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Cosmic cloaking of rich extra dimensions*

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We present arguments that show why it is difficult to see *rich* extra dimensions in the universe. Conditions are found where significant size and variation of the extra dimensions in a Kaluza–Klein compactification lead to a black hole in the lower-dimensional theory. The idea is based on the hoop conjecture concerning black hole existence, as well as on the observation that dimensional reduction on macroscopically large, twisted, or highly dynamical extra dimensions contributes positively to the energy density in the lower-dimensional theory and can induce gravitational collapse. A threshold for the size is postulated on the order of 10^{-19} m, whereby extra dimensions of length above this level must lie inside black holes, thus cloaking them from the view of outside observers. The threshold depends on the size of the universe, leading to speculation that in the early stages of evolution truly macroscopic and large extra dimensions would have been visible.

Keywords: Kaluza–Klein theory; hoop conjecture; large extra dimensions.

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According to our everyday experience, there are three spatial dimensions in addition to a dimension which tracks the passage of time. Nonetheless, the idea that there are additional spatial dimensions has persisted for a century. This is chiefly because the richer geometry provides an elegant way to encode the various fundamental fields we observe. For example, in the simplest such model, classical five-dimensional vacuum general relativity gives rise to Einstein–Maxwell theory in four dimensions with an additional scalar field. Presently, extra dimensions play a key role in string theory and the gauge theory-gravity correspondence. From a phenomenological perspective, extra "large" micron-sized dimensions have been suggested as an elegant resolution to the hierarchy problem,[1,2] an explanation for the origins of dark matter,[3] and even as a mechanism supplying the small but nonzero mass of neutrinos.[4] Thus, it appears likely that the possibility of extra dimensions is more than a mere mathematical abstraction.

Given their strong explanatory potential, an obvious question arises as to why there has yet to be any experimental evidence for their existence. Put simply, why do we not see these extra dimensions? A typical answer is that they are just too small for the current generation of experiments. While placing bounds on their size is highly model-dependent, the length scales probed by the LHC suggest that the size of such extra dimensions can be no larger than $10^{-18}\,\mathrm{m}$.[5] This begs a further question: why are extra dimensions so small? In this essay, we put forward potential answers to these questions. In a nutshell, if the extra dimensions were too large and twisted, or changed rapidly over a given four-dimensional spacetime region, then they must be hidden inside a four-dimensional black hole. Thus, the cosmos cloaks *rich extra dimensions*. The underlying mechanism is that the size, and the amount that extra dimensions warp and twist, produces an effective energy–momentum density concentration in the four-dimensional spacetime that leads to horizon formation. We also produce a rough estimate on the threshold geometry at which collapse occurs.

According to the *hoop conjecture*,[6] if enough matter is enclosed in a region of fixed size, a self-gravitating system will collapse to form a black hole. A more refined formulation is furnished by the *trapped surface conjecture*,[7] which asserts that a trapped surface will form near a body U with mass $\mathbf{m}(U)$ and size $\mathcal{R}(U)$ (in units of length) provided the following inequality holds:

$$\mathcal{R}(U) \lesssim \frac{G}{c^2}\mathbf{m}(U), \tag{1}$$

where $\lesssim$ indicates that there is a suppressed universal constant dependent on the definitions of $\mathbf{m}(U)$ and $\mathcal{R}(U)$, and G and c are Newton's constant and the speed of light, respectively. The trapped surface conjecture has been rigorously established in spherical symmetry[8,9] with $\mathcal{R}(U)$ given by the radius, and without symmetry hypotheses,[10] where $\mathcal{R}(U)$ is determined from the largest embedded torus within U.

An inequality of the form (1) is precisely stated[10] in terms of the initial value formulation of general relativity, which is the proper setting for studying dynamics.

