EQUIVALENCE RELATIONS AND DETERMINACY

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ABSTRACT. We introduce the notion of $(\mathbf{\Gamma}, E)$-determinacy for $\mathbf{\Gamma}$ a pointclass and E an equivalence relation on a Polish space X. A case of particular interest is the case when $E = E_G$ is the (left) shift-action of G on S^G where $S = 2 = \{0, 1\}$ or $S = \omega$. We show that for all shift actions by countable groups G, and any "reasonable" pointclass $\mathbf{\Gamma}$, that $(\mathbf{\Gamma}, E_G)$-determinacy implies $\mathbf{\Gamma}$-determinacy. We also prove a corresponding result when E is a subshift of finite type of the shift map on $2^{\mathbb{Z}}$.

1. INTRODUCTION

For $X = 2^\omega$ or $X = \omega^\omega$, E any equivalence relation on X, and $\mathbf{\Gamma}$ any pointclass (a collection of subsets of Polish spaces closed under continuous preimages, the reader can consult [8] and [3] for background on the basic notions of descriptive set theory which we use throughout), there is a natural notion of $(\mathbf{\Gamma}, E)$-determinacy. Namely, this asserts that any $A \subseteq X$ in $\mathbf{\Gamma}$ which is E-invariant is determined. Similarly, if G is a countable group, and we fix an enumeration of G, then there is a natural notion of $(\mathbf{\Gamma}, E)$-determinacy for sets $A \subseteq 2^G$ or $A \subseteq \omega^G$ in $\mathbf{\Gamma}$ (under the natural identification of 2^G with 2^ω via the enumeration of G). We will give a more general definition of $(\mathbf{\Gamma}, E)$-determinacy for arbitrary Polish spaces X and equivalence relations E on X in §5. However, even the special cases just mentioned have risen in various contexts. For example, when E is the Turing equivalence relation on 2^ω, then the question of when $\mathbf{\Gamma}$ Turing-determinacy implies full $\mathbf{\Gamma}$-determinacy has been an important question in modern logic. Harrington [2] showed that $\mathbf{\Sigma}_1^1$ Turing-determinacy is equivalent to $\mathbf{\Sigma}_1^1$-determinacy. Woodin showed that in $L(\mathbb{R})$ Turing determinacy implies full determinacy. It is open in general for which pointclasses $\mathbf{\Gamma}$ we have that $\mathbf{\Gamma}$ Turing-determinacy implies $\mathbf{\Gamma}$-determinacy.

In another direction, in recent years arguments involving Borel determinacy have had fruitful applications to the theory of Borel equivalence relations. The determinacy of Borel games is a fundamental result of Martin [6], [7]. Despite the central significance of this result in modern logic, this result has until recently found relatively few applications as a proof technique. Recently, however, Marks [5] uses Borel determinacy arguments to get lower-bounds on the Borel chromatic number χ_B for free actions of free products of groups, in particular, the lower-bound that $\chi_B(2^{F_n}) \geq 2n + 1$ for the chromatic number for the free part of the shift action (defined below) of the free group F_n on the space 2^{F_n}. See also [4] for a detailed account of recent advances in the theory of descriptive graph combinatorics including the use of Borel determinacy. There are currently no other proofs of this result, and so the introduction of determinacy methods into the subject represents an important connection.

In this paper we begin to investigate the general question of when $(\mathbf{\Gamma}, E)$-determinacy implies $\mathbf{\Gamma}$-determinacy. Again, we formulate the notion of $(\mathbf{\Gamma}, E)$-determinacy more generally for arbitrary Polish spaces and equivalence relations in §5. First, however, we investigate the special case where $X = 2^G$ (or $X = \omega^G$) and E is the equivalence relation induced by the shift action of G on X.

One of our main results is that for any countable group G and any pointclass $\mathbf{\Gamma}$ satisfying some reasonable closure properties (a "reasonable pointclass") that $(\mathbf{\Gamma}, E_G)$-determinacy implies $\mathbf{\Gamma}$-determinacy. The proof passes through a property of G which we call *weak amenability* which may be of interest elsewhere. In §2 and §3 we introduce the notion of weak amenability and establish some basic properties. In §4 we use weak amenabilty to give a combinatorial proof of the determinacy result.

In §5 we give the general definition of $(\mathbf{\Gamma}, E)$-determinacy and then in §6 we prove our second main result that for equivalence relations E induced by subshifts of $2^{\mathbb{Z}}$ of finite type and reasonable pointclasses

$\mathbf{\Gamma}$ that $(\mathbf{\Gamma}, E)$-determinacy implies $\mathbf{\Gamma}$-determinacy. The proof follows the outline of the first main theorem, but has extra complications involving the combinatorics of the subshift.

2. Shift Actions

Let G be a countable group. The (left) shift action of G on S^G is the action defined by $g \cdot x(h) = x(g^{-1}h)$. The cases of primary interest are when $S = 2 = \{0, 1\}$ and $S = \omega$. In either of these cases, we refer to S^G as the shift space, and note that the action of G on S^G is continuous (with the usual product of the discrete topologies on S^G). Let E_G denote the equivalence relation on S^G induced by the shift action.

Let $\pi \colon \omega \to G$ be a bijection, which we view as an enumeration of the group G, $G = \{\pi(0), \pi(1), \dots\}$. We also write g_i for $\pi(i)$ to denote the ith group element. The enumeration π induces a homeomorphism, which we also call π, between S^ω and S^G, namely $\pi(x) = y$ where $y(g_n) = x(n)$.

Definition 1. Let $\mathbf{\Gamma}$ be a pointclass. Let G be a countable group and $S = 2$ or $S = \omega$, and let E_G be the shift equivalence relation on S^G. We say $(\mathbf{\Gamma}, E_G)$-determinacy holds if for all $A \subseteq S^G$ which are in $\mathbf{\Gamma}$ and E_G-invariant we have that $\pi^{-1}(A) \subseteq S^\omega$ is determined, for all enumerations π of G.

Our main theorem will require a mild closure hypothesis on the pointclass $\mathbf{\Gamma}$ which we now state.

Definition 2. We say a pointclass $\mathbf{\Gamma}$ is *reasonable* if
  (1) $\mathbf{\Gamma}$ is closed under unions and intersections with $\mathbf{\Delta}_6^0$ sets.
  (2) $\mathbf{\Gamma}$ is closed under substitutions by $\mathbf{\Delta}_6^0$-measurable functions.

We note that all levels of the Borel hierarchy past the finite levels are reasonable, as are all levels of the projective hierarchy.

The next result is our main result connecting $(\mathbf{\Gamma}, E_G)$-determinacy with full $\mathbf{\Gamma}$-determinacy.

Theorem 3. *For any countable group G and any reasonable pointclass $\mathbf{\Gamma}$, $(\mathbf{\Gamma}, E_G)$-determinacy implies $\mathbf{\Gamma}$-determinacy.*

The proof of Theorem 3 will involve a weak form of amenabilty of groups which we simply call *weak amenability*. We give this definition next.

Definition 4. Let G be a countable group. We say G is *weakly amenable* if either G is finite, or if there is an equivalence relation $\sim$ on G such that
  (1) $G/\sim$ is infinite
  (2) $\forall g \in G \ \exists b(g) \in \mathbb{N} \ \forall C \in G/\sim |\{C' \in G/\sim \colon gC \cap C' \neq \emptyset\}| \leq b(g)$
and an increasing sequence of finite sets $A_n \subseteq G/\sim$, such that $G/\sim = \bigcup_{n \in \omega} A_n$ such that for any $g \in G$

$$(1) \qquad \lim_{n \to \infty} \frac{|\{C \in A_n \colon gC \subseteq \cup A_n\}|}{|A_n|} = 1$$

We call an equivalence relation $\sim$ on a group G which satisfies both conditions (1) and (2) of Definition 4 *appropriate*. We note that the equivalence classes C in an appropriate equivalence relation need not be finite themselves, but in the definition of weak amenability, the sets A_n are finite (i.e., they are finite sets of equivalence classes).

We note that every amenable group G is weakly amenable. This follows taking $\sim$ to be the equality equivalence relation on G. Note that the equality equivalence relation on G is an appropriate equivalence relation (with $b(g) = 1$ for every $g \in G$).

We will prove Theorem 3 by showing two separate results, one of which is a purely algebraic result concerning weak amenability, and the other a pure game argument. The algebraic result is the following:

Theorem 5. *Every infinite group has an infinite weakly amenable subgroup.*

The game argument is given in the following theorem.

Theorem 6. *If G is a countable group which has an infinite weakly amenable subgroup, then for every reasonable pointclass $\mathbf{\Gamma}$ we have that $(\mathbf{\Gamma}, E_G)$-determinacy implies $\mathbf{\Gamma}$-determinacy.*

3. Weak Amenability

In this section we establish that certain classes of groups are weakly amenable, including all the amenable groups and free groups, and prove Theorem 5.

