EXACT LOWER-TAIL LARGE DEVIATIONS OF THE KPZ EQUATION

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Abstract. Consider the Hopf–Cole solution $h(t,x)$ of the KPZ equation with the narrow wedge initial condition. Regarding $t \to \infty$ as a scaling parameter, we provide the first rigorous proof of the Large Deviation Principle (LDP) for the lower tail of $h(2t,0) + \frac{t}{12}$, with speed t^2 and an explicit rate function $\Phi_-(z)$. This result confirms existing physics predictions [SMP17, CGK+18, KLDP18]. Our analysis utilizes the formula from [BG16] to convert the LDP for the KPZ equation to calculating an exponential moment of the Airy point process. To estimate this exponential moment, we invoke the stochastic Airy operator, and use the Riccati transform, comparison techniques, and certain variational characterizations of the relevant functional.

1. Introduction

In this article, we study the lower-tail probability of the Kardar–Parisi–Zhang (KPZ) equation:

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}(\partial_x h)^2 + \xi, \qquad (t,x) \in [0,\infty) \times \mathbb{R},$$

where $\xi = \xi(t,x)$ is the spacetime white noise. Introduced in [KPZ86], the KPZ equation is a paradigm for random surface growth, which has links to a host of different physical phenomena. Via the Hopf–Cole transform and the Feynman–Kac formula, this equation connects to directed polymers in random environments [HHF85]. The spatial derivative $\partial_x h$ satisfies the stochastic Burgers equation, which is a model for randomly stirred fluids [FNS77], interacting particle systems, and driven lattice gases [vBKS85]. In additional to being a phenomenological model, the KPZ equation has been fertile ground for mathematical study. Being a nonlinear equation and an irreversible Markov process, the KPZ equation has been a prototype for the study of Stochastic Partial Differential Equations (SPDEs) and weakly irreversible interacting particle systems. Along with a vast host of (discrete and continuous) models, the KPZ equation enjoys exact solvability originating from combinatorics, representation theory, and Bethe ansatz. We refer to [FS11, Qua11, Cor12, QS15, CW17] and the references therein.

We say that h is a **Hopf–Cole solution** of the KPZ equation if $h(t,x) = \log Z(t,x)$, and the process $Z(t,x)$ solves the Stochastic Heat Equation (SHE)

$$\partial_t Z = \tfrac{1}{2}\partial_{xx} Z + \xi Z, \qquad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{1.1}$$

Throughout this article we will consider the **narrow wedge** initial condition

$$Z^{\mathrm{ic}}(x) = \delta(x). \tag{1.2}$$

Such a notion of solution is motivated by informally exponentiating the KPZ equation, and has been observed in various regularization schemes and particle systems, e.g., [BC95, BG97]. Also, for certain classes of continuous initial conditions, the Hopf–Cole solution agrees with the ones constructed from regularity structures [Hai14], paracontrolled distributions [GIP15], and energy solutions [GJ14, GP18]. A slight generalization of the standard theory [Wal86, BC95] asserts that there exists a unique $C((0,\infty),\mathbb{R})$-valued process Z that solves (1.1)–(1.2) in the mild sense, i.e.,

$$Z(t,x) = p(t,x) + \int_0^t \int_{\mathbb{R}} p(t-s,x-y) Z(s,y) \xi(s,y)\, \mathrm{d}s\mathrm{d}y,$$

where $p(t,x) := (2\pi t)^{-\frac{1}{2}} \exp(-\frac{x^2}{2t})$ denotes the standard heat kernel. Further, [Mue91, MF14] showed that for almost surely for all $t > 0$, the solution is strictly positive, i.e., $Z(t,x) > 0$, for all $x \in \mathbb{R}$ and $t > 0$. This defines the Hopf–Cole solution $h(t,x) := \log Z(t,x)$ with the initial condition (1.2).

2010 *Mathematics Subject Classification.* Primary 60F10, Secondary 60H25.

Key words and phrases. Kardar–Parisi–Zhang equation, large deviations, Airy point process, random operators, stochastic Airy Operator.

Under the initial condition (1.2), for large t, the height develops an average (downward) growth with velocity $-\frac{1}{24}$, and, after centering, fluctuates at $O(t^{\frac{1}{3}})$ and scales to the GUE Tracy–Widom distribution [ACQ11, CLDR10, Dot10, SS10]

$$t^{-\frac{1}{3}}\big(h(2t,0)+\tfrac{t}{12}\big) \Longrightarrow \text{GUE Tracy–Widom}, \quad \text{as } t\to\infty.$$

The results in [ACQ11, SS10] are based on [TW08, TW09]. Here, instead of typical behaviors of h, we focus on Large Deviations (LDs), namely the rare events that $h(2t,0)$ deviates distance $O(t)$ from its center $-\frac{t}{12}$. Regarding $t\to\infty$ as a scaling parameter, we aim at extracting the leading order of the tail probability:

$$\mathbf{P}\big[h(2t,0)+\tfrac{t}{12} > zt\big] \approx \exp\big(-t^{a_+}\Phi_+(z)\big), \quad z>0, \qquad \text{(Upper Tail)}$$

$$\mathbf{P}\big[h(2t,0)+\tfrac{t}{12} < zt\big] \approx \exp\big(-t^{a_-}\Phi_-(z)\big), \quad z<0, \qquad \text{(Lower Tail)}$$

as $t\to\infty$. More precisely, (Upper Tail) means that $\lim_{t\to\infty}\frac{1}{t^{a_+}}\log\mathbf{P}\big[h(2t,0)+\frac{t}{12}>zt\big] = -\Phi_+(z)$ for fixed $z>0$, and similarly for (Lower Tail). We refer to $t^{a_\pm}$ as the speed of deviations, and $\Phi_\pm(z)$ as the rate function.

Put in a broader context of random growth, directed polymers, and particle systems, the upper- and lower-tail LDs considered here probe excess growth and massive die out, respectively. Whereas excess growth originates from *locally* favorable environment, massive die out occurs only when a widespread area of environment *jointly* becomes unfavorable. This distinction results in asymmetric speed: $t^{a_+}=t^1$ and $t^{a_-}=t^2$. These observations and predictions of the speeds were made in [LDMS16]. The upper tail is accessible from Fredholm determinants [CQ13, Proposition 10], and it is predicted [LDMS16, SMP17] that $\Phi_+(z)=\frac{4}{3}z^{\frac{3}{2}}$, a single $\frac{3}{2}$-power. (The prediction of [LDMS16] was based on a short-time analysis, while [SMP17] analyzed the long-time regime considered here.) This predicted upper tail rate function has been recently proved in [DT21].

On the other hand, the lower-tail rate function is predicted by [KK07, MKV16] to exhibit a crossover from cubic power law $(-z)^3$ for small $|z|$ to $\frac{5}{2}$-power law $(-z)^{\frac{5}{2}}$ for large $|z|$. While the $\frac{3}{2}$-power law is seen also in zero temperature polymer models, the crossover behavior for the lower tail distinguishes the KPZ equation, as a positive temperature polymer model, from zero temperature polymers.

Given the known Fredholm determinant formula ([ACQ11, CLDR10, Dot10, SS10], see [BG16, Eq. (7)]), extracting the upper tail boils down to a perturbative analysis. This is so because, the relevant operator converges to zero (in the the trace-class norm) as $t\to\infty$. By contrast, for the lower tail, one faces the situation where an operator does not converge to zero yet the determinant does. This is a well-known issue in random matrix theory, and has since prompted the development for much more involved machineries. For example, extracting the lower tail of the GUE Tracy–Widom distribution is done by the method of commuting operators [TW94], via Riemann–Hilbert problems [BBD08, DIK08], via the Stochastic Airy Operator [RRV11], or non-rigorously via Coulomb gas [DM06].

The first result regarding the lower tail of the KPZ equation is the aforementioned almost-sure positivity of Z [Mue91]. Motivated in part by showing the existence of probability density of $Z(t,x)$, various works [MN08, MF14, HL18] have investigated the negative moments and the positivity of Z. These results mostly concern finite time behaviors of Z, and, in view of the $-\frac{t}{12}$ average growth, are not well-adapted to the $t\to\infty$ regime.

Recently, there has been much development around accessing the lower tail in the $t\to\infty$ regime. In [CG+20], *rigorous* upper and lower bounds on the lower-tail probability are obtained. The bounds hold for all sufficiently large t, and capture the aforementioned crossover behavior. The upper and lower bounds do not match as $t\to\infty$, and hence do not yield the rate function Φ_-. In the physics literature, much attention has been devoted to obtaining the the rate function. In [SMP17], an explicit rate function Φ_- (see (1.3)) was predicted. This is done by analyzing a generalization of Painlevé II, introduced in [ACQ11], through a WKB approximation, along with a self-consistency ansatz. Later, based on a formula from [BG16], [CGK+18] employed a Coulomb gas heuristic to derive the rate function Φ_-, which agrees with the result in [SMP17]. More recently, based on certain conjectural forms of expansions, [KLDP18] developed a scheme of calculating cumulants under the Airy point process, and, through resummation, produced the same rate function Φ_- previously predicted.

The aforementioned physics results provide much insight in the lower-tail Large Deviation Principle (LDP). They, however, assume certain conjectural formulas or approximations, or are based on certain infinite dimensional settings that go beyond existing theories. In this work, we give the *first rigorous proof* of the lower-tail LDP for the KPZ equation, by invoking the stochastic Airy operator, and using the Riccati transform, comparison techniques, and certain variational characterizations.

Theorem 1.1. *Let $h(t,x)$ denote the Hopf–Cole solution of the KPZ equation with narrow wedge initial condition $Z(0,x)=\delta(x)$, and fix $\zeta\in(0,\infty)$. Then*

$$\lim_{t\to\infty}\frac{1}{t^2}\log\big(\mathbf{P}\big[h(2t,0)+\tfrac{t}{12}<-\zeta t\big]\big)=-\Phi_-(-\zeta),$$

with the rate function

$$\Phi_-(z):=\tfrac{4}{15\pi^6}(1-\pi^2 z)^{\frac{5}{2}}-\tfrac{4}{15\pi^6}+\tfrac{2}{3\pi^4}z-\tfrac{1}{2\pi^2}z^2,\quad z\le 0. \tag{1.3}$$

Remark 1.2. After the first version of this article was posted, there are more works on this lower-tail LDP problem. The physics work [KLD19] shows that the four different (rigorous and non-rigorous) methods used in [SMP17, CGK⁺18, KLDP18] and this article are closely related at the level of variational problem. Two new methods of deriving Φ_- have been recently obtained, in the rigorous work [CC19] (which also produces quantitative bounds) and the physics work [LD20].

The starting point of our analysis is a formula of [BG16] that expresses the previously known Fredholm determinant formula [ACQ11, CLDR10, Dot10, SS10] in terms of the Airy Point Process (PP). Even though only the $\beta=2$ Airy PP will enter the formula, to demonstrate the generality of our approach, we will consider general $\beta>0$. Let $B(x)$, $x\ge 0$, denote a standard Brownian motion. Recall from [RRV11] that the Stochastic Airy Operator (SAO)

$$\mathcal{A}_\beta := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x+\frac{2}{\sqrt{\beta}}B'(x) \tag{1.4}$$

with Dirichlet boundary boundary condition at $x=0$ defines an unbounded, self-adjoint operator on $L^2(0,\infty)$ (see Section 2 for more details on the construction of $\mathcal{A}_\beta$). Further, $\mathcal{A}_\beta$ has a pure-point spectrum that is bounded below and has no limit points:

$$-\infty<\boldsymbol{\lambda}_1(\mathcal{A}_\beta)\le\boldsymbol{\lambda}_2(\mathcal{A}_\beta)\le\boldsymbol{\lambda}_3(\mathcal{A}_\beta)\ldots\to\infty.$$

The β Airy PP $\{\mathfrak{a}_{k,\beta}\}_{k=1}^\infty$ is simply this spectrum of $\mathcal{A}_\beta$ up to a space reversal, i.e., $\mathfrak{a}_{k,\beta}:=-\boldsymbol{\lambda}_k(\mathcal{A}_\beta)$. In [BG16, Theorem 2.1], substituting $(\frac{T}{2},\mathfrak{a}_k,u)\mapsto(t,-\boldsymbol{\lambda}_k(\mathcal{A}_2),e^{t\zeta})$, we have

$$\mathbf{E}\Big[\exp\Big(-e^{h(2t,0)+\frac{t}{12}+t\zeta}\Big)\Big]=\mathbf{E}\Big[\exp\Big(-\sum_{k=1}^\infty\phi_t\big(\boldsymbol{\lambda}_k(\mathcal{A}_2)-t^{\frac{2}{3}}\zeta\big)\Big)\Big], \tag{1.5}$$

where

$$\phi_t(\lambda):=\log\big(1+\exp(-t^{\frac{1}{3}}\lambda)\big). \tag{1.6}$$

The formula (1.5) links two *distinct* objects: the KPZ equation on the left, and the Airy PP process on the right. The identity (1.5) of [BG16] shows that specific observables of them match algebraically.

It is readily checked that the double exponential function e^{-e^x} well approximates the indicator function $\mathbf{1}_{\{x<0\}}$ except in a neighborhood of $x=0$. As $t\to\infty$, it is conceivable that the l.h.s. of (1.5) becomes a good proxy for the tail probability $\mathbf{P}[h(2t,0)+\frac{t}{12}<-\zeta t]$, and that proving Theorem 1.1 amounts to proving

Theorem 1.3. *For fixed $\zeta,\beta,L\in(0,\infty)$, we have*

$$\lim_{t\to\infty}\frac{1}{t^2}\log\Big(\mathbf{E}\Big[\exp\Big(-L\sum_{k=1}^\infty\phi_t\big(\boldsymbol{\lambda}_k(\mathcal{A}_\beta)-t^{\frac{2}{3}}\zeta\big)\Big)\Big]\Big)=-L\Big(\frac{2L}{\beta}\Big)^5\Phi_-\Big(-\Big(\frac{\beta}{2L}\Big)^2\zeta\Big). \tag{1.7}$$

Theorem 1.3 can be regarded as a result on a type of LDs of the β Airy PP. For related processes (the Sine_β and Sch_τ processes), the overcrowding LDPs were obtained in [HV15]. See also [HV17].

The relevant parameters corresponding to the r.h.s. of (1.5) are $\beta=2$ and $L=1$. Here, we state and prove Theorem 1.3 for general $\beta,L\in(0,\infty)$ to demonstrate the generality of our method. Further, it has an application in a different setup. Referring to [BBCW18, Definition 7.1], we let $h^{\mathrm{hf}}(t,x):=\log Z^{\mathrm{hf}}(t,x)$ denote the Hopf–Cole solution of the KPZ equation on the half-line $[0,\infty)$ with boundary parameter $A=-\frac{1}{2}$, with initial condition $Z^{\mathrm{hf}}(0,x)=\delta(x)$. The result [BBCW18, Theorem B] together with the convergence result of the half-space ASEP [Par19] (which generalizes the result in [CS18]) yields the identity

$$\mathbf{E}\Big[\exp\Big(-\frac{1}{4}e^{h^{\mathrm{hf}}(2t,0)+\frac{t}{12}+t\zeta}\Big)\Big]=\mathbf{E}\Big[\exp\Big(-\frac{1}{2}\sum_{k=1}^{\infty}\phi_t(\boldsymbol{\lambda}_k(\mathcal{A}_1)-t^{\frac{2}{3}}\zeta)\Big)\Big]. \tag{1.8}$$

Indeed, the r.h.s. of (1.8) corresponds to $\beta=1$ and $L=\frac{1}{2}$. As a corollary of Theorem 1.3, we have

Corollary 1.4. *Refer to* [BBCW18, Definition 7.1]. *Let $h^{hf}(t,x):=\log Z^{hf}(t,x)$ denote the Hopf–Cole solution of the KPZ equation on the half-line $[0,\infty)$ with the boundary parameter $A=-\frac{1}{2}$, with the initial condition $Z^{hf}(0,x)=\delta(x)$. Then, for any fixed $\zeta>0$,*

$$\lim_{t\to\infty}\tfrac{1}{t^2}\log\big(\mathbf{P}\big[h^{hf}(2t,0)+\tfrac{t}{12}<-t\zeta\big]\big)=-\tfrac{1}{2}\Phi_-(-\zeta).$$

Passing from Theorem 1.3 to Theorem 1.1 and Corollary 1.4 is simple, which we do in Section 4. In addition to Theorem 1.1 and Corollary 1.4, there may be further connection to the processes considered in [GS18], but we do not pursue this direction here.

The preceding discussion reduces the LDP for the KPZ equation to calculating an exponential moment of the Airy PP. This observation was first used in [CG+20], along with certain bounds on the Airy PP, to derive bounds on the lower-tail probability. Further, it was noted [CG+20, Section 2.3] that the rate function Φ_- can be derived by developing an LDP for the Airy PP from the known LDP for the Gaussian β ensemble [BAG97]. This scheme was adopted in [CGK+18]. Non-rigorously taking an edge scaling of the rate function of the Gaussian β ensemble [BAG97, Theorem 1.3], the work [CGK+18] derived an explicit rate function I_{Airy} [CGK+18, Section A, Supplementary Material] for the Airy PP, and solved a corresponding variational problem to obtain Φ_-.

The non-rigorous edge scaling from the rate function of the β Gaussian ensemble to I_{Airy} is backed by the known weak convergence [RRV11] of the Gaussian β ensemble to the Airy PP. However, justifying this passage at the LDP level requires convergence up to *exponentially small* probability, which remains an open problem. Partial results in this direction have been recently obtained in [Zho19]. Here, we take a different approach, completely bypassing the need for taking edge scaling from the Gaussian β ensemble.

1.1. **A heuristic of the proof.** We give a heuristic of the ideas behind our proof. The discussion in this subsection is informal, serves only as a conceptual guideline, and will not be used in the rest of the article.