When (1) holds, it can be shown that an apparent horizon must be present within the initial data. The existence of an apparent horizon signals that a black hole must be contained in the spacetime. In fact, once a trapped surface or apparent horizon is detected, the Hawking–Penrose singularity theorems together with weak cosmic censorship[11] imply that a horizon will form. The advantage of this approach is that the formation of black holes can be detected without knowledge of the full evolution.

We will demonstrate our proposal in the setting of standard Kaluza–Klein theory, in which there is a single extra spatial circle direction "twisted" over the usual four-dimensional spacetime. The ambient five-dimensional spacetime is taken to satisfy the vacuum Einstein equations. The length L of this circle, and the amount that it twists over the four-dimensional spacetime, vary as one moves along this base space. Roughly, when the length of the circle and the amount that it twists is too large or changes in a sufficiently rapid fashion over a domain of fixed size, a trapped region must form. We will present precise conditions on these geometric quantities that produce apparent horizons, and argue that therefore such a *rich* extra dimension would be inaccessible to experimental detection.

Consider the standard warped product with $U(1)$ bundle fibration, in which the vacuum five-dimensional spacetime metric g_5 takes the form

$$\mathrm{g}_5 = e^{2\alpha\phi}(dz + 2\mathbf{A})^2 + e^{2\beta\phi}\mathbf{g}, \tag{2}$$

where $\mathbf{g}$ represents the four-dimensional spacetime metric. The z coordinate parameterizes the S^1 direction and is understood to be periodically identified, $z \sim z+2\pi\ell$, where ℓ is a canonical length scale which we may choose to be a meter. $\mathbf{A}$ is a 1-form on spacetime which can be interpreted as measuring how the extra S^1 dimension is twisted above the base spacetime, and geometrically determines the connection of the $U(1)$ bundle. The circumference of the S^1 direction is $2\pi L$ where $L = e^{\alpha\phi}\ell$. It is well known that imposing the vacuum Einstein equations $\mathrm{Ric}(g_5) = 0$ and choosing α, β appropriately imply that the four-dimensional spacetime $(N, \mathbf{g})$ together with the 2-form field strength $F = d\mathbf{A}$, and dilaton ϕ satisfy the Einstein–Maxwell equations coupled to a charged scalar field.

Now, let M be a compact spacelike hypersurface in N with unit timelike normal vector field n, an induced positive definite metric g and extrinsic curvature k, as well as induced "electric" and "magnetic" spatial vector fields $E = F(n,\cdot)$ and $B = \star F(n,\cdot)$. An initial data set (M, g, k, E, B, ϕ) for the Einstein–Maxwell-scalar field system must satisfy the constraint equations. These consist of relations between k as well as the scalar curvature of g, and the energy and momentum densities μ, J of the slice, in addition to the analogues of Gauss's law involving the Maxwell and scalar fields. The energy and momentum densities may be obtained from the effective four-dimensional stress-energy tensor and are given by

$$\kappa\mu = \dot{\phi}^2 + |\nabla\phi|^2 + e^{3\alpha\phi}(|E|^2 + |B|^2), \quad \kappa J = 2\dot{\phi}\nabla\phi - 2e^{3\alpha\phi}E \times B, \tag{3}$$

where the $\dot{\phi} = n \cdot d\phi$ and $\nabla\phi$ represent initial time and spatial derivatives evaluated on M, and $\kappa = \frac{8\pi G}{c^4}$. Observe that the extra dimension contributes positive energy density from the four-dimensional perspective. We define the following *characteristic constants*

$$\begin{aligned}\mathcal{E}_1 &= \fint_M L\,\ell^{2/3}(|E|_g - |B|_g)^{2/3}\,d\mathrm{vol}_g,\\ \mathcal{E}_2 &= \fint_M \ell^{5/3}(|\dot{L}L^{-1}| - |\nabla \log L|)^{2/3}\,d\mathrm{vol}_g,\end{aligned} \tag{4}$$

associated with the initial data, which are normalized to have dimensions of length. The "slash" indicates that an average is taken over M.

