A NON-BOREL SPECIAL ALPHA-LIMIT SET IN THE SQUARE

STEPHEN JACKSON, BILL MANCE, AND SAMUEL ROTH

Abstract. We consider the complexity of special α-limit sets, a kind of backward limit set for non-invertible dynamical systems. We show that these sets are always analytic, but not necessarily Borel, even in the case of a surjective map on the unit square. This answers a question posed by Kolyada, Misiurewicz, and Snoha.

A *topological dynamical system* is a pair (X, f) where X is a compact metric space and $f \in C(X, X)$ is a continuous map of X to itself. Limit sets and backward limit sets provide some of the most important tools in understanding the behavior of a topological dynamical system, since they provide information about the long-term behavior of the orbits of the system. One notion, in particular, of a backward limit set is the notion of a *special α-limit set*, which has played an important role in one-dimensional dynamics (the notion is defined and discussed below).

Descriptive set theory provides a means for calibrating the complexity of sets in Polish spaces, and this in turn is important for understanding the impact of possible theorems about the objects under consideration. To give an example of this phenomenon, consider the case of normal numbers for some base $b \geq 2$. Let $\mathcal{N}_b$ denote the set of real numbers which have a normal base-b expansion. Let $\mathcal{N}_b^\perp$ denote the set of real numbers γ such that $\gamma + N_b \subseteq \mathcal{N}_b$, that is, they preserve base-b normality under addition. By its definition, this set does not at first even appear to be a Borel set as it involves a universal quatification over $\mathcal{N}_b$ (the definition shows it to be a coanalytic, or $\mathbf{\Pi}^1_1$ set; see [18] for the definition). However, a deep theorem of Rauzy characterizes the normality-preserving numbers as those with zero upper noise [20]. Since the upper noise of γ is defined by taking limits (and lim sup's) of sequences of continuous functions of γ, it follows immediately that $\mathcal{N}_b^\perp$ is Borel, in fact a $\mathbf{\Pi}^0_3$ set. In fact, in [1] it was shown that $\mathcal{N}_b^\perp$ is a $\mathbf{\Pi}^0_3$-complete set, which shows that it is no simpler than this.

The present paper shows that the special α-limit set of a point in a topological dynamical system, in fact even for the case of the unit square in $\mathbb{R}^2$, can be a $\mathbf{\Sigma}^1_1$-complete set, and thus not Borel. The significance of this is that it tells us that there are no such "hidden theorems" for special α-limit sets, even in compact metric spaces. Thus, the definition involving an existential quantification over all backward orbits cannot be simplified.

We remark that it is unusual in our experience for a dynamically-defined set to be non-Borel. In fact, most of the important sets studied in a topological dynamical system occur at the lowest few levels of the Borel hierarchy. Examples we can think of include the basin of an attracting cycle (an open set), the non-wandering set (a closed set), the set of periodic points (an F_σ set), or the set of points with a dense orbit (a G_δ set). The reader can surely

2020 *Mathematics Subject Classification.* Primary: 03E15; 37E99 Secondary: 37B10.

Key words and phrases. special alpha limit set, triangular map of the square, non-Borel analytic set.

The first author was supported by NSF grant DMS-1800323. The second author was supported by grant 2019/34/E/ST1/00082 for the project "Set theoretic methods in dynamics and number theory," NCN (The National Science Centre of Poland). The third author was supported by RVO (Czech Republic) funding for IČ47813059.

think of many more examples. On the other hand, a result of Foreman, Rudolph, and Weiss [10] says that the set of invertible, ergodic, measure preserving transformations (on the unit interval with Lebesgue measure) which are isomorphic to their inverses is a complete analytic set. In particular, the isomorphism relation between invertible, ergodic, measure preserving transformations is not Borel. G. Hjorth [16] had previously shown the the isomorphism relation is not Borel for the larger class of invertible measure preserving transformations.

The authors have recently shown that special α-limit sets can in fact occur at any level in the Borel hierarchy, but this will be part of a forthcoming paper.

We now discuss these notions more precisely. Throughout this paper $\mathbb{N} = \{0, 1, \dots\}$ will denote the set of natural numbers. The *ω-limit set* of a point $x \in X$ under a map f, denoted $\omega(x)$ or $\omega(x, f)$, consists of all accumulation points of the *forward orbit* $(f^n(x))_{n\in\mathbb{N}}$ of the point x. These sets are always closed, and their topological properties are well understood, see eg. [2, 3, 5, 6, 17, 21].

The backward limit sets of a point x play a dual role to the ω-limit set. The idea is that to describe the "source" of a trajectory, we should reverse the direction of time. For invertible dynamical systems (homeomorphisms) this leads to a well-defined α-limit set $\alpha(x) = \omega(x, f^{-1})$. This idea came into discrete dynamics from the study of flows, where α-limit sets are fundamental in defining such objects as unstable manifolds, homoclinic and heteroclinic trajectories, and the Morse decompositions at the heart of Conley theory [7, 12].

