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Splitting Brauer classes using the universal Albanese

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Abstract. We prove that every Brauer class over a field splits over a torsor under an abelian variety. If the index of the class is not congruent to 2 modulo 4, we show that the Albanese variety of any smooth curve of positive genus that splits the class also splits the class, and there exist many such curves splitting the class. We show that this can be false when the index is congruent to 2 modulo 4, but adding a single genus 1 factor to the Albanese suffices to split the class.

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1. Introduction

Our main goal in this note is to construct principal homogeneous spaces (often called torsors) for abelian varieties that split Brauer classes over fields. Before formulating the main theorem, we recall some relevant definitions.

The Brauer group $\mathrm{Br}(K)$ of a field K is a classical invariant that may be described in any of the following equivalent ways:

(1) the set of isomorphism classes of central division algebras over K

(2) the group of Morita-equivalence classes of central simple K-algebras (i.e., Azumaya algebras over K), under the tensor product operation

(3) the Galois cohomology group $\mathrm{H}^2(\mathrm{Gal}(K^s/K), (K^s)^*)$, where K^s is the separable closure of K

(4) the set of isomorphism classes of Brauer–Severi varieties over K (a K-variety is *Brauer–Severi* if it is isomorphic to some projective space over $\overline{K}$).

The equivalence between these four interpretations is nontrivial, see, e.g., [9] for more details. Generalizing the perspective in (2), Grothendieck defined the Brauer group $\mathrm{Br}(X)$ of a scheme X as the group of Morita-equivalence classes of Azumaya algebras over X; there are analogues of (3) and (4) in this setting as well. In particular, for

a quasi-projective variety X over a field K, one can identify $\mathrm{Br}(X)$ as the torsion subgroup of the etale cohomology group $\mathrm{H}^2(X, \mathbf{G}_m)$, and if X is smooth, then in fact $\mathrm{Br}(X) \cong \mathrm{H}^2(X, \mathbf{G}_m)$ since $\mathrm{H}^2(X, \mathbf{G}_m)$ is torsion. See [10–12] for a thorough treatment of Brauer groups of schemes.

For a variety X over a field K and a Brauer class $\alpha \in \mathrm{Br}(K)$, we let $\alpha_X \in \mathrm{Br}(X)$ denote the pullback of α to $\mathrm{Br}(X)$ via the structure map $X \to \operatorname{Spec} K$. In particular, note that $\mathrm{Br}(-)$ is a contravariant functor via pullback of Azumaya algebras (or in the quasi-projective case, via pullback of cohomology classes). We say that X *splits* α if α_X is trivial. If X is regular, then this is equivalent to the existence of a rational map from X to any Brauer–Severi variety with cohomology class α.

Note that a K-variety with a rational K-point does not split any nontrivial Brauer class. In particular, since an abelian variety over K has at least one K-point, it cannot split nontrivial Brauer classes; however, a nontrivial torsor (i.e., a principal homogeneous space) under an abelian variety does not have rational points.[1] We thus may ask whether torsors under abelian varieties split a given Brauer class. Our main theorem is to answer this question in the affirmative:

Theorem 1.0.1. *Given a field K and a Brauer class $\alpha \in \mathrm{Br}(K)$, there exists a torsor T under an abelian variety over K such that $\alpha_T = 0$. Equivalently, the torsor T admits a rational map to any Brauer–Severi variety V associated to α.*

In fact, we show that there are many such torsors splitting a given Brauer class by studying Albanese torsors. In general, the existence of the Albanese (variety, torsor, and morphism) of a geometrically integral variety X over any field K is due to Wittenberg [25, Appendix A] extending Serre's results [22] for algebraically closed fields (and thus by Galois descent, perfect fields). For the purposes of this paper, however, we need only the Albanese torsor and morphisms for nice curves. For a smooth proper geometrically connected curve C over K, let $C \to \mathrm{Alb}_C := \mathrm{Pic}^1_{C/K}$ denote the Albanese morphism for C (taking a point on C to the corresponding degree 1 divisor), so the Albanese torsor Alb_C is clearly a torsor under the Jacobian variety $\mathrm{Jac}_C := \mathrm{Pic}^0_{C/K}$.

In the following theorem, we split into cases based on the index of the Brauer class; recall that the *index* of $\alpha \in \mathrm{Br}(K)$ is the degree of a division algebra that represents the class.

[1] Recall that a *torsor* T over K for a group variety A is a K-variety T with an A-action that is simply transitive over $\overline{K}$. For example, a proper smooth geometrically connected genus one curve C is a torsor under its Jacobian elliptic curve. If T has a K'-point for any field $K' \supseteq K$, then it is isomorphic to A over K' via the A-action on such a point.

Theorem 1.0.2. *Let K be a field and $\alpha \in \mathrm{Br}(K)$. Write $\mathrm{ind}(\alpha)$ for the index of α.*

(1) *If $\mathrm{ind}(\alpha) \not\equiv 2 \pmod 4$, then for any smooth proper geometrically connected curve C over K of positive genus such that $\alpha_C = 0$, we have that $\alpha_{\mathrm{Alb}_C} = 0$.*

(2) *If $\mathrm{ind}(\alpha) \equiv 2 \pmod 4$, then for any smooth proper geometrically connected curve C over K of positive genus such that $\alpha_C = 0$, there exists a genus 1 curve C' over K such that $\alpha_{C' \times \mathrm{Alb}_C} = 0$.*

Note that there are many curves splitting any given Brauer class α of index m. For example, for $m \geq 3$, any complete intersections of $m-2$ sections of the anticanonical sheaf of an associated $(m-1)$-dimensional Brauer–Severi variety will split the class, and a general such complete intersection is a smooth proper geometrically connected curve by Bertini's theorem. (This argument needs to be slightly tweaked if K is finite, but in that case the theorems above are trivial since $\mathrm{Br}(K) = 0$.) Thus, Theorem 1.0.1 follows immediately from Theorem 1.0.2.

