Extremizers for Adjoint Restriction to Pairs of Translated Paraboloids

James Tautges

November 8, 2021

Abstract

Consider the adjoint restriction inequality associated with the hypersurface $\{(\tau,\xi) \in \mathbb{R}^{d+1} : \tau = |\xi|^2\} \cup \{(\tau,\xi) \in \mathbb{R}^{d+1} : \tau - \tau_0 = |\xi - \xi_0|^2\}$ for any $(\tau_0, \xi_0) \neq 0$. We prove that extremizers do not exist for this inequality and fully characterize extremizing sequences in terms of extremizers for paraboloid adjoint restriction inequality.

1 Introduction

Fix $d \in \mathbb{N}$ and define

$$\mathcal{E}f(t,x) = \int e^{i(t,x)(|\xi|^2,\xi)} f(\xi) d\xi. \tag{1}$$

It is conjectured that

$$\sup_{f \in L^p} \frac{\|\mathcal{E}f\|_q}{\|f\|_p} = A_p < \infty \tag{2}$$

for $q > p$ and $q = \frac{d+2}{d} p'$. We will call p and q for which (2) holds "valid."

There are many recent results about the class of functions f for which $\|\mathcal{E}f\|_q = A_p \|f\|_p$ and the related question of sequences $\{f_n\}$ such that $\lim \|\mathcal{E}f_n\|_q / \|f_n\|_p = A_p$. We will call f an extremizer for $\mathcal{E}$ and $\{f_n\}$ an extremizing sequence. Extremizers were shown to exist in the case $p = 2$ in all dimensions by Shao ([8]) and this was extended to all valid p, q in the interior of the set of all valid p, q by Stovall ([9]). Those results also prove that extremizing sequences are precompact modulo the symmetries of $\mathcal{E}$, which will be an important tool in this paper.

As for the value of A_p and the extremizers themselves. Foschi proved that Gaussians are the unique extremizers for $\mathcal{E}$ for $p = 2$ and $d \in \{1, 2\}$, conjecturing that this was the case for all dimensions ([3]). Christ and Quilodran ([1]) showed that this conjecture is essentially sharp by proving that Gaussians are only critical points for the functional $f \mapsto \|\mathcal{E}f\|_q / \|f\|_p$ if $p = 2$. There are similar results known for restriction to the sphere (e.g. [2], [4], [6]). See Foschi and e Silva's survey ([5]) for a more comprehensive collection of known results.

This paper will deal with the related operator

$$\mathcal{E}f(t,x) + \mathcal{E}_{(\tau_0,\xi_0)}g(t,x) := \int e^{i(t,x)\cdot(|\xi|^2,\xi)}f(\xi)d\xi + \int e^{i(t,x)\cdot(|\xi-\xi_0|^2+\tau_0,\xi)}g(\xi)d\xi$$

Symmetries and Definitions

We define subgroups $\mathbf{S} \subset \mathrm{Iso}(L^p(\mathbb{R}^d))$ and $\mathbf{T} \subset \mathrm{Iso}(L^q(\mathbb{R}^{d+1}))$, which are related by $\mathcal{E} \circ \mathbf{S} = \mathbf{T} \circ \mathcal{E}$. Let $\mathbf{S}$ and $\mathbf{T}$ be the subgroups generated by the following isometries, which are distinguished by the fact that they generate non-compact subgroups in the larger groups:

	$Sf(\xi)$	$T\mathcal{E}f(t,x)$
Scaling	$\lambda^{d/p}f(\lambda\xi)$	$\lambda^{-(d+2)/q}\mathcal{E}f(\lambda^{-2}t,\lambda^{-1}x)$
Frequency Translation	$f(\xi-\xi')$	$e^{i(t\|\xi'\|^2+x\cdot\xi')}\mathcal{E}f(t,x+2t\xi')$
Spacetime Translation	$e^{i(t_0,x_0)(\|\xi\|^2,\xi)}f(\xi)$	$\mathcal{E}f(t+t_0,x+x_0)$.

We can write any symmetry $S \in \mathbf{S}$ as

$$Sf(\xi) = \lambda^{d/p}e^{i(t_0,x_0)(|\lambda\xi-\xi'|^2,\lambda\xi-\xi')}f(\lambda\xi-\xi'), \tag{3}$$

and the corresponding $T \in \mathbf{T}$ as

$$TF(t,x) = \lambda^{-(d+2)/q}e^{i(\lambda^{-2}t|\xi'|^2+\lambda^{-1}x\cdot\xi')}F(\lambda^{-2}t+t_0,\lambda^{-1}x+x_0+2\lambda^{-2}t\xi'), \tag{4}$$

for some $\lambda \in \mathbb{R}^+$, $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}^d$, and $\xi' \in \mathbb{R}^d$.

