GLOBAL RESTRICTION ESTIMATES FOR ELLIPTIC HYPERBOLOIDS

BENJAMIN BAKER BRUCE

Abstract. We prove global Fourier restriction estimates for elliptic, or two-sheeted, hyperboloids of arbitrary dimension $d \geq 2$, extending recent joint work with D. Oliveira e Silva and B. Stovall. Our results are unconditional in the (adjoint) bilinear range, $q > \frac{2(d+3)}{d+1}$, and extend conditionally upon further progress toward the local restriction conjecture for elliptic surfaces.

1. Introduction

In this article, we establish global Fourier restriction estimates for elliptic, or two-sheeted, hyperboloids of arbitrary dimension $d \geq 2$. These surfaces take the general form

$$\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d \colon (\tau - \tau_0, \xi - \xi_0) \cdot A(\tau - \tau_0, \xi - \xi_0) = 1\}, \tag{1.1}$$

where $(\tau_0, \xi_0) \in \mathbb{R} \times \mathbb{R}^d$ and A is an invertible $(d+1) \times (d+1)$ matrix with exactly one positive eigenvalue. Due to the affine invariance of restriction estimates and time-reversal symmetry, we may (and will) restrict our attention to the surface

$$\Sigma := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d \colon \tau = \langle \xi \rangle\}, \qquad \langle \xi \rangle := \sqrt{|\xi|^2 + 1},$$

which is the "upper sheet" of (1.1) with $(\tau_0, \xi_0) = (0, 0)$ and $A = \mathrm{diag}(1, -1, \dots, -1)$.

We begin by describing the context for this project. While certain aspects of the restriction theory for hyperboloids have already been studied, see e.g. [16], [13], [4], [5], the question of the optimal range of global estimates was only recently taken up by Oliveira e Silva, Stovall, and the author in [2]. There, the hyperbolic, or one-sheeted, hyperboloid in three ambient dimensions was studied. The present article generalizes certain techniques from [2] to obtain global restriction estimates for higher-dimensional hyperboloids. As noted above, our results will be stated and proved for elliptic hyperboloids, whose local restriction theory has been well studied (see e.g. [17], [8]); however, similar methods could potentially yield purely conditional results for hyperbolic hyperboloids.

Hyperboloids are geometrically interesting from the viewpoint of restriction theory. Historically, a significant proportion of the work on restriction has focused on compact surfaces, such as the unit sphere or the truncated paraboloid. Indeed, a general form of the restriction conjecture asserts, in part, that every smooth compact surface with at least one nonvanishing principal curvature admits some nontrivial restriction estimate. As is well known, however, homogeneity can sometimes be substituted for compactness. Paraboloids and cones are prototypical examples of noncompact surfaces that obey homogeneity relations and admit nontrivial restriction estimates. While hyperboloids are not homogeneous in this sense, they come close by "interpolating" paraboloids and cones: the surface Σ, for example, resembles the paraboloid $\tau = \frac{1}{2}|\xi|^2 + 1$ as $|\xi| \to 0$ and the cone $\tau = |\xi|$ as $|\xi| \to \infty$. As we will show, hyperboloids appear to admit a range of restriction estimates that interpolates (in a precise sense) the restriction conjectures for paraboloids and cones, and our proof will rest on adaptations of the bilinear restriction theories associated to those surfaces. Also crucial to our arguments will be the invariance of hyperboloids under appropriately defined Lorentz transformations. Certain of these transformations, sometimes termed "orthochronous," additionally preserve each sheet of the two-sheeted hyperboloid. The orthochronous Lorentz group that acts (transitively) on Σ consists of linear maps on $\mathbb{R} \times \mathbb{R}^d$ that preserve the quadratic form $(\tau, \xi) \mapsto \tau^2 - |\xi|^2$ as well as $(\tau, \xi) \mapsto \mathrm{sign}\, \tau$. Throughout this article, "Lorentz" will always mean "orthochronous Lorentz," so that Lorentz transformations are symmetries of Σ.

Date: January 17, 2022.

We now turn to the basic definitions and statements of our results. To begin, we equip Σ with its unique Lorentz-invariant measure μ, given by

$$\int_\Sigma f\,d\mu := \int_{\mathbb{R}^d} f(\langle\xi\rangle,\xi)\frac{d\xi}{\langle\xi\rangle};$$

μ also coincides with the so-called affine surface measure on Σ. The role of μ is twofold: to commute with the symmetries of our surface and to compensate for the surface's degenerating curvature. As is standard, we will formulate our results in terms of the adjoint restriction, or extension, operator. Having equipped Σ with μ, this operator takes the form

$$\mathcal{E}f(t,x) := \widecheck{f\mu}(t,x) = \int_\Sigma e^{i(t,x)\cdot(\tau,\xi)} f(\tau,\xi)\,d\mu(\tau,\xi)$$

for f continuous and compactly supported on Σ. Henceforth, the dual terms "restriction" and "extension" will be used interchangeably. We denote by

$$\Sigma_0 := \{(\tau,\xi)\in\Sigma\colon |\xi|\le 2\} \qquad \text{and} \qquad \mathcal{E}_0 f := \mathcal{E}(f\chi_{\Sigma_0})$$

the low-frequency "parabolic" region of Σ and the corresponding local extension operator. Given $p,q\in[1,\infty]$, we let $\mathcal{E}(p\to q)$ denote the statement that $\mathcal{E}$ extends to a bounded linear operator from $L^p(\Sigma,\mu)$ to $L^q(\mathbb{R}\times\mathbb{R}^d)$, and we write $\mathcal{E}_0(p\to q)$ if the analogous statement holds for $\mathcal{E}_0$. We can now state a conjecture on the complete range of exponent pairs (p,q) for which $\mathcal{E}(p\to q)$ is valid (see Figure 1).

Conjecture 1.1. *If $(\frac{d}{d+2}q)' \le p \le \min\{(\frac{d-1}{d+1}q)', q\}$ and $(p,q)\ne(\frac{2d}{d-1},\frac{2d}{d-1}),(\frac{2(d+1)}{d},\frac{2(d+1)}{d})$, then $\mathcal{E}(p\to q)$ holds.*

The main results of this article are the following:

Theorem 1.2. *If $q>\max\{\frac{2(d+3)}{d+1},p\}$ and $(\frac{d}{d+2}q)'\le p\le(\frac{d-1}{d+1}q)'$, then $\mathcal{E}(p\to q)$ holds. If $d=2$, then additionally $\mathcal{E}(q\to q)$ holds for $\frac{10}{3}<q<4$.*

Theorem 1.3. *If $\mathcal{E}_0(p_0\to q_0)$ holds with $p_0'=\frac{d}{d+2}q_0$ for some $q_0<\frac{2(d+3)}{d+1}$, then $\mathcal{E}(p\to q)$ holds for all (p,q) obeying $\max\{q_0,p\}<q<\frac{2(d+3)}{d+1}$ and $(\frac{d}{d+2}q)'\le p\le(\frac{d-1}{d+1}q)'$ and*

$$\frac{1}{p} > \frac{2}{(d-1)(d+3)}\left(\frac{\frac{1}{q}-\frac{d+1}{2(d+3)}}{\frac{1}{q_0}-\frac{d+1}{2(d+3)}} + \frac{d^2+2d-7}{4}\right). \tag{1.2}$$

The conditional result, Theorem 1.3, may warrant some explanation. It implies, in particular, that any optimal local extension estimate beyond the bilinear range, i.e. $\mathcal{E}_0(p_0\to q_0)$ with $p_0'=\frac{d}{d+2}q_0$ for some $q_0<\frac{2(d+3)}{d+1}$, would lead to an improvement of our unconditional result, Theorem 1.2. The requirement that $p_0'=\frac{d}{d+2}q_0$ is a limitation, but it seems necessary (given our techniques) for obtaining any additional global estimates on the parabolic scaling line $p'=\frac{d}{d+2}q$. If we assume only that $\mathcal{E}_0(p_0\to q_0)$ holds for some $p_0>(\frac{d}{d+2}q_0)'$, then a slight modification of the proof of Theorem 1.3 would likely yield a range of global estimates that excludes the line $p'=\frac{d}{d+2}q$ beyond the bilinear range but nevertheless improves Theorem 1.2.

The rest of the article is organized as follows: In Section 2, we prove a negative result, that Conjecture 1.1 cannot be improved. In Section 3, we present the bilinear restriction theory for the "conic" portion of our surface Σ, and we deduce uniform linear extension estimates on dyadic frusta from bilinear estimates between certain thin sectors. (These results resemble the bilinear restriction theory for cones.) In Section 4, we use a Strichartz inequality for the Klein–Gordon equation to sum the uniform estimates on frusta, and consequently we obtain Theorem 1.2. In Section 5, we use the conic decoupling theorem of [1] to convert conditional local extension estimates into uniform estimates on frusta, and then, appealing to Section 4 again, we obtain Theorem 1.3. Finally, in Section 6, we discuss possible improvements to our results by means of the state-of-the-art local extension estimates for elliptic surfaces.

Notation. We use the standard notations $A\lesssim B$ and $A=O(B)$ to mean that $A\le CB$ for some constant $C>0$. If this constant depends on some parameter ε, then we might write $A\lesssim_\varepsilon B$. Typically, constants may only depend on the dimension d and relevant Lebesgue space exponents.

Acknowledgments. The author thanks his advisor Betsy Stovall for suggesting this project and acknowledges support from NSF grant DMS-1653264.