This result grew out of considering the following well-known question.

Question 1.0.3. Given a field K and a Brauer class $\alpha \in \mathrm{Br}(K)$, is there a genus 1 curve C over K such that $\alpha_C = 0$?

This question was asked explicitly by Pete L. Clark on his website and in [21], and an affirmative answer was given when α has index 3 by Swets [24], index at most 5 in [5], and index 6 under some assumptions on K by Auel [2]. The techniques in [2,5] are unlikely to generalize to higher index, however, and the general question seems quite difficult. The question has garnered much interest lately, partly because of the simplicity; understanding which fields or varieties split Brauer classes is fundamental to understanding Brauer groups, and genus one curves are perhaps the simplest nontrivial positive-dimensional cases to consider.

To prove Theorem 1.0.2, we use the basic theory of big monodromy to deform a curve splitting the class α to a curve whose Jacobian has minimal Néron–Severi group. When the Néron–Severi group is minimal, it is relatively easy to compare obstruction classes for sections of the Picard scheme of the curve with sections of the Picard scheme of its Albanese, giving the result for the general curve. Specializing back to the Albanese of the original curve finishes the proof.

Assumption 1.0.4. Since the Brauer group of a finite field is trivial, we assume from now on that K is an infinite field.

2. The Jacobian of the general curve

Our overarching goal in this section is to describe a proof of the following folk theorem. Write $\mathcal{M}_g$ for the stack of smooth proper geometrically connected curves of genus $g \geq 2$. This is a finite type separated smooth Deligned–Mumford algebraic stack over $\operatorname{Spec} \mathbf{Z}$ with irreducible geometric fibers (see, e.g., [6]).

Proposition 2.0.1. *Suppose C is a smooth proper geometrically connected curve of genus $g \geq 2$ over an algebraically closed field k such that the induced map $\operatorname{Spec} k \to \mathcal{M}_g$ sends the point of $\operatorname{Spec} k$ to the generic point of a fiber of $\mathcal{M}_g \to \operatorname{Spec} \mathbf{Z}$. Then we have that $\mathrm{NS}(\mathrm{Jac}_C) = \mathbf{Z}\Theta$, where Θ is the class of the Θ-divisor on Jac_C.*

The proof can be achieved using the theory of big monodromy in the Hodge or ℓ-adic context. Since we assume that the reader may not be intimately familiar with the theory, we briefly sketch both arguments. The ℓ-adic proof (Corollary 2.3.3) applies in all characteristics, while the Hodge-theoretic proof (Corollary 2.4.3) only applies in characteristic 0. (For Theorem 1.0.2, the characteristic 0 case suffices.) We also refer the reader to another exposition via ℓ-adic methods by Moonen in the appendix to [3]. There is also a direct proof using results of Zarhin [26,27] (building on earlier work of Mori); see Section 2.5.

2.1. Representation theory and notation. We will use the following notation and results in this section.

(1) Given a ring R, we write $\mathrm{Sp}(2g, R)$ for the symplectic group associated to the standard symplectic form of dimension $2g$. We write $\mathrm{GL}(n, R)$ for the algebraic general linear group over R (not just the R-points). Given a field L and an L-vector space V, we will write $\mathrm{GL}(V)$ for the algebraic group of automorphisms of V (not just the L-points). This is non-canonically isomorphic to $\mathrm{GL}(\dim_L V, L)$.

(2) Given an abstract group π and a representation $\rho\colon \pi \to \mathrm{GL}(n, L)$ over a field L, we will write $G(\rho) \subset \mathrm{GL}(n, L)$ for the connected component of the identity of the Zariski closure of the image of ρ. If $\mathbf{V} \to B$ is a local system of L-vector spaces on a topological space, we will write $G(\mathbf{V})$ for $G(\rho)$, where $\rho\colon \pi_1(B, b) \to \mathrm{GL}(V)$ is the monodromy representation attached to $\mathbf{V}$ and $V = \mathbf{V}_b$.

(3) Let V be a vector space over a field L of dimension $2g$ equipped with the standard symplectic form. The pairing defines an $\mathrm{Sp}(2g, L)$-invariant map $\bigwedge^2 V \to L$. The kernel $V_2 \subset \bigwedge^2 V$ of the pairing map is an absolutely irreducible representation of $\mathrm{Sp}(2g, L)$; see [8, Theorem 17.5].

2.2. The Néron–Severi sheaf. Suppose $X \to \operatorname{Spec} k$ is a smooth proper geometrically connected variety over a field k. (The theory we describe here generalizes, but we

avoid such a digression.) Let $\mathrm{Pic}_{X/k}$ be the Picard scheme of X over k and $\mathrm{Pic}^0_{X/k}$ the connected component of the identity. When X is a curve, the Jacobian variety Jac_X is identified with $\mathrm{Pic}^0_{X/k}$.