Let $\mathcal{G}_t:=L\sum_{k=1}^{\infty}\phi_t\big(\boldsymbol{\lambda}_k(\mathcal{A}_\beta)-t^{\frac{2}{3}}\zeta\big)$ denote the relevant quantity on the r.h.s. of (1.7). By Varadhan's lemma, analyzing the $t\to\infty$ behavior of $\mathbf{E}[\exp(-\mathcal{G}_t)]$ amounts to characterizing the LDs of $\mathcal{G}_t$. With B being the only random component in $\mathcal{A}_\beta$ (see (1.4)), the quantity $\mathcal{G}_t$ is a functional of B. Therefore, the questions about the LDs of $\mathcal{G}_t$ is ultimately a question on the LDs of a functional of the Brownian motion B. To better express $\mathcal{G}_t$ as a functional of B, we use the Riccati transform. Let

$$N(\lambda):=\#\{k\in\mathbb{N}:\boldsymbol{\lambda}_k(\mathcal{A}_\beta)\leq\lambda\}$$

denote the number of eigenvalues of $\mathcal{A}_\beta$ at most λ, i.e., the counting function, and consider the solution of the following ODE

$$f'(x)=x-\lambda-f^2(x)+\tfrac{2}{\sqrt{\beta}}B'(x),\ x>0,\quad f(0)=+\infty. \tag{1.9}$$

Due to the negative, quadratic drift $-f^2$, the solution may undergo a few explosions to $-\infty$, whence f is immediately restarted at $+\infty$. The Riccati transform asserts (see Section 2 for more details) that $N(\lambda)=\#\{\text{explosions of } f(x)\}$. We hence view f and $N(\lambda)$ as functionals of B through (1.9), and this gives $\mathcal{G}_t$ as a functional of B through

$$\mathcal{G}_t=L\int_{\mathbb{R}}\phi_t(\lambda-t^{\frac{3}{2}}\zeta)\mathrm{d}N(\lambda)=-L\int_{\mathbb{R}}\phi_t'(\lambda-t^{\frac{3}{2}}\zeta)N(\lambda)\mathrm{d}\lambda. \tag{1.10}$$

We now need to analyze how deviations of B affect f and $N(\lambda)$. To this end, it is instructive to first introduce a few scales. Straightforward differentiations from (1.6) shows that $\phi_t'(\lambda-t^{\frac{2}{3}}\zeta)\approx-t^{\frac{1}{3}}\mathbf{1}_{\{\lambda<t^{2/3}\zeta\}}$ for $t\gg1$. Using this in (1.10), we see that the relevant λ should be of order $t^{\frac{2}{3}}$, i.e., $\lambda=O(t^{\frac{2}{3}})$. In (1.9), if we ignore the Brownian term $\frac{2}{\sqrt{\beta}}B'(x)$, explosions of f occur only when $x\leq\lambda$. This suggests that $x=O(\lambda)=O(t^{\frac{2}{3}})$. Now, consider a generic $v\in C[0,\infty)$. We postulate that the relevant deviation is $B(x)$ behaving like a drifted Brownian motion with drift $t^{\frac{2}{3}}v(t^{-\frac{2}{3}}x)$. Here, the $(t^{-\frac{2}{3}}x)$ scaling ensures that the

drift varies at scale comparable to $x = O(t^{\frac{2}{3}})$ in (1.9), and the multiplicative factor $t^{\frac{2}{3}}$ guarantees that the drift competes at the same level as $x - \lambda = O(t^{\frac{2}{3}})$.

We henceforward regard v as the control function of the LDs in question. The LDP on sample paths of Brownian motion suggests that

$$\mathbf{P}\big[B'(x) \approx t^{\frac{2}{3}} v(t^{-\frac{2}{3}}x)\big] \approx \exp\Big(-\int_0^\infty \frac{1}{2} t^{\frac{4}{3}} v^2(t^{-\frac{2}{3}}x)\mathrm{d}x \Big) = \exp\Big(-t^2 \int_0^\infty \frac{1}{2} v^2(x)\mathrm{d}x \Big).$$

Indeed, B is not differentiable, and $B'(x) \approx t^{\frac{2}{3}} v(t^{-\frac{2}{3}}x)$ merely means that $t^{-\frac{4}{3}} B(t^{\frac{2}{3}}x)$ approximates $\int_0^x v(y)\mathrm{d}y$ uniformly in x over compact subsets. Here, however, we *informally* equate $B'(x)$ with $t^{\frac{2}{3}} v(t^{-\frac{2}{3}}x)$ in (1.9) and write

$$f_v'(x) = -t^{\frac{2}{3}}\big(-t^{-\frac{2}{3}}x + t^{-\frac{2}{3}}\lambda - \tfrac{2}{\sqrt{\beta}} v(t^{-\frac{2}{3}}x)\big) - f_v^2(x), \ x > 0, \quad f_v(0) = +\infty.$$

This equation can be solved approximately by regarding $b(x) := -t^{-\frac{2}{3}}x + t^{-\frac{2}{3}}\lambda - 2v(t^{-\frac{2}{3}}x)/\sqrt{\beta}$ as a locally constant function. Consider a generic $b > 0$ and solve for a function f_{loc} that satisfies $f_{\mathrm{loc}}' = -t^{\frac{2}{3}} b - f_{\mathrm{loc}}^2$. This gives $f_{\mathrm{loc}}(x) = \tan(t^{\frac{1}{3}} b^{\frac{1}{2}} x + c)$, $c \in \mathbb{R}$, which explodes over a period of $\pi t^{-\frac{1}{3}} b^{-\frac{1}{2}}$. Hence the time lapse between explosions of f_v near a given point x is roughly

$$\tau_v(x) \approx \pi t^{-\frac{1}{3}} \big(\big(-t^{-\frac{2}{3}}x + t^{-\frac{2}{3}}\lambda - \tfrac{2}{\sqrt{\beta}} v(t^{-\frac{2}{3}}x) \big)_+ \big)^{-\frac{1}{2}},$$

where $y_\pm := (\pm y) \vee 0$ and $1/0 := \infty$. Integrating the reciprocal time lapse $1/\tau_v(x)$ over $x \geq 0$ gives the total number of explosions:

$$N(\lambda) = N_v(\lambda) \approx \frac{t^{\frac{1}{3}}}{\pi} \int_0^\infty \big(\big(-t^{-\frac{2}{3}}x + t^{-\frac{2}{3}}\lambda - \tfrac{2}{\sqrt{\beta}} v(t^{-\frac{2}{3}}x) \big)_+ \big)^{\frac{1}{2}} \mathrm{d}x.$$

Now, substituting this approximate expression of $N(\lambda)$ in (1.10), together with the aforementioned approximation $\phi_t'(\lambda - t^{\frac{2}{3}}\zeta) \approx -t^{\frac{1}{3}} \mathbf{1}_{\{\lambda < t^{2/3}\zeta\}}$, we arrive at

$$\begin{aligned} \mathcal{G}_t = \mathcal{G}_{t,v} &\approx \frac{t^{\frac{2}{3}} L}{\pi} \int_{-\infty}^{t^{\frac{2}{3}}\zeta} \int_0^\infty \big((-t^{-\frac{2}{3}}x + t^{-\frac{2}{3}}\lambda - \tfrac{2}{\sqrt{\beta}} v(t^{-\frac{2}{3}}x))_+ \big)^{\frac{1}{2}} \mathrm{d}\lambda \mathrm{d}x \\ &= t^2 \frac{2L}{3\pi} \int_0^\infty \big((-x + \zeta - \tfrac{2}{\sqrt{\beta}} v(x))_+ \big)^{\frac{3}{2}} \mathrm{d}x. \end{aligned}$$

So far we have derived an approximate expression of $\mathcal{G}_t = \mathcal{G}_{t,v}$ as a functional of the control v, and the 'cost' for realizing a given v is $\frac{t^2}{2} \int_0^\infty v^2(x)\mathrm{d}x$. These discussions suggest that

$$\log\big(\mathbf{E}[\mathcal{G}_t]\big) \approx -t^2 \min_v \Big\{ \int_0^\infty \Big(\frac{2L}{3\pi} \big((-x + \zeta - \tfrac{2}{\sqrt{\beta}} v(x))_+ \big)^{\frac{3}{2}} + \frac{1}{2} v^2(x) \Big) \mathrm{d}x \Big\}.$$

The minimizer $v = v_*$ is solved by straightforward variation, giving

$$v_*(x) = 4L^2 \pi^{-2} \beta^{-\frac{3}{2}} \big(-1 + \sqrt{1 + (\tfrac{\beta\pi}{2L})^2 (\zeta - x)_+} \big). \tag{1.11}$$

Substitute in $v = v_*$. After straightforward but tedious calculations, we get
$\log(\mathbf{E}[\mathcal{G}_t]) \approx -t^2 L (\frac{2L}{\beta})^5 \Phi_-(-(\frac{\beta}{2L})^2 \zeta)$.

1.2. **Overview of the proof.** The crucial assumption behind the preceding heuristic is having locally constant drifts. That is, we postulate that the 'optimal strategy' is achieved by having a drift $t^{\frac{2}{3}} v(t^{-\frac{2}{3}}x)$ that is locally constant, and varies at the macroscopic scale $O(t^{\frac{2}{3}})$. It is *far from clear* why this is the case. Indeed, with B' being rough (not function-valued), local behaviors of B at scales $\lesssim t^{-\frac{1}{3}}$ could have dramatic effects on the spectrum of $\mathcal{A}_\beta$.

Our proof proceeds through a localization procedure. That is, we partition $(0, \infty)$ into intervals of length t^α: $I_i := (\eta_{i-1}, \eta_i]$, $\eta_i := i t^\alpha$, and counts the number of explosions of the Riccati ODE within each interval I_i. Our analysis works for *any* fixed exponent $\alpha \in (-\frac{1}{3}, \frac{2}{3})$. Note that this range exhausts all mesoscopic scales. As seen in Section 1.1, $t^{\frac{2}{3}}$ is the macroscopic scale of x and λ in (1.9), while $t^{-\frac{1}{3}}$ the microscopic scale of typical time lapse $\tau_v(x)$ between explosions.

To prove Theorem 1.3, we separately establish upper and lower bounds on the l.h.s. of (1.7). For the lower bound, within each interval I_i, we perform a change-of-measure (via Girsanov's theorem) so that the Brownian motion has drift $V_i := t^{\frac{2}{3}} v_*(t^{\frac{2}{3}}\eta_{i-1})$. Within I_i, the change in the linear potential x is negligible, and can be well-approximated by the constant η_{i-1}. This being the case, the number of explosions (after the change-of-measure) can be estimated by spectral comparison to the shifted Laplace operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \eta_{i-1} + \frac{2}{\sqrt{\beta}} V_i$. Doing so eventually yields the desired lower bound.

The harder part of the proof is to obtain a matching upper bound. This is where we address the aforementioned issue — that the 'best strategy' is achieved by a locally constant drift. More precisely, we show that the 'best strategy' is to have B constantly drifted within each interval I_i. To this end, we first use $\phi_t(\lambda) \approx -t^{\frac{1}{3}}\lambda_-$ to approximate the relevant quantity as a truncated sum of eigenvalues of certain Hill-type operators (see (3.33)). Next, we show in Proposition 3.4 (after passing to periodic boundary condition as done in Lemma 3.2) that the truncated sum is dominated by the one with $B'(x)$ replaced by its average $\frac{B(\eta_i)-B(\eta_{i-1})}{|I_i|}$. Key ingredients behind the proof of Proposition 3.4 are the variational characterizations built in Lemma 2.3 and (3.41).

We note here that most part of our proof works even if $\phi_t(\lambda)$ were replaced by a smooth compactly supported function. However, the aforementioned variational characterizations (Lemma 2.3 and (3.41)) are tailored to a truncated sum of eigenvalues, and hence apply only for the specific cost function $\phi_t(\lambda) \approx -t^{\frac{1}{3}}\lambda_-$.

1.3. **Quantitative bounds.** In this article, we focus on the $t \to \infty$ asymptotic of the lower-tail probability, and extract the leading order term, i.e., the rate function Φ_-. Our analysis, however, allows much room for more quantitative estimates. As mentioned in Section 1.2, the partition can take any size t^α with $\alpha \in (-\frac{1}{3}, \frac{2}{3})$. Optimizing over α (and a few other parameters within our analysis) should lead to a quantitative estimate on the tail probability in a similar spirit as [CG+20]. We do not pursue this direction here.

Acknowledgements. I thank Ivan Corwin, Promit Ghosal, and Pei-Ken Hung for useful discussions, thank Ivan Corwin and Yao-Yuan Mao for their suggestions that improve the presentation of this article, and thank the anonymous referees for their careful reading and many useful comments. Tsai's research was partially supported by a Junior Fellow award from the Simons Foundation, and by the NSF through DMS-1712575 and DMS-1953407.

Outline. In Section 2, we prepare a few basic tools. Based on these tools, in Section 3 we prove Theorem 1.3. In Section 4, we settle Theorem 1.1 and Corollary 1.4.

2. Basic tools

Hereafter throughout the rest of the article, we fix $L, \zeta, \beta \in (0,\infty)$, and drop dependence on these variables. For example $\mathcal{A} := \mathcal{A}_\beta$.

Below we will recall a general construction, via functional analysis, of a class of self-adjoint operators, and then specialize to the special case of Hill's operator and the SAO. One can also find the constructions of these random operators in [FN77] and in [AGZ10, Section 4.5.2] and [RRV11, Section 2].

We begin by recalling the classical construction of self-adjoint operators via sesquilinear forms. Consider Hilbert spaces $\mathscr{H}$ and $\mathscr{V}$, both over $\mathbb{C}$, equipped with inner products $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and $\langle \cdot, \cdot \rangle_{\mathscr{V}}$ and the thus induced norms $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{V}}$, and assume the embedding $\mathscr{V} \subset \mathscr{H}$ as vector spaces. Consider also a symmetric sesquilinear form $Q : \mathscr{V} \times \mathscr{V} \to \mathbb{C}$. The **associated operator** $T = (T, D(T))$ of Q has domain $D(T)$ consisting of $v \in \mathscr{V}$ such that

$$\exists u \in \mathscr{H} \text{ such that } Q(v,v') = \langle u, v' \rangle_{\mathscr{H}}, \ \forall v' \in \mathscr{V}, \tag{2.1}$$

and, for each $v \in D(T)$, $Tv := u$ is defined to be the (necessarily unique) vector $u \in \mathscr{H}$ that satisfies (2.1); see [Gru08, Definition 12.14]. Recall that Q is **coercive** with respect to $\mathscr{V} \subset \mathscr{H}$ if, for some fixed constant $c < \infty$,

$$\|v\|^2_{\mathscr{V}} \le c\,(\|v\|^2_{\mathscr{H}} + Q(v,v)), \quad \forall v \in \mathscr{V}.$$

Recall that $\mathscr{V}$ is **compactly embedded** in $\mathscr{H}$ if $\|v\|_{\mathscr{V}} \le c\|v\|_{\mathscr{H}}$, for some fixed constant $c < \infty$ and all $v \in \mathscr{V}$, and if any $\|\cdot\|_{\mathscr{V}}$-bounded sequence has a $\|\cdot\|_{\mathscr{H}}$-convergent subsequence. It is known (c.f., [Gru08, Corollary 12.19]) that if $\mathscr{V} \subset \mathscr{H}$ compactly and densely and if Q is coercive, then the associated operator $(T, D(T))$ is self-adjoint and closed, with $D(T) \subset \mathscr{V}$ being dense in $\mathscr{H}$. Furthermore, since Q is coercive and since $\mathscr{V} \subset \mathscr{H}$ compactly and densely, T necessarily has a pure-point spectrum that is bounded below and has no limit points, i.e., $-\infty < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$, with the corresponding eigenvectors forming a complete basis (i.e., dense orthonormal set) of $\mathscr{H}$. We will call such self-adjoint operators **standard**.

In the following, we will consider quadruples $(T, Q, \mathscr{V} \subset \mathscr{H})$, where Q is a symmetric sesquilinear form on $\mathscr{V}$ and T is the associated operator. The preceding discussion is summarized as follows.

Proposition 2.1. *Fix a quadruple $(T, Q, \mathscr{V} \subset \mathscr{H})$ described as in the preceding. If $\mathscr{V} \subset \mathscr{H}$ compactly and densely, and if Q is coercive, then T is* ***standard****: self-adjoint and has a pure-point spectrum that is bounded below and has no limit points, i.e., $-\infty < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$, with the corresponding eigenvectors forming a complete basis of $\mathscr{H}$.*

Now, to construct the SAO (1.4), we let $\mathscr{H} = L^2[0, \infty)$, and

$$\mathscr{V} = L_* := \Big\{ f \in H^1[0,\infty) : f(0) = 0,\ \int_0^\infty \big(|f'(x)|^2 + (1+x)|f(x)|^2 \big)\, \mathrm{d}x < \infty \Big\}, \tag{2.2}$$

equipped with the inner product $\langle f, g \rangle_{L_*} := \int_0^\infty (f'(x)\overline{g}'(x) + (1+x) f(x) \overline{g}(x)) \mathrm{d}x$. It is standard to check that $L_* \subset L^2[0, \infty)$ compactly and densely. Now define the symmetric sesquilinear form $Q_{\mathrm{SAO}} : L_* \times L_* \to \mathbb{C}$,

$$Q_{\mathrm{SAO}}(f, g) := \int_0^\infty \Big(f'(x)\overline{g}'(x) + \Big(x + \frac{2}{\sqrt{\beta}}\Big) f(x) \overline{g}(x) B'(x) \Big)\, \mathrm{d}x, \tag{2.3}$$

where, with $f, g \in L_*$, the integral against $B'(x)$ is understood in the integration-by-parts sense. Recall from [RRV11] (see also [AGZ10, Lemma 4.5.44 (b)]) that, almost surely, Q_{SAO} is coercive with respect to $L_* \subset L^2[0, \infty)$. Given these properties, we let $\mathcal{A}$ be the associated operator of Q_{SAO}, which, by Proposition 2.1, is standard.

Aside from the SAO, we will also consider operators of the form $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{\sqrt{\beta}} J'(x)$, on $x \in [a, b]$, for $J \in C[a, b]$, and with Dirichlet boundary condition at $x = a, b$. To define such an operator, take $\mathscr{H} = L^2[a, b]$ and $\mathscr{V} = H_0^1[a, b] := \{f \in H^1[a, b] : f(a) = f(b) = 0\}$, and define

$$Q_J(f, g) := \int_a^b \Big(f'(x)\overline{g}'(x) + \frac{2}{\sqrt{\beta}} f(x) \overline{g}(x) J'(x) \Big)\, \mathrm{d}x, \tag{2.4}$$

where, for $f, g \in H_0^1[a, b]$, the integral against $J'(x)$ is understood in the integration-by-parts sense. Indeed, $H^1[a, b] \subset L^2[a, b]$ compactly and densely. For continuous J, we show in (2.10) in the following that Q_J is coercive with respect to $H_0^1[a, b] \subset L^2[a, b]$. Given these properties, we let

$$S := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{\sqrt{\beta}} J'(x), \quad x \in (a, b),\ \text{with Dirichlet BC} \tag{2.5}$$

be the operator associated to Q_J, which, by Proposition 2.1, is standard. One particular J we will consider is $J(x) = B(x)$, which gives the Hill operator:

$$\mathcal{H}_{[a,b]} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{\sqrt{\beta}} B'(x), \quad x \in (a, b),\ \text{with Dirichlet BC.} \tag{2.6}$$

For a standard operator T, we will often adopt the notation $\boldsymbol{\lambda}_k(T)$ for its k-th eigenvalue, starting with index $k = 1$. For $(T, Q, \mathscr{V} \subset \mathscr{H})$ satisfying the properties of Proposition 2.1, we have the minimax principle:

$$\boldsymbol{\lambda}_k(T) = \min \Big\{ \max_{v \in \mathscr{E}, \|v\|_{\mathscr{H}} = 1} \{Q(v, v)\} : \mathscr{E}\ k\text{-dimensional subspace of } \mathscr{V} \Big\}. \tag{2.7}$$

This principle yields a useful comparison for the spectra of operators of the type (2.5).

