Geometry & Topology 26 (2022) 937–987

Discrete conformal geometry of polyhedral surfaces and its convergence

FENG LUO
JIAN SUN
TIANQI WU

We prove a result on the convergence of discrete conformal maps to the Riemann mappings for Jordan domains. It is a counterpart of Rodin and Sullivan's theorem on convergence of circle packing mappings to the Riemann mapping in the new setting of discrete conformality. The proof follows the same strategy that Rodin and Sullivan used by establishing a rigidity result for regular hexagonal triangulations of the plane and estimating the quasiconformal constants associated to the discrete conformal maps.

52C26, 53C44, 58E30

1. Introduction 937
2. Polyhedral metrics, vertex scaling and a variational principle 942
3. A maximum principle, a ratio lemma and spiral hexagonal triangulations 947
4. Rigidity of hexagonal triangulations of the plane 956
5. Existence of discrete uniformization metrics on polyhedral disks with special equilateral triangulations 961
6. A proof of the convergence theorem 979
7. A convergence conjecture on discrete uniformization maps 980
References 985

1 Introduction

W Thurston's conjecture on the convergence of circle packing mappings to the Riemann mapping is a constructive and geometric approach to the Riemann mapping theorem. The conjecture was solved in important work by Rodin and Sullivan [25] in 1987. There

Published: 3 August 2022 DOI: 10.2140/gt.2022.26.937

have been many research works inspired by the work of Thurston, Rodin and Sullivan since then. This paper addresses a counterpart of Thurston's convergence conjecture in the setting of discrete conformal change of polyhedral surfaces associated to the notion of vertex scaling (Definition 1.1). We prove a weak version of Rodin and Sullivan's theorem in this new setting. There are still many problems to be resolved in order to prove the full convergence conjecture.

Let us begin with a recall of Thurston's conjecture and Rodin and Sullivan's solution. Given a bounded simply connected domain Ω in the complex plane $\mathbb{C}$, one constructs a sequence of approximating triangulated polygonal disks $(D_n, \mathcal{T}_n)$ whose triangles are equilateral and whose edge lengths tend to zero such that the D_n converge to Ω. For each such polygonal disk, by the Köbe–Andreev–Thurston existence theorem, there exists a circle packing of the unit disk $\mathbb{D}$ such that the combinatorics (or the nerve) of circle packing is isomorphic to the 1–skeleton of the triangulation $\mathcal{T}_n$. This produces a piecewise linear homeomorphism f_n, called the circle packing mapping, from the polygonal disk D_n to a polygonal disk inside $\mathbb{D}$ associated to the circle packing. Thurston conjectured in 1985 that, under appropriate normalizations, the sequence $\{f_n\}$ converges uniformly on compact subsets of Ω to the Riemann mapping for Ω. Here the normalization condition is given by choosing a point $p \in \Omega$ and a sequence of vertices v_n in $(D_n, \mathcal{T}_n)$ such that $\lim_n v_n = p$ and $f_n(v_n) = 0$ such that f_n sends a small interval $[v_n, v_n + \epsilon_n]$ from v_n to $v_n + \epsilon_n$ with $\epsilon_n > 0$ into the positive x–axis. The Riemann mapping f for Ω sends p to 0 and $f'(p) > 0$. Rodin and Sullivan's proof of Thurston's conjecture is elegant and goes in two steps. In the first step, they show that the circle packing mappings f_n are K–quasiconformal for some constant K independent of the indices. In the second step, they show that there is only one hexagonal circle packings of the complex plane up to Möbius transformations. This implies that the limit of the sequence $\{f_n\}$ is conformal.

Circle packing metrics introduced by Thurston [27] can be considered as a discrete conformal geometry of polyhedral surfaces. In recent times, there have been many works on discretization of 2–dimensional conformal geometry (see Luo [18], Bobenko, Pinkall and Springborn [3], Hersonsky [14], Gu, Luo, Sun and Wu [10], Glickenstein [9] and others). We consider the counterpart of Thurston's conjecture in the setting of discrete conformal change defined by vertex scaling.

To state our main results, let us recall some related material and notation. A compact topological surface S together with a nonempty finite subset of points $V \subset S$ will be called a *marked surface*. A triangulation $\mathcal{T}$ of a marked surface (S, V) is a topological triangulation of S such that the vertex set of $\mathcal{T}$ is V. We use $E = E(\mathcal{T})$ and $V = V(\mathcal{T})$ to denote the sets of all edges and vertices in $\mathcal{T}$, respectively. A *polyhedral metric* d on (S, V), to be called a *PL metric* on (S, V) for simplicity, is a flat cone metric on (S, V) whose cone points are contained in V. We call the triple (S, V, d) a polyhedral surface. The *discrete curvature*, or simply *curvature*, of a PL metric d is the function $K: V \to (-\infty, 2\pi)$ sending an interior vertex v to 2π minus the cone angle at v and a boundary vertex v to π minus the sum of the angles at v. All PL metrics are obtained by isometric gluing of Euclidean triangles along pairs of edges. If $\mathcal{T}$ is a triangulation of a polyhedral surface (S, V, d) for which all edges in $\mathcal{T}$ are geodesic, we say $\mathcal{T}$ is *geometric* in d and d is a PL metric on $(S, \mathcal{T})$. In this case, we can represent d by the length function $l_d : E(\mathcal{T}) \to \mathbb{R}_{>0}$ sending each edge to its length. Thus, the polyhedral surface (S, V, d) can be represented by $(S, \mathcal{T}, l_d)$, where $l_d \in \mathbb{R}^E_{>0}$. We will also call $(S, \mathcal{T}, l_d)$ or l_d a PL metric on $\mathcal{T}$.

Definition 1.1 (vertex scaling change of PL metrics [18]) Two PL metrics l and l^* on a triangulated surface $(S, \mathcal{T})$ are related by a *vertex scaling* if there exists a map $w: V(\mathcal{T}) \to \mathbb{R}$ such that, if e is an edge in $\mathcal{T}$ with endpoints v and v', then the edge lengths $l(e)$ and $l^*(e)$ are related by

$$l^*(e) = e^{w(v)+w(v')} l(e). \tag{1}$$

We denote l^* by $w * l$ if (1) holds and call l^* obtained from l by a vertex scaling and w a discrete conformal factor.

Condition (1) was proposed in [18] as a discrete conformal equivalence between PL metrics on triangulated surfaces. There are three basic problems related to the vertex scaling. The first is the existence problem. Namely, given a PL metric l on a triangulated closed surface $(S, \mathcal{T})$ and a function $K: V(\mathcal{T}) \to (-\infty, 2\pi)$ satisfying the Gauss–Bonnet condition, is there a PL metric l^* of the form $w * l$ whose curvature is K? Unlike the Köbe–Andreev–Thurston theorem, which guarantees the existence of circle packing metrics, the answer to the above existence problem is negative in general. This makes the convergence of discrete conformal mappings a difficult problem. Secondly, on the other hand, the uniqueness of the vertex scaled PL metric l^* with prescribed curvature holds. This was established in an important paper by Bobenko, Pinkall and Springborn [3]. The third is the convergence problem. Namely, assuming the existence of PL metrics with prescribed curvatures, can these discrete conformal polyhedral surfaces approximate a given Riemann surface? Our main result gives a solution to the convergence problem for the simplest case of Jordan domain.

The convergence theorem that we proved is the following. Let Ω be a Jordan domain with three points p, q and r specified in the boundary. By Carathéodory's extension theorem (see Pommerenke [23]), the Riemann mapping from Ω to the unit disk $\mathbb{D}$ extends to a homeomorphism from the closure $\overline{\Omega}$ to the closure $\overline{\mathbb{D}}$. Therefore, there exists a unique homeomorphism g from $\overline{\Omega}$ to an equilateral Euclidean triangle ΔABC with vertices A, B and C such that p, q and r are sent to A, B and C and g is conformal in Ω. For simplicity, we call g and g^{-1} the Riemann mappings for $(\Omega, (p,q,r))$.

Given an oriented triangulated polygonal disk $(D, \mathcal{T}, l)$ and three boundary vertices $p, q, r \in V$, suppose there exists a PL metric $l^* = w * l$ on $(D, \mathcal{T})$ for some $w\colon V \to \mathbb{R}$ such that its discrete curvature at all vertices except $\{p,q,r\}$ are zero and the curvatures at p, q and r are $\frac{2\pi}{3}$. Then the associated flat metric on $(D, \mathcal{T}, l^*)$ is isometric to an equilateral triangle ΔABC, ie there is a geometric triangulation $\mathcal{T}'$ of ΔABC such that $(\Delta ABC, \mathcal{T}', l_{\mathrm{st}})$ is isometric to $(D, \mathcal{T}, l^*)$. Here and below, if $\mathcal{T}$ is a geometric triangulation of a domain in the plane, we use $l_{\mathrm{st}}\colon E(\mathcal{T}) \to \mathbb{R}$ to denote the length of edges e in $\mathcal{T}$ in the standard metric on $\mathbb{C}$. Let $f : D \to \Delta ABC$ be the piecewise linear orientation-preserving homeomorphism sending V to the vertex set $V(\mathcal{T}')$ of $\mathcal{T}'$, and p, q and r to A, B and C, respectively, and being linear on each triangle of $\mathcal{T}$. We call f the *discrete uniformization map* associated to $(D, \mathcal{T}, l, \{p,q,r\})$. Note that f may not exist due to the lack of an existence theorem.

Theorem 1.2 *Suppose Ω is a Jordan domain in the complex plane with three distinct points $p, q, r \subset \partial\Omega$. Then there exists a sequence $(\Omega_n, \mathcal{T}_n, l_{\mathrm{st}}, (p_n, q_n, r_n))$ of simply connected triangulated polygonal disks in $\mathbb{C}$, where $\mathcal{T}_n$ are triangulations by equilateral triangles and p_n, q_n and r_n are three boundary vertices such that*

(a) $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ *with* $\Omega_n \subset \Omega_{n+1}$, *and* $\lim_n p_n = p$, $\lim_n q_n = q$ *and* $\lim_n r_n = r$,

(b) *discrete uniformization maps associated to $(\Omega_n, \mathcal{T}_n, l_{\mathrm{st}}, (p_n, q_n, r_n))$ exist and converge uniformly to the Riemann mapping for $(\Omega, (p,q,r))$.*

In Rodin and Sullivan's convergence theorem, any sequence of approximating circle packing maps associated to the approximation triangulated polyhedral disks Ω_n such that $\Omega_n \subset \mathrm{int}(\Omega_{n+1})$ and $\Omega = \bigcup_n \Omega_n$ converges to the Riemann mapping. Theorem 1.2 is less robust in this aspect since discrete conformal maps may not exist if the triangulations $\mathcal{T}_n$ are not carefully selected. A stronger version of convergence is conjectured in Section 7. The conformality of the limit of the discrete conformal maps in Theorem 1.2 is a consequence of the following result. Recall that a geometric triangulation $\mathcal{T}$ of a polyhedral surface is called *Delaunay* if the sum of two angles facing each interior edge is at most π. Delaunay triangulations always exist for each PL metric on compact surfaces.

Theorem 1.3 *Suppose $\mathcal{T}$ is a Delaunay geometric triangulations of the complex plane $\mathbb{C}$ such that its vertex set is a lattice and $l_{\mathrm{st}}\colon E(\mathcal{T}) \to \mathbb{R}$ is the edge length function of $\mathcal{T}$. If $(\mathbb{C}, \mathcal{T}, w * l_{\mathrm{st}})$ is a Delaunay triangulated surface isometric to an open set in the Euclidean plane $\mathbb{C}$, then w is a constant function.*

We remark that the same result as above for the standard hexagonal lattice has been proved independently by Dai, Ge and Ma [7].

Using an important result, in [3], that vertex scaling is closely related to hyperbolic 3–dimensional geometry and the work of [10], one sees that Theorem 1.3 implies the following rigidity result on convex hyperbolic polyhedra:

Theorem 1.4 *Suppose $L = \mathbb{Z} + \tau\mathbb{Z}$ is a lattice in the plane $\mathbb{C}$ and $V \subset \mathbb{C}$ is a discrete set such that there exists an isometry between the boundaries of the convex hulls of L and V in the hyperbolic 3–space $\mathbb{H}^3$ preserving cell structures. Then V and L differ by a complex affine transformation of $\mathbb{C}$.*

This prompts us to propose the following conjecture. A closed set X in the Riemann sphere is said to be of *circle type* if each connected component of X is either a point or a round disk. Consider the Riemann sphere $\mathbb{C} \cup \{\infty\}$ as the infinity of the (upper-half-space model of) hyperbolic 3–space $\mathbb{H}^3$.

Conjecture 1.5 *For any genus zero connected complete hyperbolic surface Ω, there exists a circle type closed set $X \subset \mathbb{C} \cup \{\infty\}$ such that Ω is isometric to the boundary of the convex hull of X in $\mathbb{H}^3$.*

Conjecture 1.6 *Suppose X and Y are circle type closed sets in $\mathbb{C}$ such that the boundaries of the convex hulls of X and Y in $\mathbb{H}^3$ are isometric. Then X and Y differ by a Möbius transformation.*

Many results on convergence of discrete maps to Riemann mapping have been established since the work of Thurston and Rodin and Sullivan [25]. He and Schramm [13] studied the approximation of conformal maps by circle packing with arbitrary combinatorics. In [5], Bücking considered a boundary value problem. She used the Riemann mapping f of the Jordan domain and looked for a discrete conformal map with scaling factor $u = \log|f|$ on boundary vertices and proved the existence and convergence using regular hexagonal lattices. In fact, she is able to prove that the convergence is C^∞. Other related works on convergence of circle packing maps can be found in Matthes [20], Lan and Dai [16], Bücking [4] and Hersonsky [14; 15].

The paper is organized as follows. Section 2 recalls the basic material for discrete conformal geometry of polyhedral surfaces. Sections 3 and 4 are devoted to proving Theorem 1.3. The main tools used are a maximum principle, a variational principle for discrete conformal geometry of polyhedral surfaces, and spiral hexagonal triangulations derived from linear conformal factors. Section 5 investigates the existence of flat metrics with prescribed boundary curvature on polygonal disks. The main result (Theorem 5.1) is an existence result for vertex scaling equivalence if triangulations of a polyhedral disk are sufficiently finely subdivided. The basic tools used are discrete harmonic functions, their gradient estimates and solutions to ordinary differential equations. We prove the convergence (Theorem 1.2) in Section 6 using the results obtained in Sections 4–5 and Rado and Palka's theorem on uniform convergence of Riemann mappings and quasiconformal mappings. Section 7 discusses a strong version of the convergence of discrete uniformization maps and the motivation for Conjecture 1.5.

Acknowledgement We thank Michael Freedman for discussions which led to the formulation of Conjectures 1.5 and 1.6. The work is partially supported by the NSF grants DMS 1405106, DMS 1760527, DMS 1811878 and DMS 1760471 of the United States and a grant from the NSF of China.

2 Polyhedral metrics, vertex scaling and a variational principle

We begin with some notation. Let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ be the sets of complex, real and integers, respectively. $\mathbb{R}_{>0} = \{t \in \mathbb{R} \mid t > 0\}$, $\mathbb{Z}_{\geq k} = \{n \in \mathbb{Z} \mid n \geq k\}$ and $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. We use $\mathbb{D}$ to denote the open unit disk in $\mathbb{C}$ and $\mathbb{H}^n$ to denote the n–dimensional hyperbolic space.

Given that X is a compact surface with boundary, its interior is denoted by $\mathrm{int}(X)$. A graph with vertex set V and edge set E is denoted by (V, E). Two vertices i and j in a graph (V, E) are *adjacent*, denoted by $i \sim j$, if they are the endpoints of an edge. If $i \sim j$, we use $[ij]$ (respectively ij) to denote an oriented (respectively unoriented) edge from i

to j. An *edge path* joining $i, j \in V$ is a sequence of vertices $\{v_0 = i, v_1, \dots, v_m = j\}$ such that $v_k \sim v_{k+1}$. The length of the path is m. The *combinatorial distance* $d_c(i, j)$ between two vertices in a connected graph (V, E) is the length of the shortest edge path joining i and j. Suppose $(S, \mathcal{T})$ is a triangulated surface with possibly nonempty boundary ∂S and possibly noncompact S. Let $E = E(\mathcal{T})$ and $V = V(\mathcal{T})$ be the sets of edges and vertices, respectively, and $\mathcal{T}^{(1)} = (V, E)$ be the associated graph. A vertex $v \in V(\mathcal{T}) \cap \partial S$ (resp. $v \in V \cap (S - \partial S)$) is called a boundary (resp. interior) vertex. Boundary and interior edges are defined in the same way. A PL metric on $(S, \mathcal{T})$ or simply on $\mathcal{T}$ can be represented by a length function $l: E(\mathcal{T}) \to \mathbb{R}_{>0}$ such that if e_i, e_j and e_k are three edges forming a triangle in $\mathcal{T}$, then the *strict triangle inequality* holds:

$$l(e_i) + l(e_j) > l(e_k). \tag{2}$$

We will use limits of PL metrics. To this end, we introduce the notion of *generalized PL metrics* on $(S, \mathcal{T})$. Take three pairwise distinct points v_1, v_2 and v_3 in the plane. The convex hull of $\{v_1, v_2, v_3\}$ is a *generalized triangle* with vertices v_1, v_2 and v_3. We denote it by $\Delta v_1 v_2 v_3$. If v_1, v_2 and v_3 are not in a line, then $\Delta v_1 v_2 v_3$ is a (usual) triangle. If v_1, v_2 and v_3 lie in a line, then $\Delta v_1 v_2 v_3$ is a *degenerate triangle* with the flat vertex at v_i if $|v_j - v_i| + |v_k - v_i| = |v_j - v_k|$ for $\{i, j, k\} = \{1, 2, 3\}$. Let $l_i = |v_j - v_k| \in \mathbb{R}_{>0}$ be the *edge length* and $a_i \in [0, \pi]$ be the *angle* at v_i. Then $l_i + l_j \geq l_k > 0$ and the angles are given by

$$a_i = \arccos\left(\frac{l_j^2 + l_k^2 - l_i^2}{2 l_j l_k}\right). \tag{3}$$

Furthermore, the angle $a_i = a_i(l_1, l_2, l_3) \in [0, \pi]$ is continuous in (l_1, l_2, l_3). Degenerate triangles are characterized by either having an angle π or the lengths satisfying $l_i = l_j + l_k$ for some i, j and k.