The following two lemmas are easy and well-known.

Lemma 7. If G is a non-torsion group, then G has an infinite weakly amenable subgroup.

Proof. This is immediate as an element of infinite order generates an infinite cyclic subgroup, which is amenable and so weakly amenable. $\square$

Lemma 8. If G is locally finite then G is amenable, and so weakly amenable.

Proof. We may write $G = \bigcup G_n$, an increasing union of finite subgroups. The G_n can be used as Følner sets to witness the amenability of G. $\square$

Proof of Theorem 5. We may assume without loss of generality that G is an infinite countable group. By Lemma 7 we may assume that G is a torsion group. By Lemma 8 we may assume that G is not locally finite. Then G contains an infinite subgroup $H = \langle F, g \rangle$ generated by a finite subgroup $F \leq G$ and an element $g \in G$ of finite order. If suffices to show that any such group H is weakly amenable.

Every element $h \in H$ can be written (not uniquely) in the form $h = f_1 g^{a_1} f_2 g^{a_2} \cdots f_n g^{a_n}$ where $f_i \in F$ and a_i are positive integers less than the order of g. We call n the length of this representation of h. We let $|h|$ denote the minimum length of a representation of h. Note that $|h^{-1}| \leq |h| + 1$ for any $h \in H$. We easily have that $|h_1 h_2| \leq |h_1| + |h_2|$ and also $|h_1 h_2| \geq ||h_1| - |h_2|| - 1$.

We let $\sim$ be the equivalence relation on H given by $h_1 \sim h_2$ iff $|h_1| = |h_2|$. Each equivalence class is finite as F is finite and g has finite order. So, $H/\sim$ is infinite. Let $h \in H$. By the above observations we have that for any $k \in H$ that $|k| - |h| - 1 \leq |hk| \leq |k| + |h|$ and we may take $b(h) = 2|h| + 2$ to satisfy (2) of Definition 4. Thus $\sim$ is an appropriate equivalence relation on H.

Let $A_n = \{[h]_\sim : |h| \leq n\}$, so $|A_n| = n + 1$. For $h \in H$ we have that $\{C \in A_n : hC \subseteq \cup A_n\} \supseteq \{[k] \in A_n : |k| \leq n - |h|\}$, and so $|\{C \in A_n : hC \subseteq \cup A_n\}| \geq n - |h| + 1$, and Equation 1 follows. $\square$

The argument above for the proof of Theorem 5 in fact shows the following.

Theorem 9. *Every finitely generated group is weakly amenable.*

sketch of proof. Assume G is infinite and finitely generated. Let $S = \{g_1, \ldots, g_n\}$ be a finite generating set for G. For $g \in G$, let $|g|$ be the minimal length of a word representing g using the symbols g_i, g_i^{-1}, for $g_i \in S$ (we use only g_i and g_i^{-1} here, not other powers of the g_i). We define $\sim$ in the same way as before (i.e., $g \sim h$ iff $|g| = |h|$). Note that $G/\sim$ is still infinite with this modification. The rest of the argument proceeds as before. $\square$

In fact, as pointed out to us by Simon Thomas, this argument shows the following even more general fact.

Theorem 10. *If G is a countable group which has a Cayley graph with an infinite diameter, then G is weakly amenable.* $\square$

In particular, the free group F_ω on infinitely many generators is weakly amenable.

Question 11. Is every countable group weakly amenable?

4. Proof of Theorem 3

In this section we use a game argument to prove Theorem 6, which in view of Theorem 5 implies Theorem 3. For convenience we restate Theorem 6:

Theorem. *If G is a countable group which has an infinite weakly amenable subgroup, then for every reasonable pointclass $\mathbf{\Gamma}$ we have that $(\mathbf{\Gamma}, E_G)$-determinacy implies $\mathbf{\Gamma}$-determinacy*

Proof. Let $H \leq G$ be an infinite weakly amenable subgroup. Let the equivalence relation $\sim$ on H and the the sequence of sets $\{A_n\}_{n\in\omega}$ witness this. We recall here that the sets A_n are finite subsets of the quotient $H/\sim$. Without loss of generality, assume that the sequence $\{A_n\}$ is such that $\lim_{n\to\infty} \frac{|A_{n+1}\setminus A_n|}{|A_{n+1}|} = 1$.

Define H_1 and H_2 subsets of H by

$$H_1 = \cup \left(A_0 \cup \bigcup_{n\in\omega} A_{2n+2} \setminus A_{2n+1} \right)$$

$$H_2 = \cup \left(\bigcup_{n\in\omega} A_{2n+1} \setminus A_{2n} \right)$$

Let $\{g_k H : k \in \omega\}$ enumerate the cosets of H in G. Let $G_{\boldsymbol{I}} = \bigcup_{k\in\omega} g_k H_1$ and $G_{\boldsymbol{II}} = \bigcup_{k\in\omega} g_k H_2$. Clearly $G_{\boldsymbol{I}} \cap G_{\boldsymbol{II}} = \emptyset$ and $G = G_{\boldsymbol{I}} \cup G_{\boldsymbol{II}}$.

Let $A \subseteq S^\omega$ be in $\boldsymbol{\Gamma}$. We will define an alternate payoff set $\tilde{A} \subseteq S^G$ which is E_G-invariant and simulate the game A by the game $\tilde{A}$ in which player $\boldsymbol{I}$ makes moves corresponding to $g \in G_{\boldsymbol{I}}$ and player $\boldsymbol{II}$ makes moves corresponding to $g \in G_{\boldsymbol{II}}$ (we assume the enumeration π satisfies $\pi^{-1}(g)$ is even for $g \in G_{\boldsymbol{I}}$, and $\pi^{-1}(g)$ is odd for $g \in G_{\boldsymbol{II}}$).

We will define sets of rules, which if followed by both players will enforce that each player in the game $\tilde{A}$ eventually specifies moves in A to play. First, we partition the positive even integers into infinitely many disjoint subsequences $\left\{ \{c_{n,j}\}_{j\in\omega} \right\}_{n\in\omega}$, and the odd integers also into $\left\{ \{d_{n,j}\}_{j\in\omega} \right\}_{n\in\omega}$. Let $B_{n,j}^{\boldsymbol{I}}$ denote $A_{c_{n,j}} \setminus A_{c_{n,j}-1}$ and $B_{n,j}^{\boldsymbol{II}}$ denote $A_{d_{n,j}} \setminus A_{d_{n,j}-1}$. Note that $\lim_{j\to\infty} \frac{|B_{n,j}^{\boldsymbol{I}}|}{|A_{c_{n,j}}|} = 1$, and likewise for $B_{n,j}^{\boldsymbol{II}}$.

We will have the players specify in $\tilde{A}$ their nth move in A by playing more of those moves (by proportion) on the rounds corresponding to $B_{n,j}^{\boldsymbol{I}}$ (or $B_{n,j}^{\boldsymbol{II}}$ respectively). In order to successfully specify a move, they must have that the limit as $j \to \infty$ of the proportion of *classes* C for which *all* the moves in C are the same goes to 1.

Now we give the formal definition of the rules which will enforce the correct encoding of moves from A into $\tilde{A}$.

$$R_n^{\boldsymbol{I}} = \left\{ x \in X^G : \exists m \forall k \lim_{j\to\infty} \frac{\left|\{C \in B_{n,j}^{\boldsymbol{I}} : \forall h \in C \; x(g_k h) = m\}\right|}{\left|B_{n,j}^{\boldsymbol{I}}\right|} = 1 \right\}$$

$$R_n^{\boldsymbol{II}} = \left\{ x \in X^G : \exists m \forall k \lim_{j\to\infty} \frac{\left|\{C \in B_{n,j}^{\boldsymbol{II}} : \forall h \in C \; x(g_k h) = m\}\right|}{\left|B_{n,j}^{\boldsymbol{II}}\right|} = 1 \right\}$$

We claim these rules are invariant. Suppose $x \in R_n^{\boldsymbol{I}}$ and $g \in G$. Let m witness the fact that $x \in R_n^{\boldsymbol{I}}$, and fix $k \in \omega$. We want to show that m also witnesses $g \cdot x \in R_n^{\boldsymbol{I}}$, or in other words that

$$\lim_{j\to\infty} \frac{\left|\{C \in B_{n,j}^{\boldsymbol{I}} : \forall h \in C \; g \cdot x(g_k h) = m\}\right|}{\left|B_{n,j}^{\boldsymbol{I}}\right|} = 1$$

Let $\ell \in \omega$ and $h' \in H$ be so that $g^{-1}g_k = g_\ell h'$ and notice that, by definition of the shift, we are attempting to show that the following set is large.

$$\{C \in B_{n,j}^{\boldsymbol{I}} : \forall h \in C \; g \cdot x(g_k h) = m\}$$
$$= \{C \in B_{n,j}^{\boldsymbol{I}} : \forall h \in C \; x(g^{-1}g_k h) = m\}$$
$$= \{C \in B_{n,j}^{\boldsymbol{I}} : \forall h \in C \; x(g_\ell h' h) = m\}$$

Now define S_j by the formula

$$S_j = \{C \in B_{n,j}^{\boldsymbol{I}} : \forall h \in C \; x(g_\ell h' h) = m\}$$

and T_j by

$$T_j = B_{n,j}^{\boldsymbol{I}} \setminus S_j.$$

It will suffice to show that T_j is small compared to $B_{n,j}^{\boldsymbol{I}}$ as $j \to \infty$. To see this, recall that $B_{n,j}^{\boldsymbol{I}} = A_{c_{n,j}} \setminus A_{c_{n,j}-1}$ and observe that each class C in T_j is of one of the following three types:

(1) C is moved by h' to intersect some class in $A_{c_{n,j}-1}$,
(2) C is moved by h' to intersect some class outside $A_{c_{n,j}}$,
(3) or C is moved by h' to hit some other class C' which fails to specify m properly.

