2} - \cot \frac{y_m - x_j}{2} \right) E_j \\
&\quad + O(n^{1/2}\|E\|_\infty + \delta n \|E\|_\infty) \\
&\le \sum_{|j-m|\gtrsim n^{1/2}} O\left(\frac{n^2}{|m-j|^2} \cdot n^{-1} \cdot \|E\|_\infty \right) + O(n^{1/2}\|E\|_\infty + \delta n \|E\|_\infty) \\
&= O(\delta n \|E\|_\infty + n^{1/2}\|E\|_\infty).
\end{aligned}
$$

Here, the $O(n^{1/2}\|E\|_\infty)$ term in the second equality encodes the boundary terms from the telescoped sum. In the third step, we use the bound $|y_{m+1}-y_m| = O(n^{-1})$.

Now, for $A_{2,m+1} - A_{2,m}$, we start with a simplification on $A_{2,m}$ by replacing $y_m - \tilde{x}_j$ in (4.29) by $\frac{m-j}{2nu(\bar{x}_m)}$. The difference is $y_m - x_m = O(n^{-1})$, and it leads to an estimate

$$
\begin{aligned}
A_{2,m} &= \frac{1}{\pi} \left(\sum_{j=j_-}^{m-1} \sum_{l=j}^{m-1} \frac{E_l}{(m-j)(y_m - x_j)} + \sum_{j=m+1}^{j_+-1} \sum_{l=m}^{j-1} \frac{E_l}{(j-m)(y_m - x_j)} \right) \\
&\quad + n^{-1} \sum_{j\in S_m\setminus\{m\}} O\left(n^{-1} \cdot |m-j| \, \|E\|_\infty \cdot \frac{n^3}{|m-j|^3} \right) \\
&= \frac{1}{\pi} \left(\sum_{l=j_-}^{m-1} \sum_{j=j_-}^{l} \frac{E_l}{(m-j)(y_m - x_j)} + \sum_{l=m}^{j_+-2} \sum_{j=l+1}^{j_+-1} \frac{E_l}{(j-m)(y_m - x_j)} \right) + O(n\|E\|_\infty).
\end{aligned}
\tag{5.11}
$$

We are ready to estimate the difference $A_{2,m+1} - A_{2,m}$. Let us work on the first sum and telescope

$$
\begin{aligned}
&\sum_{l=j_-+1}^{m} E_l \sum_{j=j_-}^{l-1} \frac{1}{(m-j)(y_{m+1}-x_{j+1})} - \sum_{l=j_-}^{m-1} E_l \sum_{j=j_-}^{l} \frac{1}{(m-j)(y_m - x_j)} \\
&= \sum_{l=j_-+1}^{m-1} E_l \sum_{j=j_-}^{l-1} \frac{-(y_{m+1}-y_m - x_{j+1} + x_j)}{(m-j)(y_m - x_j)(y_{m+1}-x_{j+1})} + O(n\|E\|_\infty) \\
&= \sum_{l=j_-+1}^{m-1} E_l \sum_{j=j_-}^{l-1} O\left(\frac{n^{-1}}{n^{-2}|m-j|^3} \right) + O(n\|E\|_\infty) = O(n\|E\|_\infty).
\end{aligned}
$$

The second sum can be treated similarly. Note that the estimates above on $A_{1,m+1} - A_{1,m}$ and $A_{2,m+1} - A_{2,m}$ are not optimal, but they, along with (5.8), (5.9), (4.32) and (5.7), are sufficient to obtain the following bound on $\cot z$:

$$
\cot z = -\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)} + A_{1,m} + A_{2,m} + O(n^{-1/2} + n\|E\|_\infty).
$$

Here we also replaced y_m with $\bar{x}_m$ in the argument of the Hilbert transform generating $O(n^{-1})$ error. Then, using (3.4), (4.10) and (4.30), we have

$$\begin{aligned}\sin^2 z &= \frac{1}{1+\cot^2 z}\\ &= \frac{1}{1+\left(\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right)^2 - 2\frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}(A_{1,m}+A_{2,m}) + O(n^{-1/2}+n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2)}\\ &= \frac{u(\bar{x}_m)^2}{u(\bar{x}_m)^2+Hu(\bar{x}_m)^2} + \frac{2u(\bar{x}_m)^3 Hu(\bar{x}_m)}{(u(\bar{x}_m)^2+Hu(\bar{x}_m)^2)^2}(A_{1,m}+A_{2,m})\\ &\quad + O(n^{-1/2}+n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2). \end{aligned} \tag{5.12}$$

Now we derive from (5.5), (5.8) and (5.12) that

$$\begin{aligned} y_{m+1} - y_m - x_{m+1} + x_m &= -\frac{u(\bar{x}_{m+1})-u(\bar{x}_m)}{u(\bar{x}_m)}(y_{m+1}-x_{m+1})\\ &+ \frac{u(\bar{x}_m)}{2\pi n(u(\bar{x}_m)^2+Hu(\bar{x}_m)^2)}\left(\frac{Hu(\bar{x}_{m+1})}{u(\bar{x}_{m+1})} - \frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right)\\ &+ \frac{u(\bar{x}_m)^2 Hu(\bar{x}_m)}{\pi n(u(\bar{x}_m)^2+Hu(\bar{x}_m)^2)^2}(A_{1,m}+A_{2,m})\left(\frac{Hu(\bar{x}_{m+1})}{u(\bar{x}_{m+1})} - \frac{Hu(\bar{x}_m)}{u(\bar{x}_m)}\right)\\ &+ O(\delta n^{-1}|y_{m+1}-y_m-x_{m+1}+x_m| + \delta n^{-3} + \delta n^{-1}\|E\|_\infty)\\ &+ O\Big((n^{-1/2}+n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty)\cdot n^{-1}\cdot \delta n^{-1}\Big) + \frac{\sin^2 z}{8\pi^2 n^2 u(\bar{x}_m)^2}(G_m - G_{m+1}). \end{aligned}$$

Here in the penultimate term on the right hand side we used (5.9). Using (5.6), we can further simplify

$$\begin{aligned} y_{m+1} - y_m - x_{m+1} + x_m &= -\frac{\partial_x u(\bar{x}_m)}{2nu(\bar{x}_m)^2}(y_{m+1}-x_{m+1})\\ &+ \frac{1}{4\pi n^2(u(\bar{x}_m)^2+Hu(\bar{x}_m)^2)}\partial_x\left(\frac{Hu}{u}\right)(\bar{x}_m)\\ &+ \frac{u(\bar{x}_m)Hu(\bar{x}_m)}{2\pi n^2(u(\bar{x}_m)^2+Hu(\bar{x}_m)^2)^2}(A_{1,m}+A_{2,m})\partial_x\left(\frac{Hu}{u}\right)(\bar{x}_m)\\ &+ O(\delta n^{-1}|y_{m+1}-y_m-x_{m+1}+x_m| + \delta n^{-5/2} + \delta n^{-1}\|E\|_\infty + \delta(\log n)^2\|E\|_\infty^2)\\ &+ \frac{\sin^2 z}{8\pi^2 n^2 u(\bar{x}_m)^2}(G_m - G_{m+1}). \end{aligned} \tag{5.13}$$

The equation (5.13) will be our starting point in derivation of the error propagation estimates. The key task is to obtain sufficiently strong control over $G_m - G_{m+1}$ (recall that G_m is defined in (5.3)). Before starting this task, let us record the following corollary of (5.13) that we will need later.

Lemma 5.1 *We have*

$$|y_{m+1}-y_m-x_{m+1}+x_m| = O(n^{-3/2} + \log n\|E\|_\infty + (\log n)^2\|E\|_\infty^2). \tag{5.14}$$

Proof Let us recall that $A_{1,m}$ and $A_{2,m}$ have order $O(n\log n\|E\|_\infty)$, and note that it follows from (5.3), (4.31) and (4.32) that

$$G_m = 4\pi nu(\bar{x}_m)(A_{1,m} + A_{2,m}) + O(n^{1/2} + n^2\|E\|_\infty) = O(n^{1/2} + n^2\log n\|E\|_\infty).$$

Similarly, $G_{m+1} = O(n^{1/2} + n^2\log n\|E\|_\infty)$. Inserting these estimates into (5.13), we obtain

$$\begin{aligned} y_{m+1} - y_m - x_{m+1} + x_m &= O(\delta n^{-2}) + O(\delta n^{-1}\log n\|E\|_\infty) \\ &\quad + O(\delta n^{-1}|y_{m+1} - y_m - x_{m+1} + x_m| + \delta n^{-5/2} + \delta n^{-1}\|E\|_\infty + \delta(\log n)^2\|E\|_\infty^2) \\ &\quad + O(n^{-3/2} + \log n\|E\|_\infty). \end{aligned}$$

Move the $O(\delta n^{-1}|y_{m+1} - y_m - x_{m+1} + x_m)|$ term to the left hand side and simplify the equation. This yields the desired bound

$$\begin{aligned} y_{m+1} - y_m - x_{m+1} + x_m &= \left(1 + O(\delta n^{-1})\right) \cdot O\left(n^{-3/2} + \log n\|E\|_\infty + (\log n)^2\|E\|_\infty^2\right) \\ &= O\left(n^{-3/2} + \log n\|E\|_\infty + (\log n)^2\|E\|_\infty^2\right). \end{aligned}$$

□

The estimate (5.14) is suboptimal, and cannot be used to establish sufficient control over propagation of error. We will only use it as an ingredient for deriving more precise estimates. The improvements are possible because in the proof of Lemma 5.1, we did not make use of cancellations in the difference $G_m - G_{m+1}$. In the next two sections, we focus on obtaining improved error estimates of the major term $G_m - G_{m+1}$.

6 Estimates for pairs of roots: near field errors

Let us first estimate $I_{1,m} - I_{1,m+1}$; recall (4.11) which defined

$$I_{1,m} = -\sum_{j\in S_m}\left(\cot\frac{y_m - x_j}{2} - \frac{2}{y_m - x_j}\right).$$

Lemma 6.1 *We have*

$$I_{1,m} - I_{1,m+1} = O\big(n^{1/2}|y_{m+1} - y_m - x_{m+1} + x_m| + \delta n^{-1} + n^{1/2}\|E\|_\infty\big). \tag{6.1}$$

Proof We start with a Laurent series expansion for cot near zero:

$$\cot\frac{y_m - x_j}{2} = \sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k} B_{2k}}{(2k)!}\left(\frac{y_m - x_j}{2}\right)^{2k-1}, \tag{6.2}$$

where B_{2k} are the Bernoulli numbers,

$$B_{2k} \sim (-1)^{k-1} 4\sqrt{\pi k}\left(\frac{k}{\pi e}\right)^{2k}$$

for large k. Observe that for any integer $p > 0$,

$$\begin{aligned}
&\left|\sum_{j\in S_m}\left((y_m - x_j)^p - (y_{m+1} - x_{j+1})^p\right)\right| \le \sum_{j\in S_m} |y_m - x_j - y_{m+1} + x_{j+1}| p C^{p-1}\left(\frac{|j-m|^{p-1}}{n^{p-1}}\right)\\
&= pC^{p-1}\sum_{j\in S_m}\left|(y_m - x_m - y_{m+1} + x_{m+1}) - \frac{1}{2nu(\bar{x}_m)} - E_m + \frac{1}{2nu(\bar{x}_j)} + E_j\right|\left(\frac{|j-m|^{p-1}}{n^{p-1}}\right)\\
&\qquad \le C^{p-1}\left(n^{-\frac{p}{2}+1}|y_{m+1} - y_m - x_{m+1} + x_m| + \delta n^{-\frac{p+1}{2}} + n^{-\frac{p}{2}+1}\|E\|_\infty\right),
\end{aligned}$$

where in the second step the constant C comes from Lemma 4.1 and may only depend on u_0. Therefore,

$$\begin{aligned}
&\left|I_{1,m} - I_{1,m+1}\right| = \left|\sum_{j\in S_m}\sum_{k=1}^{\infty}\frac{(-1)^k 2B_{2k}}{(2k)!}\left((y_m - x_j)^{2k-1} - (y_{m+1} - x_{j+1})^{2k-1}\right)\right|\\
&\le 2\sum_{k=1}^{\infty}\frac{C^{2k-2}|B_{2k}|}{(2k)!}\left(n^{-\frac{2k-3}{2}}|y_{m+1} - y_m - x_{m+1} + x_m| + \delta n^{-k} + n^{-\frac{2k-3}{2}}\|E\|_\infty\right)\\
&\qquad \le C_1\left(n^{1/2}|y_{m+1} - y_m - x_{m+1} + x_m| + \delta n^{-1} + n^{1/2}\|E\|_\infty\right)
\end{aligned}$$

for all sufficiently large $n \ge n_0(u_0)$, and with a constant C_1 that may only depend on u_0. □

Next, for $I_{2,m} - I_{2,m+1}$, we denote $\{\tilde{x}_j\}$ and $\{\hat{x}_j\}$ equally distributed points centered at x_m and x_{m+1}, respectively. Recall that $I_{2,m}$ is defined by (4.22), and that

$$\tilde{x}_j = x_m + \frac{j-m}{2nu(\bar{x}_m)}, \quad \hat{x}_j = x_{m+1} + \frac{j-(m+1)}{2nu(\bar{x}_{m+1})}.$$

Then, we can express

$$\begin{aligned}
I_{2,m} - I_{2,m+1} &= \sum_{j\in\mathbb{Z}\setminus S_m}\frac{2}{y_m - \tilde{x}_j} - \sum_{j\in\mathbb{Z}\setminus S_{m+1}}\frac{2}{y_{m+1} - \hat{x}_j}\\
&= \sum_{j\in\mathbb{Z}\setminus S_m}\left(\frac{2}{y_m - \tilde{x}_j} - \frac{2}{y_{m+1} - \hat{x}_{j+1}}\right)\\
&= \sum_{j\in\mathbb{Z}\setminus S_m}\frac{2(y_{m+1} - y_m - x_{m+1} + x_m) + \frac{j-m}{n}\left(\frac{1}{u(\bar{x}_m)} - \frac{1}{u(\bar{x}_{m+1})}\right)}{(y_m - \tilde{x}_j)(y_{m+1} - \hat{x}_{j+1})}
\end{aligned} \tag{6.3}$$

Lemma 6.2 *The following estimate holds:*

$$I_{2,m} - I_{2,m+1} = O\big(n^{3/2}|y_{m+1} - y_m - x_{m+1} + x_m| + \delta n^{-1/2} + \delta n\|E\|_\infty\big). \tag{6.4}$$

Proof We estimate the two parts in (6.3) one by one. The first part can be estimated by

$$\begin{aligned}\sum_{j\in\mathbb{Z}\setminus S_m} \frac{2(y_{m+1} - y_m - x_{m+1} + x_m)}{(y_m - \tilde{x}_j)(y_{m+1} - \tilde{x}_{j+1})} &= (y_{m+1} - y_m - x_{m+1} + x_m) \sum_{j\in\mathbb{Z}\setminus S_m} O(n^2|m-j|^{-2}) \\ &= O\left(n^{3/2}|y_{m+1} - y_m - x_{m+1} + x_m|\right).\end{aligned}$$

For the second part in (6.3), we match the $j = m \pm k$ terms over the summation range, similarly to (4.25):

$$\begin{aligned}&\frac{k}{(y_m - \tilde{x}_{m+k})(y_{m+1} - \hat{x}_{m+k+1})} + \frac{-k}{(y_m - \tilde{x}_{m-k})(y_{m+1} - \hat{x}_{m-k+1})} \\ &= k \cdot \frac{2(y_m - x_m)\frac{k}{2nu(\bar{x}_{m+1})} + 2(y_{m+1} - x_{m+1})\frac{k}{2nu(\bar{x}_m)}}{(y_m - \tilde{x}_{m+k})(y_{m+1} - \hat{x}_{m+k+1})(y_m - \tilde{x}_{m-k})(y_{m+1} - \hat{x}_{m-k+1})} \\ &= O(k^2 \cdot n^{-2} \cdot n^4 k^{-4}) = O(n^2 k^{-2}).\end{aligned}$$

Sum over k and get

$$\frac{1}{n}\left(\frac{1}{u(\bar{x}_m)} - \frac{1}{u(\bar{x}_{m+1})}\right) \sum_{k \gtrsim n^{1/2}} O(n^2 k^{-2}) = O(\delta n^{-2}) \cdot O(n^{3/2}) = O(\delta n^{-1/2}).$$

As before (see (4.20)), there could be at most $O(1 + n^{3/2}\|E\|_\infty)$ mismatched terms, leading to the error

$$O(1 + n^{3/2}\|E\|_\infty)) \cdot O(\delta n^{-2}) \cdot O\left(n^{1/2} \cdot (n^{1/2})^2\right) = O(\delta n^{-1/2} + \delta n\|E\|_\infty)).$$

Collecting all the estimates, we conclude the proof of (6.4). □

Finally, let us estimate $I_{3,m} - I_{3,m+1}$. Recall that (4.23) defined

$$I_{3,m} = \sum_{j\in S_m} \left(\frac{2}{y_m - \tilde{x}_j} - \frac{2}{y_m - x_j}\right).$$

Lemma 6.3 *Let $I_{3,m}$ be as defined in (4.23). Then,*

$$\begin{aligned}I_{3,m} - I_{3,m+1} = D_{1,m} + B_{1,m} + B_{2,m} &+ O\big((n^3\|E\|_\infty + \delta n \log n)|y_{m+1} - y_m - x_{m+1} + x_m|\big) \\ &+ O(\delta n^{-1/2} + \delta n\|E\|_\infty + \delta n^2 \log n\|E\|_\infty^2),\end{aligned} \tag{6.5}$$

where $D_{1,m}$, $B_{1,m}$, $B_{2,m}$ are defined below in (6.8), (6.13) and (6.20) respectively.

Remark 6.4 $D_{1,m}$, $B_{1,m}$, $B_{2,m}$ all have form

$$D_{1,m} = \sum_{j\in S_m} d_1(j,m)(E_j - E_m), \quad B_{i,m} = \sum_{j\in S_m} b_i(j,m)(E_j - E_m).$$

In particular, one can see from (6.9) that $d_1(j,m) > 0$. Therefore, $D_{1,m}$ is a dissipative term. We will show later in Section 8 that $B_{1,m}$ and $B_{2,m}$ are supercritical but can be absorbed into the dissipation.