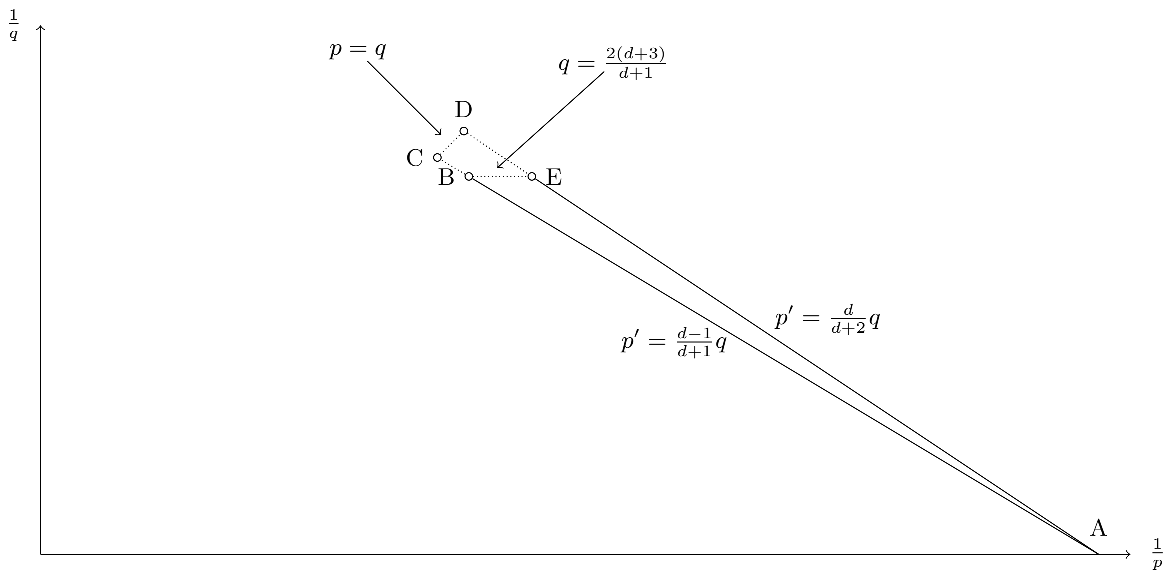

FIGURE 1. Conjecture 1.1 asserts that $\mathcal{E}(p \to q)$ holds for exponent pairs $(\frac{1}{p}, \frac{1}{q})$ lying within triangle ACD, excluding the points C and D. Theorem 1.2 states that, within the triangle, $\mathcal{E}(p \to q)$ holds unconditionally below line segment BE. Theorem 1.3 brings the range of estimates (slightly) above BE on the condition that $\mathcal{E}_0(p_0 \to q_0)$ holds for some pair $(\frac{1}{p_0}, \frac{1}{q_0})$ in the interior of line segment DE. This particular diagram was created using $d = 4$.

2. Optimality of Conjecture 1.1

In this section, we demonstrate that Conjecture 1.1 cannot be improved. Although the counterexamples we consider are well known, we include all necessary details for the reader's convenience.

Proposition 2.1. *If $\mathcal{E}(p \to q)$ holds, then (p, q) satisfies the hypotheses of Conjecture 1.1.*

Proof. Assume that $\mathcal{E}(p \to q)$ holds. To show that $p \geq (\frac{d}{d+2}q)'$, we use the standard Knapp example. Fixing $\delta \in (0, 1]$, let us consider the cap

$$C := \{(\tau, \xi) \in \Sigma \colon |\xi| \leq \delta\}$$

and the tube

$$T := \{(t, x) \in \mathbb{R} \times \mathbb{R}^d \colon |t| \leq c\delta^{-2},\ |x| \leq c\delta^{-1}\},$$

where c is a small positive constant. If $(t, x) \in T$ and c is sufficiently small, then

$$|\mathcal{E}\chi_C(t, x)| = \left| \int_C e^{i(t,x)\cdot(\tau-1,\xi)} d\mu(\tau, \xi) \right| \geq \left| \int_C \cos(t(\tau - 1) + x \cdot \xi) d\mu(\tau, \xi) \right| \sim \mu(C) \sim \delta^d.$$

Therefore,

$$\delta^{d - \frac{d+2}{q}} \sim \delta^d |T|^{\frac{1}{q}} \lesssim \|\mathcal{E}\chi_C\|_q \lesssim \mu(C)^{\frac{1}{p}} \sim \delta^{\frac{d}{p}}$$

by the validity of $\mathcal{E}(p \to q)$. Letting $\delta \to 0$, we conclude that $p \geq (\frac{d}{d+2}q)'$.

The necessity of $p \leq (\frac{d-1}{d+1}q)'$ follows by a similar argument, utilizing the decay of μ. Indeed, fixing $\lambda \geq 1$, let

$$D := \{(\tau, \xi) \in \Sigma \colon |\xi| \leq \lambda\}.$$

If $|(t, x)| \leq c\lambda^{-1}$ and c is sufficiently small, then

$$|\mathcal{E}\chi_D(t, x)| \sim \mu(D) \sim \lambda^{d-1}.$$

Thus,

$$\lambda^{d-1-\frac{d+1}{q}} \lesssim \|\mathcal{E}\chi_D\|_q \lesssim \mu(D)^{\frac{1}{p}} \sim \lambda^{\frac{d-1}{p}}$$

by $\mathcal{E}(p \to q)$, and sending $\lambda \to \infty$ gives the required inequality.

To show that $p \le q$, we consider a randomized sum of bump functions. By interpolation, we may assume that $q \ge 2$. Let ϕ be a nonzero bump function on Σ and N a positive integer. For each $j \in \{1, \dots, N\}$, let L_j be a Lorentz boost and set $\phi_j := \phi \circ L_j$. Choosing the boosts L_j appropriately, the functions ϕ_j have pairwise disjoint supports. Let $f := \sum_{j=1}^N \varepsilon_j \phi_j$, where $\varepsilon_1, \dots, \varepsilon_N$ are independent random variables with the Rademacher distribution. On one hand, Khintchine's inequality and the Lorentz invariance of μ imply that

$$\begin{aligned}
\mathbb{E}\|\mathcal{E}f\|_q^q &= \iint_{\mathbb{R}\times\mathbb{R}^d} \mathbb{E}\bigg|\sum_{j=1}^N \varepsilon_j \mathcal{E}\phi_j(t,x)\bigg|^q dtdx \\
&\gtrsim \iint_{\mathbb{R}\times\mathbb{R}^d} \bigg(\sum_{j=1}^N |\mathcal{E}\phi_j(t,x)|^2\bigg)^{\frac{q}{2}} dtdx \\
&\ge \sum_{j=1}^N \iint_{\mathbb{R}\times\mathbb{R}^d} |\mathcal{E}\phi_j(t,x)|^q dtdx \\
&= N\|\mathcal{E}\phi\|_q^q.
\end{aligned}$$

On the other hand,

$$\mathbb{E}\|\mathcal{E}f\|_q^q \lesssim \mathbb{E}\|f\|_p^q = N^{\frac{q}{p}}\|\phi\|_p^q$$

by $\mathcal{E}(p \to q)$, the fact that the ϕ_j have disjoint supports, and Lorentz invariance. Letting $N \to \infty$ shows that $p \le q$.

Finally, setting $p_1 := \frac{2d}{d-1}$ and $p_2 := \frac{2(d+1)}{d}$, we need to show that the estimates $\mathcal{E}(p_1 \to p_1)$ and $\mathcal{E}(p_2 \to p_2)$ are false. We could proceed by rescaling known counterexamples for the cone and paraboloid. Indeed, (p_1, p_1) lies on the conic scaling line $p' = \frac{d-1}{d+1}q$ and the extension operator associated to the cone is known not to be bounded on L^{p_1}; likewise, (p_2, p_2) lies on the parabolic scaling line $p' = \frac{d}{d+2}q$ and the extension operator for the paraboloid is not bounded on L^{p_2}. We will instead present direct counterexamples to $\mathcal{E}(p_1 \to p_1)$ and $\mathcal{E}(p_2 \to p_2)$, using longer but self-contained arguments.

We start with the disproof of $\mathcal{E}(p_1 \to p_1)$. Fixing $\lambda \ge 1$, let $f\colon \Sigma \to \mathbb{C}$ be defined by $f(\tau, \xi) := \psi(|\xi|)\langle\xi\rangle$, where ψ is a bump function satisfying $\chi_{[2\lambda,3\lambda]} \le \psi \le \chi_{[\lambda,4\lambda]}$ and $|\psi'| \lesssim \lambda^{-1}$. Using polar coordinates, we see that

$$\mathcal{E}f(t,x) = \int_\lambda^{4\lambda}\int_{\mathbb{S}^{d-1}} e^{i(t,x)\cdot(\langle r\rangle, r\theta)}\psi(r)r^{d-1}d\sigma(\theta)dr = \int_\lambda^{4\lambda} \check{\sigma}(rx)e^{it\langle r\rangle}\psi(r)r^{d-1}dr,$$

where $\langle r\rangle := \sqrt{r^2+1}$ and σ is the standard measure on the sphere $\mathbb{S}^{d-1}$. By a well-known stationary phase argument (see e.g. [20]), $\check{\sigma}$ obeys the asymptotic formula

$$\check{\sigma}(y) = a|y|^{-\frac{d-1}{2}}\cos(|y|+b) + O(|y|^{-\frac{d+1}{2}}) \qquad \text{as } |y| \to \infty$$

for some $a, b \in \mathbb{R}$ with $a > 0$. Thus,

$$|\mathcal{E}f(t,x)| = a|x|^{-\frac{d-1}{2}}\int_\lambda^{4\lambda}\cos(r|x|+b)e^{i(t\langle r\rangle+b)}\psi(r)r^{\frac{d-1}{2}}dr + O(\lambda^{\frac{d-1}{2}}|x|^{-\frac{d+1}{2}}),$$

provided that $|x| \ge 1$ and λ is sufficiently large. The absolute value of the integral is at least that of its real part. Using the identity $2\cos(\theta)\cos(\nu) = \cos(\theta-\nu)+\cos(\theta+\nu)$, we get the bound

$$|\mathcal{E}f(t,x)| \ge \frac{a}{2}|x|^{-\frac{d-1}{2}}(|I_1|-|I_2|) + O(\lambda^{\frac{d-1}{2}}|x|^{-\frac{d+1}{2}}), \tag{2.1}$$

where

$$\begin{aligned}
I_1 &:= \int_\lambda^{4\lambda}\cos(r|x|-t\langle r\rangle)\psi(r)r^{\frac{d-1}{2}}dr, \\
I_2 &:= \int_\lambda^{4\lambda}\cos(r|x|+t\langle r\rangle+2b)\psi(r)r^{\frac{d-1}{2}}dr.
\end{aligned}$$

