# MAXIMAL OPERATORS AND FOURIER RESTRICTION ON THE MOMENT CURVE

MICHAEL JESURUM

ABSTRACT. We bound certain $r$-maximal restriction operators on the moment curve.

## 1. INTRODUCTION

Let $\gamma(t) = (t, \frac{1}{2}t^2, \ldots, \frac{1}{d}t^d)$, and let $\Gamma$ be the image of this curve for $t \in \mathbb{R}$. Drury [2] proved the Fourier restriction estimate

$$||\hat{f}||_{L^q(\Gamma)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}$$

for $1 \leq p < \frac{d^2+d+2}{d^2+d}$ and $p' = \frac{d(d+1)}{2}q$. In the spirit of [7], we study the $r$-maximal form of this restriction operator: $M_r\hat{f}|_\Gamma$, where

$$M_r h(x) = \left( \sup_{s>0} \fint_{B(x,s)} |h|^r \right)^{1/r}$$

and $1 \leq r < \infty$. For $d \geq 3$, we have the following maximal restriction theorem:

**Theorem 1.1.** *For $q = \frac{2}{d(d+1)}p' > p$ and $r$ satisfying*

$$\begin{cases} r \leq \frac{p'}{d}, & \text{if} \quad 1 \leq p \leq \frac{d^2+2d}{d^2+2d-2}; \\ r < p' - \frac{d^2+d-2}{2}, & \text{if} \quad \frac{d^2+2d}{d^2+2d-2} < p < \frac{d^2+d+2}{d^2+d}, \end{cases}$$

*we have the following estimate for every $f \in L^p(\mathbb{R}^d)$:*

$$||M_r\hat{f}||_{L^q(\Gamma)} \leq C_{p,r}||f||_{L^p(\mathbb{R}^d)}. \tag{1.1}$$

Müller, Ricci, and Wright [5] introduced maximal restriction theorems to obtain a pointwise interpretation of the restriction operator associated to $C^2$ curves in $\mathbb{R}^2$. After proving bounds for a two-parameter maximal restriction operator, they introduced the operator

$$M_2 h(x) = (M|h|^2)^{1/2}(x) \tag{1.2}$$

to aid with bounds for a strong maximal restriction operator. Following their logic, for the restriction operator $\mathcal{R}$ associated to the moment curve $\gamma$, the case $r = 2$ in Theorem 1.1 implies:

**Corollary 1.2.** *Let $f \in L^p(\mathbb{R}^d)$ and $1 \leq p < \frac{d^2+d+2}{d^2+d}$. With respect to arclength measure, almost every $x \in \Gamma$ is a Lebesgue point for $\hat{f}$ and the regularized value of $\hat{f}$ at $x$ coincides with $\mathcal{R}f(x)$.*

Later, Vitturi [9] proved similar maximal restriction estimates in the case of the unit sphere in any dimension $d \geq 3$. Ramos [8] improved the known results on spheres in all dimensions, and then in [7] focused on dimensions $d = 2$ and $d = 3$. In particular, he generalized the operator (1.2) to

$$M_r h = (M|h|^r)^{1/r}(x)$$

for $1 \leq r < \infty$, and Theorem 2 in that paper was a maximal restriction result for this operator on the unit circle for $p < \frac{4}{3}$ and $r \leq 2$. Thus, in the case $d = 2$, Theorem 1.1 is due to Ramos [7], since the arguments that apply to the circle also apply to the parabola.

Kovač [4] took a more general approach, proving maximal and variational restriction estimates using restriction inequalities as a black box. Theorem 1, Remark 2, and Remark 3 in that paper combine with Drury's [2] restriction estimate to show that

$$||M_2 \hat{f}||_{L^q(\Gamma)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}$$

for $1 \leq p < \frac{d^2+d+2}{d^2+d+1}$ and $p' = \frac{d(d+1)}{2} q$. Theorem 1.1 extends this range of $p$ to the full Drury range for $M_2$ and gives estimates of the form (1.1) for $r > 2$. See also [3] for more on variational restriction theorems.

In the case $d \geq 3$, the first two cases of Theorem 1.1 are distinct. When $r = 2$, we obtain the full range given by Drury: $p < \frac{d^2+d+2}{d^2+d}$. For $r = \frac{d+2}{2}$, the range of $p$ corresponds to the Christ-Prestini $p < \frac{d^2+2d}{d^2+2d-2}$ (see [1] and [6]). Figure 1 illustrates these ranges.