Proof Let us consider the difference between the expressions appearing in the $(j+1)$st term in $I_{3,m+1}$ and the jth term in $I_{3,m}$. Note that $j = m$ summands vanish identically. We have

$$\frac{2}{y_{m+1} - x_{j+1}} - \frac{2}{y_m - x_j} = \frac{-2(y_{m+1} - y_m - x_{m+1} + x_m) + \frac{1}{n}\left(\frac{1}{u(\bar{x}_j)} - \frac{1}{u(\bar{x}_m)}\right) + 2(E_j - E_m)}{(y_{m+1} - x_{j+1})(y_m - x_j)}, \tag{6.6}$$

and

$$-\frac{2}{y_{m+1} - \hat{x}_{j+1}} + \frac{2}{y_m - \tilde{x}_j} = \frac{2(y_{m+1} - y_m - x_{m+1} + x_m) - \frac{j-m}{n}\left(\frac{1}{u(\bar{x}_{m+1})} - \frac{1}{u(\bar{x}_m)}\right)}{(y_{m+1} - \hat{x}_{j+1})(y_m - \tilde{x}_j)}. \tag{6.7}$$

The last summand of (6.6) produces a dissipative term that is crucial for the entire argument. We denote it as

$$D_{1,m} = \sum_{j\in S_m\backslash\{m\}} \frac{2(E_j - E_m)}{(y_{m+1} - x_{j+1})(y_m - x_j)} =: \sum_{j\in S_m\backslash\{m\}} d_1(j,m)(E_j - E_m), \tag{6.8}$$

where from the estimates (4.5) and (4.13), we know that the coefficients $d_1(j,m)$ satisfy

$$d_1(j,m) = \frac{2}{(y_{m+1} - x_{j+1})(y_m - x_j)} \sim \frac{n^2}{|m-j|^2} > 0, \quad \forall\, j \in S_m\backslash\{m\}. \tag{6.9}$$

Here $\sim$ as usual means lower and upper bounds with constants that may only depend on u_0.

We combine the first parts on the right hand side in (6.6) and (6.7). The corresponding term is

$$T_{1,m}^{j} = 2(y_{m+1} - y_m - x_{m+1} + x_m)\left(\frac{1}{(y_{m+1} - \hat{x}_{j+1})(y_m - \tilde{x}_j)} - \frac{1}{(y_{m+1} - x_{j+1})(y_m - x_j)}\right).$$

Now, we estimate the difference (using the summation convention (4.16)):

$$
\begin{aligned}
&\frac{1}{(y_{m+1}-\hat{x}_{j+1})(y_m-\tilde{x}_j)}-\frac{1}{(y_{m+1}-x_{j+1})(y_m-x_j)}\pm\frac{1}{(y_{m+1}-\hat{x}_{j+1})(y_m-x_j)}\\
&=\frac{\tilde{x}_j-x_j}{(y_{m+1}-\hat{x}_{j+1})(y_m-\tilde{x}_j)(y_m-x_j)}+\frac{\hat{x}_{j+1}-x_{j+1}}{(y_{m+1}-\hat{x}_{j+1})(y_{m+1}-x_{j+1})(y_m-x_j)}\\
&=-\frac{\sum_{l=m}^{j-1}E_l+O(\delta n^{-2}|m-j|^2)}{(y_{m+1}-\hat{x}_{j+1})(y_m-\tilde{x}_j)(y_m-x_j)}-\frac{\sum_{l=m+1}^{j}E_l+O(\delta n^{-2}|m-j|^2)}{(y_{m+1}-\hat{x}_{j+1})(y_{m+1}-x_{j+1})(y_m-x_j)}\\
&\lesssim\frac{|m-j|\|E\|_\infty+\delta(|j-m|/n)^2}{(|m-j|/n)^3}=O\left(n^3|m-j|^{-2}\|E\|_\infty+\delta n|m-j|^{-1}\right). \qquad (6.10)
\end{aligned}
$$

Here, we use the lower bound estimates (4.5), (4.13) for $|y_m-x_j|$ and $|y_{m+1}-x_{j+1}|$, and a variant of (4.27) for $x_j-\tilde{x}_j$ and $x_{j+1}-\hat{x}_{j+1}$ taking into account the presence of the δ factor. Now, summing up over $j\in S_m\backslash\{m\}$ we get

$$
\sum_{j\in S_m\backslash\{m\}}T_{1,m}^j=O\big((n^3\|E\|_\infty+\delta n\log n)|y_{m+1}-y_m-x_{m+1}+x_m|\big).
$$

Next, we combine the second parts on the right hand side of (6.6) and (6.7). The corresponding term is

$$
\begin{aligned}
T_{2,m}^j&=\frac{\frac{1}{n}\left(\frac{1}{u(\bar{x}_j)}-\frac{1}{u(\bar{x}_m)}\right)}{(y_{m+1}-x_{j+1})(y_m-x_j)}-\frac{\frac{j-m}{n}\left(\frac{1}{u(\bar{x}_{m+1})}-\frac{1}{u(\bar{x}_m)}\right)}{(y_{m+1}-\hat{x}_{j+1})(y_m-\tilde{x}_j)}\\
&=\frac{\frac{1}{n}\left(\frac{1}{u(\bar{x}_j)}-\frac{1}{u(\bar{x}_m)}\right)-\frac{j-m}{n}\left(\frac{1}{u(\bar{x}_{m+1})}-\frac{1}{u(\bar{x}_m)}\right)}{(y_{m+1}-x_{j+1})(y_m-x_j)}\\
&\quad+\frac{j-m}{n}\left(\frac{1}{u(\bar{x}_{m+1})}-\frac{1}{u(\bar{x}_m)}\right)\left(\frac{1}{(y_{m+1}-x_{j+1})(y_m-x_j)}-\frac{1}{(y_{m+1}-\hat{x}_{j+1})(y_m-\tilde{x}_j)}\right)\\
&=:T_{2,m}^{j,(1)}+T_{2,m}^{j,(2)}. \qquad (6.11)
\end{aligned}
$$

For the first term $T_{2,m}^{j,(1)}$, let us estimate the numerator

$$
\begin{aligned}
&\frac{1}{n}\left(\frac{1}{u(\bar{x}_j)}-\frac{1}{u(\bar{x}_m)}-\frac{j-m}{u(\bar{x}_{m+1})}+\frac{j-m}{u(\bar{x}_m)}\right)\\
&=\frac{1}{n}\Bigg[-\frac{\partial_x u(\bar{x}_m)}{u(\bar{x}_m)^2}(\bar{x}_j-\bar{x}_m)+O(\delta|\bar{x}_j-\bar{x}_m|^2)\\
&\qquad+\frac{\partial_x u(\bar{x}_m)(j-m)}{u(\bar{x}_m)^2}(\bar{x}_{m+1}-\bar{x}_m)+O(\delta|m-j|(\bar{x}_{m+1}-\bar{x}_m)^2)\Bigg]\\
&=-\frac{\partial_x u(\bar{x}_m)}{nu(\bar{x}_m)^2}\big(\bar{x}_j-\bar{x}_m-(j-m)(\bar{x}_{m+1}-\bar{x}_m)\big)+O(\delta n^{-3}|m-j|^2).
\end{aligned}
$$

Observe that

$$\begin{aligned}
&\bar{x}_j - \bar{x}_m - (j-m)(\bar{x}_{m+1} - \bar{x}_m) \\
&= \sum_{l=m}^{j-1} \left(\frac{1}{4nu(\bar{x}_l)} + \frac{1}{4nu(\bar{x}_{l+1})} + \frac{E_l + E_{l+1}}{2} \right) \\
&\quad - (j-m) \left(\frac{1}{4nu(\bar{x}_m)} + \frac{1}{4nu(\bar{x}_{m+1})} + \frac{E_m + E_{m+1}}{2} \right) \\
&= \sum_{l=m}^{j-1} \left(O(\delta n^{-1} \cdot n^{-1}|l-m|) + \frac{(E_l - E_m) + (E_{l+1} - E_{m+1})}{2} \right) \\
&= O(\delta n^{-2}|m-j|^2) + \sum_{l=m}^{j-1} \left(\frac{(E_l - E_m) + (E_{l+1} - E_{m+1})}{2} \right).
\end{aligned}$$

Note that the summation can be replaced by $\sum_{l=m+1}^{j-1}$ as the term $l = m$ is zero. When $j \le m-1$, the sum represents $-\sum_{j}^{m-1}$, in accordance with the summation convention (4.16).

Then for $T_{2,m}^{(1)} = \sum_{j \in S_m \setminus \{m\}} T_{2,m}^{j,(1)}$ we obtain

$$\begin{aligned}
T_{2,m}^{(1)} &= \sum_{j \in S_m \setminus \{m\}} \left[O\left(n^2|m-j|^{-2} \cdot \delta n^{-3}|m-j|^2\right) - \sum_{l=m+1}^{j-1} \frac{\partial_x u(\bar{x}_m)(E_l - E_m + E_{l+1} - E_{m+1})}{2nu(\bar{x}_m)^2 (y_{m+1} - x_{j+1})(y_m - x_j)} \right]. \\
&= B_{1,m} + O(\delta n^{-1/2}),
\end{aligned}$$

where the term $B_{1,m}$ can be expressed as

$$\begin{aligned}
B_{1,m} = \frac{-\partial_x u(\bar{x}_m)}{2nu(\bar{x}_m)^2} \Bigg[&\sum_{j \in S_m \setminus \{m\}} \sum_{l=m+1}^{j-1} \frac{(E_l - E_m) + (E_{l+1} - E_m)}{(y_{m+1} - x_{j+1})(y_m - x_j)} \\
&- (E_{m+1} - E_m) \sum_{j \in S_m \setminus \{m\}} \sum_{l=m+1}^{j-1} \frac{1}{(y_{m+1} - x_{j+1})(y_m - x_j)} \Bigg] \qquad (6.12) \\
=: \sum_{l \in S_m \setminus \{m\}} b_1(l,m)(E_l - E_m). \qquad (6.13)
\end{aligned}$$

Let us estimate the coefficients $\{b_1(l,m)\}$. For $l \ge m+2$, we change the order of summation:

$$\sum_{j=m+1}^{j_+-1} \sum_{l=m+1}^{j-1} \frac{1}{(y_{m+1} - x_{j+1})(y_m - x_j)} = \sum_{l=m+1}^{j_+-2} \sum_{j=l+1}^{j_+-1} \frac{1}{(y_{m+1} - x_{j+1})(y_m - x_j)}. \qquad (6.14)$$

Using the lower bound estimate (4.5), we can estimate the sum in (6.14) and then the coefficient $b_1(l,m)$ by

$$O(\delta n^{-1}) \sum_{j=l+1}^{j_+-1} \frac{1}{|m-j|^2 n^{-2}} = O(\delta n |m-l|^{-1}).$$

A similar calculation can be done for $l \leq m-1$. For $l = m+1$, there is a contribution similar to (6.14) (where we now also use (4.13)), and an additional summand from (6.12) that is equal to

$$\frac{\partial_x u(\bar{x}_m)}{2nu(\bar{x}_m)^2} \left(\sum_{j=j_-}^{m-1} \frac{j-m}{(y_{m+1}-x_{j+1})(y_m - x_j)} + \sum_{j=m+1}^{j_+-1} \frac{j-m-1}{(y_{m+1}-x_{j+1})(y_m-x_j)} \right)$$
$$= O(\delta n^{-1}) \sum_{1 \leq |j-m| \lesssim n^{1/2}} O\left(\frac{|m-j|}{|m-j|^2 n^{-2}} \right) = O(\delta n \log n).$$

To summarize, the coefficients

$$b_1(l,m) = \begin{cases} O(\delta n |m-l|^{-1}), & l \in S_m \backslash \{m, m+1\} \\ O(\delta n \log n), & l = m+1. \end{cases}$$

Comparing with (6.9), one sees that

$$|b_1(j,m)| \lesssim n^{-1/2} d_1(j,m), \quad \forall\, j \in S_m \backslash \{m\}. \tag{6.15}$$

Hence, for all sufficiently large n, $B_{1,m}$ can be absorbed into $D_{1,m}$ with bounds (6.9) remaining valid.

For the second term $T_{2,m}^{j,(2)}$ in (6.11), we apply a calculation as in (6.10) and obtain

$$T_{2,m}^{j,(2)} = \sum_{j \in S_m \backslash \{m\}} \frac{j-m}{n} \left(\frac{1}{u(\bar{x}_{m+1})} - \frac{1}{u(\bar{x}_m)} \right) \times \tag{6.16}$$
$$\times \left(\frac{\sum_{l=m}^{j-1} E_l + O(\delta n^{-2}|j-m|^2)}{(y_{m+1}-\hat{x}_{j+1})(y_m - \tilde{x}_j)(y_m - x_j)} + \frac{\sum_{l=m+1}^{j} E_l + O(\delta n^{-2}|j-m|^2)}{(y_{m+1}-\hat{x}_{j+1})(y_{m+1}-x_{j+1})(y_m - x_j)} \right).$$

The parts of this expression corresponding to $O(\delta n^{-2}|j-m|^2)$ in the numerator lead after summation in j to the error of the order

$$\lesssim \delta \sum_{j \in S_m \backslash \{m\}} \frac{|j-m|}{n^2} \frac{n^3}{|j-m|^3} n^{-2} |j-m|^2 = O(\delta n^{-1/2}).$$

To further simplify the remaining part, we first replace $y_{m+1} - \hat{x}_{j+1}$ by $x_{m+1} - \hat{x}_{j+1} = \frac{m-j}{2nu(\bar{x}_{m+1})}$. Note that by (4.5), (4.13) we have

$$\frac{1}{y_{m+1} - \hat{x}_{j+1}} - \frac{1}{\frac{m-j}{2nu(\bar{x}_{m+1})}} = \frac{-(y_{m+1} - x_{m+1})}{(y_{m+1} - \hat{x}_{j+1}) \cdot \frac{m-j}{2nu(\bar{x}_{m+1})}} = O\left(\frac{n}{|m-j|^2}\right). \tag{6.17}$$

The difference in the remaining part of (6.16) introduced by making such change does not exceed

$$\sum_{1 \le |j-m| \lesssim n^{1/2}} O\left(\frac{|j-m|}{n} \cdot \delta n^{-1} \cdot |m-j| \|E\|_\infty \cdot \frac{n}{|m-j|^2} \cdot \frac{n^2}{|m-j|^2}\right) = O(\delta n \|E\|_\infty). \tag{6.18}$$

We can similarly replace $y_m - \tilde{x}_j$ with $\frac{m-j}{2nu(\bar{x}_m)}$ incurring the error of the same order. Therefore, $T_{2,m}^{(2)} = \sum_{j \in S_m \setminus \{m\}} T_{2,m}^{j,(2)}$ after a straightforward calculation can be represented as

$$\begin{aligned} T_{2,m}^{(2)} =& \frac{\partial_x u(\bar{x}_m)}{nu(\bar{x}_m)^2} \sum_{j \in S_m \setminus \{m\}} \left(\frac{\sum_{l=m}^{j-1} E_l}{\frac{m-j}{2nu(\bar{x}_m)} \cdot (y_m - x_j)} + \frac{\sum_{l=m+1}^{j} E_l}{(y_{m+1} - x_{j+1})(y_m - x_j)} \right) \\ &+ O(\delta n^{-1/2} + \delta n \|E\|_\infty + \delta n^2 \log n \|E\|_\infty^2). \end{aligned} \tag{6.19}$$

To absorb the remaining supercritical term into $D_{1,m}$, we replace E_l by $E_l - E_m$, and define

$$\begin{aligned} B_{2,m} =& \frac{\partial_x u(\bar{x}_m)}{nu(\bar{x}_m)^2} \sum_{j \in S_m \setminus \{m\}} \left(\frac{\sum_{l=m}^{j-1} (E_l - E_m)}{\frac{m-j}{2nu(\bar{x}_m)} \cdot (y_m - x_j)} + \frac{\sum_{l=m+1}^{j} (E_l - E_m)}{(y_{m+1} - x_{j+1})(y_m - x_j)} \right) \\ =:& \sum_{j \in S_m \setminus \{m\}} b_2(j, m)(E_j - E_m), \end{aligned} \tag{6.20}$$

where the coefficients $b_2(j, m)$ can be estimated similarly to $b_1(j, m)$, yielding

$$b_2(j, m) = O(\delta n |m-j|^{-1}) \lesssim n^{-1/2} d_1(j, m), \quad \forall\, j \in S_m \setminus \{m\}. \tag{6.21}$$

Therefore, $B_{2,m}$ can be absorbed by $D_{1,m}$ as well. Notice that there is a partial cancellation between $B_{2,m}$ and $B_{1,m}$ but we do not need to pursue it here, as it does not eliminate the supercritical term.