A *generalized PL metric* on a triangulated surface (S, T) is represented by an edge length function $l: E(\mathcal{T}) \to \mathbb{R}_{>0}$ such that if e_i, e_j and e_k are three edges forming a triangle in $\mathcal{T}$, then the triangle inequality holds:

$$l(e_i) + l(e_j) \geq l(e_k). \tag{4}$$

We will abuse terminology and call l a generalized PL metric on $(S, \mathcal{T})$ or $\mathcal{T}$. The *discrete curvature* $K: V(\mathcal{T}) \to (-\infty, 2\pi]$ of a generalized PL metric $(S, \mathcal{T}, l)$ is defined as follows. If $v \in V(\mathcal{T})$ is an interior vertex, $K(v)$ is 2π minus the sum of the angles (of generalized triangles) at v; if v is a boundary vertex, $K(v)$ is π minus the sum of the angles at v. Note that the Gauss–Bonnet theorem, $\sum_{v \in V(\mathcal{T})} K(v) = 2\pi \chi(S)$, still holds for a compact surface S with a generalized PL metric. Clearly the curvature K and inner angles depend continuously on the length vector $l \in \mathbb{R}_{>0}^{E(\mathcal{T})}$. A generalized PL metric is called *flat* if its curvatures are zero at all interior vertices v. A generalized PL metric $(S, \mathcal{T}, l)$ (sometimes written as $(\mathcal{T}, l)$) is called *Delaunay* if, for each *interior* edge $e \in E(\mathcal{T})$, the sum of the two angles α and α' facing e is at most π. Suppose τ and τ' are the two triangles adjacent to e. If τ is a degenerate triangle such that $\alpha = \pi$, then the Delaunay condition implies that $\alpha' = 0$. Therefore, τ' is also degenerate and e is not the longest edge in τ'. This shows that if $(S, \mathcal{T}, l)$ is a closed generalized Delaunay PL metric surface, then no triangle in $\mathcal{T}$ is degenerate. Indeed, each degenerate triangle is adjacent to another degenerate triangle of larger diameter. However, this is not the case for infinite triangulations. For instance, there exists a generalized Delaunay PL surface homeomorphic to the plane which contains both degenerate and nondegenerate triangles.

If $(S, \mathcal{T}, l)$ is a Delaunay generalized PL metric such that each angle facing a boundary edge is at most $\frac{\pi}{2}$, then the metric double of $(S, \mathcal{T}, l)$ along its boundary is a Delaunay triangulated generalized PL metric surface. Two generalized PL metrics l and $\tilde{l}$ on $(S, \mathcal{T})$ are related by a *vertex scaling* if there is $w \in \mathbb{R}^V$ such that

$$\tilde{l}(vv') = e^{w(v)+w(v')} l(vv')$$

for all edges $vv' \in E(\mathcal{T})$. We write $\tilde{l} = w * l$ and call w a *discrete conformal factor*.

Two generalized triangles $\Delta v_1 v_2 v_3$ and $\Delta u_1 u_2 u_3$ are *equivalent* if there exists an isometry sending v_i to u_i for $i = 1, 2, 3$. The space of all equivalence classes of generalized triangles can be identified with $\{(l_1, l_2, l_3) \in \mathbb{R}^3_{>0} \mid l_i + l_j \geq l_k\}$. It contains the space of all equivalence classes of triangles $\{(l_1, l_2, l_3) \in \mathbb{R}^3_{>0} \mid l_i + l_j > l_k\}$. Given two generalized triangles $l = (l_1, l_2, l_3)$ and $\tilde{l} = (\tilde{l}_1, \tilde{l}_2, \tilde{l}_3)$, there exists $w = (w_1, w_2, w_3) \in \mathbb{R}^3$ such that $\tilde{l}_i = l_i e^{w_j + w_k}$.

The following result was proved in [18, Theorem 2.1] for Euclidean triangles. The extension to generalized triangles is straightforward.

Proposition 2.1 [18] *Let $\Delta v_1 v_2 v_3$ be a fixed generalized triangle with edge length vector $l = (l_1, l_2, l_3)$ and $w * l$ be the edge length vector of a vertex scaled generalized triangle whose inner angle at v_i is $a_i = a_i(w)$.*

(a) *For any two constants c_i and c_j, the set*

$$\{(w_1, w_2, w_3) \in \mathbb{R}^3 \mid w * l \text{ is a generalized triangle and } w_i = c_i,\ w_j = c_j\}$$

is either connected or empty.

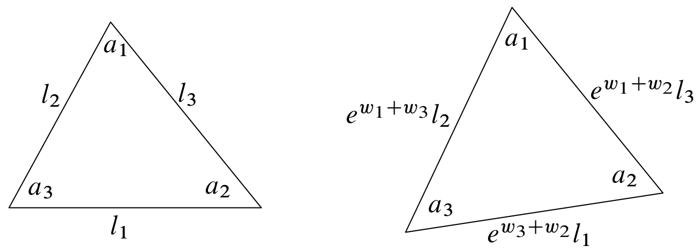

Figure 1: Vertex scaling of a triangle.

(b) *If* $(\Delta v_1 v_2 v_3, l)$ *is a nondegenerate triangle and* i, j *and* k *distinct, then*

$$\begin{aligned} \frac{\partial a_i}{\partial w_i}\bigg|_{w=0} &= -\frac{\sin(a_i)}{\sin(a_j)\sin(a_k)} < 0, \\ \frac{\partial a_i}{\partial w_j}\bigg|_{w=0} &= \frac{\partial a_j}{\partial w_i}\bigg|_{w=0} = \cot(a_k), \\ \sum_{j=1}^{3} \frac{\partial a_i}{\partial w_j} &= 0. \end{aligned} \tag{5}$$

The matrix $-[\partial a_r/\partial w_s]_{3\times 3}$ *is symmetric, positive semidefinite with null space spanned by* $(1,1,1)^T$.

(c) *If* $(\Delta v_1 v_2 v_3, l)$ *is a degenerate triangle having* v_3 *as the flat vertex, then, for small* $t > 0$, $(\Delta v_1 v_2 v_3, (0,0,t) * l)$ *is a nondegenerate triangle. The angle* $a_3(0,0,t)$ *is strictly decreasing in* t *for all* t *for which* $(0,0,t)*l$ *is a generalized triangle. The angles* $a_i(0,0,t)$ *for* $i = 1,2$ *are strictly increasing in* $t \in [0,\epsilon)$ *for some* $\epsilon > 0$.

Proof To see part (a), without loss of generality, we may assume c_1 and c_2 are the given constants. Then the variable w_3 is defined by the inequalities $e^{w_3}(e^{c_1}l_2 + e^{c_2}l_1) \geq e^{c_1+c_2}l_3$ and $e^{c_1+c_2}l_3 \geq e^{w_3}(e^{c_1}l_2 - e^{c_2}l_1) \geq -e^{c_1+c_2}l_3$. Each of these inequalities defines an interval in the w_3 variable. Therefore the solution space is either the empty set or a connected set.

Part (b) is in [18, Theorem 2.1].

To see (c), since $l_3 = l_1 + l_2$, we have $(0,0,t) * l = (e^t l_1, e^t l_2, l_1 + l_2) \in \mathbf{\Delta} := \{(x_1,x_2,x_3) \in \mathbb{R}^2_{>0} \mid x_i + x_j \geq x_k\}$ for small $t > 0$. Now, by (5) and the sine law, $\partial a_3/\partial w_3(0,0,t) = -\sin(a_3)/\sin(a_1)\sin(a_2) < 0$. Together with part (a), the angle $a_3(0,0,t)$ as a function of t is defined on an interval and is strictly decreasing in t.

Since $\lim_{t\to 0^+} \partial a_1/\partial w_3 = \lim_{t\to 0^+} \cot(a_2) = \infty$ due to $\lim_{t\to 0^+} a_2(t,0,0) = 0$, the result holds for a_1. By the same argument, the result holds for a_2. □

As a consequence:

Corollary 2.2 *Under the same assumption as in Proposition 2.1, if*

$$w(t) = (w_1(t), w_2(t), w_3(t)) \in \mathbb{R}^3$$

is smooth in t such that $w(t) * l$ is the edge length vector of a triangle with inner angle $a_i(t) = a_i(w(t) * l)$ at v_i, then

$$\frac{da_i(t)}{dt} = \sum_{j\sim i} \cot(a_k)\left(\frac{dw_j}{dt} - \frac{dw_i}{dt}\right), \tag{6}$$

where $j \sim i$ means v_j is adjacent to v_i and $\{i,j,k\} = \{1,2,3\}$.

Write $w_j'(t) = dw_j/dt$. Indeed, by the chain rule and (5), we have

$$\begin{aligned}\frac{da_i(t)}{dt} &= \frac{\partial a_i}{\partial w_i} w_i' + \sum_{j\neq i} \frac{\partial a_i}{\partial w_j} w_j' \\ &= -\sum_{j\neq i} \cot(a_k) w_i' + \sum_{j\neq i} \cot(a_k) w_j' = \sum_{j\sim i} \cot(a_k)(w_j' - w_i').\end{aligned}$$

Suppose $(S, \mathcal{T}, l)$ is a geometrically triangulated compact polyhedral surface and $w(t) \in \mathbb{R}^V$ is a smooth path in the parameter t such that $w(t) * l$ is a PL metric on $(S, \mathcal{T})$. Let $K_i = K_i(t)$ be the discrete curvature at $i \in V$ and $\theta^i_{jk} = \theta^i_{jk}(t)$ be the inner angle at the vertex i in Δijk in the metric $w(t) * l$. For an edge $[ij]$ in the triangulation $\mathcal{T}$, define η_{ij} to be $\cot(\theta^k_{ij}) + \cot(\theta^l_{ij})$ if $[ij]$ is an interior edge facing two angles θ^k_{ij} and θ^l_{ij}, and $\eta_{ij} = \cot(\theta^k_{ij})$ if $[ij]$ is a boundary edge. If $[ij]$ is an interior edge, then $\eta_{ij} \geq 0$ if and only if $\theta^k_{ij} + \theta^l_{ij} \leq \pi$, ie the Delaunay condition holds at $[ij]$.

The curvature variation formula is the following:

Proposition 2.3 *We have*

$$\frac{dK_i(t)}{dt} = \sum_{j\sim i} \eta_{ij}\left(\frac{dw_i}{dt} - \frac{dw_j}{dt}\right). \tag{7}$$

This follows directly from Corollary 2.2 since $K_i = c\pi - \sum_{r,s\in V} \theta^i_{rs}$, where $c = 1$ or 2 and θ^i_{rs} are the angles at i. Since $dK_i(t)/dt = -\sum_{r,s\in V} d\theta^i_{rs}/dt$, equation (7) follows from (6) and the definition of η_{ij}.

3 A maximum principle, a ratio lemma and spiral hexagonal triangulations

Let P_n be a star-shaped n–sided polygon having vertices $v_1, \dots, v_n$ labeled cyclically. A triangulation $\mathcal{T}$ of P_n with vertices $v_0, \dots, v_n$ with $v_0 \in \mathrm{int}(P_n)$ and triangles $\Delta v_0 v_i v_{i+1}$ (with $v_{n+1} = v_1$) is called a *star triangulation* of P_n. See Figure 2.

Theorem 3.1 (maximum principle) *Let $\mathcal{T}$ be a star triangulation of P_n and $l : E(\mathcal{T}) \to \mathbb{R}_{>0}$ be a generalized Delaunay polyhedral metric on $\mathcal{T}$. If $w : \{v_0, v_1, \dots, v_n\} \to \mathbb{R}$ satisfies that*

(a) *$w * l$ is a generalized Delaunay polyhedral metric on $\mathcal{T}$,*

(b) *the curvatures $K_0(w * l)$ of $w * l$ and $K_0(l)$ of l at the vertex v_0 satisfy $K_0(w * l) \le K_0(l)$, and*

(c) *$w(v_0) = \max\{w(v_i) \mid i = 0, 1, \dots, n\}$,*

then $w(v_i) = w(v_0)$ for all i.

As a convention, if $x = (x_0, \dots, x_m)$ and $y = (y_0, y_1, \dots, y_m)$ are in $\mathbb{R}^{m+1}$, then $x \ge y$ means $x_i \ge y_i$ for all i. Given $w : \{v_0, \dots, v_m\} \to \mathbb{R}$, we use $w_i = w(v_i)$ and identify w with $(w_0, w_1, \dots, w_m) \in \mathbb{R}^{m+1}$. The cone angle of $w * l$ at v_0 will be denoted by $\alpha(w)$. Thus, Theorem 3.1(b) says $\alpha(w) \ge \alpha(0)$.

The proof of Theorem 3.1 depends on the following lemma:

Lemma 3.2 *If $w : \{v_0, v_1, \dots, v_n\} \to \mathbb{R}$ satisfies* (a)–(c) *in Theorem 3.1 and there is $w_{i_0} < w_0$, then there exists $\widehat{w} \in \mathbb{R}^{n+1}$ such that*

(a) *$\widehat{w}_i \ge w_i$ for $i = 1, 2, \dots, n$,*

(b) *$\widehat{w}_i \le \widehat{w}_0 = w_0$ for $i = 1, 2, \dots, n$,*

(c) *$\widehat{w} * l$ is a generalized Delaunay polyhedral metric on $\mathcal{T}$, and*

(d) *we have*

$$\alpha(\widehat{w}) > \alpha(w). \tag{8}$$

Let us first prove Theorem 3.1 using Lemma 3.2.

Proof By replacing w by $w - w(v_0)(1, 1, \dots, 1)$, we may assume that $w(v_0) = 0$. Suppose the result does not hold, ie there exists w such that $w_0 = 0$, $w_i \le 0$ for

$i = 1, 2, \dots, n$ with one $w_{i_0} < 0$, and $w * l$ is a generalized Delaunay PL metric on $\mathcal{T}$ such that $\alpha(w) \geq \alpha(0)$. We will derive a contradiction as follows. By Lemma 3.2, we may assume, after replacing w by $\widehat{w}$, that

$$\alpha(w) > \alpha(0). \tag{9}$$

Consider the set

$$X = \{x \in \mathbb{R}^{n+1} \mid w \leq x \leq 0,\ x_0 = 0,\ x * l \text{ is a generalized Delaunay polyhedral metric on } \mathcal{T}\}.$$

Clearly $w \in X$ and therefore $X \neq \varnothing$ and X is bounded. Since inner angles are continuous in edge lengths, we see that X is a closed set in $\mathbb{R}^{n+1}$. Therefore, X is compact. Let $t \in X$ be a maximum point of the continuous function $f(x) = \alpha(x)$ on X. We claim that $t = 0$. To prove this, we assume $t \neq 0$ and $t \leq 0$. Then, by Lemma 3.2, we can find $\hat{t} \geq t$ such that $\hat{t}_0 = 0$ and $\hat{t} \leq 0$, and $\hat{t} * l$ is a generalized Delaunay polyhedral metric on $\mathcal{T}$ with $\alpha(\hat{t}) > \alpha(t)$. This contradicts the maximality of t. Now, for $t = 0$, we have

$$\alpha(0) = \alpha(t) \geq \alpha(w) > \alpha(0),$$

where the last inequality follows from (9). This is a contradiction. □

Now back to the proof of Lemma 3.2:

Proof After replacing w by $w - w_0(1, 1, \dots, 1)$, we way assume $w_0 = 0$. Let $a_i = a_i(w) = a_i(w_0, w_i, w_{i+1})$, $b_i = b_i(w) = b_i(w_0, w_{i-1}, w_i)$ and $c_i = c_i(w) = c_i(w_0, w_i, w_{i+1})$ be the inner angles $\angle v_0 v_{i+1} v_i$, $\angle v_0 v_{i-1} v_i$ and $\angle v_i v_0 v_{i+1}$ in the metric $w * l$, respectively. See Figure 2. Let $l_i = l(v_0 v_i)$ and $l_{i,i+1} = l(v_i v_{i+1})$ be the edge lengths in the metric l.

Let us begin the proof with the simplest case, where all triangles in $w * l$ are nondegenerate (ie $w * l$ is a PL metric) and $w_i < 0$ for all $i \geq 1$. Let $j \in \{1, 2, 3, \dots, n\}$ be the index such that $w * l(v_0 v_j) = \min\{w * l(v_0 v_k) \mid k = 1, 2, \dots, n\}$. It is well known that in, a Euclidean triangle $\triangle ABC$, $\angle A < \frac{\pi}{2}$ if $\overline{BC}$ is not the unique largest edge. Hence, due to $w * l(v_0 v_j) \leq w * l(v_0 v_{j\pm 1})$, in the triangles $\Delta v_0 v_j v_{j\pm 1}$, we have

$$a_j(w) < \tfrac{\pi}{2}, \quad b_j(w) < \tfrac{\pi}{2} \quad \text{and} \quad a_j(w) + b_j(w) < \pi. \tag{10}$$

Now consider $\widehat{w} = (w_0, w_1, \dots, w_{j-1}, w_j + t, w_{j+1}, \dots, w_n)$. For small $t > 0$, $\widehat{w(t)} * l$ is still a PL metric since $w * l$ is. We claim $\widehat{w} * l$ is still Delaunay for small t. Indeed,

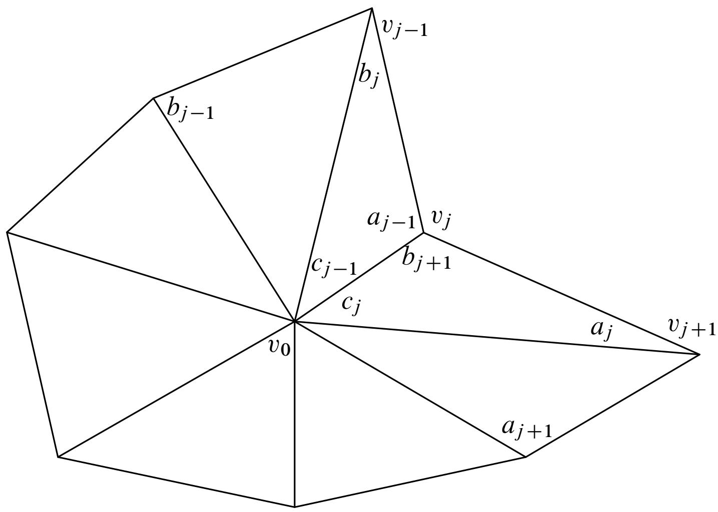

Figure 2: Star triangulation of an n–sided polygon.

by Proposition 2.1, both angles a_{j-1} and b_{j+1} decrease in t. On the other hand, $a_{j+1}(w) = a_{j+1}(\hat{w})$ and $b_{j-1}(w) = b_{j-1}(\hat{w})$. Therefore, the Delaunay conditions $b_{j-1} + a_{j-1} \le \pi$ and $b_{j+1} + a_{j+1} \le \pi$ hold for the edges $v_0 v_{j\pm1}$. The Delaunay condition on the edge $v_0 v_j$ follows from the choice of j that $a_j + b_j < \pi$. Finally, by Proposition 2.1(b), $d\alpha(\hat{w})/dt = \cot(a_j) + \cot(b_j) = \sin(a_j + b_j)/\sin(a_j)\sin(b_j) > 0$. Therefore, for small $t > 0$, we have $\alpha(\hat{w}) > \alpha(w)$.