If at least one of the two characteristic constants of an initial data set for the Einstein–Maxwell-scalar field system satisfies the *richness* condition

$$\mathcal{E}_i \gtrsim \left(\ell^5\,\frac{\mathrm{Rad}(M)}{\mathrm{Vol}(M)}\right)^{1/3}, \quad i = 1 \text{ or } 2, \tag{5}$$

then there exists an apparent horizon within M.

To see that this statement follows from the trapped surface conjecture, one may argue by contradiction. According to Eq. (1) and using the following definition of mass,

$$\mathbf{m}(M) = \frac{1}{c^2}\int_M (\mu - |J|_g)\,d\mathrm{vol}_g, \tag{6}$$

we conclude that the contrapositive of the trapped surface conjecture can be expressed as follows: if the initial data set is devoid of apparent horizons then

$$\mathrm{Rad}(M) \gtrsim \kappa \int_M (\mu - |J|_g)\,d\mathrm{vol}_g, \tag{7}$$

where we have chosen the measure of size to be in terms of the radius (one-half the furthest distance between two points). Observe that the Cauchy–Schwarz inequality yields

$$\kappa(\mu - |J|_g) \geq e^{3\alpha\phi}(|E|_g - |B|_g)^2 + (|\dot{\phi}| - |\nabla\phi|_g)^2. \tag{8}$$

Dividing (7) by the volume $\mathrm{Vol}(M)$, combining with (8), and applying Jensen's inequality then produce

$$\mathcal{E}_i \leq \mathcal{E}_1 + \mathcal{E}_2 \lesssim \left(\ell^5\,\frac{\mathrm{Rad}(M)}{\mathrm{Vol}(M)}\right)^{1/3}, \tag{9}$$

for $i = 1, 2$. Therefore, the reverse of this inequality implies that there must exist an apparent horizon within M.

Schoen and Yau have established a rigorous mathematical formulation and proof of the result that sufficient concentration of matter leads to horizon formation,[10] a result that naturally arose from their arguments establishing the positive mass theorem. Our choice of mass in (6) is motivated by their usage of the quantity $\mu - |J|_g$ which is related to the dominant energy condition. An analogous statement to that above involving (5) may be proved, following the methods of Ref. 10, with the main difference that the characteristic constants $\mathcal{E}_i$ are replaced with weighted integrals involving the principal eigenfunction of a certain differential operator on M. We also point out that the definition of mass (6) and usage of the radius to measure size have appeared in other formulations of the trapped surface conjecture, see, for example, Refs. 8 and 9.

Let us now examine some consequences of these conclusions. In particular, we will consider the case in which the initial data set encompasses the entire universe. The radius of the observable universe has been observed to be approximately 10^{26} m.[12] According to the most recent analysis of the cosmic microwave background,[13] the universe does not exhibit any known topological features, and thus its time slices may be approximated by the simply connected constant curvature model of a round 3-sphere (see also Ref. 14) or flat Euclidean 3-space in which case M is taken to be a large ball. We are thus able to compute the ratio of radius to volume appearing in the theorem, namely,

$$\left(\ell^5 \frac{\mathrm{Rad}(M)}{\mathrm{Vol}(M)}\right)^{1/3} \sim 10^{-19}\mathrm{m}. \tag{10}$$

It follows that if $\mathcal{E}_i \gtrsim 10^{-19}$ m for either $i = 1$ or 2, then a black hole must form due to concentration of the geometry, or richness, of the extra dimension. Although our result does not determine exactly where in M the apparent horizon is located, it is reasonable to surmise that the regions where the concentration is highest should be where the black holes form.