We are left with the difference between the sum in (6.19) and $B_{2,m}$, which is given by

$$E_m \cdot \frac{\partial_x u(\bar{x}_m)}{nu(\bar{x}_m)^2} \sum_{j \in S_m \setminus \{m\}} \left(-\frac{2nu(\bar{x}_m)}{y_m - x_j} + \frac{j-m}{(y_{m+1} - x_{j+1})(y_m - x_j)} \right). \tag{6.22}$$

For the first term of the sum in (6.22), we make use of the cancellation between the $j = m \pm k$ terms, as described in (4.18) and (4.21). This leads to the error

$$O\left(\|E\|_\infty \cdot \delta n^{-1} \cdot n \cdot (n + n^2 \log n \|E\|_\infty)\right) = O(\delta n \|E\|_\infty + \delta n^2 \log n \|E\|_\infty^2).$$

The second summand in (6.22) becomes the same as the first one if we replace $y_{m+1} - x_{j+1}$ by $y_{m+1} - \hat{x}_{j+1}$, and then by $x_{m+1} - \hat{x}_{j+1}$. The latter replacement produces the same error as in (6.17), (6.18). We calculate an error generated by the former replacement as follows. Using an estimate parallel to (4.27) we get

$$\begin{aligned} &\frac{1}{y_{m+1} - \hat{x}_{j+1}} - \frac{1}{y_{m+1} + x_{j-1}} = \frac{\hat{x}_{j+1} - x_{j+1}}{(y_{m+1} - \hat{x}_{j+1})(y_{m+1} - x_{j+1})} \\ &= O\left(\frac{|m-j| \|E\|_\infty + \delta n^{-2} |m-j|^2}{(|m-j|/n)^2}\right) = O(|m-j|^{-1} n^2 \|E\|_\infty + \delta). \end{aligned}$$

This leads to an error

$$O\left(\|E\|_\infty \cdot \delta n^{-1} \cdot \sum_{1 \le |j-m| \lesssim n^{1/2}} \left(\frac{n^2 \|E\|_\infty}{|m-j|} + \delta\right) \cdot \frac{n|j-m|}{|j-m|}\right) = O\left(\delta n^{1/2} \|E\|_\infty + \delta n^2 \log n \|E\|_\infty^2\right).$$

Collecting all the estimates, we arrive at (6.5). □

In summary, collecting the bounds (6.1), (6.4) and (6.5), the contribution into $G_m - G_{m+1}$ from the near field is

$$\begin{aligned} &(I_{1,m} - I_{1,m+1}) + (I_{2,m} - I_{2,m+1}) + (I_{3,m} - I_{3,m+1}) \\ &= D_{1,m} + B_{1,m} + B_{2,m} + O\left((n^{3/2} + n^3 \|E\|_\infty) |y_{m+1} - y_m - x_{m+1} + x_m|\right) \\ &\quad + O(\delta n^{-1/2} + \delta n \|E\|_\infty + \delta n^2 \log n \|E\|_\infty^2). \end{aligned} \tag{6.23}$$

7 Estimates for pairs of roots: far field errors

We start with $II_{1,m} - II_{1,m+1}$. Recall (4.1):

$$II_{1,m} = 2n P.V. \int_{|y-y_m| \le n^{-1/2}} u(y) \cot \frac{y_m - y}{2} \, dy.$$

Lemma 7.1 *Let $II_{1,m}$ be defined as in (4.1). Then,*

$$II_{1,m} - II_{1,m+1} = O(\delta n^{-1/2}). \tag{7.1}$$

Proof We have

$$II_{1,m} - II_{1,m+1} = 2n\, P.V. \int_{-n^{-1/2}}^{n^{-1/2}} \Big(u(y_{m+1}+z) - u(y_m+z)\Big) \cot\frac{z}{2}\, dz$$
$$= 2n \int_0^{n^{-1/2}} \Big(u(y_{m+1}+z) - u(y_{m+1}-z) - u(y_m+z) + u(y_m-z)\Big)\cot\frac{z}{2}\, dz$$
$$= 2n \int_0^{n^{-1/2}} \Big(2u'(y_{m+1})z - 2u'(y_m)z + O(\delta z^3)\Big)\cot\frac{z}{2}\, dz$$
$$= 2n \cdot \Big((y_{m+1}-y_m)\cdot O(\delta n^{-1/2}) + O(\delta n^{-3/2})\Big) = O(\delta n^{-1/2}).$$

□

Next, we focus on $II_{2,m} - II_{2,m+1}$. The estimate of this term is the most involved one, and will take us a while to complete. Let us decompose $II_{2,m}$ as in (4.8). We will discuss the matching summands $II^j_{2,m}$ and the mismatched part one by one. The definition of $II^j_{m,2}$ is recalled in (7.3) right after the statement of the lemma.

Lemma 7.2 *Let $II^j_{2,m}$ be defined as in (4.3). Then,*

$$\sum_{j\in S_m^c}\Big(II^j_{2,m} - II^{j+1}_{2,m+1}\Big) = D_{2,m} + H_{1,m} + H_{2,m} + H_{3,m}$$
$$+ nu(\bar{x}_{j_+-1})\cot\frac{y_m - x_{j_+}}{2}(E_{j_+-1} - E_m) + nu(\bar{x}_{j_+})\cot\frac{y_m - x_{j_++1}}{2}(E_{j_+} - E_m)$$
$$- nu(\bar{x}_{j_--})\cot\frac{y_m - x_{j_--1}}{2}(E_{j_-} - E_m) - nu(\bar{x}_{j_-1})\cot\frac{y_m - x_{j_--2}}{2}(E_{j_--1} - E_m)$$
$$+ O\big((n^{5/2}\|E\|_\infty + n^{1/2})|y_{m+1} - y_m - x_{m+1} + x_m|\big)$$
$$+ O(\delta n^{-1/2} + \delta n\|E\|_\infty + \delta n^2\log n\|E\|_\infty^2) + O(n^{1/2}\|E\|_\infty + n^2\|E\|_\infty^2), \quad (7.2)$$

where $D_{2,m}$, $H_{1,m}$, $H_{2,m}$ and $H_{3,m}$ are defined below in (7.16), (7.9), (7.26) and (7.37) respectively.

Remark 7.3 $D_{2,m}$, $H_{1,m}$, $H_{2,m}$ and $H_{3,m}$ will all have form

$$D_{2,m} = \sum_j d_2(j,m)(E_j - E_m), \quad H_{i,m} = \sum_j h_i(j,m)(E_j - E_m).$$

In particular, we will discover (7.17) that $d_2(j,m) > 0$. Therefore, $D_{2,m}$ is a dissipative term. We will show later that the supercritical terms $H_{1,m}$, $H_{2,m}$, $H_{3,m}$ can be absorbed into the dissipation.

Proof We first recall the definition of $II^j_{2,m}$ in (4.3):

$$\cot\frac{y_m - x_j}{2}\left(2n\int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y)dy - 1\right) + 2n\int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y)\left(\cot\frac{y_m - y}{2} - \cot\frac{y_m - x_j}{2}\right)dy. \quad (7.3)$$

We will obtain more precise estimates on $II_{2,m}^{j}$, and consider the difference $II_{2,m}^{j} - II_{2,m+1}^{j+1}$ only when we need to take advantage of the additional cancellation between the two terms. In simpler contributions where we do not need to consider the difference, the mth and $(m+1)$st estimates are identical and we only provide the arguments for $II_{2,m}^{j}$.

We start with the first term in (7.3). A refined compared with (4.4) calculation yields

$$\begin{aligned}\int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y)dy &= \int_{\bar{x}_{j-1}}^{x_j} (u(\bar{x}_{j-1}) + u'(\bar{x}_{j-1})(z-\bar{x}_{j-1}) + O\big(\delta(z-\bar{x}_{j-1})^2)\big)\,dz \\ &\quad + \int_{x_j}^{\bar{x}_j} \big(u(\bar{x}_j) + u'(\bar{x}_j)(z-\bar{x}_j) + O(\delta(z-\bar{x}_j)^2)\big)\,dz \\ &= \frac{1}{2n} + \frac{u(\bar{x}_{j-1})}{2}E_{j-1} + \frac{u(\bar{x}_j)}{2}E_j \\ &\quad + u'(\bar{x}_{j-1})\left(\frac{x_j - x_{j-1}}{2}\right)^2 - u'(\bar{x}_j)\left(\frac{x_{j+1}-x_j}{2}\right)^2 + O(\delta n^{-3}).\end{aligned}$$

Apply the estimate to the first term of (7.3), and sum over $j \in S_m^c$. We obtain

$$\begin{aligned}\sum_{j\in S_m^c} \cot\frac{y_m - x_j}{2}\left(2n\int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y)dy - 1\right) &= n\sum_{j\in S_m^c}\cot\frac{y_m-x_j}{2}\big(u(\bar{x}_{j-1})E_{j-1} + u(\bar{x}_j)E_j\big) \\ + \frac{n}{2}\sum_{j\in S_m^c}\cot\frac{y_m-x_j}{2}\Big(u'(\bar{x}_{j-1})(x_j-x_{j-1})^2 &- u'(\bar{x}_j)(x_{j+1}-x_j)^2\Big) + O(\delta n^{-1}\log n). \qquad (7.4)\end{aligned}$$

For the second sum on the right hand side of (7.4), we have the following cancellation:

$$\begin{aligned}&u'(\bar{x}_{j-1})(x_j - x_{j-1})^2 - u'(\bar{x}_j)(x_{j+1}-x_j)^2 \\ &= u'(\bar{x}_j)\left[\left(\frac{1}{2nu(\bar{x}_{j-1})} + E_{j-1}\right)^2 - \left(\frac{1}{2nu(\bar{x}_j)} + E_j\right)^2\right] + O(\delta n^{-3}) \\ &= u'(\bar{x}_j)\left[\frac{1}{4n^2}\left(\frac{1}{u(\bar{x}_{j-1})^2} - \frac{1}{u(\bar{x}_j)^2}\right) + (E_{j-1}^2 - E_j^2) + \left(\frac{E_{j-1}}{nu(\bar{x}_{j-1})} - \frac{E_j}{nu(\bar{x}_j)}\right)\right] + O(\delta n^{-3}) \\ &= \frac{u'(\bar{x}_j)}{nu(\bar{x}_j)}(E_{j-1} - E_j) + O(\delta n^{-3} + \delta\|E\|_\infty^2 + \delta n^{-2}\|E\|_\infty).\end{aligned}$$

This leads to the estimate

$$\begin{aligned}&\frac{n}{2}\sum_{j\in S_m^c}\cot\frac{y_m-x_j}{2}\big(u'(\bar{x}_{j-1})(x_j-x_{j-1})^2 - u'(\bar{x}_j)(x_{j+1}-x_j)^2\big) \\ &= \sum_{j\in S_m^c}\frac{u'(\bar{x}_j)}{2u(\bar{x}_j)}\cot\frac{y_m-x_j}{2}(E_{j-1}-E_j) + O(\delta n^{-1}\log n + \delta\log n\|E\|_\infty + \delta n^2\log n\|E\|_\infty^2).\end{aligned} \qquad (7.5)$$

For the remaining sum on the right side of (7.5), we can telescope and get

$$\begin{aligned}
&\sum_{j\in S_m^c}\frac{u'(\bar x_j)}{2u(\bar x_j)}\cot\frac{y_m-x_j}{2}(E_{j-1}-E_j)\\
&=\sum_{j\in S_m^c\setminus\{j_+\}}\left(\frac{u'(\bar x_{j+1})}{2u(\bar x_{j+1})}\cot\frac{y_m-x_{j+1}}{2}-\frac{u'(\bar x_j)}{2u(\bar x_j)}\cot\frac{y_m-x_j}{2}\right)E_j+O(\delta n^{1/2}\|E\|_\infty)\\
&=\sum_{|m-j|\gtrsim n^{1/2}}\left(O(\delta n^{-1}\cdot n|m-j|^{-1})+O(\delta n|m-j|^{-2})\right)E_j+O(\delta n^{1/2}\|E\|_\infty)\\
&=O(\delta n^{1/2}\|E\|_\infty),
\end{aligned}$$

where we have used (4.6) and a variant of (4.7) in the second equality.

Now, we focus on the first term on the right in (7.4),

$$n\sum_{j\in S_m^c}\cot\frac{y_m-x_j}{2}\big(u(\bar x_{j-1})E_{j-1}+u(\bar x_j)E_j\big).$$

This is a major term. We need to work on the difference with the corresponding term in II_{m+1}^{j+1} :

$$\begin{aligned}
&n\sum_{j\in S_m^c}\left[\cot\frac{y_m-x_j}{2}\big(u(\bar x_{j-1})E_{j-1}+u(\bar x_j)E_j\big)-\cot\frac{y_{m+1}-x_{j+1}}{2}\big(u(\bar x_j)E_j+u(\bar x_{j+1})E_{j+1}\big)\right]\\
&=n\sum_{j\in S_m^c}\left(\cot\frac{y_m-x_j}{2}-\cot\frac{y_{m+1}-x_{j+1}}{2}\right)\big(u(\bar x_j)E_j+u(\bar x_{j+1})E_{j+1}\big)\\
&\quad+n\sum_{j\in S_m^c}\cot\frac{y_m-x_j}{2}\big(u(\bar x_{j-1})E_{j-1}-u(\bar x_{j+1})E_{j+1}\big).
\end{aligned}\tag{7.6}$$

For the first term on the right hand side of (7.6), an estimate using mean value theorem yields

$$\begin{aligned}
&\cot\frac{y_m-x_j}{2}-\cot\frac{y_{m+1}-x_{j+1}}{2}=\frac{1}{2\sin^2\frac{z_j}{2}}\cdot(y_{m+1}-y_m-x_{j+1}+x_j)\\
&=\frac{1}{2\sin^2\frac{z_j}{2}}\cdot\left[(y_{m+1}-y_m-x_{m+1}+x_m)+\frac{1}{2n}\left(\frac{1}{u(\bar x_m)}-\frac{1}{u(\bar x_j)}\right)+(E_m-E_j)\right]
\end{aligned}\tag{7.7}$$

where $z_j\in(y_m-x_j,y_{m+1}-x_{j+1})$. This further decomposes the first term on the right in (7.6) into three parts. The first part is of the order

$$O\Big(n^{5/2}\|E\|_\infty|y_{m+1}-y_m-x_{m+1}+x_m|\Big).$$

Direct estimates of contributions corresponding to the second and third summands in (7.7) yield a supercritical $O(\delta n \log n \|E\|_\infty)$ and a dangerous quadratic $O(n^{5/2}\|E\|_\infty^2)$ errors respectively. These estimates are not sufficient and need further treatment.

The contribution corresponding to the second summand in (7.7) can be represented as follows:

$$\sum_{j\in S_m^c} \frac{1}{4\sin^2\frac{z_j}{2}}\left(\frac{1}{u(\bar{x}_m)}-\frac{1}{u(\bar{x}_j)}\right)\Big(u(\bar{x}_j)(E_j-E_m)+u(\bar{x}_{j+1})(E_{j+1}-E_m)\Big)$$
$$+\,E_m\sum_{j\in S_m^c} \frac{1}{4\sin^2\frac{z_j}{2}}\left(\frac{1}{u(\bar{x}_m)}-\frac{1}{u(\bar{x}_j)}\right)\Big(u(\bar{x}_j)+u(\bar{x}_{j+1})\Big) \tag{7.8}$$

We combine the first sum in (7.8) and the contribution corresponding to the third summand in (7.7) that is given by

$$\sum_{j\in S_m^c} \frac{-n(u(\bar{x}_j)E_j+u(\bar{x}_{j+1})E_{j+1})}{2\sin^2\frac{z_j}{2}}(E_j-E_m),$$

and denote the resulting expression $H_{1,m}$. It takes form

$$H_{1,m}=\sum_{j\in S_m^c\cup\{j_-\}} h_1(j,m)(E_j-E_m), \tag{7.9}$$

with the coefficients

$$h_1(j,m)=\begin{cases}\frac{u(\bar{x}_j)-u(\bar{x}_m)}{4u(\bar{x}_m)\sin^2\frac{z_j}{2}}+\frac{(u(\bar{x}_{j-1})-u(\bar{x}_m))u(\bar{x}_j)}{4u(\bar{x}_m)u(\bar{x}_{j-1})\sin^2\frac{z_{j-1}}{2}}-\frac{n(u(\bar{x}_j)E_j+u(\bar{x}_{j+1})E_{j+1})}{2\sin^2\frac{z_j}{2}}, & j\in S_m^c\setminus\{j_+\}\\ \frac{u(\bar{x}_{j_+})-u(\bar{x}_m)}{4u(\bar{x}_m)\sin^2\frac{z_{j_+}}{2}}-\frac{n(u(\bar{x}_{j_+})E_{j_+}+u(\bar{x}_{j_++1})E_{j_++1})}{2\sin^2\frac{z_{j_+}}{2}}, & j=j_+\\ \frac{(u(\bar{x}_{j_--1})-u(\bar{x}_m))u(\bar{x}_{j_-})}{4u(\bar{x}_m)u(\bar{x}_{j_--1})\sin^2\frac{z_{j_--1}}{2}}, & j=j_-\end{cases}$$
$$=O(n^3|m-j|^{-2}\|E\|_\infty+\delta n|m-j|^{-1}). \tag{7.10}$$

We now consider the second term in (7.8). We argue that this term is harmless. To see this, we shall make several simplifying replacements. First, replace $u(\bar{x}_j)+u(\bar{x}_{j+1})$ by $2u(\bar{x}_m)$. The error introduced by such change is

$$E_m\sum_{j\in S_m^c} O\left(\frac{n^2}{|m-j|^2}\cdot\frac{\delta|m-j|}{n}\cdot\frac{\delta|m-j|}{n}\right)=O(\delta n\|E\|_\infty).$$

Next, replace $\frac{1}{u(\bar{x}_m)}-\frac{1}{u(\bar{x}_j)}$ by $\frac{u'(\bar{x}_m)}{u^2(\bar{x}_m)}(\bar{x}_j-\bar{x}_m)$, using that

$$\frac{1}{u(\bar{x}_m)}-\frac{1}{u(\bar{x}_j)}=\frac{u'(\bar{x}_m)}{u^2(\bar{x}_m)}(\bar{x}_j-\bar{x}_m)+O(\delta n^{-2}|m-j|^2)$$

The contribution from such change to the error is

$$E_m \sum_{j \in S_m^c} O\left(\frac{n^2}{|m-j|^2} \cdot \frac{\delta |m-j|^2}{n^2}\right) = O(\delta n \|E\|_\infty).$$

We also replace $\frac{1}{4\sin^2\frac{z_j}{2}}$ by $\frac{1}{(\bar{x}_j - \bar{x}_m)^2}$. Note that $z_j \in (y_m - x_j, y_{m+1} - x_{j+1})$, and $|(y_{m+1} - x_{j+1}) - (y_m - x_j)| = O(n^{-1})$. Then, $|z_j - (y_m - x_j)| = O(n^{-1})$. Since $|(y_m - x_j) - (\bar{x}_j - \bar{x}_m)| = O(n^{-1})$, we get $|z_j - (\bar{x}_j - \bar{x}_m)| = O(n^{-1})$ as well. Therefore, we have

$$\begin{aligned} \frac{1}{4\sin^2\frac{z_j}{2}} &= \frac{1}{4\sin^2\left(\frac{\bar{x}_j - \bar{x}_m}{2} + O(n^{-1})\right)} = \frac{1}{4\sin^2\frac{\bar{x}_j - \bar{x}_m}{2}} + O(n^2|m-j|^{-3}) \\ &= \frac{1}{(\bar{x}_j - \bar{x}_m)^2} + O(1) + O(n^2|m-j|^{-3}). \end{aligned}$$

Thus difference created by such substitution leads to an error

$$E_m \sum_{j \in S_m^c} O\left(\left(1 + \frac{n^2}{|m-j|^3}\right) \cdot \frac{\delta |m-j|}{n}\right) = O(\delta n \|E\|_\infty).$$

Hence, we brought the second term in (7.8) to the form

$$\frac{u'(\bar{x}_m) E_m}{u(\bar{x}_m)} \sum_{j \in S_m^c} \frac{1}{\bar{x}_j - \bar{x}_m} + O(\delta n \|E\|_\infty).$$