In the general case, the above arguments still work.

Let $J = \{j \in V \mid w_j < 0\}$. By assumption, $J \neq \varnothing$.

Claim 1 *If $j \in J$, then $c_j(w) < \pi$ and $c_{j-1}(w) < \pi$.*

We prove $c_{j-1}(w) < \pi$ by contradiction. Suppose otherwise that $c_{j-1}(w) = \pi$. Then the triangle $\Delta v_0 v_j v_{j-1}$ is degenerate in the $w * l$ metric, ie $e^{w_j + w_{j-1}} l_{j,j-1} = e^{w_j} l_j + e^{w_{j-1}} l_{j-1}$. Due to $w_j < 0$ and $w_{j-1} \le 0$, we have

$$\begin{aligned} e^{w_j + w_{j-1}} l_{j,j-1} &= e^{w_j} l_j + e^{w_{j-1}} l_{j-1} \\ &> e^{w_j + w_{j-1}} l_j + e^{w_j + w_{j-1}} l_{j-1} = e^{w_j + w_{j-1}} (l_j + l_{j-1}). \end{aligned}$$

This shows $l_{j,j-1} > l_j + l_{j-1}$, which contradicts the triangle inequality for the l metric. Therefore, $c_{j-1}(w) < \pi$. By the same argument, we have $c_j(w) < \pi$. This proves Claim 1.

Let $I = \{i > 0 \mid w_i = 0\}$ and

$$\beta(w) = \sum_{i \in I} (b_i(w) + a_i(w))$$

and

$$\gamma(w) = \sum_{j \in J} (b_j(w) + a_j(w)).$$

Note that the cone angle at v_0 is

$$\alpha(w) = \sum_{i=1}^{n} (\pi - a_i(w) - b_{i+1}(w)) = \pi n - \beta(w) - \gamma(w).$$

By the assumption that $\alpha(w) \geq \alpha(0)$, we have

$$\beta(w) + \gamma(w) \leq \beta(0) + \gamma(0). \tag{11}$$

Claim 2 *If $I \neq \varnothing$, then $\beta(w) > \beta(0)$.*

Indeed, if $i \in I$, ie $w_i = 0$, then, in the triangle $\Delta v_0 v_i v_{i\pm1}$, we have $w_0 = 0$, $w_i = 0$ and $w_{i\pm1} \leq 0$. Since the $\Delta v_0 v_i v_{i-1}$ are generalized triangles in both the l and $w * l$ metrics, by Proposition 2.1(a), we see that $\Delta v_0 v_i v_{i-1}$ is a generalized triangle in $(w_0, \ldots, w_{i-2}, t w_{i-1}, w_i, \ldots, w_n) * l$ for $t \in [0, 1]$. By Proposition 2.1 and since $w_{i-1} \leq 0$, $b_i(w_0, \ldots, w_{i-2}, t w_{i-1}, w_i, \ldots, w_n)$ is increasing in $t \geq 0$ and is strictly increasing in $t \geq 0$ if $w_{i-1} < 0$. Therefore,

$$\begin{aligned} b_i(w) &= b_i(w_0, w_{i-1}, w_i) \\ &\geq b_i(w_0, 0, w_i) = b_i(w_0, w_1, \ldots, w_{i-2}, 0, w_i, \ldots, w_n) = b_i(0), \end{aligned}$$

and $b_i(w) > b_i(0)$ if $w_{i-1} < 0$. Applying the same argument to $\Delta v_0 v_i v_{i+1}$ and a_i, we have $a_i(w) \geq a_i(0)$ and $a_i(w) > a_i(0)$ if $w_{i+1} < 0$. Therefore, $\beta(w) \geq \beta(0)$. On the other hand, since $J \neq \varnothing$, there exists an $i \in I$ such that either $i-1$ or $i+1$ is in J. Say $i - 1 \in J$, ie $w_{i-1} < 0$. Then we have $b_i(w) > b_i(0)$ and $\beta(w) > \beta(0)$.

By Claim 2 and (11), if $I \neq \varnothing$, we conclude that

$$\gamma(w) = \sum_{j \in J} (a_j(w) + b_j(w)) < \gamma(0). \tag{12}$$

Since $w * l$ and l are Delaunay, we have $a_i(w) + b_i(w) \leq \pi$ and $a_i(0) + b_i(0) \leq \pi$ for all $i = 1, 2, \ldots, n$. This implies, by (12), that there exists $j \in J$ such that

$$a_j(w) + b_j(w) < \pi. \tag{13}$$

If $I = \varnothing$, let $j \in J = \{1, 2, 3, \dots, n\}$ be the index such that

$$w * l(v_0 v_j) = \min\{w * l(v_0 v_k) \mid k = 1, 2, \dots, n\}.$$

Then the same argument used in showing (10) and Claim 1 imply (13) still holds. (Here Claim 1 is used to show that $(w_0, \dots, w_{j-1}, w_j + t, w_{j+1}, \dots, w_n) * l$ is a generalized PL metric for small $t > 0$.)

Fix $j \in J$ as above. To finish the proof, we will show that there exists a small $t > 0$ such that, for $\hat{w} = (w_0, w_1, \dots, w_{j-1}, w_j + t, w_{j+1}, \dots, w_n) \in \mathbb{R}^{n+1}_{\leq 0}$, the following hold:

(i) $\hat{w} * l$ is a generalized polyhedral metric on $\mathcal{T}$.

(ii) $\hat{w} * l$ satisfies the Delaunay condition.

(iii) $\alpha(\hat{w}) > \alpha(w)$.

Since $w_j < 0$, any $t \in (0, -w_j)$ will make $\hat{w} \in \mathbb{R}^{n+1}_{\leq 0}$.

To see part (i), by Claim 1 and (13), which imply $a_j(w), b_j(w), c_j(w), c_{j-1}(w) < \pi$, the triangle $(\Delta v_0 v_j v_{j+1}, w * l)$ (or $(\Delta v_0 v_j v_{j-1}, w * l)$) is either nondegenerate or is degenerate with angle π at v_j, ie $b_{j+1}(w) = \pi$ (or $a_{j-1}(w) = \pi$, respectively). Therefore, by Proposition 2.1(c), for small $t > 0$, $\hat{w} * l$ is still a generalized PL metric.

To see part (ii), we check the sum of opposite angles at the edges $v_0 v_{j-1}$, $v_0 v_{j+1}$ and $v_0 v_j$. At the edge $v_0 v_j$, due to (13) and continuity, we see $a_j(\hat{w}) + b_j(\hat{w}) < \pi$ for small $t > 0$. At the edge $v_0 v_{j-1}$ (or similarly $v_0 v_{j+1}$), by Proposition 2.1(c), which says that $a_{j-1}(\hat{w} * l)$ and $b_{j+1}(\hat{w} * l)$ are strictly decreasing functions in $t > 0$ and $b_{j-1}(\hat{w}) = b_{j-1}(w)$, we have

$$a_{j-1}(\hat{w}) + b_{j-1}(\hat{w}) < a_{j-1}(w) + b_{j-1}(w) \leq \pi.$$

Similarly, we have the Delaunay condition for $\hat{w} * l$ at the edge $v_0 v_{j+1}$.

Finally, to see (iii), by Proposition 2.1 and (13), we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \alpha(\hat{w}) &= \frac{d}{dt}\Big|_{t=0} (c_j(\hat{w})) + \frac{d}{dt}\Big|_{t=0} (c_{j-1}(\hat{w})) \\ &= \cot(b_j(\hat{w})) + \cot(a_j(\hat{w})) > 0. \end{aligned}$$

Therefore, for small $t > 0$, $\alpha(\hat{w}) > \alpha(w)$. □

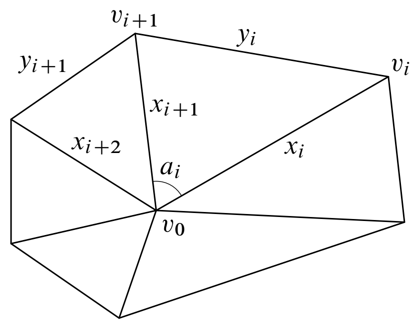

Figure 3: Triangulated hexagon and length ratio.

Lemma 3.3 *Let $(P_N, \mathcal{T})$ be a star triangulation of an N–gon with boundary vertices $v_1, \dots, v_N$ labeled cyclically and one interior vertex v_0, and $l\colon E(\mathcal{T}) \to \mathbb{R}_{>0}$ be a flat generalized PL metric on $\mathcal{T}$. There is a constant $\lambda(l)$ depending on l such that, if $(P_N, \mathcal{T}, w * l)$ with $w\colon \{v_0, \dots, v_N\} \to \mathbb{R}$ is a generalized PL metric with zero curvature at v_0, then the ratio of edge lengths satisfies*

$$\frac{w * l(v_i v_0)}{w * l(v_i v_{i+1})} \le \lambda(l) \tag{14}$$

for all indices.

Proof Let $x_i(w) = w * l(v_0 v_i)$ and $y_i(w) = w * l(v_i v_{i+1})$ be the edge lengths in the metric $w * l$, where $v_{N+1} = v_1$. By definition,

$$\frac{x_{i+2}}{y_{i+1}} = \lambda_i \frac{x_i}{y_i}, \tag{15}$$

where $\lambda_i > 0$ depends on l. Then

$$\frac{x_{i+1}}{y_{i+1}} \ge \frac{x_{i+2} - y_{i+1}}{y_{i+1}} = \frac{x_{i+2}}{y_{i+1}} - 1 = \lambda_i \frac{x_i}{y_i} - 1. \tag{16}$$

We prove by contradiction. If the result of the lemma is not true, then there exists a sequence of conformal factors $w^{(n)}$ such that

$$\frac{x_i(w^{(n)})}{y_i(w^{(n)})} \to \infty$$

for some i. Without loss of generality, assume $i = 1$; then, by (16), inductively we have

$$\frac{x_2(w^{(n)})}{y_2(w^{(n)})} \to \infty, \quad \frac{x_3(w^{(n)})}{y_3(w^{(n)})} \to \infty, \quad \dots, \quad \frac{x_N(w^{(n)})}{y_N(w^{(n)})} \to \infty.$$

Then the angles $a_i(w^{(n)})$ at v_0 in the triangle $\Delta v_0 v_i v_{i+1}$ (in the $w^{(n)} * l$ metric) converge to 0 for any i. But that contradicts the fact that the curvature $2\pi - \sum_{i=1}^{N} a_i(w^{(n)})$ at v_0 is zero. □

The next result concerns linear discrete conformal factor and spiral hexagonal triangulations. It is a counterpart of Doyle spiral circle packing in the discrete conformal setting. Unlike Doyle spiral circle packing, not all choices of linear functions produce generalized PL metrics.

We begin by recalling the developing maps. If $(S, \mathcal{T}, l)$ is a flat generalized PL metric on a simply connected surface S (ie $K_v = 0$ for all interior vertices v), then a *developing map* $\phi\colon (S, \mathcal{T}, l) \to \mathbb{C}$ for $(\mathcal{T}, l)$ is an isometric immersion determined by $|\phi(v) - \phi(v')| = l(vv')$ for $v \sim v'$. It is constructed as follows. Fix a generalized triangle $t \in \mathcal{T}$ and isometrically embeds t to $\mathbb{C}$. This defines $\phi|_t$. If s is a generalized triangle sharing a common edge e with t, we can extend $\phi|_t$ to $\phi|_{t\cup s}$ by isometrically embedding s to $\phi(s) \subset \mathbb{C}$ sharing the edge $\phi(e)$ with $\phi(t)$ so that $\phi(s)$ and $\phi(t)$ are on different sides of $\phi(e)$. Since the surface is simply connected, by the monodromy theorem, we can keep extending ϕ to all triangles in $\mathcal{T}$ and produce a well-defined isometric immersion. As a convention, if τ is a triangle in $\mathcal{T}$ and l is a generalized PL metric on $\mathcal{T}$, we use (τ, l) to denote the induced generalized PL metric on τ.

Given a lattice L in $\mathbb{C}$, there exists a Delaunay triangulation $\mathcal{T}_{\mathrm{st}} = \mathcal{T}_{\mathrm{st}}(L)$ of $\mathbb{C}$ with vertex set L such that $\mathcal{T}_{\mathrm{st}}$ is invariant under the translation action of L. In particular, $\mathcal{T}_{\mathrm{st}}$ descends to a 1–vertex triangulation of the torus $\mathbb{C}/L$. Therefore, the degree of each vertex $v \in \mathcal{T}_{\mathrm{st}}$ is 6, ie this triangulation is topologically the same as the standard hexagonal triangulation of $\mathbb{C}$. Let $l_0\colon E(\mathcal{T}_{\mathrm{st}}) \to \mathbb{R}_{>0}$ be the edge length function of $(\mathbb{C}, \mathcal{T}_{\mathrm{st}}(L), d_{\mathrm{st}})$, where d_{st} is the standard flat metric on $\mathbb{C}$. Let τ be a triangle in $\mathcal{T}_{\mathrm{st}}$ with vertices 0, u_1 and u_2. Then $L = u_1\mathbb{Z} + u_2\mathbb{Z}$ and $\{u_1, u_2\}$ is called a *geometric basis* of L. Note that two vertices $v, v' \in L$ are joined by an edge $e \in \mathcal{T}_{\mathrm{st}}$ if and only if $v - v' \in \{\pm u_1, \pm u_2, \pm(u_1 - u_2)\}$.

Proposition 3.4 *Suppose $(\mathbb{C}, \mathcal{T}_{\mathrm{st}}, l_0)$ is a hexagonal Delaunay triangulation of the plane with vertex set a lattice $V = u_1\mathbb{Z} + u_2\mathbb{Z}$, where $\{u_1, u_2\}$ is a geometric basis. Let $w\colon V \to \mathbb{R}$ be a nonconstant linear function $w(nu_1 + mu_2) = n\ln(\lambda) + m\ln(\mu)$ for $m, n, \in \mathbb{Z}$ such that $w * l_0$ is a generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$. Then the following hold*:

(a) *The generalized PL metric $(\mathcal{T}_{\mathrm{st}}, w * l_0)$ is flat.*

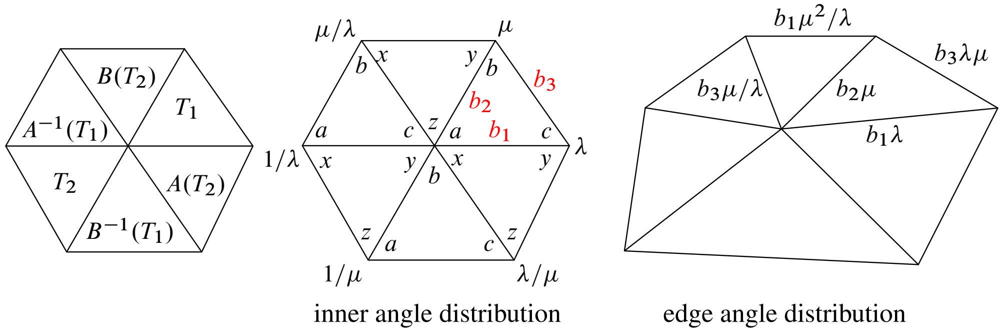

Figure 4: Flatness of spiral hexagonal triangulations.

Let ϕ be the developing map for the flat metric $(\mathcal{T}_{\text{st}}, w * l_0)$.

(b) *If there exists a nondegenerate triangle in the generalized PL metric $w * l_0$, then there are two distinct nondegenerate triangles σ_1 and σ_2 in $(\mathcal{T}_{\text{st}}, w * l_0)$ such that $\phi(\text{int}(\sigma_1)) \cap \phi(\text{int}(\sigma_2)) \neq \varnothing$.*

(c) *Suppose all triangles in $w * l_0$ are degenerate. Then there exists an automorphism ψ of the triangulation $\mathcal{T}_{\text{st}}$ such that $w(\psi(nu_1 + mu_2)) = n\ln(\gamma_1(u_1, u_2)) + m\ln(\gamma_2(u_1, u_2))$, where $\gamma_i(u_1, u_2)$ are two explicit numbers depending on u_1 and u_2* (see (17)).

We remark that parts (a) and (b) for the lattice $\mathbb{Z} + e^{2\pi i/3}\mathbb{Z}$ were proved in [28].

Proof Consider two automorphisms A and B of the topological triangulation $\mathcal{T}_{\text{st}}$ defined by $A(v) = v + u_1$ and $B(v) = v + u_2$ for $v \in V$. By definition, we have $AB = BA$ and A and B generate the group $\langle A, B\rangle \cong \mathbb{Z}^2$ acting on $\mathcal{T}_{\text{st}}$. Any triangle in $\mathcal{T}_{\text{st}}$ is equivalent, under the action of $\langle A, B\rangle$, to exactly one of the two triangles T_1 or T_2, where the vertices of T_1 are 0, u_1 and u_2 and the vertices of T_2 are 0, $-u_1$ and $-u_2$. In the generalized PL metric $w * l_0$, the maps A and B satisfy $w * l_0(A(e)) = \lambda^2 w * l_0(e)$ and $w * l_0(B(e)) = \mu^2 w * l_0(e)$ for each edge $e \in \mathcal{T}$. It follows that, for any triangle $\tau \in \mathcal{T}_{\text{st}}$, the generalized triangle $(A(\tau), w * l_0)$ (resp. $(B(\tau), w * l_0)$) is the scalar multiplication of $(\tau, w * l_0)$ by λ^2 (resp. by μ^2). Hence, there are only two similarity types of triangles in $(\mathbb{C}, \mathcal{T}_{\text{st}}, w * l_0)$. For each $v \in V$, the six angles at v are congruent to the six inner angles in T_1 and T_2 in the $w * l_0$ metric. Therefore, $(\mathcal{T}, w * l_0)$ is a flat metric. See Figure 4, center.

