Quantization Rate and AoI-Induced Distortion Trade-off Analysis with Application to Remote Agents

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***Abstract*— In this paper, we consider a communication system where an agent is receiving update information from a source/controller through a wireless channel. In particular, we investigate the problem of optimal quantization considering a unified distortion measure caused by the quantization and the age of information (AoI). To this end, we propose an upperbound for the quantization distortion given an arbitrarily distributed stochastic process. We then provide a specific example of the above problem where the question is whether to send the commands (actions) or the command generator (action distribution) to the agent. We show that a command generator which is a probability distribution function (PDF) can be a more efficient policy when the cost of communication is of interest. We analyze the existing tradeoff between the rate of change of the update and the channel use. We show that since the faster the process varies, the larger the distortion becomes, it requires a higher quantization rate to meet the same level of distortion.**

***Index Terms*—Age of Information (AoI), quantization, rate-distortion, optimal policy, cost of communication.**

I. INTRODUCTION

Transmitting timely and fresh update information from sources (e.g., sensors or remote controllers) to the destinations (e.g., cloud or robots) is being vital due to the increasing demand for real-time communications in different applications from smart transportation to smart healthcare networks [1, 2]. From its advent in [3], age of information (AoI) has been recognized as a powerful metric to assess the freshness of the update data [4–8]. Ever since a rich state of the art has been developed towards finding the optimal tradeoffs with other performance metrics [9–19].

In the case of transmitting real-valued data, an interesting problem is to balance the tradeoff between the AoI and the distortion caused by the quantization. The tradeoff here is due to the fact that a larger distortion (small quantization levels and shorter codes) leads to faster communications and hence, smaller AoI and vice versa. So far, a few works have studied this problem [20–26].

In this regard, a basic problem was considered in [20] where distortion was simply modeled as a monotonically decreasing function of the processing time. Through a multi-objective optimization, the work in [21] considers minimizing time-averaged AoI and mean squared error (MSE) to obtain the optimal channel block length. The authors in [22, 23] consider a bit allocation problem in an ON/OFF channel to minimize the average receiver-side AOI under the distortion constraint. A triple tradeoff between AoI, distortion, and energy was considered in [24] where packet scheduling among multiple sources was investigated in order to maximize a linear combination of the three metrics. The AoI-distortion tradeoff was also investigated in [25] for an energy-harvesting sensor where the goal was to schedule the transmit power such that a weighted sum of AoI and distortion is minimized. Considering partial information rather than the original information, [26] aims to minimize the AoI at the receiver side while maintaining the mutual information between the partial and the original information at an acceptable level.

In this paper, we consider a system in which a remote agent is receiving update information regarding its path direction through a wireless channel. Instead of considering the AoI itself, we are interested in the distortion caused by the AoI so that to obtain a unified distortion measure. The contributions of this paper are as follows:

- In order to obtain a unified distortion measure, we are interested in the distortion caused by the AoI along with the distortion caused by the quantization. In particular, we consider the rate-of-change of the update information assumed to be a continuous stochastic process.
- We obtain an upperbound for the quantization distortion given a general distribution for the update information. Using this upperbound, we then obtain the minimum transmission rate, since the goal is to minimize the channel use while an acceptable level of distortion is guaranteed.
- After providing a general framework, we then consider a specific problem in which the question is whether to send a command or a command generator to a remote agent. For the command generator (called the stochastic policy) assumed to be uniformly distributed, using the bounds from rate-distortion theory, we obtain a tradeoff between

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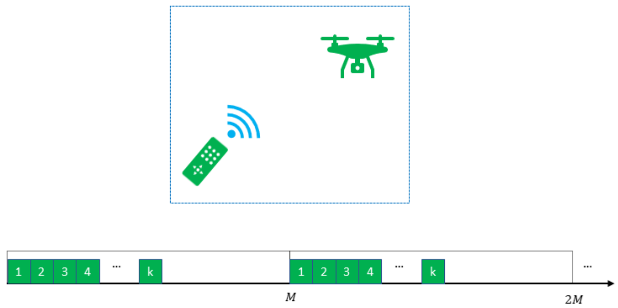

Fig. 1: A simple illustration of the system model and data transmission policy.