Suppose that $1 \le |x| \le c\lambda$ and $|t - |x|| \le c\lambda^{-1}$, where c is a positive constant. If c is sufficiently small, then $|r|x| - t\langle r\rangle| \le 1$ for all $r \in [\lambda, 4\lambda]$, leading to the bound

$$|I_1| \gtrsim \lambda^{\frac{d+1}{2}}.$$

To estimate I_2, we first write $I_2 = I_{21} + I_{22}$, where

$$I_{21} := \int_\lambda^{4\lambda} \cos(r(|x|+t)+2b)\psi(r)r^{\frac{d-1}{2}}\,dr,$$
$$I_{22} := \int_\lambda^{4\lambda} (\cos(r|x|+t\langle r\rangle+2b) - \cos(r(|x|+t)+2b))\psi(r)r^{\frac{d-1}{2}}\,dr.$$

The assumptions on (t,x) imply that $||x|+t| \ge 1$, giving the bound

$$|I_{21}| = \left|\int_\lambda^{4\lambda} \frac{\sin(r(|x|+t)+2b)}{|x|+t}\frac{d}{dr}(\psi(r)r^{\frac{d-1}{2}})dr\right| \lesssim \lambda^{\frac{d-1}{2}}.$$

Estimating the integrand of I_{22} using the mean value theorem, we find that

$$|I_{22}| \lesssim c\lambda^{\frac{d+1}{2}}.$$

Thus, $|I_1| - |I_2| \ge \frac{1}{2}|I_1| \gtrsim \lambda^{\frac{d+1}{2}}$ if λ is sufficiently large and c sufficiently small. Plugging this bound into (2.1), we conclude that

$$|\mathcal{E}f(t,x)| \gtrsim \lambda^{\frac{d+1}{2}}|x|^{-\frac{d-1}{2}}$$

for all (t,x) satisfying $1 \le |x| \le c\lambda$ and $|t-|x|| \le c\lambda^{-1}$. If $\mathcal{E}(p_1 \to p_1)$ were true, then it would follow that

$$\lambda^{\frac{d^2+1}{d-1}}\log\lambda \lesssim \|\mathcal{E}f\|_{p_1}^{p_1} \lesssim \|f\|_{p_1}^{p_1} \sim \lambda^{\frac{d^2+1}{d-1}},$$

and sending $\lambda \to \infty$ would give a contradiction.

The disproof of $\mathcal{E}(p_2 \to p_2)$ is similar but simpler. We will follow an argument from [14, Chapter VIII]. Define $f\colon \Sigma \to \mathbb{C}$ by $f(\tau,\xi) := \psi(\xi)\langle\xi\rangle$, where ψ is a bump function satisfying $\psi(\xi) = 1$ for $|\xi| \le c$ and $\psi(\xi) = 0$ for $|\xi| \ge 2c$, with c a positive constant. Fix $(t,x) \in \mathbb{R}\times\mathbb{R}^d \setminus \{(0,0)\}$, and let $\lambda := |(t,x)|$ and $(t_0,x_0) := \lambda^{-1}(t,x)$. Then

$$\mathcal{E}f(t,x) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(\xi;t_0,x_0)}\psi(\xi)d\xi,$$

where $\Phi(\xi;s,y) := s\langle\xi\rangle + y\cdot\xi$. Since $\nabla_\xi\Phi(0;1,0) = 0$ and $\det\nabla_\xi^2\Phi(0;1,0) = 1$, the implicit function theorem implies the existence of a neighborhood U of $(1,0)$ such that for each $(s,y) \in U$ there exists a unique $\xi(s,y)$ such that $\nabla_\xi\Phi(\xi(s,y);s,y) = 0$. Making U smaller if necessary, we may assume that $|\xi(s,y)| \le c$ and $|\det\nabla_\xi^2\Phi(\xi(s,y);s,y)| \gtrsim 1$ for all $(s,y) \in U$. Suppose that $(t_0,x_0) \in U$, so that $\xi(t_0,x_0)$ is a nondegenerate critical point of the function $\Phi(\cdot;t_0,x_0)$. If c is sufficiently small (not depending on (t,x)), then

$$\left|\int_{\mathbb{R}^d} e^{i\lambda\Phi(\xi;t_0,x_0)}\psi(\xi)d\xi\right| = a|\det\nabla_\xi^2\Phi(\xi(t_0,x_0);t_0,x_0)|^{-\frac{1}{2}}\lambda^{-\frac{d}{2}} + O(\lambda^{-\frac{d+1}{2}})$$

for some constant $a > 0$, by a standard stationary phase result (see e.g. [14, Chapter VIII, Proposition 6]). It follows that

$$|\mathcal{E}f(t,x)| \gtrsim |(t,x)|^{-\frac{d}{2}}$$

whenever t is sufficiently large and $t^{-1}|x|$ sufficiently small. Consequently, we find that $\mathcal{E}f \notin L^{p_2}(\mathbb{R}\times\mathbb{R}^d)$, and thus $\mathcal{E}(p_2 \to p_2)$ cannot hold. ∎

3. Bilinear and linear restriction on frusta

Now we begin working toward our main results. In this section, we divide the "conic" portion of our surface, $\Sigma \setminus \Sigma_0$, into frusta Σ_N of width 2^N for which we prove uniform extension estimates. Our proof will combine the bilinear restriction theory for the frusta Σ_N (resembling that for conic frusta), the bilinear theory for Σ_0 (resembling that for the paraboloid), and the bilinear-to-linear argument found in [18].

We turn to the details. For each integer $N \geq 1$, let

$$\Sigma_N := \{(\tau, \xi) \in \Sigma \colon 2^N \leq |\xi| \leq 2^{N+1}\}.$$

Given $1 \leq k \leq N$, we cover the frustum Σ_N by sectors of angular width 2^{-k} by defining

$$\Sigma_{N,k}^{\omega} := \left\{(\tau, \xi) \in \Sigma_N \colon \left|\frac{\xi}{|\xi|} - \omega\right| \leq 2^{-k}\right\}$$

for each $\omega \in \mathbb{S}^{d-1}$. We refer to these sets as (N,k)-*sectors*. Two (N,k)-sectors $\Sigma_{N,k}^{\omega}$ and $\Sigma_{N,k}^{\omega'}$ are *related* if (i) $2^{2-k} \leq |\omega - \omega'| \leq 2^{4-k}$ and $k < N$, or (ii) $|\omega - \omega'| \leq 2^{4-N}$ and $k = N$. The conic-type bilinear estimate that we require is the following:

Lemma 3.1. *Let $\Sigma_{N,k}^{\omega}$ and $\Sigma_{N,k}^{\omega'}$ be related (N,k)-sectors with $k < N$, and let $q > \frac{2(d+3)}{d+1}$. Then*

$$\|\mathcal{E}f_1 \mathcal{E}f_2\|_{q/2} \lesssim 2^{(N-k)(d-1-\frac{2(d+1)}{q})} \|f_1\|_2 \|f_2\|_2$$

whenever $\operatorname{supp} f_1 \subseteq \Sigma_{N,k}^{\omega}$ *and* $\operatorname{supp} f_2 \subseteq \Sigma_{N,k}^{\omega'}$.

Proof. After dividing $\Sigma_{N,k}^{\omega}$ and $\Sigma_{N,k}^{\omega'}$ into a bounded number of subsets with sufficiently small radial and angular width, one can directly apply [3, Theorem 1.10]. □

Next, we perform our bilinear-to-linear deduction. To make it work, we will need additional bilinear estimates corresponding to related (N,N)-sectors, since these thinnest sectors are absent from Lemma 3.1. Via a Lorentz boost, the local extension theory will be sufficient:

Lemma 3.2. *If $\mathcal{E}_0(p \to q)$ holds, then $\|\mathcal{E}f\|_q \lesssim \|f\|_p$ whenever f is supported in an (N,N)-sector.*

Proof. Given an (N,N)-sector, a suitable Lorentz boost maps it into Σ_0. □

Lemma 3.3. *Suppose that p, q, r, and α relate in the following ways:* (i) $r \leq p \leq q \leq 4$, (ii) $\alpha < 0$, (iii) $\alpha \leq (d-1)(\frac{2}{p} - \frac{2}{r})$, *and* (iv) $\alpha \neq (d-1)(\frac{2}{q} - \frac{2}{r})$ *or $p < q$. Additionally, suppose that $\mathcal{E}_0(p \to q)$ holds and that*

$$\|\mathcal{E}f_1 \mathcal{E}f_2\|_{q/2} \lesssim 2^{(N-k)\alpha} \|f_1\|_r \|f_2\|_r \tag{3.1}$$

whenever f_1 and f_2 are supported in related (N,k)-sectors with $k < N$. Then

$$\|\mathcal{E}f\|_q \lesssim \|f\|_p$$

whenever $|f| \sim \chi_\Omega$ for some Ω contained in some Σ_N with $N \geq 0$.