By the assumption that w is not a constant, we have $(\lambda, \mu) \neq (1, 1)$. Say $\lambda \neq 1$. Using the developing map ϕ, there exist two complex affine maps α and β of the complex plane $\mathbb{C}$ such that $\phi A = \alpha\phi$ and $\phi B = \beta\phi$. Since A is a scaling by the factor $\lambda^2 \neq 1$ and ϕ is a local isometry, the affine map α is of the form $\alpha(z) = \lambda^* z + c$, where $|\lambda^*| = \lambda^2 \neq 1$ and α has a unique fixed point $p \in \mathbb{C}$. By $AB = BA$, it follows that $\alpha\beta = \beta\alpha$. Therefore, from $\beta(p) = \beta\alpha(p) = \alpha\beta(p)$, we conclude $\beta(p) = p$. After replacing the developing map ϕ by $\rho \circ \phi$ for an isometry ρ of $\mathbb{C}$, we may assume that α and β both fix 0, ie $\alpha(z) = \lambda^* z$ and $\beta(z) = \mu^* z$ are both scalar multiplications. Let $G = \langle \alpha, \beta \rangle$ be the abelian group generated by α and β which acts on $\mathbb{C}$ by scalar multiplication.

To see part (b), let Ω be the image $\phi(\mathbb{C})$ of the developing map which is invariant under the action of G. By the assumption that there are nondegenerate triangles in $(\mathcal{T}_{\mathrm{st}}, w * l_0)$, the image Ω has nonempty interior. There are two cases we have to consider. In the first case, there exists a pair of integers $(n, m) \neq (0, 0)$ such that $\alpha^n \beta^m$ is the identity element in the group G. In this case, we take σ_1 to be any nondegenerate triangle and $\sigma_2 = A^n B^m(\sigma_1)$. By definition, we have $\phi(\sigma_1) = \phi(\sigma_2)$. Therefore, the result holds. In the second case, for all $(n, m) \neq (0, 0)$, $\alpha^n \beta^m \neq \mathrm{id}$, ie the group G is isomorphic to $\mathbb{Z}^2$. Since both $\alpha(z)$ and $\beta(z)$ are scalar multiplications, this implies that the action of the group G on $\mathrm{int}(\Omega)$ is not discontinuous. In particular, for any nonempty open set $U \subset \Omega$, there is $\alpha^n \beta^m \in G - \{\mathrm{id}\}$ such that $\alpha^n \beta^m(U) \cap U \neq \varnothing$. Take σ_1 to be a nondegenerate triangle, $U = \phi(\mathrm{int}(\sigma_1))$ and $\sigma_2 = A^n B^m(\sigma_1)$. Then we have $\phi(\mathrm{int}(\sigma_1)) \cap \phi(\mathrm{int}(\sigma_2)) \neq \varnothing$.

To see part (c), since each triangle is degenerate, the inner angles a, b and c and x, y and z of the two triangles T_1 and T_2 are 0 or π, as shown in Figure 4, center. Composing with an automorphism of $\mathcal{T}_{\mathrm{st}}$, we may assume that $a = \pi$, and then, by the Delaunay condition, $y = 0$.

There are two cases, depending on $(x, y, z) = (\pi, 0, 0)$ or $(0, 0, \pi)$. The two cases differ by the automorphism ρ of the lattice $u_1\mathbb{Z} + u_2\mathbb{Z}$ and of $\mathcal{T}_{\mathrm{st}}$ such that $\rho(u_1) = u_2$, $\rho(u_2) = u_2 - u_1$ and $\rho(0) = 0$. Thus, it suffices to consider the case $z = \pi$. Let the lengths of u_1, u_2 and $u_2 - u_1$ in the l_0 metric be b_1, b_2 and b_3, respectively. The lengths of the corresponding edges in the $w * l_0$ metric are λb_1, μb_2 and $\lambda\mu b_3$. By the same computation, one works out the edge lengths of the triangle with vertices 0, u_2 and $u_2 - u_1$ in the $w * l_0$ metric to be $(\mu^2/\lambda) b_1$, μb_2 and $(\mu/\lambda) b_3$. See Figure 4, right.

We obtain two equations for the edge lengths of degenerate triangles: $\lambda b_1 + \mu b_2 = \lambda\mu b_3$ (due to $a = \pi$) and $(\mu^2/\lambda) b_1 = \mu b_2 + (\mu/\lambda) b_3$ (due to $z = \pi$). See Figure 4, right.

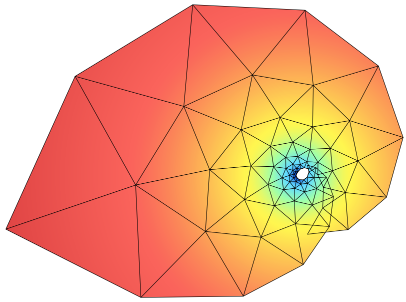

Figure 5: Spiral hexagonal triangulations.

These are same as $\lambda b_1 + \mu b_2 = \lambda\mu b_3$ and $\mu b_1 = \lambda b_2 + b_3$. Solving μ in terms of λ, we obtain a quadratic equation in λ,

$$b_2 b_3 \lambda^2 + (b_3^2 - b_1^2 - b_2^2)\lambda - b_2 b_3 = 0. \tag{17}$$

Since $b_i > 0$, this equation has a unique positive solution, which we call $\gamma_1(u_1, u_2)$. The solution in $\mu = (\gamma_1(u_1,u_2)b_2 + b_3)/b_1$ is $\gamma_2(u_1,u_2)$. □

4 Rigidity of hexagonal triangulations of the plane

We begin with:

Definition 4.1 A flat generalized PL metric on a simply connected surface $(X, \mathcal{T}, l)$ with developing map ϕ is said to be *embeddable* into $\mathbb{C}$ if, for every simply connected finite subcomplex P of $\mathcal{T}$, there exists a sequence of flat PL metrics on P whose developing maps ϕ_n converge uniformly to $\phi|_P$ and $\phi_n\colon P \to \mathbb{C}$ is an embedding.

For instance, all geometric triangulations of open sets in $\mathbb{C}$ are embeddable. However, the spiral flat triangulations produced in Proposition 3.4 are not embeddable. The main result in this section works for embeddable flat PL metrics only.

The following lemma is a consequence of the definition:

Lemma 4.2 *Suppose $(X, \mathcal{T}, l)$ is a flat generalized PL metric on a simply connected surface with a developing map ϕ.*

(a) *Suppose ϕ is embeddable. If t_1 and t_2 are two distinct nondegenerate triangles or two distinct edges in $\mathcal{T}$, then $\phi(\mathrm{int}(t_1)) \cap \phi(\mathrm{int}(t_2)) = \varnothing$.*

(b) *If ϕ is the pointwise convergent limit $\lim_{n\to\infty} \psi_n$ of the developing maps ψ_n of embeddable flat generalized PL metrics $(X, \mathcal{T}, l_n)$, then $(X, \mathcal{T}, l)$ is embeddable.*

Proof To see (a), suppose the contrary. Take P to be a finite simply connected subcomplex containing t_1 and t_2; then the developing maps ϕ_n defined on P which converge uniformly to $\phi|_P$ must satisfy $\phi_n(\mathrm{int}(t_1)) \cap \phi_n(\mathrm{int}(t_2)) \neq \varnothing$ for n large. This contradicts that the ϕ_n are embeddings.

Part (b) follows from the fact that the ψ_n converge to ϕ uniformly on compact subsets and the fact that, if $\lim_{n\to\infty} a_n = a$ and $\lim_{m\to\infty} b_{n,m} = a_n$, then $a = \lim_{j\to\infty} b_{j,n_j}$ for some subsequence. □

Let $\mathcal{T}_{\mathrm{st}}$ be a hexagonal Delaunay triangulation of the plane $S = \mathbb{C}$ with vertex set the lattice $V = \{u_1 n + u_2 m \mid n, m \in \mathbb{Z}\}$ and $l_0 \colon E(\mathcal{T}_{\mathrm{st}}) \to \mathbb{R}_{>0}$ be the edge length function associated to $(S, \mathcal{T}_{\mathrm{st}}, d_{\mathrm{st}})$. Given a flat generalized PL metric $(S, \mathcal{T}_{\mathrm{st}}, l)$, its normalized developing map $\phi = \phi_l \colon S \to \mathbb{C}$ is a developing map such that $\phi(0) = 0$ and $\phi(u_1)$ is in the positive x–axis. Suppose $\{u_1, u_2\}$ is a geometric basis of the lattice $u_1\mathbb{Z} + u_2\mathbb{Z}$. Two vertices v and v' are adjacent in $\mathcal{T}_{\mathrm{st}}$, ie $v \sim v'$, if and only if $v = v' + \delta$ for some $\delta \in \{\pm u_1, \pm u_2, \pm(u_1 - u_2)\}$. Given two vertices $v, v' \in V$, the combinatorial distance $d_c(v, v')$ between v and v' is the length of the shortest edge path joining them.

The goal of this section is to prove the following stronger version of Theorem 1.3:

Theorem 4.3 *Suppose $(S, \mathcal{T}_{\mathrm{st}}, l_0)$ is a hexagonal Delaunay triangulation whose vertex set is a lattice in $\mathbb{C}$ and $(S, \mathcal{T}_{\mathrm{st}}, w * l_0)$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$. If $(S, \mathcal{T}_{\mathrm{st}}, w * l_0)$ is embeddable into $\mathbb{C}$, then w is a constant function.*

Theorem 1.3 is clearly a special case of Theorem 4.3. Theorem 4.3 will be proved using several lemmas.

4.1 Limits of discrete conformal factors

The following lemma is a corollary of Theorem 3.1:

Lemma 4.4 *Suppose $(S, \mathcal{T}_{\mathrm{st}}, w * l_0)$ is a flat generalized Delaunay PL metric surface, $\delta \in V$ and $f \colon V \to \mathbb{R}$ is defined by $f(v) = w(v + \delta) - w(v)$. Then $f * (w * l_0) = (f + w) * l_0$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$. In particular, if there exists a vertex v_0 such that $f(v_0) = \max\{f(v) \mid v \in V\}$, then f is constant.*

We next show how to produce discrete conformal factors w such that $w(v + \delta) - w(v)$ are constants:

Lemma 4.5 *Suppose $w * l_0$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$. Then, for any $\delta \in \{\pm u_1, \pm u_2, \pm(u_1 - u_2)\}$, there exist $v_n \in V$ such that $w_n \in \mathbb{R}^V$ defined by $w_n(v) = w(v + v_n) - w(v_n)$ satisfies:*

(a) *For all $v \in V$, the following limit exists:*
$$w_\infty(v) = \lim_{n\to\infty} w_n(v) \in \mathbb{R}.$$

(b) *$w_n * l_0$ and $w_\infty * l_0$ are flat generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$.*

(c) *$w_\infty(v+\delta) - w_\infty(v) = a$ for all $v \in V$, where $a = \sup\{w(v+\delta) - w(v) \mid v \in V\}$.*

(d) *The normalized developing maps $\phi_{w_n * l_0}$ of $w_n * l_0$ converges uniformly on compact sets in S to the normalized developing map ϕ_∞ of $w_\infty * l_0$. In particular, if $(S, \mathcal{T}_{\mathrm{st}}, w * l_0)$ is embeddable, then $(S, \mathcal{T}_{\mathrm{st}}, w_\infty * l_0)$ is embeddable.*

Proof By Lemma 3.3, there is a constant $M = M(V)$, depending only on the lattice $V = u_1\mathbb{Z} + u_2\mathbb{Z}$, such that $a = \sup\{w(v+\delta) - w(v) \mid v \in V\} \le M(V)$. Take $v_n \in V$ such that
$$w(v_n + \delta) - w(v_n) \ge a - \frac{1}{n}.$$
By definition,
$$w_n(0) = 0, \quad w_n(\delta) \ge a - \frac{1}{n}, \quad w_n(v+\delta) - w_n(v) \le a \tag{18}$$
and
$$\sup\{|w_n(v) - w_n(v')| \mid v \sim v'\} < \infty. \tag{19}$$
By Lemma 3.3, if $v \in V$ is of combinatorial distance m to 0, then, using $w_n(0) = 0$, we have
$$|w_n(v)| \le m M(V). \tag{20}$$
By (20) and the diagonal argument, we see that there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$ for simplicity, such that w_n converges to $w_\infty \in \mathbb{R}^V$ in the pointwise convergence topology. By construction, each $w_n * l_0$ is a flat generalized Delaunay PL metric. By $\lim_{n\to\infty} w_n = w_\infty$ and continuity, we conclude that $w_\infty * l_0$ is again a flat generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$. By (18),
$$w_\infty(\delta) - w_\infty(0) = \max\{w_\infty(v+\delta) - w_\infty(v) \mid v \in V\}.$$
By Lemma 4.4, we see that conclusion (c) holds. Since the developing map $\phi_{w * l_0}$ depends continuously on $w \in \mathbb{R}^V$, $\lim_{n\to\infty} \phi_{w_n * l_0}(v) = \phi_\infty(v)$ for each vertex $v \in V$.

On the other hand, a developing map ϕ is determined by its restriction to V. We see that $\phi_{w_n * l_0}$ converges to ϕ_∞ uniformly on compact subsets of the plane. The last statement follows from Lemma 4.2(b) since each $\phi_{w_n * l_0}$ is embeddable by definition. □

4.2 Proof of Theorem 4.3

Suppose $w * l_0$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$ with an embeddable developing map ϕ. Our goal is to show that $w\colon V \to \mathbb{R}$ is a constant. Suppose otherwise; we will derive a contradiction by showing that the developing map ϕ is not embeddable.

Since w is not a constant, we can choose $\delta_1 \in \{\pm u_1, \pm u_2, \pm(u_1 - u_2)\}$ such that $a_1 = \sup\{w(v+\delta_1) - w(v) \mid v \in V\} > 0$. By Lemma 4.5 applied to $w * l_0$ and $\delta = \delta_1$, we produce a function $w_\infty\colon V \to \mathbb{R}$ such that $w_\infty * l_0$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$ and $w_\infty(v+\delta_1) = w_\infty(v) + a_1$ for all $v \in V$. Now, applying Lemma 4.5 to $w_\infty * l_0$ with $\delta_2 \in \{\pm u_1, \pm u_2, \pm(u_1 - u_2)\} - \{\pm\delta_1\}$, we obtain a second function $\widehat{w} = (w_\infty)_\infty\colon V \to \mathbb{R}$ and $b_1 \in \mathbb{R}$ such that $\widehat{w} * l_0$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{\mathrm{st}}$ and

$$\widehat{w}(v+\delta_1) = \widehat{w}(v) + a_1, \quad \widehat{w}(v+\delta_2) = \widehat{w}(v) + b_1$$

for all $v \in V$. This shows that $\widehat{w}\colon V \to \mathbb{R}$ is a nonconstant affine function, ie $\widehat{w}(n + me^{\pi i/3}) = a_2 n + b_2 m + c_2$ for some $a_2, b_2, c_2 \in \mathbb{R}$.

Let $\widehat{\phi}$, ϕ_∞ and ϕ be the normalized developing maps for $\widehat{w} * l_0$, $w_\infty * l_0$ and $w * l_0$, respectively. Since ϕ is embeddable, by Lemma 4.5, $\widehat{\phi}$ and ϕ_∞ are embeddable.

If $\widehat{w} * l_0$ contains a nondegenerate triangle, then, by Proposition 3.4, there exist two nondegenerate triangles t_1 and t_2 in $(\mathcal{T}_{\mathrm{st}}, \widehat{w} * l_0)$ such that $\widehat{\phi}(\mathrm{int}(t_1)) \cap \widehat{\phi}(\mathrm{int}(t_2)) \neq \varnothing$. By Lemma 4.2(a), this contradicts that $\widehat{w} * l_0$ is embeddable.

Therefore, all triangles in the generalized PL metric $\widehat{w} * l_0$ are degenerate, ie all angles in triangles are either 0 or π. We will use the same notation used in the proof of Proposition 3.4. By Proposition 3.4(c) and Figure 6, we may assume, after composing with an automorphism of $\mathcal{T}_{\mathrm{st}}$ and subtracting by a constant, that $\widehat{w}(nu_1 + mu_2) = n\ln(\gamma_1(V)) + m\ln(\gamma_2(V))$, where $(\gamma_1(V), \gamma_2(V))$ are given by the solutions of (17) and the angles a, b, c, x, y and z of T_1 and T_2 are $(a,b,c,x,y,z) = (\pi,0,0,0,0,\pi)$.

Let $P_1 = u_2 - 2u_1$, $P_2 = u_2 - u_1$, $P_3 = 0$ and $P_4 = u_1$ in V. See Figure 6, bottom left. In the case of $a = z = \pi$, we claim that the length $(\mu/\lambda)b_3$ of the edge P_2P_3 is

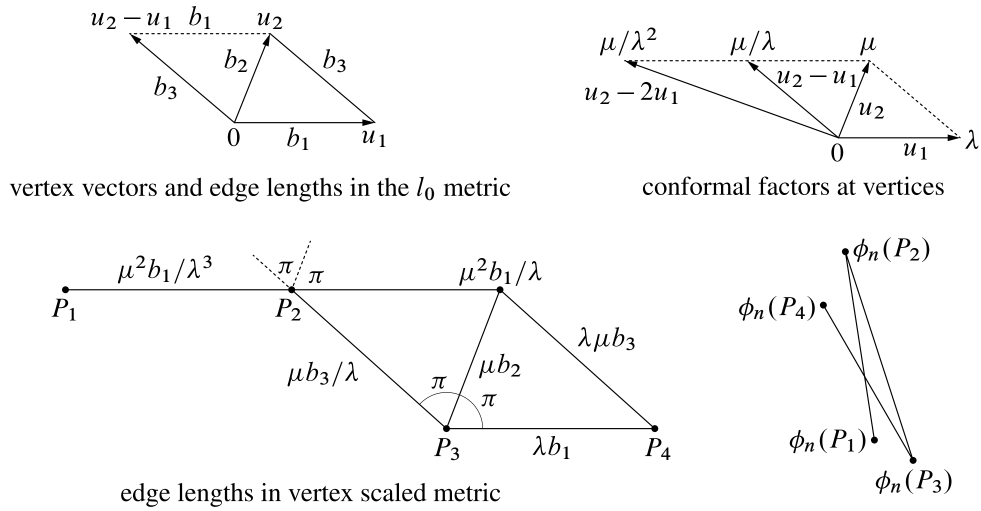

Figure 6: Angles a and z are zero in $\widehat{w} * l_0$. Top right is the developing image of corresponding set in $w * l_0$.

strictly less than the sum of the lengths λb_1 of the edge P_3P_4 and $(\mu^2/\lambda^3)b_1$ of the edge P_1P_2, ie

$$\frac{\mu}{\lambda}b_3 < \lambda b_1 + \frac{\mu^2}{\lambda^3}b_1. \tag{21}$$

Indeed, by the equations $\lambda b_1 + \mu b_2 = \lambda\mu b_3$ and $\mu b_1 = \lambda b_2 + b_3$ derived in the proof of Proposition 3.4, we obtain

$$\frac{b_3}{b_1} = \frac{\lambda^2+\mu^2}{(1+\lambda^2)\mu}.$$

Equation (21) says

$$\frac{b_3}{b_1} < \frac{\lambda^4+\mu^2}{\lambda^2\mu}.$$

Thus, it suffices to show that $(\lambda^2+\mu^2)/(1+\lambda^2)\mu < (\lambda^4+\mu^2)/\lambda^2\mu$. This is the same as $\lambda^2(\lambda^2+\mu^2) < (1+\lambda^2)(\lambda^4+\mu^4)$, ie $\lambda^4+\lambda^2\mu^2 < \lambda^4+\lambda^2\mu^2+\lambda^6+\mu^2$. The last inequality clearly holds since both λ and μ are positive.