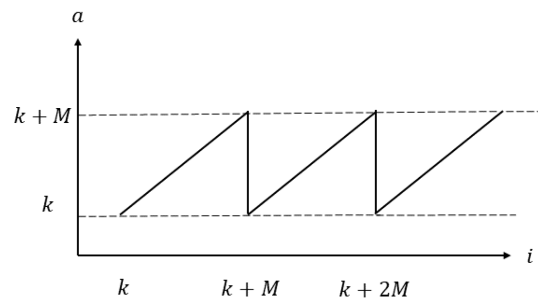

Fig. 2: AoI Evolution.

the communication cost and the overall distortion caused by quantization and AoI.

The organization of this paper is as follows: In Section II we provide the system model and preliminaries. Then, in Section III we present the distortion analysis and the rate minimization problem. Section IV investigates the example problem and Section V concludes the paper.

II. System Model

Figure 1 shows a simple illustration of the system model. We consider a time-slot based transmission where a controller sends the direction information or commands to an agent through wireless channels. We assume that the information at the n-th transmission is a stochastic process denoted by $X_n, n = 1, \dots \infty$ where $|X_n| \leq 1$[1]. We define the rate of change for X_n as $\Delta_X(m,n) = \mathbb{E}[(X_n - X_m)^2]$ where $\mathbb{E}[.]$ is the expectation function. Furthermore, the information is undergone quantization before being transmitted. Hence, at the receiver, X_n is estimated as $\hat{X}_n$. We assume that X_n is generated at the beginning of each $M \in \mathbb{R}$ time-slot and considering k bits quantization, it takes k time-slots for X_n to be completely received by the agent. Therefore, the AoI of X_n at time-slot j can be obtained as

$$a_{X_n}(j) = j - (n-1)M,$$
$$\forall j \in \{(n-1)M + k, nM + k\}$$

in which $(n-1)M$ is the generation time of the X_n.

Note that when the symbol X_n has been replaced by X_{n+1}, AoI of the system is obtained based on the age of X_{n+1} and not X_n anymore. Furthermore, we assume no delay in the wireless channel. Figure 2 shows the AoI evolution for the proposed system model.

We assume a noiseless channel where X_n is decoded correctly at the receiver. After transmitting one update, the transmitter then waits for $M - k$ time-slots to be passed and then starts transmitting another update. Hence, the data transmission is performed at each M time units. Now, with k bits transmission at each M time-slots, we can write the communication rate as

$$R = \frac{k}{M}. \tag{1}$$

The distortion due to the k-bit quantization is defined as

$$E_k = \mathbb{E}\left[(\hat{X}_n - X_n)^2\right]. \tag{2}$$

As noted earlier, the quantized data received at the receiver is estimated as $\hat{X}_n$. We show the process in which the data changes as below

$$X_n \xrightarrow{\text{Quantization \& Estimation}} \hat{X}_n \xrightarrow{\text{Aged}} \hat{X}_{n-a} = Y_n, \tag{3}$$

where $k \leq a \leq M$ is the age of symbol in progress. Now considering both the quantization and delay, we define an overall distortion measure denoted by D, as below

$$\begin{aligned} D &= \mathbb{E}\left[(Y_n - X_n)^2\right] \\ &= \mathbb{E}\left[(\hat{X}_{n-a} - X_n)^2\right] \\ &= \mathbb{E}\left[(\hat{X}_{n-a} - X_{n-a})^2\right] + \mathbb{E}\left[(X_{n-a} - X_n)^2\right] \\ &= E_k + \Delta_X(n, n-a), \end{aligned}$$

where $\Delta_X(n, n-a)$ is a non-decreasing function of a. In this paper, we assume that X_n changes in time according to a Brownian motion process with variance σ_B^2, i.e., $X_{n_1} - X_{n_2} \sim \mathcal{N}(0, \sigma_B^2(n_1 - n_2))$ [27], we have

$$\Delta_X(n, n-a) = a\sigma_B^2. \tag{4}$$

III. Distortion Analysis and Optimal Rate

In this section, we provide initial analysis on the overall distortion, D. To this end, we first obtain an upperbound for E_k through the following lemma.