Definition 2.2.1. The *Néron–Severi sheaf* of X is the fppf quotient sheaf

$$\mathrm{NS}_{X/k} = \mathrm{Pic}_{X/k} / \mathrm{Pic}^0_{X/k}.$$

The sheaf $\mathrm{NS}_{X/k}$ is representable by an étale group scheme over k. Indeed, it suffices to prove this when k is algebraically closed. The sheaves $\mathrm{Pic}_{X/k}$ and $\mathrm{Pic}^0_{X/k}$ are representable (see, e.g., [14, Corollary 4.18.3]), so the quotient is also representable. Moreover, the tangent space to the quotient sheaf at the identity section is trivial, since $\mathrm{Pic}^0_{X/k} \to \mathrm{Pic}_{X/k}$ gives an isomorphism of tangent spaces at the identity. It follows that $\mathrm{NS}_{X/k}$ is an étale group scheme. The classical Néron–Severi group $\mathrm{NS}(X)$ is defined as $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. With this notation, we see that $\mathrm{NS}(X) = \mathrm{NS}_{X/k}(k)$ if k is algebraically closed.

Observation 2.2.2. Since $\mathrm{NS}_{X/k}$ is étale, we have that for any extension $L \subset L'$ of separably closed extension fields of k, the induced map $\mathrm{NS}_{X/k}(L) \to \mathrm{NS}_{X/k}(L')$ is an isomorphism. In particular, given a separable closure k^s contained in an algebraic closure $\overline{k}$ of k, we have $\mathrm{NS}_{X/k}(k^s) \cong \mathrm{NS}_{X/k}(\overline{k})$. In particular, any Néron–Severi class on $X_{\overline{k}}$ is defined over some finite separable extension of k. Under the additional assumption that $\mathrm{Pic}^0_{X/k}$ is smooth (e.g., for X a curve or an abelian variety), we have that any Néron–Severi class defined over k^s is induced by an invertible sheaf on $X \otimes_k k^s$. (Indeed, in this case the fppf and étale cohomology of $\mathrm{Pic}^0_{X/k}$ agree by Grothendieck's theorem [12, Théorème 11.7], so we have that $\mathrm{H}^1_{\mathrm{fppf}}(\mathrm{Spec}\, k^s, \mathrm{Pic}^0_{X/k}) = 0$.)

2.3. Big monodromy: ℓ-adic realization. We first show that the Néron–Severi group of the Jacobian of the geometric generic fiber for a curve is generated by the class of the Θ-divisor if the relevant Galois representation has large image. We then use results of Katz–Sarnak to find families of curves over finite fields with large monodromy.

Proposition 2.3.1. *Let k be any field. Fix a prime ℓ invertible in k and assume k contains all ℓ-power roots of unity, and fix an isomorphism $\overline{\mathbf{Q}}_\ell \cong \overline{\mathbf{Q}}_\ell(1)$ of Galois modules. Let C over k be a smooth proper geometrically connected curve of genus g such that the identity component of the Zariski closure of the image of the Galois representation*

$$\rho_0 : \mathrm{Gal}(k^s/k) \to \mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$$

induced by the Galois action on $\mathrm{H}^1(C_{\overline{k}}, \overline{\mathbf{Q}}_\ell)$ is all of $\mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$. Then the base change $\overline{C} := C_{k^s}$ (and hence also the geometric fiber) has the property that $\mathrm{NS}(\mathrm{Jac}_{\overline{C}}) = \mathbf{Z}\Theta$.

Proof. The first Chern class defines a morphism

$$c : \mathrm{NS}(\mathrm{Jac}_{\overline{C}}) \to \mathrm{H}^2(\mathrm{Jac}_{\overline{C}}, \overline{\mathbf{Q}}_\ell),$$

which is injective (modulo torsion, but the Néron–Severi group for abelian varieties is torsion-free). (Note that we may ignore Tate twists in this proof since we fixed an isomorphism $\overline{\mathbf{Q}}_\ell \cong \overline{\mathbf{Q}}_\ell(1)$ above.) Because any class of $\mathrm{NS}(\mathrm{Jac}_{\overline{C}})$ is defined over a finite separable extension of k by Observation 2.2.2, the image of c is contained in the union of all the subspaces $\mathrm{H}^2(\mathrm{Jac}_{\overline{C}}, \overline{\mathbf{Q}}_\ell)^\Gamma$, where Γ ranges over open subgroups of the absolute Galois group $G_k := \mathrm{Gal}(k^s/k)$.

By assumption $G(\rho_0) = \mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$, but also for any open subgroup $\Gamma \subset G_k$, the group $G(\rho_0|_\Gamma)$ is $\mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$, since passage to finite index subgroups does not change the identity component of the Zariski closure.

The cup product pairing defines a Galois-invariant map

$$p : \mathrm{H}^2(\mathrm{Jac}_{\overline{C}}, \overline{\mathbf{Q}}_\ell) = \bigwedge^2 \mathrm{H}^1(\mathrm{Jac}_{\overline{C}}, \overline{\mathbf{Q}}_\ell) = \bigwedge^2 \mathrm{H}^1(\overline{C}, \overline{\mathbf{Q}}_\ell) \to \overline{\mathbf{Q}}_\ell$$

with kernel V. For an open subgroup $\Gamma \subset G_k$, the space $\mathrm{H}^2(\mathrm{Jac}_{\overline{C}}, \overline{\mathbf{Q}}_\ell)^\Gamma \cap V$ is a subspace stable under Γ, hence under $G(\rho_0|_\Gamma) = \mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$. Since V is an irreducible representation of $\mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$ by Section 2.1(3), the intersection must be 0.