Proof. As noted above, we will adapt the bilinear-to-linear argument found in [18]. Since $\mathcal{E}_0(p \to q)$ holds, we may fix $N \geq 1$. Our first step is to construct a Whitney decomposition of $\Sigma_N \times \Sigma_N$. For each $k \in \{1, \ldots, N\}$, choose a (finite) set $\Lambda_k \subset \mathbb{S}^{d-1}$ satisfying

$$\mathbb{S}^{d-1} = \bigcup_{\omega \in \Lambda_k} \{\omega' \in \mathbb{S}^{d-1} \colon |\omega - \omega'| \leq 2^{-k}\}$$

and $|\omega - \omega'| \gtrsim 2^{-k}$ for all distinct $\omega, \omega' \in \Lambda_k$. Given $\omega, \omega' \in \Lambda_k$, we write $\omega \sim \omega'$ if (i) $2^{2-k} \leq |\omega - \omega'| \leq 2^{4-k}$ and $k < N$, or (ii) $|\omega - \omega'| \leq 2^{4-N}$ and $k = N$. (That is, $\omega \sim \omega'$ exactly when $\Sigma_{N,k}^{\omega}$ and $\Sigma_{N,k}^{\omega'}$ are related.)

We claim that

$$\Sigma_N \times \Sigma_N = \bigcup_{k=1}^{N} \bigcup_{\substack{\omega, \omega' \in \Lambda_k \colon \\ \omega \sim \omega'}} \Sigma_{N,k}^{\omega} \times \Sigma_{N,k}^{\omega'}. \tag{3.2}$$

Indeed, fix $(\tau, \xi), (\tau', \xi') \in \Sigma_N$, and let $\zeta := \frac{\xi}{|\xi|}$ and $\zeta' := \frac{\xi'}{|\xi'|}$. First, suppose that $|\zeta - \zeta'| \geq 12 \cdot 2^{-N}$. Then there exists $k \in \{1, \ldots, N-1\}$ and $\omega, \omega' \in \Lambda_k$ such that $6 \cdot 2^{-k} \leq |\zeta - \zeta'| \leq 12 \cdot 2^{-k}$ and $|\omega - \zeta|, |\omega' - \zeta'| \leq 2^{-k}$.

It follows that $(\tau,\xi) \in \Sigma_{N,k}^{\omega}$, $(\tau',\xi') \in \Sigma_{N,k}^{\omega'}$, and $\omega \sim \omega'$. The case when $|\zeta-\zeta'| \le 12\cdot 2^{-N}$ can be treated similarly using $k = N$, so the claim is proved.

The pieces of the decomposition (3.2) are (unfortunately) not disjoint. An easy argument shows, however, that two such pieces, $\Sigma_{N,k_1}^{\omega_1}\times\Sigma_{N,k_1}^{\omega_1'}$ and $\Sigma_{N,k_2}^{\omega_2}\times\Sigma_{N,k_2}^{\omega_2'}$, overlap only if (i) $|k_1-k_2| \lesssim 1$ or (ii) $k_1 = k_2$ and $|\omega_1-\omega_2|, |\omega_1'-\omega_2'| \lesssim 2^{-k_1}$. Let

$$\mathcal{I} := \{(k,\omega,\omega') \colon 1 \le k \le N;\ \omega,\omega' \in \Lambda_k;\ \omega \sim \omega'\},$$

and let C be a large constant. We can partition $\mathcal{I}$ into $O(1)$ sets $\mathcal{I}_1,\dots,\mathcal{I}_n$ with the following separation property: If $(k_1,\omega_1,\omega_1'),(k_2,\omega_2,\omega_2') \in \mathcal{I}_j$, then either (i) $(k_1,\omega_1,\omega_1') = (k_2,\omega_2,\omega_2')$, or (ii) $k_1 = k_2$ and $\max\{|\omega_1-\omega_2|, |\omega_1'-\omega_2'|\} \ge C2^{-k_1}$, or (iii) $|k_1-k_2| \ge C$. Thus, if C is sufficiently large, then

$$\Sigma_N \times \Sigma_N = \bigcup_{j=1}^{n} \dot{\bigcup_{(k,\omega,\omega')\in\mathcal{I}_j}} \Sigma_{N,k}^{\omega}\times\Sigma_{N,k}^{\omega'}, \tag{3.3}$$

where the dot indicates a disjoint union.

From (3.3), it follows that

$$\|\mathcal{E}f\|_q^2 = \|(\mathcal{E}f)^2\|_{q/2} \lesssim \max_{1\le j\le n}\Big\| \sum_{(k,\omega,\omega')\in\mathcal{I}_j} \mathcal{E}(f\chi_{\Sigma_{N,k}^{\omega}})\mathcal{E}(f\chi_{\Sigma_{N,k}^{\omega'}})\Big\|_{q/2}. \tag{3.4}$$

By elementary geometry, there exist rectangular boxes $R_{k,\omega,\omega'}$ such that $\Sigma_{N,k}^{\omega}+\Sigma_{N,k}^{\omega'} \subseteq R_{k,\omega,\omega'}$ and, for each k, the collection $\{2R_{k,\omega,\omega'}\}_{\omega,\omega'\in\Lambda_k\colon\omega\sim\omega'}$ has bounded overlap. This fact allows us to exploit almost orthogonality in the form of [18, Lemma 6.1]. We obtain the bound

$$\Big\| \sum_{(k,\omega,\omega')\in\mathcal{I}_j} \mathcal{E}(f\chi_{\Sigma_{N,k}^{\omega}})\mathcal{E}(f\chi_{\Sigma_{N,k}^{\omega'}})\Big\|_{q/2} \lesssim \sum_{k=1}^{N}\Big(\sum_{\substack{\omega,\omega'\in\Lambda_k\colon\\ \omega\sim\omega'}} \|\mathcal{E}(f\chi_{\Sigma_{N,k}^{\omega}})\mathcal{E}(f\chi_{\Sigma_{N,k}^{\omega'}})\|_{q/2}^{q/2}\Big)^{\frac{2}{q}} \tag{3.5}$$

for each j. To help us estimate (3.5), we set

$$\tilde{\Sigma}_{N,k}^{\omega} := \bigcup_{\substack{\omega'\in\Lambda_k\colon\\ \omega'\sim\omega \text{ or } \omega'=\omega}} \Sigma_{N,k}^{\omega'}$$

and note that, for each k, the collection $\{\tilde{\Sigma}_{N,k}^{\omega}\}_{\omega\in\Lambda_k}$ has bounded overlap. We first consider the terms in (3.5) with $k < N$. Using (3.1) and assuming that $|f| \sim \chi_\Omega$ for some $\Omega \subseteq \Sigma_N$, we see that

$$\begin{aligned}\sum_{k=1}^{N-1}\Big(\sum_{\substack{\omega,\omega'\in\Lambda_k\colon\\ \omega\sim\omega'}} \|\mathcal{E}(f\chi_{\Sigma_{N,k}^{\omega}})\mathcal{E}(f\chi_{\Sigma_{N,k}^{\omega'}})\|_{q/2}^{q/2}\Big)^{\frac{2}{q}} &\lesssim \sum_{k=1}^{N-1} 2^{(N-k)\alpha}\Big(\sum_{\substack{\omega,\omega'\in\Lambda_k\colon\\ \omega\sim\omega'}} \mu(\Omega\cap\Sigma_{N,k}^{\omega})^{\frac{q}{2r}}\mu(\Omega\cap\Sigma_{N,k}^{\omega'})^{\frac{q}{2r}}\Big)^{\frac{2}{q}}\\ &\lesssim \sum_{k=1}^{N-1} 2^{(N-k)\alpha}\max_{\omega\in\Lambda_k}\mu(\Omega\cap\tilde{\Sigma}_{N,k}^{\omega})^{\frac{2}{r}-\frac{2}{q}}\Big(\sum_{\omega\in\Lambda_k}\mu(\Omega\cap\tilde{\Sigma}_{N,k}^{\omega})\Big)^{\frac{2}{q}}\\ &\lesssim \sum_{k=1}^{N-1} 2^{(N-k)\alpha}\min\{\mu(\Omega), 2^{(N-k)(d-1)}\}^{\frac{2}{r}-\frac{2}{q}}\mu(\Omega)^{\frac{2}{q}}.\end{aligned} \tag{3.6}$$

If $\mu(\Omega) \le 2^{d-1}$, then the hypotheses that $\alpha < 0$ and $r \le p$ imply that (3.6) is $O(\mu(\Omega)^{\frac{2}{p}})$. Thus, we may assume that $\mu(\Omega) \ge 2^{d-1}$, and (3.6) becomes

$$\sum_{k=1}^{N-\lceil \log_2\mu(\Omega)^{\frac{1}{d-1}}\rceil} 2^{(N-k)\alpha}\mu(\Omega)^{\frac{2}{r}} + \sum_{k=N-\lceil \log_2\mu(\Omega)^{\frac{1}{d-1}}\rceil+1}^{N-1} 2^{(N-k)(\alpha-(d-1)(\frac{2}{q}-\frac{2}{r}))}\mu(\Omega)^{\frac{2}{q}}.$$