Thus,

$$T_j \subseteq \left\{C \in B_{n,j}^{\boldsymbol{I}} \colon \exists C' \in A_{c_{n,j}-1} \ h'C \cap C' \neq \emptyset\right\}$$
$$\cup \left\{C \in B_{n,j}^{\boldsymbol{I}} \colon h'C \not\subseteq \cup A_{c_{n,j}}\right\}$$
$$\cup \left\{C \in B_{n,j}^{\boldsymbol{I}} \colon h'C \subseteq \cup B_{n,j}^{\boldsymbol{I}} \wedge \ \exists C'(h'C \cap C' \neq \emptyset \wedge \exists \hat{h} \in C' \ x(g_\ell \hat{h}) \neq m)\right\}.$$

We want to show that

$$\lim_{j \to \infty} \frac{|T_j|}{|B_{n,j}^{\boldsymbol{I}}|} = 0$$

and we will do so by showing that the limit is 0 for each of the three sets above whose union contains T_j.

The first set has size at most $b(h')\left|A_{c_{n,j}-1}\right|$, which is small compared to $\left|B_{n,j}^{\boldsymbol{I}}\right|$ as $j \to \infty$. The second set is small compared to $|A_{c_{n,j}}|$ by the weak amenability hypothesis, and so also small compared to $|B_{n,j}^{\boldsymbol{I}}|$. The third set has size at most $b(h'^{-1})\left|T_j'\right|$, where $T_j' = \left\{C' \in B_{n,j}^{\boldsymbol{I}} \colon \exists \hat{h} \in C' \ x(g_\ell \hat{h}) \neq m\right\}$, which is small compared to $\left|B_{n,j}^{\boldsymbol{I}}\right|$ as $j \to \infty$ since m witnesses $x \in R_n^{\boldsymbol{I}}$.

Thus the rule sets $R_n^{\boldsymbol{I}}$ and $R_n^{\boldsymbol{II}}$ are all invariant.

Now we define the payoff for the auxiliary game. Via the bijection $\pi \colon \omega \to G$ we have a natural bijection between S^ω and S^G. The auxiliary game is officially a subset of S^ω, but we view it as a subset of S^G with this bijection. Thus, for a position n in the game, the move $y(n)$ is viewed as giving the value $\tilde{x}(\pi(n))$, where $\tilde{x} \in S^G$ is the function the players are jointly building. The payoff $\tilde{A} \subseteq S^G$ for player $\boldsymbol{I}$ in the auxiliary game is given by:

$$\tilde{A} = \bigcup_{n \in \omega} \left(\bigcap_{i \leq n} R_i^{\boldsymbol{I}} \setminus R_n^{\boldsymbol{II}}\right) \cup \left(\bigcap_{n \in \omega} (R_n^{\boldsymbol{I}} \cap R_n^{\boldsymbol{II}}) \cap f^{-1}(A)\right)$$

where f is the following decoding function with domain $\bigcap_{n \in \omega} \left(R_n^{\boldsymbol{I}} \cap R_n^{\boldsymbol{II}}\right)$. For $\tilde{x} \in \bigcap_{n \in \omega} \left(R_n^{\boldsymbol{I}} \cap R_n^{\boldsymbol{II}}\right)$, define

$$f(\tilde{x})(2n) = m \Leftrightarrow \forall k \ \lim_{j \to \infty} \frac{\left|\left\{C \in B_{n,j}^{\boldsymbol{I}} \colon \forall h \in C \ \tilde{x}(g_k h) = m\right\}\right|}{\left|B_{n,j}^{\boldsymbol{I}}\right|} = 1$$

$$f(\tilde{x})(2n+1) = m \Leftrightarrow \forall k \ \lim_{j \to \infty} \frac{\left|\left\{C \in B_{n,j}^{\boldsymbol{II}} \colon \forall h \in C \ \tilde{x}(g_k h) = m\right\}\right|}{\left|B_{n,j}^{\boldsymbol{II}}\right|} = 1$$

Since all the rule sets $R_n^{\boldsymbol{I}}$ and $R_n^{\boldsymbol{II}}$ are invariant, and the function f is invariant, $\tilde{A}$ is invariant. We want to show that whichever player has a winning strategy in the game $\tilde{A}$ has a winning strategy for A.

The rule sets $R_n^{\boldsymbol{I}}$, $R_n^{\boldsymbol{II}}$ are easily $\boldsymbol{\Pi}_3^0$ if S is finite, and $\boldsymbol{\Sigma}_4^0$ if $S = \omega$. This easily gives that $\tilde{A}$ is the union of a $\boldsymbol{\Sigma}_5^0$ set with the intersection of a $\boldsymbol{\Pi}_5^0$ set and $f^{-1}(A)$. A simple computation gives that f is $\boldsymbol{\Delta}_6^0$-measurable. Since $\boldsymbol{\Gamma}$ is reasonable, $\tilde{A} \in \boldsymbol{\Gamma}$.

Suppose now $\tilde{\tau}$ is a winning strategy for player $\boldsymbol{II}$ in $\tilde{A}$ (the case for player $\boldsymbol{I}$ is similar but slightly easier). We will define a winning strategy τ for player $\boldsymbol{II}$ in A.

We call a class C *declared* (at position p) if $C \cap \mathrm{dom}(p) \neq \emptyset$. At any position p in the game $\tilde{A}$ only finitely many digits of the resulting real $\tilde{x} \in S^G$ have been determined, and thus only finitely many classes C have been declared. For each position p and declared class C relative to p, we have exactly one of the following:

(1) for some $m \in S$, for all moves $p(g)$ so far played with $g \in C$ we have $p(g) = m$, (we say in this case that C is a *m-class*)
(2) or there are moves $p(g)$, $p(h)$ played so far with $g, h \in C$ for which $p(g) \neq p(h)$. (we say in this case that C is an *invalid class*).

For every position p of the game, any class C is either undeclared, an m-class for some unique m, or an invalid class. Over the course of the game, a class can change from an undeclared class to an m-class, and will then either remain an m-class or become an invalid class at some point. Note that invalid classes can never change. Thus the players' progress towards following the rules can actually be measured at a finite position p.

We say a ring $B_{n,j}^{I}$ or $B_{n,j}^{II}$ is declared relative to a position p if all of the classes C in this ring are declared relative to p. Consider one of the sets $B_{n,j}^{I}$ (or $B_{n,j}^{II}$). Suppose p is a position of the auxiliary game and $B_{n,j}^{I}$ is a declared ring. We say (relative to the position p) that $B_{n,j}^{I}$ is an invalid ring if at least $1/10$ of the classes $C \in B_{n,j}^{I}$ are invalid at position p. We say $B_{n,j}^{I}$ is an m-ring if at least $1/2$ of the classes $C \in B_{n,j}^{I}$ are m-classes. If $B_{n,j}^{I}$ is invalid at some position p, and q is a position which extends p, then $B_{n,j}^{I}$ is also invalid with respect to q. If $B_{n,j}^{I}$ (or B_{n-j}^{II}) is declared but not an m-ring with respect to p, then it is not an m-ring with respect to any extension q of p.

Consider the first round of the game A. Suppose I makes first move m_0 in this game, and we define $\tau(m_0)$. Consider the set P_{m_0} of positions p of even length in the auxiliary game $\tilde{A}$ satisfying:

(1) p is consistent with $\tilde{\tau}$.
(2) For every j, every class $C \in B_{0,j}^{I}$, every k and every $g \in C$, if $g_k g \in \mathrm{dom}(p)$, then we have that $p(g_k g) = m_0$.