For the remaining sum, we can pair the $j = m \pm k$ summands and make use of the cancellation similar to (4.17)

$$\frac{1}{\bar{x}_{m+k} - \bar{x}_m} + \frac{1}{\bar{x}_{m-k} - \bar{x}_m} = \frac{(\bar{x}_{m+k} - \bar{x}_m) + (\bar{x}_{m-k} - \bar{x}_m)}{(\bar{x}_{m+k} - \bar{x}_m)(\bar{x}_{m-k} - \bar{x}_m)} = O\left(\frac{k^2 n^{-2} + k\|E\|_\infty}{k^2 n^{-2}}\right), \tag{7.11}$$

which after summation in k yields $O(n + n^2 \log n \|E\|_\infty)$. There is at most $O(1 + n^{3/2}\|E\|_\infty)$ mismatched terms at the boundary, leading to $O(n^{1/2} + n^2\|E\|_\infty)$ contribution to the sum. Finally, we arrive at a desired bound on the second term in (7.8):

$$O\left(\delta \|E\|_\infty \cdot (n + n^2 \log n \|E\| + n^{1/2})\right) + O(\delta n \|E\|_\infty) = O(\delta n \|E\|_\infty + \delta n^2 \log n \|E\|_\infty^2). \tag{7.12}$$

To sum up, the bound we obtained for the first term in (7.6) is given by

$$H_{1,m} + O(n^{5/2}\|E\|_\infty |y_{m+1} - y_m - x_{m+1} + x_m| + \delta n \|E\|_\infty + \delta n^2 \log n \|E\|_\infty^2).$$

Now, let us turn to the second term in (7.6). Replace $E_{j\pm1}$ in this term by $E_{j\pm1}-E_m$. The difference leads to an error

$$nE_m \sum_{j\in S_m^c} \cot\frac{y_m - x_j}{2}\big(u(\bar{x}_{j-1}) - u(\bar{x}_{j+1})\big). \tag{7.13}$$

Again, we shall argue that the error (7.13) is harmless. Compute

$$\begin{aligned}
u(\bar{x}_{j-1}) - u(\bar{x}_{j+1}) &= -u'(\bar{x}_j)(\bar{x}_{j+1} - \bar{x}_{j-1}) + O(\delta n^{-2}) \\
&= -u'(\bar{x}_m)(\bar{x}_{j+1} - \bar{x}_{j-1}) + O(\delta n^{-2}|m-j|) \\
&= -u'(\bar{x}_m)\left(\frac{1}{nu(\bar{x}_m)} + O(\delta n^{-2}|m-j| + \|E\|_\infty)\right) + O(\delta n^{-2}|m-j|) \\
&= -\frac{u'(\bar{x}_m)}{nu(\bar{x}_m)} + O(\delta n^{-2}|m-j| + \delta\|E\|_\infty).
\end{aligned}$$

Then the error term (7.13) becomes

$$-\frac{u'(\bar{x}_m)E_m}{u(\bar{x}_m)} \sum_{j\in S_m^c} \cot\frac{y_m - x_j}{2} + O\big(\delta n\|E\|_\infty + \delta n^2 \log n\|E\|_\infty^2\big).$$

Using (6.2) and then pairing up the $j = m \pm k$ leading terms similarly to (7.11), estimating the rest of the series and any mismatched terms, we have

$$\sum_{j\in S_m^c} \cot\frac{y_m - x_j}{2} = O(n + n^2\log n\|E\|_\infty). \tag{7.14}$$

It follows that the error term (7.13) has the same order as in (7.12).

The remainder of the second term in (7.6) can be telescoped

$$\begin{aligned}
& n\sum_{j\in S_m^c} \cot\frac{y_m - x_j}{2}\big(u(\bar{x}_{j-1})(E_{j-1} - E_m) - u(\bar{x}_{j+1})(E_{j+1} - E_m)\big) \\
&= \sum_{j\in S_m^c\setminus\{j_+, j_- - 1\}} nu(\bar{x}_j)\left(\cot\frac{y_m - x_{j+1}}{2} - \cot\frac{y_m - x_{j-1}}{2}\right)(E_j - E_m) \\
&\quad + nu(\bar{x}_{j_+-1})\cot\frac{y_m - x_{j_+}}{2}(E_{j_+-1} - E_m) + nu(\bar{x}_{j_+})\cot\frac{y_m - x_{j_++1}}{2}(E_{j_+} - E_m) \\
&\quad - nu(\bar{x}_{j_-})\cot\frac{y_m - x_{j_--1}}{2}(E_{j_-} - E_m) - nu(\bar{x}_{j_--1})\cot\frac{y_m - x_{j_--2}}{2}(E_{j_--1} - E_m).
\end{aligned} \tag{7.15}$$

Denote the first term on the right hand side of (7.15) as

$$D_{2,m} = \sum_{j\in S_m^c\setminus\{j_+, j_- - 1\}} d_2(j,m)(E_j - E_m), \tag{7.16}$$

with positive coefficients

$$d_2(j,m) = nu(\bar{x}_j)\left(\cot\frac{y_m - x_{j+1}}{2} - \cot\frac{y_m - x_{j-1}}{2}\right) = \frac{nu(\bar{x}_j)}{2\sin^2\frac{\tilde{z}_j}{2}}(x_{j+1} - x_{j-1})$$
$$= \frac{1}{2\sin^2\frac{y_m - x_j}{2}}\left(1 + O(n^{-1/2} + n\|E\|_\infty)\right) \sim \frac{n^2}{|m-j|^2}, \tag{7.17}$$

where $\tilde{z}_j \in (y_m - x_{j+1}, y_m - x_{j-1})$ and we performed several elementary simplifications in the third step. The expression $D_{2,m}$ is our main dissipative term in the far field. The remaining four boundary terms in (7.15) are kept in (7.2), and will be handled later. This completes the treatment of the first term in (7.3), and summing up all the bounds we see that it has been represented as the sum of the last two lines in (7.15) and

$$D_{2,m} + H_{1,m} + O(n^{5/2}\|E\|_\infty|y_{m+1} - y_m - x_{m+1} + x_m| + \delta n\|E\|_\infty + \delta n^2 \log n\|E\|_\infty^2).$$

Next, we consider the second term in (7.3),

$$2n\int_{\bar{x}_{j-1}}^{\bar{x}_j} u(y)\left(\cot\frac{y_m - y}{2} - \cot\frac{y_m - x_j}{2}\right)dy. \tag{7.18}$$

First, we can replace $u(y)$ by $u(x_j)$, or any $u(z)$ where z is $O(n^{-1})$ distance from y, for instance, $u(\bar{x}_{j-1})$ or $u(\bar{x}_j)$. The resulting difference sums up to

$$\sum_{|j-m|\gtrsim n^{1/2}} O(n\cdot n^{-1}\cdot \delta n^{-1}\cdot n|m-j|^{-2}) = O(\delta n^{-1/2}).$$

Then, we expand $\cot\frac{y_m - y}{2}$ into Taylor series around $\frac{y_m - x_j}{2}$:

$$\cot\frac{y_m - y}{2} = \sum_{k=0}^{\infty}\frac{(-1)^k\cot^{(k)}\frac{y_m - x_j}{2}}{2^k k!}(y - x_j)^k. \tag{7.19}$$

We have

$$\int_{\bar{x}_{j-1}}^{\bar{x}_j}(y - x_j)^k\,dy = \frac{1}{k+1}\left(\left(\frac{x_{j+1} - x_j}{2}\right)^{k+1} - \left(\frac{x_{j-1} - x_j}{2}\right)^{k+1}\right)$$
$$= \frac{1}{(k+1)2^{k+1}}\left((x_{j+1} - x_j)^{k+1} + (-1)^k(x_j - x_{j-1})^{k+1}\right). \tag{7.20}$$

When k is odd, the expression in brackets appearing in (7.20) for all sufficiently large n can be estimated by

$$
\begin{aligned}
(k+1)(x_{j+1}+x_{j-1}-2x_j)O(n^{-k}) &= (k+1)\left(\frac{1}{2nu(\bar{x}_j)} - \frac{1}{2nu(\bar{x}_{j-1})} + E_j - E_{j-1}\right)O(n^{-k}) \\
&= (k+1)O(\delta n^{-k-2} + \|E\|_\infty n^{-k}). \qquad (7.21)
\end{aligned}
$$

As usual, the constants involved in O may only depend on u_0, but not on n or k. Using the Laurent series for cot (6.2) and (4.5), it is not difficult to show that

$$
\left|\cot^{(k)}\frac{y_m - x_j}{2}\right| \le \frac{C^k k! n^{k+1}}{|m-j|^{k+1}} \qquad (7.22)
$$

with some constant $C > 0$ that may only depend on u_0. Taking into account the factor n in front of the integral in (7.18), we obtain that a contribution to (7.18) from any odd k in (7.19) can be estimated as

$$
O\left(\frac{(k+1)C^k n^{k+2}}{|m-j|^{k+1}} \cdot (\delta n^{-k-2} + \|E\|_\infty n^{-k})\right) = O(C^k \delta |m-j|^{-k-1} + C^k n^2 |m-j|^{-k-1}\|E\|_\infty),
$$

where we can absorb $k+1$ into C^k by slightly adjusting the constant. Summing up over $j \in S_m^c$ we get the bound of the order

$$
O(C^k \delta n^{-\frac{k}{2}} + C^k n^{-\frac{k}{2}+2}\|E\|_\infty). \qquad (7.23)
$$

For all n sufficiently large, the first expression in (7.23) can be summed up over $k \ge 1$, yielding $O(\delta n^{-1/2})$. The second expression in (7.23) can be summed up over $k \ge 3$ leading to $O(n^{1/2}\|E\|_\infty)$ contribution. The case $k = 1$ for the second error will now be considered in more detail.

Going back to (7.19), (7.20) and (7.21), we see that the part of the linear term $k = 1$ leading to the large error is given by

$$
\begin{aligned}
\frac{nu(x_j)}{8\sin^2\frac{y_m - x_j}{2}}(E_j - E_{j-1})(x_{j+1} - x_{j-1}) &= \frac{1}{8\sin^2\frac{y_m - x_j}{2}}\Big((E_j - E_{j-1}) + nu(x_j)(E_j^2 - E_{j-1}^2)\Big) \\
&\quad + O(n|j-m|^{-2}\|E\|_\infty). \qquad (7.24)
\end{aligned}
$$

For the second summand in the brackets on the right side of (7.24), we telescope when summing over j and get

$$
\begin{aligned}
\sum_{j\in S_m^c} \frac{nu(x_j)}{8\sin^2\frac{y_m - x_j}{2}}(E_j^2 - E_{j-1}^2) &= \sum_{j\in S_m^c\setminus\{j_+\}}\left(\frac{nu(x_j)}{8\sin^2\frac{y_m - x_j}{2}} - \frac{nu(x_{j+1})}{8\sin^2\frac{y_m - x_{j+1}}{2}}\right)E_j^2 + O(n^2\|E\|_\infty^2) \\
= \sum_{j\in S_m^c\setminus\{j_+\}} O\left(\frac{n^3}{|m-j|^2}\cdot\delta n^{-1} + \frac{n^4}{|m-j|^3}\cdot n^{-1}\right)E_j^2 &+ O(n^2\|E\|_\infty^2) = O(n^2\|E\|_\infty^2).
\end{aligned}
$$

Summing up the rest of (7.24) in j, and adding the earlier bounds, we obtain that summation over all odd k in (7.19) results in the following contribution to (7.18):

$$\sum_{j\in S_m^c} \frac{E_j - E_{j-1}}{8\sin^2\frac{y_m - x_j}{2}} + O(\delta n^{-1/2} + n^{1/2}\|E\|_\infty + n^2\|E\|_\infty^2). \tag{7.25}$$

Telescoping in the sum that remains in (7.25) would yield a critical error of the order $O(n\|E\|_\infty)$ without δ factor, which should be avoided if we want to control the evolution of errors for all times. Let us consider the difference of the sum in (7.25) with the corresponding sum in $II_{2,m+1}^{j+1}$:

$$H_{2,m} := \sum_{j\in S_m^c} \frac{E_j - E_{j-1}}{8\sin^2\frac{y_m - x_j}{2}} - \sum_{j\in S_{m+1}^c} \frac{E_j - E_{j-1}}{8\sin^2\frac{y_{m+1} - x_j}{2}} = \sum_{j\in S_m^c\cup\{j_+-1,j_-\}} h_2(j,m)(E_j - E_m). \tag{7.26}$$

Here we write $E_j - E_{j-1} = (E_j - E_m) - (E_{j-1} - E_m)$ and reorganize the sum. The coefficients $h_2(j,m)$ are given by

$$h_2(j,m) = \begin{cases} \frac{1}{8\sin^2\frac{y_m - x_j}{2}} - \frac{1}{8\sin^2\frac{y_m - x_{j+1}}{2}} - \frac{1}{8\sin^2\frac{y_{m+1} - x_j}{2}} + \frac{1}{8\sin^2\frac{y_{m+1} - x_{j+1}}{2}}, & j \in S_m^c\backslash\{j_- - 1, j_+\} \\ \frac{1}{8\sin^2\frac{y_m - x_{j_- -1}}{2}} - \frac{1}{8\sin^2\frac{y_{m+1} - x_{j_- -1}}{2}} + \frac{1}{8\sin^2\frac{y_{m+1} - x_{j_-}}{2}}, & j = j_- - 1 \\ -\frac{1}{8\sin^2\frac{y_{m+1} - x_{j_-}}{2}}, & j = j_- \\ -\frac{1}{8\sin^2\frac{y_m - x_{j_+}}{2}}, & j = j_+ - 1 \\ \frac{1}{8\sin^2\frac{y_m - x_{j_+}}{2}} - \frac{1}{8\sin^2\frac{y_m - x_{j_++1}}{2}} + \frac{1}{8\sin^2\frac{y_{m+1} - x_{j_++1}}{2}}, & j = j_+ \end{cases} \tag{7.27}$$

In the bulk when $j \in S_m^c\backslash\{j_- - 1, j_+\}$, we have

$$\frac{1}{8\sin^2\frac{y_m - x_j}{2}} - \frac{1}{8\sin^2\frac{y_m - x_{j+1}}{2}} = O(n^3|m-j|^{-3}\cdot n^{-1}) = O(n^2|m-j|^{-3}),$$

and similarly for the summands containing y_{m+1}. Therefore, the coefficients satisfy

$$h_2(j,m) = O(n^2|m-j|^{-3}), \quad \text{for } j \in S_m^c\backslash\{j_- - 1, j_+\}. \tag{7.28}$$

At the boundary, there is no such cancellation. Since $x_{j_+} - y_m = n^{-1/2} + O(n^{-1})$, we have

$$\sin^2\frac{x_{j_+} - y_m}{2} = \frac{1}{4n} + O(n^{-3/2}). \tag{7.29}$$

Similar estimates hold for all terms in (7.27) at the boundary. Therefore, we have

$$h_2(j,m) = \begin{cases} -\frac{n}{2} + O(n^{1/2}), & j = j_-, j_+ - 1 \\ \frac{n}{2} + O(n^{1/2}), & j = j_- - 1, j_+ \end{cases} \tag{7.30}$$

We now consider the case of even k in (7.19). In this case, we will be matching the mth and $(m+1)$st contributions. From (7.19) and (7.20), we see that the key expression to estimate is

$$\cot^{(k)} \frac{y_m - x_j}{2} \left((x_{j+1} - x_j)^{k+1} + (x_j - x_{j-1})^{k+1} \right) - \\ \cot^{(k)} \frac{y_{m+1} - x_{j+1}}{2} \left((x_{j+2} - x_{j+1})^{k+1} + (x_{j+1} - x_j)^{k+1} \right). \tag{7.31}$$

Let us telescope, and first observe that

$$\left| \cot^{(k)} \frac{y_m - x_j}{2} - \cot^{(k)} \frac{y_{m+1} - x_{j+1}}{2} \right| \leq \frac{1}{2} |\cot^{(k+1)} z_j| \cdot |y_m - x_j - y_{m+1} + x_{j+1}|,$$

where $z_j \in (\frac{y_m - x_j}{2}, \frac{y_{m+1} - x_{j+1}}{2})$. But

$$y_{m+1} - x_{j+1} - y_m + x_j = y_{m+1} - x_{m+1} - y_m + x_m + \frac{1}{2nu(\bar{x}_m)} - \frac{1}{2nu(\bar{x}_j)} + E_m - E_j,$$

and taking into account (7.22) and (4.5), we obtain that the contribution to (7.18) coming from the difference of cotangents in (7.31) has the order of

$$O\left(n \cdot \frac{C^{k+1}(k+1)! n^{k+2}}{(k+1)! |m-j|^{k+2}} \cdot n^{-k-1} \cdot \left(|y_{m+1} - x_{m+1} - y_m + x_m| + \delta |m-j| n^{-2} + \|E\|_\infty \right) \right).$$

Simplifying this expression and summing over $j \in S_m^c$, we obtain

$$\sum_{j \in S_m^c} O\left(C^{k+1} n^2 |m-j|^{-k-2} (|y_{m+1} - x_{m+1} - y_m + x_m| + \delta |m-j| n^{-2} + \|E\|_\infty) \right) \\ = O\left(C^{k+1} \left(n^{-\frac{k-3}{2}} |y_{m+1} - x_{m+1} - y_m + x_m| + \delta n^{-\frac{k}{2}} + n^{-\frac{k-3}{2}} \|E\|_\infty \right) \right). \tag{7.32}$$

Given that the zeroth term in (7.19) is cancelled identically, we sum (7.32) over even $k \geq 2$, obtaining the contribution of the order

$$O(n^{1/2} |y_{m+1} - x_{m+1} - y_m + x_m| + \delta n^{-1} + n^{1/2} \|E\|_\infty).$$

Next, let us estimate the second telescoped difference

$$\cot^{(k)} \frac{y_m - x_j}{2} \cdot \left((x_{j+1} - x_j)^{k+1} + (x_j - x_{j-1})^{k+1} - (x_{j+2} - x_{j+1})^{k+1} - (x_{j+1} - x_j)^{k+1} \right). \tag{7.33}$$

Note that the factor in brackets in (7.33) equals

$$\begin{aligned}(x_j - x_{j-1})^{k+1} - (x_{j+2} - x_{j+1})^{k+1} &= (x_j - x_{j-1} - x_{j+2} + x_{j+1}) \cdot O(kn^{-k}) \\ &= O\left(kn^{-k} \cdot \left(\frac{1}{2nu(\bar{x}_{j-1})} - \frac{1}{2nu(\bar{x}_{j+1})} + E_{j-1} - E_{j+1}\right)\right) \\ &= O(kn^{-k}(\delta n^{-2} + \|E\|_\infty)). \end{aligned} \tag{7.34}$$

Combining (7.34) with (7.22) and (7.19), we find that the contribution of the difference of mth and $(m+1)$st kth summand from (7.19) for even k has the order

$$O\left(n\frac{C^k k! n^{k+1}}{k!|m-j|^{k+1}} \cdot kn^{-k}(\delta n^{-2} + \|E\|_\infty)\right) = O(C^k|m-j|^{-k-1}(\delta + n^2\|E\|_\infty)).$$

Summing up over $j \in S_m^c$, we obtain

$$\sum_{j \in S_m^c} O\left(C^k|m-j|^{-k-1}(\delta + n^2\|E\|_\infty)\right) = O\left(C^k(\delta n^{-\frac{k}{2}} + n^{-\frac{k}{2}+2}\|E\|_\infty)\right). \tag{7.35}$$

We can sum the first expression on the right hand side of (7.35) over even $k \geq 2$, and the second one over $k \geq 4$, yielding the contribution of the order $O(\delta n^{-1} + \|E\|_\infty)$. For the $k = 2$ case of the second expression in (7.35), we get the contribution of the order $O(n\|E\|_\infty)$. This is a critical error without δ factor and so we will consider it in more detail. This error arises from the difference $E_{j-1} - E_{j+1}$ in (7.34), and equals

$$\sum_{j \in S_m^c} \frac{nu(x_j)\cos\frac{y_m - x_j}{2}}{\sin^3\frac{y_m - x_j}{2}} \cdot \big((E_{j-1} - E_m) - (E_{j+1} - E_m)\big) \cdot O(n^{-2}), \tag{7.36}$$

where $O(n^{-2})$ contains a constant factor and a factor coming from (7.34). We represent (7.36) as

$$H_{3,m} := \sum_{j \in S_m^c \cup \{j_-, j_+ - 1\}} h_3(j,m)(E_j - E_m), \tag{7.37}$$

where the coefficients $h_3(j,m)$ can be estimated by

$$h_3(j,m) = O\left(\frac{n^4}{|m-j|^3}\right) \cdot O(n^{-2}) = O\left(\frac{n^2}{|m-j|^3}\right). \tag{7.38}$$

A better estimate is possible in the bulk, taking into account the cancellation - but we do not need it.