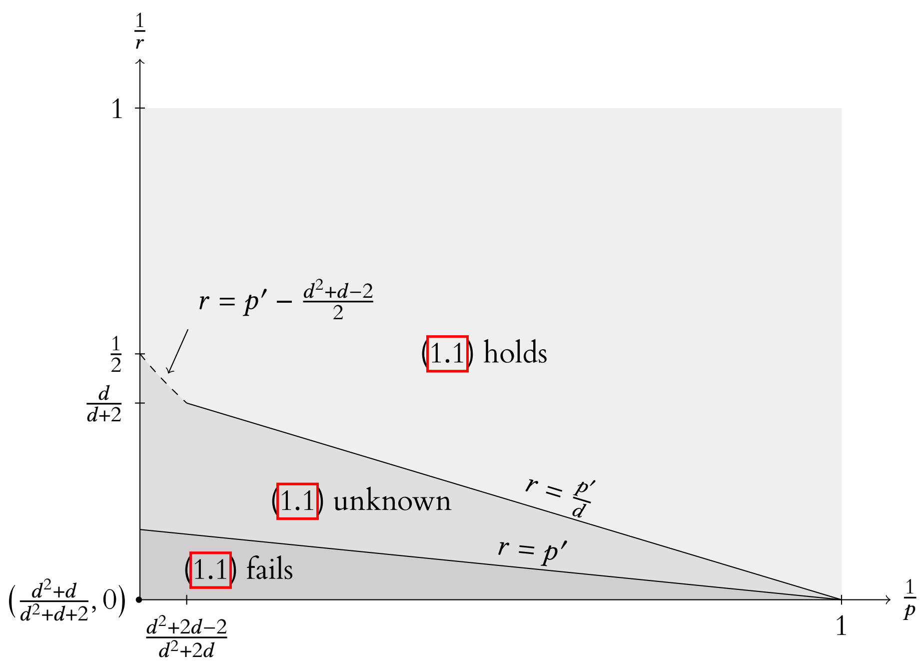


FIGURE 1. *Range of r and p for which the r-maximal restriction operator is bounded from $L^p$ to $L^q$, where $q = \frac{2}{d(d+1)} p'$.*

**Outline of Proof.** The overall structure follows Drury's induction scheme from [2]. To accomodate this, we prove the superficially stronger result:

**Proposition 1.3.** *Let $2 \le r < \frac{d+2}{2}$. Denote by $M_r^k$ the $k$-fold composition of $M_r$ with itself. Then for each $k \in \mathbb{N}$, $1 \le p < \frac{d^2+d+2(r-1)}{d^2+d+2(r-2)}$, and $p' = \frac{1}{2}d(d+1)q$, we have*

$$||M_r^k \hat{f}||_{L^q(\Gamma)} \le C_{p,r}||f||_{L^p(\mathbb{R}^d)}.$$

Indeed, for $r \ge \frac{d+2}{2}$, we can interpolate the above result with the bound

$$||M_\infty \hat{f}||_{L^\infty(\Gamma)} \le ||f||_{L^1(\mathbb{R}^d)},$$

which follows from Hausdorff-Young. For $1 \le r < 2$, we can apply Hölder's inequality to see that

$$M_r \hat{f}(x) \le M_2 \hat{f}(x).$$

Thus, Theorem 1.1 follows from Proposition 1.3. To prove Proposition 1.3, we first linearize the operator $f \mapsto M_r^k \hat{f}|_\Gamma$ (Section 2). Then, in Section 3, we apply the induction hypothesis to prove a mixed-norm estimate for the $d$-fold power of the linear operator from Section 2. We interpolate this estimate with an $L^2$ bound for that same operator that comes from Plancherel. This interpolation allows us to increase the value of $p$, which completes the induction.

**Acknowledgements.** The author would like to thank Betsy Stovall for suggesting this project and for advising throughout the process. He also thanks the anonymous referee for their comments and suggestions. This work was supported by NSF grant DMS-1653264.

## 2. Kolmogorov-Seliverstov-Plessner Linearization

We first fix $2 \le r < \frac{d+2}{2}$. This value of $r$ will remain fixed throughout this section and the next. The first step is to linearize the maximal operator given in Proposition 1.3. The technique here is similar to [7], which built on the techniques of [5], [9], and [8].

Let $\chi_a(x)$ be the $L^1$-normalized characteristic function of the ball of radius $a$; that is,

$$\chi_a(x) = \frac{1}{|B_a|}\chi\left(\frac{x}{a}\right),$$

where $B_a$ is the ball centered at 0 with radius $a$. Let $\rho_1, \dots, \rho_k : \mathbb{R}^d \to \mathbb{R}_{>0}$ be measurable and $\eta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a measurable function such that

$$\fint_{B_{\rho_1(x)}(x)} \cdots \fint_{B_{\rho_k(y_{k-1})}(y_{k-1})} |\eta(x, y_k)|^{r'} \mathrm{d}y_k \dots \mathrm{d}y_1 \le 1 \tag{2.1}$$

for every $x \in \mathbb{R}^d$. Set

(2.2)

$$\mathcal{A}_x^k(z_1, \dots, z_k) = \eta(x, x - z_1 - \dots - z_k)\chi_{\rho_1(x)}(z_1)\chi_{\rho_2(x-z_1)}(z_2)\dots\chi_{\rho_k(x-z_1-\dots-z_{k-1})}(z_k).$$

and

$$M_{r,\eta,\rho,k}f(x) = \int_{\mathbb{R}^{kd}} \hat{f}(x - z_1 - \dots - z_k)\mathcal{A}_x^k(z_1, \dots, z_k)\mathrm{d}z_1 \dots \mathrm{d}z_k, \tag{2.3}$$

or, equivalently,

$$M_{r,\eta,\rho,k}f(x) = \fint_{B_{\rho_1(x)}(x)} \cdots \fint_{B_{\rho_k(z_{k-1})}(z_{k-1})} \hat{f}(z_k)\eta(x, z_k)\mathrm{d}z_k \dots \mathrm{d}z_1. \tag{2.4}$$