First note that we cannot have a sequence $p_0, p_1, \ldots$ in P_{m_0} with p_n extending p_{n-1} for all n, and for each n there is a j such that $B_{0,j}^{II}$ is invalid with respect to p_n but either not invalid or not declared with respect to p_{n-1}. For otherwise the limit of the p_n would give a run by $\tilde{\tau}$ for which I has satisfied the rule R_0^{I} but II has not satisfied R_0^{II}, contradicting that $\tilde{\tau}$ is winning for II. Let $q_0 \in P_{m_0}$ be such that there is no extension of q_0 in P_{m_0} which a new ring $B_{0,j}^{II}$ becomes invalid. So, for all sufficiently large j and any q extending q_0, the ring $B_{0,j}^{II}$ is not invalid. Likewise we cannot have a sequence $q_1 \subseteq q_2 \subseteq \cdots$ of positions extending q_0 such that for each n there is a j so that $B_{0,j}^{II}$ is declared and not an m-ring (for any m) at q_n, but is not declared at q_{n-1}. For in this case each of these rings $B_{0,j}^{II}$ would remain not m-rings in the limiting run, which again violates R_0^{II}. By extending q_0 we may assume that for all sufficiently large j and all q extending q_0, $B_{0,j}^{II}$, if it is declared at q, is an m-ring for some m at q. Note that this ring will remain an m-ring for all further extension r of q, since it cannot change to become an invalid ring or an m' ring for any $m' \neq m$. Finally, a similar argument shows that we cannot have a sequence $q_1 \subseteq q_2 \subseteq \cdots$ extending q_0 such that for each n there are two rings $B_{0,j}^{II}$ and $B_{0,j'}^{II}$ declared at q_n but not declared at q_{n-1} with $B_{0,j}^{II}$ an m-ring and $B_{0,j'}^{II}$ an m'-ring with $m \neq m'$. By extending q_0 further we may assume that we have a declared m_1-ring $B_{0,j}^{II}$ with respect to q_0, and such that for all extensions q of q_0 and all $j' > j$, if $B_{0,j'}^{II}$ is declared at q then it is also an m_1-ring We define $\tau(m_0) = m_1$.

In general, suppose I has played $m_0, m_2, \ldots, m_{2k}$ in A. Suppose inductively we have defined positions $q_0 \subseteq q_1 \subseteq \cdots \subseteq q_{k-1}$. Let $P_{m_0,\ldots,m_{2i}}$, for $i \leq k$, be the set of positions p in $\tilde{A}$ extending q_{i-1} such that for all $i' \leq i$, $C \in B_{i',j}^{I}$ and $g \in C$, and all k, if $g_k g \in (\mathrm{dom}(p) \setminus \mathrm{dom}(q_{i-1}))$ then we have $p(g_k g) = m_{2i'}$. We inductively assume that for all sufficiently large j and any q extending q_{k-1} in $P_{m_0,\ldots,m_{2k-2}}$, and for any $i < k$, if $B_{i,j}^{II}$ is declared at q then it is an m_{2i+1}-ring, where $m_{2i+1} = \tau(m_0, \ldots, m_{2i})$. We now consider extensions of q_{k-1} in $P_{m_0,\ldots,m_{2k}}$. By the same arguments as above, there is a $q_k \in P_{m_0,\ldots,m_{2k}}$ extending q_{k-1} such that for all large enough q and all extensions q of q_k, if $B_{k,j}^{II}$ is declared at q, then it is an m_{2k+1}-ring for some fixed integer m_{2k+1}. We let $\tau(m_0, \ldots, m_{2k}) = m_{2k+1}$. This complete the definition of the strategy τ.

To see that τ is winning, note that for any run x according to τ, we have a sequence of positions $q_0, q_1, \ldots$ which give us a run $\tilde{x}$ consistent with $\tilde{\tau}$ which follows all the rules. From the definition of the q_i we have that $f(\tilde{x}) = x$. Thus since $\tilde{\tau}$ is winning in $\tilde{A}$, we know that $\tilde{x} \notin f^{-1}(A)$, and so $x \notin A$, resulting in a win for player II in the game A. $\qquad \square$

5. $(\mathbf{\Gamma}, E)$-DETERMINACY

We now present a notion of $(\mathbf{\Gamma}, E)$-determinacy for more general equivalence relations on Polish spaces.

Definition 12. Let $\mathbf{\Gamma}$ be a pointclass and E an equivalence relation on a Polish space X. We say $(\mathbf{\Gamma}, E)$-determinacy holds if for all continuous, onto $\pi\colon \omega^\omega \to X$ and all $A \subseteq X$ which are E-invariant $\mathbf{\Gamma}$ sets, we have that $\pi^{-1}(A)$ is determined.

First we note that the restriction that the coding maps π be onto is necessary to avoid trivialities. For by the Silver dichotomy, for every Borel equivalence relation E on X with uncountably many classes, there is a continuous map $\pi\colon \omega^\omega \to X$ such that $x \neq y$ implies $\neg\, \pi(x) E \pi(y)$. Given $A \subseteq \omega^\omega$ in some pointclass $\mathbf{\Gamma}$, let $B \subseteq X$ be the E-saturation of $\pi(A)$. Then B is an E-invariant subset of X which also lies in $\mathbf{\Gamma}$ for most pointclasses (in particular for all pointclasses closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$). Since $\pi^{-1}(B) = A$ we see that if we allow non-surjective maps π in Definition 12 then $(\mathbf{\Gamma}, E)$-determinacy trivially implies $\mathbf{\Gamma}$-determinacy (even restricting the maps to be continuous).

In the case where $X \subseteq S^G$ ($S = 2$ or ω), it is immediate that $(\mathbf{\Gamma}, E)$-determinacy by Definition 12 implies $(\mathbf{\Gamma}, E)$-determinacy by Definition 1. In the case where $E = E_G$ is the equivalence relation generated by the shift action of the group G on S^G or ω^G, Theorem 3 shows that for any reasonable pointclass that $(\mathbf{\Gamma}, E_G)$-determinacy as in Definition 1 implies full $\mathbf{\Gamma}$-determinacy and thus $(\mathbf{\Gamma}, E)$-determinacy as in Definition 12. So, for all reasonable pointclasses, the two definitions are equivalent for shift actions on S^G. For more general shift actions on $X \subseteq S^G$ (such as subshifts of finite type considered in §6) it is not clear that the definitions are equivalent, and the stronger Definition 12 seems to be the more reasonable definition, and as well as the natural generalization to arbitrary equivalence relations.

Another possible variation of Definition 12 would be to allow Borel onto maps $\pi\colon \omega^\omega \to X$. Although we do not see that this version trivializes the notion, it seems more natural to require the coding maps to be as effective as possible.

A common situation is that we wish to impose a set of "rules" on the players in the game $\pi^{-1}(A)$. We next show that a more general version of $(\mathbf{\Gamma}, E)$-determinacy which allows for rules imposed on the game is in fact equivalent to $(\mathbf{\Gamma}, E)$-determinacy as in Definition 12.

Definition 13. Let $T \subseteq \omega^{<\omega}$ be a pruned tree (i.e., T has no terminal nodes). We say $(\mathbf{\Gamma}, E, T)$-determinacy holds if for every continuous onto map $\pi\colon [T] \to X$ and every E-invariant $\mathbf{\Gamma}$ set $A \subseteq X$, we have that game $G(\pi^{-1}(A), T)$ with payoff set $\pi^{-1}(A)$ and rule set T is determined.

Theorem 14. *For every pointclass $\mathbf{\Gamma}$, for every equivalence relation E, we have that $(\mathbf{\Gamma}, E)$-determinacy implies $(\mathbf{\Gamma}, E, T)$-determinacy for every T.*

Proof. Assume $(\mathbf{\Gamma}, E)$-determinacy and let $T \subseteq \omega^{<\omega}$ be a pruned tree, and let $\pi\colon [T] \to X$ be a continuous, onto map. Fix an E-invariant $\mathbf{\Gamma}$ set $A \subseteq X$. We define a continuous onto map $\pi'\colon \omega^\omega \to X$ which extends π. Let $x \in \omega^\omega \setminus [T]$, and we define $\pi'(x)$. Let s be the least initial segment of x with $s \notin T$. Let $s' = s{\upharpoonright}(|s|-1)$. We consider two cases. First suppose that $N_{s'} \cap \pi^{-1}(A) \neq \emptyset$ and $N_{s'} \cap \pi^{-1}(X \setminus A) \neq \emptyset$. Fix x_s, y_s in $[T]$ with $\pi(x_s) \in A$ and $\pi(y_s) \notin A$. If $|s| - 1$ is even (i.e., player $\boldsymbol{I}$ was responsible for first violating the rules) then we set $\pi'(x) = \pi(y_s)$. Likewise, if $|s| - 1$ is odd, we set $\pi'(s) = \pi(x_s)$. Next suppose that $N_{s'} \cap [T] \subseteq \pi^{-1}(A)$ or $N_{s'} \cap [T] \subseteq \pi^{-1}(X \setminus A)$. Note that the game is essentially decided at this point, so our intention is to ignore further violations of the rules. In this case let $\pi'(x) = \pi(\ell(x))$ where $\ell\colon \omega^\omega \to [T]$ is a fixed Lipschitz continuous retraction of ω^ω to $[T]$. We clearly have that π' is continuous and extends π.