The first sum is $O(\mu(\Omega)^{\frac{2}{p}})$ by the hypotheses that $\alpha < 0$ and $\alpha \le (d-1)(\frac{2}{p}-\frac{2}{r})$. Treating separately the cases where α is strictly less than, strictly greater than, or equal to $(d-1)(\frac{2}{q}-\frac{2}{r})$, the second sum is similarly seen to be $O(\mu(\Omega)^{\frac{2}{p}})$. Thus, altogether the terms in (3.5) with $k < N$ contribute $O(\mu(\Omega)^{\frac{2}{p}})$. Now we estimate the terms with $k = N$. By the Cauchy–Schwarz inequality, the hypothesis $\mathcal{E}_0(p \to q)$, Lemma 3.2, and the hypothesis that $p \le q$, we find that

$$\Big(\sum_{\substack{\omega,\omega' \in \Lambda_N : \\ \omega \sim \omega'}} \| \mathcal{E}(h\chi_{\Sigma^\omega_{N,N}}) \mathcal{E}(h\chi_{\Sigma^{\omega'}_{N,N}}) \|_{q/2}^{q/2} \Big)^{\frac{2}{q}} \lesssim \Big(\sum_{\substack{\omega,\omega' \in \Lambda_N : \\ \omega \sim \omega'}} \mu(\Omega \cap \Sigma^\omega_{N,N})^{\frac{q}{2p}} \mu(\Omega \cap \Sigma^{\omega'}_{N,N})^{\frac{q}{2p}} \Big)^{\frac{2}{q}}$$
$$\lesssim \Big(\sum_{\omega \in \Lambda_N} \mu(\Omega \cap \tilde{\Sigma}^\omega_{N,N})^{\frac{q}{p}} \Big)^{\frac{2}{q}}$$
$$\lesssim \mu(\Omega)^{\frac{2}{p}}.$$

Thus, we have shown that (3.5) is $O(\mu(\Omega)^{\frac{2}{p}})$. Inserting this bound into (3.4) and noting that $\|f\|_p \sim \mu(\Omega)^{\frac{1}{p}}$, the proof is complete. ☐

Corollary 3.4. *If $q > \frac{2(d+3)}{d+1}$ and $(\frac{d}{d+2}q)' \le p \le \min\{(\frac{d-1}{d+1}q)', q\}$ and $(p,q) \ne (\frac{2d}{d-1}, \frac{2d}{d-1})$, then $\|\mathcal{E}f\|_q \lesssim \|f\|_p$ whenever f is supported in Σ_N for some $N \ge 0$.*

Proof. We will apply Lemma 3.3. By interpolation, we may assume that $q \le \frac{2(d+2)}{d}$. Then conditions (i)–(iv) in the lemma are satisfied with $r = 2$ and $\alpha = d - 1 - \frac{2(d+1)}{q}$. The estimate $\mathcal{E}_0(p \to q)$ is a consequence of the techniques in [17], and the bilinear estimate (3.1) is valid by Lemma 3.1. Thus, Lemma 3.3 gives the restricted strong type analogue of the required estimate, and real interpolation completes the proof. ☐

4. Summing bounds on frusta and proof of Theorem 1.2

Let $\mathcal{E}_{\text{fru}}(p \to q)$ denote the statement that $\|\mathcal{E}f\|_q \lesssim \|f\|_p$ whenever f is supported in Σ_N for some $N \ge 0$. We have shown, by Corollary 3.4, that $\mathcal{E}_{\text{fru}}(p \to q)$ holds for (p,q) in (a superset of) the range required by Theorem 1.2. In this section, we sum these uniform bounds and consequently prove Theorem 1.2. Our argument will utilize the following Strichartz estimate for the Klein–Gordon equation (see [10] and references therein): If $r \in [2,\infty]$, $s \in [2, \frac{2d}{d-2}]$ (with $\frac{2d}{d-2} := \infty$ when $d = 2$), $(r,s) \ne (2,\infty)$, and $\frac{1}{r} = \frac{d-1+\theta}{2}(\frac{1}{2} - \frac{1}{s})$ for some $\theta \in [0,1]$, then

$$\|\mathcal{E}f\|_{L^r_t L^s_x} \lesssim \|\langle\cdot\rangle^{\frac{1}{r}-\frac{1}{s}} f\|_2. \tag{4.1}$$

Lemma 4.1. *If $\mathcal{E}_{\text{fru}}(p_0 \to q_0)$ holds for some $(\frac{d}{d+2}q_0)' \le p_0 \le \min\{(\frac{d-1}{d+1}q_0)', q_0\}$, then $\mathcal{E}(p \to q)$ holds whenever $q > q_0$ and $p' = \frac{p_0'}{q_0}q$.*

Proof. We will show that the hypothesis of the lemma implies the following bilinear estimate: Given $q > q_0$ and $p' = \frac{p_0'}{q_0}q$, there exists a positive constant c such that

$$\|\mathcal{E}f_1 \mathcal{E}f_2\|_{q/2} \lesssim 2^{-c|N_1-N_2|} \|f_1\|_p \|f_2\|_p \tag{4.2}$$

whenever $\operatorname{supp} f_1 \subseteq \Sigma_{N_1}$ and $\operatorname{supp} f_2 \subseteq \Sigma_{N_2}$ for some $N_1, N_2 \ge 0$. Assuming the validity of (4.2), we now demonstrate how $\mathcal{E}(p \to q)$ follows. The case $q = \infty$ is trivial, so we assume that $q < \infty$ and set $n := \lceil q/2 \rceil$. Fixing f, we have

$$\|\mathcal{E}f\|_q^q = \Big\| \sum_{N=0}^\infty \mathcal{E}(f\chi_{\Sigma_N}) \Big\|_q^q \le \sum_{N_1,\dots,N_{2n} \ge 0} \Big\| \prod_{j=1}^{2n} \mathcal{E}(f\chi_{\Sigma_{N_j}}) \Big\|_{\frac{q}{2n}}^{\frac{q}{2n}}$$

since $q \le 2n$. Given $\boldsymbol{N} \in \{0,1,2\dots\}^{2n}$, let $p(\boldsymbol{N}) = (p_j(\boldsymbol{N}))_{j=1}^{2n}$ be a permutation of $\boldsymbol{N}$ such that $\|f\|_{L^p(\Sigma_{p_1(\boldsymbol{N})})}$ is maximal among $\|f\|_{L^p(\Sigma_{p_j(\boldsymbol{N})})}$ and $|p_1(\boldsymbol{N}) - p_2(\boldsymbol{N})|$ is maximal among $|p_1(\boldsymbol{N}) - p_j(\boldsymbol{N})|$. Then, by Hölder's inequality, estimate (4.2), and the fact that $p \le q$ (which follows from our hypothesis), we see that

$$\sum_{N_1,\dots,N_{2n} \ge 0} \Big\| \prod_{j=1}^{2n} \mathcal{E}(f\chi_{\Sigma_{N_j}}) \Big\|_{\frac{q}{2n}}^{\frac{q}{2n}} \lesssim \sum_{\boldsymbol{N} :\, p(\boldsymbol{N}) = \boldsymbol{N}} \Big\| \prod_{j=1}^{2n} \mathcal{E}(f\chi_{\Sigma_{N_j}}) \Big\|_{\frac{q}{2n}}^{\frac{q}{2n}}$$
$$\le \sum_{\boldsymbol{N} :\, p(\boldsymbol{N}) = \boldsymbol{N}} \prod_{j=1}^{n} \|\mathcal{E}(f\chi_{\Sigma_{p_{2j-1}(\boldsymbol{N})}}) \mathcal{E}(f\chi_{\Sigma_{p_{2j}(\boldsymbol{N})}})\|_{\frac{q}{2}}^{\frac{q}{2n}}$$

$$\begin{aligned}
&\lesssim \sum_{\boldsymbol{N}:\, p(\boldsymbol{N})=\boldsymbol{N}} 2^{-\frac{cq}{2n}|p_1(\boldsymbol{N})-p_2(\boldsymbol{N})|}\|f\|^q_{L^p(\Sigma_{p_1(\boldsymbol{N})})} \\
&\lesssim \sum_{N_1,N_2\geq 0} |N_1-N_2|^{2n-2}2^{-\frac{cq}{2n}|N_1-N_2|}\|f\|^q_{L^p(\Sigma_{N_1})} \\
&\lesssim \sum_{N_1\geq 0} \|f\|^q_{L^p(\Sigma_{N_1})} \\
&\leq \|f\|_p^q.
\end{aligned}$$

Thus, we have shown that $\mathcal{E}(p\to q)$ holds.

We turn to the proof of (4.2). If $N_1=N_2$, then the desired estimate is a consequence of $\mathcal{E}_{\mathrm{fru}}(p_0\to q_0)$, the Cauchy–Schwarz inequality, and interpolation. Thus, we may assume that $N_1<N_2$; in particular, we have $N_2\geq 1$. Now, let $q_1 := \frac{2q_0}{p_0'}$ and choose r_1,s_1,r_2,s_2 obeying the conditions $r_i\in[2,\infty]$, $s_i\in[2,\frac{2d}{d-2}]$ (with $\frac{2d}{d-2}:=\infty$ when $d=2$), $(r_i,s_i)\neq(2,\infty)$, and $\frac{2}{r_i}+\frac{2p_0'}{(q_0-p_0')s_i}=\frac{p_0'}{q_0-p_0'}$, as well as $r_1<s_1$ and $\frac{2}{q_1}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{s_1}+\frac{1}{s_2}$. (For example, fixing an arbitrary $r_1\in[\frac{d(q_0-p_0')}{p_0'},q_1)$ determines such a choice.) The Strichartz estimate (4.1) and our hypothesis imply that $\|\mathcal{E}f\|_{L_t^{r_i}L_x^{s_i}}\lesssim\|\langle\cdot\rangle^{\frac{1}{r_i}-\frac{1}{s_i}}f\|_2$ for every f. Thus, by the mixed-norm Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|\mathcal{E}f_1\mathcal{E}f_2\|_{q_1/2} &\lesssim \|\mathcal{E}f_1\|_{L_t^{r_1}L_x^{s_1}}\|\mathcal{E}f_2\|_{L_t^{r_2}L_x^{s_2}} \\
&\lesssim 2^{N_1(\frac{1}{r_1}-\frac{1}{s_1})}2^{N_2(\frac{1}{r_2}-\frac{1}{s_2})}\|f_1\|_2\|f_2\|_2 = 2^{-(\frac{1}{r_1}-\frac{1}{s_1})|N_1-N_2|}\|f_1\|_2\|f_2\|_2. \qquad (4.3)
\end{aligned}$$