For non-invertible mappings $f : X \to X$, the situation is more complicated, since a point may have several preimages (or none at all). There are several ways to define backward limit sets in this setting:

(1) The *α-limit set of a point*, denoted $\alpha(x)$, consists of all accumulation points of all *preimage sequences* $(x_n)_{n\in\mathbb{N}}$, where $x_n \in f^{-n}(\{x\})$ for all n. These sets are especially useful in dimension one. Coven and Nitecki proved that a point is non-wandering for an interval map if and only if it belongs to its own α-limit set [9], and Cui and Ding related the α-limit sets of a unimodal interval map to its renormalizations [8].
(2) The *α-limit set of a backward orbit*, denoted $\alpha((x_n)_{n=0}^{\infty})$, consists of all accumulation points of a single *backward orbit*, i.e. a sequence $(x_n)_{n\in\mathbb{N}}$, where $f(x_{n+1}) = x_n$ for all n. Hirsch et al. showed that the α-limit set of a backward orbit is always internally chain transitive [15]. A converse result holds when f is expansive and has the shadowing property [11]. For maps on the interval, the α-limit set of a backward orbit satisfies a local transportation condition which makes it simultaneously an ω-limit set of f [4].
(3) The *special α-limit set of a point*, denoted $s\alpha(x)$, is the union $\bigcup \alpha((x_n)_{n=0}^{\infty})$ taken over all *backward orbits of x*, i.e. sequences $(x_n)_{n\in\mathbb{N}}$ such that $f(x_{n+1}) = x_n$ for all n and $x_0 = x$. These sets were defined by Hero, who showed that for interval maps, a point is in the attracting center if and only if it belongs to its own special α-limit set [14]. Generalizations of this result to other one-dimensional spaces were given in [22, 23].

Kolyada et al. pointed out that special α-limit sets need not be closed, and asked whether they are necessarily Borel or at least analytic [19]. The difficulty arises when x has uncountably many backward orbit branches, since we are then taking an uncountable union of their (closed) accumulation sets.

The complexity of special α-limit sets (or $s\alpha$-limit sets, for short) for maps of the interval $I = [0, 1]$ will be addressed in a forthcoming paper, where it will be shown that $s\alpha(x)$ is always Borel, and in fact both F_σ and G_δ [13].

This paper is concerned with the complexity of $s\alpha$-limit sets in other compact metric spaces. We start in Section 1 by showing that $s\alpha$-limit sets are always analytic. The proof is short

and uses the fact that backward orbits occur in a well-known compact metric space related to (X, f), namely, the natural extension.

The main construction in this paper gives a map on the unit square $I^2 = [0,1]^2$ with a $s\alpha$-limit set which is $\mathbf{\Sigma}^1_1$-complete, i.e. analytic but not Borel. Our construction starts in Section 2 at the symbolic level with a one-sided shift space $X \subset \{0,1,2,3,4\}^{\mathbb{N}}$, in which the set of all ill-founded trees on a countable set have been suitably encoded into the backward orbit branches of a given point x. We show that $s\alpha(x) \subset X$ is not Borel. Then in Section 3 we embed this shift space as a totally invariant subsystem for a map on the square $F : I^2 \to I^2$, and show that the $s\alpha$-limit set of the corresponding point is not a Borel subset of I^2. We remark that the map F in our counterexample is surjective, piecewise monotone, and triangular, i.e. a skew product map of the form $F(x,y) = (f(x), g_x(y))$.

1. Special α-limit sets are analytic

A *Polish space* is any separable, completely metrizable topological space. A subset B of a Polish space Y is called *analytic* (or a $\mathbf{\Sigma}^1_1$ set) if there exist a Polish space X, a Borel subset $A \subseteq X$, and a continuous map $f : X \to Y$ such that $B = f(A)$. The class of analytic sets in an uncountable Polish space strictly contains the class of Borel sets. A set is called $\mathbf{\Sigma}^1_1$-*complete* if it is analytic but not Borel, or equivalently, if it is analytic but its complement is not.

Recall that a topological dynamical system is a pair (X, f) where X is a compact metric space and $f : X \to X$ is continuous. In this section we show that $s\alpha$-limit sets of topological dynamical systems are always analytic.

Theorem 1. *Every $s\alpha$-limit set of a topological dynamical system (X, f) is analytic.*

Proof. Give $X^{\mathbb{N}}$ the product topology. It is compact and metrizable. For example, if d is the metric on X, then $\widehat{d}(x_0x_1\cdots, y_0y_1\cdots) = \sum_n 2^{-n}\min(1, d(x_n, y_n))$ is one compatible metric on $X^{\mathbb{N}}$. Now let $\widehat{X} \subseteq X^{\mathbb{N}}$ be the closed subspace

$$\widehat{X} = \{x_0x_1\cdots \mid f(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N}\}.$$

$\widehat{X}$ is a topological space whose points are the backward orbits of (X, f). We remark that the space $\widehat{X}$ equipped with the map $x_0x_1\cdots \mapsto f(x_0)x_0x_1\cdots$ is a well-known object in topological dynamics called the natural extension of (X, f).