By the assumption of $(\mathbf{\Gamma}, E)$-determinacy, the game $\pi'^{-1}(A) \subseteq \omega^\omega$ is determined. Say without loss of generality that σ' is a winning strategy for $\boldsymbol{I}$ in $\pi'^{-1}(A)$. Let $\sigma = \ell \circ \sigma'$ (as ℓ is Lipschitz, we may view σ' as defined on sequences $s \in \omega^{<\omega}$). We show that σ is winning for $\boldsymbol{I}$ in $G(\pi^{-1}(A), T)$. Since $\sigma = \ell \circ \sigma'$, $\boldsymbol{I}$ following σ will never first move off of the tree T. So we assume therefore $\boldsymbol{II}$ always moves in the tree T. Consider a run x of $G(\pi^{-1}(A), T)$ where $\boldsymbol{I}$ follows σ and both players move in T. If for every even n we have that $\sigma'(x{\upharpoonright}n) = \sigma(x{\upharpoonright}n)$, then x is also a run of σ' and so $x \in \pi^{-1}(A)$. Suppose that there is a least (even) n so that $\sigma'(x{\upharpoonright}n) \neq \sigma(x{\upharpoonright}n)$, that is $(x{\upharpoonright}n)^\frown \sigma'(x{\upharpoonright}n) \notin T$. Let $s' = x{\upharpoonright}n$ and $s = s'^\frown \sigma'(s')$. We cannot be in the first case above (that is, both $N_{s'} \cap \pi^{-1}(A)$ and $N_{s'} \cap \pi^{-1}(X \setminus A)$ are non-empty), as otherwise $\pi'(x) = y_{s'} \notin A$, and would be a loss of $\boldsymbol{I}$ in $\pi'^{-1}(A)$, a contradiction. In the second case we have either $N_{s'} \cap [T] \subseteq \pi^{-1}(A)$ or $N_{s'} \cap [T] \subseteq \pi^{-1}(X \setminus A)$. We cannot have that $N_{s'} \cap [T] \subseteq \pi^{-1}(X \setminus A)$ since then
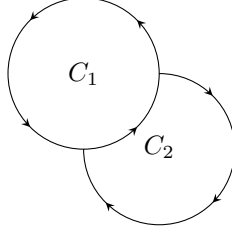
FIGURE 1. a double loop

$N_{s'} \subseteq \pi'^{-1}(X \setminus A)$, which contradicts σ' being winning for $\boldsymbol{I}$. So we have $N_{s'} \cap [T] \subseteq \pi^{-1}(A)$ and so since $\sigma = \ell \circ \sigma'$, $x \in N_{s'} \cap [T] \subseteq \pi^{-1}(A)$, and so $\boldsymbol{I}$ has won the run following σ. $\qquad\square$

6. SUBSHIFTS OF FINITE TYPE

In this section we consider $(\boldsymbol{\Gamma}, E)$-determinacy where E is the equivalence relation corresponding to a subshift $X \subseteq 2^{\mathbb{Z}}$ of finite type. Recall this means that there is a finite set of "forbidden words" $w_1, \ldots, w_e \in 2^{<\omega}$ and $X = \{x \in 2^{\mathbb{Z}} : \forall k \in \mathbb{Z} \ \forall \ell \leq e \ x{\restriction}[k, k + |w_\ell|] \neq w_\ell\}$, where $|w|$ denotes the length of the word w.

Theorem 15. *Let E be the shift equivalence relation on a subshift $X \subseteq 2^{\mathbb{Z}}$ of finite type, and assume E has uncountably many classes. Then for all reasonable pointclasses $\boldsymbol{\Gamma}$, $(\boldsymbol{\Gamma}, E)$-determinacy implies $\boldsymbol{\Gamma}$-determinacy.*

Proof. Let $w_1, \ldots, w_e$ be the forbidden words of the subshift X. Fix $N \geq \max\{|w_i| : 1 \leq i \leq e\}$. Let G be the finite directed graph, the de Bruijn graph, corresponding to the forbidden words and N. That is, the vertices of G are elements of 2^N which don't contain any forbidden words, and (u, v) is an edge in G iff $v{\restriction}[0, N-1] = u{\restriction}[1, N]$.

Throughout the rest of this section, G will denote this fixed de Bruijn graph (and not a countable group).

Definition 16. Let $u \in G$. We say u is *good to the right* there are uncountably many directed paths $p = (u, u_1, u_2, \ldots)$ in G starting from u. Likewise we say u is *good to the left* if there are uncountably many $(\ldots, u_2, u_1, u)$ paths starting from u and moving in the reverse direction in G (i.e., (u_{n+1}, u_n) is an edge in G).

If v is good to the right and there is a path $u = u_0, u_1, \ldots, u_n = v$ from u to v in the graph G, then u is good to the right as well. Likewise in this case, if u is good to the left, then v is also good to the left. This simple observation will be used throughout.

Note that an element of X can be identified with a bi-infinite path through G.

Definition 17. A *double loop* in a directed graph G is a directed subgraph consisting of the union of two cycles with vertex sets C_1 and C_2 such that $C_1 \cap C_2 \neq \emptyset$ and $C_1 \neq C_2$. See Figure 1.

Lemma 18. *If G is a finite directed graph with uncountably many paths, then G contains a double loop.*

Proof. Let u_0 be a vertex in G for which there are uncountably many directed paths in G starting from u_0. Inductively define $(u_0, u_1, \ldots, u_n)$, a directed path in G, such that there are uncountably many directed paths in G starting from u_n. We can clearly continue this construction until we reach a least n so that $u_n \in \{u_0, \ldots, u_{n-1}\}$ (since G is finite). Let $j \leq n - 1$ be such that $u_n = u_j$, Let p_0 denote this directed path $(u_0, u_1, \ldots, u_n)$. Let $\ell_0 = (u_j, u_{j+1}, \ldots, u_{n-1})$ be the "loop" portion of p_0. There must be a vertex $v_0 = u_k$ of ℓ_0 such that there is an edge (v_0, v_1) in G where $v_1 \neq u_{k+1}$ and such that there are uncountably many directed paths in G starting from v_1. For if not, then the only directed paths in G starting from v_0 would be, except for a countable set, those which continually follow the loop ℓ_0. This is a contradiction to the definition of v_0 as there are only countably many such paths. If $v_1 \in p_0$ then we have a double loop in the graph. If not, then we repeat the process starting at v_1 forming a path $p_1 = (v_0, v_1, \ldots, v_{m-1}, v_m)$ where $v_m \in p_0 \cup (v_0, \ldots, v_{m-1})$. If $v_m \in p_0$, then we have a double loop in G. Otherwise $v_m = v_i$ for some
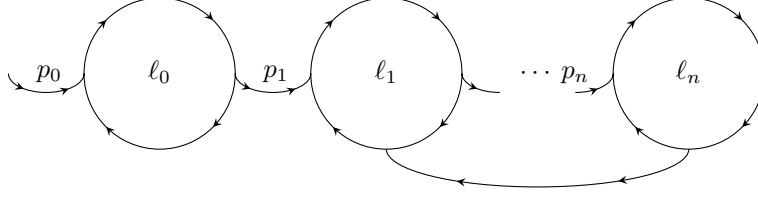
FIGURE 2

$i \leq m - 1$, and $\ell_1 = (v_i, \ldots, v_{m-1})$ gives another loop in G. Since G is finite, we must eventually produce a double loop in G.

$\square$

Note that every vertex in a double loop is good to the right and left. Returning to the proof of the theorem, by our assumptions, the de Bruijn graph G for the subshift has a double loop, and thus G has a vertex which is good to the right and left.

We now define the continuous onto map $\pi \colon [T] \to X$, where T will be a pruned tree on ω which we will be implicitly defining as we describe π. Let $x \in \omega^\omega$, and we describe the conditions on x which give $x \in [T]$, and in this case describe $\pi(x) \in X \subseteq 2^{\mathbb{Z}}$. First, view every digit i as coding a binary sequence u_i of length N (recall N was maximum size of the forbidden words, and was used to construct the de Bruijn graph G). Fix a fast growing sequence $0 = b_0 < b_1 < \cdots$, with say $\lim_i \frac{\sum_{j<i} b_j}{b_i} = 0$, and with $N | (b_i - b_{i-1})$ for all i. Let $A_i = [b_{i-1}, b_i)$ for $i \in \omega^+$. Let also $A_{-i} = -A_i - 1 = \{-a - 1 \colon a \in A_i\}$.