Lemma 1. For any stochastic process X_n where $|X_n| \leq 1$, E_k is upperbounded as below:

$$E_k \leq 2^{-2k}. \tag{5}$$

Proof. For the proof, we use the uniform quantization shown in Figure 3 where assuming k bits quantization,

[1] In general, we can assume that the random variable is normalized.

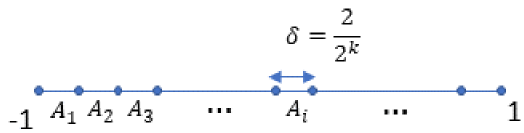

Fig. 3: The uniform quantization of a random process between [-1, 1].

we have 2^k quantization levels in the interval $[-1, 1]$. Hence, we have an interval lengths of $\delta = \frac{2}{2^k} = 2^{1-k}$. We also note that $\hat{X}_n = \mathbb{E}\left[X_n \mid A_i\right]$. Now, we calculate E_k as below:

$$
\begin{aligned}
E_k &= \mathbb{E}_i\left[\mathbb{E}\left[\left(X_n - \hat{X}_n\right)^2 \mid A_i\right]\right] \\
&= \mathbb{E}_i\left[\mathbb{E}\left[\left(X_n - \mathbb{E}\left[X_n \mid A_i\right]\right)^2 \mid A_i\right]\right] \\
&= \mathbb{E}_i\left[\mathrm{Var}((X_n - \hat{X}_n) \mid A_i)\right] \\
&\overset{(a)}{\leq} \mathbb{E}_i\left[\left(\frac{1}{2^k}\right)^2\right] \\
&\overset{(b)}{=} 2^{-2k},
\end{aligned}
$$

where (a) results from the fact that if $a_1 \leq X \leq a_2$ is an arbitrary random variable, then $\mathrm{Var}(X) \leq \frac{(a_2 - a_1)^2}{4}$ [27], where Var(.) is the variance function and (b) comes from the law of total expectation. □

Therefore, considering Lemma 1 and Equation (4), the overall distortion is written as

$$D \leq a\sigma_B^2 + 2^{-2k}. \tag{6}$$

We wish to obtain the optimal k and M such that an acceptable level of distortion is met. To do so, we set up an optimization problem to obtain the minimum data rate with the constraint of maximum allowable distortion as below:

$$
\begin{aligned}
&\min_{k,M} R, \\
&\text{s.t.} \quad D \leq d_{\text{th}}, \\
&\qquad\quad k \leq M,
\end{aligned} \tag{7}
$$

where d_{th} is the maximum allowable distortion and the last inequality comes from the system model assumption.

Before solving Problem (7), we first rewrite the first condition as below:

$$D \leq (M + k)\sigma_B^2 + 2^{-2k} \leq d_{\text{th}}, \tag{8}$$

since in the worst case scenario, we have $a = M + k$. Now we first choose the maximum allowable M from (8) as below

$$M^* = \left[\frac{d_{\text{th}} - 2^{-2k}}{\sigma_B^2} - k\right]^+, \tag{9}$$

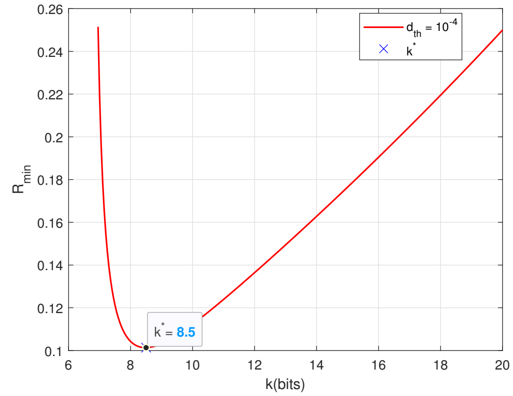

Fig. 4: R_{min} versus k, $d_{\text{th}} = 10^{-4}$ and $\sigma_B = 0.001$.