Collecting all the estimates, we obtain (7.2). □

Next, we work on the mismatched part in (4.8)

$$II_{\mathrm{Mis},m} := 2n \int_{\Omega_m} u(y) \cot \frac{y_m - y}{2}\, dy = 2n \int_{\Omega_m^+} u(y) \cot \frac{y_m - y}{2}\, dy + 2n \int_{\Omega_m^-} u(y) \cot \frac{y_m - y}{2}\, dy. \tag{7.39}$$

Recall that

$$\Omega_m = \Omega_m^- \cup \Omega_m^+, \quad \text{where} \quad \Omega_m^- = [\bar{x}_{j_- -1}, y_m - n^{-1/2}],\ \Omega_m^+ = [y_m + n^{-1/2}, \bar{x}_{j_+ -1}].$$

The endpoints in the definition of the intervals $\Omega_m^\pm$ record the accurate limits of integration in (7.39) (which influences the sign of the contributions). The correct order of the limits of integration can be inferred from (4.1), (4.3) and (4.8).

Lemma 7.4 *Let $II_{Mis,m}$ be as defined in (7.39). Then,*

$$\begin{aligned} II_{Mis,m} - II_{Mis,m+1} &= nu(\bar{x}_{j_+}) \cot \frac{z_+}{2} \big((E_{j_+ -1} - E_m) + (E_{j_+} - E_m) \big) \\ &+ nu(\bar{x}_{j_-}) \cot \frac{z_-}{2} \big((E_{j_- -1} - E_m) + (E_{j_-} - E_m) \big) \\ &+ O\big(n^{3/2} (y_{m+1} - y_m - x_{m+1} + x_m) + \delta n^{-1/2} + \delta n \|E\|_\infty \big), \end{aligned} \tag{7.40}$$

where z_+, z_- are such that $z_\pm = n^{-1/2} + O(n^{-1})$.

Proof We focus our discussion on the first integral (over Ω_m^+) in (7.39). The second integral can be treated using the analogous argument. Decompose the integral as follows:

$$2nu(\bar{x}_{j_+ -1}) \int_{\Omega_m^+} \cot \frac{y_m - y}{2}\, dy + 2n \int_{\Omega_m^+} \big(u(y) - u(\bar{x}_{j_+ -1}) \big) \cot \frac{y_m - y}{2}\, dy. \tag{7.41}$$

The second term in (7.41) can be estimated by

$$O\big(n \cdot n^{-1} \cdot \delta n^{-1} \cdot n^{1/2} \big) = O(\delta n^{-1/2}).$$

Here, we have used the fact that $|\Omega_m^+| = O(n^{-1})$, and $y_m - y = n^{-1/2} + O(n^{-1})$ for any $y \in \Omega_m^+$. Let us offset the first term in (7.41) by its counterpart from $II_{2,m+1}$:

$$\begin{aligned} &2nu(\bar{x}_{j_+ -1}) \int_{y_m + n^{-1/2}}^{\bar{x}_{j_+ -1}} \cot \frac{y_m - y}{2}\, dy - 2nu(\bar{x}_{j_+}) \int_{y_{m+1} + n^{-1/2}}^{\bar{x}_{j_+}} \cot \frac{y_{m+1} - y}{2}\, dy \\ &= -2nu(\bar{x}_{j_+ -1}) \int_{n^{-1/2}}^{\bar{x}_{j_+ -1} - y_m} \cot \frac{z}{2}\, dz + 2nu(\bar{x}_{j_+}) \int_{n^{-1/2}}^{\bar{x}_{j_+} - y_{m+1}} \cot \frac{z}{2}\, dz. \end{aligned} \tag{7.42}$$

Let us replace $u(\bar{x}_{j_+-1})$ in the first summand in (7.42) with $u(\bar{x}_{j_+})$. This generates an error of the order

$$O\big(\delta n^{-1} \cdot n \cdot n^{-1} \cdot n^{1/2}\big) = O(\delta n^{-1/2}).$$

Then the difference in (7.42) can be written as

$$2nu(\bar{x}_{j_+}) \int_{\bar{x}_{j_+-1}-y_m}^{\bar{x}_{j_+}-y_{m+1}} \cot\frac{z}{2}\,dz = 2nu(\bar{x}_{j_+}) \cot\frac{z_+}{2} \cdot (\bar{x}_{j_+} - y_{m+1} - \bar{x}_{j_+-1} + y_m), \tag{7.43}$$

where $z_+ \in (\bar{x}_{j_+-1} - y_m, \bar{x}_{j_+} - y_{m+1})$, and clearly $z_+ = n^{-1/2} + O(n^{-1})$. Further decomposition yields

$$\bar{x}_{j_+} - y_{m+1} - \bar{x}_{j_+-1} + y_m = -(y_{m+1} - y_m - x_{m+1} + x_m) + \left(\frac{E_{j_+-1} + E_{j_+}}{2} - E_m\right) + \frac{1}{2n}\left(\frac{1}{2u(\bar{x}_{j_+-1})} + \frac{1}{2u(\bar{x}_{j_+})} - \frac{1}{u(\bar{x}_m)}\right). \tag{7.44}$$

The first term in (7.44) leads to an error

$$O\big(n^{3/2}|y_{m+1} - y_m - x_{m+1} + x_m|\big).$$

The second term in (7.44) produces dissipation at the boundary

$$nu(\bar{x}_{j_+}) \cot\frac{z_+}{2}(E_{j_+-1} - E_m) + nu(\bar{x}_{j_+}) \cot\frac{z_+}{2}(E_{j_+} - E_m).$$

We keep this term in (7.40). The last term in (7.44) leads to a contribution

$$u(\bar{x}_{j_+}) \cot\frac{z_+}{2} \cdot \left(\frac{1}{2u(\bar{x}_{j_+-1})} + \frac{1}{2u(\bar{x}_{j_+})} - \frac{1}{u(\bar{x}_m)}\right). \tag{7.45}$$

Let us make the following simplifications here. Replace z_+ by $n^{-1/2}$. As $z_+ - n^{-1/2} = O(n^{-1})$, the error generated by such substitution is

$$O\big(n \cdot n^{-1} \cdot \delta n^{-1/2}\big) = O(\delta n^{-1/2}).$$

Replace $u(\bar{x}_{j_+-1})$ by $u(\bar{x}_{j_+})$, the error is

$$O\big(n^{1/2}\delta n^{-1}\big) = O(\delta n^{-1/2}).$$

Then (7.45) becomes

$$\cot\frac{n^{-1/2}}{2} \cdot \frac{u(\bar{x}_m) - u(\bar{x}_{j_+})}{u(\bar{x}_m)}. \tag{7.46}$$

Such contribution produces a supercritical error of order $O(\delta)$. We will group (7.46) with the analogous term coming from the integral in Ω_m^-, leading to

$$\cot\frac{n^{-1/2}}{2}\cdot\frac{u(\bar{x}_m)-u(\bar{x}_{j_+})}{u(\bar{x}_m)}+\cot\frac{n^{-1/2}}{2}\cdot\frac{u(\bar{x}_m)-u(\bar{x}_{j_-})}{u(\bar{x}_m)}. \tag{7.47}$$

To explain the contribution from integration over Ω_m^- to (7.47), let us quickly trace through the corresponding computations. The analogs of the estimates in (7.41), (7.42), (7.43) lead to the principal contribution

$$2nu(\bar{x}_{j_-})\int_{y_{m+1}-\bar{x}_{j_-}}^{y_m-\bar{x}_{j_--1}}\cot\frac{z}{2}\,dz=2nu(\bar{x}_{j_-})\cot\frac{z_-}{2}\cdot(y_m-\bar{x}_{j_--1}+\bar{x}_{j_-}-y_{m+1})$$

where $z_-\in(y_{m+1}-\bar{x}_{j_-},\,y_m-\bar{x}_{j_--1})$. In the analog of (7.44), the second term leads to the dissipative contribution

$$nu(\bar{x}_{j_-})\cot\frac{z_-}{2}(E_{j_--1}-E_m)+nu(\bar{x}_{j_-})\cot\frac{z_-}{2}(E_{j_-}-E_m),$$

that appears in (7.40). The analog of the last term in (7.44) leads to a contribution

$$u(\bar{x}_{j_-})\cot\frac{z_-}{2}\cdot\left(\frac{1}{2u(\bar{x}_{j_--1})}+\frac{1}{2u(\bar{x}_{j_-})}-\frac{1}{u(\bar{x}_m)}\right)$$

that after simplifications yields the second term in (7.47).

Coming back to (7.47), we aim to exploit additional cancellations in the numerator:

$$\big(u(\bar{x}_m)-u(\bar{x}_{j_+})\big)+\big(u(\bar{x}_m)-u(\bar{x}_{j_-})\big)=\partial_x u(\bar{x}_m)\big((\bar{x}_m-\bar{x}_{j_+})+(\bar{x}_m-\bar{x}_{j_-})\big)+O(\delta n^{-1}). \tag{7.48}$$

The higher order expression $O(\delta n^{-1})$ in (7.48) leads to an error $O(\delta n^{-1/2})$. For the remaining first summand on the right side of (7.48), write

$$\bar{x}_m-\bar{x}_{j_+}=-\frac{1}{4nu(\bar{x}_m)}-\sum_{l=1}^{j_+-m-1}\frac{1}{2nu(\bar{x}_{m+l})}-\frac{1}{4nu(\bar{x}_{j_+})}-\frac{E_m}{2}-\sum_{l=1}^{j_+-m-1}E_{m+l}-\frac{E_{j_+}}{2}, \tag{7.49}$$

$$\bar{x}_m-\bar{x}_{j_-}=\frac{1}{4nu(\bar{x}_m)}+\sum_{l=1}^{m-j_--1}\frac{1}{2nu(\bar{x}_{m-l})}+\frac{1}{4nu(\bar{x}_{j_-})}+\frac{E_m}{2}+\sum_{l=1}^{m-j_--1}E_{m-l}+\frac{E_{j_-}}{2}. \tag{7.50}$$

We can estimate the sum

$$\sum_{l=1}^{\sim n^{1/2}}\left(-\frac{1}{2nu(\bar{x}_{m+l})}+\frac{1}{2nu(\bar{x}_{m-l})}\right)=\sum_{l=1}^{\sim n^{1/2}}O\big(n^{-1}\cdot\delta l n^{-1}\big)=O(\delta n^{-1}). \tag{7.51}$$

Compared with (7.51), in the corresponding sums in (7.49), (7.50) there could be at most $O(1 + n^{3/2}\|E\|_\infty))$ mismatched summands. Each of these mismatched summands is of the order $O(n^{-1})$, leading to the error $O(\delta n^{-1/2} + \delta n\|E\|_\infty)$ due to the $\partial_x u$ factor in (7.48) and cot factor in (7.47). The rest of the summands in (7.49), (7.50) involving $\{E_j\}$ add up to $O(n^{1/2}\|E\|_\infty)$, leading to $O(\delta n\|E\|_\infty)$ error. Putting everything together, we obtain

$$\cot \frac{n^{-1/2}}{2} \cdot \frac{u(\bar{x}_m) - u(\bar{x}_{j_+})}{u(\bar{x}_m)} + \cot \frac{n^{-1/2}}{2} \cdot \frac{u(\bar{x}_m) - u(\bar{x}_{j_-})}{u(\bar{x}_m)} = O\big(\delta n^{-1/2} + \delta n\|E\|_\infty\big).$$

Collecting all bounds, we obtain (7.40). ◻

Combining (7.2) and (7.40), we conclude that

$$\begin{aligned} II_{2,m} - II_{2,m+1} &= D_{2,m} + D_{3,m} + H_{1,m} + H_{2,m} + H_{3,m} \\ &+ O\big((n^{5/2}\|E\|_\infty + n^{3/2})|y_{m+1} - y_m - x_{m+1} + x_m|\big) \\ &+ O(\delta n^{-1/2} + \delta n\|E\|_\infty + \delta n^2 \log n\|E\|_\infty^2) + O(n^{1/2}\|E\|_\infty + n^2\|E\|_\infty^2), \end{aligned} \tag{7.52}$$

where we collect the boundary terms and define $D_{3,m}$ as

$$\begin{aligned} D_{3,m} :=& nu(\bar{x}_{j_+}) \cot \frac{z_+}{2}\big((E_{j_+-1} - E_m) + (E_{j_+} - E_m)\big) \\ &+ nu(\bar{x}_{j_-}) \cot \frac{z_-}{2}\big((E_{j_--1} - E_m) + (E_{j_-} - E_m)\big) \\ &- nu(\bar{x}_{j_+}) \cot \frac{x_{j_+} - y_m}{2}(E_{j_+-1} - E_m) - nu(\bar{x}_{j_+}) \cot \frac{x_{j_++1} - y_m}{2}(E_{j_+} - E_m) \\ &- nu(\bar{x}_{j_-}) \cot \frac{y_m - x_{j_--1}}{2}(E_{j_-} - E_m) - nu(\bar{x}_{j_-}) \cot \frac{y_m - x_{j_--2}}{2}(E_{j_--1} - E_m) \\ =& \sum_{j\in\{j_--1,\ j_-,\ j_+-1,\ j_+\}} d_3(j, m)(E_j - E_m). \end{aligned} \tag{7.53}$$

We adjusted some arguments of u for simplicity; these changes generate a subcritical error of the order $O(\delta n^{1/2}\|E\|_\infty)$.

The following lemma shows that the coefficients $d_3(j, m)$ are positive, and hence $D_{3,m}$ produces dissipation at the interface. This will turn out to be only a part of dissipation on the interface; we will bring all parts together later in Section 8.

Lemma 7.5 *The coefficients $d_3(j, m)$ in (7.53) are positive, and they satisfy*

$$d_3(j, m) = \begin{cases} \frac{n}{2} + O(n^{1/2} + n^2 \log n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2), & j = j_-, j_+ - 1, \\ \frac{3n}{2} + O(n^{1/2} + n^2 \log n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2), & j = j_- - 1, j_+. \end{cases} \tag{7.54}$$

Proof Let us first check $d_3(j_+ - 1, m)$ (a similar argument applies to $d_3(j_-, m)$). From (7.53), we have

$$d_3(j_+ - 1, m) = nu(\bar{x}_{j_+}) \cot \frac{z_+}{2} - nu(\bar{x}_{j_+}) \cot \frac{x_{j_+} - y_m}{2}$$

$$= = \frac{nu(\bar{x}_{j_+})}{2\sin^2\frac{x_{j_+}-y_m}{2}}(x_{j_+}-y_m-z_+)+O(n^{1/2}). \tag{7.55}$$

Now, we show $x_{j_+}-y_m-z_+>0$. Recall that $z_+\in(\bar{x}_{j_+-1}-y_m,\bar{x}_{j_+}-y_{m+1})$. Let us check the two endpoints. Clearly,

$$(x_{j_+}-y_m)-(\bar{x}_{j_+-1}-y_m)=\frac{1}{4nu(\bar{x}_{j_+-1})}+\frac{E_{j_+-1}}{2}=\frac{1}{4nu(\bar{x}_{j_+})}+O(\|E\|_\infty+n^{-2})>0 \tag{7.56}$$

for all n large enough due to (3.4). We adjusted the argument of u from $\bar{x}_{j_+-1}$ to $\bar{x}_{j_+}$ incurring $O(n^{-2})$ error.

For the other endpoint, we apply the rough estimate (5.14) on $y_{m+1}-y_m-x_{m+1}+x_m$ in (7.44) and get

$$|\bar{x}_{j_+}-y_{m+1}-\bar{x}_{j_+-1}+y_m|=O(n^{-3/2}+\log n\|E\|_\infty$$
$$+(\log n)^2\|E\|_\infty^2)+O(\|E\|_\infty)+O(\delta n^{-3/2}).$$

Then, we deduce

$$\begin{aligned}(x_{j_+}&-y_m)-(\bar{x}_{j_+}-y_{m+1})\\ &=\big((x_{j_+}-y_m)-(\bar{x}_{j_+-1}-y_m)\big)-(\bar{x}_{j_+}-y_{m+1}-\bar{x}_{j_+-1}+y_m)\\ &=\frac{1}{4nu(\bar{x}_{j_+})}+O(n^{-3/2}+\log n\|E\|_\infty+(\log n)^2\|E\|_\infty^2).\end{aligned} \tag{7.57}$$

Combining (7.56) and (7.57), we conclude

$$x_{j_+}-y_m-z_+=\frac{1}{4nu(\bar{x}_{j_+})}+O(n^{-3/2}+\log n\|E\|_\infty+(\log n)^2\|E\|_\infty^2). \tag{7.58}$$

Moreover, recall (7.29):

$$\sin^2\frac{x_{j_+}-y_m}{2}=\frac{1}{4n}+O(n^{-3/2}).$$

Substituting these estimates into (7.55), we get

$$\begin{aligned}d_3(j_+-1,m)&=\frac{n^2u(\bar{x}_{j_+})}{2nu(\bar{x}_{j_+})}+O(n^{1/2}+n^2\log n\|E\|_\infty+(\log n)^2\|E\|_\infty^2)\\ &=\frac{n}{2}+O(n^{1/2}+n^2\log n\|E\|_\infty+n^2(\log n)^2\|E\|_\infty^2),\end{aligned}$$

exactly as claimed in (7.54).