Let $n \mapsto u_n$ be an onto map from ω to 2^N (binary sequences of length N). We first define a preliminary map π'. Given $x \in \omega^\omega$ we define $y' = \pi'(x)$ as follows. Given the first $2n$ digits of x, that is, $x{\restriction}2n$, we define $y'{\restriction}[-nN, nN)$. Assume $y'{\restriction}[-nN, nN)$ has been defined, and from $x(2n)$, $x(2n + 1)$ we extend to $y'{\restriction}[-(n+1)N, (n+1)N)$. If $nN \in A_i$ for i odd, then $y'{\restriction}[nN, (n+1)N) = u_{x(2n)}$ and $y'{\restriction}[-(n+1)N, -nN) = u_{x(2n+1)}$. If i is even, then we let $y'{\restriction}[nN, (n+1)N) = u_{x(2n+1)}$ and $y'{\restriction}[-(n+1)N, -nN) = u_{x(2n)}$. Let $\tilde{T} \subseteq \omega^{<\omega}$ be the tree of sequences $x{\restriction}2n$ such that the corresponding sequence $y'{\restriction}[-nN, nN)$ contains no forbidden words. Note that $[\tilde{T}] = \pi'^{-1}(X)$. Let $F = [\tilde{T}]$ be the closed set corresponding to $\tilde{T}$, and let $T \subseteq \tilde{T}$ be the pruned tree with $[T] = [\tilde{T}] = F$.

We now define $y = \pi(x)$. If $\mathbf{I}'s$ first move $u_{x(0)}$ is bad in both directions, we set $\pi(x) = \pi'(x)$. If for all n we have that $u_{x(n)}$ is good to the left and right, then we also set $\pi(x) = \pi'(x)$. Otherwise, say n_0 is least such that $u_{x(2n_0)}$ or $u_{x(2n_0+1)}$ is not good to both the left and right.

Suppose that $u_{x(2n_0)}$ is not good in both directions, that is, $\mathbf{I}$ has made the first such move. Note that $u_{x(2n_0)}$ must be good in one direction as we are assuming that $\mathbf{I}$'s first move $u_{x(0)}$ is good in at least one direction. In this case we ignore the remainder of $\mathbf{I}$'s moves in the game, and we alternate concatenating $\mathbf{II}$'s moves on the left and right. This produces $y = \pi(x)$. If $\mathbf{II}$ first makes the move $u_{x(2n_0+1)}$ which is not good in both direction, we similarly ignore the future moves of $\mathbf{II}$ and alternating concatenating $\mathbf{I}$'s move to the left and right. This defines $y = \pi(x)$ in all cases.

Let $A \subseteq 2^\omega$ be a $\mathbf{\Gamma}$ set. We define the auxiliary payoff set $\tilde{A} \subseteq X$ to which we will apply the hypothesis of $(\mathbf{\Gamma}, E)$-determinacy for the π constructed above. The set $\tilde{A}$ will be a shift-invariant $\mathbf{\Gamma}$ set. The construction of $\tilde{A}$ will be similar to that of Theorem 6. Say $x \in [T] \subseteq \omega^\omega$ is the play of the game $\pi^{-1}(\tilde{A})$ where both players have followed the rules (i.e., no forbidden words appear in $y = \pi(x)$).

First, if there exists a subword of length N in y which is bad to right and also a subword which is bad to the left, then $\mathbf{I}$ loses (this is part of the definition of $\tilde{A}$). In the remaining cases we assume that there is at least one direction so that all subwords of y are good in that direction.

Recall the sets A_i have been defined for $i \in \mathbb{Z} \setminus \{0\}$. Let A'_i be defined (for $i \in \mathbb{Z} \setminus \{0\}$) by $A'_i = A_i$ if every subword of y is good to the right, and otherwise let $A'_i = A_{-i-1}$. As in the proof of Theorem 6 we partition ω into ω-many pairwise disjoint sets $B^{\mathbf{I}}_{n,j}$, $B^{\mathbf{II}}_{n,j}$. These sets are defined exactly as before using the sets A'_i for $i > 0$, except (for notational convenience) we let the $B^{\mathbf{I}}_{n,j}$ correspond to odd i, and the $B^{\mathbf{II}}_{n,j}$ to even i.

In order to define the rule sets for the game we will make use of the following notion.

Definition 19. Given $y \in X$ and a double loop (C_0, C_1) in G, and given integers $a < b$, we say $y{\restriction}[a, b)$ *traces the double loop with pattern* $s \in 2^{<\omega}$ if $v_0 = y{\restriction}[a, a + N) \in C_0 \cap C_1$, and if $v_i = y{\restriction}[a + i, a + i + N)$, then the sequence $v_0, v_1, \ldots, v_{b-a-N+1}$ of nodes in G is a path in G of the form $C'_{s(0)}{}^\frown C'_{s(1)}{}^\frown \cdots C'_{s(|s|-1)}$ where C'_0, C'_1 are the same cycles as C_0, C_1 except we start at the vertex v_0. If $|s^{-1}(i)| > |s^{-1}(1 - i)|$, we call C_i the *majority loop* and C_{1-i} the *minority loop*.

We define the conditions $R^{\mathbf{I}}_n$, $R^{\mathbf{II}}_n$ and the decoding function $f \colon X \to 2^\omega$ as follows. We fix an ordering on the cycles of G (the de Bruijn graph) as so write each double loop in G as (C_0, C_1) where $C_0 < C_1$. This makes the representation of each double loop unique. In the definition of $R^{\mathbf{I}}_n$ (and likewise for $R^{\mathbf{II}}_n$) we require that the following hold:

(2) $y \in R^{\mathbf{I}}_n \leftrightarrow \exists i \in \{0, 1\} \; \forall \epsilon > 0 \; \exists j_0 \; \forall j \geq j_0 \; \exists$ double loop (C_0, C_1) such that the following hold:

$$\exists [a, b) \subseteq B^{\mathbf{I}}_{n,j} \; (b - a) > (1 - \epsilon)|B^{\mathbf{I}}_{n,j}| \text{ and } y{\restriction}[a, b) \text{ traces } (C_0, C_1) \text{ with pattern } s \in 2^{<\omega}$$

$$|s^{-1}(i)| \geq (1 - \epsilon)\frac{2}{3}|s|$$

$$|s^{-1}(1 - i)| \geq (1 - \epsilon)\frac{1}{10}|s|$$

Likewise we define $R^{\mathbf{II}}_n$ using the $B^{\mathbf{II}}_{n,j}$ sets. If there is some direction so that every subword of y is good in that direction and $y \in \bigcap_n R^{\mathbf{I}}_n \cap \bigcap_n R^{\mathbf{II}}_n$, then we define the decoding map f at y by $f(y)(2n)$ is the witness $i \in \{0, 1\}$ to $y \in R^{\mathbf{I}}_n$, and likewise $f(y)(2n + 1)$ is the witness to $y \in R^{\mathbf{II}}_n$.

To ensure f is well-defined, we note that if i witnesses $y \in R^{\mathbf{I}}_n$, then $1 - i$ does not. This is because for small enough ϵ, for all large enough j if y traces (C_0, C_1) with pattern s over a subinterval I of length $(1 - \epsilon)|B|$ of some fixed $B = B^{\mathbf{I}}_{n,j}$ then the double loop (C_0, C_1) is unique (for this j). Say C_i ($i \in \{0, 1\}$) is the majority loop. The majority loop C_i is traced say M times, where $M \geq (1 - \epsilon)\frac{2}{3}|s|$ many times. The minority loop is traced say m times, where $m \leq (\frac{1}{3} + \frac{2}{3}\epsilon)|s| + \epsilon|B|$. The condition $m \leq M$ becomes $\epsilon|B| \leq (\frac{1}{3} - \frac{4}{3}\epsilon)|s|$. Since $|s| \geq (1 - \epsilon)\frac{|B|}{|G|}$ it suffices to have $\epsilon < (\frac{1}{3} - \frac{4}{3}\epsilon)\frac{(1-\epsilon)}{|G|}$. Clearly this is satisfied for ϵ small enough. On the other hand, $m \geq (1 - \epsilon)\frac{1}{10}|s| \geq (1 - \epsilon)^2\frac{1}{10}\frac{|B|}{|G|}$. Any other loop can be traced only in $B \setminus I$, and so can be traced at most $\epsilon|B|$ many times. But $(1 - \epsilon)^2\frac{1}{10}\frac{|B|}{|G|} \geq \epsilon|B|$ holds for all small enough ϵ. So, there is a fixed ϵ, independent of j, so that if $y{\restriction}B^{\mathbf{I}}_{n,j}$ satisfies Equation 2 for this ϵ then the double loop (C_0, C_1) is well-defined as is the integer $i \in \{0, 1\}$ with C_i being the majority loop.