Let

$$\mathcal{E}_{(\tau_0,\xi_0)}f(t,x) = \int e^{i(t,x)(|\xi-\xi_0|^2+\tau_0,\xi)}f(\xi)d\xi$$

be the extension operator associated with the surface $P_{(\tau_0,\xi_0)} = \{(\tau,\xi) : \tau = |\xi-\xi_0|^2+\tau_0\}$. Then, although $\mathcal{E}_{(\tau_0,\xi_0)} \circ \mathbf{S} \neq \mathbf{T} \circ \mathcal{E}_{(\tau_0,\xi_0)}$, the generators of $\mathbf{T}$ pass through $\mathcal{E}_{(\cdot,\cdot)}$ as

$$\begin{aligned}
\lambda^{-(d+2)/q}\mathcal{E}_{(\tau_0,\xi_0)}g(\lambda^{-2}t,\lambda^{-1}x) &= \mathcal{E}_{(\lambda^{-2}\tau_0,\lambda^{-1}\xi_0)}(\lambda^{d/p}g(\lambda\xi))(t,x)\\
e^{i(t|\xi'|^2+x\cdot\xi')}\mathcal{E}_{(\tau_0,\xi_0)}g(t,x+2t\xi') &= \mathcal{E}_{(\tau_0+2\xi_0\cdot\xi',\xi_0)}(g(\xi-\xi'))(t,x)\\
\mathcal{E}_{(\tau_0,\xi_0)}g(t+t_0,x+x_0) &= \mathcal{E}_{(\tau_0,\xi_0)}(e^{i(t_0,x_0)(|\xi-\xi_0|^2+\tau_0,\xi)}g(\xi))(t,x).
\end{aligned} \tag{5}$$

Hence, for every $T \in \mathbf{T}$ written in the form of (4) we have

$$\begin{aligned}
&T\mathcal{E}_{(\tau_0,\xi_0)}g(t,x)\\
&= \mathcal{E}_{(\lambda^{-2}(\tau_0+2\xi_0\cdot\xi'),\lambda^{-1}\xi_0)}[\lambda^{d/p}e^{i(t_0,x_0)\cdot(|\lambda\xi-\xi'-\xi_0|^2+\tau_0,\lambda\xi-\xi')}g(\lambda\xi-\xi')](t,x).
\end{aligned} \tag{6}$$

Main Results

From now on we will assume p,q are exponents such that $A_p < \infty$.

Theorem 1.1. Assume $|\tau_0| + |\xi_0| \neq 0$.

1.
$$\sup_{f,g\in L^p} \frac{\|\mathcal{E}f + \mathcal{E}_{(\tau_0,\xi_0)}g\|_q}{(\|f\|_p^p + \|g\|_p^p)^{1/p}} = 2^{1/p'}A_p$$

2. For all $f, g \in L^p$,
$$\frac{\|\mathcal{E}f + \mathcal{E}_{(\tau_0,\xi_0)}g\|_q}{(\|f\|_p^p + \|g\|_p^p)^{1/p}} < 2^{1/p'}A_p.$$

   In other words, $\mathcal{E} + \mathcal{E}_{(\tau_0,\xi_0)}$ has no extremizers.

3. If the sequence $(f_n, g_n) \subset L^p \times L^p$ extremizes $\mathcal{E} + \mathcal{E}_{(\tau_0,\xi_0)}$, then there exists a subsequence in n along which
$$f_n(\xi) = S_n f(\xi) + r_n(\xi) \quad \text{and} \quad g_n(\xi) = S_n f(\xi) + w_n(\xi)$$
   such that f extremizes $\mathcal{E}$, $\{S_n\} \subset \mathbf{S}$ written in the form of (3) satisfy

   (a) $\lambda_n \to \infty$,

   (b) $\lambda_n^{-2}\xi_0 \cdot \xi_n' \to 0$, and

   (c) $\lambda_n t_n \xi_0 \to 0$,

   and $\|r_n\|_p + \|w_n\|_p \to 0$.

The proof is straightforward, essentially consisting of an exercise in distribution theory and geometric considerations related to the separation of the two surfaces.

We prove Theorem 1.1 part 1 by considering a sequence of dilates of an extremizer for $\mathcal{E}$. Next, we use uniform convexity to prove that for any extremizing sequence $\{(f_n, g_n)\}$, $\|\mathcal{E}f_n - \mathcal{E}_{(\tau_0,\xi_0)}g_n\|_q \to 0$. Using the separation and transversality of the paraboloids, we also prove that $f_n \rightharpoonup 0$ and $g_n \rightharpoonup 0$ and deduce Theorem 1.1 part 2. Finally, we use the results in [9] to find symmetries $\{S_n\}$ such that $S_n f_n \to f$ in L^p. Theorem 1.1 part 3 follows from a direct computation.

Acknowledgements

This project was suggested and overseen by Betsy Stovall and supported in part by NSF DMS-1653264. The author would like to thank her for many helpful conversations and invaluable guidance in the writing of this paper.

2 Operator Norm

First, we prove boundedness.