Next, we check $d_3(j_+,m)$ (a similar argument applies to $d_3(j_--1,m)$). The analog of the calculation in (7.55) is

$$d_3(j_+,m)=nu(\bar{x}_{j_+})\cot\frac{z_+}{2}-nu(\bar{x}_{j_+})\cot\frac{x_{j_++1}-y_m}{2}$$

$$= \frac{nu(\bar{x}_{j_+})}{2\sin^2 \frac{x_{j_+ +1} - y_m}{2}} (x_{j_+ +1} - y_m - z_+) + O(n^{1/2}).$$

From (7.58), it follows that

$$x_{j_+ +1} - y_m - z_+ = \frac{3}{4nu(\bar{x}_{j_+})} + O(n^{-3/2} + \log n \|E\|_\infty + (\log n)^2 \|E\|_\infty^2).$$

Hence,

$$\begin{aligned} d_3(j_+, m) &= 2n^2 u(\bar{x}_{j_+}) \cdot \frac{3}{4nu(\bar{x}_{j_+})} + O(n^{1/2} + n^2 \log n \|E\|_\infty + (\log n)^2 \|E\|_\infty^2) \\ &= \frac{3n}{2} + O(n^{1/2} + n^2 \log n \|E\|_\infty + n^2 (\log n)^2 \|E\|_\infty^2). \end{aligned}$$

□

8 Propagation of errors equation

Now we are going to put together our estimates and bring the propagation of errors equation to the form that is most convenient for further analysis. Let us recall the estimates in (6.23), (7.1) and (7.52). These bounds lead to the characterization

$$\begin{aligned} G_m - G_{m+1} &= D_{1,m} + D_{2,m} + D_{3,m} + B_{1,m} + B_{2,m} + H_{1,m} + H_{2,m} + H_{3,m} \\ &\quad + O\big((n^3 \|E\|_\infty + n^{3/2}) |y_{m+1} - y_m - x_{m+1} + x_m|\big) \\ &\quad + O\big(\delta(n^{-1/2} + n\|E\|_\infty + n^2 \log n \|E\|_\infty^2)\big) + O(n^{1/2} \|E\|_\infty + n^2 \|E\|_\infty^2). \end{aligned}$$

Substituting this into (5.13), we have

$$\begin{aligned} & y_{m+1} - y_m - x_{m+1} + x_m \\ &= -\frac{\partial_x u(\bar{x}_m)}{2nu(\bar{x}_m)^2} (y_{m+1} - x_{m+1}) + \frac{1}{4\pi n^2 (u(\bar{x}_m)^2 + Hu(\bar{x}_m)^2)} \partial_x \left(\frac{Hu}{u}\right)(\bar{x}_m) \\ &+ \frac{u(\bar{x}_m) Hu(\bar{x}_m)}{2\pi n^2 (u(\bar{x}_m)^2 + Hu(\bar{x}_m)^2)^2} (A_{1,m} + A_{2,m}) \partial_x \left(\frac{Hu}{u}\right)(\bar{x}_m) \\ &+ \frac{\sin^2 z}{8\pi^2 n^2 u(\bar{x}_m)^2} (D_{1,m} + D_{2,m} + D_{3,m} + B_{1,m} + B_{2,m} + H_{1,m} + H_{2,m} + H_{3,m}) \\ &+ O\big((n\|E\|_\infty + n^{-1/2}) |y_{m+1} - y_m - x_{m+1} + x_m|\big) \\ &+ O\big(\delta n^{-5/2} + \delta n^{-1} \|E\|_\infty\big) + O(n^{-3/2} \|E\|_\infty + (\log n)^2 \|E\|_\infty^2). \end{aligned}$$

The term in the penultimate line can be absorbed into the left hand side. Provided that n is sufficiently large and (3.4) holds, we obtain

$$y_{m+1} - y_m - x_{m+1} + x_m = \big(1 + O(n\|E\|_\infty + n^{-1/2})\big) \times$$

$$\times \Bigg[-\frac{\partial_x u(\bar{x}_m)}{2nu(\bar{x}_m)^2}(y_{m+1} - x_{m+1}) + \frac{1}{4\pi n^2(u(\bar{x}_m)^2 + Hu(\bar{x}_m)^2)}\partial_x\left(\frac{Hu}{u}\right)(\bar{x}_m)$$
$$+ \frac{u(\bar{x}_m)Hu(\bar{x}_m)}{2\pi n^2(u(\bar{x}_m)^2 + Hu(\bar{x}_m)^2)^2}(A_{1,m} + A_{2,m})\partial_x\left(\frac{Hu}{u}\right)(\bar{x}_m)$$
$$+ \frac{\sin^2 z}{8\pi^2 n^2 u(\bar{x}_m)^2}(D_{1,m} + D_{2,m} + D_{3,m} + B_{1,m} + B_{2,m} + H_{1,m} + H_{2,m} + H_{3,m})$$
$$+ O\big(\delta n^{-5/2} + \delta n^{-1}\|E\|_\infty\big) + O\left(n^{-3/2}\|E\|_\infty + (\log n)^2\|E\|_\infty^2\right)\Bigg]. \tag{8.1}$$

Now, we link the estimate above to the propagation of the error $\{E_m^t\}$. Let $\Delta t = \frac{1}{2n}$. To obtain sufficiently strong estimates, we need the following lemma that will let us control decay in time of the time derivatives of u. Let us recall that $u(x, t)$ solves

$$\partial_t u = -\frac{u^2}{\pi(u^2 + Hu^2)}\,\partial_x\left(\frac{Hu}{u}\right) = -\frac{1}{\pi}\frac{u\Lambda u - Hu\partial_x u}{u^2 + Hu^2}. \tag{8.2}$$

Lemma 8.1 *Let $u(x, t)$ be the solution of (8.2) corresponding to $u_0 \in H^s(\mathbb{S})$, $s > 5/2$. Then for every time t,*

$$\|\partial_t u(\cdot, t)\|_{L^\infty} + \|\partial_t^2 u(\cdot, t)\|_{L^\infty} + \|\partial_{tx}^2 u(\cdot, t)\|_{L^\infty} \leq C\delta(t),$$

where δ is defined in (5.2), and the constant C that may only depend on u_0 and s.

Proof The bound for $\partial_t u$ follows directly from (8.2), global regularity, the bound $\|Hu\|_{L^\infty} \leq C\|\partial_x u\|_{L^\infty}$, as well as

$$\|\Lambda u\|_{L^\infty} = \|H\partial_x u\|_{L^\infty} \leq C(\gamma)\|\partial_x u\|_{C^\gamma}. \tag{8.3}$$

Note that $\|\partial_x u\|_{C^\gamma}$ can be controlled by $\|\partial_x u\|_{L^\infty}$ and $\|\partial_x^2 u\|_{L^\infty}$, and hence is bounded by δ.

The bound on $\|\partial_{tx}^2 u\|_{L^\infty}$ is proved by differentiating (8.2) and carrying out estimates very similar to the above one. Additional term that we need to estimate for $\|\partial_t^2 u\|_{L^\infty}$ takes form

$$\|\partial_t \Lambda u\|_{L^\infty} = \frac{1}{\pi}\left\|\Lambda\left(\frac{u\Lambda u - Hu\partial_x u}{u^2 + (Hu)^2}\right)\right\|_{L^\infty}. \tag{8.4}$$

This term can be controlled by using (8.3) as well as

$$\|\Lambda f\|_{C^\gamma} = \|H\partial_x f\|_{C^\gamma} \leq C(\gamma)\|\partial_x f\|_{C^\gamma},$$

where we can set f equal to the expression in the brackets in (8.4), and elementary calculations. □

By the definition of the error (1.3), we have

$$E_m^{t+\Delta t} - E_m^t = (y_{m+1} - y_m - x_{m+1} + x_m) - \frac{1}{2nu(\bar{y}_m, t+\Delta t)} + \frac{1}{2nu(\bar{x}_m, t)} \quad (8.5)$$

Using Lemma 8.1, we find that

$$\begin{aligned}
&\frac{1}{u(\bar{y}_m, t+\Delta t)} - \frac{1}{u(\bar{x}_m, t)} \\
&= \partial_t \left(\frac{1}{u}\right)(\bar{x}_m, t) \cdot \Delta t + O(\delta \Delta t^2) + \partial_x \left(\frac{1}{u}\right)(\bar{x}_m, t+\Delta t) \cdot (\bar{y}_m - \bar{x}_m) + O(\delta |\bar{y}_m - \bar{x}_m|^2) \\
&= -\frac{\partial_t u(\bar{x}_m, t)}{u(\bar{x}_m, t)^2} \Delta t - \frac{\partial_x u(\bar{x}_m, t)}{u(\bar{x}_m, t)^2} (\bar{y}_m - \bar{x}_m) + O(\delta n^{-2}) \\
&= \frac{1}{\pi (u(\bar{x}_m, t)^2 + Hu(\bar{x}_m, t)^2)} \partial_x \left(\frac{Hu}{u}\right)(\bar{x}_m, t) \cdot \frac{1}{2n} \\
&\quad - \frac{\partial_x u(\bar{x}_m, t)}{u(\bar{x}_m, t)^2} (y_{m+1} - x_{m+1}) + \frac{\partial_x u(\bar{x}_m, t)}{2u(\bar{x}_m, t)^2} (y_{m+1} - y_m - x_{m+1} + x_m) + O(\delta n^{-2}), \qquad (8.6)
\end{aligned}$$

where we have used (8.2) in the last equality. Let us substitute the estimate (8.6) into (8.5) and get

$$\begin{aligned}
E_m^{t+\Delta t} - E_m^t &= (y_{m+1} - y_m - x_{m+1} + x_m) \left(1 - \frac{\partial_x u}{4nu^2}\right) \\
&- \frac{1}{4\pi n^2 (u^2 + Hu^2)} \partial_x \left(\frac{Hu}{u}\right) + \frac{\partial_x u}{2nu^2} \cdot (y_{m+1} - x_{m+1}) + O(\delta n^{-3}). \qquad (8.7)
\end{aligned}$$

Here, for simplicity we no longer indicate the spatial and time dependence, as all quantities are all evaluated at $(\bar{x}_m, t)$.

Deploying the estimate (8.1) in (8.7), we observe that the leading terms of the order $O(n^{-2})$ cancel. We obtain

$$\begin{aligned}
&E_m^{t+\Delta t} - E_m^t \\
&= \left[-\frac{\partial_x u}{2nu^2}(y_{m+1} - x_{m+1}) + \frac{1}{4\pi n^2 (u^2 + Hu^2)} \partial_x \left(\frac{Hu}{u}\right)\right] \cdot O\big(n\|E\|_\infty + n^{-1/2}\big) \\
&+ \frac{uHu}{2\pi n^2 (u^2 + Hu^2)^2} \partial_x \left(\frac{Hu}{u}\right) (A_{1,m} + A_{2,m}) \cdot \big(1 + O(n\|E\|_\infty + n^{-1/2})\big) \\
&+ \big(1 + O(n\|E\|_\infty + n^{-1/2})\big) \cdot \frac{\sin^2 z}{8\pi^2 n^2 u^2} \times \\
&\quad \times (D_{1,m} + D_{2,m} + D_{3,m} + B_{1,m} + B_{2,m} + H_{1,m} + H_{2,m} + H_{3,m}) \\
&+ O\big(\delta n^{-5/2} + \delta n^{-1} \|E\|_\infty\big) + O(\delta n^{-3} + n^{-3/2}\|E\|_\infty + (\log n)^2 \|E\|_\infty^2 + n(\log n)^2 \|E\|_\infty^3). \qquad (8.8)
\end{aligned}$$

Let us examine (8.8). The first term on the right side of (8.8) can be estimated by

$$O\left(\delta n^{-2} \cdot (n\|E\|_\infty + n^{-1/2})\right) = O(\delta n^{-5/2} + \delta n^{-1}\|E\|_\infty).$$

The second term on the right side of (8.8) looks like

$$O\left(\delta n^{-2}\cdot n\log n\|E\|_\infty\cdot(1+n\|E\|_\infty+n^{-1/2})\right)$$
$$=O(\delta n^{-1}\log n\|E\|_\infty+\delta\log n\|E\|_\infty^2+\delta n^{-3/2}\log n\|E\|_\infty). \tag{8.9}$$

The first error in (8.9) is supercritical and requires additional treatment. We write the corresponding term explicitly:

$$\frac{uHu}{2\pi n^2(u^2+Hu^2)^2}\partial_x\left(\frac{Hu}{u}\right)(A_{1,m}+A_{2,m}). \tag{8.10}$$

Below, we will prove estimates that will allow to absorb this part into the dissipation, namely into the $D_{i,m}$ expressions in (8.8).

For $A_{1,m}$, decompose the sum in (5.10) into two parts and use the estimate similar to (7.14):

$$\begin{aligned}A_{1,m}&=\frac{1}{4\pi}\sum_{j\in S_m^c}\cot\frac{y_m-x_j}{2}\big((E_{j-1}-E_m)+(E_j-E_m)\big)\\&\quad+\frac{E_m}{2\pi}\sum_{j\in S_m^c}\cot\frac{y_m-x_j}{2}+O(\delta n\|E\|_\infty)\\&=H_{4,m}+O(n\|E\|_\infty+n^2\log n\|E\|_\infty^2),\end{aligned}\tag{8.11}$$

where $H_{4,m}$ has the form

$$H_{4,m}=\sum_{j\in S_m^c\cup\{j_+-1\}}h_4(j,m)(E_j-E_m),$$

with coefficients

$$h_4(j,m)=\begin{cases}\frac{1}{4\pi}\left(\cot\frac{y_m-x_j}{2}+\cot\frac{y_m-x_{j+1}}{2}\right), & j\in S_m^c\backslash\{j_--1\}\\ \frac{1}{4\pi}\cot\frac{y_m-x_{j_--1}}{2}, & j=j_--1\\ \frac{1}{4\pi}\cot\frac{y_m-x_{j_+}}{2}, & j=j_+-1\end{cases}=O(n|m-j|^{-1}). \tag{8.12}$$

Now, for $A_{2,m}$, decompose the sum in (5.11) into two parts and use the estimate (4.18) and a straightforward bound on at most $O(1+n^{3/2}\|E\|_\infty)$ mismatched summands to arrive at

$$A_{2,m}=\frac{1}{\pi}\left(\sum_{j=j_-}^{m-1}\sum_{l=j}^{m-1}\frac{E_l-E_m}{(m-j)(y_m-x_j)}+\sum_{j=m+1}^{j_+-1}\sum_{l=m}^{j-1}\frac{E_l-E_m}{(j-m)(y_m-x_j)}\right)$$

$$+\frac{E_m}{\pi}\left(\sum_{j=j_-}^{m-1}\frac{1}{y_m-x_j}+\sum_{j=m+1}^{j_+-1}\frac{1}{y_m-x_j}\right)+O(n\|E\|_\infty)$$
$$=B_{3,m}+O(n\|E\|_\infty+n^2\log n\|E\|_\infty^2), \tag{8.13}$$

where $B_{3,m}$ has the form

$$B_{3,m}=\sum_{l\in S_m\setminus\{m\}}b_3(l,m)(E_l-E_m),$$

with coefficients

$$b_3(l,m)=\begin{cases}\sum_{j=j_-}^{l}\frac{1}{(m-j)(y_m-x_j)}, & j_-\le l\le m-1\\ \sum_{j=l+1}^{j_+-1}\frac{1}{(j-m)(y_m-x_j)}, & m+1\le l\le j_+-2\end{cases}=O\big(n|m-l|^{-1}\big). \tag{8.14}$$

Observe that after multiplication by a factor in (8.10), the error terms in (8.11) and (8.13) become of the order $O(\delta n^{-1}\|E\|_\infty+\delta\log n\|E\|_\infty^2)$.

We combine (8.10) with the term on the fourth and fifth lines in (8.8). Let us introduce the final dissipation term, covering all scales. It equals

$$\sum_{j\neq m}\kappa(j,m)(E_j-E_m),$$

with the coefficients

$$\kappa(j,m)=\big(1+O(n\|E\|_\infty+n^{-1/2})\big)\cdot\frac{\sin^2 z}{8\pi^2n^2u^2}\left[\sum_{i=1}^{3}d_i(j,m)+\sum_{i=1}^{2}b_i(j,m)+\sum_{i=1}^{3}h_i(j,m)\right]$$
$$+\frac{uHu}{2\pi n^2(u^2+Hu^2)^2}\cdot\partial_x\left(\frac{Hu}{u}\right)\big(b_3(j,m)+h_4(j,m)\big). \tag{8.15}$$

Then, according to our estimates, the evolution of E_m^t in (8.8) can be summarized as

$$E_m^{t+\Delta t}-E_m^t=\sum_{j\neq m}\kappa^t(j,m)(E_j^t-E_m^t)+O\big(\delta n^{-5/2}+\delta n^{-1}\|E^t\|_\infty\big)$$
$$+O(n^{-3/2}\|E^t\|_\infty+(\log n)^2\|E^t\|_\infty^2+n(\log n)^2\|E^t\|_\infty^3). \tag{8.16}$$

Finally, let us examine more carefully the coefficients $\kappa^t(j,m)$.