Lemma 2.2. *Fix a finite interval $[a, b]$ and continuous functions $J_i \in C[a, b]$, $i = 1, 2$. Let S_i be the operators as in (2.5) with J_i in place of J. We have*

$$\boldsymbol{\lambda}_n(S_1) \leq \big(1 + \tfrac{\kappa+1}{\kappa^3}\big)\, \boldsymbol{\lambda}_n(S_2) + \tfrac{4}{\beta} \big(\tfrac{(\kappa+1)^2}{\kappa^3} U_2^2 + \kappa^2 U_{12}^2 \big), \quad \kappa > 0,\ n = 1, 2, \ldots,$$

where $U_2 := \sup_{x \in [a,b]} |J_2(x)|$ and $U_{12} := \sup_{x \in [a,b]} |J_1(x) - J_2(x)|$.

Proof. To simplify notation, we write $H_0^1 := H_0^1[a,b]$ and $L^2 := L^2[a,b]$. For $J \in C[a,b]$, we write $U_J := \sup_{x\in[a,b]} |J(x)|$. Let $f \in H_0^1[a,b]$ and $r > 0$. Applying the inequality $2|a_1 a_2| \le |a_1|^2 + |a_2|^2$ for $a_1 = r^{-\frac12} f'(x)$ and $a_2 = \frac{2}{\sqrt{\beta}} r^{\frac12} f(x)J(x)$, we have

$$\frac{2}{\sqrt{\beta}}\Big|\int_a^b |f(x)|^2 J'(x)\mathrm{d}x\Big| := \frac{2}{\sqrt{\beta}}\Big| - \int_a^b \big(f(x)\overline{f}'(x) + \overline{f}(x)f'(x)\big)J(x)\mathrm{d}x\Big|$$
$$\le \int_a^b \Big(r^{-1}|f'(x)|^2 + \frac{4r}{\beta}\,|J(x)f(x)|^2\Big)\,\mathrm{d}x \le r^{-1}\,\|f'\|_{L^2}^2 + \frac{4r}{\beta}\,U_J^2\,\|f\|_{L^2}^2.$$

Setting $(J,r) = (J_1 - J_2, \kappa^2)$ and $(J,r) = (J_2, \kappa+1)$ gives

$$Q_{J_1}(f,f) \le Q_{J_2}(f,f) + \kappa^{-2}\|f'\|_{L^2}^2 + \tfrac{4\kappa^2}{\beta} U_{12}^2 \|f\|_{L^2}^2, \tag{2.8}$$

$$Q_{J_2}(f,f) \ge \|f'\|_{L^2}^2 - \tfrac{1}{\kappa+1}\|f'\|_{L^2}^2 - \tfrac{4(\kappa+1)}{\beta} U_2^2 \|f\|_{L^2}^2. \tag{2.9}$$

The inequality (2.9) is rearranged as

$$\|f'\|_{L^2}^2 \le \tfrac{\kappa+1}{\kappa} Q_{J_2}(f,f) + \tfrac{4(\kappa+1)^2}{\beta\kappa} U_2^2 \|f\|_{L^2}^2. \tag{2.10}$$

Inserting (2.10) into (2.8) gives

$$Q_{J_1}(f,f) \le \big(1 + \tfrac{\kappa+1}{\kappa^3}\big) Q_{J_2}(f,f) + \tfrac{4}{\beta}\,\big(\tfrac{(\kappa+1)^2}{\kappa^3} U_2^2 + \kappa^2 U_{12}^2\big)\|f\|_{L^2}^2.$$

This together with the minimax principle (2.7) yields the desired result. □

We will also use the following variational characterization of sums of eigenvalues.

Lemma 2.3. *For $(T, Q, \mathscr{V} \subset \mathscr{H})$ satisfying the properties of Proposition 2.1, we have*

$$\sum_{k=1}^n \boldsymbol{\lambda}_k(T) = \min\Big\{ \sum_{k=1}^n Q(v_k, v_k) : \{v_1, \ldots, v_n\} \subset \mathscr{V} \text{ orthonormal in } \mathscr{H} \Big\}.$$

Proof. To simplify notation we write $\boldsymbol{\lambda}_k(T) = \lambda_k$ throughout this proof. Let $u_1, u_2, \ldots$ denote the corresponding orthonormal eigenvectors. Since $\lambda_1 > -\infty$, by shifting $T \mapsto T + c$ and $Q(v,v') \mapsto Q(v,v') + c\langle v, v'\rangle_{\mathscr{H}}$, we may assume without lost of generality that T is positive and Q is elliptic, i.e., $\|v\|_{\mathscr{V}}^2 \le c' Q(v,v)$. Given any set $\{v_1, \ldots, v_n\} \subset \mathscr{V}$ that is orthonormal in $\mathscr{H}$, expand each vector into the eigenbasis $v_k = \sum_{i=1}^\infty a_k^i u_i$, $a_k^i := \langle v_k, u_i\rangle_{\mathscr{H}}$. Using this, we have

$$\sum_{k=1}^n Q(v_k, v_k) = \sum_{k=1}^n Q\Big(\sum_{i=1}^\infty a_k^i u_i, \sum_{i'=1}^\infty a_k^{i'} u_{i'}\Big) = \sum_{k=1}^n \sum_{i,i'=1}^\infty a_k^i \overline{a}_k^{i'} Q(u_i, u_{i'}) = \sum_{i=1}^\infty \sum_{k=1}^n |a_k^i|^2 \lambda_i, \tag{2.11}$$

where, in the second equality we exchanged infinite sums with Q, which is justified by Q being elliptic. Put differently, (2.11) states that $\sum_{k=1}^n Q(v_k, v_k)$ is given by a weighted average of the eigenvalues, with weight $w_n := \sum_{k=1}^n |a_k^i|^2$. Moreover, the total amount of weight is fixed:

$$\sum_{n=1}^\infty w_n = \sum_{k=1}^n \sum_{i=1}^\infty |a_k^i|^2 = \sum_{k=1}^n \|v_i\|_{\mathscr{H}}^2 = n.$$

Given this constraint, to minimize (2.11), it is desirable to allocate more weights to smaller eigenvalues. On the other hand, each eigenvalue cannot receive weight more than 1:

$$w_n = \sum_{k=1}^n \big|\langle v_k, u_i\rangle_{\mathscr{H}}\big|^2 \le \|u_i\|_{\mathscr{H}}^2 = 1,$$

where the inequality follows because $\{v_1, \ldots, v_n\}$ is orthonormal. Combining the preceding properties, we see that the quantity in (2.11) cannot be smaller than $\sum_{k=1}^n \lambda_k$. Conversely, for $v_k = u_k$, $k = 1, \ldots, n$, we indeed have $\sum_{k=1}^n Q(u_k, u_k) = \sum_{k=1}^n \lambda_k$. □

A useful tool for analyzing the eigenvalue distribution is the Riccati transform. We refer to [CRR07] and [AD14, Sections 2–3] for expositions on the Riccati transform of Hill's operator and the SAO. The starting point of the Riccati transform is the eigenvalue problem for $\mathcal{A}$:

$$g''(x) = \tfrac{2}{\sqrt{\beta}} g(x) B'(x) + (x - \boldsymbol{\lambda}) g(x), \;\; x > 0, \tag{2.12}$$

understood in the integration-by-parts sense. Namely, we say that $g \in L_*$ (defined in (2.2)) solves (2.12) if it holds upon integrating against any test function $p(x) \in C_c^\infty[0,\infty)$, under the interpretation $\int_0^\infty p(x)g(x)B'(x)\mathrm{d}x := -\int_0^\infty (p'(x)g(x)+p(x)g'(x))B(x)\mathrm{d}x$. The Riccati transform $f(x) := g'(x)/g(x)$ brings the second order equation (2.12) into a first order one:

$$f'(x) = x - \boldsymbol{\lambda} - f^2(x) + \tfrac{2}{\sqrt{\beta}} B'(x),\ x > 0.$$

More generally, instead of taking an eigenvalue $\boldsymbol{\lambda}$ of $\mathcal{A}$, we consider a generic $\lambda \in \mathbb{R}$, regarded as a tunable parameter of the first order equation:

$$f'(x) = x - \lambda - f^2(x) + \tfrac{2}{\sqrt{\beta}} B'(x),\ x > 0. \tag{2.13}$$

With $B'(x)$ not being function-valued, we make sense of (2.13) by integrating in x. Note that, due to the negative, quadratic drift $-f^2(x)$, the solution $f(x)$ may undergo explosions to $-\infty$, so we integrate only over intervals that does not contain such explosions:

$$\begin{aligned} f(x)\big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \big(x - \lambda - f^2(x)\big)\mathrm{d}x + \tfrac{2}{\sqrt{\beta}} B(x)\big|_{x_1}^{x_2}, \\ [x_1, x_2] \subset [0,\infty) \text{ such that no explosions occur in } [x_1, x_2]. \end{aligned} \tag{2.13'}$$

For a given initial condition $f_0 \in \mathbb{R}$, it is readily checked that (2.13') permits a unique $C([0,\tau_1))$-valued solution f with $f(0) = f_0$ until the first explosion time τ_1 of f. We will also consider $f_0 = +\infty$, which is understood as $\lim_{x\to 0^+} f(x) = +\infty$. It is not hard to show that, existence and uniqueness (up to first explosion) holds also for $f_0 = \infty$. At each explosion τ_n to $-\infty$, we immediately restart f at $f(\tau_n) = +\infty$.

Given the prescribed explosion structure, it is convenient to view f as taking values in a countable disjoint union of $\mathbb{R}$, i.e.,

$$f \in \mathbb{R}_{-1} \cup \mathbb{R}_{-2} \cup \mathbb{R}_{-3} \cup \ldots := \mathbb{R}_{-\mathbb{N}},$$

with each component $\mathbb{R}_{-i}$ keeping track of the value of f between the $(i-1)$-th and i-th explosions. To define the topology and ordering on $\mathbb{R}_{-\mathbb{N}}$, take an order-preserving homeomorphism $u : \mathbb{R} \to (0,1)$ (e.g., $u(x) := (\arctan(x) + \pi/2)/\pi$), and consider the map $\widetilde{u} : \mathbb{R}_{-\mathbb{N}} \to (0,\infty)$: $\widetilde{u}(x,n) := u(x) - n - 1$. That is, each $\mathbb{R}_{-i}$ is mapped into $(i-1, i)$ in an order-preserving and homeomorphic manner. We endow the space $\mathbb{R}^*_{-\mathbb{N}}$ with the pull-back topology and ordering through $\widetilde{u}$. Indeed, the latter is simply lexicographical ordering, i.e., $(x,n) > (x',n')$ if $n > n' \in -\mathbb{N}$, and $(x,n) \geq (x',n)$ if $x \geq x' \in \mathbb{R}$.

We now recall known properties on the Riccati transform that will be used subsequently. Hereafter, for a standard operator T, we let $N(\lambda, T)$ denote the counting function of eigenvalues:

$$N(\lambda, T) = \#\big\{n \in \mathbb{N} : \boldsymbol{\lambda}_n(T) \leq \lambda\big\}.$$

Proposition 2.4 ([RRV11]). *Under the prescribed ordering and topology, we have the following.*

(a) Fix $\lambda \in \mathbb{R}$ and an initial condition $f(0) \in \mathbb{R} \cup \{+\infty\}$, Equations (2.13)–(2.13') admits a unique, continuous solution $f(x) = f(x,\lambda)$. Further, $f(x,\lambda)$ is decreasing in λ for each x.

(b) Equations (2.13)–(2.13') preserves ordering. That is, given any continuous solutions $f_1(x)$ and $f_2(x)$ of (2.13) with $f_1(0) \geq f_2(0)$, we have $f_1(x) \geq f_2(x)$ for all $x \geq 0$.

(c) Almost surely for all λ, $N(\lambda, \mathcal{A}) = \#\{$explosions of $f(\bullet, \lambda)$ in $(0,\infty)\}$.

Parts (a) and (c) are stated in [RRV11, Fact 3.1, Proposition 3.5], and Part (b) follows immediately from Part (a). Let us emphasize that, our discussions regarding the Riccati transform are *pathwise*, and in particular hold if B is replaced by any $w \in C([0,\infty))$ with sublinear growth: $\lim_{x\to\infty} |g(x)|x^{-a} = 0$, for some $a < 1$.

As for the Hill operator, similarly consider the Riccati transform:

$$f'(x) = -\lambda - f^2(x) + \tfrac{2}{\sqrt{\beta}} B'(x),\ x \in (a,b). \tag{2.14}$$

Just like in the preceding, we interpret (2.14) in the integrated sense

$$\begin{aligned} f(x)\big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \big(-\lambda - f^2(x)\big)\mathrm{d}x + \tfrac{2}{\sqrt{\beta}} B(x)\big|_{x_1}^{x_2}, \\ [x_1, x_2] \subset [a,b] \text{ such that no explosions occur in } [x_1, x_2], \end{aligned} \tag{2.14'}$$

and whenever an explosion occurs f is immediately restarted at $+\infty$. It is standard to show (see [FN77]) that the following analogue of Proposition 2.4 holds.

Proposition 2.5. *Under the prescribed ordering and topology, we have the following.*

(a) Fix $\lambda \in \mathbb{R}$ and an initial condition $f(0) \in \mathbb{R} \cup \{+\infty\}$, Equations (2.14)–(2.14') admits a unique, continuous solution $f(x) = f(x, \lambda)$. Further, $f(x, \lambda)$ is decreasing in λ for each x.

(b) Equations (2.14)–(2.14') preserves ordering. That is, given any continuous solutions $f_1(x)$ and $f_2(x)$ of (2.13) with $f_1(0) \geq f_2(0)$, we have $f_1(x) \geq f_2(x)$ for all $x \geq 0$.

(c) Almost surely for all λ, $N(\lambda, \mathcal{H}_{[a,b]}) = \#\{$explosions of $f(\bullet, \lambda)$ in $(a, b]\}$.

As mentioned previously in Section 1.2, our proof of Theorem 1.3 proceeds by a localization procedure. To setup notation for it, fix $\alpha \in (-\frac{1}{3}, \frac{2}{3})$, and partition $(0, \infty)$ into intervals of length t^α, i.e., $\eta_i := i t^\alpha$, and

$$I_i := (\eta_{i-1}, \eta_i],\ i = 1, 2, \ldots i_*, \quad I_{i*+1} := [\eta_{i*}, \infty). \tag{2.15}$$

To decide what i_* should be, referring back to (1.11), we see that $v_*(x)|_{x \geq \zeta} \equiv 0$. It is natural to choose $i_* \geq \zeta t^{\frac{2}{3} - \alpha}$ so that $t^{2/3} v_*(t^{-2/3} x)|_{x \geq \eta_{i_*}} \equiv 0$. We choose

$$i_* := \lceil \zeta t^{\frac{2}{3} - \alpha} + t^{\frac{2}{3} - \alpha} \rceil, \tag{2.16}$$

where the $+t^{\frac{2}{3} - \alpha}$ factor makes room for subsequent analysis. Counting the number of explosions of (2.13) on each subinterval gives

$$N_i(\lambda, \mathcal{A}) := \#\{x \in I_i : \lim_{y \to x^-} f(y, \lambda) = -\infty\},$$

where $f(x, \lambda)$ solves (2.13) with the initial condition $f(0, \lambda) = +\infty$. Then,

$$N(\lambda, \mathcal{A}) = \sum_{i=1}^{i_*+1} N_i(\lambda, \mathcal{A}). \tag{2.17}$$

Note that we have omitted the dependence on t in the notation I_i, η_i, and so on. Similar conventions will be frequently adopted without explicitly stating so.

Indeed, $N_i(\lambda, \mathcal{A})$ depends on the entrance value $f(\eta_{i-1}, \lambda)$ of f at the start η_{i-1} of the interval I_i. As a result the processes $N_i(\bullet, \mathcal{A})$, $i = 1, \ldots, i_* + 1$ are mutually dependent. This being the case, it will often be more convenient to consider

$$N(\lambda, \mathcal{H}_{I_i}),\ i = 1, \ldots, i_*, \quad N(\lambda, \mathcal{A}_*) := \#\big\{k \in \mathbb{N} : \boldsymbol{\lambda}_k(\mathcal{A}_*) \leq \lambda\big\},$$

where $\mathcal{A}_*$ is the SAO restricted to $[\eta_{i_*}, \infty)$:

$$\mathcal{A}_* := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \frac{2}{\sqrt{\beta}} B'(x), \quad x \geq \eta_{i_*}, \text{ with Dirichlet BC at } x = \eta_{i_*}, \tag{2.18}$$

constructed in a similar way as the SAO. Recall from Proposition 2.5(c) that $N(\lambda, \mathcal{H}_{I_i})$ counts the number of explosions within $x \in I_i$ of the solution $f_i(x) = f_i(x, \lambda)$ of

$$f_i'(x) = -\lambda - f_i^2(x) + \tfrac{2}{\sqrt{\beta}} B'(x),\ x \in I_i, \quad f_i(\eta_{i-1}) = +\infty. \tag{2.19}$$

Similarly, $N(\lambda, \mathcal{A}_*)$ counts the number of explosions within $x \in I_{i_*+1}$ of the solution $f_*(x) = f_*(x, \lambda)$ of

$$f_*'(x) = x - \lambda - f_*^2(x) + \tfrac{2}{\sqrt{\beta}} B'(x),\ x \in I_{i_*+1}, \quad f_*(\eta_{i_*}) = +\infty. \tag{2.13*}$$

From the preceding descriptions, we see that $N(\lambda, \mathcal{H}_{I_i})$ depends only on the increment $B(x) - B(\eta_{i-1})$ of the Brownian motion within $x \in I_i$, and $N(\lambda, \mathcal{A}_*)$ depends only on $B(x) - B(\eta_{i*})$ for $x \in I_{i_*+1}$. Hence, the processes $N(\bullet, \mathcal{H}_{I_i})$, $i = 1, \ldots, i_*$, and $N(\bullet, \mathcal{A}_*)$ are independent.

To relate the processes $N(\bullet, \mathcal{H}_{I_i})$ and $N(\bullet, \mathcal{A}_*)$ back to $N_i(\bullet, \mathcal{A})$, we establish the following inequalities.

Lemma 2.6. *Couple the processes $N_i(\bullet, \mathcal{A})$, $N(\bullet, \mathcal{H}_{I_i})$, $N(\bullet, \mathcal{A}_*)$ by having the same spatial white noise $B'(x)$ for the operators in (1.4), (2.6), and (2.18). Almost surely for all $\lambda \in \mathbb{R}$ and $i = 1, \ldots, i_*$, we have*

$$N(\lambda - \eta_i, \mathcal{H}_{I_i}) \leq N_i(\lambda, \mathcal{A}) \leq N(\lambda - \eta_{i-1}, \mathcal{H}_{I_i}) + 1, \quad N_{i_*+1}(\lambda, \mathcal{A}) \leq N(\lambda, \mathcal{A}_*) + 1.$$

Proof. Fix i and λ. Let $f(x) = f(x,\lambda)$ be the solution of (2.13) with $f(0) = +\infty$. Restricting (2.13) to the relevant interval $x \in I_i$, we write

$$f'(x) = x - \lambda - f^2(x) + \tfrac{2}{\sqrt{\beta}} B'(x),\ x \in I_i, \quad f(\eta_{i-1}) \in \mathbb{R} \cup \{+\infty\},\ \text{given.} \tag{2.20}$$

Let $g(x) = f_i(x, \lambda - \eta_i)$ be the solution of (2.19) with $\lambda \mapsto \lambda - \eta_i$, i.e.,

$$g'(x) = -(\lambda - \eta_i) - g^2(x) + \tfrac{2}{\sqrt{\beta}} B'(x),\ x \in I_i, \quad g(\eta_{i-1}) = +\infty. \tag{2.21}$$

By definition, $N_i(\lambda, \mathcal{A})$ is the number of explosions of f on $I_i = (\eta_{i-1}, \eta_i]$, and recall that $N(\lambda - \eta_i, \mathcal{H}_{I_i})$ is equal to the number of explosions of g in I_i. Since $x - \lambda \le -(\lambda - \eta_i)$ on $x \in I_i$ and since $f(\eta_{i-1}) \le g(\eta_{i-1}) = +\infty$, by comparison we have $f(x) \le g(x)$, $x \in I_i$, under the ordering of $\mathbb{R}_{-\mathbb{N}}$. This gives the first inequality $N(\lambda - \eta_i, \mathcal{H}_{I_i}) \le N_i(\lambda, \mathcal{A})$.