Consider the relation $R \subseteq \widehat{X} \times X$ given by

$$\begin{aligned} R &= \{(x_0x_1\cdots, y) \mid x_{n_i} \to y \text{ along some subsequence } n_i \to \infty\} \\ &= \bigcap_{j=1}^{\infty} \bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} \left\{(x_0x_1\cdots, y) \mid d(x_n, y) < \tfrac{1}{j}\right\}. \end{aligned}$$

Then R is a countable intersection of open sets, so it is Borel.

Let $\pi_0 : \widehat{X} \to X$ be the projection on the zeroth coordinate, $x_0x_1\cdots \mapsto x_0$. Now fix an arbitrary point $x \in X$ and let $A = \pi_0^{-1}(\{x\})$. Then A is closed in $\widehat{X}$. Finally, let $\pi : \widehat{X} \times X \to X$ be the map $(x_0x_1\cdots, y) \mapsto y$. By the definition of $s\alpha$-limit sets we have

$$s\alpha(x) = \pi(R \cap (A \times X)).$$

Thus we have expressed $s\alpha(x)$ as the continuous image of a Borel subset of the compact metric space $\widehat{X} \times X$. It follows that $s\alpha(x)$ is analytic. □

2. A non-Borel $s\alpha$-limit set in a shift space

In order to show that a set B in a Polish space Y is not Borel, it suffices to start with a known non-Borel set A in a Polish space X and find a *reduction* of A to B, that is, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$. We follow this strategy using the well-known example of the ill-founded trees as a non-Borel subset of the Cantor space.

To describe those trees let $\mathbb{N}^{<\mathbb{N}}$ denote the set of all finite words in the countable alphabet $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then a tree is any subset $T \subseteq \mathbb{N}^{<\mathbb{N}}$ which contains along with any word all of its initial segments. The nodes of the tree are just the individual words $v \in T$, and if v has length $n \geq 1$, then its parent node is its initial segment of length $n - 1$. By passing to shorter and shorter initial segments we traverse the branch which connects the node v back to the empty word $\emptyset$, which forms the root of the whole tree.

A tree T is said to be *ill-founded* if it has an *infinite branch*, that is, an infinite word $v \in \mathbb{N}^{\mathbb{N}}$ such that every initial segment of v belongs to T. Let $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ denote the power set, that is, the collection of all subsets $A \subseteq \mathbb{N}^{<\mathbb{N}}$. Let $\mathcal{T}$ and $\mathcal{T}_{\text{IF}}$ denote the set of trees and the set of ill-founded trees, respectively. $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ becomes the Cantor space when we identify it with the countable product $\{0, 1\}^{\mathbb{N}^{<\mathbb{N}}}$ by identifying each subset $A \subseteq \mathbb{N}^{<\mathbb{N}}$ with its characteristic function $\chi_A : \mathbb{N}^{<\mathbb{N}} \to \{0, 1\}$. A well-known result in descriptive set theory is that $\mathcal{T}_{\text{IF}} \subset \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ is analytic but not Borel, and in fact, this is regarded as the prototypical example of a $\mathbf{\Sigma}^1_1$-complete set, see [18, Theorem 27.1].

The following notation will be useful. We write $\text{len}(w) \in \mathbb{N} \cup \{\infty\}$ for the length of a finite or infinite word, and we write w_i for the ith symbol of w for $0 \leq i < \text{len}(w)$. We write $w|n = w_0 \ldots w_{n-1}$ for the initial segment of w of length n. We write $v^\frown w$ for the concatenation of the words v, w, provided $\text{len}(v) < \infty$. We use exponents to indicate repetition of a symbol, so a^n is the symbol a repeated n times, while a^∞ is the infinite sequence $aaa\cdots$. If we write a word explicitly as a string of digits $w = 4530\cdots$ the reader may assume that each w_i is a single-digit number, otherwise we will use a longer notation such as $w = (4, 53, 0, \cdots)$ to avoid confusion.

We recall a few facts from the standard proof that $\mathcal{T}_{\text{IF}} \subset \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ is analytic, as we will need them later on.

Lemma 2. *The set of trees $\mathcal{T}$ is closed in $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$. The set*

$$R = \{(T, y) \in \mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}} \mid T \text{ is a tree and } y \text{ is an infinite branch of } T\}$$

is closed in $\mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}}$. In particular, this implies that $\mathcal{T}_{\text{IF}} \subset \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ is analytic.

Proof. We include the proof for completeness. Suppose that $A \subseteq \mathbb{N}^{<\mathbb{N}}$ is not a tree. Then there is a word v and an initial segment w of v such that $v \in A$ but $w \notin A$. Consider the set $\{A' \subseteq \mathbb{N}^{<\mathbb{N}} \mid v \in A', w \notin A'\}$. Translating to characteristic functions this is the set of $\chi_{A'} \in \{0, 1\}^{\mathbb{N}^{<\mathbb{N}}}$ which take the value 1 in position v and 0 in position w. But specifying values in finitely many positions clearly defines an open set in the product topology on $\{0, 1\}^{\mathbb{N}^{<\mathbb{N}}}$ (such sets are usually called cylinder sets). This shows that the complement of $\mathcal{T}$ is open in $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$, so $\mathcal{T}$ is closed.