We thus find that the composition map

$$\mathrm{NS}(\mathrm{Jac}_{\overline{C}}) \otimes \overline{\mathbf{Q}}_\ell \xrightarrow{c} \bigcup_{\Gamma \subset G_k} \mathrm{H}^2(\mathrm{Jac}_{\overline{C}}, \overline{\mathbf{Q}}_\ell)^\Gamma \hookrightarrow \mathrm{H}^2(\mathrm{Jac}_{\overline{C}}, \overline{\mathbf{Q}}_\ell) \xrightarrow{p} \overline{\mathbf{Q}}_\ell$$

is an isomorphism (since $\mathrm{NS}(\mathrm{Jac}_{\overline{C}})$ is not 0).

As a consequence, we have an injection $\mathrm{NS}(\mathrm{Jac}_{\overline{C}}) \hookrightarrow \mathrm{NS}(\overline{C})$. It is well known that the pullback of the Θ-divisor class to $\overline{C}$ has degree g (see, e.g., [20, Theorem 17.4]), and since Θ is a principal polarization, it is indivisible in $\mathrm{NS}(\mathrm{Jac}_{\overline{C}})$. We therefore have $\mathrm{NS}(\mathrm{Jac}_{\overline{C}}) = \mathbf{Z}\Theta$; furthermore, the injection $\mathrm{NS}(\mathrm{Jac}_{\overline{C}}) \hookrightarrow \mathrm{NS}(\overline{C})$ is identified with $\mathbf{Z} \to g\mathbf{Z} \hookrightarrow \mathbf{Z}$. ∎

In [13, Chapter 10], Katz and Sarnak produce families of curves over finite fields with large monodromy groups. Recall that, given a family of curves $f : C \to B$ over a finite field $\mathbf{F}_q$ with ℓ an invertible prime, the *geometric monodromy* group G_{geom} of f is the $\overline{\mathbf{Q}}_\ell$-algebraic group $G(\rho)$ associated to the representation

$$\rho : \pi_1(B \otimes_{\mathbf{F}_q} \overline{\mathbf{F}_q}) \to \mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$$

attached to the lisse sheaf $\mathbf{R}^1 f_* \overline{\mathbf{Q}}_\ell$ (as in Section 2.1(2)). The following theorem is a summary of [13, Theorem 10.1.16 and Theorem 10.2.2].

Theorem 2.3.2 (Katz–Sarnak). *For any genus g and any finite field $\mathbf{F}_q$ with ℓ an invertible prime, there is an open subset $U \subset \mathbf{A}^1_{\mathbf{F}_q}$ and a family $\mathcal{C} \to U$ of smooth proper geometrically connected genus g curves such that $G_{\mathrm{geom}} = \mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$.*

Corollary 2.3.3. *Let $\mathcal{C} \to U$ be a family as in Theorem 2.3.2. The geometric generic fiber $\overline{C} := \mathcal{C} \times_U \operatorname{Spec} \overline{\mathbf{F}_q(t)}$ has the property that $\mathrm{NS}(\mathrm{Jac}_{\overline{C}}) = \mathbf{Z}\Theta$.*

Proof. Let $k = \overline{\mathbf{F}_q(t)}$. Theorem 2.3.2 gives that $G_{\mathrm{geom}} = G(\rho) = \mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$, and the natural surjection $G_k \cong \pi_1(\operatorname{Spec} k) \to \pi_1(U \otimes_{\mathbf{F}_q} \overline{\mathbf{F}_q})$ implies that the composition map $\rho_0 : G_k \to \pi_1(U \otimes_{\mathbf{F}_q} \overline{\mathbf{F}_q}) \to \mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$ also has the property that $G(\rho_0) = \mathrm{Sp}(2g, \overline{\mathbf{Q}}_\ell)$. We thus apply Proposition 2.3.1 to obtain the desired result. ∎

2.4. Big monodromy: Hodge realization. We give a briefer sketch of the Hodge version of the argument, as it seems to be more widespread in the literature. For example, [4, Theorem 17.5.2] and the discussion leading up to it are a valuable source.

Let $\mathbf{V}$ be a local system on a connected space B with monodromy representation $\rho\colon \pi_1(B, b) \to \mathrm{GL}(V)$, where $V = \mathbf{V}_b$. Recall that $\mathbf{V}$ is said to have *big monodromy* if $G(\mathbf{V})$ acts irreducibly on $V_{\mathbf{C}}$.

Given a family of principally polarized abelian varieties $g\colon A \to B$, the polarization defines a quotient sheaf $\mathbf{R}^2 g_* \mathbf{Q} = \bigwedge^2 \mathbf{R}^1 g_* \mathbf{Q} \to \underline{\mathbf{Q}}$. The kernel of this map is a local system $\mathbf{V}_2(A)$. Fiberwise, it is invariant under the symplectic group and is itself an irreducible representation (Section 2.1(3)). We will say that $g\colon A \to B$ has *big monodromy for* H^2 if $\mathbf{V}_2(A)$ has big monodromy.