Turning to the second inequality, we consider $\widetilde{g}(x) = f_i(x, \lambda - \eta_{i-1})$, which solves

$$\widetilde{g}'(x) = -(\lambda - \eta_{i-1}) - \widetilde{g}^2(x) + \tfrac{2}{\sqrt{\beta}} B'(x),\ x \in I_i, \quad \widetilde{g}(\eta_{i-1}) = +\infty, \tag{2.22}$$

and consider the first explosion time of f on I_i. If f does not explode within I_i, then $N_i(\lambda, \mathcal{A}) = 0$, whence the desired inequality $N_i(\lambda, \mathcal{A}) \le N(\lambda - \eta_{i-1}, \mathcal{H}_{I_i}) + 1$ follows trivially. Otherwise let $b \in [\eta_{i-1}, \eta_i]$ denote the first explosion. We then have $x - \lambda \ge -(\lambda - \eta_{i-1})$ on $x \in [b, \eta_i]$ and $+\infty = f(b) \ge \widetilde{g}(b)$. Comparison applied to f and $\widetilde{g}$ over the interval $x \in [b, \eta_i]$ yields $f(x) \ge \widetilde{g}(x)$, $x \in [b, \eta_i]$. Taking into account the explosion of f at $x = b$, we obtain $N_i(\lambda, \mathcal{A}) \le N(\lambda - \eta_{i-1}, \mathcal{H}_{I_i}) + 1$.

The last inequality concerning $N_{i_*+1}(\lambda, \mathcal{A})$ and $N(\lambda, \mathcal{A}_*)$ follows by the same comparison argument applied to solutions of (2.13*) and (2.20) for $i = i_* + 1$. □

3. Proof of Theorem 1.3

Our proof of Theorem 1.3 breaks into lower and upper bounds. That is, we establish matching bounds on the l.h.s. of (1.7) to obtain the desired result. Hereafter, we use $c = c(a, b, \ldots)$ to denote a generic, deterministic, finite positive constant that may change from line to line, but depend only on the designated variables. As declared previously, $\beta, \zeta, L \in (0, \infty)$ are fixed throughout this article, so their dependence will not be designated.

3.1. Lower bound.

To simplify notation, set

$$G := \mathbf{E}\Big[\exp\Big(-L \sum_{k=1}^{\infty} \phi_t(\boldsymbol{\lambda}_k(\mathcal{A}) - t^{\frac{2}{3}}\zeta)\Big)\Big]. \tag{3.1}$$

Our goal is to establish a desired lower bound on $t^{-2} \log G$. The proof is carried out in steps.

Step 1: localization. Recall the partition (2.15) introduced previously. By definition, $N(\lambda, \mathcal{A})$ counts the number of eigenvalues $\boldsymbol{\lambda}_k(\mathcal{A})$ of $\mathcal{A}$ at most λ. Using this fact, together with the decomposition (2.17), we rewrite the infinite sum in (3.1) as

$$\begin{aligned} -\sum_{k=1}^{\infty} \phi_t(\boldsymbol{\lambda}_k(\mathcal{A}) - t^{\frac{2}{3}}\zeta) &= -\int_{\mathbb{R}} \phi_t(\lambda - t^{\frac{2}{3}}\zeta)\, \mathrm{d}N(\lambda, \mathcal{A}) \\ &= \int_{\mathbb{R}} N(\lambda, \mathcal{A}) \phi_t'(\lambda - t^{\frac{2}{3}}\zeta)\, \mathrm{d}\lambda = \sum_{i=1}^{i_*+1} \int_{\mathbb{R}} N_i(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}) \phi_t'(\lambda)\, \mathrm{d}\lambda, \end{aligned} \tag{3.2}$$

where d acts on the variable $\lambda \in \mathbb{R}$. In the second inequality in (3.2) we used integration by parts and

$$\lim_{\lambda \to \infty} \phi_t(\lambda) N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}) = 0, \qquad \text{almost surely.} \tag{3.3}$$

We postpone the proof of (3.3) until the end of Step 3 to streamline the presentation, since the proof uses arguments similar to those in Step 3. Recall the Hill operator $\mathcal{H}_{I_i}$ from (2.6) and $\mathcal{A}_*$ from (2.18). Our goal here is to pass from the operator $\mathcal{A}$ to $\mathcal{H}_{I_i}$ for $i = 1, \ldots, i_*$ and to $\mathcal{A}_*$ for $i = i_* + 1$. To simplify notation, set $\mathcal{N}_i(\lambda) := N(\lambda + t^{\frac{2}{3}}\zeta - \eta_{i-1}, \mathcal{H}_{I_i})$ for $i = 1, \ldots, i_*$, and $\mathcal{N}_{i_*+1} := N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}_*)$. Consider the event

$\Omega_1 := \{\boldsymbol{\lambda}_1(\mathcal{A}) > -t^{\frac{2}{3}}\}$ that the groundstate eigenvalue of $\mathcal{A}$ lies above $-t^{\frac{2}{3}}$. It is readily checked from (1.6) that $\phi_t'(\lambda) < 0$. Using this and the bounds from Lemma 2.6 in (3.2), we write

$$\begin{aligned} G &\geq \mathbf{E}\Big[\mathbf{1}_{\Omega_1} \cdot \prod_{i=1}^{i_*+1} \exp\Big(L \int_{-t^{\frac{2}{3}}(1+\zeta)}^{\infty} N_i(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A})\phi_t'(\lambda)\,\mathrm{d}\lambda\Big)\Big] \\ &\geq \mathbf{E}\Big[\mathbf{1}_{\Omega_1} \cdot \prod_{i=1}^{i_*+1} \exp\Big(L \int_{-t^{\frac{2}{3}}(1+\zeta)}^{\infty} (1 + \mathcal{N}_i(\lambda))\phi_t'(\lambda)\,\mathrm{d}\lambda\Big)\Big]. \end{aligned} \tag{3.4}$$

Within the last expression, separate the 1's from the $\mathcal{N}_i$'s and evaluate the contribution of the former

$$L \int_{-t^{\frac{2}{3}}(1+\zeta)}^{\infty} 1 \cdot \phi_t'(\lambda)\mathrm{d}\lambda = -L\,\phi_t(-t^{\frac{2}{3}}(1+\zeta)) = -L\,\log(1 + e^{t(1+\zeta)}) \geq -ct.$$

With $i_* + 1 \leq ct^{\frac{2}{3}-\alpha}$, we bound

$$\prod_{i=1}^{i_*+1} \exp\Big(L \int_{-t^{\frac{2}{3}}(1+\zeta)}^{\infty} 1 \cdot \phi_t'(\lambda)\,\mathrm{d}\lambda\Big) \geq e^{-ct^{\frac{5}{3}-\alpha}}.$$

Use this bound in (3.4), and for the remaining integral of $\mathcal{N}_i\phi_t'$ (which is negative), release the range of integration from $\lambda \in (-t^{\frac{2}{3}}(1+\zeta), \infty)$ to $\lambda \in \mathbb{R}$. We get

$$G \geq e^{-ct^{\frac{5}{3}-\alpha}} \mathbf{E}\Big[\mathbf{1}_{\Omega_1} \cdot \prod_{i=1}^{i_*+1} \exp\Big(L \int_{\mathbb{R}} \mathcal{N}_i(\lambda)\phi_t'(\lambda)\,\mathrm{d}\lambda\Big)\Big]. \tag{3.5}$$

Step 2: change of measure. Write $y_\pm := (\pm y) \vee 0$ for the positive/negative part, and consider

$$v_*(x) := 4L^2\pi^{-2}\beta^{-\frac{3}{2}} \Big(-1 + \sqrt{1 + \big(\tfrac{\pi\beta}{2L}\big)^2(\zeta - x)_+}\Big), \tag{3.6}$$

and set

$$V_i := t^{\frac{2}{3}} v_*(t^{-\frac{2}{3}}\eta_{i-1}), \quad V(x) := \sum_{i=1}^{i_*} V_i \mathbf{1}_{I_i}(x). \tag{3.7}$$

Girsanov's theorem asserts that

$$\mathbf{E}[\,\bullet\,] = \widetilde{\mathbf{E}}\big[e^{-\int_0^\infty V(x)\mathrm{d}B(x) + \frac{1}{2}\int_0^\infty V^2(x)\mathrm{d}x}\,(\,\bullet\,)\big], \tag{3.8}$$

and, under $\widetilde{\mathbf{E}}$, B is distributed as a drifted Brownian motion, i.e., $B \overset{\text{law}}{=} \widetilde{B} + \int_0^\bullet V(y)\mathrm{d}y$, where $\widetilde{B}$ is a standard Brownian motion. Let $\widetilde{\mathcal{A}}_* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \frac{2}{\sqrt{\beta}}\widetilde{B}'(x)$, $x \geq \eta_{i_*}$, and let $\widetilde{\mathcal{H}}_{I_i} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{\sqrt{\beta}}\widetilde{B}'(x)$, $x \in I_i$, denote the analogous operators. On the r.h.s. of (3.5), apply (3.8), and express each B in terms of $\widetilde{B}$ and V for the result. We obtain

$$G \geq e^{-ct^{\frac{5}{3}-\alpha} - \frac{1}{2}\int_0^\infty V^2(x)\mathrm{d}x} \cdot \widetilde{\mathbf{E}}\Big[\mathbf{1}_{\widetilde{\Omega}_2} e^{-\int_0^\infty V(x)\mathrm{d}\widetilde{B}(x)} \cdot \prod_{i=1}^{i_*+1} \exp\Big(L \int_{\mathbb{R}} \widetilde{\mathcal{N}}_i(\lambda)\phi_t'(\lambda)\,\mathrm{d}\lambda\Big)\Big], \tag{3.9}$$

where

$$\widetilde{\mathcal{N}}_i(\lambda) := N(\lambda + t^{\frac{2}{3}}\zeta - \tfrac{2}{\sqrt{\beta}}V_i - \eta_{i-1}, \widetilde{\mathcal{H}}_{I_i}),\ i = 1, \ldots, i_*, \quad \widetilde{\mathcal{N}}_{i_*+1}(\lambda) := N(\lambda + t^{\frac{2}{3}}\zeta, \widetilde{\mathcal{A}}_*),$$

and $\widetilde{\Omega}_2 := \{\boldsymbol{\lambda}_1(\widetilde{\mathcal{A}} + \frac{2}{\sqrt{\beta}}V) > -t^{\frac{2}{3}}\}$. In the last expression we interpreted V as a multiplicative operator $L^2[0,\infty) \to L^2[0,\infty)$, which is a bounded, Hermitian operator. From this point onward, we will always operate under the transformed measure $\widetilde{\mathbf{E}}$. To alleviate heavy notation, we dropped all the tildes and rewrite (3.9) as

$$G \geq e^{-ct^{\frac{5}{3}-\alpha} - \frac{1}{2}\int_0^\infty V^2(x)\mathrm{d}x} \cdot \mathbf{E}\Big[\mathbf{1}_{\Omega_2} e^{-\int_0^\infty V(x)\mathrm{d}B(x)} \cdot \prod_{i=1}^{i_*+1} \exp\Big(L \int_{\mathbb{R}} \mathcal{M}_i(\lambda)\phi_t'(\lambda)\,\mathrm{d}\lambda\Big)\Big]. \tag{3.9'}$$

where $\Omega_2 := \{\boldsymbol{\lambda}_1(\mathcal{A} + \frac{2}{\sqrt{\beta}}V) > -t^{\frac{2}{3}}\}$, and

$$\mathcal{M}_i(\lambda) := N(\lambda + t^{\frac{2}{3}}\zeta - \tfrac{2}{\sqrt{\beta}}V_i - \eta_{i-1}, \mathcal{H}_{I_i}),\ i = 1, \ldots, i_*, \quad \mathcal{M}_{i_*+1}(\lambda) := N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}_*). \tag{3.10}$$

Step 3: bounding terms on the r.h.s. of (3.9'). We begin with the term $\mathcal{M}_i(\lambda)$, $i = 1, \ldots, i_*$. To bound $\mathcal{M}_i(\lambda)$, we will apply spectral comparison of the Hill operator $\mathcal{H}_{I_i}$ and the Laplace operator

$$-\Delta_{I_i} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \text{with Dirichlet BC.}$$

Set $U_i := \max_{x\in I_i} |B(x) - B(\eta_{i-1})|$, fix $i = 1, \ldots, i_*$, and let $\kappa \geq 1$ be an auxiliary parameter. Apply Lemma 2.2 with $(J_1(x), J_2(x)) = (0, B(x) - B(\eta_{i-1}))$ to get

$$\big(1 + \tfrac{\kappa+1}{\kappa^3}\big)\, \boldsymbol{\lambda}_n(\mathcal{H}_{I_i}) \geq \boldsymbol{\lambda}_n(-\Delta_{I_i}) - c\,(\kappa+1)^2 U_i^2. \tag{3.11}$$

From this we deduce, for $r = t^{\frac{2}{3}}\zeta - \frac{2}{\sqrt{\beta}}V_i - \eta_{i-1}$,

$$\begin{aligned} \mathcal{M}_i(\lambda) = \#\big\{n\in\mathbb{N} : \boldsymbol{\lambda}_n(\mathcal{H}_{I_i}) \leq \lambda + r\big\} &\leq \#\big\{n\in\mathbb{N} : \boldsymbol{\lambda}_n(-\Delta_{I_i}) \leq \big(1+\tfrac{\kappa+1}{\kappa^3}\big)\big(\lambda + r\big) + c\,(\kappa+1)^2 U_i^2\big\} \\ &= N\big(\big(1+\tfrac{\kappa+1}{\kappa^3}\big)\big(\lambda+r\big) + (\kappa+1)^2 c_\star U_i^2, -\Delta_{I_i}\big), \end{aligned} \tag{3.12}$$

for some fixed constant $c_\star < \infty$. Fix $\delta \in (0, \frac{2}{3} - \alpha)$, and consider the event

$$\Omega_3(\kappa) := \big\{(\kappa+1)^2 c_\star U_i^2 \leq t^{\delta+\alpha_+}, \tag{3.13a}$$

$$U_i \leq t^{\frac{1}{2}(\delta+\alpha_+)},\ i = 1, \ldots, i_* \big\}. \tag{3.13b}$$

Given that the interval I_i has length $|I_i| = t^\alpha$, it is straightforward to verify $\mathbf{P}[\Omega_3(\kappa)] \to 1$, for fixed $\kappa \in [1,\infty)$ as $t \to \infty$. Under the condition (3.13a), we have

$$\mathbf{1}_{\Omega_3(\kappa)}\mathcal{M}_i(\lambda) \leq M_i(\lambda,\kappa), \quad i = 1, \ldots, i_*, \tag{3.14}$$

where

$$M_i(\lambda,\kappa) := N\big(\big(1+\tfrac{\kappa+1}{\kappa^3}\big)\big(\lambda+r\big) + t^{\delta+\alpha_+}, -\Delta_{I_i}\big), \quad r_i := t^{\frac{2}{3}}\zeta - \tfrac{2}{\sqrt{\beta}}V_i - \eta_{i-1}. \tag{3.15}$$

We now turn to bounding $\mathcal{M}_{i_*+1}(\lambda) = N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}_*)$. Shifting the operator $\mathcal{A}_*$ (defined in (2.18)) by $x \mapsto x - \eta_{i_*}$, we see that $\{\boldsymbol{\lambda}_n(\mathcal{A}_*)\}_{n=1}^\infty \overset{\text{law}}{=} \{\boldsymbol{\lambda}_n(\mathcal{A}) + \eta_{i_*}\}_{n=1}^\infty$, or equivalently

$$\mathcal{M}_{i_*+1}(\bullet) \overset{\text{law}}{=} N(\bullet + t^{\frac{2}{3}}\zeta - \eta_{i_*}, \mathcal{A}). \tag{3.16}$$

Our next step is to compare the spectrum of $\mathcal{H}$ to that of the Airy operator $\mathsf{A} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x$, in a way similarly to Lemma 2.2. Recall that $\mathcal{A}$ is the associated operator of the form (2.3), with $\mathscr{V} = L_*$ given in (2.2) and $\mathscr{H} = L^2[0,\infty)$. For the Airy operator, we take the same Hilbert spaces $\mathscr{V} = L_* \subset \mathscr{H} = L^2[0,\infty)$, with the form $Q_{\mathsf{A}}(f,g) := \int_0^\infty (f'(x)\overline{g}'(x) + xf(x)\overline{g}(x))\mathrm{d}x$. We seek to apply [AGZ10, Lemma 4.5.44 (b)]. To this end, note that the $\|f\|_*^2$ defined in [AGZ10, p 308] is equal to $Q_{\mathsf{A}}(f,f) + \|f\|^2_{L^2[0,\infty)}$ here, and the $\langle f,f\rangle_{\mathbf{H}_\beta}$ defined in [AGZ10, Equation (4.5.15)] is equal to $Q_{\mathcal{A}}(f,f)$ here. By [AGZ10, Lemma 4.5.44 (b)], there exists a $[0,\infty)$-valued random variable U such that,

$$Q_{\mathcal{A}}(f,f) \geq \tfrac{1}{2}Q_{\mathsf{A}}(f,f) - U\|f\|^2_{L^2[0,\infty)}, \quad \forall f \in L_*.$$

The minimax principle (2.7) hence gives $\boldsymbol{\lambda}_n(\mathcal{A}) \geq \frac{1}{2}\boldsymbol{\lambda}_n(\mathsf{A}) - U$. From this we conclude that $N(\lambda + t^{\frac{2}{3}}\zeta - \eta_{i_*}, \mathcal{A}) \leq N(2(\lambda + t^{\frac{2}{3}}\zeta - \eta_{i_*}) + 2U, \mathsf{A})$. Recall i_* from (2.16). We have $\eta_{i_*} = i_* t^\alpha \geq t^{\frac{2}{3}}\zeta + t^{\frac{2}{3}}$, so

$$N(\lambda + t^{\frac{2}{3}}\zeta - \eta_{i_*}, \mathcal{A}) \leq N(2(\lambda - t^{\frac{2}{3}} + U), \mathsf{A}). \tag{3.17}$$

The spectrum of the Airy operator is exactly the zero set of the Airy function on $\mathbb{R}$ up to a spatial reversal, and the real zeros of the Airy function admit precise asymptotic expansions (see, e.g., [Olv97, Section 11.5]). In particular, $N(\lambda, \mathsf{A}) \leq c\,(\lambda_+)^{3/2}$, for all $\lambda \in \mathbb{R}$. Combining this with (3.16) and (3.17), we have that

$$\mathcal{M}_{i_*+1}(\lambda) \leq N(2(\lambda - t^{\frac{2}{3}} + U_*), \mathsf{A}), \tag{3.18}$$

for some $U_* \overset{\text{law}}{=} U$. Therefore,

$$\exp\Big(L\int_{\mathbb{R}} \mathcal{M}_{i_*+1}(\lambda)\phi_t'(\lambda)\mathrm{d}\lambda\Big) \geq \exp\Big(c\int_{\mathbb{R}} (\lambda - t^{\frac{2}{3}} + U_*)_+^{\frac{3}{2}}\phi_t'(\lambda)\mathrm{d}\lambda\Big). \tag{3.19}$$

Consider the event $\Omega_4 := \{U_* \leq t^{\frac{2}{3}}\}$. Indeed, since $U_* \stackrel{\text{law}}{=} U$ is $[0,\infty)$-valued, we have $\mathbf{P}[\Omega_4] \to 1$, as $t \to \infty$. On the r.h.s. of (3.19), use $\phi'_t(\lambda) \geq -t^{\frac{1}{3}} e^{-t^{\frac{1}{3}}\lambda}$ (verified from (1.6)) and perform the change of variables $\lambda - t^{\frac{2}{3}} + U_* \mapsto \lambda$. Under the condition $\Omega_4 := \{U_* \leq t^{\frac{2}{3}}\}$, we have

$$\begin{aligned}\mathbf{1}_{\Omega_4} \cdot \exp\Big(L \int_{\mathbb{R}} \mathcal{M}_{i_*+1}(\lambda)\phi'_t(\lambda)\mathrm{d}\lambda\Big) &\geq \mathbf{1}_{\Omega_4} \cdot \exp\Big(-c\int_0^\infty \lambda^{\frac{3}{2}} t^{\frac{1}{3}} e^{-t^{\frac{1}{3}}(\lambda + t^{\frac{2}{3}} - U_*)} \mathrm{d}\lambda\Big) \\ &\geq \exp\Big(-c\int_0^\infty \lambda^{\frac{3}{2}} t^{\frac{1}{3}} e^{-t^{\frac{1}{3}}\lambda}\mathrm{d}\lambda\Big) \geq \frac{1}{2},\end{aligned} \tag{3.20}$$

for all t large enough.