For the second statement of the lemma, since $\mathcal{T}$ is closed it suffices to show that R is relatively closed in $\mathcal{T} \times \mathbb{N}^{\mathbb{N}}$. So suppose that (T, y) belongs to the complement of R in $\mathcal{T} \times \mathbb{N}^{\mathbb{N}}$. This means that y is not an infinite branch of the tree T, so there must be some n such that $y|n \notin T$. Consider the set of pairs $(T', y') \in \mathcal{T} \times \mathbb{N}^{\mathbb{N}}$ such that $y|n \notin T'$ and $y_i = y'_i$ for all $i < n$. Again, since we've specified the values of T', y' in only finitely many positions, we get a relatively open set in $\mathcal{T} \times \mathbb{N}^{\mathbb{N}}$. This shows that the complement of R is open, so R is closed.

Finally, note that $\mathcal{T}_{\mathrm{IF}}$ is just the projection of R on the first coordinate, and the continuous image of a Borel subset in a Polish space is analytic by definition. □

We wish to "encode" the set $\mathcal{T}_{\mathrm{IF}}$ into a $s\alpha$-limit set. Let $2^{\mathbb{N}}$ be the standard Cantor space. Choose an enumeration (s_i) of $\mathbb{N}^{<\mathbb{N}}$ and define a homeomorphism $h : \mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \to 2^{\mathbb{N}}$ by letting $h(T)$ be the point x such that $x_i = 1$ whenever $s_i \in T$ and $x_i = 0$ whenever $s_i \notin T$. Let $5^{\mathbb{N}} = \{0,1,2,3,4\}^{\mathbb{N}}$ be the one-sided shift space in five symbols (it is also a Cantor space) and let $\sigma : 5^{\mathbb{N}} \to 5^{\mathbb{N}}$ be the shift map defined by $(\sigma x)_i = x_{i+1}$. A *subshift* in $5^{\mathbb{N}}$ is any closed subset $X \subseteq 5^{\mathbb{N}}$ such that $\sigma(X) \subseteq X$. For T an ill-founded tree and y an infinite branch of T we define inductively points $\omega_n(T,y) \in 5^{\mathbb{N}}$ by

$$\begin{aligned} &\omega_0(T,y) = 3^\frown 4^\frown 0^\infty, \\ &\omega_n(T,y) = 3^\frown (x|n)^\frown 2^{y_{n-1}\frown}\omega_{n-1}(T,y), \qquad \text{for } n \geq 1, \text{ where } x = h(T). \end{aligned} \tag{1}$$

Then define a subset $X \subset 5^{\mathbb{N}}$ by

$$X := \{0,1,2,3\}^{\mathbb{N}} \cup \{\sigma^j(\omega_i(T,y)) \mid (T,y) \in R, i,j \in \mathbb{N}\}, \tag{2}$$

where we continue to use the set R from Lemma 2.

Example 3. Let T be the tree of strictly increasing finite words $s \in \mathbb{N}^{<\mathbb{N}}$ and let $y = (2,3,5,7,11,\ldots) \in \mathbb{N}^{\mathbb{N}}$ be an infinite word in the body of T. Let (s_i) be an enumeration of $\mathbb{N}^{<\mathbb{N}}$ whose first few terms are $s_0 = \emptyset$, $s_1 = 0$, $s_2 = 00$, so that $s_0, s_1 \in T$ but $s_2 \notin T$. Then

$$\omega_3(T,y) = 3\ 110\ 22222\ 3\ 11\ 222\ 3\ 1\ 22\ 3\ 4\ 00000\cdots$$

Thus, from $\omega_n(T,y)$ we can read off the first n symbols of $x = h(T)$ and y. Each block of 2's encodes a digit of y. The blocks of 0's and 1's record the initial segments of x.

Lemma 4. *The set $X \subset 5^{\mathbb{N}}$ is a subshift and $\sigma(X) = X$.*

Proof. Invariance $\sigma(X) \subseteq X$ is immediate from (2). Surjectivity of $\sigma|_X$ follows from (1), (2), and the observation that $\sigma^{t_n}(\omega_n(T,y)) = \omega_{n-1}(T,y)$, where $t_n = 1 + n + y_{n-1}$.

Equations (1) and (2) implicitly determine a set of forbidden words F in the alphabet $\{0,1,2,3,4\}$. Any word with an initial 4 and a non-zero symbol occuring after it is forbidden. Any word which ends with a 4 is forbidden if it is not an initial segment of some $\sigma^j(\omega_i(T,y))$, $i,j \in \mathbb{N}$, $(T,y) \in R$. A point $x \in 5^{\mathbb{N}}$ is in X if and only if it contains none of those forbidden words. This shows that X is closed. □

Now consider the dynamical system given by the subshift $\sigma|_X : X \to X$ and fix the point $\omega_0 = 3^\frown 4^\frown 0^\infty \in X$. The backward orbit branches of ω_0 correspond to the ill-founded trees, allowing us to prove the following result.