On the invariant set of y such that there is at least one direction so that all subwords are good in that direction we define $\tilde{A}$ by:

$$\tilde{A} = \bigcup_{n \in \omega} \left(\bigcap_{i \leq n} R^{\mathbf{I}}_i \setminus R^{\mathbf{II}}_n \right) \cup \left(\bigcap_{n \in \omega} (R^{\mathbf{I}}_n \cap R^{\mathbf{II}}_n) \cap f^{-1}(A) \right)$$

This completes the definitions of π, and the auxiliary game $\tilde{A}$.

We next observe that the auxiliary game $\tilde{A} \subseteq 2^{\mathbb{Z}}$ is shift invariant. Given $y \in 2^{\mathbb{Z}}$, the case split as to whether there is a direction so that all all subwords of y are good in that direction is clearly shift invariant (and the set of such directions is also invariant). In the case where there is at least one such good direction, whether $y \in \tilde{A}$ is decided by putting down the sets A'_i for $i \in Z \setminus \{0\}$, using these to define the sets $B^{\mathbf{I}}_{n,j}$, $B^{\mathbf{II}}_{n,j}$, then defining the sets $R^{\mathbf{I}}_n$, $R^{\mathbf{II}}_n$ as in Equation 2 which gives the decoding function f and finally asking whether $f(y) \in A$. It suffices to show that the sets $R^{\mathbf{I}}_n$, $R^{\mathbf{II}}_n$ are invariant (in that $y \in R^{\mathbf{I}}_n$ iff $m \cdot y \in R^{\mathbf{I}}_n$ for all $m \in \mathbb{Z}$), as this implies that the decoding function f is also invariant. The intervals $B^{\mathbf{I}}_{n,j}(m \cdot y)$ as defined for the shift $m \cdot y$ are just the shifts $m \cdot B^{\mathbf{I}}_{n,j}(y)$ of the corresponding sets $B^{\mathbf{I}}_{n,j}(y)$ for y. In particular, $|B^{\mathbf{I}}_{n,j}(y) \cap B^{\mathbf{I}}_{n,j}(m \cdot y)|/|B^{\mathbf{I}}_{n,j}(y)|$ tends to 1 as j goes to infinity. Thus the asymptotic condition of Equation 2 holds for y iff it holds for $m \cdot y$. The value of i in Equation 2 is therefore the same for both y and $m \cdot y$. This shows that $f(y) = f(m \cdot y)$ and so $y \in \tilde{A}$ iff $m \cdot y \in \tilde{A}$.

By the assumption of $(\mathbf{\Gamma}, E)$-determinacy the game $\pi^{-1}(\tilde{A})$ on ω is determined. First consider the case where $\mathbf{I}$ has a winning strategy σ in $\pi^{-1}(\tilde{A})$.

Claim 20. If $x(0), x(1), \ldots, x(2n)$ is a position of the game $\pi^{-1}(\tilde{A})$ consistent with σ in which all of $\mathbf{II}$'s moves $u_{x(2k+1)}$, $k < n$, are good in both directions, then $u_{x(2n)}$ is good in both directions.

Proof. We first note that $u_{x(2n)}$ is not bad in both directions. If $n > 0$ this is clear as $u_{x(0)}$ is good in both directions and the last move is legal, and so good in (at least) the direction pointing back to $u_{x(0)}$. If $n = 0$, and $u_{x(0)}$ is bad in both directions, then there is no direction for which every subword of the resulting y is good, which is a loss for $\mathbf{I}$, a contradiction. Suppose now that $u_{x(2n)}$ were bad in exactly one direction, say bad to the right. If the definition of π we gave full control of all future moves to $\mathbf{II}$ (we ignored $\mathbf{I}$'s moves after this point). But $\mathbf{II}$ can now play moves to violate $R_0^{\mathbf{I}}$ and thus produce a loss for $\mathbf{I}$, a contradiction. For example, $\mathbf{II}$ can move (for each of the two directions) to a cycle of G, and simply trace this cycle forever. $\qquad\square$

We will construct a position p_0 of odd length which is consistent with σ, so that σ is committed to a particular witness i_0 to its following of $R_n^{\mathbf{I}}$, which will be our first move. Fix $\bar{\epsilon}$ small enough so that for all large enough j, if a double loop is traced in $B_{n,j}^{\mathbf{I}}$ meeting the conditions of Equation 2 for this $\bar{\epsilon}$, then that double loop is unique.

Now consider the tree of positions of odd length which

(1) are consistent with σ,
(2) in which player $\mathbf{II}$ has made only moves which are good in both directions,
(3) there is some j such that for the partial sequence $y\restriction[c,d)$ constructed so far, we have $[c,d) \cap B_{0,j}^{\mathbf{I}} = B_{0,j}^{\mathbf{I}}$ and $[c,d) \cap B_{0,j+1}^{\mathbf{I}} = \emptyset$,
(4) and that $y\restriction[c,d)$ doesn't satisfy the existence of a double loop as in Equation 2 on $B_{0,j}^{\mathbf{I}}$ with $\epsilon = \bar{\epsilon}$.

This tree must be wellfounded, as a branch could be used to produce a loss for $\mathbf{I}$ which is consistent with σ, a contradiction. Let p_0' be a terminal position in this tree. Next, we consider the tree of positions of odd length extending p_0' which

(1) are consistent with σ,
(2) in which player $\mathbf{II}$ has made only moves which are good in both directions,
(3) there is some j so that for the partial sequence $y\restriction[c,d)$ constructed so far, we have $[c,d) \cap B_{0,j+1}^{\mathbf{I}} = B_{0,j+1}^{\mathbf{I}}$ and $[c,d) \cap B_{0,j+2}^{\mathbf{I}} = \emptyset$,
(4) if $y\restriction[c,d)$ traces (C_0, C_1) in $B_{0,j}^{\mathbf{I}}$ and (C_0', C_1') in $B_{0,j+1}^{\mathbf{I}}$ satisfying the conditions of Equation 2 with $\epsilon = \bar{\epsilon}$ and the majority loop which $y\restriction[c,d)$ traces in $B_{0,j}^{\mathbf{I}}$ is C_i, then the majority loop $y\restriction[c,d)$ traces in $B_{0,j+1}^{\mathbf{I}}$ is C_{1-i}'.

Again, this tree must be wellfounded, since a branch would result in a y which fails to satisfy $R_0^{\mathbf{I}}$. Let p_0 be any terminal node of this tree and let j_0 be such that if $y\restriction[c_0, d_0)$ is the portion of y constructed at p_0, then $[c_0, d_0) \cap B_{0,j_0}^{\mathbf{I}} = B_{0,j_0}^{\mathbf{I}}$ and $[c_0, d_0) \cap B_{0,j_0+1}^{\mathbf{I}} = \emptyset$. Notice that for any extension q of p_0 which is consistent with σ and in which $\mathbf{II}$'s moves are good in both directions, with $y\restriction[c,d)$ the portion of y constructed at q, we will have that $y\restriction[c,d)$ traces a double loop in each $B_{0,j}^{\mathbf{I}}$ for all $j \geq j_0$, satisfying the conditions in Equation 2 with $\epsilon = \bar{\epsilon}$. Furthermore, there will be some $i_0 \in \{0, 1\}$ so that for any $j \geq j_0$, if $y\restriction[c,d)$ traces (C_0, C_1) in $B_{0,j}^{\mathbf{I}}$, then C_{i_0} will be the majority loop and C_{1-i_0} will be the minority loop.

Our first move in the game A will be to play this i_0. Next suppose our opponent plays i_1. We will again consider a tree of positions consistent with σ. We consider only positions of odd length extending p_0 which

(1) are consistent with σ,
(2) in which player $\mathbf{II}$ has made only moves which are good in both directions,
(3) in each new $B_{0,j}^{\mathbf{II}}$ player $\mathbf{II}$ has moved as quickly as possible to the nearest double loop (C_0, C_1) in G, and in $B_{0,j}^{\mathbf{II}}$ declares C_{i_1} the majority loop and C_{1-i_1} the minority loop (say by tracing the minority loop sufficiently many times first, then the majority loop for the rest of the time) to satisfy the conditions in Equation 2,

(4) for the partial sequence $y{\restriction}[c,d)$ constructed so far, we have for some j that $[c,d) \cap B^{I}_{1,j} = B^{I}_{1,j}$ and $[c,d) \cap B^{I}_{1,j+1} = \emptyset$,

(5) and that $y{\restriction}[c,d)$ doesn't satisfy the conditions of Equation 2 on $B^{I}_{1,j}$ with $\epsilon = \bar{\epsilon}$.