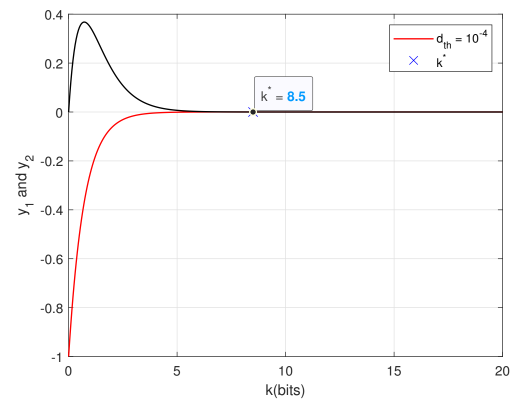

Fig. 5: Solution of Equation (11) for $d_{\text{th}} = 10^{-4}$.

where $[.]^+ = \max\left[., 0\right]$. We note that we should have $M^* > k$ for the sake of feasibility. Now substituting (9) in the objective function, i.e., $R = \frac{k}{M}$, we obtain

$$R_{\text{min}}(k) = \frac{k}{M^*} = \frac{k\sigma_B^2}{d_{\text{th}} - 2^{-2k} - k\sigma_B^2}. \tag{10}$$

Figure 4 is the schematic representation of R_{min} in (10) versus k for $d_{\text{th}} = 10^{-4}$. According to this figure, we can see that given M^*, there is an optimum k, denoted by k^*, as well that minimizes R. To obtain k^*, we take the derivative of (10) with respect to k and we obtain

$$2k2^{-2k}\ln 2 = d_{th} - 2^{-2k}. \tag{11}$$

Since, Equation (11) is not analytically solvable, in Figure 5 we plot the curves in the two sides of equation to show the value of k^* for d_{th}. In this figure, $k^* = 8.5$ shows the value of k for which the two curves intersect. This is consistent with the k^* shown in Figure 4.

Theorem 1. Problem (7) has a unique solution $\lceil k^* \rceil$ or

$\lfloor k^* \rfloor$ obtained from (11) for $d_{th} \geq d_{\min}$, where

$$d_{\min} = \sigma_B^2 \left[\frac{1}{\ln 2} - \log_2 \frac{\sigma_B^2}{\ln 2} \right]^+ . \quad (12)$$

Proof. We note that $R_{\min}(k)$ is convex for $k > -\frac{1}{2}\log_2 d_{\text{th}}$, for which $R_{\min} > 0$. Hence, the k^* obtained from (11) is the optimal k and, since $k \in \mathbb{N}$, we need to check that which one of the $\lceil k^* \rceil$ or $\lfloor k^* \rfloor$ results in the less value in $R_{\min}$. Finally, since in Problem (7), we have $k \leq M$, the minimum distortion with respect to M in Equation (8) is obtained when $M = k$. Therefore, (12) is obtained by substituting M with k in (8) that insures for any σ_B^2, there exists a minimum d_{th} for the larger values of which the solution is feasible. □

The problem investigated here was developed for a general random process. In the next section, provide an example in which we analytically answer a very basic question.

IV. Optimal Communication of Commands

In this section, we investigate a specific example of the above scenario. Motivated by the work in [28], where an autonomous agent is aimed to be controlled by a controller through transmitting a moving policy i.e., distribution of the directions, while considering the cost of communications, we investigate the following basic problem: From the cost of communication perspective, what is the efficient method to control the agent? Action commands or action distribution?

Consider a remote agent is being used in a mission by a controller. There are N actions that an agent can operate them at each time instant in order to accomplish its mission. The controller can either send the exact command (deterministic policy) or the distribution of the actions (stochastic policy). In the former, the controller sends the index of the action, $i \in \{1, 2, 3, \ldots, N\}$, while in the latter it sends the probability of actions, for example $p_i, 0 < p_i < 1, i = 1, 2, 3, \ldots, N$. The total mission time is assumed to be T seconds.