Proposition 2.1.
$$\left\|\mathcal{E}f + \mathcal{E}_{(\tau_0,\xi_0)}g\right\|_q \leq 2^{1/p'}A_p\left(\|f\|_p^p + \|g\|_p^p\right)^{1/p} \tag{7}$$

Proof. Since $\|\mathcal{E}_{(\tau_0,\xi_0)}g(t,x)\|_q = \|e^{i(t,x)\cdot(\tau_0,\xi_0)}\mathcal{E}[g(\cdot+\xi_0)](t,x)\|_q \le A_p\|g\|_p$,

$$\begin{aligned}\left\|\mathcal{E}f + \mathcal{E}_{(\tau_0,\xi_0)}g\right\|_q = \left\|\mathcal{E}f + \mathcal{E}_{(\tau_0,\xi_0)}g\right\|_q &\le \|\mathcal{E}f\|_q + \left\|\mathcal{E}_{(\tau_0,\xi_0)}g\right\|_q \\ &\le A_p\left(\|f\|_p + \|g\|_p\right) \le 2^{1/p'}A_p\left(\|f\|_p^p + \|g\|_p^p\right)^{1/p}.\end{aligned} \tag{8}$$

□

Next, we construct an extremizing sequence for $\mathcal{E} + \mathcal{E}_{(\tau_0,\xi_0)} : L^p \times L^p \to L^q$ to show that $2^{1/p'}A_p$ is the sharp constant.

Proposition 2.2. Let $f \in L^p$ be such that $\|\mathcal{E}f\|_q = A_p\|f\|_p$ and let $f_\lambda(\xi) := \lambda^{d/p}f(\lambda\xi)$. Then

$$\lim_{\lambda\to 0}\frac{\|\mathcal{E}f_\lambda + \mathcal{E}_{(\tau_0,\xi_0)}f_\lambda\|_q}{2^{1/p}\|f\|_p} = 2^{1/p'}A_p.$$

Proof. Let $f \in L^p$ be such that $\|f\|_p = 1$ and $\|\mathcal{E}f\|_q = A_p$. By the scaling properties of $\mathcal{E}$ we know that $\|f_\lambda\|_p = \|f\|_p$ and $\|\mathcal{E}f_\lambda\|_q = \|\mathcal{E}f\|_q$ for all $\lambda > 0$. The identity

$$\mathcal{E}_{(\tau_0,\xi_0)}f_\lambda(t,x) = \lambda^{-(d+2)/q}\mathcal{E}_{(\lambda^2\tau_0,\lambda\xi)}f(\lambda^{-2}t,\lambda^{-1}x).$$

applys to the translated paraboloid, so $\left\|\mathcal{E}_{(\tau_0,\xi_0)}f_\lambda + \mathcal{E}f_\lambda\right\|_q = \left\|\mathcal{E}_{(\lambda^2\tau_0,\lambda\xi_0)}f + \mathcal{E}f\right\|_q$. Moreover,

$$\mathcal{E}_{(\lambda^2\tau_0,\lambda\xi_0)}f(t,x) = e^{it\lambda^2(|\xi_0|^2+\tau_0)}\mathcal{E}f(t,x-2\lambda t\xi_0).$$

By approximating $\mathcal{E}f$ in C^∞_{cpct} and applying dominated convergence, it is clear that $\|\mathcal{E}_{(\lambda^2\tau_0,\lambda\xi_0)}f - \mathcal{E}f\|_q \to 0$. Since f is an extremizer, this implies that $\|\mathcal{E}_{(\tau_0,\xi_0)}f + \mathcal{E}f\|_q \to 2A_p$. □

Proof of Theorem 1.1 part 1. This follows from Proposition 2.2 and Proposition 2.1. □

3 Non-Existence of Extremizers

Now we use the operator norm to understand extremizing sequences. We begin by proving that extremizing sequences of functions must converge weakly to zero.

Proposition 3.1. Assume that $|\tau_0| + |\xi_0| > 0$. Then for every bounded extremizing sequence $\{(f_n, g_n)\}$ for $\mathcal{E} + \mathcal{E}_{(\tau_0,\xi_0)}$,

1. $\{f_n\}$ extremizes $\mathcal{E}$ and $\{g_n\}$ extremizes $\mathcal{E}_{(\tau_0,\xi_0)}$;
2. $\|f_n\|_p - \|g_n\|_p \to 0$ as $n \to \infty$;
3. $\|\mathcal{E}f_n - \mathcal{E}_{(\tau_0,\xi_0)}g_n\|_q \to 0$ as $n \to \infty$; and
4. $f_n, g_n \rightharpoonup 0$ weakly.

Proof. Since the constant in (7) is sharp, extremizing sequences must approach equality for every line of the computation

$$\begin{aligned}\left\|\mathcal{E}f_n + \mathcal{E}_{(\tau_0,\xi_0)}g_n\right\|_q &\le \left\|\mathcal{E}f_n\right\|_q + \left\|\mathcal{E}_{(\tau_0,\xi_0)}g_n\right\|_q \\ &\le A_p\left(\|f_n\|_p + \|g_n\|_p\right) \le 2^{1/p'}A_p\left(\|f_n\|_p^p + \|g_n\|_p^p\right)^{1/p}.\end{aligned}$$

Let $\{(f_n, g_n)\} \subset \ell^p(L^p \times L^p)$ such that $(\|f_n\|_p^p + \|g_n\|_p^p)^{1/p} = 1$ for all n and

$$\lim_{n\to\infty} \|\mathcal{E}f_n + \mathcal{E}_{(\tau_0,\xi_0)}g_n\|_q = 2^{1/p'}A_p.$$