Next, we examine the individual cases in which each characteristic constant satisfies the richness condition (5). In particular, with regard to the first characteristic constant, if the twisting of the extra dimension is generically on the order of $(|E|_g - |B|_g)^{2/3} \sim 10^{-p}\ell^{-2/3}$, then the richness condition will be satisfied when, on the scale of the universe, the average circumference of the extra dimension satisfies $\mathrm{avg}(L) \gtrsim 10^{-19+p}$ m. We then have the existence of black holes due to the excessive average size of the extra dimension, and as argued the horizons should be located in regions where this size is greatest. In this way, large extra dimensions are hidden from the view of outside observers. On the other hand, if the size is generically $L \sim 10^{-p}$ m, then the richness condition is satisfied when the average twisting surpasses the threshold $10^{-19+p}\ell^{-1}$, in which case sufficiently twisted extra dimensions are hidden behind horizons.

Consider now the second characteristic constant, which concerns a measurement of rate of change in time and space of the size of the circle fibers. The richness condition will be satisfied if on average, throughout a time slice of the universe, the rate of change of the extra dimension's size is greater than 10^{-19} m. Thus, highly dynamical extra dimensions or those with extreme spatial size variations are enclosed inside black holes. In a different direction, Penrose [15, Sec. 31.12] has sketched an instability argument that suggests the emergence of singularities in supersymmetric compactifications due to Planck-sized extra dimensions.

It should be pointed out that the threshold of 10^{-19} m is tied to the diameter of the universe in the current epoch. Thus, we speculate that at earlier times, when the diameter was significantly smaller the threshold would be much larger. In fact, in the early stages of the universe truly macroscopic and even large extra dimensions would have been visible, and allowed to change rapidly without being trapped behind horizons. Conversely, as the universe expands into the future, the threshold will drop and eventually achieve a level on par with the Planck length, making it virtually impossible to detect the extra dimensions.

The conclusions of this note suggest that there is a fundamental tension between the "richness" — the twisting, warping and size — of extra dimensions and the ability to explore these dimensions experimentally. We have restricted attention to the simplest model of extra dimensions to emphasize key features of the arguments. It should be the case, however, that additional (possibly curved) spatial dimensions should produce further positive contributions to the energy density of the effective theory, leading to similar effects. The same is also true if additional matter fields are present and satisfy the dominant energy condition, or if there is a nonnegative cosmological constant. Thus, we expect that the results demonstrated here can be generalized to more complicated models in a robust manner.

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References

1. N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *Phys. Lett. B* **429** (1998) 263.
2. L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83** (1999) 3370.
3. A. Carmona, J. Ruiz and M. Neubert, *Eur. Phys. J. C* **81** (2021) 58.
4. D. Caldwell, R. Mohapatra and S. Yellin, Neutrinos, large extra dimensions and solar neutrino puzzle, *Workshop on Lepton Flavor Violation* (Honolulu, October 2000), arXiv:hep-ph/0101043.
5. A. Hoecker, Physics at the LHC Run-2 and beyond, *Lecture Notes from the 2016 European School of High Energy Physics* (Skeikampen, Norway, 2016), 61 p.
6. K. Thorne, *Magic without Magic: John Archibald Wheeler*, ed. J. Klauder (Freeman, San Francisco, 1972), p. 231.
7. H. Seifert, *Gen. Relativ. Gravit.* **10** (1979) 1065.
8. M. Khuri, *Phys. Rev. D* **80** (2009) 124025.
9. E. Malec and N. O'Murchadha, *Phys. Rev. D* **50** (1994) R6033.
10. R. Schoen and S. T. Yau, *Comm. Math. Phys.* **90** (1983) 575.
11. R. Penrose, *Nuovo Cimento Rivista Ser.* **1** (1969) 252.
12. P. Halpern and N. Tomasello, *Adv. Astrophys.* **1** (2016) 135.
13. P. Ade *et al.*, *Astron. Astrophys. A* **571** (2014) 26.
14. G. Galloway, M. Khuri and E. Woolgar, arXiv:2010.06739.
15. R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (Jonathan Cape London, 2004).