The estimate (4.2) now follows by interpolating (4.3) with either the trivial inequality $\|\mathcal{E}f_1\mathcal{E}f_2\|_\infty\lesssim\|f_1\|_1\|f_2\|_1$, if $q\geq q_1$, or the estimate $\|\mathcal{E}f_1\mathcal{E}f_2\|_{q_0/2}\lesssim\|f_1\|_{p_0}\|f_2\|_{p_0}$ (a consequence of our hypothesis), if $q<q_1$. ☐

Proof of Theorem 1.2. Together, Corollary 3.4 and Lemma 4.1 imply the theorem, except for the estimates $\mathcal{E}(q\to q)$ with $\frac{10}{3}<q<4$ when $d=2$. The latter bounds can be obtained by (straightforwardly) adapting the proof of [2, Lemma 8.2]. ☐

5. Conic decoupling and proof of Theorem 1.3

In this section, we prove our conditional result, Theorem 1.3. We will argue as follows: To prove global extension estimates, it suffices to obtain uniform estimates on dyadic frusta, according to Lemma 4.1. By Lemma 3.3, these bounds would follow from appropriate bilinear estimates between (N,k)-sectors. Lemma 3.1 provides one such bilinear estimate, with a very favorable constant (relative to the hypotheses of Lemma 3.3) but valid only for q in the bilinear range. As we will show, the conic decoupling theorem of [1] and the hypothesis of Theorem 1.3 together imply a second bilinear estimate, with a worse constant but a smaller value of q. Interpolation then leads to a compromise, wherein Lemma 3.3 may be applied for a small set of exponents that nevertheless improves on the bilinear range. After some arithmetic, the admissible exponents work out to be those satisfying (1.2).

We now turn to the details, beginning with the following elementary fact:

Lemma 5.1. *If $u\in\mathbb{R}^n$ for some $n\geq 1$ and $x,y\in\mathbb{R}^2\setminus\{0\}$ with $|x|,|y|\geq|u|$, then*

$$\left|\frac{(x,u)}{|(x,u)|}-\frac{(y,u)}{|(y,u)|}\right|\geq\frac{1}{4}\left|\frac{x}{|x|}-\frac{y}{|y|}\right|.$$

Proof. By a rotation of $\mathbb{R}^2$, we may assume that $y_1>0$ and $y_2=0$. Let $\theta\in[-\pi,\pi]$ denote the angle between the vectors x and $(1,0)$, so that $x=|x|(\cos\theta,\sin\theta)$. Noting that $|\theta|\geq|\frac{x}{|x|}-\frac{y}{|y|}|$, it suffices to show that

$$\left|\frac{(x,u)}{|(x,u)|}-\frac{(y,u)}{|(y,u)|}\right|\geq\frac{|\theta|}{4}. \qquad (5.1)$$

We find, by a bit of algebra, that

$$\left|\frac{(x,u)}{|(x,u)|}-\frac{(y,u)}{|(y,u)|}\right|^2=\frac{2(|(x,u)||(y,u)|-|x||y|\cos(\theta)-|u|^2)}{|(x,u)||(y,u)|}.$$

Due to the bound $\cos\theta \le 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$ (which follows from Taylor's theorem), the Cauchy–Schwarz inequality, and the hypothesis that $|x|, |y| \ge |u|$, the right-hand side is bounded below by

$$\frac{2(|(x,u)||(y,u)| - |x||y| - |u|^2)}{|(x,u)||(y,u)|} + \frac{2|x||y|}{|(x,u)||(y,u)|}\left(\frac{\theta^2}{2} - \frac{\theta^4}{24}\right) \ge \frac{\theta^2}{2} - \frac{\theta^4}{24} \ge \frac{\theta^2}{16},$$

completing the proof. □

The following consequence of conic decoupling is the technical heart of this section:

Lemma 5.2. *Suppose that $\mathcal{E}_0(p \to q)$ holds for some $p \ge 2$ and $2 \le q \le \frac{2(d+1)}{d-1}$. Then*

$$\|\mathcal{E}f\|_q \lesssim_\varepsilon 2^{(N-k)((d-1)(\frac{1}{2}-\frac{1}{p})+\varepsilon)}\|f\|_p$$

for every $\varepsilon > 0$ whenever f is supported in an (N,k)-sector.

Proof. By rotational symmetry, it suffices to prove the lemma for functions f supported in the sector $\Sigma_{N,k}^{e_1}$. We may also assume that $k > C$ and $N - k > C$, where C is a positive integer of our choice. Indeed, if $k \le C$, then we can cover $\Sigma_{N,k}^{e_1}$ by a bounded number of (N,C)-sectors. Similarly, if $N - k \le C$, then $\Sigma_{N,k}^{e_1}$ is covered by a bounded number of (N,N)-sectors, and the required estimate is a consequence of the hypothesis $\mathcal{E}_0(p \to q)$ and Lemma 3.2.

We proceed by rescaling the extension estimate on $\Sigma_{N,k}^{e_1}$ to one on a nearly conic set of unit size. There, the conic decoupling theorem, [1, Theorem 1.2], can be directly applied. Let $e_1, \dots, e_d$ denote the standard basis vectors in $\mathbb{R}^d$, and let $M := LD$, where D is the conic dilation $D(\tau,\xi) := 2^{-N}(\tau,\xi)$ and L is the linear map satisfying

$$\begin{aligned} L(0,e_j) &= 2^{(k-C)}(0,e_j), \qquad 2 \le j \le d,\\ L(1,e_1) &= (1,e_1),\\ L(-1,e_1) &= 2^{2(k-C)}(-1,e_1). \end{aligned}$$

One easily checks that $D(\Sigma_{N,k}^{e_1})$ lies in an $O(2^{-2N})$-neighborhood of the conic sector

$$\Gamma := \left\{(|\xi|,\xi)\colon 1 \le |\xi| \le 2,\ \left|\frac{\xi}{|\xi|} - e_1\right| \le 2^{-k}\right\}.$$

The vectors $(0,e_2),\dots,(0,e_d)$ are angularly tangent to the cone at the point $(1,e_1)$, while the vector $(1,e_1)$ is radially tangent and $(-1,e_1)$ is normal. The map L preserves the cone and expands Γ to a sector of (roughly) unit angular width contained in the frustum

$$\tilde\Gamma := \{(|\xi|,\xi)\colon 1 \le |\xi| \le 3\}.$$

Setting $\delta := 2^{2(k-N)}$ and assuming C is sufficiently large, $M(\Sigma_{N,k}^{e_1})$ lies in the δ-neighborhood of $\tilde\Gamma$. Let $M_*\mu$ be the pushforward of μ by M, that is,

$$\int_{M(\Sigma)} g\, dM_*\mu := \int_\Sigma g \circ M\, d\mu,$$

and let $\mathcal{E}^M g := \widetilde{gM_*\mu}$. Let $\mathcal{P}$ be a partition of the δ-neighborhood of $\tilde\Gamma$ into plates of angular width $\delta^{1/2}$, thickness δ, and length 1, as in [1, Theorem 1.2]. Then

$$\|\mathcal{E}^M g\|_q \lesssim_\varepsilon \delta^{-\varepsilon}\left(\sum_{\theta\in\mathcal{P}} \|\mathcal{E}^M(g\chi_{\theta'})\|_q^2\right)^{\frac{1}{2}} \tag{5.2}$$

for all g supported in $M(\Sigma_{N,k}^{e_1})$, where $\theta' := \theta \cap M(\Sigma_{N,k}^{e_1})$. Let $\mathcal{Q}$ be a covering of $\Sigma_{N,k}^{e_1}$ by (N,N)-sectors having bounded overlap, and let $\{\psi_\kappa\}_{\kappa\in\mathcal{Q}}$ be partition of unity with $\operatorname{supp}\psi_\kappa \subseteq \kappa$. We claim that each $\theta \in \mathcal{P}$ obeys the bound

$$\#\{\kappa \in \mathcal{Q}\colon \kappa \cap M^{-1}(\theta') \ne \emptyset\} \lesssim 1. \tag{5.3}$$

If (5.3) holds, then taking $g = f \circ M^{-1}$ in (5.2), rescaling, applying the hypothesis $\mathcal{E}_0(p \to q)$ using Lemma 3.2, and finally applying Hölder's inequality and summing, we find that

$$\begin{aligned}\|\mathcal{E}f\|_q &\lesssim_\varepsilon \delta^{-\varepsilon}\Big(\sum_{\theta\in\mathcal{P}} \|\mathcal{E}(f\chi_{M^{-1}(\theta')})\|_q^2\Big)^{\frac12} \\ &\lesssim \delta^{-\varepsilon}\Big(\sum_{\theta\in\mathcal{P}}\sum_{\kappa\in\mathcal{Q}} \|\mathcal{E}(f\psi_\kappa\chi_{M^{-1}(\theta')})\|_q^2\Big)^{\frac12} \\ &\lesssim \delta^{-\varepsilon}\Big(\sum_{\theta\in\mathcal{P}}\sum_{\kappa\in\mathcal{Q}} \|f\psi_\kappa\|_{L^p(M^{-1}(\theta'))}^2\Big)^{\frac12} \\ &\lesssim 2^{(N-k)((d-1)(\frac12-\frac1p)+2\varepsilon)}\|f\|_p.\end{aligned}$$

Since ε is arbitrary, the proof is complete modulo the claim (5.3).