By the computation in the proof of Proposition 2.1, we must have

$$\lim_{n\to\infty} \frac{\left\|\mathcal{E}f_n + \mathcal{E}_{(\tau_0,\xi_0)}g_n\right\|_q}{\left\|\mathcal{E}f_n\right\|_q + \left\|\mathcal{E}_{(\tau_0,\xi_0)}g_n\right\|_q} = 1; \tag{9}$$

$$\lim_{n\to\infty} \frac{\left\|\mathcal{E}f_n\right\|_q + \left\|\mathcal{E}_{(\tau_0,\xi_0)}g_n\right\|_q}{A_p\left(\|f_n\|_p + \|g_n\|_p\right)} = 1; \tag{10}$$

and

$$\lim_{n\to\infty} \frac{A_p\left(\|f_n\|_p + \|g_n\|_p\right)}{2^{1/p'}A_p\left(\|f_n\|_p^p + \|g_n\|_p^p\right)^{1/p}} = 1. \tag{11}$$

By (11) and the sharp Hölder inequality, $\|f_n\|_p, \|g_n\|_p \to 2^{-1/p}$ proving claim 2. Combining this with (10) implies claim 1 and, in particular, $\|\mathcal{E}f_n\|_q, \|\mathcal{E}_{(\tau_0,\xi_0)}g_n\|_q \to 2^{-1/p}A_p$. In light of (9), we see that $\|\mathcal{E}f_n + \mathcal{E}_{(\tau_0,\xi_0)}g_n\|_q - \|\mathcal{E}f_n\|_q - \|\mathcal{E}_{(\tau_0,\xi_0)}g_n\|_q \to 0$. Since L^q is uniformly convex ([7, Theorem 2.5]), this proves claim 3.

Turning to claim 4, let $\psi = \widehat{\widehat{\phi}} \in \mathcal{S}$ and let $d\sigma(\tau, \xi)$ be the measure on the paraboloid centered at the origin given by the pullback of Lebesgue measure via the projection map $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. By definition, $\mathcal{E}f = \widehat{fd\sigma}$ and since $L^q \subset \mathcal{S}'$, we may compute

$$\langle \mathcal{E}f_n, \phi\rangle = \langle f_n d\sigma, \widehat{\phi}\rangle = \int f_n(\xi)\psi(|\xi|^2, \xi)d\xi.$$

In the same way, let $d\sigma'(\tau, \xi)$ be the measure on the translated paraboloid so that

$$\langle \mathcal{E}_{(\tau_0,\xi_0)}g_n, \phi\rangle = \langle g_n d\sigma', \widehat{\phi}\rangle = \int g_n(\xi)\psi(|\xi - \xi_0|^2 + \tau_0, \xi)d\xi.$$

Let $\eta \in \mathbb{R}^d$. As long as $(|\eta|^2, \eta) \in P$ isn't on the intersection of the two paraboloids, there exists $r > 0$ sufficiently small that $B((|\eta|^2, \eta), r)$ is disjoint from $P_{(\tau_0,\xi_0)}$. Let $\psi : \mathbb{R}^{d+1} \to \mathbb{R}$ be any smooth function supported on $B((|\eta|^2, \eta), r)$. Since $\psi(|\xi - \xi_0|^2 + \tau_0, \xi) = 0$ for all $\xi \in \mathbb{R}^d$, $\langle \mathcal{E}_{(\tau_0,\xi_0)}g_n, \phi\rangle = 0$ for all n so we can apply claim 3 to see that

$$\int f_n(\xi)\psi(|\xi|^2, \xi)d\xi \to 0.$$

Since $\psi(|\cdot|^2, \cdot)$ ranges over all smooth functions supported on $B(\eta, r)$, $f_n \rightharpoonup 0$ weakly on a neighborhood of almost every point. Hence $f_n \rightharpoonup 0$ weakly on all of $\mathbb{R}^d$ and the statement for g_n follows similarly. This proves claim 4. □

Proof of Theorem 1.1 part 2. An extremizing pair (f, g) is a constant extremizing sequence. Therefore non-zero extremizers do not exist. $\square$

4 Characterization of Extremizing Sequences

Let $\{(f_n, g_n)\} \subset \ell^p(L^p \times L^p)$ be an extremizing sequence for $\mathcal{E} + \mathcal{E}_{(\tau_0,\xi_0)}$ such that $\|f_n\|_p^p + \|g_n\|_p^p = 1$ for all n. By Proposition 3.1, $\{f_n\}$ is an extremizing sequence for $\mathcal{E}$ so by [9, Theorem 1.1] there exist $\{S_n\} \subset \mathbf{S}$ such that $S_n f_n \to f$ in L^p along some subsequence where f is an extremizer for $\mathcal{E}$. Let $\{T_n\} \subset \mathbf{T}_+$ be such that $T_n \circ \mathcal{E} = \mathcal{E} \circ S_n$ for all n. Then

$$\begin{aligned} 2^{1/p'} A_p &= \lim_{n\to\infty} \|\mathcal{E} f_n + \mathcal{E}_{(\tau_0,\xi_0)} g_n\|_q \\ &= \lim_{n\to\infty} \|T_n \mathcal{E} f_n + T_n \mathcal{E}_{(\tau_0,\xi_0)} g_n\|_q = \lim_{n\to\infty} \|\mathcal{E} f + T_n \mathcal{E}_{(\tau_0,\xi_0)} g_n\|_q. \end{aligned}$$

Since $\|\mathcal{E} f\|_q = A_p \|f\|_p$ and T_n are L^q symmetries, uniform convexity implies that $T_n \mathcal{E}_{(\tau_0,\xi_0)} g_n \to \mathcal{E} f$ in L^q. From here we can deduce the behavior of $S_n g_n$.