For the sake of simplicity, we assume that we only have two commands, for example, "right" and "left". As mentioned above, in the deterministic scenario, the controller sends the exact order. Therefore, assuming a binary scheme for transmission, the communication bit rate per unit of time for deterministic policy is simply

$$R_d = 1. \quad (13)$$

In the stochastic policy, we apply a different transmission regime: we consider a transmission period of M time units where $M \in \mathbb{R}$ in general. The reason is that when the distribution is sent, the agent can make decisions based on that for a while until it is being updated by the controller. Therefore, it is expected that the stochastic policy is transmitted less frequently than the deterministic commands. Specifically, as $M \to \infty$, it

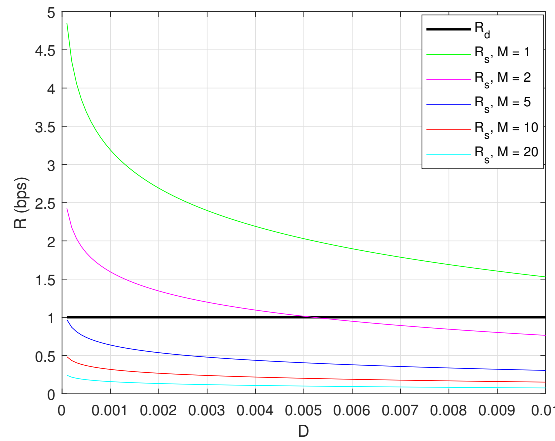

Fig. 6: Comparison of R_s and R_d versus the distortion where R_s is plotted for different values of M.

will be more efficient in terms of channel usage to send the distribution while as M decreases, at some point, the deterministic policy outperforms. Note that in the stochastic policy, the controller sends $N-1$ symbols of the distribution since the last value will be obtained by $p_N = 1 - \sum_{i=1}^{N-1} p_i$.

Lets denote the stochastic policy at time-slot n by X_n. Then, we assume a mean squared error of $E[(X_n - \hat{X}_n)^2] \leq D$, where $\hat{X}_n$ is the representation of X_n at the receiver side. Hence, considering the optimal quantization for uniform random variable $X_n \sim U[0,1]$, which is the uniform quantization with L levels with $L \geq \sqrt{\frac{1}{12D}}$, we have the following bounds for $R(D)$

$$-\frac{1}{2}\log_2 2\pi e D \leq R(D) \leq -\frac{1}{2}\log_2 12D. \quad (14)$$

Therefore, in this scenario, $R(D)$ bits are transmitted every M time units which means the bit rate can be obtained as

$$R_s = \frac{1}{\tau_s} = \frac{R(D)}{M}. \quad (15)$$

Considering the upperbound of $R(D)$, we have

$$R(D) = \frac{M}{\tau_s} \leq -\frac{1}{2}\log_2 12 D_X(M, \tau_s), \quad (16)$$

which gives us an upperbound for $D_X(M, \tau_s)$ as below:

$$D_X(M, \tau_s) \leq \frac{1}{12} 2^{-2\frac{M}{\tau_s}} \leq D, \quad (17)$$

in which D is the desired distortion. Figure 6 shows a comparison of the rate performance between the two policies. The bit rates, R_d and R_s, are plotted versus the maximum tolerable distortion. According to this figure and as intuitively expected, as M increases, R_s decreases, resulting to a less channel use in comparison to R_d.

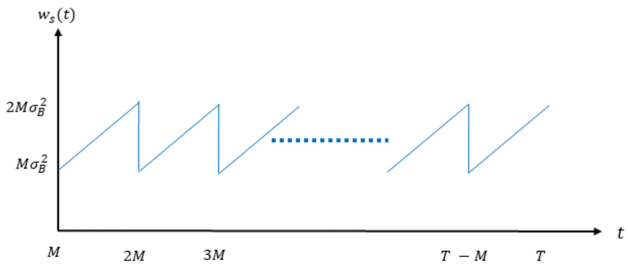

Fig. 7: Maximum distortion caused by the AoI.