**Lemma 2.1.** *Suppose there is $C > 0$ such that for every $\eta_x$ and $\rho_1, \ldots, \rho_k$ as above, and for all $f$ Schwartz,*

$$||M_{r,\eta,\rho,k} f||_{L^q(\Gamma)} \leq C||f||_{L^p(\mathbb{R}^d)}.$$

*Then*

$$||M_r^k \hat{f}||_{L^q(\Gamma)} \leq C||f||_{L^p(\mathbb{R}^d)}.$$

*for all Schwartz functions $f$.*

*Proof.* Let $f$ be a Schwartz function, and let

$$\eta(x,y) = \frac{\overline{\hat{f}(y)}|\hat{f}(y)|^{r-2}}{\left(⨍_{B_{\rho_1(x)}(x)} \cdots ⨍_{B_{\rho_k(z_{k-1})}(z_{k-1})} |f(z_k)|^r \mathrm{d}z_k \ldots \mathrm{d}z_1\right)^{\frac{1}{r'}}}.$$

Then for any $x \in \mathbb{R}^d$,

$$\begin{aligned} &⨍_{B_{\rho_1(x)}(x)} \cdots ⨍_{B_{\rho_k(y_{k-1})}(y_{k-1})} |\eta(x,y_k)|^{r'} \mathrm{d}y_k \ldots \mathrm{d}y_1 \\ &= \frac{⨍_{B_{\rho_1(x)}(x)} \cdots ⨍_{B_{\rho_k(y_{k-1})}(y_{k-1})} |f(y_k)|^{r'(r-1)} \mathrm{d}y_k \ldots \mathrm{d}y_1}{⨍_{B_{\rho_1(x)}(x)} \cdots ⨍_{B_{\rho_k(z_{k-1})}(z_{k-1})} |f(z_k)|^r \mathrm{d}z_k \ldots \mathrm{d}z_1}. \end{aligned}$$

Since $r'(r-1) = r$, the numerator and denominator are equal and hence $\eta$ satisfies (2.1). Moreover, using (2.4), we have

$$M_{r,\eta,\rho,k} f(x) = \frac{⨍_{B_{\rho_1(x)}(x)} \cdots ⨍_{B_{\rho_k(z_{k-1})}(z_{k-1})} |\hat{f}(z_k)|^r \mathrm{d}z_k \ldots \mathrm{d}z_1}{\left(⨍_{B_{\rho_1(x)}(x)} \cdots ⨍_{B_{\rho_k(z_{k-1})}(z_{k-1})} |f(z_k)|^r \mathrm{d}z_k \ldots \mathrm{d}z_1\right)^{\frac{1}{r'}}}.$$

Thus, we obtain

$$M_{r,\eta,\rho,k} f(x) = \left(⨍_{B_{\rho_1(x)}(x)} \cdots ⨍_{B_{\rho_k(z_{k-1})}(z_{k-1})} |\hat{f}(z_k)|^r \mathrm{d}z_k \ldots \mathrm{d}z_1\right)^{\frac{1}{r}}.$$

For well-chosen $\rho_1, \ldots, \rho_k$, this can be made arbitrarily close to $M_r^k \hat{f}(x)$, so the claim holds. □

Hereafter, we will use the form of $M_{r,\eta,\rho,k}$ given in (2.3). As is often the case, it will be more convenient to work with an extension operator rather than the restriction. Given $g \colon \mathbb{R}^d \to \mathbb{C}$ and $f \colon \mathbb{R} \to \mathbb{C}$,

$$\begin{aligned} \langle M_{r,\eta,\rho,k} g, f\rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}^{kd}} \hat{g}(\gamma(t) - z_1 - \ldots - z_k) \mathcal{A}^k_{\gamma(t)}(z_1, \ldots, z_k) \overline{f(t)} \mathrm{d}z_1 \ldots \mathrm{d}z_k \mathrm{d}t \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{kd}} \int_{\mathbb{R}^d} e^{-2\pi i \xi(\gamma(t) - z_1 - \ldots - z_k)} g(\xi) \mathcal{A}^k_{\gamma(t)}(z_1, \ldots, z_k) \overline{f(t)} \mathrm{d}\xi \mathrm{d}z_1 \ldots \mathrm{d}z_k \mathrm{d}t. \end{aligned}$$

Hence the adjoint is given by

$$\begin{aligned} M^*_{r,\eta,\rho,k} f(\xi) &= \int_{\mathbb{R}} \int_{\mathbb{R}^{kd}} e^{2\pi i \xi(\gamma(t) - z_1 - \ldots - z_k)} \overline{\mathcal{A}^k_{\gamma(t)}}(z_1, \ldots, z_k) f(t) \mathrm{d}z_1 \ldots \mathrm{d}z_k \mathrm{d}t \\ &= \int_{\mathbb{R}} e^{2\pi i \xi \gamma(t)} \overline{\widehat{\mathcal{A}^k_{\gamma(t)}}}(\xi, \ldots, \xi) f(t) \mathrm{d}t. \end{aligned}$$

Setting $\vec{\xi}^k = (\xi, \ldots, \xi)$, we have

$$M^*_{r,\eta,\rho,k} f(\xi) = \int_{\mathbb{R}} e^{2\pi i \xi \gamma(t)} \widehat{\mathscr{A}^k_{\gamma(t)}}(\vec{\xi}^k) f(t) \mathrm{d}t.$$