Theorem 5. *The set $s\alpha(\omega_0) \subset X$ is $\mathbf{\Sigma}^1_1$-complete.*

Proof. We know from Theorem 1 that $s\alpha(\omega_0)$ is analytic. It remains to show that it is not Borel. Define a map $f : \mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \to X$ by $f(T) = 3^\frown h(T)$. The map is well-defined since X contains $\{0,1,2,3\}^{\mathbb{N}}$. It is continuous since h is. We will show that $T \in \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ is an ill-founded tree if and only if $f(T) \in s\alpha(\omega_0)$. This means $\mathcal{T}_{\mathrm{IF}} = f^{-1}(s\alpha(\omega_0))$, and since the preimage of a Borel set by a continuous map is always Borel, it follows that $s\alpha(\omega_0)$ is not a Borel subset of X.

Suppose first that $T \in \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ is an ill-founded tree. Let $y \in \mathbb{N}^{\mathbb{N}}$ be an infinite branch of T. Continuing to use the notation from Lemma 2 we see that $(T,y) \in R$ and so each of the points $\omega_n(T,y)$, $n \in \mathbb{N}$, belongs to X. Clearly there is a backward orbit branch of ω_0 with $(\omega_n(T,y))_n$ as a subsequence. But $\omega_n(T,y) \to 3^\frown h(T) = f(T)$ as $n \to \infty$.

Conversely, choose any $T \in \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ such that $f(T) \in s\alpha(\omega_0)$. Then we can choose a backward orbit branch of ω_0 with a subsequence converging to $f(T)$, so we get points $\omega_i \in X$ and times $t_i \geq 1$, $i \in \mathbb{N}$, such that $\omega_i \to 3^\frown h(T)$ as $i \to \infty$ and

$$\sigma^{t_i}(\omega_{i+1}) = \omega_i, \qquad i \in \mathbb{N}. \tag{3}$$

We may suppose without loss of generality that each ω_i begins with the symbol 3. By the definition of X we may write $\omega_i = \omega_{n_i}(T_i, y_i)$ for some $(T_i, y_i) \in R$, $n_i \in \mathbb{N}$. We warn the reader that $y_i \in \mathbb{N}^\mathbb{N}$ is a whole infinite word; the subscript here does not refer to the symbol in position i, but to the ith infinite word in the sequence. Since n_i counts the number of 3's in ω_i (except the last 3 before the 4), equations (1) and (3) imply that $n_{i+1} > n_i$. Thus the numbers n_i form an increasing sequence, and in particular $n_i \geq i$ for all i. Write $x_i = h(T_i)$ for all i. Again from (1) and (3) we get $x_i|n_i = x_{i+1}|n_i$ and $y_i|n_i = y_{i+1}|n_i$ for all i. Therefore we may pass to the limits $x = \lim x_i$, $y = \lim y_i$, and since $\omega_i \to 3^\frown h(T)$ we get $x = h(T)$. But h is a homeomorphism and therefore $T = \lim T_i$. Since each $(T_i, y_i) \in R$ and R is closed, we conclude $(T, y) \in R$, that is, T is an ill-founded tree. This concludes the proof. □

3. A non-Borel $s\alpha$-limit set in the square

The *square* is the Cartesian product $I^2 = [0,1] \times [0,1]$ with the usual Euclidean topology. For a mapping $f : X \to X$ a subset $A \subseteq X$ is called *invariant* if $f(A) \subseteq A$ and *totally invariant* if $f^{-1}(A) = A$. Two topological dynamical systems (X, f), (Y, g) are called *conjugate* if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. The main goal of this section is to prove the following theorem.

Theorem 6. *There is a continuous surjective map $F : I^2 \to I^2$ and a point $x_0 \in I^2$ such that $s\alpha(x_0) \subset I^2$ is $\mathbf{\Sigma}^1_1$-complete.*

The proof proceeds by constructing the map F. It has the form of a skew product $F(x, y) = (f(x), g_x(y))$. There is a closed subset S in the "x-axis" of the square, $S \subset I \times \{0\} \subset I^2$, such that $F^{-1}(S) = F(S) = S$. The subsystem (S, F) is conjugate to the subshift (X, σ) constructed in the previous section.

Proposition 7. *There is a surjective continuous map $F : I^2 \to I^2$ and a closed subset $S \subset I^2$ such that $F^{-1}(S) = F(S) = S$ and (S, F) is conjugate to (X, σ).*

Proof. Start with the embedding $e : 5^\mathbb{N} \to I$ and the map $f : I \to I$ given by

$$e(x_0 x_1 x_2 \cdots) = \sum_{j=0}^{\infty} \frac{2x_j}{9^{j+1}},$$

$$f(x) = \begin{cases} 9x - 2i, & \text{for } x \in \left[\frac{2i}{9}, \frac{2i+1}{9}\right], i \in \{0,1,2,3,4\} \\ (2i+2) - 9x & \text{for } x \in \left[\frac{2i+1}{9}, \frac{2i+2}{9}\right], i \in \{0,1,2,3\}. \end{cases}$$

Then f is the standard 9-horseshoe with 5 increasing laps and 4 decreasing laps and $C = e(5^\mathbb{N}) \subset I$ is the Cantor set of points whose trajectories stay in the increasing laps, see Figure 1. Clearly $f \circ e = e \circ \sigma$, and so (C, f) is conjugate to $(5^\mathbb{N}, \sigma)$. Henceforth we will write $I_i = [\frac{2i}{9}, \frac{2i+1}{9}]$ for the ith increasing lap of f.