Lemma 2.4.1. *Let B be a smooth $\mathbf{C}$-scheme. If $f : X \to B$ is a family of smooth proper curves with $G(\mathbf{R}^1 f_* \mathbf{Q}) = \mathrm{Sp}(2g, \mathbf{Q})$, then the Jacobian family $g\colon \mathrm{Jac}_X = \mathrm{Pic}^0_{X/B} \to B$ has big monodromy for both H^1 and H^2.*

Proof. Recall that $\mathbf{R}^1 g_* \mathbf{Q} = \mathbf{R}^1 f_* \mathbf{Q}$ as local systems, so we can identify $\mathbf{R}^2 g_* \mathbf{Q}$ with $\bigwedge^2 \mathbf{R}^1 f_* \mathbf{Q}$. The rest follows from the definitions. ∎

Corollary 2.4.2. *Suppose B is a smooth $\mathbf{C}$-scheme. If $f : X \to B$ is a smooth proper family of curves with $G(\mathbf{R}^1 f_* \mathbf{Q}) = \mathrm{Sp}(2g, \mathbf{Q})$, then for any very general $b \in B(\mathbf{C})$, the Néron–Severi group $\mathrm{NS}(\mathrm{Jac}_{X_b})$ is isomorphic to $\mathbf{Z}$ and generated by the theta divisor Θ.*

Proof. As the Néron–Severi group for abelian varieties is torsion-free, it is enough to show the result after tensoring with $\mathbf{Q}$. For any point $b \in B(\mathbf{C})$, the Néron–Severi group $\mathrm{NS}(\mathrm{Jac}_{X_b}) \otimes \mathbf{Q}$ is a trivial $\mathbf{Q}$-Hodge structure contained in the $\mathbf{Q}$-Hodge structure $\mathrm{H}^2(\mathrm{Jac}_{X_b}, \mathbf{Q})$. (In fact, by the Lefschetz (1,1) theorem, it is the maximal such structure.)

By [19, Theorem 10.20], for a very general point b, the sub-Hodge structure $(\mathbf{V}_2)_b \subset \mathrm{H}^2(\mathrm{Jac}_{X_b}, \mathbf{Q})$ has no rational Hodge substructures. By the exact sequence

$$0 \to (\mathbf{V}_2)_b \to \mathrm{H}^2(\mathrm{Jac}_{X_b}, \mathbf{Q}) \to \mathbf{Q} \to 0,$$

it follows that $\mathrm{NS}(\mathrm{Jac}_{X_b}) \otimes \mathbf{Q}$ is one-dimensional. ∎

Corollary 2.4.3. *Let k be a field of characteristic 0 and $X \to \operatorname{Spec} k$ be a smooth proper geometrically connected curve such that the image of the induced map $\operatorname{Spec} k \to \mathcal{M}_g$ is the generic point. Then we have that $\mathrm{NS}(\mathrm{Jac}_{X_{\overline{k}}}) = \mathbf{Z}\Theta$.*

Proof. The statement is invariant under extension of k. It is well known that for the universal curve over $\mathcal{M}_g$ the geometric monodromy group attached to H^1 is $\mathrm{Sp}(2g)$; see, e.g., [6, §5] or [7, Theorem 6.4]. Thus, if m is a very general complex point of $\mathcal{M}_g$ corresponding to a curve C, then by Corollary 2.4.2 we have that $\mathrm{NS}(\mathrm{Jac}_C) = \mathbf{Z}\Theta$. Since Néron–Severi groups can only grow under specialization, it follows that the geometric generic point must have the same property, giving the desired result. ∎

2.5. Appeal to example. Since Néron–Severi rank can only increase under specialization, another proof of Proposition 2.0.1 follows from showing that there exists a single curve of every genus at least 2, in any characteristic, whose Jacobian has Néron–Severi rank exactly 1. Results of Zarhin [26, 27] on endomorphism rings of hyperelliptic Jacobians imply that many such curves exist over most characteristics; in fact, Zarhin shows that for any hyperelliptic curve $C : y^2 = f(x)$, where $f(x)$ is an irreducible separable degree $n \geq 5$ polynomial with Galois group either S_n or A_n, in any characteristic $p > 3$, the endomorphism ring over the algebraic closure is $\mathbf{Z}$, implying that $\mathrm{NS}(\mathrm{Jac}_C)$ also has rank 1. These results themselves are quite subtle and rely on very different techniques than those sketched in this paper.

2.6. General deformations of smoothable curves. In this section we describe how to put any smoothable curve in a family with the generic curve. This will be useful for studying the splitting of Brauer classes, as we explain in Section 3. (We will only apply this to smooth curves, but we suspect that the full statement for smoothable curves may be useful in the future, so we record it here.)

Corollary 2.6.1. *Suppose C is a smoothable proper geometrically connected curve over a field K such that $\mathrm{Ext}^2(L_{C/K}, \mathcal{O}_C) = 0$ (for example, a proper nodal curve). Let W be a complete dvr with residue field K. There is a proper flat family $\mathcal{C} \to \operatorname{Spec} W$ such that*

(1) $\mathcal{C} \otimes_W K \cong C$, *and*

(2) *if* $\eta \to \operatorname{Spec} W$ *is a geometric generic point, then we have* $\mathrm{NS}(\mathrm{Jac}_{\mathcal{C}_\eta}) = \mathbf{Z}\Theta$, *where* Θ *is the usual theta-divisor class associated to* $\mathcal{C}_\eta$.

Proof. Write Q for the fraction field of W. The assumptions on C ensure that the universal formal deformation of C is represented by a proper morphism of schemes $\mathfrak{C} \to \operatorname{Spec} W[\![x_1, \ldots, x_n]\!]$ with a smooth fiber. Since the stack of proper curves is an Artin stack locally of finite presentation, Artin's algebraization theorem tells us that there is a pair (X, x) with X a smooth W-scheme and $x \in X(K)$ and a family $\mathcal{C} \to X$ such that the restriction of $\mathcal{C} \to X$ to $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to the universal deformation. Let X° denote the locus over which $\mathcal{C}$ is smooth and let $\mathbf{F}$ be the prime field of Q. There is an induced map $\mu\colon X^\circ \to \mathcal{M}_{g,\mathbf{F}}$, and by the openness of versality we know that this map is dominant.