Proposition 4.1. Let $f \in L^p(\mathbb{R}^d)$ such that $\|f\|_p = 1$ and $\{(\tau_n, \xi_n)\} \subset \mathbb{R}^{d+1}$. If $\lim \tau_n = \tau_0$ and $\lim \xi_n = \xi_0$, then

$$\lim \|\mathcal{E}_{(\tau_0,\xi_0)} f - \mathcal{E}_{(\tau_n,\xi_n)} f\|_q = 0.$$

Conversely, if $f \in L^p(\mathbb{R}^d)$ and $\{g_n\} \subset L^p(\mathbb{R}^d)$ are such that $\|f\|_p = \|g_n\|_p = 1$ for all n and

$$\lim \|\mathcal{E}_{(\tau_0,\xi_0)} f - \mathcal{E}_{(\tau_n,\xi_n)} g_n\|_q = 0,$$

then $\lim \tau_n = \tau_0$ and $\lim \xi_n = \xi_0$.

Proof. Assume that $\lim \tau_n = \tau_0$ and $\lim \xi_n = \xi_0$. First, we rearrange the expression:

$$\begin{aligned} &\|\mathcal{E}_{(\tau_0,\xi_0)} f - \mathcal{E}_{(\tau_n,\xi_n)} f\|_q \\ &= \left\| \int \left[e^{i(t,x)\cdot(|\xi-\xi_0|^2+\tau_0,\xi)} - e^{i(t,x)\cdot(|\xi-\xi_n|^2+\tau_n,\xi)} \right] f(\xi) d\xi \right\|_q \\ &= \left\| \int \left[e^{it(-2\xi\cdot\xi_0+|\xi_0|^2+\tau_0)} - e^{it(-2\xi\cdot\xi_n+|\xi_n|^2+\tau_n)} \right] e^{i(t,x)\cdot(|\xi|^2,\xi)} f(\xi) d\xi \right\|_q \\ &= \Bigg\| \int e^{it(-2\xi\cdot\xi_0+|\xi_0|^2+\tau_0)} \\ &\qquad \left[1 - e^{it(-2\xi\cdot(\xi_n-\xi_0)+|\xi_n|^2-|\xi_0|^2+\tau_n-\tau_0)} \right] e^{i(t,x)\cdot(|\xi|^2,\xi)} f(\xi) d\xi \Bigg\|_q \end{aligned}$$

Let $\varepsilon > 0$. Take $g \in C^\infty_{cpct}(\mathbb{R}^d)$ such that $\|f - g\|_p = O(\varepsilon)$. Let $R > 0$ be large enough that $\operatorname{supp} g \subset B(0, R)$ and

$$\left| \|\mathcal{E}_{(\tau_0,\xi_0)} g - \mathcal{E}_{(\tau_n,\xi_n)} g\|_{L^q(\mathbb{R}^{d+1})} - \|\mathcal{E}_{(\tau_0,\xi_0)} g - \mathcal{E}_{(\tau_n,\xi_n)} g\|_{L^q(B(0,R))} \right| = O(\varepsilon).$$

Then by Minkowski's integral inequality,

$$\begin{aligned}
&\|\mathcal{E}_{(\tau_0,\xi_0)}f - \mathcal{E}_{(\tau_n,\xi_n)}f\|_{L^q(\mathbb{R}^{d+1})} \\
&= \|\mathcal{E}_{(\tau_0,\xi_0)}g - \mathcal{E}_{(\tau_n,\xi_n)}g\|_{L^q(\mathbb{R}^{d+1})} + O(\varepsilon) \\
&= \|\mathcal{E}_{(\tau_0,\xi_0)}g - \mathcal{E}_{(\tau_n,\xi_n)}g\|_{L^q(B(0,R))} + O(\varepsilon) \\
&= \Bigg\| \int e^{it(-2\xi\cdot\xi_0+|\xi_0|^2+\tau_0)} \\
&\left[1 - e^{it(-2\xi\cdot(\xi_n-\xi_0)+|\xi_n|^2-|\xi_0|^2+\tau_n-\tau_0)}\right] e^{i(t,x)\cdot(|\xi|^2,\xi)} g(\xi)d\xi \Bigg\|_{L^q(B(0,R))} + O(\varepsilon) \\
&\leq \|g\|_1 \sup_{|\xi|<R} \|1 - e^{it(-2\xi\cdot(\xi_n-\xi_0)+|\xi_n|^2-|\xi_0|^2+\tau_n-\tau_0)}\|_{L^q(B(0,R))} + O(\varepsilon).
\end{aligned}$$