Next we turn to the exponential martingale in (3.9'). Recall that $U_i := \max_{x\in I_i}|B(x) - B(\eta_{i-1})|$, and that $V(x)$ takes the constant value V_i on I_i, and note from (3.7) that $|V_i| \leq ct^{\frac{2}{3}}$. From these properties we have

$$\Big|\int_0^\infty V(x)\mathrm{d}B(x)\Big| \leq \sum_{i=1}^{i_*} |V_i||B(\eta_i) - B(\eta_{i-1})| \leq ct^{\frac{2}{3}} \sum_{i=1}^{i_*} U_i.$$

Using the condition (3.13b) together with $i_* \leq ct^{\frac{2}{3}-\alpha}$ gives

$$\mathbf{1}_{\Omega_3(\kappa)} e^{-\int_0^\infty V(x)\mathrm{d}B(x)} \geq \exp(-ct^{\frac{4}{3}+\frac{1}{2}(\delta+\alpha_+)}). \tag{3.21}$$

On the r.h.s. of (3.9') within the expectation, multiply by $\mathbf{1}_{\Omega_3(\kappa)\cap\Omega_4}$ to get

$$G \geq e^{-ct^{\frac{5}{3}-\alpha} - \frac{1}{2}\int_0^\infty V^2(x)\mathrm{d}x} \cdot \mathbf{E}\Big[\mathbf{1}_{\Omega_2\cap\Omega_3(\kappa)\cap\Omega_4} e^{-\int_0^\infty V(x)\mathrm{d}B(x)} \cdot \prod_{i=1}^{i_*+1} \exp\Big(L\int_{\mathbb{R}} \mathcal{M}_i(\lambda)\phi'_t(\lambda)\,\mathrm{d}\lambda\Big)\Big].$$

On the r.h.s., insert the bounds (3.14) and (3.20)–(3.21) (noting that $M_i(\lambda,\kappa)$ is deterministic), take the logarithm, and divide the result by t^2. We obtain

$$\begin{aligned} t^{-2}\log G \geq &- ct^{-\frac{1}{3}-\alpha} - \frac{1}{2}\int_0^\infty t^{-2}V^2(x)\,\mathrm{d}x - ct^{-\frac{2}{3}+\frac{1}{2}(\delta+\alpha_+)} + L\sum_{i=1}^{i_*}\int_{\mathbb{R}} t^{-2}M_i(\lambda,\kappa)\phi'_t(\lambda)\,\mathrm{d}\lambda \\ &- t^{-2}\log 2 + t^{-2}\log \mathbf{P}[\Omega_2\cap\Omega_3(\kappa)\cap\Omega_4]. \end{aligned} \tag{3.22}$$

As has been argued previously, $\mathbf{P}[\Omega_3(\kappa)], \mathbf{P}[\Omega_4] \to 1$, for fixed $\kappa \in (0,\infty)$ as $t\to\infty$. As for Ω_2, with $V(x) \geq 0$, a comparison argument similarly to the preceding one gives $\boldsymbol{\lambda}_1(\mathcal{A} + \frac{2}{\sqrt{\beta}}V) \geq \boldsymbol{\lambda}_1(\mathcal{A})$. This being the case, we necessarily have $\mathbf{P}[\Omega_2] = \mathbf{P}[\boldsymbol{\lambda}_1(\mathcal{A} + \frac{2}{\sqrt{\beta}}V) > -t^{\frac{2}{3}}] \geq \mathbf{P}[\boldsymbol{\lambda}_1(\mathcal{A}) > -t^{\frac{2}{3}}] \to 1$, as $t\to\infty$. Consequently, $\mathbf{P}[\Omega_2\cap\Omega_3(\kappa)\cap\Omega_4] \to 1$. Now, for fixed $\kappa\in(0,\infty)$, sending $t\to\infty$ in (3.22), together with $\alpha > -\frac{1}{3}$ and $\delta + \alpha_+ < \frac{2}{3}$, we arrive at

$$\liminf_{t\to\infty}(t^{-2}\log G) \geq \liminf_{t\to\infty}\Big(-\frac{1}{2}\int_0^\infty t^{-2}V^2(x)\,\mathrm{d}x\Big) + \liminf_{t\to\infty}\Big(L\sum_{i=1}^{i_*}\int_{\mathbb{R}} t^{-2}M_i(\lambda,\kappa)\phi'_t(\lambda)\,\mathrm{d}\lambda\Big). \tag{3.23}$$

Proof of (3.3). The proof of (3.3) amounts to bounding $N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A})$. Apply (2.17) to decompose $N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A})$ into a sum, and apply Lemma 2.6 to bound the result as

$$N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}) \leq \sum_{i=1}^{i_*}\big(N(\lambda + t^{\frac{2}{3}}\zeta - \eta_{i-1}, \mathcal{H}_{I_i}) + 1\big) + N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}_*) + 1. \tag{3.24}$$

For the second to last term in (3.24), recalling that $N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}_*) =: \mathcal{M}_{i_*+1}(\lambda)$, we have the bound (3.19). As for the summand in (3.24), recall the definition of $\mathcal{M}_i(\lambda)$ from (3.10) and rerun the arguments that lead up to (3.12) but with $V_i \mapsto 0$ and $\kappa = 1$. We have

$$N(\lambda + t^{\frac{2}{3}}\zeta - \eta_{i-1}, \mathcal{H}_{I_i}) \leq N\big(3\big(\lambda + t^{\frac{2}{3}}\zeta - \tfrac{2}{\sqrt{\beta}} - \eta_{i-1}\big) + \widetilde{U}_i, -\Delta_{I_i}\big), \tag{3.25}$$

for some $[0,\infty)$-valued random variable $\widetilde{U}_i$ that does *not* depend on λ. Insert the bounds (3.19) and (3.25) into (3.24). Having in mind the goal of proving (3.3), we view t,ζ as being fixed, let λ vary, and note that

η_i, i_* does *not* depend on λ. We have

$$N(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}) \leq \sum_{i=1}^{i_*} \big(N(2\lambda + c(t,\zeta) + \widetilde{U}_i, -\Delta_{I_i}) + 1\big) + N(2\lambda + c(t,\zeta) + U_*, \mathsf{A}) + 1, \tag{3.26}$$

where $\mathsf{A} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x$. As mentioned previously, for the Airy operator A we have $N(\widetilde{\lambda}, \mathsf{A}) \leq c\,(\widetilde{\lambda}_+)^{3/2}$; for the Laplace operator $-\Delta_{I_i}$, it is standard to show that $N(\widetilde{\lambda}, -\Delta_{I_i}) \leq c(t,\zeta)\,(\widetilde{\lambda}_+)^{1/2}$. Use these bounds in (3.26), recall that $\widetilde{U}_i, U_*, i_*$ are λ-independent, and note that the factor $\phi_t(\lambda)$ in (3.3) decays exponentially as $\lambda \to \infty$. We conclude (3.3). □

Step 4: evaluating the limit. The last step is to evaluate the limits on the r.h.s. of (3.23). For the first term, recall the definition of $v_*(x)$ and $V(x)$ from (3.6)–(3.7). Substituting in $|I_i| = t^\alpha$, we have

$$\frac{1}{2}\int_0^\infty t^{-2}V^2(x)\,\mathrm{d}x = \frac{t^{-2}}{2}\sum_{i=1}^{i_*} t^{\frac{4}{3}} v_*^2(\eta_{i-1}t^{-\frac{2}{3}})\,|I_i| = \frac{1}{2}\sum_{i=1}^{i_*} v_*^2((i-1)t^{\alpha-\frac{2}{3}})\,t^{\alpha-\frac{2}{3}}.$$

The last expression is indeed a Riemann sum of the integral $\frac{1}{2}\int_0^\infty v_*^2(x)\mathrm{d}x$. Since v_* is continuous and compactly supported, we have

$$\lim_{t\to\infty}\frac{1}{2}\int_0^\infty t^{-2}V^2(x)\,\mathrm{d}x = \int_0^\infty \frac{1}{2}v_*^2(x)\mathrm{d}x. \tag{3.27}$$

Next, recall the definition of $M_i(\lambda,\kappa)$ and r_i from (3.15). Indeed, the spectrum of the Laplace operator $-\Delta_{I_i}$ is simply $\{\boldsymbol{\lambda}_n(-\Delta_{I_i})\}_{n=1}^\infty = \{n^2\pi^2|I_i|^{-2}\}_{n=1}^\infty$. Substituting in $|I_i| = t^\alpha$, we obtain

$$M_i(\lambda,\kappa) \leq \frac{t^\alpha}{\pi}\sqrt{\Big(\big(1+\tfrac{\kappa+1}{\kappa^3}\big)\big(\lambda+r_i\big)+t^{\delta+\alpha_+}\Big)_+}. \tag{3.28}$$

Apply $\sum_{i=1}^{i_*}\int_{\mathbb{R}} t^{-2}(\,\bullet\,)\phi_t'(\lambda)\,\mathrm{d}\lambda$ to both sides of (3.28). With $\phi_t' < 0$, the resulting equality flips sides, giving

$$L\sum_{i=1}^{i_*}\int_{\mathbb{R}} t^{-2}M_i(\lambda,\kappa)\phi_t'(\lambda)\,\mathrm{d}\lambda \geq \int_{\mathbb{R}}\frac{t^{\alpha-2}L}{\pi}\sum_{i=1}^{i_*}\sqrt{\Big(\big(1+\tfrac{\kappa+1}{\kappa^3}\big)\big(\lambda+r_i\big)+t^{\delta+\alpha_+}\Big)_+}\,\phi_t'(\lambda)\,\mathrm{d}\lambda.$$

Substitute in $r_i = t^{\frac{2}{3}}\zeta - \frac{2}{\sqrt{\beta}}V_i - \eta_{i-1}$, $V_i = t^{\frac{2}{3}}v_*((i-1)t^{\alpha-\frac{2}{3}})$, $\eta_{i-1} = (i-1)t^\alpha$, $\phi_t'(\lambda) = -t^{\frac{1}{3}}e^{-t^{\frac{1}{3}}\lambda}/(1+e^{-t^{\frac{1}{3}}\lambda})$, and perform a change of variables $t^{-\frac{2}{3}}\lambda \mapsto \lambda$. We then obtain

$$\begin{aligned} L\sum_{i=1}^{i_*}\int_{\mathbb{R}} t^{-2}M_i(\lambda,\kappa)\phi_t'(\lambda)\,\mathrm{d}\lambda &\geq -\frac{L}{\pi}\int_{\mathbb{R}}\frac{e^{-t\lambda}}{1+e^{-t\lambda}} \\ &\sum_{i=1}^{i_*}\sqrt{\Big(\big(1+\tfrac{\kappa+1}{\kappa^3}\big)\Big(\lambda+\zeta-\tfrac{2}{\sqrt{\beta}}v_*((i-1)t^{\alpha-\frac{2}{3}})-(i-1)t^{\alpha-\frac{2}{3}}\Big)+t^{-\frac{2}{3}+\delta+\alpha_+}\Big)_+}\;t^{\alpha-\frac{2}{3}}\,\mathrm{d}\lambda. \end{aligned}$$

As $t\to\infty$, the factor $\frac{e^{-t\lambda}}{1+e^{-t\lambda}} \to \mathbf{1}_{(-\infty,0)}(\lambda)$ for all $\lambda \neq 0$. Within the last sum, given that $\delta+\alpha_+ < \frac{2}{3}$, the term $t^{-\frac{2}{3}+\delta+\alpha_+}$ vanishes as $t\to\infty$. Ignoring this term, we recognize the sum as a Riemann sum of $\int_0^\infty \sqrt{(1+\frac{\kappa+1}{\kappa^3})(\lambda+\zeta-\frac{2}{\sqrt{\beta}}v_*(x)-x)_+}\,\mathrm{d}x$. Hence, upon taking the limit $t\to\infty$, we have

$$\begin{aligned} \liminf_{t\to\infty}\sum_{i=1}^{i_*}\int_{\mathbb{R}} t^{-2}M_i(\lambda,\kappa)\phi_t'(\lambda)\,\mathrm{d}\lambda &\geq -\frac{L}{\pi}\Big(1+\frac{\kappa+1}{\kappa^3}\Big)^{\frac{1}{2}}\int_{-\infty}^0\Big(\int_0^\infty\sqrt{(\lambda+\zeta-\tfrac{2}{\sqrt{\beta}}v_*(x)-x)_+}\,\mathrm{d}x\Big)\mathrm{d}\lambda \\ &= -\Big(1+\frac{\kappa+1}{\kappa^3}\Big)^{\frac{1}{2}}\int_0^\infty\frac{2L}{3\pi}\big((\zeta-\tfrac{2}{\sqrt{\beta}}v_*(x)-x)_+\big)^{\frac{3}{2}}\,\mathrm{d}x. \end{aligned} \tag{3.29}$$

Insert (3.27) and (3.29) into (3.23), and send $\kappa\to\infty$. We thus obtain

$$\liminf_{t\to\infty}(t^{-2}\log G) \geq -\int_0^\infty\big(\tfrac{1}{2}v_*^2(x)+\tfrac{2L}{3\pi}\big((\zeta-\tfrac{2}{\sqrt{\beta}}v_*(x)-x)_+\big)^{\frac{3}{2}}\big)\,\mathrm{d}x. \tag{3.30}$$

It is readily checked from (3.6) that $(\zeta - \frac{2}{\sqrt{\beta}} v_*(x) - x)_+ = (\frac{\sqrt{\beta}\pi}{2L} v_*(x))^2$. Using this to substitute the $\frac{3}{2}$-power in (3.30), after straightforward but tedious calculations, we arrive at the desired lower bound:

$$\liminf_{t\to\infty} (t^{-2} \log G) \geq -\int_0^\infty \Big(\frac{1}{2} v_*^2(x) + \frac{2L}{3\pi} \Big(\frac{\sqrt{\beta}\pi}{2L} v_*(x) \Big)^3 \Big) \, dx = -L \Big(\frac{2L}{\beta} \Big)^5 \Phi_- \Big(- \Big(\frac{\beta}{2L} \Big)^2 \zeta \Big). \tag{3.31}$$

3.2. **Upper bound.** First, from (1.6), it is readily checked that $\phi_t(\lambda) \geq t^{\frac{1}{3}} \lambda_-$. Using this, in (3.1) we replace $\phi_t(\boldsymbol{\lambda}_k(\mathcal{A}) - t^{\frac{2}{3}}\zeta)$ with $t^{\frac{1}{3}}(t^{\frac{2}{3}}\zeta - \boldsymbol{\lambda}_k(\mathcal{A}))_+$ to get

$$G \leq \mathbf{E}\Big[\exp\Big(-L \sum_{k=1}^\infty t^{\frac{1}{3}} (t^{\frac{2}{3}}\zeta - \boldsymbol{\lambda}_k(\mathcal{A}))_+ \Big) \Big] = \mathbf{E}\Big[\exp\Big(-L \int_{\mathbb{R}} t^{\frac{1}{3}} (t^{\frac{2}{3}}\zeta - \lambda)_+ \, dN(\lambda, \mathcal{A}) \Big) \Big].$$

In the integral in the last expression, perform integration by parts in λ and use $((t^{2/3}\zeta - \lambda)_+ N(\lambda, \mathcal{A}))|_{\lambda > t^{2/3}\zeta} = 0$ and $((t^{2/3}\zeta - \lambda)_+ N(\lambda, \mathcal{A}))|_{\lambda < \boldsymbol{\lambda}_1(\mathcal{A})} = 0$. The integral becomes $-t^{1/3} L \int_{-\infty}^{t^{2/3}\zeta} N(\lambda, \mathcal{A}) \, d\lambda$. Perform the change of variables $\lambda \mapsto \lambda + t^{2/3}\zeta$ and the decomposition (2.17). We have

$$G \leq \mathbf{E}\Big[\exp\Big(-t^{\frac{1}{3}} L \sum_{i=1}^{i_*+1} \int_{-\infty}^0 N_i(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}) \, d\lambda \Big) \Big] \leq \mathbf{E}\Big[\exp\Big(-t^{\frac{1}{3}} L \sum_{i=1}^{i_*} \int_{-\infty}^0 N_i(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A}) \, d\lambda \Big) \Big].$$