Proposition 1.3 now follows from the following lemma:

**Lemma 2.2.** *Let $1 \le p < \frac{d^2+d+2(r-1)}{2(r-1)}$, $q = \frac{d(d+1)}{2} p'$, and $2 \le r < \frac{d+2}{2}$. There is $C > 0$ such that for $\rho_1, \ldots, \rho_k$ and $\eta$ measurable satisfying (2.1), and for all Schwartz functions $f$,*

$$\|M^*_{r,\eta,\rho,k} f\|_{L^q(\mathbb{R}^d)} \le C \|f\|_{L^p(\Gamma)}. \tag{2.5}$$

## 3. The Induction Argument

The proof of Lemma 2.2 proceeds by induction. The base case is $p = 1$ and $q = \infty$. Here,

$$|M^*_{r,\eta,\rho,k} f(\xi)| = \left| \int_{\mathbb{R}} e^{2\pi i \xi \gamma(t)} \widehat{\mathscr{A}^k_{\gamma(t)}}(\vec{\xi}) f(\xi) \mathrm{d}t \right| \le \int_{\mathbb{R}} \sup_{x \in \mathbb{R}^d} \|\mathscr{A}^k_x\|_{L^1(\mathbb{R}^{kd})} |f(t)| \mathrm{d}t.$$

By (2.1), $\|\mathscr{A}^k_x\|_{L^1(\mathbb{R}^d)} \le 1$ for all $x$. Thus,

$$|M^*_{r,\eta,\rho,k} f(\xi)| \le \int_{\mathbb{R}} |f(t)| \mathrm{d}t = \|f\|_{L^1(\Gamma)}.$$

This completes the base case. The following lemma, along with a little arithmetic, establishes the claimed range of $p$ and $q$:

**Lemma 3.1.** *Assume for every $1 \le p < p_0 < \frac{d^2+d+2(r-1)}{2(r-1)}$, $q = \frac{d(d+1)}{2} p'$, there is $C > 0$ such that (2.5) holds for all $k$, all measurable $\rho_1, \ldots, \rho_k \colon \mathbb{R}^d \to \mathbb{R}_{>0}$, all measurable $\eta \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfying (2.1) for every $x \in \mathbb{R}^d$, and all $f$. Then (2.5) holds for all such $\eta$, $\rho$, $k$, and $f$, and for all $p$ satisfying*

$$\frac{d}{p} > \frac{2}{(d+2)p_0'(r'-1)} + \frac{d}{(d+2)p_0},$$

*and $q = \frac{d(d+1)}{2} p'$.*

To prove Lemma 3.1, we adapt Drury's argument in [2]. Thus, we rewrite the left-hand side of (2.5) as

$$\|M^*_{r,\eta,\rho,k} f\|_{L^q(\mathbb{R}^d)} = \|(M^*_{r,\eta,\rho,k} f)^d\|^{1/d}_{L^{q/d}(\mathbb{R}^d)}.$$

Expanding gives

$$(M^*_{r,\eta,\rho,k} f)^d(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \xi \sum_{j=1}^d \gamma(x_j)} \prod_{j=1}^d \widehat{A_{\gamma(x_j)}}(\vec{\xi}^k) f(x_j) \mathrm{d}x.$$

Define

$$TG(\xi) = \int_{\Gamma + \cdots + \Gamma} e^{2\pi i \xi y} \prod_{j=1}^d \widehat{\mathscr{A}^k_{\gamma(x_j)}}(\vec{\xi}^k) G(y) \mathrm{d}y, \tag{3.1}$$

where $x = x(y)$ is uniquely determined by $x_1 < x_2 < \cdots < x_d$ and $y = \sum_{j=1}^d \gamma(x_j)$. Let $v$ be the Vandermonde determinant,

$$|v(x)| = \prod_{1 \le i < j \le d} |x_i - x_j|.$$

Then with the choice

$$G(y) = \prod_{j=1}^{d} f(x_j)|v(x)|^{-1},$$

we have

$$TG(\xi) = \frac{1}{d!}(M^*_{r,\eta,\rho,k}f)^d(\xi). \tag{3.2}$$

To apply the induction hypothesis, we will need to work with another change of variables. For $h = (h_1, h')$ with $h_1 = 0 < h_2 < \cdots < h_d$, set $x_j = t + h_j$ and $\gamma_h(t) = \sum_{j=1}^d \gamma(x_j)$. Define the auxiliary function

$$\tilde{G}(t,h) = G(\gamma_h(t)) = \prod_{j=1}^{d} f(t+h_j)|v(h)|^{-1}. \tag{3.3}$$

Finally, fix $0 < \varepsilon < \frac{d+2}{2} - r$.