By Proposition 2.0.1, it suffices to show that there is a map $\operatorname{Spec} W \to X$ whose closed point lands at x and whose generic point maps to the generic point of $\mathcal{M}_{g,\mathbf{F}}$. Since $\mathbf{F}$ is countable, $\mathcal{M}_{g,\mathbf{F}}$ has only countably many closed substacks. Consider the polycylinder $B^n = \{(a_0, \ldots, a_n) \mid a_i \in Q, |a_i| < 1\}$ parametrizing all W-points of $\widehat{\mathcal{O}}_{X,x}$. Since μ is dominant, no closed subscheme of $\mathcal{M}_{g,\mathbf{F}}$ contains all of B^n. Since Q is uncountable it follows (e.g., by induction on n) that there is a point of B^n not in the pullback of any closed substack of $\mathcal{M}_{g,\mathbf{F}}$. This gives a W-point of X with the desired properties. ∎

3. Proof of Theorem 1.0.2

Lemma 3.0.1. *Suppose C is a smooth proper geometrically connected curve of genus g over a field K and $C \to X$ is an Albanese morphism. Suppose further that the morphism $\mathrm{NS}(X_{\overline{K}}) \to \mathrm{NS}(C_{\overline{K}})$ is injective with image $g\,\mathrm{NS}(C_{\overline{K}})$. If $\alpha \in \mathrm{Br}(K)$ is a class of order prime to g such that $\alpha_C = 0$, then $\alpha_X = 0$.*

Proof. By the Leray spectral sequence, we have $\alpha_C = 0$ if and only if α is the obstruction class for a global section of $\mathrm{Pic}_{C/K}$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0_{X/K} & \longrightarrow & \mathrm{Pic}_{X/K} & \longrightarrow & \mathrm{NS}_{X/K} & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0_{C/K} & \longrightarrow & \mathrm{Pic}_{C/K} & \longrightarrow & \mathrm{NS}_{C/K} & \longrightarrow & 0 \end{array}$$

of sheaves on $\operatorname{Spec} K$. The Snake Lemma applied to the diagram yields an exact sequence of sheaves

$$0 \to \mathrm{Pic}_{X/K} \to \mathrm{Pic}_{C/K} \to \mathbf{Z}/g\mathbf{Z} \to 0.$$

Thus, the map on global sections

$$\mathrm{Pic}_{X/K}(K) \to \mathrm{Pic}_{C/K}(K)$$

is injective with cokernel annihilated by g. Since g is prime to the order of α and the obstruction map for sections of the Picard scheme is a group homomorphism, we see that $\alpha_X = 0$ if and only if $g\alpha_X = 0$. If $s \in \mathrm{Pic}_{C/K}(K)$ is a section with obstruction α, then the preimage of gs in $\mathrm{Pic}_{X/K}(K)$ is a section with obstruction $g\alpha$, and the desired result follows. ∎

Proof of Theorem 1.0.2. Suppose C is a smooth proper geometrically connected curve over K of genus $g \geq 1$ such that $\alpha_C = 0$. If $g = 1$, the desired conclusion holds because $\mathrm{Alb}_C = C$. Thus, we assume $g \geq 2$. We now show that to prove the theorem, it suffices to prove that Alb_C splits α under the additional assumption that g is relatively prime to $\mathrm{ind}(\alpha)$ – and therefore relatively prime to the order of α, which is also known as the *period* of α (since the period and the index of α have the same prime factors [18, Lemma 2.1.1.3]).

Observe that $\mathrm{ind}(\alpha)$ divides $2g - 2 = 2(g - 1)$, since the canonical divisor of C has degree $2g - 2$. As a result, all odd divisors of $\mathrm{ind}(\alpha)$ visibly cannot divide g, and if $4 | \mathrm{ind}(\alpha)$, then g is odd. Hence, if $\mathrm{ind}(\alpha) \not\equiv 2 \pmod 4$, we find that g is relatively prime to $\mathrm{ind}(\alpha)$. If $\mathrm{ind}(\alpha) \equiv 2 \pmod 4$, then we can write $\alpha = \alpha_2 + \alpha'$ with α_2 of index 2 and α' of odd order. Since α_2 is split by a conic (namely, the Brauer–Severi variety of the associated division algebra) and any conic admits a cover by a genus 1 curve (namely, the branched cover over a general divisor of degree 4), we see upon taking products that it suffices to prove that Alb_C splits α' (which has odd index relatively prime to g).

Let W be a complete dvr with residue field K. By [10, Theorem 6.1], since W is Henselian (being complete), the natural restriction map induces a bijection from the set of isomorphism classes of Azumaya algebras of degree r over W to the corresponding set over K. It follows that there is a unique Brauer class $\widetilde{\alpha} \in \mathrm{Br}(W)$ lifting α, and that $\widetilde{\alpha}$ has the same period and index as α.