Within the last expression, apply the bounds from Lemma 2.6 to pass from $N_i(\lambda + t^{\frac{2}{3}}\zeta, \mathcal{A})$ to $N(\lambda - \eta_i + t^{\frac{2}{3}}\zeta, \mathcal{H}_{I_i})$. Since the processes $N(\bullet, \mathcal{H}_{I_i})$, $i = 1, \ldots, i_*$, are independent, the resulting bound factorizes

$$G \leq \prod_{i=1}^{i_*} G_i, \quad G_i := \mathbf{E}\Big[\exp\Big(-t^{\frac{1}{3}} L \int_{-\infty}^0 N(\lambda - \eta_i + t^{\frac{2}{3}}\zeta, \mathcal{H}_{I_i}) \, d\lambda \Big) \Big]. \tag{3.32}$$

Our next step is to bound each G_i in (3.32). Fix hereafter $i \in \{1, \ldots, i_*\}$, and, to simplify notation, we will often omit dependence on i in notation, e.g., $I = I_i$. To begin with, using

$$-t^{\frac{1}{3}} L \int_{-\infty}^0 N(\lambda + r, \mathcal{H}_I) \, d\lambda = -t^{\frac{1}{3}} L \int_{\mathbb{R}} (r - \lambda)_+ \, dN(\lambda, \mathcal{H}_I) = -t^{\frac{1}{3}} L \sum_{n=1}^\infty \big(r - \boldsymbol{\lambda}_n(\mathcal{H}_I) \big)_+, \tag{3.33}$$

we rewrite the term G_i as

$$G_i = \mathbf{E}\Big[\exp\Big(-t^{\frac{1}{3}} L \sum_{n=1}^\infty \big(t^{\frac{2}{3}}\zeta - \eta_i - \boldsymbol{\lambda}_n(\mathcal{H}_I) \big)_+ \Big]. \tag{3.34}$$

Recall that $\mathcal{H}_I$ is constructed with Dirichlet boundary condition. We will also need to consider operators with *periodic* and *Neumann* boundary conditions. To setup notation for this, identify $I = (\eta_{i-1}, \eta_i]$ with the torus $\mathbb{T} := \mathbb{R}/(|I|\mathbb{Z})$, and consider the Hilbert spaces $H^1(\mathbb{T})$ and $H^1(I)$. It is standard to check that Q_B (defined in (2.4) for $J = B$) defines a coercive form, both with respect to $H^1(\mathbb{T}) \subset L^2(I)$ and with respect to $H^1(I) \subset L^2(I)$. Given this, we let $\mathcal{H}_{\mathbb{T}}$ and $\mathcal{H}_{\mathrm{Neu}}$ be the associated operators of Q_B with respect to $H^1(\mathbb{T}) \subset L^2(I)$ and $H^1(I) \subset L^2(I)$, respectively:

$$\mathcal{H}_{\mathbb{T}} := -\frac{d^2}{dx^2} + \frac{2}{\sqrt{\beta}} B'(x), \quad x \in \mathbb{T},$$
$$\mathcal{H}_{\mathrm{Neu}} := -\frac{d^2}{dx^2} + \frac{2}{\sqrt{\beta}} B'(x), \quad x \in I, \text{ with Neumann B.C.}$$

Remark 3.1. At first glance, it may seem that the Hilbert space $\mathscr{V} = H^1(I)$ for $\mathcal{H}_{\mathrm{Neu}}$ does not capture Neumann boundary condition, but in fact any eigenfunction g of $\mathcal{H}_{\mathrm{Neu}}$ does satisfy $g'(\eta_{i-1}) = g'(\eta_i) = 0$. To see this, consider an eigenvalue problem for $\mathcal{H}_{\mathrm{Neu}}$: a given function $g \in H^1(I)$ and $\lambda \in \mathbb{R}$ satisfying

$$\int_I \Big(\frac{1}{2} g'(x) p'(x) + \frac{2}{\sqrt{\beta}} g(x) p(x) B'(x) - \lambda g(x) p(x) \Big) dx = 0, \quad \forall p \in H^1(I). \tag{3.35}$$

Given that B is a-Hölder continuous for $a < \frac{1}{2}$, it is standard to show that g' is also a-Hölder continuous for $a < \frac{1}{2}$, so in particular $g'(\eta_{i-1})$ and $g'(\eta_i)$ are well-defined. Now, for the test function $p(x) = p_\delta(x) := (1 - \delta^{-1}(x - \eta_{i-1}))_+$, using $g, g' \in C(I)$, it is readily checked that

$$\lim_{\delta\to 0} \int_I g'(x) p_\delta'(x) dx = -g'(\eta_{i-1}),$$

$$\lim_{\delta\to 0}\int_I g(x)p_\delta(x)\mathrm{d}x = 0,$$

$$\lim_{\delta\to 0}\int_I g(x)p_\delta(x)B'(x)\mathrm{d}x := \lim_{\delta\to 0}\Big(g(x)p_\delta(x)B(x)\big|_{\eta_{i-1}}^{\eta_i} - \int_I \big(g'(x)p_\delta(x)+g(x)p'_\delta(x)\big)B(x)\mathrm{d}x\Big)$$
$$= -g(\eta_{i-1})B(\eta_{i-1}) + g(\eta_{i-1})B(\eta_{i-1}) = 0.$$

Combining these properties with (3.35) yields $g'(\eta_{i-1}) = 0$. A similar procedure applied to the test function $(1-\delta^{-1}(\eta_i - x))_+$ yields $g'(\eta_i) = 0$.

To bound the r.h.s. of (3.34), our first step is to pass from $\mathcal{H}_I$ to $\mathcal{H}_{\mathbb{T}}$ and $\mathcal{H}_{\mathrm{Neu}}$.

Lemma 3.2. *Almost surely for all $r \in \mathbb{R}$,*

$$-\sum_{n=1}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_I)\big)_+ \le \big(r - \boldsymbol{\lambda}_1(\mathcal{H}_{Neu})\big)_+ - \sum_{n=1}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})\big)_+. \tag{3.36}$$

Remark 3.3. The following proof actually shows that

$$-\sum_{n=1}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_I)\big)_+ \le \big(r - \boldsymbol{\lambda}_1(\mathcal{H}_{\mathbb{T}})\big)_+ - \sum_{n=1}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})\big)_+,$$

and then uses $\boldsymbol{\lambda}_1(\mathcal{H}_{\mathbb{T}}) \ge \boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}})$ (explained in the proof) to get (3.36). The reason for going from $\boldsymbol{\lambda}_1(\mathcal{H}_{\mathbb{T}})$ to $\boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}})$ is because the latter is easier to bound (by the Riccati transform, as done in Lemma 3.5).

Proof. Fix a mollifier q, namely $q \in C^\infty(\mathbb{R})$, supported in $(-1,1)$, $q \ge 0$, and $\int_{\mathbb{R}} q(x)\mathrm{d}x = 1$. For $\varepsilon > 0$, mollify the Brownian motion $B_\varepsilon(x) := \int_{\mathbb{R}} q(\varepsilon^{-1}y)B(x-y)\varepsilon^{-1}\mathrm{d}y \in C^\infty(I)$. Accordingly, let $\mathcal{H}_{I,\varepsilon}$ and $\mathcal{H}_{\mathbb{T},\varepsilon}$ be the associated operators of Q_{B_ε} with respect to $H^1(\mathbb{T}) \subset L^2(I)$ and $H^1(I) \subset L^2(I)$, respectively. A classical result [CL55, Equation (3.15), Proof of Theorem 8.3.1] of Sturm–Liouville theory asserts that, for operators of the form (2.5) with piecewise continuous $J'(x)$, the eigenvalues under Dirichlet and under periodic boundary conditions interlace. Applying this result with $J = B_\varepsilon$ gives

$$-\infty < \boldsymbol{\lambda}_1(\mathcal{H}_{\mathbb{T},\varepsilon}) \le \boldsymbol{\lambda}_1(\mathcal{H}_{I,\varepsilon}) \le \boldsymbol{\lambda}_2(\mathcal{H}_{\mathbb{T},\varepsilon}) \le \boldsymbol{\lambda}_2(\mathcal{H}_{I,\varepsilon}) \le \boldsymbol{\lambda}_3(\mathcal{H}_{\mathbb{T},\varepsilon}) \le \boldsymbol{\lambda}_3(\mathcal{H}_{I,\varepsilon}) \le \ldots \to \infty. \tag{3.37}$$

Our next step is to pass (3.37) to the limit $\varepsilon \to 0$. Indeed, almost surely for all $\varepsilon \in (0,1)$, we have $\sup_{x\in I}|B_\varepsilon(x)| \le \sup_{x\in[\eta_{i-1}-1,\eta_i+1]}|B(x)| < \infty$. Also, as $\varepsilon \to 0$, we have $\sup_{x\in I}|B_\varepsilon(x) - B(x)| \to_{\mathrm{P}} 0$. Given these properties, apply the bounds from Lemma 2.2 with $(J_1, J_2) = (B, B_\varepsilon)$ and with $(J_1, J_2) = (B_\varepsilon, B)$. Sending $\varepsilon \to 0$ and $\kappa \to \infty$ in order, we obtain that $\boldsymbol{\lambda}_n(\mathcal{H}_{I,\varepsilon}) \to_{\mathrm{P}} \boldsymbol{\lambda}_n(\mathcal{H}_I)$, for any $n \in \mathbb{N}$ as $\varepsilon \to \infty$. A similar argument applied to periodic boundary condition gives $\boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T},\varepsilon}) \to_{\mathrm{P}} \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})$. Now taking the limit $\varepsilon \to 0$ in (3.37) gives

$$-\infty < \boldsymbol{\lambda}_1(\mathcal{H}_{\mathbb{T}}) \le \boldsymbol{\lambda}_1(\mathcal{H}_I) \le \boldsymbol{\lambda}_2(\mathcal{H}_{\mathbb{T}}) \le \boldsymbol{\lambda}_2(\mathcal{H}_I) \le \boldsymbol{\lambda}_3(\mathcal{H}_{\mathbb{T}}) \le \boldsymbol{\lambda}_3(\mathcal{H}_I) \le \ldots \to \infty. \tag{3.38}$$

The interlacing condition (3.38) gives, for any $r \in \mathbb{R}$,

$$-\sum_{n=1}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_I)\big)_+ \le -\sum_{n=2}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})\big)_+ = \big(r - \boldsymbol{\lambda}_1(\mathcal{H}_{\mathbb{T}})\big)_+ - \sum_{n=1}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})\big)_+. \tag{3.39}$$

On the other hand, since $H^1(\mathbb{T}) \subset H^1(I)$, applying the minimax principle (2.7) for $k = 1$ and for $T = \mathcal{H}_{\mathbb{T}}, \mathcal{H}_{\mathrm{Neu}}$, we have $\boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}}) \le \boldsymbol{\lambda}_1(\mathcal{H}_{\mathbb{T}})$. Using this in (3.39) to bound $(r - \boldsymbol{\lambda}_1(\mathcal{H}_{\mathbb{T}}))_+ \le (r - \boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}}))_+$, we conclude the desired result. □

We now direct our attention to the last sum in (3.36). The next proposition is the key step of the proof, c.f., the first and fourth paragraphs in Section 1.2.

Proposition 3.4. *Set $\lambda^*_n := (2\pi|I|^{-1}\lfloor\frac{n}{2}\rfloor)^2$. Almost surely for all $r \in \mathbb{R}$,*

$$-\sum_{n=1}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})\big)_+ \le -\sum_{n=1}^{\infty}\Big(r - \frac{2}{\sqrt{\beta}}\frac{B(\eta_i) - B(\eta_{i-1})}{|I|} - \lambda^*_n\Big)_+. \tag{3.40}$$

Proof. The readily checked identity that 'removes the +' will be useful:

$$-\sum_{n=1}^{\infty}(x_n)_+ = -\sup_{m\in\mathbb{Z}_{\geq 0}}\Big\{\sum_{n=1}^{m}x_n\Big\} = \inf_{m\in\mathbb{Z}_{\geq 0}}\Big\{-\sum_{n=1}^{m}x_n\Big\}, \qquad \text{for any } \infty > x_1 \geq x_2 \geq x_3 \geq \ldots, \tag{3.41}$$

with the convention that the empty sum is zero. Now, consider the Fourier basis of $L^2(\mathbb{T})$:

$$f_1(x) := |I|^{-\frac{1}{2}}, \quad f_{2k}(x) := |I|^{-\frac{1}{2}}e^{\mathbf{i}\frac{2\pi x}{|I|}},\ f_{2k+1}(x) := |I|^{-\frac{1}{2}}e^{-\mathbf{i}\frac{2\pi x}{|I|}}, \quad k = 1, 2, \ldots.$$

Set $b := \frac{1}{|I|}(B(\eta_i) - B(\eta_{i-1}))$ to simplify notation. Insert these vectors f_n into the form Q_B (defined in (2.4) for $J = B$) and sum the result over $n = 1, \ldots, m$. Within the result, recognize $|f_n(x)|^2 \equiv \frac{1}{|I|}$ and $\int_{\eta_{i-1}}^{\eta_i}|f_n(x)|^2 B'(x)\mathrm{d}x = \frac{1}{|I|}\int_{\eta_{i-1}}^{\eta_i}B'(x)\mathrm{d}x = b$. We have

$$\sum_{n=1}^{m}Q_{\mathcal{H}_{\mathbb{T}}}(f_n, f_n) = \sum_{n=1}^{m}\Big(\int_{\mathbb{T}}|f_n'(x)|^2\mathrm{d}x + \frac{2}{\sqrt{\beta}}\int_{\eta_{i-1}}^{\eta_i}|f_n(x)|^2B'(x)\mathrm{d}x\Big) = \sum_{n=1}^{m}\big(\lambda_n^* + \tfrac{2}{\sqrt{\beta}}b\big).$$

Since $\{f_1, \ldots, f_m\} \subset H^1(\mathbb{T})$ is orthonormal in $L^2(\mathbb{T})$, Lemma 2.3 gives $\sum_{n=1}^{m}\boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}}) \leq \sum_{n=1}^{m}(\lambda_n^* + \frac{2}{\sqrt{\beta}}b)$, or equivalently

$$-\sum_{n=1}^{m}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})\big) \leq -\sum_{n=1}^{m}\Big(r - \tfrac{2}{\sqrt{\beta}}b - \lambda_n^*\Big).$$

Applying (3.41) with $x_n = r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})$, we have

$$-\sum_{n=1}^{\infty}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})\big)_+ \leq -\sum_{n=1}^{m}\big(r - \boldsymbol{\lambda}_n(\mathcal{H}_{\mathbb{T}})\big) \leq -\sum_{n=1}^{m}\Big(r - \tfrac{2}{\sqrt{\beta}}b - \lambda_n^*\Big),$$

for any $m \in \mathbb{Z}_{\geq 0}$. Since this holds for all $m \in \mathbb{Z}_{\geq 0}$, optimizing over m, and then applying (3.41) with $x_n = r - \frac{2}{\sqrt{\beta}}b - \lambda_n^*$ in reverse, we conclude the desired result. □

Write $|I|^{-1}(B(\eta_i) - B(\eta_{i-1})) := t^{-\frac{\alpha}{2}}Z$, so that Z is a standard Gaussian. Recall the given expression (3.34) of G_i. Combine Lemma 3.2 with Proposition 3.4 for $r = t^{\frac{2}{3}}\zeta - \eta_i$. Multiply the result by $t^{\frac{1}{3}}L$, exponentiate, and take $\mathbf{E}[\,\cdot\,]$. With $(r - \boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}}))_+ \leq r_+ + (\boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}}))_- \leq ct^{2/3} + (\boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}}))_-$, we have

$$G_i \leq \mathbf{E}\Big[\exp\Big(t^{\frac{1}{3}}L\Big(ct^{2/3} + (\boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}}))_-\Big) - t^{\frac{1}{3}}L\sum_{n=1}^{\infty}\Big(t^{\frac{2}{3}}\zeta - \eta_i - \tfrac{2}{\sqrt{\beta}}t^{-\frac{\alpha}{2}}Z - \lambda_n^*\Big)_+\Big)\Big].$$

Fix an auxiliary parameter $\kappa \in [1, \infty)$. To separate terms within the last expression, we apply Hölder's inequality with exponents $\kappa + 1$ and $\frac{\kappa+1}{\kappa}$ to get

$$G_i \leq e^{ctL}G_{i,1}^{\frac{1}{\kappa+1}}G_{i,2}^{\frac{\kappa}{\kappa+1}}, \tag{3.42}$$

where

$$G_{i,1} := \mathbf{E}\Big[\exp\Big(L\,t^{\frac{1}{3}}(\kappa+1)(\boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}}))_-\Big)\Big],$$

$$G_{i,2} := \mathbf{E}\Big[\exp\Big(-t^{\frac{1}{3}}L\frac{\kappa+1}{\kappa}\sum_{n=1}^{\infty}\big(t^{\frac{2}{3}}\zeta - \eta_i - \tfrac{2}{\sqrt{\beta}}t^{-\frac{\alpha}{2}}Z - \lambda_n^*\big)_+\Big)\Big].$$

We now proceed to bound the terms $G_{i,1}$ and $G_{i,2}$.

Lemma 3.5. *For all* $t \geq 1$, *we have* $\log(G_{i,1}) \leq c\,(\kappa+1)^3 t$.

The proof of Lemma 3.5 goes through a series of comparison argument for Riccati-type ODE's. As the argument is somewhat disconnected from the rest of the proof, to avoid breaking the flow, we postpone proving Lemma 3.5 until the end of this subsection. As for the term $G_{i,2}$, recall the definition of v_* from (3.6).