Now consider the oriented edge path $P_1P_2P_3P_4$ (oriented from P_1 to P_4) in $\mathcal{T}_{\mathrm{st}}$ and its image under the developing map $\widehat{\phi}$ of $\widehat{w} * l_0$ in $\mathbb{C}$. By the assumption that $a = z = \pi$, the angles of the polygonal path $\widehat{\phi}(P_1P_2P_3P_4)$ at $\widehat{\phi}(P_2)$ and $\widehat{\phi}(P_3)$ are 2π. See Figure 6, bottom left. Also the sum of the lengths of $\widehat{\phi}(P_1P_2)$ and $\widehat{\phi}(P_2P_4)$ is larger than the length of $\widehat{\phi}(P_2P_3)$ by the claim above. On the other hand, since $\widehat{\phi}$ is embeddable, there exists a sequence of flat PL metrics on $\mathcal{T}_{\mathrm{st}}$ whose developing maps ϕ_n are embeddings that converge uniformly on compact sets to $\hat{\phi}$. This implies that, for n large, the two line segments $\phi_n(P_1P_2)$ and $\phi_n(P_3P_4)$ intersect in their interiors. This contradicts the assumption that ϕ_n is an embedding.

This ends the proof of Theorem 4.3. □

Remark 4.6 The above argument also gives a new proof of Rodin and Sullivan's hexagonal circle packing theorem.

The following will be used to show that the limit of discrete uniformization maps is conformal. Let $B_n(v) = \{i \in V(\mathcal{T}_{\mathrm{st}}) \mid d_c(i,v) \le n\}$ and $\mathcal{B}_n(v)$ be the subcomplex of $\mathcal{T}_{\mathrm{st}}$ whose simplices have vertices in $B_n(v)$.

Lemma 4.7 *Take the standard hexagonal lattice $V = \mathbb{Z} + e^{2\pi/3}\mathbb{Z}$ and its associated standard hexagonal triangulation, whose edge length function is $l_{\mathrm{st}}\colon V \to \{1\}$. There is a sequence s_n of positive numbers decreasing to zero with the following property: For any integer n and a vertex v, there exists $N = N(n,v)$ such that, if $m \ge N$ and $(\mathcal{B}_m(v), w * l_{\mathrm{st}})$ is a flat Delaunay triangulated PL surface with embeddable developing map, then the ratio of the lengths of any two edges sharing a vertex in $\mathcal{B}_m(v)$ is at most $1 + s_n$.*

The proof of the lemma is exactly the same as that of Rodin and Sullivan [25, pages 353–354] since we have Lemma 3.3 and Theorem 4.3, which play the roles of Rodin and Sullivan's ring lemma and rigidity of hexagonal circle packing in [25, pages 352–353].

5 Existence of discrete uniformization metrics on polyhedral disks with special equilateral triangulations

By a *polygonal disk* we mean a flat PL surface $(\mathcal{P}, V, d)$ which is isometrically embedded in the complex plane $\mathbb{C}$ with $\mathcal{P}$ homeomorphic to the closed disk. The goal of this section is to prove the existence of a discrete conformal metric by regular subdividing of the given triangulations.

An *equilateral triangulation* $\mathcal{T}$ of a polyhedral surface is a geometric triangulation whose triangles are equilateral. The edge length function of an equilaterally triangulated connected polyhedral surface will be denoted by the constant function $l_{\mathrm{st}}\colon E(\mathcal{T}) \to \mathbb{R}$.

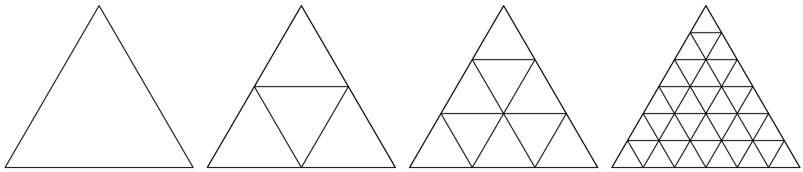

Figure 7: The standard subdivisions.

Given an equilateral Euclidean triangle $\Delta \subset \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 1}$, the n^{th} *standard subdivision* of Δ is the equilateral triangulation of Δ by n^2 equilateral triangles. See Figure 7. If $\mathcal{T}$ is an equilateral triangulation of a polyhedral surface, its n^{th} *standard subdivision*, denoted by $\mathcal{T}_{(n)}$, is the equilateral triangulation obtained by replacing each triangle in $\mathcal{T}$ by its n^{th} standard subdivision. We use $V_{(n)}$ to denote $V(\mathcal{T}_{(n)})$.

The main result of this section is the following theorem:

Theorem 5.1 *Suppose $(\mathcal{P}, \mathcal{T}, l_{\text{st}})$ is a flat polygonal disk with an equilateral triangulation $\mathcal{T}$ such that exactly three boundary vertices p, q and r have curvature $\frac{2\pi}{3}$. Then, for sufficiently large n, there is a discrete conformal factor $w_n \colon V_{(n)} \to \mathbb{R}$ for the n^{th} standard subdivision $(\mathcal{P}, \mathcal{T}_{(n)}, l_{\text{st}})$ such that the discrete curvature K of $w_n * l_{\text{st}}$ satisfies:*

(a) $K_i = 0$ *for all* $i \in V_{(n)} - \{p, q, r\}$.

(b) $K_i = \frac{2\pi}{3}$ *for all* $i \in \{p, q, r\}$.

(c) *There is a constant $\epsilon_0 > 0$, independent of n, such that all inner angles of triangles in $(\mathcal{T}_{(n)}, w_n * l_{\text{st}})$ are in the interval $\left[\epsilon_0, \frac{\pi}{2} + \epsilon_0\right]$, the sum of two angles facing each interior edge is at most $\pi - \epsilon_0$, and each angle facing a boundary edge is at most $\frac{\pi}{2} - \epsilon_0$.*

Conditions (a) and (b) imply that the underlying metric space of $(\mathcal{P}, \mathcal{T}_{(n)}, w_n * l_{\text{st}})$ is an equilateral triangle. Condition (c) says that the metric doubles of $(\mathcal{P}, \mathcal{T}_{(n)}, l_{\text{st}})$ and $(\mathcal{P}, \mathcal{T}_{(n)}, w_n * l_{\text{st}})$ are two Delaunay triangulated polyhedral 2–spheres differing by a vertex scaling.

There are two steps involved in the construction of the discrete conformal factor w_n in Theorem 5.1. In the first step, we produce a discrete conformal factor $w^{(1)} \colon V_{(n)} \to \mathbb{R}$ such that $w^{(1)}$ vanishes outside the union of combinatorial balls of radius $\left[\frac{1}{3}n\right]$ (the integral part of $\frac{1}{3}n$) centered at nonflat vertices $v \neq p, q, r$ and the discrete curvature satisfies $K_i(w^{(1)} * l_{\text{st}}) = 0$ if $d_c(i, v) < \left[\frac{1}{3}n\right]$ and $K_i(w^{(1)} * l_{\text{st}}) = O(1/\sqrt{\ln(n)})$ if

$d_c(i,v) = \left[\frac{1}{3}n\right]$. This step diffuses the nonzero discrete curvatures $\frac{\pi}{3}$, $-\frac{\pi}{3}$ and $-\frac{2\pi}{3}$ (at nonflat vertices v) to small curvatures at vertices defined by $d_c(i,v) = \left[\frac{1}{3}n\right]$. In the second step, by choosing n large such that all curvatures are very small, we use a perturbation argument to show that there is $w^{(2)}\colon V_{(n)} \to \mathbb{R}$ such that $w^{(2)} * (w^{(1)} * l_{\mathrm{st}})$ satisfies the conditions in Theorem 5.1. The required discrete conformal factor w_n is $w^{(1)} + w^{(2)}$ since $(w^{(2)} + w^{(1)}) * l_{\mathrm{st}} = w^{(2)} * (w^{(1)} * l_{\mathrm{st}})$.

The basic tools to be used for proving Theorem 5.1 are discrete harmonic functions, their gradient estimates and ordinary differential equations (ODEs). We begin by recalling the related material.

5.1 Laplace operator on a finite graph

Given a graph (V,E), the set of all oriented edges in (V,E) is denoted by $\bar{E}$. If $i \sim j$ in V, we use $[ij] \in \bar{E}$ to denote the oriented edge from i to j. If $x \in \mathbb{R}^V$ and $y \in \mathbb{R}^{\bar{E}}$, we use x_i and y_{ij} to denote $x(i)$ and $y([ij])$, respectively. A *conductance* on G is a function $\eta\colon \bar{E} \to \mathbb{R}_{\geq 0}$ such that $\eta_{ij} = \eta_{ji}$.

Definition 5.2 Given a finite graph (V,E) with a conductance η, the gradient $\nabla\colon \mathbb{R}^V \to \mathbb{R}^{\bar{E}}$ is the linear map

$$(\nabla f)_{ij} = \eta_{ij}(f_i - f_j),$$

the Laplace operator associated to η is the linear map $\triangle\colon \mathbb{R}^V \to \mathbb{R}^V$ defined by

$$(\triangle f)_i = \sum_{j \sim i} \eta_{ij}(f_i - f_j),$$

and the Dirichlet energy of $f \in \mathbb{R}^V$ on (V,E,η) is

$$\mathcal{E}(f) = \frac{1}{2}\sum_{i \sim j} \eta_{ij}(f_i - f_j)^2.$$

The following is well known (see [6]):

Proposition 5.3 (Green's identity) *Given a finite graph (V,E) with a conductance η,*

(a) *for any subset $V_0 \subset V$,*

$$\sum_{i \in V_0} f_i(\triangle g)_i - g_i(\triangle f)_i = \sum_{i \in V_0,\, j \sim i,\, j \notin V_0} \eta_{ij}(g_i f_j - f_i g_j);$$

(b) $\sum_{i \in V}(\triangle f)_i = 0$.

Given a set $V_0 \subset V$ and $g\colon V_0 \to \mathbb{R}$, the *Dirichlet problem* asks for a function $f : V \to \mathbb{R}$ such that

$$(\Delta f)_i = 0 \quad \text{for all } i \in V - V_0 \qquad \text{and} \qquad f|_{V_0} = g. \tag{22}$$

The *Dirichlet principle* states that solutions f to the Dirichlet problem (22) are the same as minimum points of the Dirichlet energy function restricted to the affine subspace $\{h \in \mathbb{R}^V \mid h|_{V_0} = g\}$, ie

$$\mathcal{E}(f) = \min\{\mathcal{E}(h) \mid h \in \mathbb{R}^V \text{ and } h|_{V_0} = g\}. \tag{23}$$

In particular, the Dirichlet problem (22) is always solvable.

A subset $U \subset V$ in a graph (V, E) is called *connected* if any two vertices $i, j \in U$ can be joined by an edge path whose vertices are in U. For instance, a connected graph (V, E) means V is a connected. The following is well known (see [6]):

Proposition 5.4 *Suppose (V, E) is a finite connected graph with a conductance $\eta_{ij} > 0$ for all edges $[ij]$ and $V_0 \subset V$. Let f be a solution to the Dirichlet problem* (22). *Then*:

(a) ***Maximum principle*** *For $V_0 \neq \varnothing$,*

$$\max_{i \in V} f_i = \max_{i \in V_0} f_i.$$

(b) ***Strong maximum principle*** *If $V - V_0$ is connected and $\max_{i \in V - V_0} f_i = \max_{i \in V_0} f_i$, then $f|_{V - V_0}$ is a constant function.*

5.2 A system of ODEs associated to discrete conformal change

Let $(S, \mathcal{T}, l)$ be a compact connected polyhedral surface with discrete curvature K^0. Given a subset $V_0 \subset V$ and a function $K^*\colon V - V_0 \to (-\infty, 2\pi)$, we try to find a function $w\colon V \to \mathbb{R}$ such that $w * l$ is a PL metric whose curvature $K(w)$ is equal to K^* on $V - V_0$ and $w|_{V_0} = 0$. In the PL metric $w * l$, let $\theta^i_{jk} = \theta^i_{jk}(w)$ be the angle at vertex i in the triangle Δijk and $\eta_{ij} = \eta_{ij}(w)$ be $\cot(\theta^k_{ij}) + \cot(\theta^l_{ij})$ if $[ij]$ is an interior edge and $\eta_{ij} = \cot(\theta^k_{ij})$ if $[ij]$ is a boundary edge. The associated Laplacian $\Delta\colon \mathbb{R}^V \to \mathbb{R}^V$ is $(\Delta f)_i = \sum_{j \sim i} \eta_{ij}(f_i - f_j)$. We will construct w by choosing a smooth 1–parameter family $w(t) \in \mathbb{R}^V$ such that $w(0) = 0$ and $w(t) * l$ is a PL metric whose curvature $K_i(t) = K_i(w(t) * l_{\mathrm{st}})$ satisfies

$$K_i(t) = (1-t)K^0_i + tK^*_i \quad \text{for all } i \in V - V_0, \qquad w_i(t) = 0 \quad \text{for all } i \in V_0. \tag{24}$$

The required vector w is defined to be $w(1)$. Note that, by definition, $K(0) = K^0$. Due to the curvature evolution equation (7), that $dK_i(t)/dt = \sum_{j\sim i} \eta_{ij}(w(t))(w_i' - w_j')$, where $w_i'(t) = dw_i(t)/dt$, we obtain the system of ODEs in $w(t)$, equivalent to (24),

$$\begin{aligned} \sum_{j\sim i} \eta_{ij}(w_i' - w_j') &= K_i^* - K_i^0 && \text{for all } i \in V - V_0, \\ w_i'(t) &= 0 && \text{for all } i \in V_0, \\ w(0) &= 0. \end{aligned} \tag{25}$$

Using $\triangle f$, we can write (25) as

$$\begin{aligned} (\Delta w')_i &= K_i^* - K_i^0 && \text{for all } i \in V - V_0, \\ w_i'(t) &= 0 && \text{for all } i \in V_0, \\ w(0) &= 0. \end{aligned} \tag{26}$$

We will show, under some assumptions on $(\mathcal{T}, l)$, that the solution to (25) exists for all $t \in [0, 1]$.

Let $W \subset \mathbb{R}^V$ be the open set

(27) $W = \{w \in \mathbb{R}^V \mid w * l \text{ is a PL metric on } \mathcal{T} \text{ and } \eta_{ij}(w) > 0 \text{ for all edges } [ij]\}.$

Lemma 5.5 *Suppose $V_0 \neq \varnothing$ and $0 \in W$. The initial valued problem* (25) *defined on W has a unique solution in a maximum interval $[0, t_0)$ with $t_0 > 0$ such that, if $t_0 < \infty$, then either $\liminf_{t\to t_0^-} \theta^i_{jk}(w(t)) = 0$ for some angle θ^i_{jk} or $\liminf_{t\to t_0^-} \eta_{ij}(w(t)) = 0$ for some edge $[ij]$.*

Proof Indeed, (25) can be written as $Y(w) \cdot w'(t) = \beta$ and $w(0) = 0$, where $Y(w)$ is a square matrix–valued smooth function of $w \in W$ and $w'(t)$ is considered as a column vector. We claim that $Y(w)$ is an invertible matrix for $w \in W$. If $Y(w)$ is invertible, then (25) can be written as $w'(t) = Y(w)^{-1}\beta$ and, by Picard's existence theorem, there exists an interval on which the ODE (25) has a solution. Now $Y(w)$ is invertible if and only if the following system of linear equations has only the trivial solution $x = 0$:

$$Y(w) \cdot x = 0. \tag{28}$$

By (25), equation (28) is the same as $(\Delta x)_i = 0$ for $i \in V - V_0$ and $x_i = 0$ for $i \in V_0$. Furthermore, $w \in W$ implies $\eta_{ij}(w) > 0$ for all edges $[ij]$. By the maximum principle (Proposition 5.4), we see that $x = 0$.