Furthermore, since $\frac{d}{d\theta}(1-e^{i\theta}) = -i$,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{|\xi|<R} \|1 - e^{it(-2\xi\cdot(\xi_n-\xi_0)+|\xi_n|^2-|\xi_0|^2+\tau_n-\tau_0)}\|_{L^q(|(t,x)|<R)} \\
&\lesssim \lim R^{\frac{d+1}{q}} \sup_{|\xi|<R,\ |(t,x)|<R} |1 - e^{it(-2\xi\cdot(\xi_n-\xi_0)+|\xi_n|^2-|\xi_0|^2+\tau_n-\tau_0)}| \\
&\leq \lim R^{\frac{d+1}{q}} \sup |t(-2\xi\cdot(\xi_n-\xi_0)+|\xi_n|^2-|\xi_0|^2+\tau_n-\tau_0)| \\
&\leq \lim R^{\frac{d+1}{q}} R(2R|\xi_n-\xi_0| + ||\xi_n|^2-|\xi_0|^2| + |\tau_n-\tau_0|) \\
&= 0.
\end{aligned}$$

Hence $\limsup_{n\to\infty} \|\mathcal{E}_{(\tau_0,\xi_0)}f - \mathcal{E}_{(\tau_n,\xi_n)}f\|_{L^q(\mathbb{R}^{d+1})} = O(\varepsilon)$ and, taking $\varepsilon \to 0$,

$$\lim_{n\to\infty} \|\mathcal{E}_{(\tau_0,\xi_0)}f - \mathcal{E}_{(\tau_n,\xi_n)}f\|_{L^q(\mathbb{R}^{d+1})} = 0.$$

Conversely, assume $\lim \tau_n \neq \tau_0$ or $\lim \xi_n \neq \xi_0$.

Consider the signed vertical (τ) distance between the two paraboloids, $h(\xi) := |\xi-\xi_0|^2 - |\xi-\xi_n|^2 + \tau_0 - \tau_n$. We rearrange to find $h(\xi) = 2\xi\cdot(\xi_n-\xi_0) + |\xi_0|^2 + \tau_0 - |\xi_n|^2 - \tau_n$. Let $A_n = \{\xi : h(\xi) = 0\}$. We claim that for any fixed d-ball $R > 0$,

$$\limsup_n \inf_{\{\xi:|\xi|<R,\ \mathrm{dist}(\xi,A_n)>s\}} |h(\xi)| =: c_{s,R} > 0. \tag{12}$$

Indeed, $c_{s,R} \geq \limsup_n 2s|\xi_n - \xi_0|$ by differentiating h. If this quantity is zero, then $\lim \xi_n = \xi_0$ and therefore $\lim \tau_n \neq \tau_0$ in this case. Let $\varepsilon > 0$. There exists an $N > 0$ such that

$$\sup_{\{\xi:|\xi|<R\}} \left||\xi-\xi_0|^2 - |\xi-\xi_n|^2\right| < \varepsilon$$

for all $n > N$, so

$$c_{s,R} \geq \limsup_n \inf_{\{\xi:|\xi|<R\}} |h(\xi)| \geq \limsup_n |\tau_n - \tau_0| - \varepsilon.$$

Since $\limsup_n |\tau_n - \tau_0| > 0$ in the case we're considering, we can take ε small enough to show that $c_{s,R} > 0$, proving (12).

We would like to construct a function $\Psi \in C^\infty_{cpct}(\mathbb{R}^{d+1})$ such that after passing to a subsequence in n,

1. $|\langle f d\sigma, \Psi \rangle| > \frac{1}{2}$, and
2. $\lim_n \sup_{\xi \in B(0,R)} |\Psi(\xi, |\xi - \xi_n|^2 + \tau_n)| = 0$.

Let $\eta \in \mathbb{C}^\infty_{cpct}(\mathbb{R})$ be a non-negative bump function with $\operatorname{supp} \eta \subset B(0,1)$ and $\eta|_{B(0,1/2)} \equiv 1$. Also let $\Phi \in C^\infty_{cpct}(\mathbb{R}^d)$ be such that $\|\Phi\|_{p'} = 1$ and $|\langle f, \Phi \rangle| > \frac{3}{4}$, and take $R > 0$ so that $\operatorname{supp} \Phi \subset B(0,R)$. Note that, since the zero set of h is a $(d-1)$-hyperplane,

$$|\{\xi : \operatorname{dist}(\xi, A_n) < s\} \cap B(0,R)| \leq c_{d-1} R^{d-1} s$$

where c_{d-1} is a dimensional constant. Therefore, there exists an $s_0 > 0$ such that

$$\|\Phi - \left[1 - \eta\left(\frac{\operatorname{dist}(\xi, A_n)}{2s_0}\right)\right] \Phi\|_{p'} < \frac{1}{4}$$

and hence

$$|\langle f, \left[1 - \eta\left(\frac{\operatorname{dist}(\xi, A_n)}{2s_0}\right)\right] \Phi \rangle| > \frac{1}{2}$$

for all n. Now let

$$\Psi_n(\tau, \xi) := \eta\left(3\frac{\tau - \tau_0 - |\xi - \xi_0|^2}{c_{s_0,R}}\right) \left[1 - \eta\left(\frac{\operatorname{dist}(\xi, A_n)}{2s_0}\right)\right] \Phi(\xi).$$

By (12), $\Psi_n(|\xi - \xi_n|^2 + \tau_n, \xi) = 0$ for all $\xi \in B(0,R)$. In addition, $|\langle f d\sigma, \Psi_n \rangle| > \frac{1}{2}$ by construction.