Lemma 3.6. *For all* $\kappa > 0$ *and* $t < \infty$,

$$\log G_{i,2} \leq -t^{\alpha+\frac{4}{3}}\Big(\frac{2L}{3\pi}\big(\zeta - t^{-\frac{2}{3}}\eta_i - \tfrac{2}{\sqrt{\beta}}v_*(t^{-\frac{2}{3}}\eta_i)\big)^{\frac{3}{2}} + \frac{1}{2}v_*^2(t^{-\frac{2}{3}}\eta_i)\Big) + c\,\kappa^{-1}(\kappa+1)^3t^{\frac{1}{3}-3\alpha}.$$

Proof. Recall that $\lambda^*_n := (2\pi|I|^{-1}\lfloor\frac{n}{2}\rfloor)^2$. Forgoing the first eigenvalue λ^*_1, we write

$$-\sum_{n=1}^{\infty}(r-\lambda^*_n)_+ \leq -\sum_{n=2}^{\infty}(r-\lambda^*_n)_+ = -\sum_{k=1}^{\infty}2(r-4\pi^2|I|^{-2}k^2)_+.$$

Since $(r-4\pi^2|I|^{-2}x^2)_+$ is a decreasing function of x for $x\geq 0$, comparing sums to integrals gives, for $y_0 := 4\pi|I|^{-1}$,

$$-\sum_{n=1}^{\infty}(r-\lambda^*_n)_+ \leq -2\int_2^{\infty}(r-4\pi^2|I|^{-2}x^2)_+\mathrm{d}x = \frac{|I|}{\pi}\Big(-\frac{2}{3}r^{\frac{3}{2}}+ry_0-\frac{1}{3}y_0^3\Big)\mathbf{1}_{\{r>y_0^2\}}.$$

Within the last expression, drop the $-\frac{1}{3}y_0^3$ term, and split $-\frac{2}{3}r^{\frac{3}{2}}$ into 'two pieces' to get

$$-\sum_{n=1}^{\infty}(r-\lambda^*_n)_+ \leq \frac{|I|}{\pi}\Big(-\frac{2\kappa}{3(1+\kappa)}r^{\frac{3}{2}}-\frac{2}{3(1+\kappa)}r^{\frac{3}{2}}+ry_0\Big)\mathbf{1}_{\{r>y_0^2\}}.$$

Consider separately the cases $\frac{2}{3(1+\kappa)}r^{\frac{3}{2}}\geq ry_0$ and $y_0^2<\frac{2}{3(1+\kappa)}r^{\frac{3}{2}}<ry_0$. In the former case $-\frac{2}{3(1+\kappa)}r^{\frac{3}{2}}+ry_0\leq 0$; in the latter case $r<c\,(1+\kappa)^2y_0^2$, which gives $(-\frac{2}{3(1+\kappa)}r^{\frac{3}{2}}+ry_0)\mathbf{1}_{\{r>y_0^2\}}\leq c\,(1+\kappa^2)y_0^3$. Hence,

$$-\sum_{n=1}^{\infty}(r-\lambda^*_n)_+ \leq \frac{|I|}{\pi}\Big(-\frac{2\kappa}{3(1+\kappa)}r_+^{\frac{3}{2}}+c\,(1+\kappa)^2y_0^3\Big) \leq -\frac{2\kappa|I|}{3(1+\kappa)\pi}r_+^{\frac{3}{2}}+c\,(1+\kappa)^2|I|^{-3}. \tag{3.43}$$

Within (3.43), substitute $r=t^{\frac{2}{3}}\zeta-\eta_i-\frac{2}{\sqrt{\beta}}t^{-\frac{\alpha}{2}}Z$ and $|I|=t^\alpha$, multiply the result by $t^{\frac{1}{3}}\frac{\kappa+1}{\kappa}L$, exponentiate, and take $\mathbf{E}[\,\cdot\,]$. We have

$$G_{i,2}\leq e^{c\,\kappa^{-1}(\kappa+1)^3t^{\frac{1}{3}-3\alpha}}\mathbf{E}\Big[\exp\Big(-\frac{2L}{3\pi}t^{\frac{1}{3}+\alpha}\big(t^{\frac{2}{3}}\zeta-\eta_i-\tfrac{2}{\sqrt{\beta}}t^{-\frac{\alpha}{2}}Z\big)_+^{\frac{3}{2}}\Big)\Big]. \tag{3.44}$$

Recall that Z is a standard Gaussian. We then evaluate the expectation on the r.h.s. of (3.44) as

$$\int_{\mathbb{R}}\frac{e^{-F(y)}}{\sqrt{2\pi}}\mathrm{d}y, \quad F(y):=\frac{2L}{3\pi}t^{\frac{1}{3}+\alpha}\big(t^{\frac{2}{3}}\zeta-\eta_i-\tfrac{2}{\sqrt{\beta}}t^{-\frac{\alpha}{2}}y\big)_+^{\frac{3}{2}}+\frac{1}{2}y^2.$$

Indeed, F is C^∞ except at $y=y_{\mathrm{c}}$ where $t^{\frac{2}{3}}\zeta-\eta_i-\frac{2}{\sqrt{\beta}}t^{-\frac{\alpha}{2}}y_{\mathrm{c}}=0$, and at y_{c}, F is still C^1. Given these properties, straightforward differentiations show that $F(y)$ reaches its global minimum at $y_*:=t^{\frac{2}{3}+\frac{\alpha}{2}}v_*(t^{-\frac{2}{3}}\eta_i)$, and $F''(y)\geq 1$ expect at $y=y_{\mathrm{c}}$. Consequently, $F(y)\geq F(y_*)+\frac{1}{2}(y-y_*)^2$, which gives

$$\int_{\mathbb{R}}\frac{e^{-F(v)}}{\sqrt{2\pi}}\mathrm{d}v\leq\exp(-F(v_*))=\exp\Big(-t^{\alpha+\frac{4}{3}}\Big(\frac{2L}{3\pi}\big(\zeta-t^{-\frac{2}{3}}\eta_i-\tfrac{2}{\sqrt{\beta}}v_*(t^{-\frac{2}{3}}\eta_i)\big)_+^{\frac{3}{2}}+\frac{1}{2}v_*^2(t^{-\frac{2}{3}}\eta_i)\Big)\Big).$$

Combining this with (3.44) gives the desired result. □

Now, rewrite (3.32) and (3.42) as $\log G\leq\sum_{i=1}^{i_*}\log G_i\leq cLti_*+\sum_{i=1}^{i_*}(\frac{1}{\kappa+1}\log G_{i,1}+\frac{\kappa}{\kappa+1}\log G_{i,2})$. Then, insert the bounds from Lemmas 3.5–3.6, and divide the result by t^2. With $i_*\leq ct^{\frac{2}{3}-\alpha}$ and L being a fixed constant, we arrive at

$$t^{-2}\log G\leq c\Big(t^{-\frac{1}{3}-\alpha}+(\kappa+1)^2t^{-\frac{1}{3}-\alpha}+(\kappa+1)^2t^{-1-3\alpha}\Big) \tag{3.45a}$$

$$-\sum_{i=1}^{i_*}\frac{\kappa}{\kappa+1}\Big(\frac{2L}{3\pi}\big(\zeta-t^{-\frac{2}{3}}\eta_i-\tfrac{2}{\sqrt{\beta}}v_*(t^{-\frac{2}{3}}\eta_i)\big)_+^{\frac{3}{2}}+\frac{1}{2}v_*^2(t^{-\frac{2}{3}}\eta_i)\Big)t^{-\frac{2}{3}+\alpha}. \tag{3.45b}$$

Given that $\alpha\in(-\frac{1}{3},\frac{2}{3})$, the r.h.s. of (3.45a) vanishes as $t\to\infty$. Recognizing the term in (3.45b) as a Riemann sum (as done in Section 3.1), sending $t\to\infty$ and $\kappa\to\infty$ in order, we obtain

$$\limsup_{t\to\infty}\big(t^{-2}\log G\big)\leq-\int_0^{\infty}\frac{2L}{3\pi}\Big(\big(\zeta-x-\tfrac{2}{\sqrt{\beta}}v_*(x)\big)_+^{\frac{3}{2}}+\frac{1}{2}v_*^2(x)\Big)\mathrm{d}x.$$

The last expression matches the previously established lower bound (3.30). The proof is now completed upon settling Lemma 3.5.

Proof of Lemma 3.5. Throughout the proof, we write $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_1(\mathcal{H}_{\mathrm{Neu}})$ to simplify notation. Recall that i indexes which interval $I = I_i$ we are considering. The law of $\boldsymbol{\lambda}_1$ is clearly independent of i, so, without lost of generality, we take $i = 1$, and $I = I_1 = (0, \eta_1]$.

The proof amounts to establishing a suitable tail bound on $(\boldsymbol{\lambda}_1)_-$. We achieve this by a series of comparison arguments of the Riccati equation (2.14). Recall that our discussion regarding (2.14) in Section 2 is *pathwise*, and holds for every realization (i.e., any $C[0, \eta_1]$ function) of B. On the other hand, within this proof we will also regard (2.14) as a Stochastic Differential Equation (SDE)

$$\mathrm{d}f(x) = (-\lambda - f^2(x))\mathrm{d}x + \tfrac{2}{\sqrt{\beta}}\mathrm{d}B(x), \tag{3.46}$$

and, accordingly, sometimes view f as a process. It is standard to check that f satisfies the strong Markov property. That is, letting $\mathscr{F}(x) := \sigma(B(y) : y \geq 0)$ denote the canonical filtration of B, and letting $f^a(x)$ denote the solution of (3.46) with initial condition $f(0) = a$, for any $\mathscr{F}$-stopping time τ, we have

$$f(\bullet + \tau) \stackrel{\text{law}}{=} f^{f(\tau)}(\bullet).$$

Let $f(x, \lambda)$ denote the solution of (2.14) with initial condition $f(0, \lambda) = 0$, and let $\tau(\gamma; g) := \inf\{x \in [0, \eta_1] : g(x) = \gamma\}$ denote the first hitting time of a given function g at level γ, with the convention that $\inf \emptyset := \infty$. To simplify notation we write $\tau_{\pm,s} := \tau(\pm\frac{1}{2}\sqrt{s}; f(\bullet, -s))$.

The proof is carried out in steps.

Step 1: truncation. This step of the proof follows similar arguments in [DV13]. In this step we establish a useful truncation bound (3.47) that allows use to restriction our attention to the band $f(x, -s) \in [-\frac{1}{2}\sqrt{s}, \frac{1}{2}\sqrt{s}]$. To setup notation, let

$$\Omega_{-+} := \{\tau_{-,s} < \tau_{+,s}\}, \quad \Omega_{+-} := \{\tau_{+,s} < \tau_{-,s}\}.$$

For $s \geq t^{\alpha_+}$, we aim at showing that

$$\mathbf{P}\big[\tau_{-,s} < \infty\big] \leq c\,\mathbf{P}\big[\{\tau_{-,s} < \infty\} \cap \Omega_{-+}\big]. \tag{3.47}$$

Decompose the l.h.s. of (3.47) into

$$\mathbf{P}\big[\tau_{-,s} < \infty\big] = \mathbf{P}\big[\tau_{-,s} < \infty,\ \Omega_{-+}\big] + \mathbf{P}\big[\tau_{-,s} < \infty,\ \Omega_{+-}\big]. \tag{3.48}$$

The last term in (3.48) encodes the probability that $f(x, -s)$, which starts at $f(0, -s) = 0$, first hits level $\frac{1}{2}\sqrt{s}$, and then hits level $-\frac{1}{2}\sqrt{s}$. Reinitiate the process $f(x, -s)$ at $x = \tau_{+,s}$. The strong Markov property gives $f(\bullet + \tau_{+,s}) \stackrel{\text{law}}{=} f_1(\bullet)$, where f_1 solves (3.46) for $\lambda = -s$ with the initial condition $f_1(0) = \frac{1}{2}\sqrt{s}$. This gives

$$\mathbf{P}\big[\tau_{-,s} < \infty,\ \Omega_{+-}\big] \leq \mathbf{P}\big[\tau(-\tfrac{1}{2}\sqrt{s}, f_1) < \infty\big]. \tag{3.49}$$

The r.h.s. of (3.49) encodes the probability that f_1, which starts at $f_1(0) = \frac{1}{2}\sqrt{s}$, hits level $-\frac{1}{2}\sqrt{s}$ within $x \in [0, \eta_1]$. This being the case, f_1 must also have hit level 0. Reinitiate the process $f_1(x)$ at $x = \tau(0; f_1)$. By the strong Markov property, we have $f_1(\bullet + \tau(0; f_1)) \stackrel{\text{law}}{=} f(\bullet, -s)$, so

$$\mathbf{P}\big[\tau_{-,s} < \infty,\ \Omega_{+-}\big] \leq \mathbf{P}\big[\tau(0, f_1) < \infty\big] \cdot \mathbf{P}\big[\tau_{-,s} < \infty\big].$$

Combining this with (3.48)–(3.49) now gives

$$\mathbf{P}\big[\tau_{-,s} < \infty\big] = (1 - R)^{-1}\mathbf{P}\big[\tau_{-,s} < \infty,\ \Omega_{-+}\big], \tag{3.50}$$

where $R := \mathbf{P}[\tau(0, f_1) < \infty]$.

We proceed to bound R. To this end, consider the event $D_0 := \{\sup_{x \in [0,\eta_1]} \frac{2}{\sqrt{\beta}}|B(x)| \geq \frac{1}{4}\sqrt{s}\}$. Recall that $f_1(0) = \frac{1}{2}\sqrt{s}$. Let $\tau_1^* := \sup\{x \in [0, \tau(0, f_1)] : f_1(x) \geq \frac{1}{2}\sqrt{s}\}$ be the last exit time of f_1 from the region above $\frac{1}{2}\sqrt{s}$ before f_1 hits level 0. Under the occurrence of $\{\tau(0, f_1) < \infty\}$, setting $(x_1, x_2) = (\tau_1^*, \tau(0, f_1))$ in (2.14') gives

$$\text{On } \{\tau(0, f_1) < \infty\}, \quad -\frac{\sqrt{s}}{2} = f_1(x)\Big|_{x=\tau_1^*}^{x=\tau(0,f_1)} = \int_{\tau_1^*}^{\tau(0,f_1)} (s - f_1^2(x))\mathrm{d}x + \frac{2}{\sqrt{\beta}}B(x)\Big|_{x=\tau_1^*}^{x=\tau(0,f_1)}.$$

On the r.h.s., the integral is nonnegative since $(s - f_1^2(x)) \geq \frac{3}{4}s \geq 0$ for $x \in [\tau_1^*, \tau(0, f_1)]$. This gives

$$\big\{\tau(0, f_1) < \infty\big\} \subset \Big\{\tfrac{2}{\sqrt{\beta}}B(x)\big|_{x=\tau_1^*}^{x=\tau(0,f_1)} \leq -\tfrac{1}{2}\sqrt{s}\Big\} \subset D_0.$$

and hence $R := \mathbf{P}[\tau(0, f_1) < \infty] \le \mathbf{P}[D_0]$. Under the assumption $s \ge t^{\alpha+}$, together with $\eta_1 = t^\alpha$, it is readily checked that $\mathbf{P}[D_0] \le \frac{1}{c+1}$, for all $t \ge 1$. Hence $R \le \frac{1}{c+1}$. Inserting this bound into (3.50) gives (3.47).

Step 2: Reduction to Brownian exit probability. Fix $s \ge t^{\alpha + \vee(-2\alpha)}$. Our goal in this step is to bound the tail probability $\mathbf{P}[\boldsymbol{\lambda}_1 < -s]$. To begin with, consider the associated eigenfunction g_* of $\boldsymbol{\lambda}_1$. Taking the real part of g_* if necessary, we may assume that g_* is $\mathbb{R}$-valued. Referring to Remark 3.1, we have that g_* is in fact C^1 with $g_*'(0) = g_*'(\eta_1) = 0$. The Riccati transform $f_* := g_*'/g_*$ furnishes a solution of (2.14) for $\lambda = \boldsymbol{\lambda}_1$ such that $f_*(0) = f_*(\eta_1) = 0$. On the event $\{\boldsymbol{\lambda}_1 < -s\}$ under current consideration, Proposition 2.5(a) asserts that $f(x, -s) \le f_*(x)$, $\forall x \in I$, under the ordering described in Section 2. Consequently, either $f(x, -s)$ hits the level $-\frac{1}{2}\sqrt{s}$ (which gives $\tau_{-,s} < \infty$), or, if not, $f(\eta_1, -s) \le 0$. This gives

$$\mathbf{P}\big[\boldsymbol{\lambda}_1 < -s\big] = \mathbf{P}\big[\tau_{-,s} < \infty\big] + \mathbf{P}\big[\tau_{-,s} = \infty,\ f(\eta_1, -s) \le 0\big].$$

Applying (3.47) to the first term on the r.h.s., we have

$$\mathbf{P}\big[\boldsymbol{\lambda}_1 < -s\big] \le c\,\mathbf{P}\big[\Omega_1\big] + \mathbf{P}\big[\Omega_2\big], \tag{3.51}$$

where $\Omega_1 := \{\tau_{-,s} < \infty\} \cap \Omega_{-+}$ and $\Omega_2 := \{\tau_{-,s} = \infty,\ f(\eta_1, -s) \le 0\}$.

The next step is to bound the probability on the r.h.s. of (3.51). Under the occurrence of Ω_1, set $(x_1, x_2) = (0, \tau_{-,s})$ and $\lambda = -s$ in (2.14') to get

$$\text{On } \Omega_1, \quad -\frac{\sqrt{s}}{2} = f(\tau_{-,s}) = \int_0^{\tau_{-,s}} (s - f^2(x))\mathrm{d}x + \frac{2}{\sqrt{\beta}} B(\tau_{-,s}).$$

Since $|f(x)| \le \frac{1}{2}\sqrt{s}$ for all $x \le \tau_{+,s} \wedge \tau_{-,s}$, here we have $\int_0^{\tau_{-,s}} (s - f^2(x))\mathrm{d}x \ge \frac{3}{4} s \tau_{-,s}$. This gives

$$\text{On } \Omega_1, \quad -\tfrac{\sqrt{s}}{2} - \tfrac{3}{4} s \tau_{-,s} \ge \tfrac{2}{\sqrt{\beta}} B(\tau_{-,s}). \tag{3.52}$$

Consider further the sub-events $\Omega_{1,\le} := \{\tau_{-,s} \le s^{-\frac{1}{2}}\} \cap \Omega_1$ and $\Omega_{1,>} := \{s^{-\frac{1}{2}} < \tau_{-,s} < \infty\} \cap \Omega_1$. Under the occurrence of $\Omega_{1,\le}$, forgoing the term $-\frac{3}{4} s \tau_{-,s}$ in (3.52) gives

$$\Omega_{1,\le} \subset \Big\{ \sup_{x \in [0, s^{-1/2}]} \frac{2}{\sqrt{\beta}} |B(x)| \ge \frac{\sqrt{s}}{2} \Big\} := D_1(s). \tag{3.53}$$

Under the occurrence of $\Omega_{1,>}$, forgoing the term $-\frac{\sqrt{s}}{2}$ in (3.52) gives

$$\Omega_{1,\le} \subset \Big\{ \sup_{x \ge s^{-1/2}} \frac{2}{\sqrt{\beta}} \frac{|B(x)|}{|x|} \ge \frac{3}{4} s \Big\} := D_2(s). \tag{3.54}$$

Consequently,

$$\mathbf{P}\big[\Omega_1\big] \le \mathbf{P}\big[D_1(s)\big] + \mathbf{P}\big[D_2(s)\big]. \tag{3.55}$$

Next we turn to bounding $\mathbf{P}[\Omega_2]$. Consider the last exit $\tau^{*,s} := \sup\{x \in [0, \eta_1] : f(x, -s) \ge \frac{1}{2}\sqrt{s}\}$ of $f(x, -s)$ from the region above $\frac{1}{2}\sqrt{s}$, with the convention that $\sup \emptyset := -\infty$. Under the occurrence of Ω_2, set $(x_1, x_2) = (0 \vee \tau^{*,s}, \eta_1)$ and $\lambda = -s$ in (2.14') to get

$$\text{On } \Omega_2, \quad -\frac{\sqrt{s}}{2} \mathbf{1}_{\{\tau^{*,s} \ge 0\}} \ge f(x)\Big|_{0 \vee \tau^{*,s}}^{\eta_1} = \int_{0 \vee \tau^{*,s}}^{\eta_1} (s - f^2(x))\mathrm{d}x + \frac{2}{\sqrt{\beta}} B(x)\Big|_{0 \vee \tau^{*,s}}^{\eta_1}.$$