If $t_0 < \infty$ and $t \uparrow t_0$, then $w(t)$ leaves every compact set in W. For each $\delta > 0$, we claim that $W_\delta = \{w \in W \mid \theta^i_{jk} \geq \delta, |w_i| \leq 1/\delta, \eta_{ij} \geq \delta\}$ is compact. Clearly

W_δ is bounded by definition. To see that W_δ is closed in $\mathbb{R}^V$, take a sequence $x_n \in W_\delta$ such that $\lim_{n\to\infty} x_n = y \in \mathbb{R}^V$. Then $y * l$ is a generalized PL metric with all angles $\theta^i_{jk} \geq \delta$. Since each degenerate triangle has an angle which is zero, $y * l$ is a PL metric. Also, by continuity, we have $\theta^i_{jk}(y) \geq \delta$, $\eta_{ij}(y) \geq \delta$ and $|y_i| \leq 1/\delta$, ie $y \in W_\delta$. Since $w(t)$ leaves every W_δ for each $\delta > 0$, one of the following three occurs: $\liminf_{t\to t_0^-} \theta^i_{jk}(w(t)) = 0$ for some θ^i_{jk}, or $\liminf_{t\to t_0^-} \eta_{ij}(w(t)) = 0$ for some edge $[ij]$, or $\limsup_{t\to t_0^-} |w_i(t)| = \infty$ for some $i_0 \in V$. However, if $\limsup_{t\to t_0^-} |w_{i_0}(t)| = \infty$ for one vertex i_0, then $\liminf_{t\to t_0^-} \theta^l_{jk}(w(t)) = 0$ for some θ^l_{jk}. Indeed, if otherwise, $\liminf_{t\to t_0^-} \theta^l_{jk}(w(t)) \geq \delta > 0$ for all θ^l_{jk} for some δ. It is well known that in a Euclidean triangle whose angles are at least δ, the ratio of two edge lengths is at most $1/\sin(\delta)$. Therefore, in each triangle $\Delta v_i v_j v_k$ in $\mathcal{T}$, we have $e^{w_i(t)} \leq e^{w_j(t)} l(v_j v_k)/l(v_i v_k)\sin(\delta)$. Since $w_j(t) = 0$ for $j \in V_0$ and the surface S is connected, we conclude that all $w_k(t)$ for $k \in V$ are bounded for all t. This contradicts $\limsup_{t\to t_0^-} |w_{i_0}(t)| = \infty$. □

5.3 Standard subdivision of an equilateral triangle

Theorem 5.6 *Let $S = \Delta ABC$ be an equilateral triangle, $\mathcal{T}$ be the n^{th} standard subdivision of S with the associated PL metric $l_{\text{st}}\colon E = E(\mathcal{T}) \to \{1/n\}$ and $V_0 = \{v \in V \mid v$ is in the edge BC of the triangle $\Delta ABC\}$. Given any $\alpha \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, there exists a smooth family of vectors $w(t) \in \mathbb{R}^V$ for $t \in [0,1]$ such that $w(0) = 0$ and $w(t) * l_{\text{st}}$ is a PL metric on $\mathcal{T}$ with curvature $K(t) = K(w(t) * l_{\text{st}})$ satisfying:*

(a) $K_A(t) = -t\alpha + (2+t)\frac{\pi}{3}$ $\left(\text{angle at } A \text{ is } t\alpha + (1-t)\frac{\pi}{3}\right)$.

(b) $K_i(t)$=0 *for all* $i \in V - \{A\} \cup V_0$.

(c) $w_i(t) = 0$ *for all* $i \in V_0$.

(d) *All inner angles* $\theta^i_{jk}(t)$ *in metric* $w(t) * l_{\text{st}}$ *are in the interval*

$$\left[\tfrac{\pi}{3} - \left|\alpha - \tfrac{\pi}{3}\right|, \tfrac{\pi}{3} + \left|\alpha - \tfrac{\pi}{3}\right|\right] \subset \left[\tfrac{\pi}{6}, \tfrac{\pi}{2}\right].$$

(e) $\theta^i_{jk}(t) \leq \frac{59\pi}{120}$ *for* $i \neq A$.

(f) $|K_i(t) - K_i(0)| \leq 2000/\sqrt{\ln(n)}$ *for* $i \neq A$ *and*

$$\sum_{i\in V_0} |K_i(t) - K_i(0)| \leq \frac{\pi}{6}. \tag{29}$$

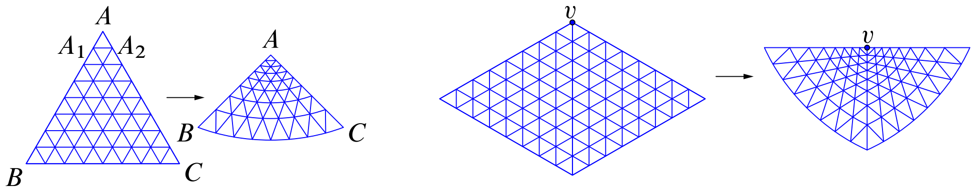

Figure 8: Discrete conformal maps of equilateral triangles and their unions.

Remark 5.7 The discrete conformal map

$$(\Delta ABC, \mathcal{T}, l_{\mathrm{st}}) \to (\Delta ABC, \mathcal{T}, w(1) * l_{\mathrm{st}})$$

is a discrete counterpart of the analytic function $f(z) = z^{3\alpha/\pi}$.

Our proof of Theorem 5.6 relies on the following two lemmas about estimates on discrete harmonic functions on $\mathcal{T}$.

Lemma 5.8 *Assume $\Delta ABC, n, \mathcal{T}$ and V_0 are as given in Theorem 5.6. Let $\tau: \mathcal{T} \to \mathcal{T}$ be the involution induced by the reflection of ΔABC about the angle bisector of $\angle BAC$ and η: $E \to \mathbb{R}_{\geq 0}$ be a conductance such that $\eta\tau = \eta$ and $\eta_{ij} = \eta_{ji}$. Let $\Delta: \mathbb{R}^V \to \mathbb{R}^V$ be the Laplace operator defined by $(\Delta f)_i = \sum_{j \sim i} \eta_{ij}(f_i - f_j)$. If $f \in \mathbb{R}^V$ satisfies $(\Delta f)_i = 0$ for $i \in V - \{A\} \cup V_0$ and $f|_{V_0} = 0$, then, for all edges $[ij]$, the gradient $(\nabla f)_{ij} = \eta_{ij}(f_i - f_j)$ satisfies*

$$|\eta_{ij}(f_i - f_j)| \leq \tfrac{1}{2}|\Delta(f)_A|. \tag{30}$$

Lemma 5.9 *Assume $\Delta ABC, n, \mathcal{T}$ and V_0 are as given in Theorem 5.6. Let $\eta: E(\mathcal{T}) \to [1/M, M]$ be a conductance function for some $M > 0$ and Δ be the Laplace operator on $\mathbb{R}^V$ associated to η. If $f: V \to \mathbb{R}$ solves the Dirichlet problem $(\Delta f)_i = 0$ for all $i \in V - \{A\} \cup V_0$, $f|_{V_0} = 0$ and $(\Delta f)_A = 1$, then, for all $u \in V_0$, $|(\Delta f)_u| \leq 20M/\sqrt{\ln n}$.*

We will prove Lemmas 5.8 and 5.9 and Theorem 5.6 in order.

The simplest way to see Lemma 5.8 is to use the theory of electric networks. We put a resistance of $1/\eta_{ij}$ Ohms at the edge $[ij]$ (if $\eta_{ij} = 0$, the resistance is ∞, or remove edge $[ij]$ from the network). Now place a one-volt battery at vertex A and ground every vertex in V_0. Then Kirchhoff's laws show that the voltage f_i at the vertex i solves the Dirichlet problem $(\Delta f)_i = 0$ for all $i \in V - \{A\} \cup V_0$, $f_A = 1$ and $f|_{V_0} = 0$. Ohm's law says $\eta_{ij}(f_i - f_j)$ is the electric current through the edge $[ij]$. Since the resistance is symmetric with respect to the symmetry τ, the currents in the network are the same as the currents in the quotient network $\mathcal{T}/\tau$. In the quotient network $\mathcal{T}/\tau$, there is only one edge e_A from the vertex A. Therefore, the current through any edge is at most the current $\frac{1}{2}|(\Delta f)_A|$ through e_A (in the network $\mathcal{T}/\tau$). This shows $|\eta_{ij}(f_i - f_j)| \le \frac{1}{2}|(\Delta f)_A|$.

Proof of Lemma 5.8 Removing all edges $[ij]$ for which $\eta_{ij} = 0$ from the graph (V, E), we obtain a finite collection of disjoint connected subgraphs $\Gamma_1, \dots, \Gamma_N$ from (V, E). By construction, the associated Laplace operators on Γ_i with conductance $\eta|_{E(\Gamma_i)}$ is the restriction of the Laplace operator Δ to $V(\Gamma_i)$. By the maximum principle (Proposition 5.4), the function $f|_{V(\Gamma_m)}$ is a constant and (30) holds unless Γ_m contains the vertex A and some vertex in V_0. Therefore, it suffices to prove the lemma for those edges $[ij]$ in the connected graph $\Gamma_m = (V', E')$ such that $A \in V'$ and $V' \cap V_0 \ne \varnothing$. Let $A_1, A_2 = \tau(A_1)$ be the vertices adjacent to A. Since $\tau(A) = A$, $\eta\tau = \eta$ and $V' \cap V_0 \ne \varnothing$, we have $\tau(\Gamma_m) = \Gamma_m$ and $A_1, A_2 \in V'$.

We will work on the graph $\Gamma_m = (V', E')$ from now on. Using the maximum principle for $f - f\tau$, we see that $f = f\tau$. By replacing f by $-f$ if necessary, we may assume that $f_A > 0$. By the maximum principle, we have that $0 \le f_i < f_A$ for all $i \in V' - \{A\}$.

Take an edge $[ij]$ in the graph Γ_m. If $\tau\{i, j\} = \{i, j\}$, then $\tau_i = j$ and $\tau_j = i$. This implies $f_i = f\tau_i = f_j$ and (30) holds. If $\tau\{i, j\} = \{i', j'\} \ne \{i, j\}$, say $\tau_i = i'$ and $\tau_j = j'$, then $f_i = f_{i'}$, $f_j = f_{j'}$. We may assume that $f_i \le f_j$. If $f_i = f_j$, then (30) holds. Hence, we may assume $f_i < f_j$. If $j = A$, then $i = A_1$ or A_2. Due to $f_{A_1} = f_{A_2}$, then (30) holds. If $j \ne A$, then, by the maximum principle applied to f on the subgraph $(V' - \{A\}, E' - \{AA_1, AA_2\})$, we conclude that $f_{A_1} \ge f_j > f_i$. Let $U = \{k \in V' - \{A\} \mid f_k > f_i\}$. By definition, $j, j', A_1, A_2 \in U$, $i, i', A \notin U$ and $V_0 \cap U = \varnothing$. This shows $(\Delta f)_k = 0$ for all $k \in U$ and hence $\sum_{k \in U}(\Delta f)_k = 0$. By Green's formula (Proposition 5.3),

$$\sum_{k \in U}(\Delta f)_k = \sum_{k \in U,\, l \notin U,\, k \sim l} \eta_{kl}(f_k - f_l) = 0.$$

If $l \notin U \cup \{A\}$, then, by definition, $f_i \ge f_l$. Therefore, if $k \in U$, $k \sim l$ and $l \notin U \cup \{A\}$, then $f_k > f_i \ge f_l$. This shows

$$\begin{aligned} 0 &= \sum_{k \in U,\, l \notin U,\, l \sim k} \eta_{kl}(f_k - f_l) \\ &= \sum_{k \in U,\, l \notin U \cup \{A\},\, l \sim k} \eta_{kl}(f_k - f_l) + \sum_{k \sim A} \eta_{kA}(f_k - f_A) \\ &\ge (\nabla f)_{ji} + (\nabla f)_{j'i'} - (\Delta f)_A. \end{aligned}$$

Therefore, $|(\Delta f)_A| \ge 2|(\nabla f)_{ij}|$ since $(\nabla f)_{ij} = (\nabla f)_{i'j'}$. □

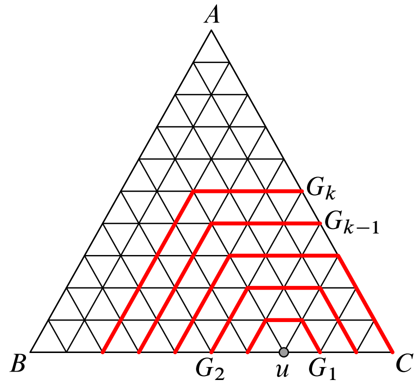

Figure 9: Layers in triangle ABC.

Proof of Lemma 5.9 For the given $u \in V_0$, construct a function $g\colon V \to \mathbb{R}$ by solving the Dirichlet problem $(\Delta g)_i = 0$ for all $i \in V - V_0$, $g_u = 1$ and $g|_{V_0 - \{u\}} = 0$. By the maximum principle (Proposition 5.4), $0 \le g_i \le 1$ for all i. Using Green's identity that $\sum_{i \in V}[f_i(\Delta g)_i - g_i(\Delta f)_i] = 0$, we obtain $g_A(\Delta f)_A + g_u(\Delta f)_u = 0$. Since $(\Delta f)_A = 1$ and $g_u = 1$, we see

$$(\Delta f)_u = -g_A.$$

Therefore, it suffices to show that $|g_A| \le 20M/\sqrt{\ln n}$. For this purpose, take $k \le \left[\frac{1}{2}n\right]$ and define $U_k = \{i \in V \mid d_c(i,u) = k\}$, where $d_c(i,j)$ is the combinatorial distance in the graph $\mathcal{T}^{(1)}$. Let G_k be the subgraph of $\mathcal{T}^{(1)}$ whose edges are $[ij]$ where $i, j \in U_k$. Due to $k \le \left[\frac{1}{2}n\right]$, $U_k \cap V_0 \ne \varnothing$, and G_k is topologically an arc. By the maximum principle applied to g on the subgraph whose edges consist of $[ij]$ with $i, j \in \{v \in V \mid d_c(v,u) \ge k\}$, we obtain $g_A \le \max_{i \in U_k} g_i$. Let $v_k \in U_k$ be such that $g_{v_k} = \max_{i \in U_k} g_i$ and edge path E_k be the shortest edge path in G_k joining v_k to a point u_k in $V_0 - \{u\}$. By construction, $g_{u_k} = 0$. Since U_k contains at most $3k+1$ vertices, the length of E_k is at most $3k$. The Dirichlet energy $\mathcal{E}(g)$ of g on $\mathcal{T}^{(1)}$ is given by

$$\mathcal{E}(g) = \frac{1}{2}\sum_{i \sim j} \eta_{ij}(g_i - g_j)^2 \ge \sum_{k=1}^{[n/2]} \mathcal{E}_k, \tag{31}$$

where

$$\mathcal{E}_k = \frac{1}{2}\sum_{[ij] \in \bar{E}_k} \eta_{ij}(g_i - g_j)^2,$$

and $\bar{E}_k$ be the set of oriented edges in E_k. Suppose $w_0 = v_k \sim w_1 \sim w_2 \sim \cdots \sim w_{l_k} = u_k$ are the vertices in the edge path E_k where $l_k \le 3k$. Using the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
\mathcal{E}_k &= \sum_{i=1}^{l_k} \eta_{w_i w_{i-1}} (g_{w_i} - g_{w_{i-1}})^2 \\
&\ge \frac{1}{M} \sum_{i=1}^{l_k} (g_{w_i} - g_{w_{i-1}})^2 \\
&\ge \frac{1}{M l_k} \left[\sum_{i=1}^{l_k} (g_{w_i} - g_{w_{i-1}}) \right]^2 \\
&\ge \frac{1}{3kM} (g_{v_k} - g_{u_k})^2 = \frac{g_{v_k}^2}{3kM} \ge \frac{g_A^2}{3kM}.
\end{aligned} \tag{32}
$$

By (31) and (32), we obtain

$$
\mathcal{E}(g) \ge \frac{g_A^2}{3M} \sum_{k=1}^{[n/2]} \frac{1}{k} \ge \frac{g_A^2 \ln(n)}{100M}. \tag{33}
$$

On the other hand, the Dirichlet principle says

$$
\mathcal{E}(g) = \min_{h \in \mathbb{R}^V} \left\{ \frac{1}{2} \sum_{i \sim j} \eta_{ij} (h_i - h_j)^2 \mid h_u = 1,\, h|_{V_0 - \{u\}} = 0 \right\}.
$$

Take $h \in \mathbb{R}^V$ to be $h_u = 1$ and $h_i = 0$ for all $i \in V - \{u\}$. We obtain

$$
\mathcal{E}(g) \le \frac{1}{2} \sum_{i \sim j} \eta_{ij} (h_i - h_j)^2 \le 4M.
$$

Combining this with (33), we obtain

$$
\frac{g_A^2 \ln(n)}{100M} \le 4M,
$$

ie

$$
g_A \le \frac{20M}{\sqrt{\ln(n)}}. \qquad \square
$$

Proof of Theorem 5.6 We construct the smooth family $w(t) \in \mathbb{R}^V$ by solving the system of ordinary differential equations (25), where $(S, \mathcal{T}, l) = (\Delta ABC, \mathcal{T}, l_{\mathrm{st}})$, $K^*|_{V - V_0 \cup \{A\}} = 0$, $K_A^* = \pi - \alpha$ and $w_i(t) = 0$ for $i \in V_0$. By the assumption that $\theta_{jk}^i(0) = \frac{\pi}{3}$ (ie $\mathcal{T}$ is an equilateral triangulation), $0 \in W$, where the space W is defined by (27). By Lemma 5.5, there exists a maximum $s > 0$ such that a solution $w(t)$ to (25) exists and condition (d) holds for all $t \in [0, s)$. We claim that $s \ge 1$, $w(1)$ exists and $w(1) * l_{\mathrm{st}}$ is a PL metric. In particular, $w(1) * l_{\mathrm{st}}$ satisfies condition (d) and $w(1) \in W$.

Without loss of generality, let us assume that $s < \infty$. By Lemma 5.5 and condition (d), we conclude

$$\text{(34)}\qquad \begin{array}{lll} \text{either} & \liminf_{t\to s^-} \eta_{ij}(w(t)) = 0 & \text{for some } [ij], \\ \text{or} & \limsup_{t\to s^-} \left|\theta^i_{jk}(w(t)) - \tfrac{\pi}{3}\right| = \left|\alpha - \tfrac{\pi}{3}\right| & \text{for some } \theta^i_{jk}. \end{array}$$

The conclusion $\liminf_{t\to s^-} \theta^i_{jk}(w(t)) = 0$ is ruled out by condition (d), which implies $\theta^i_{jk}(w(t)) \geq \frac{\pi}{6}$.