**Lemma 3.2.** *For $T$ defined as in (3.1) and $r + \varepsilon < \frac{d+2}{2}$, we have*

$$||TG||_{L^{r+\varepsilon}} \leq C||\tilde{G}||_{L_{h'}^{(r+\varepsilon)'}(L_t^{(r+\varepsilon)'};|v(h)|)}. \tag{3.4}$$

*Proof.* For any test function $H$, by (3.1) we have

$$|\langle TG, H\rangle| = \left|\int_{\mathbb{R}^d} TG(\xi)\overline{H}(\xi)\mathrm{d}\xi\right| = \left|\int_{\mathbb{R}^d}\int_{\Gamma+\cdots+\Gamma} e^{2\pi i\xi y}\prod_{j=1}^{d}\widehat{\overline{\mathscr{A}^k_{\gamma(x_j)}}}(\vec{\xi}^k)G(y)\overline{H}(\xi)\mathrm{d}y\mathrm{d}\xi\right|.$$

Changing the order of integration and applying Plancherel in $\xi$,

$$|\langle TG, H\rangle| = \left|\int_{\Gamma+\cdots+\Gamma} G(y)\int_{\mathbb{R}^{kd}} \overline{\mathop{\Large *}_{j=1}^{d}\mathscr{A}^k_{\gamma(x_j)}(w_1,\ldots,w_k)\widehat{H}(y-w_1-\ldots-w_k)}\mathrm{d}w_1\ldots\mathrm{d}w_k\mathrm{d}y\right|.$$

We can now apply the following lemma, which we will prove shortly.

**Lemma 3.3.** *Let $\mathscr{A}^k_z(w_1,\ldots,w_k)$ be defined as in (2.2), and let $\hat{H}$ be a test function. Then for each $n,k \in \mathbb{N}$ and $z_1,\ldots,z_n$,*

$$\int_{\mathbb{R}^{kd}} |(\mathscr{A}^k_{z_1} * \ldots * \mathscr{A}^k_{z_n})(w_1,\ldots,w_k)\hat{H}(y-w_1-\ldots-w_k)|\mathrm{d}w_1\ldots\mathrm{d}w_k \leq M_r^{nk}\hat{H}(y). \tag{3.5}$$

Using this lemma, we see that

$$|\langle TG, H\rangle| \leq \int_{\Gamma+\cdots+\Gamma}|G(y)|M_r^{kd}\hat{H}(y)\mathrm{d}y.$$

By Hölder's inequality, we obtain

$$|\langle TG, H\rangle| \leq ||G||_{(r+\varepsilon)'}||M_r^{kd}\hat{H}||_{r+\varepsilon}.$$

Since $r < r+\varepsilon$, we have

$$|\langle TG, H\rangle| \leq ||G||_{(r+\varepsilon)'}||\hat{H}||_{r+\varepsilon}.$$

Finally, $r+\varepsilon > 2$, so by Hausdorff-Young,

$$|\langle TG, H\rangle| \leq ||G||_{(r+\varepsilon)'}||H||_{(r+\varepsilon)'}.$$

Thus, TG is bounded from $L^{(r+\varepsilon)'}$ to $L^{r+\varepsilon}$, which, along with a change of variables, proves the lemma. □

Now we prove Lemma 3.3.

*Proof.* Fix $k \geq 1$. We proceed by induction. The base case is $n = 1$. In this case, the left-hand side of (3.5) is

$$\int_{\mathbb{R}^{kd}} |\hat{H}(y - w_1 - \ldots - w_k)\eta(z, z - w_1 - \ldots - w_k) \cdot \chi_{\rho_1(z)}(w_1) \ldots \chi_{\rho_k(z-w_1-\ldots-w_{k-1})}(w_k)| \mathrm{d}w_1 \ldots \mathrm{d}w_k.$$

Applying Hölder's inequality, this is bounded by

$$\left(\int_{\mathbb{R}^{kd}} |\hat{H}(y - w_1 - \ldots - w_k)|^r |\chi_{\rho_1(z)}(w_1)| \ldots |\chi_{\rho_k(z-w_1-\ldots-w_{k-1})}(w_k)| \mathrm{d}w_1 \ldots \mathrm{d}w_k\right)^{\frac{1}{r}}$$
$$\cdot \left(\int_{\mathbb{R}^{kd}} |\eta(z, z - w_1 - \ldots - w_k)|^{r'} |\chi_{\rho_1(z)}(w_1)| \ldots |\chi_{\rho_k(z-w_1-\ldots-w_{k-1})}(w_k)| \mathrm{d}w_1 \ldots \mathrm{d}w_k\right)^{\frac{1}{r'}}.$$

Changing variables in each integral transforms the above into

$$\left(\fint_{B_{\rho_1(z)}(y)} \ldots \fint_{B_{\rho_k(v_{k-1}+z-y)}(v_{k-1})} |\hat{H}(v_k)|^r \mathrm{d}v_k \ldots \mathrm{d}v_1\right)^{\frac{1}{r}}$$
$$\cdot \left(\fint_{B_{\rho_1(z)}(z)} \ldots \fint_{B_{\rho_k(z-v_{k-1})}(v_{k-1})} |\eta(z, v_k)|^{r'} \mathrm{d}v_k \ldots \mathrm{d}v_1\right)^{\frac{1}{r'}}$$