Since the space of all $(d-1)$-planes intersecting the d-ball $\overline{B(0, R + 10s_0)}$ is compact, either $A_n \cap \overline{B(0, R + 10s_0)} = \emptyset$ for sufficiently large n, or there exists a hyperplane A such that for a subsequence, $A_n \to A$ in the sense that

$$\lim_n \sup_{\substack{\xi \in A \cap \overline{B(0,R+10s_0)}, \\ \zeta \in A_n \cap \overline{B(0,R+10s_0)}}} |\xi - \zeta| = 0.$$

In the first case, let

$$\Psi(\tau, \xi) := \eta\left(3\frac{\tau - \tau_0 - |\xi - \xi_0|^2}{c_{s_0,R}}\right) \Phi(\xi),$$

and in the second, pass to the subsequence mentioned above and let

$$\Psi(\tau, \xi) := \eta\left(3\frac{\tau - \tau_0 - |\xi - \xi_0|^2}{c_{s_0,R}}\right) \left[1 - \eta\left(\frac{\operatorname{dist}(\xi, A)}{2s_0}\right)\right] \Phi(\xi).$$

In either case, we see that $\Psi_n \to \Psi$ in C^∞_{cpct}. Condition 1 holds since $fd\sigma \in \mathcal{S}'(\mathbb{R}^{d+1})$ and hence $\langle fd\sigma, \Psi_n\rangle \to \langle fd\sigma, \Psi\rangle$. Condition 2 holds by the triangle inequality and the fact that it's satisfied for each Ψ_n individually.

By Plancherel and condition 1, $|\langle \mathcal{E}_{(\tau_0,\xi_0)} f, \widetilde{\widehat{\Psi}}\rangle| > \frac{1}{2}$. On the other hand, by condition 2, $\lim_n |\langle \mathcal{E}_{(\tau_n,\xi_n)} g_n, \widetilde{\widehat{\Psi}}\rangle| = 0$. Thus

$$\lim_{n\to\infty} \|\mathcal{E}_{(\tau_0,\xi_0)} f - \mathcal{E}_{(\tau_n,\xi_n)} g_n\|_q \geq \frac{1}{2\|\widehat{\Psi}\|_{q'}} \neq 0,$$

which is a contradiction and proves the proposition. □

Proof of Theorem 1.1 part 3. By Proposition 3.1,

$$\|\mathcal{E} f_n\|_q, \|\mathcal{E}_{(\tau_0,\xi_0)} g_n\|_q \to A_p \quad \text{and} \quad \|\mathcal{E} f_n + \mathcal{E}_{(\tau_0,\xi_0)} g_n\|_q \to 2A_p,$$

so uniform convexity implies that $\|\mathcal{E} f_n - \mathcal{E}_{(\tau_0,\xi_0)} g_n\|_q \to 0$.

Let $\phi \in C^\infty_{cpct}(\mathbb{R}^d)$. Set

$$L = \sup_{\xi \in \operatorname{supp} \phi} \left| |\xi - \xi_0|^2 + \tau_0 - |\xi|^2 \right|$$

and let $\eta \in C^\infty_{cpct}(\mathbb{R})$ be such that $\eta|_{B(0,2L)} \equiv 1$. Now let $\psi(\tau,\xi) = \phi(\xi)\eta(\tau - |\xi|^2)$.

By construction of η,

$$\begin{aligned}\lim_{n\to\infty} |\langle f_n - g_n, \phi\rangle_{L^2(\mathbb{R}^d)}| &= \lim_n |\langle f_n d\sigma - g_n d\sigma', \psi\rangle_{L^2(\mathbb{R}^{d+1})}| \\ &= \lim_n |\langle \mathcal{E} f_n - \mathcal{E}_{(\tau_0,\xi_0)} g_n, \widehat{\psi}\rangle| \leq \lim_n \|\mathcal{E} f_n - \mathcal{E}_{(\tau_0,\xi_0)} g_n\|_q \|\widehat{\psi}\|_{q'} = 0,\end{aligned}$$

which proves that $f_n - g_n \rightharpoonup 0$. Indeed, since $\|f_n\|_p - \|g_n\|_p \to 0$ by Proposition 3.1, uniform convexity implies that the convergence is strong, $\|f_n - g_n\|_p \to 0$. Since f_n is an extremizing sequence, by [9, Theorem 1.1] there exists a subsequence in n, $\{S_n\} \subset \mathbf{S}$, and an extremizer $f \in L^p$ such that $S_n f_n \to f$ so we have $\|f - S_n g_n\|_p \to 0$ along this subsequence as well.