We prove the claim that $s \geq 1$ as follows. Since $\alpha \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, we have $\frac{\pi}{3} + \left|\alpha - \frac{\pi}{3}\right| \leq \frac{\pi}{2}$ and $\frac{\pi}{3} - \left|\alpha - \frac{\pi}{3}\right| \geq \frac{\pi}{6}$. This shows, by (d),

$$\text{(35)}\qquad \theta^i_{jk}(t) \in \left[\tfrac{\pi}{6}, \tfrac{\pi}{2}\right] \quad \text{for all } t \in [0, s).$$

In particular, $\cot(\theta^k_{ij}) \geq 0$ and $\eta_{ij} \geq \cot(\theta^k_{ij}) \geq 0$. Hence, by definition, we have

$$|(\nabla w')_{ij}| = \eta_{ij} |w'_i - w'_j| \geq \cot(\theta^k_{ij}) |w'_i - w'_j|.$$

By Lemma 5.8 and the variation formula (7) that $dK_i/dt = (\Delta w')_i$, we obtain

$$2|(\nabla w')_{ij}| \leq |(\Delta w')_A| = \left|\frac{dK_A}{dt}\right| = \left|\alpha - \frac{\pi}{3}\right|.$$

This implies, by (6),

$$\text{(36)}\qquad \begin{aligned} \left|\frac{d\theta^k_{ij}}{dt}\right| &\leq \cot(\theta^i_{jk})|w'_j - w'_k| + \cot(\theta^j_{ik})|w'_i - w'_k| \\ &\leq |(\nabla w')_{jk}| + |(\nabla w')_{ik}| \leq \left|\alpha - \tfrac{\pi}{3}\right|. \end{aligned}$$

Therefore, for all $t \in [0, s)$,

$$\text{(37)}\qquad \left|\theta^k_{ij}(t) - \tfrac{\pi}{3}\right| = |\theta^k_{ij}(t) - \theta^k_{ij}(0)| \leq \int_0^t \left|\frac{d\theta^k_{ij}(t)}{dt}\right| dt \leq t\left|\alpha - \tfrac{\pi}{3}\right| \leq s\left|\alpha - \tfrac{\pi}{3}\right|.$$

The above inequality shows that $s \geq 1$. Indeed, if otherwise $s < 1$, using (37), we conclude that $\theta^i_{jk}(t) \in \left[\frac{\pi}{3} - s\left|\alpha - \frac{\pi}{3}\right|, \frac{\pi}{3} + s\left|\alpha - \frac{\pi}{3}\right|\right]$. In particular, $\liminf_{t\to s^-} \eta_{ij}(t) \geq \cot\left(\frac{\pi}{3} + s\left|\alpha - \frac{\pi}{3}\right|\right) > 0$ and $\limsup_{t\to s^-}\left|\theta^k_{ij}(t) - \frac{\pi}{3}\right| < \left|\alpha - \frac{\pi}{3}\right|$. This contradicts (34).

To see part (e), by (37), if $t \in \left[0, \frac{1}{2}\right]$, we have

$$\left|\theta^i_{jk}(t) - \tfrac{\pi}{3}\right| \leq \tfrac{1}{2}\left|\alpha - \tfrac{\pi}{3}\right| \leq \tfrac{\pi}{12}, \quad \text{ie} \quad \theta^i_{jk}(t) \in \left[\tfrac{\pi}{4}, \tfrac{5\pi}{12}\right].$$

Now, if $[ij]$ is an interior edge, then, for $t \in \big[0, \frac{1}{2}\big]$,

$$
\begin{aligned}
(38) \qquad |(\nabla w')_{ij}| &= (\cot(\theta_{ij}^k) + \cot(\theta_{ij}^l))|w_i' - w_j'| \\
&\geq \left(1 + \frac{\cot(\theta_{ij}^l)}{\cot(\theta_{ij}^k)}\right)\cot(\theta_{ij}^k)|w_i' - w_j'| \\
&\geq \big(1 + \cot\big(\tfrac{5\pi}{12}\big)\big)\cot(\theta_{ij}^k)|w_i' - w_j'| \\
&\geq \tfrac{5}{4}\cot(\theta_{ij}^k)|w_i' - w_j'|.
\end{aligned}
$$

If θ_{jk}^i is an angle with $i \neq A$, then either one of the two edges $[ij]$ or $[ik]$ is an interior edge, or $i \in \{B, C\}$. In the first case, say $[ij]$ is an interior edge, using (38) and Lemma 5.8, for $t \in \big[0, \frac{1}{2}\big]$, we have

$$
\begin{aligned}
(39) \qquad \left|\frac{d\theta_{jk}^i}{dt}\right| &\leq \cot(\theta_{ij}^k)|w_i' - w_j'| + \cot(\theta_{ik}^j)|w_i' - w_k'| \\
&\leq \tfrac{4}{5}|(\nabla w')_{ij}| + |(\nabla w')_{ik}| \\
&\leq \big(\tfrac{4}{5} + 1\big)\cdot\tfrac{1}{2}|(\Delta w')_A| = \tfrac{9}{10}\big|\alpha - \tfrac{\pi}{3}\big| \leq \tfrac{9}{10}\cdot\tfrac{\pi}{6} = \tfrac{3\pi}{20}.
\end{aligned}
$$

In the second case, that $i \in \{B, C\}$, one of the edges $[ij]$ or $[ik]$, say $[ij]$, is in the edge BC of ΔABC, ie $w_i' = w_j' = 0$. Therefore, by Lemma 5.8, for $t \in \big[0, \frac{1}{2}\big]$, we have

$$
\begin{aligned}
(40) \qquad \left|\frac{d\theta_{jk}^i}{dt}\right| &\leq \cot(\theta_{ij}^k)|w_i' - w_j'| + \cot(\theta_{ik}^j)|w_i' - w_k'| \leq |(\nabla w')_{ik}| \\
&\leq \tfrac{1}{2}|(\Delta w')_A| = \tfrac{1}{2}\big|\alpha - \tfrac{\pi}{3}\big| \leq \tfrac{3\pi}{20}.
\end{aligned}
$$

Therefore, if θ_{jk}^i is not the angle at A and $t \in [0, 1)$, by (39) and (40), we have

$$
\begin{aligned}
\big|\theta_{jk}^i(t) - \tfrac{\pi}{3}\big| &= |\theta_{jk}^i(t) - \theta_{jk}^i(0)| \\
&\leq \int_0^t \left|\frac{d\theta_{jk}^i}{dt}\right| dt \leq \int_0^1 \left|\frac{d\theta_{jk}^i}{dt}\right| dt = \int_0^{1/2} \left|\frac{d\theta_{jk}^i}{dt}\right| dt + \int_{1/2}^1 \left|\frac{d\theta_{jk}^i}{dt}\right| dt \\
&\leq \tfrac{1}{2}\cdot\tfrac{3\pi}{20} + \tfrac{1}{2}\big|\alpha - \tfrac{\pi}{3}\big| \leq \tfrac{3\pi}{40} + \tfrac{1}{2}\cdot\tfrac{\pi}{6} = \tfrac{19\pi}{120}.
\end{aligned}
$$

Therefore, $\theta_{jk}^i(t) \in \big[\frac{21\pi}{120}, \frac{59\pi}{120}\big] \subset \big(\frac{\pi}{6}, \frac{\pi}{2}\big)$ for all $t \in [0, 1)$. Since conditions (d) and (e) hold for all $t \in [0, 1)$, by the definition of η_{ij}, we see $\liminf_{t\to 1} \eta_{ij}(w(t)) > 0$. Now we prove that $w(1)$ is defined and $w(1) * l_{\mathrm{st}}$ is a PL metric. By the estimates above, there exists $\delta > 0$ such that, for all $t \in [0, 1)$, $w(t) \in \mathcal{W}_\delta = \{w \in W \mid \theta_{ij}^i \geq \delta,\ \eta_{ij} \geq \delta\}$. By Lemma 5.5, the maximum time t_0 for which $w(t)$ exists on $[0, t_0)$ must be greater than 1. Therefore, $w(1)$ exists and $w(1) \in W$. Since (d) and (e) are closed conditions, it follows that $w(1) * l_{\mathrm{st}}$ satisfies (d) and (e).

Now we prove part (f). By parts (d) and (e), we have $\theta^i_{jk}(t) \in \left[\frac{\pi}{6}, \frac{59\pi}{120}\right]$ for $i \neq A$ and $\theta^A_{jk} \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$. Since the conductance η_{ij} is either $\cot(\theta^k_{ij})$ or a sum $\cot(\theta^k_{ij}) + \cot(\theta^l_{ij})$, we obtain, for all edges $[ij]$ in $\mathcal{T}$, $\eta_{ij}(t) \in \left[\cot\left(\frac{59\pi}{120}\right), 2\cot\left(\frac{\pi}{6}\right)\right] \subset \left[\frac{1}{100}, 100\right]$. Let $K_i(t)$ be the curvature of the metric $w(t) * l_{\mathrm{st}}$ at the vertex i. By Lemma 5.9 for $f = 1/\left|\alpha - \frac{\pi}{3}\right| \cdot dw(t)/dt$ and $M = 100$, we conclude that, for all $i \in V_0$,

$$\left|\frac{dK_i(t)}{dt}\right| = |(\Delta w')_i| \leq \frac{2000\left|\alpha - \frac{\pi}{3}\right|}{\sqrt{\ln(n)}} \leq \frac{2000}{\sqrt{\ln(n)}}.$$

Therefore,

$$|K_i(t) - K_i(0)| \leq \int_0^t \left|\frac{dK_i(t)}{dt}\right| dt \leq \int_0^1 \left|\frac{dK_i(t)}{dt}\right| dt \leq \frac{2000}{\sqrt{\ln(n)}}.$$

Finally, to prove (29), if $\alpha = \frac{\pi}{3}$, then all $w(t) = 0$ and $K(t) = K(0)$ and the result follows. If $\alpha \neq \frac{\pi}{3}$, we first claim that $w'_A(t) \neq 0$ for each t. Indeed, if otherwise $w'_A(t_1) = 0$ for some t_1, then, by the maximum principle applied to the Dirichlet problem $(\Delta w'(t_1))_i = 0$ for $i \in V - \{A\} \cup V_0$ and $w'_i(t_1) = 0$ for $i \in V_0 \cup \{A\}$, we conclude $w'_i(t_1) = 0$ for all $i \in V$. In particular, $\alpha - \frac{\pi}{3} = (\Delta w')_A = 0$ at $t = t_1$, which is a contradiction. Therefore, $w'_A(t) \neq 0$ and, by the maximum principle again, $w'_A(t) w'_i(t) \geq 0$. Now, if $i \in V_0$, then $K'_i(t) = \sum_{j \sim i} \eta_{ji}(w'_i - w'_j) = -\sum_{j \sim i} \eta_{ji} w'_j$. Since $\eta_{ij} \geq 0$, therefore $w'_A(t) K'_i(t) \leq 0$ for $i \in V_0$. It follows that $(K_i(t) - K_i(0)) w'_A(t) \leq 0$ for all $i \in V_0$. At the vertex A, $|K_A(t) - K_A(0)| = \left|t\left(\alpha - \frac{\pi}{3}\right)\right| \leq \frac{\pi}{6}$. Therefore, by the Gauss–Bonnet theorem that $K_A(t) + \sum_{i \in V_0} K_i(t) = K_A(t) + \sum_{i \in V} K_i(t) = 2\pi$ and since $K_i(t) - K_i(0)$ have the same signs for $i \in V_0$, we obtain $\sum_{i \in V_0} |K_i(t) - K_i(0)| = \left|\sum_{i \in V_0} (K_i(t) - K_i(0))\right| = |K_A(t) - K_A(0)| \leq \frac{\pi}{6}$. □

5.4 A gradient estimate of discrete harmonic functions

The proof Theorem 5.1 is based on the following estimate. Given a triangulated surface $(S, \mathcal{T})$, $v \in V(\mathcal{T})$ and $r > 0$, we use $B_r(v) = \{j \in V(\mathcal{T}) \mid d_c(j, v) \leq r\}$ to denote the combinatorial ball of radius r centered at the vertex i, where d_c is the combinatorial distance on $\mathcal{T}^{(1)}$.

Proposition 5.10 *Suppose $(\mathcal{P}, \mathcal{T}', l)$ is a polygonal disk with an equilateral triangulation and $\mathcal{T}$ is the n^{th} standard subdivision of the triangulation $\mathcal{T}'$ with $n \geq e^{10^6}$. Let $\eta\colon E = E(\mathcal{T}) \to [1/M, M]$ be a conductance function and $\Delta\colon \mathbb{R}^V \to \mathbb{R}^V$ be the associated Laplace operator. Let $V_0 \subset V(\mathcal{T})$ be a thin subset, such that for all $v \in V$ and $m \leq \frac{1}{2}n$, $|B_m(v) \cap V_0| \leq 10m$. If $f\colon V \to \mathbb{R}$ satisfies $(\Delta f)_i = 0$ for $i \in V - V_0$,*

$|(\Delta f)_i| \le M/\sqrt{\ln(n)}$ *for* $i \in V_0$ *and* $\sum_{i\in V_0}|(\Delta f)_i| \le M$, *then, for all edges* $[uv]$ *in* $\mathcal{T}$,

$$|f_u - f_v| \le \frac{200M^3}{\sqrt{\ln(\ln(n))}}.$$

Proof Fix an edge $[uv]$ in the triangulation $\mathcal{T}$. Construct a function $g\colon V = V(\mathcal{T}) \to \mathbb{R}$ by solving the Dirichlet problem $(\Delta g)_i = 0$ for $i \ne u, v$, and $g_u = 1$ and $g_v = 0$. By the maximum principle, we have $0 \le g_i \le 1$. By the identity $\sum_{i\in V}(\Delta g)_i = 0$ and since g is not a constant, we obtain $(\Delta g)_u = -(\Delta g)_v \ne 0$. Using Green's identity that $\sum_{i\in V}(f_i(\Delta g)_i - g_i(\Delta f)_i) = 0$ and the assumptions on f and g, we obtain

$$f_u(\Delta g)_u + f_v(\Delta g)_v - \sum_{i\in V_0} g_i(\Delta f)_i = 0.$$

Since $(\Delta g)_v = -(\Delta g)_u$, this shows

$$f_u - f_v = \frac{1}{(\Delta g)_u}\sum_{i\in V_0} g_i(\Delta f)_i.$$

On the other hand, by the maximum principle $g_u - g_j \ge 0$, we have $|(\Delta g)_u| = \left|\sum_{j\sim u}\eta_{ju}(g_j - g_u)\right| = \sum_{j\sim u}\eta_{ju}(g_u - g_j) \ge (1/M)(g_u - g_v) = 1/M$. Therefore,

$$|f_u - f_v| \le M\left|\sum_{i\in V_0} g_i(\Delta f)_i\right|. \tag{41}$$

To estimate the right-hand side of (41), take $r = [\sqrt[3]{\ln(n)}]$ and select $a \notin B_r(u)$. Then, using $0 = \sum_{i\in V}(\Delta f)_i = \sum_{i\in V_0}(\Delta f)_i$, $|g_i| \le 1$, (41) and Lemma 5.11 below, we obtain

$$\begin{aligned}
|f_u - f_v| &\le M\left|\sum_{i\in V_0} g_i(\Delta f)_i\right| = M\left|\sum_{i\in V_0}(g_i - g_a)(\Delta f)_i\right| \\
&\le M\sum_{i\in V_0}|(g_i - g_a)||(\Delta f)_i| \\
&\le M\left(\sum_{i\in V_0\cap B_r(u)}|g_i - g_a||(\Delta f)_i| + \sum_{i\in V_0 - B_r(u)}|g_i - g_a||(\Delta f)_i|\right) \\
&\le M\left(\frac{2M}{\sqrt{\ln(n)}}|V_0\cap B_r(u)| + \frac{100M}{\sqrt{\ln(r)}}\sum_{i\in V_0}|(\Delta f)_i|\right) \\
&\le M\left[\frac{20M\sqrt[3]{\ln(n)}}{\sqrt{\ln(n)}} + \frac{100M^2}{\sqrt{\ln(\sqrt[3]{\ln(n)})}}\right] \\
&\le \frac{200M^3}{\sqrt{\ln(\ln(n))}}.
\end{aligned}$$

In the last two steps, we have used $|V_0 \cap B_r(u)| \le 10r = 10\sqrt[3]{\ln n}$ and $n \ge e^{10^6}$ to ensure $1/\sqrt{\ln(\sqrt[3]{\ln(n)})} \ge \sqrt[3]{\ln(n)}/\sqrt{\ln(n)}$. □

Lemma 5.11 *Assume $(\mathcal{P}, \mathcal{T}', l), \mathcal{T}, E, M, \eta$ and Δ are as given in Proposition 5.10, and g is as given in the proof of Proposition 5.10, ie $(\Delta g)_i = 0$ for $i \ne u, v$, and $g_u = 1$ and $g_v = 0$. If $100 \le r \le \frac{1}{3}n$ and $\{a, b\} \cap B_r(u) = \varnothing$, then*

$$|g_a - g_b| \le \frac{100M}{\sqrt{\ln(r)}}.$$

The strategy of the proof of Lemma 5.11 is similar to that of Lemma 5.9.

Proof For $k \le \frac{1}{3}r$, let $U_k = \{i \in V \mid d_c(i,u) = k\}$. Since $\mathcal{T}$ is an equilateral triangulation of a flat surface, we have $|U_k| \le 6k$. Recall that a subset U of $V = V(\mathcal{T})$ is called connected if any two points in U can be joined by an edge path in $\mathcal{T}^{(1)}$ whose vertices are in U. Each subset $U \subset V$ is a disjoint of connected subsets, which are called connected components of U. We claim that there exists a connected component G_k of U_k such that $\{a, b\}$ lies in a connected component of $V - G_k$. To see this, note that, since $\mathcal{T}$ is the n^{th} standard subdivision of $\mathcal{T}'$, for all $k \le \frac{1}{3}r \le \frac{1}{9}n$, the set $B_k(u) = \{i \in V \mid d_c(i,u) \le k\}$ is connected and $B_k(u)^c = \{i \in V | d_c(i,u) > k\}$ has at most two connected components which are also connected components of $V - U_k$. If $B_k(u)^c$ is connected, then U_k is connected and we take $G_k = U_k$. If $B_k(u)^c$ has two connected components R_1 and R_2, then there exists a nonflat boundary vertex $v' \in R_2$ such that $d_c(u, v') \le 3k \le r$. This shows that $v' \in B_r(u)$. See Figure 10. The component R_2 is contained in $B_r(u)$ due to $d_c(v', u) \le r$. Since $a, b \notin B_r(u)$, it follows that a and b are in $\mathcal{R}_1$. We take G_k to be the connected component of U_k such that R_1 is a connected component of $V - G_k$. Therefore, the claim follows.