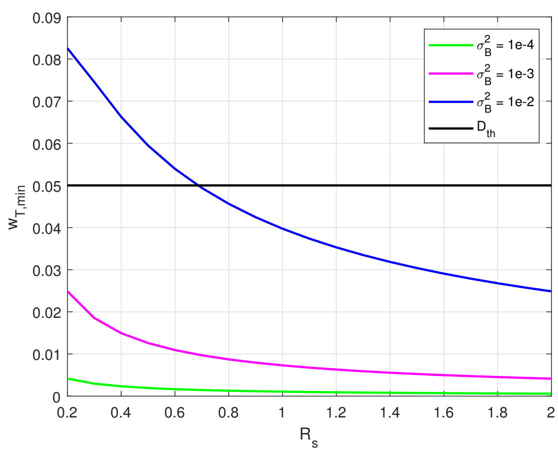

Fig. 8: Minimum values of $w_{T,\text{max}}$ as obtained in (20) vs. R_s.

Now, we notice that although the stochastic policy has the potential of decreasing the channel use, it however, comes with a few costs in terms of distortion as well. In what follows we wish to investigate this aspect of problem. As mentioned in the previous section, in the stochastic scenario, we have two sources of distortion: the distortion caused by the quantization and the distortion caused by the AoI. The main idea is to characterize the existing trade-offs between the communication cost and distortion while considering the signal variations. In order to analyze the amount of distortion caused by the outdated information, we assume that at each time unit, the probability measure p changes according to a truncated Brownian motion process with variance σ_B^2 such that it will never jump out of [0,1]. Therefore, at time-slots $t = jM, j = 1, 2, ..., \frac{T}{M}$, that the agent receives an update about the probability status, it has been already outdated by a maximum amount of $w_s = M\sigma_B^2$. Figure 7 shows the worst case AoI for this system setup. According to this figure, we see that the maximum amount of distortion due to the AoI reaches $2M\sigma_B^2$.

The overall distortion in the stochastic policy can be written as $w_T = w_s + D(M, \tau_s)$ for which we would like to find the proper τ_s such that a maximum threshold for the distortion is met:

$$\begin{aligned} w_{T,\text{max}} &= 2M\sigma_B^2 + D_X(M, \tau_s) \\ &= 2M\sigma_B^2 + \frac{1}{12}2^{-2\frac{M}{\tau_s}} \\ &\leq D. \end{aligned} \tag{18}$$

In Equation 18, we can obtain the optimum M minimizing $w_{T,\text{max}}$ as below:

$$M^* = -\frac{\tau_s}{2}\log_2\left(\frac{12\tau_s\sigma_B^2}{\ln 2}\right). \tag{19}$$

Hence, we can say that for any R_s and σ_B^2, there exists an M^* that minimizes w_T. Now substituting (19) in (18), we obtain the minimum of $w_{T,\text{max}}$ we have

$$\min_M(w_{T,\text{max}}) = -\frac{\sigma_B^2}{R_s}\log_2\left(12\frac{\sigma_B^2}{\ln 2R_s}\right) + \frac{\sigma_B^2}{\ln 2R_s}. \tag{20}$$

Figure 8, shows the minimum value of $w_{T,\text{max}}$ obtained with respect to M vs R_s for different values of σ_B^2. It can be seen that for the highly variating processes, i.e., processes with a high σ_B^2, transmitting with lower R_s results in distortion exceeding the threshold. In other words, for highly variating processes we need more channel use to meet the same level of distortion.

V. Conclusion

In this paper, we have introduced a unified distortion measure that includes both distortions from quantization and the AoI. We first developed a rate minimization problem for an arbitrary random variable. Using the proposed framework, we then considered a problem in which the question was whether to send a command deterministically or stochastically. We showed that if the communication cost is of importance, it is more efficient to send the command generator. However, we observed that for highly variating processes, we need to send more bits so that the distortion is kept below a desired threshold. The problem solved in this paper was for an agent with two policies. We wish to extend this work to a multi-agent network while policies with more than two actions are operated. Another interesting direction would be to consider a sequence of data instead of a single symbol.

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