By (2.1), the second term is bounded by 1. Moreover, the first term is bounded by $M_r^k \hat{H}(y)$, so the base case is done. Now, assume we have (3.5) for some $n$ and all functions $\hat{H}$. We want to bound

$$\int_{\mathbb{R}^{kd}} |(\mathscr{A}_{z_1}^k * \ldots * \mathscr{A}_{z_{n+1}}^k)(w_1, \ldots, w_k)\hat{H}(y - w_1 - \ldots - w_k)| \mathrm{d}w_1 \ldots \mathrm{d}w_k, \tag{3.6}$$

with the convolution performed $n+1$ times. Split up the convolution as the convolution of an $n$-fold convolution with $A_{z_{n+1}}$ to rewrite (3.6) as

$$\int_{\mathbb{R}^{kd}} |(\mathscr{A}_{z_1}^k * \ldots * \mathscr{A}_{z_n}^k) * \mathscr{A}_{z_{n+1}}^k(w_1, \ldots, w_k)\hat{H}(y - w_1 - \ldots - w_k)| \mathrm{d}w_1 \ldots \mathrm{d}w_k.$$

Expanding this convolution, we can further rewrite (3.6) as

$$\int_{\mathbb{R}^{kd}} \int_{\mathbb{R}^{kd}} |(\mathscr{A}_{z_1}^k * \ldots * \mathscr{A}_{z_n}^k)(v_1, \ldots, v_k)\mathscr{A}_{z_{n+1}}^k(w_1 - v_1, \ldots, w_k - v_k) \cdot \hat{H}(y - w_1 - \ldots - w_k)| \mathrm{d}v_1 \ldots \mathrm{d}v_k \mathrm{d}w_1 \ldots \mathrm{d}w_k.$$

With the change of variables $u_j = w_j - v_j$, (3.6) becomes

$$\int_{\mathbb{R}^{kd}} \int_{\mathbb{R}^{kd}} |(\mathscr{A}_{z_1}^k * \ldots * \mathscr{A}_{z_n}^k)(v_1, \ldots, v_k)\mathscr{A}_{z_{n+1}}^k(u_1, \ldots, u_k) \cdot \hat{H}(y - v_1 - \ldots - v_k - u_1 - \ldots - u_k)| \mathrm{d}v_1 \ldots \mathrm{d}v_k \mathrm{d}u_1 \ldots \mathrm{d}u_k.$$

By the induction hypothesis, the above is bounded by

$$\int_{\mathbb{R}^{kd}} |\mathscr{A}_{z_{n+1}}^k(v_1, \ldots, v_k) M_r^{nk} \hat{H}(y - v_1 - \ldots - v_k)| \mathrm{d}v_1 \ldots \mathrm{d}v_k.$$

Finally, another application of the induction hypothesis shows that (3.6) is bounded by $M_r^{(n+1)k} \hat{H}(y)$. □

**Lemma 3.4.** *For $T$ defined as in (3.1), there is a constant $C_{p,r}$ such that*

$$||TG||_{L^q} \le C_{p,r}||\tilde{G}||_{L^1_{h'}(L^p_t;|v(h)|)}. \tag{3.7}$$

*Proof.* By Minkowski's inequality for integrals,

$$\begin{aligned} ||TG||_{L^q} &= \left\| \int_0^\infty \int_{h_2}^\infty \cdots \int_{h_{d-1}}^\infty \int_{\mathbb{R}} e^{2\pi i \xi \gamma_h(t)} \prod_{j=1}^d \widehat{\mathscr{A}^k_{\gamma(t+h_j)}}(\vec{\xi}^k)\tilde{G}(t,h)v(h)\mathrm{d}t\mathrm{d}h' \right\|_{L^q_\xi} \\ &\le \left\| \int_{\mathbb{R}} e^{2\pi i \xi \gamma_h(t)} \prod_{j=1}^d \widehat{\mathscr{A}^k_{\gamma(t+h_j)}}(\vec{\xi}^k)\tilde{G}(t,h)v(h)\mathrm{d}t \right\|_{L^1_h L^q_\xi}. \end{aligned}$$

Define the operator

$$S_h F(\xi) = \int_{\mathbb{R}} e^{2\pi i \xi \gamma_h(t)} \prod_{j=1}^d \widehat{\mathscr{A}^k_{\gamma(t+h_j)}}(\vec{\xi}^k)F(t)\mathrm{d}t,$$

whose adjoint is given by

$$\begin{aligned} S_h^* H(t) &= \int_{\mathbb{R}^d} e^{-2\pi i \xi \gamma_h(t)} \prod_{j=1}^d \widecheck{\mathscr{A}^k_{\gamma(t+h_j)}}(\vec{\xi}^k)H(\xi)\mathrm{d}\xi \\ &= \int_{\mathbb{R}^{kd}} \mathop{\ast}_{j=1}^d \mathscr{A}^k_{\gamma(t+h_j)}(w_1,\ldots,w_k)\hat{H}(\gamma_h(t)-w_1-\ldots-w_k)\mathrm{d}w_1\ldots\mathrm{d}w_k. \end{aligned}$$

Using Lemma 3.3, we obtain the bound

$$|S_h^* H(t)| \le M_r^k \hat{H}(\gamma_h(t)).$$

Since each $\gamma_h$ is an affine transformation of the original curve, the induction hypothesis yields

$$||S_h^* H||_{L^{p'}} \le C_{p,r}||H||_{L^{q'}}.$$

Hence, we have

$$||S_h F||_{L^q_\xi} \le C_{p,r}||F||_{L^p}.$$

Setting $F(t) = \tilde{G}(t,h)v(h)$ for each $h$ and integrating in $h'$ finishes the proof. □