Theorem 8.2 *The error E_m^t propagates according to the evolution equation*

$$E^{t+\Delta t}-E^t=\Delta t\,\mathcal{L}^tE^t$$
$$+O\big(\delta n^{-5/2}+\delta n^{-1}\|E^t\|_\infty\big)+O(n^{-3/2}\|E^t\|_\infty+(\log n)^2\|E^t\|_\infty^2+n(\log n)^2\|E^t\|_\infty^3), \tag{8.17}$$

where the operator $\mathcal{L}^t$ is a diffusive operator given by

$$(\mathcal{L}^t E^t)_m = \frac{1}{\Delta t} \sum_{j, j \neq m} \kappa^t(j, m)(E^t_j - E^t_m). \tag{8.18}$$

The diffusion coefficients $\kappa^t(j, m)$ satisfy

$$\kappa^t(j, m) = \frac{1}{16\pi^2 n^2 (u^2 + Hu^2) \sin^2 \frac{x_m^{t+\Delta t} - x_j^t}{2}} \times \\ \Big(1 + O(n^{-1/2} + \delta |j - m| n^{-1} + n \log n \|E\|_\infty + (\log n)^2 (n^2 \|E\|_\infty^2 + n^3 \|E\|_\infty^3 + n^4 \|E\|_\infty^4))\Big), \tag{8.19}$$

where u and Hu can be evaluated at x_m^t, and x_j^t, $x_m^{t+\Delta t}$ are the polynomial roots at time t and $t + \Delta t$, respectively. Note that $x_m^{t+\Delta t} = y_m$ in our usual notation. In particular, assuming (3.4), for all n exceeding a threshold $n_0(\tilde{u}_0)$ that may only depend on the initial data we have

$$\kappa^t(j, m) \sim |m - j|^{-2}; \tag{8.20}$$

the constants involved in $\sim$ can be chosen uniformly for all t, m and j and depend only on $\tilde{u}_0$.

Remark 8.3 Recall the formula for the fractional Laplacian

$$\Lambda u(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^2} \, dy = \frac{1}{4\pi} P.V. \int_{\mathbb{S}} \frac{u(x) - u(y)}{\sin^2(\frac{x-y}{2})} \, dy.$$

Observe that given the formula above, $\Delta t = \frac{1}{2n}$ relationship, (8.18) and (8.19), the equation (8.17) looks like a discretization of the continuous in time and space PDE that to the main order is just a modulated fractional heat equation:

$$\partial_t E(x, t) = \frac{u(x, t)}{\pi (u^2(x, t) + Hu^2(x, t))} \Lambda E(x, t) + errors.$$

The leading term of this equation coincides with the dissipative part of (1.2). The appearance of such a simple PDE controlling the error between the evolution of roots and the PDE (1.1) is surprising.

Proof For simplicity, we will omit time dependence in notation for $\kappa(j, m)$. Let us start with the near field $j \in S_m \backslash \{m, j_-, j_+ - 1\}$, where

$$\kappa(j, m) = \big(1 + O(n\|E\|_\infty + n^{-1/2})\big) \cdot \frac{\sin^2 z}{8\pi^2 n^2 u^2} \cdot \big(d_1(j, m) + b_1(j, m) + b_2(j, m)\big) \\ + \frac{uHu}{2\pi n^2 (u^2 + Hu^2)^2} \cdot \partial_x \left(\frac{Hu}{u}\right) \cdot b_3(j, m).$$

Note that there are additional contributions from $D_{3,m}$ and other terms at the boundary $j = j_-, j_+ - 1$ that will be discussed later.

First, observe that (8.14) implies

$$\begin{aligned}\tilde{b}_{3,m}(j) &:= \frac{uHu}{2\pi n^2(u^2+Hu^2)^2}\cdot\partial_x\left(\frac{Hu}{u}\right)\cdot b_3(j,m)\\ &= O(\delta n^{-2}\cdot n|m-j|^{-1}) \lesssim |m-j|^{-2}n^{-1/2}\end{aligned} \tag{8.21}$$

for $j \in S_m$. Next, we have already shown in (6.9), (6.15) and (6.21) that

$$d_1(j,m) = \frac{2}{(y_m - x_j)(y_{m+1}-x_{j+1})} \sim \frac{n^2}{|m-j|^2};\ |b_1(j,m)|, |b_2(j,m)| \lesssim n^{-1/2}d_1(j,m). \tag{8.22}$$

To incorporate the contribution (8.21) from $\tilde{b}_{3,m}(j)$, let us analyze the pre-factor in the fourth line of (8.8). Applying (5.12), we get

$$\begin{aligned}\left(1+O(n\|E\|_\infty+n^{-1/2})\right)\cdot\frac{\sin^2 z}{8\pi^2n^2u^2} &= \frac{1}{8\pi^2n^2(u^2+Hu^2)}\\ &+O\left(n^{-5/2}+n^{-1}\log n\|E\|_\infty+(\log n)^2\|E\|_\infty^2+n(\log n)^2\|E\|_\infty^3\right).\end{aligned} \tag{8.23}$$

Then, taking into account (8.21) and (8.22), as well as (4.5) and (4.13), a short computation leads to

$$\begin{aligned}\kappa(j,m) &= \left(1+O(n\|E\|_\infty+n^{-1/2})\right)\cdot\frac{\sin^2 z}{8\pi^2n^2u^2}\cdot\left(d_1(j,m)+b_1(j,m)+b_2(j,m)\right)+\tilde{b}_{3,m}(j)\\ &= \frac{d_1(j,m)}{8\pi^2n^2(u^2+Hu^2)}\left(1+O(n^{-1/2}+n\log n\|E\|_\infty+n^2(\log n)^2\|E\|_\infty^2+n^3(\log n)^2\|E\|_\infty^3)\right)\\ &\sim |m-j|^{-2}(1+O(n^{-1/2}+n\log n\|E\|_\infty+n^2(\log n)^2\|E\|_\infty^2+n^3(\log n)^2\|E\|_\infty^3))\end{aligned} \tag{8.24}$$

for $j \in S_m\backslash\{m, j_-, j_+-1\}$.

To pass from (8.24) to the form (8.19), note first that

$$\frac{1}{y_m-x_j}-\frac{1}{y_{m+1}-x_{j+1}} = \frac{1}{y_m-x_j}\cdot O(n\|E\|_\infty+n^{-1}),$$

where we used (6.6) and an estimate $|y_{m+1}-y_m-x_{m+1}+x_m| = O(\|E\|_\infty+n^{-2})$ that follows from (8.1). Here we split the numerator in the usual manner and used that from the more precise estimate (8.1) it follows that $|y_{m+1}-y_m-x_{m+1}+x_m| = O(\|E\|_\infty+n^{-2})$. Thus we can replace $y_{m+1}-x_{j+1}$ in (8.22) by y_m-x_j and absorb the difference into the error. Then observe that for $|y_m - x_j| \lesssim n^{-1/2}$,

$$\frac{2}{(y_m - x_j)^2} - \frac{1}{2\sin^2 \frac{y_m - x_j}{2}} = \frac{1}{2\sin^2 \frac{y_m - x_j}{2}} \cdot O(|y_m - x_j|^2) = \frac{1}{2\sin^2 \frac{y_m - x_j}{2}} \cdot O(n^{-1}),$$

proving (8.19) in near field.

Next, we consider the interface $j = j_- - 1, j_-, j_+ - 1, j_+$. The analysis for $j = j_-$ and $j_+ - 1$ is similar, so let us focus on the former. According to (7.54),

$$d_3(j_-, m) = \frac{n}{2}\left(1 + O(n^{-1/2} + n\log n\|E\|_\infty + n(\log n)^2\|E\|_\infty^2)\right). \quad (8.25)$$

We also get from (7.10) that

$$h_1(j, m) = O(n^2\|E\|_\infty + \delta n^{1/2}), \quad (8.26)$$

and from (7.38), (8.12) that

$$|h_3(j, m)| + |h_4(i, m)| = O(n^{1/2}) \quad (8.27)$$

for all $j = j_- - 1, j_-, j_+ - 1, j_+$.

The term $H_{2,m}$, on the other hand, yields an essential contribution: the estimate (7.30) shows that

$$h_2(j_-, m) = -\frac{n}{2}\left(1 + O(n^{-1/2})\right). \quad (8.28)$$

Finally, since $j_- \in S_m$, there is also a contribution from $D_{1,m}$, with

$$d_1(j_-, m) = \frac{2}{(y_{m+1} - x_{j_-+1})(y_m - x_{j_-})} = \frac{2}{n^{-1} + O(n^{-3/2})} = 2n\left(1 + O(n^{-1/2})\right). \quad (8.29)$$

Then, due to (8.25), (8.28), and (8.29) we have

$$d_1(j_-, m) + d_3(j_-, m) + h_2(j_-, m) = 2n\left(1 + O(n^{-1/2} + n\log n\|E\|_\infty + n(\log n)^2\|E\|_\infty^2)\right) > 0. \quad (8.30)$$

Combining (5.12), (8.30), (8.26), (8.27), (8.21) and (8.15), we obtain

$$\kappa(j_-, m) = \frac{1}{4\pi^2(u^2 + Hu^2)n} \times \\ \times \left(1 + O\left(n^{-1/2} + n\log n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2 + n^3(\log n)^2\|E\|_\infty^3\right)\right). \quad (8.31)$$

For $j = j_+$ (and similarly for $j = j_- - 1$), from (7.54) and (7.30) we find

$$d_3(j_+, m) + h_2(j_+, m) = 2n\left(1 + O(n^{-1/2} + n\log n\|E\|_\infty + n(\log n)^2\|E\|_\infty^2)\right).$$

Taking into account (5.12), (8.26), (8.27), (8.12) and (8.15), we arrive at the estimate for $\kappa(j_+, m)$ identical to (8.31). Note also that the main term in (8.31) coincides with (8.19), since $|y_m - x_j| = n^{-1/2} + O(n^{-1})$ for $j = j_- - 1$, j_-, $j_+ - 1$, and j_+. It follows that $\kappa(j, m)$ at the interface satisfy (8.19), and in particular

$$\kappa(j, m) \sim |m - j|^{-2}$$

for all these j for all sufficiently large n.

Finally, for the far field $j \in S_m^c \backslash \{j_- - 1, j_+\}$, we have

$$\begin{aligned}\kappa(j, m) = &\left(1 + O(n\|E\|_\infty + n^{-1/2})\right) \cdot \frac{\sin^2 z}{8\pi^2 n^2 u^2} \\ &\cdot \left(d_2(j, m) + h_1(j, m) + h_2(j, m) + h_3(j, m)\right) \\ &+ \frac{uHu}{2\pi n^2(u^2 + Hu^2)^2} \cdot \partial_x\left(\frac{Hu}{u}\right) \cdot h_4(j, m).\end{aligned} \tag{8.32}$$

We have shown in (7.17), (7.10), (7.28) and (7.38) that

$$d_2(j, m) = \frac{1}{2\sin^2\frac{y_m - x_j}{2}}\left(1 + O(n^{-1/2} + n\|E\|_\infty)\right), \tag{8.33}$$

$$\sum_{i=1}^{3} h_i(j, m) = O\left(\frac{\delta n}{|m - j|} + \frac{n^3\|E\|_\infty}{|m - j|^2} + \frac{n^2}{|m - j|^3}\right). \tag{8.34}$$

The last two terms in the expression for $\sum_{i=1}^3 h_i(j, m)$ in (8.34) can be absorbed into $d_2(j, m)$ since due to (4.5), we have

$$\frac{C(u_0)n^2}{|m - j|^2} \geq d_2(j, m) \geq \frac{2u_{\min}^2 n^2}{|m - j|^2}, \tag{8.35}$$

while

$$O\left(\frac{n^3\|E\|_\infty}{|m - j|^2} + \frac{n^2}{|m - j|^3}\right) = \frac{n^2}{|m - j|^2} \cdot O(n\|E\|_\infty + n^{-1/2}), \quad \forall\, j \in S_m^c. \tag{8.36}$$

The first term $O(\delta n|m - j|^{-1})$ is trickier, since when $|m - j| \sim n$, this term is generally of the order $O(1)$, similarly to $d_2(j, m)$. Due to (8.10), (8.12), a term of the order $O(\delta n|m - j|^{-1})$ also arises from $H_{4,m}$ in (8.32). We will handle these perturbations as follows. Using (8.35), let us write

$$O(\delta n|m-j|^{-1}) = n^2|m-j|^{-2}O(\delta n^{-1}|m-j|) = d_2(j,m)O(\delta n^{-1}|m-j|). \tag{8.37}$$

Applying (8.23), (8.32), (8.33), (8.36), (8.37) and (8.34) we get that for $j \in S_m^c \backslash \{j_- - 1, j_+\}$,

$$\kappa(j,m) = \frac{1}{16\pi^2 n^2(u^2 + Hu^2)\sin^2\frac{y_m - x_j}{2}} \times \\ \times \left(1 + O(\delta n^{-1}|m-j| + n^{-1/2} + n\log n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2 + n^3(\log n)^2\|E\|_\infty^3)\right). \tag{8.38}$$

This already proves (8.19), but it remains to show (8.20). Observe that there exists $\beta(u_0) > 0$ such that if $|m-j| \leq \beta(u_0)n$, then the dangerous error term in (8.38) satisfies

$$\left|O(\delta n^{-1}|m-j|)\right| \leq \frac{1}{3},$$

for all times, as the constants depending on u_0 that are involved in O are uniformly bounded due to Theorem 3.1. On the other hand, observe that

$$\frac{1}{8\pi^2 n^2(u^2 + Hu^2)} \sum_{n \geq |m-j| > \beta(u_0)n} d_2(j,m)O(\delta n^{-1}|m-j|)(E_j - E_m) \\ \lesssim \delta n^{-1}\|E\|_\infty \sum_{n \geq |m-j| > \beta(u_0)n} |m-j|^{-1} = O(\delta n^{-1}\|E\|_\infty).$$

It follows that for j such that $|m-j| > \beta(u_0)n$, the $O(\delta n^{-1}|m-j|)$ term in (8.38) can be removed and absorbed into the favorable critical error $O(\delta n^{-1}\|E\|_\infty)$ in (8.16). Using (4.5), this implies that we can arrange so that for all sufficiently large n,

$$\kappa(j,m) \geq \frac{u_{\min}^2}{16\pi^2(u^2 + Hu^2)|m-j|^2} \sim |m-j|^{-2}.$$

For the rest of the paper, we will assume that this arrangement is made, and that in (8.16)

$$O(\delta|j-m|n^{-1}) \leq \frac{1}{3}$$

for all j, m, t. □

Next, we show an l^1 bound on $\{\kappa^t(\cdot, m)\}$. Since our evolution is discrete in time, we need control on the l^1 norm of our kernel to ensure stability that allows an argument similar to maximum principle that we will use to prove Theorem 1.1.

Lemma 8.4 *Fix any time t, and assume (3.4) holds. Then, for all $n \geq n_0(\tilde{u}_0)$ large enough, for all m, we have*

$$\sum_{\{j:j\neq m\}} \kappa^t(j,m) = S(t) < 1-\rho.$$

The constant $\rho > 0$ depends only on u_0 and may be chosen uniformly for all times, all sufficiently large $n \geq n_0(\tilde{u}_0)$, and m.

Proof From (8.20) it follows that the contribution of the sum over $j \in S_m^c$ is of the order $O(n^{-1/2})$. Then (8.22), (8.24) and (8.31) imply

$$\sum_{j\neq m} \kappa(j,m) = \frac{1}{8\pi^2 n^2 (u^2 + Hu^2)} \sum_{j\in S_m\setminus\{m\}} \frac{2}{(y_{m+1}-x_{j+1})(y_m - x_j)} + O(n^{-1/2} + n\log n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2 + n^3(\log n)^2\|E\|_\infty^3). \tag{8.39}$$

Let us break the sum in (8.39) into three regions: (i) $m+2 \leq j \leq j_+ - 1$, (ii) $j_- \leq j \leq m-1$, and (iii) $j = m+1$. For the first region ($j \geq m+2$), we have

$$\begin{aligned} x_j - y_m &= x_{m+1} - y_m + \sum_{l=m+1}^{j-1} \left(\frac{1}{2nu(\bar{x}_l)} + E_l \right) \\ &\geq \frac{j-m-1}{2nu(\bar{x}_m)} + O\left(\frac{|m-j|^2}{n^2} + |m-j|\,\|E\|_\infty \right); \\ x_{j+1} - y_{m+1} &\geq \frac{j-m-1}{2nu(\bar{x}_m)} + O\left(\frac{|m-j|^2}{n^2} + |m-j|\,\|E\|_\infty \right). \end{aligned}$$

Then,

$$\begin{aligned} &\frac{1}{8\pi^2 n^2(u^2+Hu^2)} \sum_{j=m+2}^{j_+-1} \frac{2}{(y_{m+1}-x_{j+1})(y_m-x_j)} \\ &\leq \frac{1}{8\pi^2 n^2(u^2+Hu^2)} \sum_{j=m+2}^{j_+-1} \left[\frac{2}{\left(\frac{j-m-1}{2nu}\right)^2} + O\left(\frac{n}{|m-j|} + \frac{n^3\|E\|_\infty}{|m-j|^2} + \frac{n^4\|E\|_\infty^2}{|m-j|^2} \right) \right] \\ &= \frac{u^2}{\pi^2(u^2+Hu^2)} \sum_{j=m+2}^{j_+-1} \frac{1}{(j-m-1)^2} + O\big(n^{-1}\log n + n\|E\|_\infty + n^2\|E\|_\infty^2\big) \\ &\leq \frac{u^2}{6(u^2+Hu^2)} + O\big(n^{-1}\log n + n\|E\|_\infty + n^2\|E\|_\infty^2\big). \end{aligned}$$

Here, we have used the Euler identity $\sum_{k=1}^\infty k^{-2} = \frac{\pi^2}{6}$. A similar argument can be done for the second region $j \leq m-1$, yielding an identical bound.