Since $|f(x)| \le \frac{1}{2}\sqrt{s}$ for all $x \in [0 \vee \tau^{*,s}, \tau_{-,s}]$, here we have $\int_{0 \vee \tau^{*,s}}^{\eta_1} (s - f^2(x))\mathrm{d}x \ge \frac{3}{4} s(\eta_1 - 0 \vee \tau^{*,s})$. This gives

$$\text{On } \Omega_2, \quad -\tfrac{\sqrt{s}}{2} \mathbf{1}_{\{\tau^{*,s} \ge 0\}} - \tfrac{3}{4} s(\eta_1 - 0 \vee \tau^{*,s}) \ge \tfrac{2}{\sqrt{\beta}} B(x)\big|_{0 \vee \tau^{*,s}}^{\eta_1}. \tag{3.56}$$

Consider further the sub-events $\Omega_{2,\le} := \{\tau^{*,s} \le \eta_1 - s^{-\frac{1}{2}}\} \cap \Omega_2$ and $\Omega_{2,>} := \{\tau^{*,s} > \eta_1 - s^{-\frac{1}{2}}\} \cap \Omega_2$. Under the occurrence of $\Omega_{2,\le}$, forgoing the term $-\frac{\sqrt{s}}{2} \mathbf{1}_{\{\tau^{*,s} \ge 0\}}$ in (3.56) gives

$$\Omega_{2,\le} \subset \Big\{ \sup_{x \in [0, \eta_1 - s^{-1/2}]} \frac{2}{\sqrt{\beta}} \frac{|B(x) - B(\eta_1)|}{|x - \eta_1|} \ge \frac{3}{4} s \Big\} := \widetilde{D}_2(s).$$

Recall our current assumption $s \geq t^{\alpha_+\vee(-2\alpha)}$, which ensures that $s^{-\frac{1}{2}} \leq \eta_1$. Hence under the occurrence of $\Omega_{2,>}$, we necessarily have $\tau^{*,s} \geq 0$. Forgoing the term $-\frac{3}{4}s(\eta_1 - 0 \vee \tau^{*,s})$ in (3.56) gives

$$\Omega_{2,>} \subset \Big\{ \sup_{x\in[\eta_1-s^{-1/2},\eta_1]} \frac{2}{\sqrt{\beta}}|B(x)-B(\eta_1)| \geq \frac{\sqrt{s}}{2} \Big\} := \widetilde{D}_1(s).$$

Further, since $B(\bullet) - B(\eta_1) \overset{\text{law}}{=} B(\bullet - \eta_1)$, we have $\mathbf{P}[\widetilde{D}_1(s)] = \mathbf{P}[D_1(s)]$ and $\mathbf{P}[\widetilde{D}_2(s)] \leq \mathbf{P}[D_2(s)]$. The preceding discussion gives $\mathbf{P}[\Omega_2] \leq \mathbf{P}[D_1(s)] + \mathbf{P}\big[D_2(s)\big]$. Combining this with (3.55) and (3.51) gives

$$\mathbf{P}\big[\boldsymbol{\lambda}_1 < -s\big] \leq c\,\mathbf{P}\big[D_1(s)\big] + c\,\mathbf{P}\big[D_2(s)\big], \quad s \geq t^{\alpha_+\vee(-2\alpha)}. \tag{3.57}$$

Step 3: estimating Brownian exit probability. We now proceed to bound the r.h.s. of (3.57). Referring to the definition (3.53) of $D_1(s)$, it is readily checked that $\mathbf{P}[D_1(s)] \leq \exp(-\frac{1}{c}s^{\frac{3}{2}})$. As for $D_2(s)$ (defined in (3.54)), partition $[s^{-\frac{1}{2}},\infty)$ into intervals $S_k := [ks^{-\frac{1}{2}},(k+1)s^{-\frac{1}{2}})$, $k\in\mathbb{N}$ of length $s^{-\frac{1}{2}}$.

$$\mathbf{P}\big[D_2(s)\big] \leq \sum_{k=1}^{\infty}\mathbf{P}\Big[\sup_{x\in S_k}\frac{2}{\sqrt{\beta}}\frac{|B(x)|}{x} \geq \frac{3s}{4}\Big] \leq \sum_{k=1}^{\infty}\mathbf{P}\Big[\sup_{x\in[0,(k+1)s^{-1/2}]}|B(x)| \geq \frac{ks^{\frac{1}{2}}}{c}\Big] \leq \sum_{k=1}^{\infty}\exp\Big(-\frac{k^2s^{\frac{3}{2}}}{c\,(k+1)}\Big).$$

The last sum is bounded by $\exp(-\frac{1}{c}s^{-\frac{3}{2}})$ for all $s \geq 1$. Consequently,

$$\mathbf{P}\big[\boldsymbol{\lambda}_1 \leq -s\big] \leq \exp\big(-\tfrac{1}{c}s^{\frac{3}{2}}\big), \quad s \geq t^{\alpha_+\vee(-2\alpha)}. \tag{3.58}$$

Now, write

$$G_{i,1} = \mathbf{E}\big[e^{cL\,t^{\frac{1}{3}}(\kappa+1)(\boldsymbol{\lambda}_1)_-}\big] = \mathbf{P}\big[(\boldsymbol{\lambda}_1)_- \geq 0\big] + c\,t^{\frac{1}{3}}(\kappa+1)\int_0^\infty \mathbf{P}\big[(\boldsymbol{\lambda}_1)_- \geq s\big]e^{cL\,t^{\frac{1}{3}}(\kappa+1)s}\mathrm{d}s.$$

Indeed, $\mathbf{P}[(\boldsymbol{\lambda}_1)_- \geq 0] = 1$. For the last integral, bound $\mathbf{P}[(\boldsymbol{\lambda}_1)_- \geq s] \leq 1$ for $s \in [0, t^{\alpha_+\vee(-2\alpha)}]$, and use the bound (3.58) for $s > t^{\alpha_+\vee(-2\alpha)}$. With L being a fixed constant, we have

$$G_{i,1} \leq 1 + t^{\frac{1}{3}+\alpha_+\vee(-2\alpha)}(\kappa+1)e^{c(\kappa+1)t^{\frac{1}{3}+\alpha_+\vee(-2\alpha)}} + t^{\frac{1}{3}}(\kappa+1)e^{c(\kappa+1)^3t}.$$

With $\alpha \in (-\frac{1}{3},\frac{2}{3})$, the last term $\exp(c(\kappa+1)^3t)$ dominates for large t. From this we conclude the desired result: $\log(G_{i,1}) \leq c(\kappa+1)^3t$, for all $t \geq 1$. □

4. Proof of Theorem 1.1 and Corollary 1.4

Passing from Theorem 1.3 to Theorem 1.1 and Corollary 1.4 amounts to showing

Lemma 4.1. *Let X_t, $t>0$, be a sequence of $\mathbb{R}$-valued random variables, and let $b\in(0,\infty)$, $g\in C[0,\infty)$. If, for any fixed $\zeta\in(0,\infty)$ we have*

$$\lim_{t\to\infty}\frac{1}{t^2}\log\big(\mathbf{E}\big[\exp\big(-be^{X_t+t\zeta}\big)\big]\big) = g(\zeta), \tag{4.1}$$

then, for any fixed $\zeta\in(0,\infty)$,

$$\lim_{t\to\infty}\frac{1}{t^2}\log\big(\mathbf{P}\big[X_t < -t\zeta\big]\big) = g(\zeta).$$

Indeed, given the identity (1.5), Theorem 1.1 follows by combining Theorem 1.3 for $(\beta,L)=(2,1)$ and $g(\zeta) = -\Phi_-(-\zeta)$ and Lemma 4.1 for $X_t = h(2t,0)+\frac{t}{12}$ and $b=1$. Similarly, given (1.8), Corollary 1.4 follows by combining Theorem 1.3 for $(\beta,L)=(1,2)$ and $g(\zeta) = -\frac{1}{2}\Phi_-(-\zeta)$ and Lemma 4.1 for$X_t = h^{\mathrm{hf}}(2t,0)+\frac{t}{12}$ and $b=\frac{1}{4}$.

Proof of Lemma 4.1. Write $F(x) := \exp(-be^x)$ for the double exponential function. Fix $\delta\in(0,\zeta)$. We indeed have $F(x+\delta t) \leq \mathbf{1}_{\{x<0\}} + \exp(-be^{\delta t})$ and $F(x-\delta t) \geq \exp(-be^{-\delta t})\mathbf{1}_{\{x<0\}}$. From this we conclude that

$$\mathbf{P}\big[X_t < -t\zeta\big] + \exp(-be^{\delta t}) \geq \mathbf{E}\big[F(X_t + t(\zeta+\delta))\big], \tag{4.2}$$

$$e^{-be^{-\delta t}}\mathbf{P}\big[X_t < -t\zeta\big] \leq \mathbf{E}\big[F(X_t + t(\zeta-\delta))\big]. \tag{4.3}$$

Combining the given assumption (4.1) for $\zeta\mapsto\zeta+\delta$ with (4.2) gives, for all large enough t,

$$\mathbf{P}\big[X_t < -t\zeta\big] > \tfrac{1}{2}\exp(t^2g(\zeta+\delta)) - \exp(-be^{\delta t}).$$

On the r.h.s., the first term dominates as $t \to \infty$, *regardless* of the sign of $g(\zeta + \delta)$. Consequently, for all large enough t,

$$\mathbf{P}\big[X_t < -t\zeta\big] > \tfrac{1}{2}\exp(t^2 g(\zeta + \delta)). \tag{4.4}$$

Now, apply $\frac{1}{t^2}\log(\bullet)$ to both sides of (4.3)–(4.4), take $t \to \infty$ with the aid of (4.1), and take $\delta \downarrow 0$ and use the continuity of g. We conclude the desired result. ☐

References

[ACQ11] G. Amir, I. Corwin, and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in 1+ 1 dimensions. *Comm Pure Appl Math*, 64(4):466–537, 2011.

[AD14] R. Allez and L. Dumaz. Tracy–Widom at high temperature. *J Stat Phys*, 156(6):1146–1183, 2014.

[AGZ10] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices.* Cambridge University Press, Cambridge New York, 2010.

[BAG97] G. Ben Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab Theory Related Fields*, 108(4):517–542, 1997.

[BBCW18] G. Barraquand, A. Borodin, I. Corwin, and M. Wheeler. Stochastic six-vertex model in a half-quadrant and half-line open asymmetric simple exclusion process. *Duke Math J*, 2018.

[BBD08] J. Baik, R. Buckingham, and J. DiFranco. Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function. *Commun Math Phys*, 280(2):463–497, 2008.

[BC95] L. Bertini and N. Cancrini. The stochastic heat equation: Feynman-Kac formula and intermittence. *J Stat Phys*, 78(5-6):1377–1401, 1995.

[BG97] L. Bertini and G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Commun Math Phys*, 183(3):571–607, 1997.

[BG16] A. Borodin, , and V. Gorin. Moments match between the KPZ equation and the Airy point process. *SIGMA*, 12:102, 2016.

[CC19] M. Cafasso and T. Claeys. A Riemann–Hilbert approach to the lower tail of the KPZ equation. *To appear in Comm Pure Appl Math. arXiv:1910.02493*, 2019.

[CG+20] I. Corwin, P. Ghosal, et al. Lower tail of the KPZ equation. *Duke Math J*, 169(7):1329–1395, 2020.

[CGK+18] I. Corwin, P. Ghosal, A. Krajenbrink, P. Le Doussal, and L.-C. Tsai. Coulomb-gas electrostatics controls large fluctuations of the Kardar-Parisi-Zhang equation. *Phys Rev Lett*, 121(6):060201, 2018.

[CL55] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations.* Tata McGraw-Hill Education, 1955.

[CLDR10] P. Calabrese, P. Le Doussal, and A. Rosso. Free-energy distribution of the directed polymer at high temperature. *Europhys Lett*, 90(2):20002, 2010.

[Cor12] I. Corwin. The Karder-Parisi-Zhang equation and universality class. *Random Matrices: Theory Appl*, 01(01):1130001, 2012.

[CQ13] I. Corwin and J. Quastel. Crossover distributions at the edge of the rarefaction fan. *Ann Probab*, 41(3A):1243–1314, 2013.

[CRR07] S. Cambronero, J. Ramírez, and B. Rider. The Riccati map in random Schrodinger and random matrix theory. *Math Sci Res Inst Publ*, page 77, 2007.

[CS18] I. Corwin and H. Shen. Open ASEP in the weakly asymmetric regime. *Comm Pure Appl Math*, 71(10):2065–2128, 2018.

[CW17] A. Chandra and H. Weber. Stochastic PDEs, regularity structures, and interacting particle systems. *Annales de la faculté des sciences de Toulouse Sér. 6*, 26(4):847–909, 2017.

[DIK08] P. Deift, A. Its, and I. Krasovsky. Asymptotics of the Airy-Kernel determinant. *Commun Math Phys*, 278:643–678, 2008.

[DM06] D. S. Dean and S. N. Majumdar. Large deviations of extreme eigenvalues of random matrices. *Phys Rev Lett*, 97(16):160201, 2006.

[Dot10] V. Dotsenko. Bethe ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers. *Europhys Lett*, 90(2):20003, 2010.

[DT21] S. Das and L.-C. Tsai. Fractional moments of the stochastic heat equation. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 57(2):778–799, 2021.

[DV13] L. Dumaz and B. Virág. The right tail exponent of the Tracy–Widom β distribution. *Ann Inst H Poin*, 49(4):915–933, 2013.

[FN77] M. Fukushima and S. Nakao. On spectra of the Schrödinger operator with a white Gaussian noise potential. *Probab Theory Related Fields*, 37(3):267–274, 1977.

[FNS77] D. Forster, D. R. Nelson, and M. J. Stephen. Large-distance and long-time properties of a randomly stirred fluid. *Phys Rev A*, 16(2):732, 1977.

[FS11] P. Ferrari and H. Spohn. Random growth models. In J. B. G. Akemann and P. D. Francesco, editors, *Oxford Handbook of Random Matrix Theory.* Oxford University Press, 2011.

[GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. In *Forum of Mathematics, Pi*, volume 3, 2015.

[GJ14] P. Gonçalves and M. Jara. Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Arch Ration Mech Anal*, 212(2):597–644, 2014.

[GP18] M. Gubinelli and N. Perkowski. Energy solutions of KPZ are unique. *J Amer Math Soc*, 31(2):427–471, 2018.

[Gru08] G. Grubb. *Distributions and operators*, volume 252. Springer Science & Business Media, 2008.

[GS18] V. Gorin and S. Sodin. The KPZ equation and moments of random matrices. *Math Phys Anal Geom*, 14(3):286–296, 2018.

[Hai14] M. Hairer. A theory of regularity structures. *Invent Math*, 198(2):269–504, 2014.

[HHF85] D. A. Huse, C. L. Henley, and D. S. Fisher. Huse, Henley, and Fisher respond. *Phys Rev Lett*, 55(26):2924, 1985.

[HL18] Y. Hu and K. Lê. Asymptotics of the density of parabolic Anderson random fields. *arXiv:1801.03386*, 2018.

[HV15] D. Holcomb and B. Valkó. Large deviations for the Sine_β and Sch_τ processes. *Probab Theory Related Fields*, 163(1-2):339–378, 2015.

[HV17] D. Holcomb and B. Valkó. Overcrowding asymptotics for the Sine_β process. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 53, pages 1181–1195. Institut Henri Poincaré, 2017.

[KK07] I. Kolokolov and S. Korshunov. Optimal fluctuation approach to a directed polymer in a random medium. *Phys Rev B*, 75(14):140201, 2007.

[KLD19] A. Krajenbrink and P. Le Doussal. Linear statistics and pushed Coulomb gas at the edge of β-random matrices: Four paths to large deviations. *EPL (Europhysics Letters)*, 125(2):20009, 2019.

[KLDP18] A. Krajenbrink, P. Le Doussal, and S. Prolhac. Systematic time expansion for the Kardar–Parisi–Zhang equation, linear statistics of the GUE at the edge and trapped fermions. *Nucl Phys B*, 936:239–305, 2018.

[KPZ86] M. Kardar, G. Parisi, and Y.-C. Zhang. Dynamic scaling of growing interfaces. *Phys Rev Lett*, 56(9):889, 1986.

[LD20] P. Le Doussal. Large deviations for the Kardar–Parisi–Zhang equation from the Kadomtsev–Petviashvili equation. *J Stat Mech : Theory Exp*, 2020(4):043201, 2020.

[LDMS16] P. Le Doussal, S. N. Majumdar, and G. Schehr. Large deviations for the height in 1d Kardar-Parisi-Zhang growth at late times. *Europhys Lett*, 113(6):60004, 2016.

[MF14] G. R. Moreno Flores. On the (strict) positivity of solutions of the stochastic heat equation. *Ann Probab*, 42(4):1635–1643, 2014.

[MKV16] B. Meerson, E. Katzav, and A. Vilenkin. Large deviations of surface height in the Kardar-Parisi-Zhang equation. *Phys Rev Lett*, 116(7):070601, 2016.

[MN08] C. Mueller and D. Nualart. Regularity of the density for the stochastic heat equation. *Electron J Probab*, 13:2248–2258, 2008.

[Mue91] C. Mueller. On the support of solutions to the heat equation with noise. *Stochastics: An International Journal of Probability and Stochastic Processes*, 37(4):225–245, 1991.

[Olv97] F. Olver. *Asymptotics and special functions*. AK Peters/CRC Press, 1997.

[Par19] S. Parekh. The KPZ limit of ASEP with boundary. *Commun Math Phys*, 365(2):569–649, 2019.

[QS15] J. Quastel and H. Spohn. The one-dimensional KPZ equation and its universality class. *J Stat Phys*, 160(4):965–984, 2015.

[Qua11] J. Quastel. Introduction to KPZ. *Current developments in mathematics*, 2011(1), 2011.

[RRV11] J. Ramirez, B. Rider, and B. Virág. Beta ensembles, stochastic Airy spectrum, and a diffusion. *J Amer Math Soc*, 24(4):919–944, 2011.

[SMP17] P. Sasorov, B. Meerson, and S. Prolhac. Large deviations of surface height in the 1+ 1-dimensional Kardar–Parisi–Zhang equation: exact long-time results for $\lambda h < 0$. *J Stat Mech Theory Exp*, 2017(6):063203, 2017.

[SS10] T. Sasamoto and H. Spohn. One-dimensional Kardar-Parisi-Zhang equation: an exact solution and its universality. *Phys Rev Lett*, 104(23):230602, 2010.

[TW94] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Commun Math Phys*, 159(1):151–174, 1994.

[TW08] C. A. Tracy and H. Widom. A Fredholm determinant representation in ASEP. *J Stat Phys*, 132(2):291–300, 2008.

[TW09] C. A. Tracy and H. Widom. Asymptotics in ASEP with step initial condition. *Commun Math Phys*, 290(1):129–154, 2009.

[vBKS85] H. van Beijeren, R. Kutner, and H. Spohn. Excess noise for driven diffusive systems. *Phys Rev Lett*, 54(18):2026, 1985.

[Wal86] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'Été de Probabilités de Saint Flour XIV-1984*, pages 265–439. Springer, 1986.

[Zho19] C. Zhong. Large deviation bounds for the Airy point process. *arXiv:1910.00797*, 2019.

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