By interpolating (3.4) and (3.7), we obtain

$$||TG||_{L^c} \le C_{a,b,r}||\tilde{G}||_{L^a_{h'}(L^b_t;|v(h)|)} \tag{3.8}$$

for all $(a^{-1}, b^{-1})$ in the triangle with vertices $(1,1)$, $(1,p_0^{-1})$, and $(((r+\varepsilon)')^{-1}, ((r+\varepsilon)')^{-1})$, with $c$ satisfying

$$\frac{(d+2)(d-1)}{2}a^{-1} + b^{-1} + \frac{d(d+1)}{2}c^{-1} = \frac{d(d+1)}{2}.$$

Expanding out $\tilde{G}$ using (3.3), we see that

$$||\tilde{G}||_{L^a_{h'}(L^b_t;|v(h)|)} = \left( \int_{\mathbb{R}} |v(h)|^{-(a-1)} \left( \int_{\mathbb{R}^{d-1}} |f(t+h_1)\ldots f(t+h_d)|^b \mathrm{d}t \right)^{\frac{a}{b}} \mathrm{d}h' \right)^{\frac{1}{a}}.$$

As noted in [2], $v(0,h')^{-1} \in L^{\frac{d}{2},\infty}_{h'}$, so we can apply Hölder's inequality to obtain

$$\|\tilde{G}\|_{L^a_{h'}(L^b_t;|v(h)|)} \le \|f\|^d_{L^{p,1}_t} \tag{3.9}$$

for

$$\begin{cases} 1 < a < \frac{d+2}{2}, \\ a \le b < \frac{2a}{d+2-da}, \text{ and} \\ \frac{d}{p} = \frac{(d+2)(d-1)}{2}a^{-1} + b^{-1} - \frac{d(d-1)}{2}. \end{cases}$$

Plugging (3.2) and (3.9) into (3.8),

$$\|M_{r,\eta,\rho,k}f\|_{L^q} \lesssim \|f\|_{L^{p,1}}, \tag{3.10}$$

for

$$\frac{d}{p} = \frac{(d+2)(d-1)}{2}a^{-1} + b^{-1} - \frac{d(d-1)}{2}, \tag{3.11}$$

where $q = \frac{d(d+1)}{2}p'$, and $a$ and $b$ satisfy (Figure 2):

$$\begin{cases} \frac{d}{d+2} < a^{-1} < 1, \\ b^{-1} \le a^{-1}, \\ (d+2)a^{-1} - 2b^{-1} < d, \text{ and} \\ (p_0 - (r+\varepsilon)')a^{-1} + p_0((r+\varepsilon)'-1)b^{-1} \ge p_0 - 1. \end{cases}$$

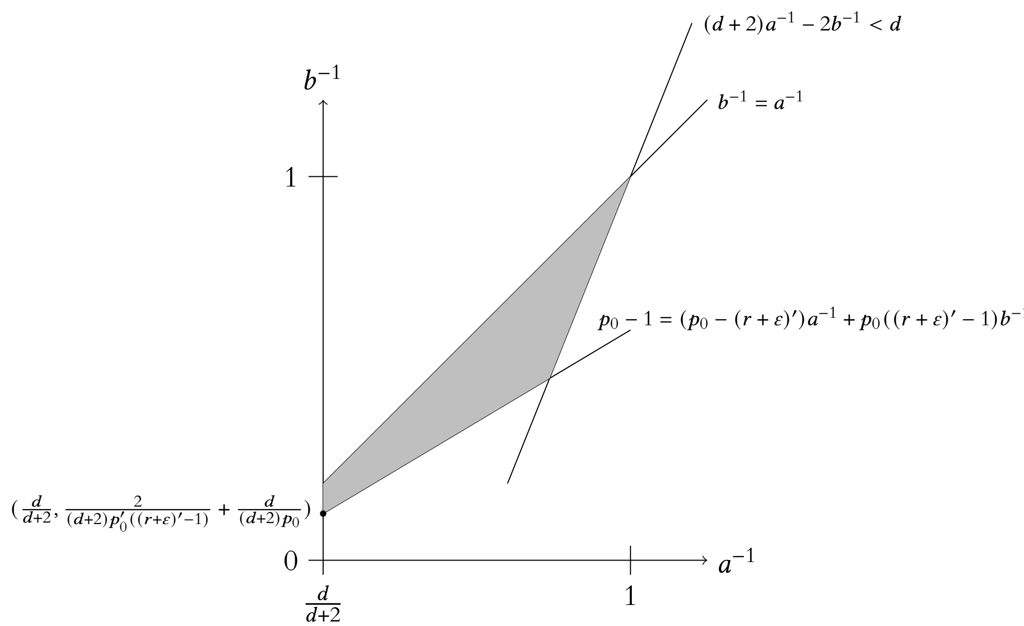


FIGURE 2. *Range of $a$ and $b$ for which (3.10) holds with $p$ satisfying (3.11) and $q = \frac{d(d+1)}{2}p'$.*

Since $r + \varepsilon \le \frac{d+2}{2}$, the point $(a^{-1}, b^{-1}) = (\frac{d}{d+2}, \frac{2}{(d+2)p_0'((r+\varepsilon)'-1)} + \frac{d}{(d+2)p_0})$ lies on the boundary of this region and satisfies

$$\frac{(d+2)(d-1)}{2}a^{-1} + b^{-1} - \frac{d(d-1)}{2} < \frac{d}{p_0}.$$

Taking $(a^{-1}, b^{-1})$ slightly inside of the region and using real interpolation, we obtain

$$||M_{r,\eta,\rho,k}f||_{L^q} \lesssim ||f||_{L^p}, \quad q = \frac{d(d+1)}{2}p', \quad \frac{d}{p} > \frac{2}{(d+2)p_0'((r+\varepsilon)'-1)} + \frac{d}{(d+2)p_0}.$$