Toward proving (5.3), fix $\theta \in \mathcal{P}$ and let

$$n := \#\{\kappa \in \mathcal{Q} \colon \kappa \cap M^{-1}(\theta') \neq \emptyset\}.$$

To avoid notational annoyances, we will assume that $d \geq 3$ for the remainder of this argument. The case $d = 2$ is similar, but easier, and essentially appears in [2]. Given two points $(\tau, \xi), (\tau', \xi') \in \mathbb{R}\times\mathbb{R}^d$, let

$$\mathrm{dist}_{\mathrm{ang}}((\tau,\xi),(\tau',\xi')) := \left|\frac{\xi}{|\xi|} - \frac{\xi'}{|\xi'|}\right|$$

denote their angular separation. Suppose that $\Sigma^{e_1}_{N,k} \cap M^{-1}(\theta)$ has angular width at most $n^{\frac{1}{2(d-1)}}2^{-N}$. Since $\mathcal{Q}$ has bounded overlap, it follows that

$$n \lesssim \frac{(n^{\frac{1}{2(d-1)}}2^{-N})^{d-1}}{2^{-N(d-1)}} = n^{\frac12},$$

and thus $n \lesssim 1$. We may assume, therefore, that there exist points $(\tau,\xi),(\tau',\xi') \in \Sigma^{e_1}_{N,k} \cap M^{-1}(\theta)$ such that $\mathrm{dist}_{\mathrm{ang}}((\tau,\xi),(\tau',\xi')) \geq n^{\frac{1}{2(d-1)}}2^{-N}$. Since θ has angular width $O(2^{k-N})$, it suffices to show that

$$\mathrm{dist}_{\mathrm{ang}}(M(\tau,\xi), M(\tau',\xi')) \gtrsim 2^k\, \mathrm{dist}_{\mathrm{ang}}((\tau,\xi),(\tau',\xi')). \tag{5.4}$$

We proceed by exploiting symmetry. First, we observe that

$$\mathrm{dist}_{\mathrm{ang}}(M(\tau,\xi), M(\tau',\xi')) = \mathrm{dist}_{\mathrm{ang}}(M(\langle\lambda^{-1}\rangle, \lambda^{-1}\xi), M(\langle(\lambda')^{-1}\rangle, (\lambda')^{-1}\xi')), \tag{5.5}$$

where $\lambda := |\xi|$, $\lambda' := |\xi'|$, and $\langle x\rangle := \sqrt{x^2+1}$. Next, we utilize rotational invariance. Let $R := I_2 \oplus S$, where I_2 is the 2×2 identity matrix and S is a rotation of $\mathbb{R}^{d-1}$ satisfying

$$S\left(\frac{\lambda^{-1}(\xi_2,\ldots,\xi_d) - (\lambda')^{-1}(\xi'_2,\ldots,\xi'_d)}{|\lambda^{-1}(\xi_2,\ldots,\xi_d) - (\lambda')^{-1}(\xi'_2,\ldots,\xi'_d)|}\right) = (1,0,\ldots,0).$$

(One can check that $\lambda^{-1}(\xi_2,\ldots,\xi_d) \neq (\lambda')^{-1}(\xi'_2,\ldots,\xi'_d)$.) The maps M and R commute, and R (and thus R^{-1}) preserves angular separation. Setting $(\rho,\zeta) := R(\langle\lambda^{-1}\rangle, \lambda^{-1}\xi)$ and $(\rho',\zeta') := R(\langle(\lambda')^{-1}\rangle, (\lambda')^{-1}\xi')$ and using (5.5), we see that

$$\mathrm{dist}_{\mathrm{ang}}(M(\tau,\xi), M(\tau',\xi')) = \mathrm{dist}_{\mathrm{ang}}(M(\rho,\zeta), M(\rho',\zeta')). \tag{5.6}$$

The definitions of $\Sigma^{e_1}_{N,k}$, λ, λ', and R imply the following:

(i) $|\zeta| = |\zeta'| = 1$ and $|\zeta - e_1|, |\zeta' - e_1| \leq 2^{-k}$;
(ii) $\mathrm{dist}_{\mathrm{ang}}((\tau,\xi),(\tau,\xi')) = |\zeta - \zeta'|$;
(iii) $\zeta_j = \zeta'_j =: a_j$ for all $j \in \{3,\ldots,d\}$;
(iv) $1 \leq \rho, \rho' \leq 1 + 2^{-2N}$;

We write $(\rho,\zeta) = (\rho, r\cos\nu, r\sin\nu, a_3,\ldots,a_d)$, where $r := \sqrt{1 - a_3^2 - \cdots - a_d^2} = \sqrt{\zeta_1^2+\zeta_2^2}$ and $\nu := \arctan(\zeta_2/\zeta_1)$, and we record that $1 - 2^{-2k} \leq r \leq 1$ and $|\nu| \leq 2^{1-k}$ if C is sufficiently large. We compute that $M(\rho,\zeta) = 2^{-N-1}(m_1(\rho,\nu), m_2(\rho,\nu), m_3(\rho,\nu), u)$, where

$$m_1(x,y) := (1+2^{2(k-C)})x + (1-2^{2(k-C)})r\cos y,$$

$$\begin{aligned}m_2(x,y) &:= (1-2^{2(k-C)})x + (1+2^{2(k-C)})r\cos y,\\ m_3(x,y) &:= 2^{k-C+1} r\sin y,\\ u &:= 2^{k-C+1}(a_3,\dots,a_d).\end{aligned}$$

One easily checks that if C is sufficiently large, then $|u| \le 1$ and $1 \le m_2(x,y) \le 2$ and $|m_3(x,y)| \le 2$ whenever $1 \le x \le 1+2^{-2N}$ and $|y| \le 2^{1-k}$. Writing, analogously, $(\rho',\zeta') = (\rho', r\cos\nu', r\sin\nu', a_3,\dots,a_d)$ and using (5.6) and Lemma 5.1, we find that

$$\begin{aligned}\operatorname{dist}_{\text{ang}}(M(\tau,\xi), M(\tau',\xi')) &= \left|\frac{(m_2(\rho,\nu), m_3(\rho,\nu), u)}{|(m_2(\rho,\nu), m_3(\rho,\nu), u)|} - \frac{(m_2(\rho',\nu'), m_3(\rho',\nu'), u)}{|(m_2(\rho',\nu'), m_3(\rho',\nu'), u)|}\right|\\ &\gtrsim \left|\frac{(m_2(\rho,\nu), m_3(\rho,\nu))}{|(m_2(\rho,\nu), m_3(\rho,\nu))|} - \frac{(m_2(\rho',\nu'), m_3(\rho',\nu'))}{|(m_2(\rho',\nu'), m_3(\rho',\nu'))|}\right|\\ &\sim |A(\rho,\nu) - A(\rho',\nu')|, \end{aligned} \tag{5.7}$$

where

$$A(x,y) := \arctan\left(\frac{m_3(x,y)}{m_2(x,y)}\right).$$

Using the mean value theorem and bounds on the m_j, we find that

$$|A(\rho,\nu) - A(\rho,\nu')| \ge |\nu-\nu'| \inf_{|y|\le 2^{1-k}} |\partial_2 A(\rho,y)| \gtrsim 2^{-C}2^k|\nu-\nu'| \tag{5.8}$$

and

$$|A(\rho,\nu') - A(\rho',\nu')| \le |\rho-\rho'| \sup_{1\le x\le 1+2^{-2N}} |\partial_1 A(x,\nu')| \lesssim 2^{-2C}2^{-2N}2^{2k} \le 2^{-3C}2^k 2^{-N}, \tag{5.9}$$

where the implicit constants do not depend on C. Since

$$2^{-N} \lesssim \operatorname{dist}_{\text{ang}}((\tau,\xi),(\tau',\xi')) = |\zeta-\zeta'| = |(\zeta_1,\zeta_2) - (\zeta_1',\zeta_2')| \sim |\nu-\nu'|,$$

(5.4) follows from (5.7)–(5.9) after fixing C sufficiently large. □

Proof of Theorem 1.3. By Lemma 4.1, it suffices to prove that $\mathcal{E}_{\text{fru}}(p\to q)$ holds for all (p,q) satisfying the hypotheses of the theorem. We have assumed that $\mathcal{E}_0(p_0\to q_0)$ holds with $p_0' = \frac{d}{d+2}q_0$ for some $q_0 < \frac{2(d+3)}{d+1}$. Necessarily $p_0 \ge 2$, so by the Cauchy–Schwarz inequality and Lemma 5.2, we have

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{q_0/2} \lesssim_\varepsilon 2^{(N-k)((d-1)(1-\frac{2}{p_0})+2\varepsilon)}\|f_1\|_{p_0}\|f_2\|_{p_0}$$

for every $\varepsilon>0$ whenever f_1 and f_2 are supported in (N,k)-sectors. Given $q_1 > \frac{2(d+3)}{d+1}$, we also have

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{q_1/2} \lesssim 2^{(N-k)(d-1-\frac{2(d+1)}{q_1})}\|f_1\|_2\|f_2\|_2$$

by Lemma 3.1, provided f_1 and f_2 are supported in related (N,k)-sectors. Interpolating these estimates, we see that

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{q_t/2} \lesssim_\varepsilon 2^{(N-k)\alpha_t}\|f_1\|_{r_t}\|f_2\|_{r_t},$$

where

$$\begin{aligned}\left(\frac{1}{r_t},\frac{1}{q_t}\right) &:= (1-t)\left(\frac{1}{p_0},\frac{1}{q_0}\right) + t\left(\frac12,\frac{1}{q_1}\right),\\ \alpha_t &:= (1-t)\left((d-1)\left(1-\frac{2}{p_0}\right)+2\varepsilon\right) + t\left(d-1-\frac{2(d+1)}{q_1}\right)\end{aligned}$$

for $t\in[0,1]$. Given $q\in(q_0,\frac{2(d+3)}{d+1})$, let t be such that $q=q_t$, and suppose that

$$\frac1p > \frac{\alpha_t}{2(d-1)} + \frac{1}{r_t}. \tag{5.10}$$

We may apply Lemma 3.3 to obtain the estimate

$$\|\mathcal{E}f\|_q \lesssim \|f\|_p \tag{5.11}$$

whenever $|f| \sim \chi_\Omega$ for some Ω contained in some Σ_N. Indeed, the hypotheses (i)–(iv) in Lemma 3.3 hold with $r = r_t$ and $\alpha = \alpha_t$, and the estimate $\mathcal{E}_0(p \to q)$ is a consequence of interpolating $\mathcal{E}_0(p_0 \to q_0)$ and $\mathcal{E}_0(1 \to \infty)$ and applying Hölder's inequality. Letting $q_1 \to \frac{2(d+3)}{d+1}$ and $\varepsilon \to 0$, the condition (5.10) becomes (1.2), and thus (5.11) extends to all p satisfying (1.2). Real interpolation now implies that $\mathcal{E}_{\text{fru}}(p \to q)$ holds in the required range. □

6. Possible improvements via local estimates for elliptic surfaces

In this final section, we discuss some likely improvements to Theorem 1.2 by means of the state-of-the-art local extension estimates for elliptic surfaces (as defined in, e.g., [18]). We will ignore some details for simplicity, and thus we do not claim any improvement definitively.