By Corollary 2.6.1, there is a family $\mathcal{C} \to \mathrm{Spec}\, W$ such that the generic fiber $\mathcal{C}_\eta$ of $\mathcal{C}$ satisfies $\mathrm{NS}_{\mathrm{Jac}_{\mathcal{C}_\eta}/\eta} = \mathbf{Z}\Theta$ (as étale sheaves). Since Θ is ample (and thus any descent datum must act trivially on the Néron–Severi sheaf, as it must preserve ample classes), the $\mathrm{Jac}_{\mathcal{C}_\eta}$-torsor structure on $\mathrm{Alb}_{\mathcal{C}_\eta}$ yields a canonical identification of sheaves $\mathcal{C}\ \mathrm{NS}_{\mathrm{Alb}_{\mathcal{C}_\eta}/\eta} = \mathbf{Z}\Theta$. By [20, Theorem 17.4], we see that $\mathcal{C}_\eta$ satisfies the conditions of Lemma 3.0.1.

Since $\alpha_C = 0$, we have by [12, Theorem 3.1] that $\widetilde{\alpha}_{\mathcal{C}_\eta} = 0$. By Lemma 3.0.1, we have that $\widetilde{\alpha}_{\mathrm{Alb}_{\mathcal{C}_\eta}} = 0$. Since $\mathrm{Alb}_{\mathcal{C}/W}$ is regular, it follows (see, for example, [11, Corollary 1.10]) that $\widetilde{\alpha}_{\mathrm{Alb}_{\mathcal{C}/W}} = 0$. Thus, $\alpha_{\mathrm{Alb}_C} = 0$ by specialization.[2] ∎

Proof of Theorem 1.0.1. For any Brauer class $\alpha \in \mathrm{Br}(K)$ of index $m \geq 3$, as explained in the introduction, there are many smooth proper geometrically connected positive genus curves C over K such that $\alpha_C = 0$ in $\mathrm{Br}(C)$. Applying Theorem 1.0.2 to such a curve C gives the desired torsor, either Alb_C itself or a product of Alb_C with a genus one curve. The case where α has index 2 is covered by the construction of a genus one curve splitting such an α, as in [5]. ∎

Antieau and Auel's proof of Theorem 1.0.2. A different proof of Theorem 1.0.2, due to Benjamin Antieau and Asher Auel [1], uses results on the stable birational geometry of symmetric powers of Brauer–Severi varieties to show that an appropriate symmetric power of a curve splitting a Brauer class also splits the class. Here is a sketch of their proof; more details may appear elsewhere in the future.

Suppose C is a smooth proper geometrically connected curve over K of genus $g \geq 1$ such that $\alpha_C = 0$. As in the first proof, we may reduce to showing that Alb_C splits α under the assumptions that α is nontrivial and that g is relatively prime to $\mathrm{ind}(\alpha)$.

The image of C in a Brauer–Severi variety V associated to α cannot be a point, so the image of the induced map $\mathrm{Sym}^{2g-1} C \to \mathrm{Sym}^{2g-1} V$ intersects the smooth locus of $\mathrm{Sym}^{2g-1} V$. By [15, Theorem 1.1 (4)] (see also [16]), the space $\mathrm{Sym}^{2g-1} V$ is stably birational to $\mathrm{Sym}^m V$, where $m = \gcd(2g-1, \mathrm{ind}(\alpha))$; here, we have $m = 1$ since $\mathrm{ind}(\alpha)$ divides $2g-2$. Thus, the smooth locus U of $\mathrm{Sym}^{2g-1} V$ is stably birational to V and splits α, which implies that $\mathrm{Sym}^{2g-1} C$ also splits α.

By the Riemann–Roch theorem, we have that $\sigma\colon \mathrm{Sym}^{2g-1} C \to \mathrm{Pic}^{2g-1}_{C/K} \cong \mathrm{Alb}_C$ is a Brauer–Severi scheme of relative dimension $g-1$. Since $\sigma^* \alpha_{\mathrm{Alb}_C} = 0$, the class α_{Alb_C} is g-torsion. But α_{Alb_C} is also killed by $\mathrm{ind}(\alpha)$, so the assumption that g is relatively prime to $\mathrm{ind}(\alpha)$ implies that $\alpha_{\mathrm{Alb}_C} = 0$, as desired. ∎

4. The conditions of Theorem 1.0.2 are necessary

In this section, we show that there are many examples of Brauer classes α with index congruent to 2 modulo 4 that split on a curve C but not on Alb_C. These examples

[2] Instead of using a very general W-point of the universal deformation ring of C over K, we could just use the universal deformation directly. Our approach avoids enlarging the fraction field of W at the expense of a small amount of extra work.

are easily constructed over local fields (and hence over many finitely generated fields, by standard approximation techniques), and we suspect one could also make similar examples over number fields.

Given a smooth proper geometrically connected curve C over a field K, recall that the *index* of C is the smallest degree of a divisor on C, and the *period* of C is the smallest degree of a divisor class. (Equivalently, the period of C is the order of the Albanese variety $\mathrm{Alb}_C = \mathrm{Pic}^1_{C/K}$ in $\mathrm{H}^1(\mathrm{Spec}\, K, \mathrm{Jac}_C)$.)