Since this is true for all $0 < \varepsilon < \frac{d+2}{d} - r$, we have

$$||M_{r,\eta,\rho,k}f||_{L^q} \lesssim ||f||_{L^p}, \quad q = \frac{d(d+1)}{2}p', \quad \frac{d}{p} > \frac{2}{(d+2)p_0'(r'-1)} + \frac{d}{(d+2)p_0},$$

which proves Lemma 2.2 and hence Theorem 1.1.

## 4. Bounds on $r$

We are not able to show, nor do we believe, that the range of $r$ is sharp. The following proposition shows that $r \leq p'$ is necessary in any bound of the form (1.1), which corresponds to $r \leq \frac{d^2+d+2}{2}$ in the full Drury range. This counterexample in dimension $d = 2$ is due to Ramos [7].

**Proposition 4.1.** *Suppose that for some $p$, $q$, and $r$, and all $f \in L^p(\mathbb{R}^d)$, we have the bound*

$$||M_r\hat{f}||_{L^q(\Gamma)} \leq C_{p,r}||f||_{L^p(\mathbb{R}^d)}. \tag{4.1}$$

*Then $r \leq p'$.*

*Proof.* For $0 < t < 1$, let $\hat{f}_t = \chi_{[-t,t]^d}$, and let $k = 1$. We first compute $f_t$.

$$f_t(x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi}\chi_{[-t,t]}(\xi)\mathrm{d}\xi = \int_{-t}^{t}\cdots\int_{-t}^{t} e^{2\pi i x\xi}\mathrm{d}\xi$$
$$= \prod_{j=1}^{d} \frac{e^{2\pi i x_j t} - e^{-2\pi i x_j t}}{2\pi i x_j} = \prod_{j=1}^{d} \frac{\sin(2\pi x_j t)}{\pi x_j}.$$

Thus, we have

$$||f_t||_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}\left|\prod_{j=1}^{d}\frac{\sin(2\pi x_j t)}{\pi x_j}\right|^p \mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^d}\prod_{j=1}^{d}\left(\left|\frac{2t\sin(y_j)}{y_j}\right|^p \cdot \frac{1}{2\pi t}\right)\mathrm{d}x\right)^{\frac{1}{p}} = Ct^{\frac{d}{p'}}.$$

For $x \in [-1,1]^d$, by taking the ball centered at $x$ with radius 10, we see that

$$M_r\hat{f}_t(x) \gtrsim t^{\frac{d}{r}}.$$

Hence, we have

$$||M_r\hat{f}_t||_{L^q(\Gamma)} \gtrsim t^{\frac{d}{r}}.$$

Combining these estimates and (4.1) gives

$$t^{\frac{d}{r}} \lesssim t^{\frac{d}{p'}}.$$

Sending $t \to 0$ shows that

$$\frac{d}{r} \geq \frac{d}{p'},$$

which means that $r \leq p'$. □

## References


[1] Michael Christ. "On the restriction of the Fourier transform to curves: endpoint results and the degenerate case". In: *Trans. Amer. Math. Soc.* 287 (1985). DOI: 10.1090/S0002-9947-1985-0766216-6.

[2] Stephen W. Drury. "Restrictions of Fourier transforms to curves". In: *Annales de L'Institut Fourier* 35.1 (1985), pp. 117–123. DOI: 10.5802/aif.1001.

[3] Vjeckoslav Kovač and Diogo Oliveria e Silva. "A variational restriction theorem". In: *Arch. Math.* 117 (2021), pp. 65–78. DOI: 10.1007/s00013-021-01604-1.

[4] Vjekoslav Kovač. "Fourier restriction implies maximal and variational Fourier restriction". In: *Journal of Functional Analysis* 277.10 (2019), pp. 3355–3372. DOI: 10.1016/j.jfa.2019.03.015.

[5] Detlef Müller, Fulvio Ricci, and James Wright. "A maximal restriction theorem and Lebesgue points of functions in $\mathcal{F}(L^p)$". In: *Revista Matemática Iberoamerican* 35.3 (2019), pp. 693–702. DOI: 10.4171/RMI/1066.

[6] Elena Prestini. "Restriction theorems for the Fourier transform to some manifolds in $\mathbb{R}^n$". In: Proceedings of Symposia in Pure Mathematics. Vol. 35.1.1. 1979, pp. 101–109. DOI: 10.1090/pspum/035.1.

[7] João Ramos. *Low-dimensional maximal restriction principles for the Fourier transform*. 2019. arXiv: 1904.10858.

[8] João Ramos. "Maximal restriction estimates and the maximal function of the Fourier transform". In: *Proc. Amer. Math. Soc.* 148 (2020), pp. 1131–1138. DOI: 10.1090/proc/14805.

[9] Marco Vitturi. *A note on maximal Fourier restriction for spheres in all dimensions*. 2017. arXiv: 1703.09495.