This tree must be wellfounded, since along a branch, $\boldsymbol{II}$ would be satisfying $R^{\boldsymbol{II}}_0$ (with i_1) but player $\boldsymbol{I}$ isn't satisfying $R^{\boldsymbol{I}}_1$, which would be a loss consistent with σ, a contradiction. Let p'_1 be a terminal position in this tree. Next, we consider the tree of positions of odd length extending p'_1 which

(1) are consistent with σ,

(2) in which player $\boldsymbol{II}$ has made only moves which are good in both directions,

(3) in each new $B^{\boldsymbol{II}}_{0,j}$ player $\boldsymbol{II}$ has moved as quickly as possible to the nearest double loop (C_0, C_1) in G, and in $B^{\boldsymbol{II}}_{0,j}$ declares C_{i_1} the majority loop and C_{1-i_1} the minority loop to satisfy the conditions in Equation 2,

(4) there is some j so that for the partial sequence $y{\restriction}[c,d)$ constructed so far, we have $[c,d) \cap B^{I}_{1,j+1} = B^{I}_{1,j+1}$ and $[c,d) \cap B^{I}_{1,j+2} = \emptyset$,

(5) if $y{\restriction}[c,d)$ traces (C_0, C_1) in $B^{I}_{1,j}$ and (C'_0, C'_1) in $B^{I}_{1,j+1}$ satisfying the conditions of Equation 2 with $\epsilon = \bar{\epsilon}$ and the majority loop which $y{\restriction}[c,d)$ traces in $B^{I}_{1,j}$ is C_i, then the majority loop $y{\restriction}[c,d)$ traces in $B^{I}_{1,j+1}$ is C'_{1-i}.

This tree also must be wellfounded, since $\boldsymbol{I}$ must commit to a particular i_2 with which to satisfy $R^{\boldsymbol{I}}_1$, and a branch through this tree would have $\boldsymbol{I}$ changing its answer infinitely often, resulting in a loss consistent with σ. Let p_1 be any terminal position in this tree, and let i_2 be the digit which $\boldsymbol{I}$ will declare in each new $B^{I}_{1,j}$ from p_1 onwards.

We play the move i_2 in A and continue playing in this manner, using each new move i_{2n+1} by our opponent to satisfy in the auxiliary game $\pi^{-1}(\tilde{A})$ an additional rule $R^{\boldsymbol{II}}_n$, and then by moving to terminal positions p_n in wellfounded trees, and use these to fix digits i_{2n} which we will play in A. By the construction of these positions p_n, we will have that if y is the resulting element of X corresponding to the sequence of moves $i_0, i_1, \cdots$ in A, then we will have $f(y)(n) = i_n$, and that $y \in \bigcap_n (R^{\boldsymbol{I}}_n \cap R^{\boldsymbol{II}}_n)$, so that our strategy is winning for $\boldsymbol{I}$ in A.

Consider now the case where $\boldsymbol{II}$ has a winning strategy τ in the game $\pi^{-1}(\tilde{A})$. The argument is similar to the case above where $\boldsymbol{I}$ had the winning strategy, so we just sketch the differences. By Lemma 18 there is a double loop in the graph G, and therefore there is a word of length N which is good in both directions. Have $\boldsymbol{I}$ play in $\tilde{A}$ such a word as their first move $u_{x(0)}$. Suppose $\boldsymbol{I}$ plays i_0 in the game A. We proceed as in the argument above having $\boldsymbol{I}$ move as quickly as possible (only making moves which are good in both directions) to a double loop within each $B^{I}_{0,j}$, and moving to encode i_0 within $B^{I}_{0,j}$. An analogous claim to Claim 20 shows that as long as $\boldsymbol{I}$ plays in this manner, τ's moves are also good in both direction. since $\boldsymbol{I}$ is satisfying $R^{\boldsymbol{I}}_0$ (with the digit i_0), a wellfoundedness argument as before will produce a position p_0 consistent with τ and a digit i_1 so that for all runs y consistent with τ in which $\boldsymbol{I}$ plays as just described we have that y satisfies $R^{\boldsymbol{II}}_0$ with i_1. Continuing in this manner, as previously, this defines a winning strategy for $\boldsymbol{II}$ in A.

7. CONCLUSION

We have introduced the general notion of (Γ, E)-determinacy for arbitrary equivalence relations E on Polish spaces and pointclasses Γ. Since the definition involves the use of continuous coding maps from ω^ω onto X (which seems necessary to have a reasonable definition), it is not immediately clear to what extent the structure of the particular Polish space X plays a role. Since all uncountable Polish spaces are Borel isomorphic, it is reasonable to ask the following:

Question 21. Suppose X, Y are uncountable Polish spaces, $\varphi \colon X \to Y$ is a Borel isomorphism, and E_X is a Borel equivalence relation on X. Let E_Y be the corresponding Borel equivalence relation on Y, that is $y_1 E_Y y_2$ iff $\varphi^{-1}(y_1) E_X \varphi^{-1}(y_2)$. Then for any pointclass Γ closed under substitution by Borel functions, countable unions and countable intersections, is it the case that (Γ, E_X)-determinacy is equivalent to (Γ, E_Y)-determinacy?

If E, F are equivalence relation on X with $E \subseteq F$, then we immediately have that $(\mathbf{\Gamma}, E)$-determinacy implies $(\mathbf{\Gamma}, F)$-determinacy (because F-invariant sets are also E-invariant). Since the shift equivalence relation $E_{\mathbb{Z}}$ on $2^{\mathbb{Z}}$ is a subset of the Turing equivalence relation on $2^{\mathbb{Z}}$, we have that $(\mathbf{\Gamma}, E_{\mathbb{Z}})$-determinacy implies $\mathbf{\Gamma}$ Turing-determinacy for any $\mathbf{\Gamma}$. We have shown that $(\mathbf{\Gamma}, E_{\mathbb{Z}})$-determinacy implies full $\mathbf{\Gamma}$-determinacy for any reasonable $\mathbf{\Gamma}$. We recall that Harrington showed that $\mathbf{\Sigma}_1^1$ Turing-determinacy implies $\mathbf{\Sigma}_1^1$-determinacy and Woodin showed that in $L(\mathbb{R})$, Turing-determinacy is equivalent to full determinacy. If E_T is the Turing equivalence relation on 2^{ω}, then our definition of $(\mathbf{\Gamma}, E_T)$-determinacy immediately implies $\mathbf{\Gamma}$ Turing determinacy in the usual sense (all E_T invariant subsets of 2^{ω} are determined) but the converse direction is not clear as we allow more general continuous onto maps for the coding. So we ask the following.

Question 22. For which $\mathbf{\Gamma}$ and E on 2^{ω} is $(\mathbf{\Gamma}, E)$-determinacy equivalent to the determinacy of all E-invariant $\mathbf{\Gamma}$ subsets of 2^{ω}?

Extending Theorem 3 to more general equivalence relations is expected to be a difficult problem. Nevertheless, in Theorem 15 we extended the result to include subshifts of $2^{\mathbb{Z}}$ of finite type.

We recall that the Feldman-Moore theorem states that every countable Borel equivalence relation E on a Polish space is generated by the Borel action of a countable group G (one can also choose the Polish topology to make the action continuous). We also recall the result [1] that every equivalence relation E generated by the action of a countable group G Borel (equivariantly) embeds into the shift action of $G \times \mathbb{Z}$ on $2^{G \times \mathbb{Z}}$. Theorem 6 applies to shift actions of arbitrary countable groups, so the problem of passing to general (not necessarily closed) subshifts embodies the general question of whether $(\mathbf{\Gamma}, E)$-determinacy implies $\mathbf{\Gamma}$-determinacy. In particular, we can ask:

Question 23. For which subshifts (closed, or more generally Borel, invariant subsets X of $2^{\mathbb{Z}}$ with the shift map) of $2^{\mathbb{Z}}$ do we have that $(\mathbf{\Gamma}, E)$-determinacy implies $\mathbf{\Gamma}$-determinacy, where E is the shift equivalence relation restricted to X.

If E is generated by the continuous action of a countable group G on a compact 0-dimensional space X, then [1] shows that (X, E) equivariantly and continuously embeds into a subshift of $2^{\mathbb{Z} \times G}$. Thus, given a positive answer to Question 21, the question of whether $(\mathbf{\Gamma}, E)$-determinacy implies $\mathbf{\Gamma}$-determinacy reduces to considering the question for subshifts of 2^G, for countable groups G.

Aside from the observation above on subequivalence relations, it is not clear how the notion of $(\mathbf{\Gamma}, E)$-determinacy interacts with other aspects of the theory of Borel equivalence relations. So we ask:

Question 24. How does the notion of $(\mathbf{\Gamma}, E)$-determinacy interact with the notions of Borel reducibility of equivalence relations, products of equivalence relations, increasing unions of equivalence relations, etc.?

$\square$

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