By Lemma 4 we have $\sigma(X) = X$. That means we can write X as the following finite union of closed sets

$$X = \bigcup_{i<5} X_i, \quad X_i := \{x \in X \mid i^\frown x \in X\}.$$

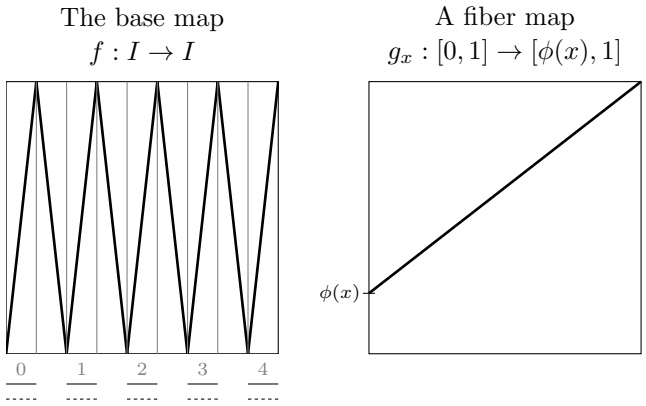

FIGURE 1. The components of the skew product mapping $F(x, y) = (f(x), g_x(y))$. The invariant Cantor set C is suggested by the line segments below the graph of f. For the definitions of ϕ and g_x see equations (5), (6).

Here X_i is the "follower set" of the symbol i in X, i.e. the set of points in X which can be preceded by the symbol i and still belong to X.

Going through the embedding e we get that the closed set $A = e(X) \subset I$ is the union of the closed sets $A_i = e(X_i)$, $i < 5$. Moreover, we have $A \subset C$ and (A, f) is conjugate to (X, σ). For any closed set $Y \subseteq I$ we write $d(x, Y) = \min\{|x - y| \mid y \in Y\}$ for the distance from a point $x \in I$ to Y. For $i < 5$ we define sets

$$B_i = \{x \in I \mid d(x, A_i) \leq d(x, A_j) \text{ for all } j < 5\}.$$

Thus B_i represents the points $x \in I$ for which A_i is as close or closer to x than any of the other sets A_j. Clearly the sets B_i are closed and their union is all of I. For points $x \in A$ we have $x \in B_i$ if and only if $x \in A_i$, because the distance to at least one of the sets A_i is zero. This shows that

$$I = \bigcup_{i<5} B_i, \text{ and for all } i < 5,\ B_i \cap A = A_i. \tag{4}$$

Now we define a function $\phi : I \to [0, \frac{1}{2}]$. Let

$$\phi(x) = \frac{1}{2} d(f(x), B_i), \quad \text{for } x \in I_i,\ i < 5. \tag{5}$$

This defines ϕ on the increasing laps of f. On the decreasing laps of f we have already defined ϕ at the endpoints, and we simply extend to any continuous function with $0 < \phi(x) \leq \frac{1}{2}$ whenever $x \in I \setminus \bigcup_{i<5} I_i$.

Now define a skew product mapping $F : I^2 \to I^2$ by

$$F(x, y) = \big(f(x), g_x(y)\big), \quad \text{where } g_x(y) = \phi(x) + y \cdot (1 - \phi(x)), \tag{6}$$

see Figures 1 and 2.

Let $E : 5^{\mathbb{N}} \to I^2$ be the embedding $E(x) = (e(x), 0)$ and let $S = E(X) = A \times \{0\}$. We claim that ϕ has the following three properties:

(i) $\phi(x) = 0$ for all $x \in A$,
(ii) $\phi(x) > 0$ if $x \notin A$ but $f(x) \in A$,
(iii) For each $y \in I$ there is a preimage $x \in f^{-1}(y)$ with $\phi(x) = 0$.

Property (i) gives $F \circ E = E \circ \sigma$, which implies that $F(S) = S$ and (S, F) is conjugate to (X, σ). Property (ii) implies that $F^{-1}(S) = S$. Property (iii) makes F surjective. This is