As described in the introduction, the validity of the local estimate $\mathcal{E}_0(p \to q)$ on the parabolic scaling line $p' = \frac{d}{d+2}q$ for some $q < \frac{2(d+3)}{d+1}$ would imply an improvement of Theorem 1.2 by a direct application of our conditional result, Theorem 1.3. Such an estimate appears to follow from known results: Let $\mathcal{E}^d_{\text{ell}}(p \to q)$ denote the statement that for every elliptic phase $\phi : [-1,1]^d \to \mathbb{R}$, the associated extension operator

$$\mathcal{E}_\phi f(t,x) := \int_{[-1,1]^d} e^{i(t,x)\cdot(\phi(\xi),\xi)} f(\xi) d\xi$$

is bounded from $L^p([-1,1]^d)$ to $L^q(\mathbb{R}\times\mathbb{R}^d)$ with operator norm depending only on p, q, d, and the parameters used to define ellipticity. (In particular, $\mathcal{E}^d_{\text{ell}}(p \to q)$ would imply $\mathcal{E}_0(p \to q)$.) Hickman and Rogers [8] have shown that for each $d \geq 2$, there exists some $q_d < \frac{2(d+3)}{d+1}$ such that $\mathcal{E}^d_{\text{ell}}(q \to q)$ holds whenever $q > q_d$. (Their result is stated for paraboloids, but an adaptation of their methods yields estimates for general elliptic surfaces; see [8, Remark 11.3] and references therein.) One can move their estimate to the scaling line $p' = \frac{d}{d+2}q$ in a standard way, but with a loss in the range of q. Namely, one interpolates the bilinear version of $\mathcal{E}^d_{\text{ell}}(q \to q)$ (from the Cauchy–Schwarz inequality) with the L^2-based bilinear extension estimate for elliptic surfaces (see [17]) and then obtains a linear estimate on the scaling line using the bilinear-to-linear method, [18, Theorem 2.2]. In the end, these steps reveal that $\mathcal{E}^d_{\text{ell}}(p \to q)$ holds with $p' = \frac{d}{d+2}q$ whenever

$$q > \tilde{q}_d,$$

for some threshold $\tilde{q}_d < \frac{2(d+3)}{d+1}$ that can be explicitly written as a function of q_d. This leads to an improvement of Theorem 1.2 in every dimension. Hickman and Rogers' exponent q_d can be computed (see [8, Footnote 5 and Figure 1]), and for most values of d it defines the best known range of local extension estimates for d-dimensional elliptic surfaces. (Some stronger results are known for the paraboloid; see Wang [19], Guth [7], and Hickman–Zahl [9].)

Additionally, the method of slicing offers a means of improving Theorem 1.2 on the conic scaling line $p' = \frac{d-1}{d+1}q$. Since the cross sections of Σ are $(d-1)$-dimensional spheres, it is possible to deduce certain extension estimates for Σ using the boundedness of the extension operator associated to $\mathbb{S}^{d-1}$. We have the following conditional result:

Proposition 6.1. *If $p' = \frac{d-1}{d+1}q$ and*

$$\left\| \int_{\mathbb{S}^{d-1}} e^{ix\cdot\theta} f(\theta) d\sigma(\theta) \right\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{S}^{d-1})} \tag{6.1}$$

for all $f \in L^p(\mathbb{S}^{d-1})$, then $\mathcal{E}(p \to q)$ holds.

Proof. We proceed along the lines of arguments in [6] and [11]. For later use, we record that $q \geq \max\{2, p\}$ due to the hypothesis that (6.1) holds. Now, in polar coordinates, our extension operator takes the form

$$\mathcal{E}f(t,x) = \int_0^\infty \int_{\mathbb{S}^{d-1}} e^{i(t,x)\cdot(\langle r\rangle, r\theta)} f(\langle r\rangle, r\theta) \frac{r^{d-1}}{\langle r\rangle} d\sigma(\theta) dr = \int_1^\infty e^{its} \int_{\mathbb{S}^{d-1}} e^{i\langle\langle s\rangle\rangle x\cdot\theta} f(s, \langle\langle s\rangle\rangle\theta) \langle\langle s\rangle\rangle^{d-2} d\sigma(\theta) ds,$$

where $\langle\langle s\rangle\rangle := \sqrt{s^2-1}$. Using the (dualized) Lorentz space version of the Hausdorff–Young inequality and a Minkowski-type inequality (see [15, Corollary 3.16] and [11, Lemma 2.1], respectively), it follows that

$$\|\mathcal{E}f\|_q \lesssim \left\| \left\| \int_{\mathbb{S}^{d-1}} e^{i\langle\langle s\rangle\rangle x\cdot\theta} f(s, \langle\langle s\rangle\rangle\theta) \langle\langle s\rangle\rangle^{d-2} d\sigma(\theta) \right\|_{L^{q',q}_s} \right\|_{L^q_x}$$

$$\lesssim \left\| \left\| \int_{\mathbb{S}^{d-1}} e^{i\langle\langle s\rangle\rangle x\cdot\theta} f(s, \langle\langle s\rangle\rangle\theta)\langle\langle s\rangle\rangle^{d-2} d\sigma(\theta) \right\|_{L^q_x} \right\|_{L^{q',q}_s}. \tag{6.2}$$

By a change of variable and the estimate (6.1), the inner norm in (6.2) obeys the bound

$$\left\| \int_{\mathbb{S}^{d-1}} e^{i\langle\langle s\rangle\rangle x\cdot\theta} f(s, \langle\langle s\rangle\rangle\theta)\langle\langle s\rangle\rangle^{d-2} d\sigma(\theta) \right\|_{L^q_x} \lesssim \langle\langle s\rangle\rangle^{d-2-\frac{d}{q}} \|f(s, \langle\langle s\rangle\rangle\theta)\|_{L^p_\theta}. \tag{6.3}$$

Due to the embedding $L^{q',p} \hookrightarrow L^{q',q}$ and the Lorentz space version of Hölder's inequality (see [12, Theorem 3.6]), we have that

$$\begin{aligned} \|\langle\langle s\rangle\rangle^{d-2-\frac{d}{q}} \|f(s, \langle\langle s\rangle\rangle\theta)\|_{L^p_\theta}\|_{L^{q',q}_s} &\lesssim \|\langle\langle s\rangle\rangle^{d-2-\frac{d}{q}} \|f(s, \langle\langle s\rangle\rangle\theta)\|_{L^p_\theta}\|_{L^{q',p}_s} \\ &\lesssim \|\langle\langle s\rangle\rangle^{-\frac{1}{\alpha}}\|_{L^{\alpha,\infty}_s} \|\langle\langle s\rangle\rangle^{\frac{1}{\alpha}+d-2-\frac{d}{q}} \|f(s, \langle\langle s\rangle\rangle\theta)\|_{L^p_\theta}\|_{L^{p,p}_s} \\ &\lesssim \|\langle\langle s\rangle\rangle^{\frac{1}{\alpha}+d-2-\frac{d}{q}} \|f(s, \langle\langle s\rangle\rangle\theta)\|_{L^p_\theta}\|_{L^p_s}, \end{aligned} \tag{6.4}$$

where $\frac{1}{\alpha} := \frac{1}{q'} - \frac{1}{p} = \frac{2}{(d-1)q}$. By the change of variable $r := \langle\langle s\rangle\rangle$ and some algebra, we find that

$$\|\langle\langle s\rangle\rangle^{\frac{1}{\alpha}+d-2-\frac{d}{q}} \|f(s, \langle\langle s\rangle\rangle\theta)\|_{L^p_\theta}\|_{L^p_s} = \left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |f(\langle r\rangle, r\theta)|^p \frac{r^{d-1}}{\langle r\rangle} d\sigma(\theta) dr \right)^{\frac{1}{p}} = \|f\|_p. \tag{6.5}$$

Combining (6.2)–(6.5), we conclude that $\mathcal{E}(p \to q)$ holds. □

Since the sphere $\mathbb{S}^{d-1}$ is elliptic, (6.1) holds in the range $q > \tilde{q}_{d-1}$ and $p' = \frac{d-1}{d+1}q$, as discussed above. Proposition 6.1 therefore yields an improvement to Theorem 1.2 on the conic scaling line whenever

$$\tilde{q}_{d-1} < \frac{2(d+3)}{d+1}. \tag{6.6}$$

The code from [8, Footnote 5] can be used to compute q_{d-1} (and thus $\tilde{q}_{d-1}$) and determine explicit values of d for which the condition (6.6) holds.

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Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC, Canada, V6T 1Z2

E-mail address: bbruce@math.ubc.ca