Finally, the summand corresponding to $j = m+1$ can be large if y_m is close to x_{m+1}. However, we established a fairly precise control over this distance in Lemma 4.4. Let us apply the lower bound estimate (4.34) and obtain

$$\begin{aligned}
&\frac{1}{8\pi^2 n^2(u^2+Hu^2)} \cdot \frac{2}{(y_{m+1}-x_{m+2})(y_m - x_{m+1})} \\
&= \frac{1}{8\pi^2 n^2(u^2+Hu^2)} \cdot \frac{2}{\left(\frac{1}{2\pi n u}\operatorname{arccot}\left(\frac{Hu}{u}\right)\right)^2} + O\big(n^{-1/2} + n\log n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2\big) \\
&= \frac{u^2}{(u^2+Hu^2)\left(\operatorname{arccot}\left(\frac{Hu}{u}\right)\right)^2} + O\big(n^{-1/2} + n\log n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2\big).
\end{aligned}$$

Here u and Hu are evaluated at $\bar{x}_m$, and error coming from the main term for $y_{m+1} - x_{m+2}$ in (4.34) is absorbed into the lager $O(n^{-3/2})$ error. Putting the three regions together, we get the following bound

$$\sum_{j, j\neq m} \kappa(j,m) \le \frac{1}{1+\left(\frac{Hu}{u}\right)^2} \cdot \left(\frac{1}{3} + \frac{1}{\left(\operatorname{arccot}\frac{Hu}{u}\right)^2}\right) + O\big(n^{-1/2} + n\log n\|E\|_\infty + n^2(\log n)^2\|E\|_\infty^2\big). \tag{8.40}$$

Set $a = \operatorname{arccot}\left(\frac{Hu}{u}\right) \in (0,\pi)$. The value of a depends on u. Since $\frac{Hu}{u}$ is uniformly bounded in time, $a \geq a(u_0) > 0$ is bounded away from zero uniformly for all times and x_m. We rewrite the bound above in terms of a as follows:

$$\sum_{j, j\neq m} \kappa(j,m) \le \sin^2 a \cdot \left(\frac{1}{3} + \frac{1}{a^2}\right) + o(1) =: F(a) + o(1).$$

The $o(1)$ includes all errors from (8.40) and is based on (3.4). Observe that $F(a)$ is a decreasing function in $(0,\pi)$, and clearly $\lim_{a\to 0} F(a) = 1$. Indeed, one can compute

$$F'(a) = \frac{2\sin a}{3a^3}\big(a(a^2+3)\cos a - 3\sin a\big).$$

It is immediate that $F'(a) < 0$ if $a \in [\frac{\pi}{2}, \pi)$. For $a \in (0, \frac{\pi}{2})$, we claim that $a(a^2 + 3)\cos a - 3\sin a < 0$, or equivalently $\tan a > a + \frac{a^3}{3}$. But using Taylor series for $\tan a$, we see that

$$\tan a - a - \frac{a^3}{3} = \sum_{i=2}^{\infty} \frac{(\tan)^{(2i+1)}(0) a^{2i+1}}{(2i+1)!} = \frac{2a^5}{15} + \frac{17a^7}{315} + \cdots > 0, \quad \forall\, a \in \left(0, \frac{\pi}{2}\right). \tag{8.41}$$

Indeed, the last inequality in (8.41) follows from $\tan^{(2i)}(0) = 0$, $\tan^{(2i+1)}(0) \geq 0$ for all i (this can be derived by induction using $f' = 1 + f^2$ for $f(x) = \tan x$). Therefore, we end up with

$$\sum_{j\neq m} \kappa(j,m) \leq F(a(u_0)) + o(1) < 1$$

for all $n \geq n_0(\tilde{u}_0)$. □

9 Proof of Theorem 1.1

It will be convenient for us to define

$$\kappa^t(m,m) := 1 - \sum_{\{j:j\neq m\}} \kappa^t(j,m) = 1 - S(t). \tag{9.1}$$

Then, we can express

$$\begin{aligned} E_m^{t+\Delta t} &= E_m^t + \Delta t\,(\mathcal{L}^t E^t)_m = E_m^t + \sum_{\{j:j\neq m\}} \kappa^t(j,m)(E_j^t - E_m^t) \\ &= (1 - S(t))E_m^t + \sum_{\{j:j\neq m\}} \kappa^t(j,m)E_j^t = \sum_{j=1}^{2n} \kappa^t(j,m)E_j^t. \end{aligned}$$

Therefore, the dynamics (8.17) becomes

$$\begin{aligned} E_m^{t+\Delta t} &= \sum_{j=1}^{2n} \kappa^t(j,m)E_j^t + O\big(\delta n^{-5/2} + \delta n^{-1}\|E^t\|_\infty\big) \\ &\quad + O(n^{-3/2}\|E^t\|_\infty + (\log n)^2\|E^t\|_\infty^2 + n(\log n)^2\|E^t\|_\infty^3). \end{aligned} \tag{9.2}$$

The diffusion coefficients $\kappa^t(j,m)$ satisfy

$$\sum_{j=1}^{2n} \kappa^t(j,m) = 1$$

due to the definition (9.1). Also, from (8.20) and Lemma 8.4, we know that $\kappa^t(j,m) \sim |m-j|^{-2} > 0$ if $j \neq m$, and $\kappa^t(m,m) \geq \rho > 0$.

We continue with the following lemma, which establishes a bound that will play for us a role similar to the mean zero condition for the Poincaré inequality.

Lemma 9.1 *Suppose $\int_0^{2\pi} u_0(x)\,dx = 1$. Then for all times t, we have*

$$\sum_{j=1}^{2n} E_j^t = O(\delta n^{-2} + \delta n\|E^t\|_\infty). \tag{9.3}$$

Proof Observe that $\int_0^{2\pi} u(x,t)\,dx = 1$ for all times. On the other hand, by midpoint rule

$$\int_0^{2\pi} u(x,t)\,dx = \sum_j \Big(u(\bar{x}_j^t, t)(x_{j+1}^t - x_j^t) + O(\delta n^{-3})\Big)$$
$$= \sum_j \left(\frac{1}{2n} + E_j^t u(\bar{x}_j^t, t) + O(\delta n^{-3})\right) = 1 + \sum_j E_j^t u(\bar{x}_j^t, t) + O(\delta n^{-2}).$$

This leads to

$$\sum_{j=1}^{2n} E_j^t u(\bar{x}_j^t, t) = O(\delta n^{-2}).$$

Then, we can compute

$$\sum_{j=1}^{2n} E_j^t \le \frac{1}{u_{\min}(t)} \sum_{\{j:E_j^t \ge 0\}} E_j^t u(\bar{x}_j^t, t) + \frac{1}{u_{\max}(t)} \sum_{\{j:E_j^t < 0\}} E_j^t u(\bar{x}_j^t, t)$$
$$= \left(\frac{1}{u_{\min}(t)} - \frac{1}{u_{\max}(t)}\right) \sum_{\{j:E_j^t \ge 0\}} E_j^t u(\bar{x}_j^t, t) + \frac{1}{u_{\max(t)}} \sum_{j=1}^{2n} E_j^t u(\bar{x}_j^t, t)$$
$$\le \frac{\pi \|\partial_x u(\cdot, t)\|_{L^\infty}}{u_{0,\min}^2} \cdot 2n u_{0,\max} \|E^t\|_\infty + \frac{1}{u_{0,\min}} \cdot O(\delta n^{-2}) = O(\delta n^{-2} + \delta n \|E^t\|_\infty).$$

A similar bound can be obtained for $-\sum_j E_j^t$. This completes the proof of (9.3). $\square$

Next we obtain an improved estimate on the dissipative term $\sum_j \kappa^t(j,m)E_j^t$, using a discrete analogue of the proof on (3.1) (see [26,Lemma 4.2]).

Proposition 9.2 *Suppose $\int_0^{2\pi} u_0(x)\,dx = 1$, and (3.4) holds. Given any $t \ge 0$ and $m \in \{1, \cdots, 2n\}$, and provided that $n \ge n_0(\tilde{u}_0)$ is sufficiently large, we have*

$$\sum_{j=1}^{2n} \kappa^t(j,m)E_j^t \le \left(1 - \frac{\sigma}{2n}\right) \|E^t\|_\infty + O\big(\delta n^{-4} + \delta n^{-1}\|E^t\|_\infty + n^{-3/2}\|E^t\|_\infty + \log n \|E^t\|_\infty^2\big), \tag{9.4}$$

where $\sigma = \frac{2}{\pi^2 \bar{u}} = \frac{4}{\pi}$.

Proof For any given $\alpha \in (0,\pi)$, we can decompose the indices $\{1, \cdots, 2n\}$ into two parts

$$J(\alpha) = \left\{ j : |x_m^{t+\Delta t} - x_j^t| \le \alpha \right\}, \quad J^c(\alpha) = \{1, \cdots, 2n\} \backslash J(\alpha).$$

Note that the sets $J(\alpha)$, $J^c(\alpha)$ depend on t and m.

From (8.19), using (3.4), we have

$$\kappa^t(j,m) = \frac{1}{16\pi^2 n^2(u^2+Hu^2)\sin^2\frac{x_m^{t+\Delta t}-x_j^t}{2}} + O\big(n^{-5/2}+\delta|j-m|n^{-3}+n^{-1}\log n\|E^t\|_\infty\big). \tag{9.5}$$

Let us denote

$$\bar{\kappa}_m^t(\alpha) = \frac{1}{16\pi^2 n^2(u^2(\bar{x}_m^t,t)+Hu^2(\bar{x}_m^t,t))\sin^2\frac{\alpha}{2}}. \tag{9.6}$$

Then, we obtain

$$\begin{aligned}\kappa^t(j,m) &\geq \bar{\kappa}_m^t(\alpha) - O\big(\delta n^{-2}+n^{-5/2}+n^{-1}\log n\|E^t\|_\infty\big), \quad j\in J(\alpha),\\ \kappa^t(j,m) &\leq \bar{\kappa}_m^t(\alpha) + O\big(\delta n^{-2}+n^{-5/2}+n^{-1}\log n\|E^t\|_\infty\big), \quad j\in J^c(\alpha).\end{aligned}$$

Now, we compute

$$\begin{aligned}\sum_{j=1}^{2n}\kappa^t(j,m)E_j^t &= \sum_{j\in J(\alpha)}\kappa^t(j,m)(E_j^t-\|E^t\|_\infty) + \|E^t\|_\infty\sum_{j\in J(\alpha)}\kappa^t(j,m)\\ &\quad + \sum_{j\in J^c(\alpha)}\kappa^t(j,m)(E_j^t+\|E^t\|_\infty) - \|E^t\|_\infty\sum_{j\in J^c(\alpha)}\kappa^t(j,m)\\ &\leq \bar{\kappa}_m^t(\alpha)\sum_{j\in J(\alpha)}(E_j^t-\|E^t\|_\infty) + \bar{\kappa}_m^t(\alpha)\sum_{j\in J^c(\alpha)}(E_j^t+\|E^t\|_\infty)\\ &\quad + O(n\|E^t\|_\infty)\cdot O\big(\delta n^{-2}+n^{-5/2}+n^{-1}\log n\|E^t\|_\infty\big)\\ &\quad + \|E^t\|_\infty\Big(\sum_{j\in J(\alpha)}\kappa^t(j,m) - \sum_{j\in J^c(\alpha)}\kappa^t(j,m)\Big)\\ &= \bar{\kappa}_m^t(\alpha)\sum_{j=1}^{2n}E_j^t + \bar{\kappa}_m^t(\alpha)\|E^t\|_\infty(2n-2|J(\alpha)|) + \|E^t\|_\infty\Big(1-2\sum_{j\in J^c(\alpha)}\kappa^t(j,m)\Big)\\ &\quad + O\big(\delta n^{-1}\|E^t\|_\infty + n^{-3/2}\|E^t\|_\infty + \log n\|E^t\|_\infty^2\big)\end{aligned} \tag{9.7}$$

For the rest of the argument, let us set $\alpha = \pi/2$. Then in the penultimate line of (9.7), the first term can be bounded by $O(\delta n^{-4})$ using (9.3) and (9.6). To estimate the second summand in the penultimate line of (9.7), observe that for any $j \in [m-\frac{n}{2}, m+\frac{n}{2}]$

$$\begin{aligned}|x_m^{t+\Delta t}-x_j^t| &\leq |x_m^{t+\Delta t}-x_m^t| + |x_m^t-x_j^t| \leq \frac{|m-j|+1}{2nu_{\min}(t)} + O(|m-j|\cdot\|E^t\|_\infty)\\ &\leq \alpha = \frac{\pi(|m-j|+1)}{n} + O(\delta+n\|E^t\|_\infty),\end{aligned}$$

where the last equality is due to (3.1). This implies

$$|J(\pi/2)| \geq n - C(\delta n + n^2\|E^t\|_\infty).$$

Therefore,

$$\bar{\kappa}_m^t(\pi/2)\|E^t\|_\infty(2n-2|J(\pi/2)|) = O(\delta n^{-1}\|E^t\|_\infty + \|E^t\|_\infty^2).$$

For the remaining third summand in the penultimate line of (9.7), we apply (9.5), (3.1), (3.2), and the following bound

$$x_m^{t+\Delta t} - x_j^t = x_m^{t+\Delta t} - x_m^t + \sum_{l=m}^{j-1}\left(\frac{1}{2nu(\bar{x}_l^t, t)} + E_l^t\right) = \frac{|m-j|}{2n\bar{u}} + O(\delta + n^{-1} + n\|E^t\|_\infty).$$

This yields

$$\begin{aligned}\sum_{j\in J^c(\pi/2)}\kappa^t(j,m) &= \sum_{j\in J^c(\pi/2)}\frac{1}{16\pi^2n^2u(\bar{x}_j^t,t)^2\sin^2\frac{|m-j|}{4n\bar{u}}} + O\big(\delta n^{-1} + n^{-2} + \|E^t\|_\infty\big)\\ &= \frac{1}{16\pi^2n^2\bar{u}^2}\sum_{j\in J^c(\alpha)}\frac{1}{\sin^2\frac{|m-j|}{4n\bar{u}}} + O\big(\delta n^{-1} + n^{-3/2} + \log n\|E^t\|_\infty\big).\end{aligned}$$

The sum can be approximated by the following integral.

$$\begin{aligned}\sum_{j\in J^c(\pi/2)}\frac{1}{\sin^2\frac{|m-j|}{4n\bar{u}}} &= 4n\bar{u}\int_{\frac{3\pi}{4}}^{\pi/4}\frac{1}{\sin^2 z}\,dz + O(1)\\ &= 8n\bar{u} + O(1) = \frac{4n}{\pi} + O(1).\end{aligned}$$

Then

$$\sum_{j\in J^c(\pi/2)}\kappa^t(j,m) = \frac{1}{2\pi^2n\bar{u}} + O\big(\delta n^{-1} + n^{-2} + \|E^t\|_\infty\big).$$

Let us insert all the estimates above to (9.7). We end up with

$$\begin{aligned}\sum_{j=1}^{2n}\kappa^t(j,m)E_j^t &\le O(\delta n^{-4} + \delta n^{-1}\|E^t\|_\infty) + \|E^t\|_\infty\left(1 - \frac{1}{\pi^2n\bar{u}} + O(\delta n^{-1} + n^{-2} + \|E^t\|_\infty)\right)\\ &\quad + O\big(\delta n^{-1}\|E^t\|_\infty + n^{-3/2}\|E^t\|_\infty + \log n\|E^t\|_\infty^2\big)\\ &= \left(1 - \frac{1}{\pi^2n\bar{u}}\right)\|E^t\|_\infty + O\big(\delta n^{-4} + \delta n^{-1}\|E^t\|_\infty + n^{-3/2}\|E^t\|_\infty + \log n\|E^t\|_\infty^2\big).\end{aligned}$$

This is exactly (9.4) if we recall that $\sigma = \frac{2}{\pi^2\bar{u}}$. □

Now we are ready to finish the proof of Theorem 1.1.

Proof of Theorem 1.1 Let us apply (9.4) to the estimate (9.2) and get

$$\|E^{t+\Delta t}\|_\infty \le \left(1 - \frac{\sigma}{2n}\right)\|E^t\|_\infty + C_1\delta(t)\left(n^{-1}\|E^t\|_\infty + n^{-5/2}\right)$$

$$+ C_2 \left(n^{-3/2} \|E^t\|_\infty + (\log n)^2 \|E^t\|_\infty^2 + n(\log n)^2 \|E^t\|_\infty^3 \right),$$

where C_1 and C_2 depend only on u_0. Given (3.1), (3.2), and (3.4), we can find a constant $C_3 = C_3(u_0)$ such that for all $n \geq n_0(\tilde{u}_0)$ and all t,

$$\begin{aligned} & C_1 \delta(t) \leq \frac{C_3}{2} e^{-\sigma t}, \\ & C_2 \left(n^{-3/2} + (\log n)^2 \|E^t\|_\infty + n(\log n)^2 \|E^t\|_\infty^2 \right) \leq \frac{C_3}{2} n^{-1-\epsilon/2}. \end{aligned}$$

Then we have that for all sufficiently large $n \geq n_0(\tilde{u}_0)$

$$\|E^{t+\Delta t}\|_\infty \leq \left(1 - \frac{\sigma}{2n} + \frac{C_3 e^{-\sigma t}}{2n} + \frac{C_3}{2n^{1+\epsilon/2}} \right) \|E^t\|_\infty + \frac{C_3 e^{-\sigma t}}{n^{5/2}} \tag{9.8}$$

for all t. By iterating (9.8), we find that

$$\begin{aligned} \|E^t\|_\infty \leq Z_0 n^{-1-\epsilon} & \prod_{i=0}^{2nt-1} \left(1 - \frac{\sigma}{2n} + \frac{C_3 e^{-\frac{\sigma i}{2n}}}{2n} + \frac{C_3}{2n^{1+\epsilon/2}} \right) \\ & + \frac{C_3}{n^{5/2}} \sum_{i=0}^{2nt-1} e^{-\frac{\sigma i}{2n}} \prod_{s=i+1}^{2nt-1} \left(1 - \frac{\sigma}{2n} + \frac{C_3 e^{-\frac{\sigma s}{2n}}}{2n} + \frac{C_3}{2n^{1+\epsilon/2}} \right) \\ & \leq Z_0 n^{-1-\epsilon} e^{-\sigma t + C_3 \left(\frac{1}{\sigma} + \frac{t}{n^{\epsilon/2}} \right)} + \frac{C_3}{n^{5/2}} \sum_{i=0}^{2nt-1} e^{-\frac{\sigma i}{2n}} \cdot e^{-\sigma \left(t - \frac{i+1}{2n} \right) + \frac{C_3}{\sigma} + \frac{C_3}{n^{\epsilon/2}} \left(t - \frac{i+1}{2n} \right)} \\ & \leq C \left(Z_0 n^{-1-\epsilon} + n^{-3/2} t \right) e^{\left(-\sigma + \frac{C_3}{n^{\epsilon/2}} \right) t}. \end{aligned} \tag{9.9}$$

where C is a constant that only depends on u_0, and (9.9) holds for all $n \geq n_0(\tilde{u}_0)$ and all t while (3.4) remains in force. In fact, we can compute explicitly that $C = (2 + 4C_3) e^{\frac{C_3}{\sigma}}$ would work. This is exactly (1.4).

Now we can go back and note that given u_0 and the initial fit controlled by $Z_0 n^{-1-\epsilon}$, we can define $Z_{\max}$ in (3.4) to be equal to

$$\max(C(u_0) Z_0, C(u_0)),$$

where $C(u_0)$ is from (9.9). With this definition of $Z_{\max}$ at hand, we can determine the threshold value $n_0(\tilde{u}_0)$ such that for all $n \geq n_0(\tilde{u}_0)$, (9.9) holds for all t. □

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