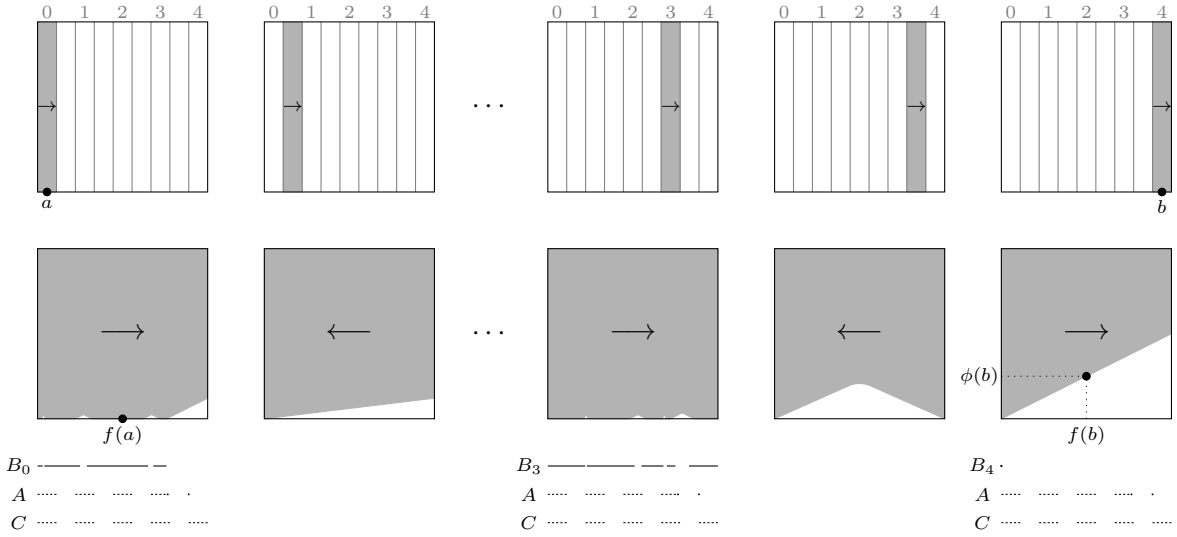

FIGURE 2. The map $F : I^2 \to I^2$ applied to nine rectangular regions. Arrows indicate orientation. C is the full Cantor set $5^{\mathbb{N}}$ embedded into I. A is the embedded copy of the shift space X. The sets B_i have gaps around the points that cannot be preceded by the symbol i. For example, B_0 is missing the embedded image of 40^∞, since this point can only be preceded by a 3, and B_4 is a singleton, since only 0^∞ can be preceded by a 4. (Note: the gaps are not drawn to scale, and many smaller gaps are not visible at all). The choice of ϕ causes the image of each rectangle $I_i \times I$ to meet the x-axis of the square only in the set B_i. The point a represents 02^∞, so $a, f(a) \in A$ and $\phi(a) = 0$. The point b represents 42^∞, so $b \notin A$, $f(b) \in A$, and $\phi(b) > 0$.

everything we needed to show about the system (S, F). It remains to establish properties (i) – (iii) of the function ϕ.

To prove property (i) choose $x \in A$ and write $x = e(x_0x_1\cdots)$ with $x_0x_1\ldots \in X$. Let $i = x_0$ so $x \in I_i$. Then $x_1x_2\cdots \in X_i$ so $f(x) \in A_i \subseteq B_i$ and therefore $\phi(x) = 0$.

To prove property (ii) we choose a point $x \notin A$ such that $f(x) \in A$. There are two cases. First, suppose $x \notin C$. If x is in an increasing lap of f then $f(x) \notin C$ either, contradicting $f(x) \in A$. But if x is not in an increasing lap of f then $\phi(x) > 0$ by the definition of ϕ, so we are done. Second, suppose $x \in C \setminus A$. Then we can write $x = e(x_0x_1\cdots)$ with $x_0x_1\cdots \in 5^{\mathbb{N}} \setminus X$. Let $i = x_0$, so I_i is the lap containing x. Then $x_1x_2\cdots \notin X_i$, so $f(x) \notin A_i$. But we supposed $f(x) \in A$, so by (4) we have $f(x) \notin B_i$. Therefore $\phi(x) > 0$ by the definition of ϕ.

To prove property (iii) pick any point $y \in I$. Since the sets B_i cover I we can find i with $y \in B_i$. Then we take $x = f|_{I_i}^{-1}(y) \in I_i$. Then $\phi(x) = 0$ by the definition of ϕ. □

Proof of Theorem 6. By Theorem 5 the subshift (X, σ) has a point ω_0 such that $s\alpha(\omega_0) \subset X$ is not Borel. By Proposition 7 there is a continuous surjection $F : I^2 \to I^2$ and a closed subset $S \subset I^2$ such that $F^{-1}(S) = F(S) = S$ and (S, F) is conjugate to (X, σ). Let $E : X \to S \subset I^2$ denote the conjugacy and let $x_0 = E(\omega_0)$. The backward orbit branches of x_0 in the system (I^2, F) are the same as the backward orbit branches in the subsystem (S, F), and these correspond through the conjugacy to the backward orbit branches of ω_0 in (X, σ). Therefore

$s\alpha(x_0) = E(s\alpha(\omega_0))$. Since E is a topological embedding we conclude that $s\alpha(x_0) \subset I^2$ is not Borel. □

Remark 8. The only property of the subshift (X, σ) which was really needed in the proof of Proposition 7 was surjectivity $\sigma(X) = X$. Thus, the same proof technique leads to the following general embedding result.