Lemma 4.0.1. *Suppose C is a curve over a field K and $\alpha \in \mathrm{Br}(K)$ is a class of order 2. If $\alpha_{\mathrm{Pic}^1_C} = 0$, then $\alpha_{\mathrm{Pic}^m_C} = 0$ for any odd number m.*

Proof. The mth power map on the Picard stack descends to a morphism

$$\lambda_m \colon \mathrm{Pic}^1_C \to \mathrm{Pic}^m_C$$

that is an étale form of the multiplication by m on Jac_C. In particular, λ_m is finite flat of degree m^{2g}, where g is the genus of C. By assumption, the class $\alpha_{\mathrm{Pic}^m_C}$ vanishes upon pullback along the morphism λ_m of odd degree. By standard calculations in Galois cohomology (see, for example, [18, Proposition 4.1.1.1]), this implies that $\alpha_{\mathrm{Pic}^m_C} = 0$, as desired. ∎

Proposition 4.0.2. *Let m be an odd positive integer and suppose C is a smooth proper geometrically connected curve over a local field K of index $2m$, period m, and genus $m+1$. Then the unique non-zero Brauer class $q \in \mathrm{Br}(K)[2]$ is killed by C but not by Alb_C. Thus, there are Brauer classes α of all even indices dividing $2m$ that are killed by C but not by Alb_C.*

Proof. Since m is odd, any class α in $\mathrm{Br}(K)[2m]$ can be written as $q + h$ with $h \in \mathrm{Br}(K)[m]$. The relative Brauer group $\mathrm{Br}(C/K)$ is precisely $\mathrm{Br}(K)[2m]$ by the theorem of Roquette–Lichtenbaum [17, Theorem 3]. In addition, period and index are equal over a local field. Thus, to prove the full statement, it suffices to prove the first part.

From Roquette–Lichtenbaum, we have that $q_C = 0$. If $q_{\mathrm{Alb}_C} = 0$ also, then $q_{\mathrm{Pic}^m_C} = 0$ by Lemma 4.0.1. Since C has period m, there is a K-point of Pic^m_C, and restricting to that point would imply that $q = 0$, which is a contradiction. ∎

By a result of Sharif [23, Theorem 2], for a local field K of characteristic not 2 and for any odd m, there exists a curve C over K of index $2m$, period m, and genus $m+1$. Proposition 4.0.2 then shows that the Albanese of C cannot kill numerous classes in $\mathrm{Br}(C/K)$, implying that the conditions of Theorem 1.0.2 are sharp.

5. Some observations

5.1. Products of genus one curves. One might attempt to answer Question 1.0.3 by first splitting the class on a family of Albanese varieties with a member that splits as a product of genus 1 curves, and hoping that this decomposition will have implications for splitting the class over a factor. As we briefly explain, there are two reasons that this is unlikely to work.

First, the results of Section 4 show that one cannot hope to use only Albanese varieties of curves from the beginning, because there are examples where the decisive role is played by a genus 1 factor added after the fact, whose presence is necessary to split a single quaternion factor of the Brauer class.

Second, we make a simple observation about products: suppose T and T' are genus 1 curves over K, with T of index 2 and T' of index 3. Any Brauer class α killed by T has order 2 and any Brauer class α' killed by T' has order 3. The natural tensoring map $\mathrm{Pic}_T \times \mathrm{Pic}_{T'} \to \mathrm{Pic}_{T\times T'}$ is additive on obstruction classes (since it is equivariant for the multiplication map $\mathbf{G}_m \times \mathbf{G}_m \to \mathbf{G}_m$ of bands for the Picard stacks). Thus, $\alpha + \alpha'$ is killed by $T \times T'$, which is a torsor under $\mathrm{Jac}_T \times \mathrm{Jac}_{T'}$, but $\alpha + \alpha'$ is not killed by either T or T'. (With these choices of indices, under mild hypotheses, Auel's result [2] would contruct a genus one curve that splits $\alpha + \alpha'$, though it would have no relation to T or T'.) In general, if α and α' are any Brauer classes split by genus one curves, it is not known if $\alpha + \alpha'$ is split by a genus one curve.

Examples of both types are easily constructed over local fields.

5.2. The universal Albanese does not do anything on its own. In light of the method used here – that is, splitting Brauer classes by splitting them on particular base changes of the Albanese map of the universal curve – one might be tempted to ask the following question.

Question 5.2.1. Given a field K and a positive integer $g > 2$, let $C \to \mathrm{Spec}\,\kappa(\mathcal{M}_{g,K})$ be the universal curve of genus g over the function field of the stack of all curves of genus g. What is the kernel of the map

$$\mathrm{Br}(K) \to \mathrm{Br}(C)?$$

Proposition 5.2.2. *The kernel of the map of Question 5.2.1 is* 0.

Proof. Over any field K, there is a curve C_0 of genus g with a K-point. Since the universal curve is regular, any class that is trivialized over C is trivialized over any specialization, such as C_0. Further specializing to the K-point shows that the class itself is 0. ∎

6. Some questions

Some natural questions arise from the results we describe here. As mentioned in the introduction, one way to produce curves splitting Brauer classes is as complete intersections of sections of the anticanonical sheaf in a Brauer–Severi variety. On the other hand, if we restrict to a single anticanonical divisor, we obtain Calabi–Yau varieties that split the class. This observation leads to several directions for further exploration.

Question 6.0.1. Is there a fixed positive integer n such that every Brauer class over a field is split by a torsor under an abelian variety of dimension n, independent of the index of the class? (Note that Theorem 1.0.2 applied to complete intersections of anti-canonical divisors gives a torsor of dimension $1 + \frac{1}{2}m^{m-1}(m-3)$ for classes of index m, but this depends on m.)

Question 6.0.2. Is there a fixed positive integer n such that every Brauer class over a field is split by a Calabi–Yau variety of dimension n?

Question 6.0.3. Is every Brauer class over a field split by a K3 surface?

Question 6.0.4. Is every Brauer class over a field split by a curve sitting in a K3 surface?

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