Recall that the symmetries $T_n \mathcal{E}_{(\tau_0,\xi_0)} g_n$ can be expressed in the form of (6) as

$$\begin{aligned}&\mathcal{E}_{(\lambda_n^{-2}(\tau_0 + 2\xi_0\cdot\xi_n'), \lambda_n^{-1}\xi_0)} \lambda_n^{d/p} e^{i(t_n,x_n)\cdot(|\lambda_n\xi - \xi_n' - \xi_0|^2 + \tau_0, \lambda_n\xi - \xi_n')} g_n(\lambda_n\xi - \xi_n')(t,x) \\ &\quad = \mathcal{E}_{(\lambda_n^{-2}(\tau_0 + 2\xi_0\cdot\xi_n'), \lambda_n^{-1}\xi_0)} (e^{it_n(-2\lambda_n\xi\cdot\xi_0 + 2\xi_0\cdot\xi_n' + |\xi_0|^2 + \tau_0)} S_n g_n)(t,x) \qquad (13)\end{aligned}$$

for $\lambda_n \in \mathbb{R}^+$, $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d$ and $\xi_n' \in \mathbb{R}^d$. Proposition 4.1 immediately implies that $(\lambda_n^{-2}(\tau_0 + 2\xi_0 \cdot \xi_n'), \lambda_n^{-1}\xi_0) \to 0$, which also implies $\lambda_n \to \infty$ and $\lambda_n^{-2}\xi_0 \cdot \xi_n' \to 0$ since we assumed $|\tau_0| + |\xi_0| \neq 0$. Since $S_n g_n \to f$ in L^p, by the triangle inequality and the forward part of Proposition 4.1,

$$\left\| \mathcal{E} f - \mathcal{E}(e^{it_n(-2\lambda_n\xi\cdot\xi_0 + 2\xi_0\cdot\xi_n' + |\xi_0|^2 + \tau_0)} f)(t,x) \right\|_q \to 0$$

as well. Since $\mathbb{R}/(2\pi\mathbb{Z})$ is compact, we may pass to a subsequence along which $\theta := \lim t_n(2\xi_0 \cdot \xi_n' + |\xi_0|^2 + \tau_0)/(2\pi\mathbb{Z})$ exists. By rearranging, applying the boundedness of $\mathcal{E}$, and invoking dominated convergence,

$$\left\| \mathcal{E} e^{i\theta} f(t, x - 2\lambda_n t_n \xi_0) - \mathcal{E}(e^{it_n(-2\lambda_n \xi\cdot\xi_0 + 2\xi_0\cdot\xi_n' + |\xi_0|^2 + \tau_0)} f) \right\|_q \to 0.$$

The two preceding limits imply that

$$\left\| \mathcal{E} e^{i\theta} f(t, x - 2\lambda_n t_n \xi_0) - \mathcal{E} f \right\|_q \to 0.$$

Hence the sequence $\{\lambda_n t_n\}$ must be bounded and there exists a subsequence along which $c := \lim \lambda_n t_n$ exists. Since $\lambda_n \to \infty$, $\lim t_n = 0$. As translation is continuous in L^q,

$$\left\| \mathcal{E} f - \mathcal{E} e^{i\theta} f(t, x - 2c\xi_0) \right\|_q = 0.$$

From this is is clear that $c = 0$ as $|\mathcal{E}f(t,x)| = |\mathcal{E}f(t, x - 2c\xi_0)|$ would imply that $\mathcal{E}f \equiv 0$ since $\mathcal{E}f \in L^q$. We now also see that $e^{i\theta} = 1$ by strict convexity.

This proves the claim. □

References

[1] M. Christ and R. Quilodrán, *Gaussians rarely extremize adjoint Fourier restriction inequalities for paraboloids*, Proc. Amer. Math. Soc. **142** (2013), no. 3, 887–896.

[2] M. Christ and S. Shao, *Existence of extremals for a Fourier restriction inequality*, Anal. PDE. **2** (2012), 261–312.

[3] D. Foschi, *Maximizers for the Strichartz inequality*, J. Eur. Math. Soc. **9** (2007), no. 4, 739–774.

[4] ______, *Global maximizers for the sphere adjoint Fourier restriction inequality*, J. Funct. Anal. **268** (2015), no. 3, 690-702.

[5] D. Foschi and D. Oliveira e Silva, *Some recent progress on sharp Fourier restriction theory*, Analysis Mathematica **43** (2017), no. 2, 241–265.

[6] R. Frank, E. H. Lieb, and J. Sabin, *Maximizers for the Stein-Tomas inequality*, Geom. Funct. Anal. **26** (2016), no. 4, 1095–1134.

[7] E. H. Lieb and M. Loss, *Analysis*, American Mathematical Society, Providence, RI, 2001.

[8] S. Shao, *Maximizers for the Strichartz inequalities and the Sobolev-Strichartz inequalities for the Schrödinger equation*, Electron. J. Differential Equations **3** (2009), 13.

[9] B. Stovall, *Extremizability of Fourier restriction to the paraboloid*, Preprint, available at `arXiv:1804.03605`.