Proposition 9. *Any subshift $X \subseteq \{0, \dots, r-1\}^{\mathbb{N}}$, $r \geq 2$, with $\sigma(X) = X$ can be embedded as a totally invariant subsystem of a surjective map on the square.*

Proof. The proof is essentially the same as the proof of Proposition 7. To accomodate a shift space with r symbols we divide the square into $2r-1$ vertical strips and use for the base map the full $2r-1$ horseshoe. No other significant changes are needed. □

Remark 10. In dimension one, the $s\alpha$-limit sets of an interval map $f : I \to I$ are always Borel and can only occur at or below the second level of the Borel hierarchy (they are simultaneously F_σ and G_δ, see [13]). But the possibility to embed subshifts into the square without the restrictions imposed by the order structure of $\mathbb{R}$ gives much more flexibility to dynamical systems in dimension two than in dimension one. This gives some explanation of why their special α-limit sets can be more complex.

References

[1] D. Airey, S. Jackson, B. Mance, *Some complexity results in the theory of normal numbers*, to appear in Canad. J. Math. Preprint: arXiv 1609.08702.

[2] S. Agronsky, A. Bruckner, J. Ceder, T. Pearson, *The structure of ω-limit sets for continuous functions* Real Anal. Exchange **15** (1989/90), no. 2, 483–510.

[3] S. Agronsky, J. Ceder, *Each Peano subspace of E^k is an ω-limit set*, Real Anal. Exchange **17** (1991/92), no. 1, 371–378.

[4] F. Balibrea, G. Dvorníková, M. Lampart, P. Oprocha, *On negative limit sets for one-dimensional dynamics*, Nonlinear Anal. **75** (2012), 3262–3267.

[5] L. S. Block, W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics 1513, Springer-Verlag, Berlin, 1992.

[6] A. Bruckner, J. Smítal, *The structure of ω-limit sets for continuous maps of the interval,* Math. Bohem. **117** (1992), no. 1, 42–47.

[7] C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Reg. Conf. Ser. Math., vol. 38, American Mathematical Society, Providence, RI, 1978.

[8] H. Cui, Y. Ding, *The α-limit sets of a unimodal map without homtervals*, Topology Appl. **157** (2010), no. 1, 22–28.

[9] E. Coven, Z. Nitecki, *Non-wandering sets of the powers of maps of the interval*, Ergod. Theor. Dyn. Syst. **1** (1981), 9–31.

[10] M. Foreman, D. Rudolph, B. Weiss, *The conjugacy problem in ergodic theory*, Ann. of Math. (2) 173 (2011), no. 3, 1529–1586.

[11] C. Good, J. Meddaugh, J. Mitchell, *Shadowing, internal chain transitivity and α-limit sets*, J. Math. Anal. Appl. **491** (2020), no. 1, 124291, 19 pp.

[12] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Appl. Math. Sci., vol. 42, Springer-Verlag, New York, 1983.

[13] J. Hantáková and S. Roth *On backward attractors of interval maps*, Preprint: arXiv 2007.10883.

[14] M. Hero, *Special α-limit points for maps of the interval*, Proc. Amer. Math. Soc. **116** (1992), 1015–1022.

[15] M. Hirsch, H. Smith, X. Zhao, *Chain transitivity, attractivity, and strong repellors for semidynamical systems*, J. Dynam. Differential Equations **13** (2001), no. 1, 107–131

[16] G. Hjorth, *On invariants for measure preserving transformations*, Fund. Math. 169 (2001), no. 1, 5184.

[17] V. Jiménez López and J. Smítal, *ω-limit sets for triangular mappings*, Fund. Math. **167** (2001), no. 1, 1–15.

[18] A. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics 156, Springer-Verlag, New York, 1995.

[19] S. Kolyada, M. Misiurewicz, and Ľ. Snoha, *Special α-limit sets*, Dynamics: topology and numbers 157–173. Contemp. Math., 744, Amer. Math. Soc., Providence, RI, 2020.
[20] G. Rauzy, Nombres normaux et processus déterministes. Acta Arith. **29** (1976), no. 3, 211–225.
[21] A. N. Sharkovsky, *Attractors of trajectories and their basins (Russian)*, Naukova Dumka, Kiev, 2013.
[22] T. Sun, H. Xi, Z. Chen, and Y. Zhang, *The attracting center and the topological entropy of a graph map*, Adv. Math. (China) **33** (2004), no. 5, 540–546.
[23] T. Sun, H. Xi, and H. Liang *Special α-limit points and unilateral γ-limit points for graph maps*, Sci. China Math. **54** (2011), no. 9, 2013–2018.

Department of Mathematics, University of North Texas, General Academics Building 435, 1155 Union Circle, #311430, Denton, Texas 76203-5017
E-mail address: stephen.jackson@unt.edu

Uniwersytet im. Adama Mickiewicza w Poznaniu, Collegium Mathematicum, ul. Umultowska 87, 61-614 Poznań, Poland
E-mail address: william.mance@amu.edu.pl

Mathematical Institute of the Silesian University in Opava, Na Rybničku 1, 74601, Opava, Czech Republic
E-mail address: samuel.roth@math.slu.cz