Let us assume without loss of generality that $g_a \le g_b$. By the maximum principle applied to g on the connected graph whose vertex set is the connected component of $V - G_k$ containing $\{a, b\}$, there exist two vertices $u_k, u'_k \in G_k$ such that

$$g_{u_k} \ge g_b \quad \text{and} \quad g_{u'_k} \le g_a.$$

Let E_k be the shortest edge path with vertices in G_k connecting u_k to u'_k and $\overline{E}_k$ be the set of all oriented edges in E_k. The length of E_k is at most $|G_k| \le 6k$. The Dirichlet energy of g on the graph $\mathcal{T}^{(1)}$ is

$$\text{(42)}\quad \mathcal{E}(g) = \frac{1}{2}\sum_{i \sim j} \eta_{ij}(g_i - g_j)^2 \ge \frac{1}{2M}\sum_{i \sim j}(g_i - g_j)^2 \ge \frac{1}{2M}\sum_{k=1}^{[r/3]}\sum_{[ij] \in \overline{E}_k}(g_i - g_j)^2.$$

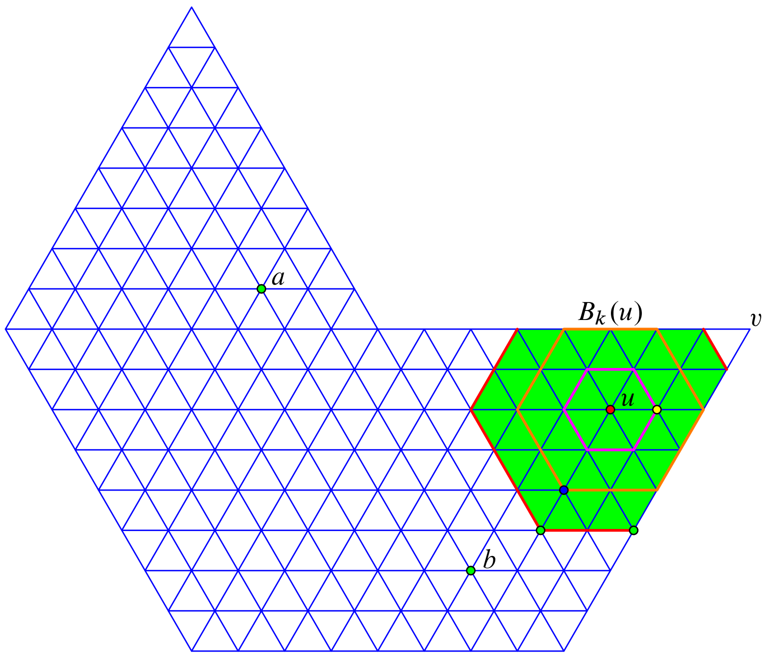

Figure 10: Triangulated polygonal disks.

Suppose $w_0 = u_k \sim w_1 \sim w_2 \sim \cdots \sim w_{l_k} = u'_k$ is the edge path E_k, where $l_k \le 6k$. Then, by the Cauchy–Schwarz inequality, we have

$$(43)\qquad \frac{1}{2}\sum_{[ij]\in \bar{E}_k}(g_i-g_j)^2 = \sum_{i=1}^{l_k}(g_{w_i}-g_{w_{i-1}})^2 \ge \frac{1}{l_k}\Bigl(\sum_{i=1}^{l_k}(g_{w_i}-g_{w_{i-1}})\Bigr)^2 \ge \frac{1}{l_k}(g_{u_k}-g_{u'_k})^2 \ge \frac{(g_a-g_b)^2}{6k}.$$

Combining (42) and (43), we obtain

$$(44)\qquad \mathcal{E}(g) \ge \frac{1}{2M}\frac{(g_a-g_b)^2}{6}\sum_{i=1}^{[r/3]}\frac{1}{k} \ge \frac{(g_a-g_b)^2\ln(r)}{100M}.$$

On the other hand, by the Dirichlet principle we have $\mathcal{E}(g) \le \frac{1}{2}\sum_{i\sim j}\eta_{ij}(h_i-h_j)^2$ for any $h \in \mathbb{R}^V$ such that $h_u = 1$ and $h_v = 0$. Take h to be $h_u = 1$ and $h_i = 0$ for all $i \in V - \{u\}$. We obtain $\mathcal{E}(g) \le \frac{1}{2}\sum_{i\sim j}\eta_{ij}(h_i-h_j)^2 \le 6M$. Therefore, $(g_b-g_a)^2\ln(r)/100M \le 6M$, which implies

$$|g_b-g_a| \le \frac{100M}{\sqrt{\ln(r)}}. \qquad \square$$

5.5 A proof of Theorem 5.1

For simplicity, a boundary vertex $v \in \mathcal{P} - \{p,q,r\}$ with nonzero curvature will be called a *corner*. Note that corners in $\mathcal{T}$ and its n^{th} standard subdivision $\mathcal{T}_{(n)}$ are the same. In particular, the total number of corners is independent of n. Let V_c be the set of all corner vertices. Since $\mathcal{P}$ is embedded in $\mathbb{C}$, given a corner $v \in V_c$, the degree m of v has to be 3, 5 or 6. Consider the combinatorial ball $B_{[n/3]}(v)$ of radius $\left[\frac{1}{3}n\right]$ centered at a corner $v \in V_c$ in $\mathcal{T}_{(n)}$. By construction, $B_{[n/3]}(v) \cap B_{[n/3]}(v') = \varnothing$ for distinct corners v and v'. Each $B_{[n/3]}(v)$ is a union of $m-1$ $\left[\frac{1}{3}n\right]^{\text{th}}$ standard subdivided equilateral triangles $\Delta_1, \dots, \Delta_{m-1}$ in $\mathcal{T}$. Applying Theorem 5.6 with $\alpha = \pi/(m-1)$ to the triangulated equilateral triangle (Δ_i, v) for each $i = 1, 2, \dots, m-1$, we produce a discrete conformal factor $w(\Delta_i) \in \mathbb{R}^{V(\Delta_i)}$ for each Δ_i such that, if a vertex $u \in V(\Delta_i) \cap V(\Delta_j)$, then $w_u(\Delta_i) = w_u(\Delta_j)$. In particular, there is a well-defined discrete conformal factor $w(B_{[n/3]}(v))$ on $B_{[n/3]}(v)$ obtained by gluing these $w(\Delta_i)$. See Figure 8. Define $w^{(1)}\colon V(\mathcal{T}_{(n)}) \to \mathbb{R}$ as follows: if $u \in \bigcup_{v \in V_c} B_{[n/3]}(v)$, then $w^{(1)}_u = w_u(B_{[n/3]}(v))$ for $u \in B_{[n/3]}(v)$ and $w^{(1)}(u) = 0$ for $u \notin \bigcup_{v \in V_c} B_{[n/3]}(v)$. Let $\hat{l} = w^{(1)} * l_{\text{st}}$ be the PL metric on $\mathcal{T}_{(n)}$ and $\widehat{K}$ be its the discrete curvature. Let $K^*\colon V_{(n)} \to \mathbb{R}$ be defined by $K^*_i = 0$ if $i \notin \{p,q,r\}$, and $K^*_i = \frac{2\pi}{3}$ if $i \in \{p,q,r\}$. By Theorem 5.6, the PL metric $\hat{l}$ and $\widehat{K}$ satisfy the following:

(a) The curvature $\widehat{K}_i = K^*_i$ at all vertices i such that $d_c(i,v) \neq \left[\frac{1}{3}n\right]$ for some corner $v \in V_c$.

(b) $w^{(1)}_i = 0$ for $i \notin \bigcup_{v \in V_c} B_{[n/3]}(v)$.

(c) All inner angles at a corner $v \in V_c$ are in $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

(d) All inner angles at a noncorner vertex are in $\left[\frac{\pi}{6}, \frac{59\pi}{120}\right]$.

(e) $|\widehat{K}_i - K^*_i| \le 4000/\sqrt{\ln(n)}$ and $\sum_{i \in V} |\widehat{K}_i - K^*_i| \le \frac{\pi}{3} N$, where N is the number of corners in $\mathcal{P}$.

We will find a discrete conformal factor $w^{(2)}\colon V_{(n)} \to \mathbb{R}$ such that $w^{(2)} * \hat{l}$ and its curvature satisfy Theorem 5.1 by solving the system of ordinary differential equations in $w(t)$

$$
\begin{aligned}
\frac{dK_i(w(t) * \hat{l})}{dt} &= K^*_i - \widehat{K}_i && \text{for all } i \in V(\mathcal{T}_{(n)}) - \{p,q,r\}, \\
w_s(t) &= 0 && \text{for all } s \in \{p,q,r\}, \\
w(0) &= 0.
\end{aligned}
\tag{45}
$$

Let $K(t) = K(w(t) * \hat{l})$. Note that (45) and the Gauss–Bonnet formula imply that $K'_p(t) = K^*_p - \widehat{K}_p$. By Lemma 5.5, the solution $w(t)$ exists on some interval $[0, \epsilon)$.

Our goal is to show that, for n large, the solution $w(t)$ exists on $[0,1]$. In this case, the conformal factor $w^{(2)}$ is taken to be $w(1)$. The required discrete conformal factor w_n in Theorem 5.1 is taken to be $w^{(1)}+w^{(2)}$.

Consider the maximum time t_0 such that the solution $w(t)$ to (45) exists for $t\in[0,t_0)$ and the PL metrics $w(t)*\hat{l}$ satisfy:

- (c′) All inner angles at a corner $v\in V_c$ are in $\left[\frac{\pi}{6}-\frac{\pi}{1000},\frac{\pi}{2}+\frac{\pi}{1000}\right]$.
- (d′) All inner angles at a noncorner vertex are in $\left[\frac{\pi}{6}-\frac{\pi}{1000},\frac{59\pi}{120}+\frac{\pi}{1000}\right]$.

Let $V_0=\bigcup_{v\in V_c}\left\{i\in V_{(n)}\mid d_c(i,v)=\left[\frac{1}{3}n\right]\right\}$. By construction, $|B_r(i)\cap V_0|\le 10r$ for all $r\le\frac{1}{3}n$. Then $\sum_{i\in V_0}|(\Delta w'(t))_i|=\sum_{i\in V_0}|K_i'(t)|\le\sum_{i\in V_0}|\widehat{K}_i-K_i^*|\le\frac{\pi}{3}N$ and $|(\Delta w')_i|=|K_i'(t)|\le|\widehat{K}_i-K_i^*|\le 4000/\sqrt{\ln(n)}$. Choose $M=\max\left\{4000,\frac{\pi}{3}N\right\}$. Then, by (c′), (d′), (e) and the formula $\cot(a)+\cot(b)=\sin(a+b)/\sin(a)\sin(b)$, for all $t\in[0,t_0)$, we have $\eta_{ij}(t)=\eta_{ij}(w(t)*\hat{l})\in\left[\frac{1}{4000},4000\right]\subset[1/M,M]$, $(\Delta w')_i=0$ for $i\in V(\mathcal{T}_{(n)})-V_0$, $|(\Delta w')_i|\le M/\sqrt{\ln(n)}$ and $\sum_{i\in V_0}|(\Delta w')_i|\le M$. In summary, $f=w'$ satisfies the conditions in Proposition 5.10 for all $t\in[0,t_0)$. By Proposition 5.10, if $i\sim j$, then

$$|w_i'(t)-w_j'(t)|\le\frac{200M^3}{\sqrt{\ln(\ln(n))}}.$$

On the other hand, by the variation of angle formula (6) and $M\ge|\cot(\theta_{ij}^k)|$, we have

$$\begin{aligned}\left|\frac{d\theta_{ij}^k}{dt}\right|&\le|\cot(\theta_{jk}^i)(w_j'-w_k')|+|\cot(\theta_{ik}^j)(w_i'-w_k')|\\&\le M(|w_j'-w_k'|+|w_i'-w_k'|)\le\frac{400M^4}{\sqrt{\ln(\ln(n))}}.\end{aligned}$$

Therefore, for $t\in[0,t_0)$ and sufficiently large n,

$$|\theta_{ij}^k(w(t))-\theta_{ij}^k(0)|\le\int_0^t\left|\frac{d\theta_{ij}^k(w(t))}{dt}\right|dt\le\frac{400M^4t_0}{\sqrt{\ln(\ln(n))}}\le\frac{\pi t_0}{2000}.\tag{46}$$

It follows that $t_0>1$ (or $t_0=\infty$) since, otherwise, by (46), the choices of angles in (c), (d), (c′), (d′) and Lemma 5.5, we can extend the solution $w(t)$ to $[0,t_0+\epsilon)$ for some $\epsilon>0$ such that (c′) and (d′) still hold. To be more precise, by Lemma 5.5 on the maximality of t_0, we have either $\limsup_{t\to t_0}\left|\theta_{jk}^i(t)-\frac{\pi}{3}\right|=\frac{\pi}{1000}$ for an inner angle θ_{jk}^i at a corner $i\in V_c$, or $\limsup_{t\to t_0}\theta_{jk}^i(t)=\frac{\pi}{6}-\frac{\pi}{1000}$ or $\limsup_{t\to t_0}\theta_{jk}^i(t)=\frac{59\pi}{120}+\frac{\pi}{1000}$ for an angle θ_{jk}^i at a noncorner vertex i. But, due to (46), none of these conditions holds if $t_0\le 1$. Therefore the solution $w(1)$ exists. By construction, the curvature $K(1)$ of $w(1)*\hat{l}$ is $K(0)+\int_0^1K'(t)\,dt=\widehat{K}+K^*-\widehat{K}=K^*$. Furthermore, condition (c) in Theorem 5.1 follows from (c′) and (d′).

6 A proof of the convergence theorem

We will prove the following theorem:

Theorem 6.1 *Let Ω be a Jordan domain in the complex plane and $\{p,q,r\} \subset \partial\Omega$. There exists a sequence of triangulated polygonal disks $(\Omega_n, \mathcal{T}_n, d_{\mathrm{st}}, (p_n, q_n, r_n))$, where $\mathcal{T}_n$ is an equilateral triangulation and p_n, q_n and r_n are three boundary vertices such that*

- (a) $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ *with* $\Omega_n \subset \Omega_{n+1}$, *and* $\lim_n p_n = p$, $\lim_n q_n = q$ *and* $\lim_n r_n = r$,
- (b) *discrete uniformization maps f_n associated to $(\Omega_n, \mathcal{T}_n, d_{\mathrm{st}}, (p_n, q_n, r_n))$ exist and converge uniformly to the Riemann mapping associated to $(\Omega, (p,q,r))$.*

Before giving the proof, let us recall Rado's theorem and its generalization to quasiconformal maps. If $\phi\colon \mathbb{D} \to \Omega$ is a K–quasiconformal map onto a Jordan domain Ω, then ϕ extends continuously to a homeomorphism $\overline{\phi}\colon \overline{\mathbb{D}} \to \overline{\Omega}$ between their closures (see [1, Corollary on page 30]). If $K = 1$, $\overline{\phi}$ is the Carathéodory extension of the Riemann mapping. A sequence of Jordan curves J_n in $\mathbb{C}$ is said to converge uniformly to a Jordan J curve in $\mathbb{C}$ if there exist homeomorphisms $\phi_n\colon \mathbb{S}^1 \to J_n$ and $\phi\colon \mathbb{S}^1 \to J$ such that ϕ_n converges uniformly to ϕ. Rado's theorem [23] and its extension by Palka [21, Corollary 1] states that:

Theorem 6.2 (Rado, Palka) *Suppose Ω_n is a sequence of Jordan domains such that $\partial\Omega_n$ converges uniformly to $\partial\Omega$. If $f_n\colon \mathbb{D} \to \Omega_n$ is a K–quasiconformal map for each n such that the sequence $\{f_n\}$ converges to a K–quasiconformal map $f\colon \mathbb{D} \to \Omega$ uniformly on compact sets of $\mathbb{D}$, then $\overline{f_n}$ converges to $\overline{f}$ uniformly on $\overline{\mathbb{D}}$.*

The following compactness result is a consequence of Palka's theorem [21, Corollary 1] and Lehto and Virtanen's work [17, Theorems 5.1 and 5.5].

Theorem 6.3 *Suppose Ω_n is a sequence of Jordan domains such that $\partial\Omega_n$ converges uniformly to $\partial\Omega$ and $K > 0$ is a constant. Let $p_n, q_n, r_n \in \partial\Omega_n$ and $p,q,r \in \partial\Omega$ be distinct points such that $\lim_n p_n = p$, $\lim_n q_n = q$, $\lim_n r_n = r$ and $h_n\colon \mathbb{D} \to \Omega_n$ be K–quasiconformal maps such that $\bar{h}_n$ sends $(1, \sqrt{-1}, -1)$ to (p_n, q_n, r_n). Then there exists a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ converging uniformly on $\overline{\mathbb{D}}$ to a K–quasiconformal map $h\colon \mathbb{D} \to \Omega$ sending $(1, \sqrt{-1}, -1)$ to (p,q,r).*

Now we prove Theorem 6.1.

Proof Given a Jordan domain Ω with three distinct points p, q and r in $\partial\Omega$, construct a sequence of approximating polygonal disks Ω_n such that

(1) each Ω_n is triangulated by equilateral triangles of side lengths tending to 0,

(2) $\partial\Omega_n$ converges uniformly to the Jordan curve $\partial\Omega$ such that $\Omega_n \subset \Omega_{n+1}$,

(3) there are three boundary vertices $p_n, q_n, r_n \subset \partial\Omega_n$ such that $\lim_n p_n = p$, $\lim_n q_n = q$ and $\lim_n r_n = r$, and

(4) the curvatures of Ω_n at p_n, q_n and r_n are $\frac{2\pi}{3}$ and curvatures of Ω_n at all other boundary vertices are not $\frac{2\pi}{3}$.

By Theorem 5.1, we produce a standard subdivision $\mathcal{T}_n$ of Ω_n and $w_n \in \mathbb{R}^{V(\mathcal{T}_n)}$ such that $(\Omega_n, \mathcal{T}_n, w_n * l_{\mathrm{st}})$ is isometric to the equilateral triangle $(\Delta ABC, \mathcal{T}_n', l_n')$ with a Delaunay triangulation $\mathcal{T}_n'$ and A, B and C correspond to p_n, q_n and r_n. Let $f_n\colon (\Delta ABC, \mathcal{T}_n', (A, B, C)) \to (\Omega_n, \mathcal{T}_n, (p_n, q_n, r_n))$ be the associated discrete conformal map and $\bar{f}\colon (\Delta ABC, (A, B, C)) \to (\bar{\Omega}, (p, q, r))$ be the Riemann mapping. We claim that f_n converges uniformly to $\bar{f}$ on ΔABC.

To establish the claim, first, by Theorem 5.1, we know all angles of triangles in the triangulated PL surface $(\Delta ABC, \mathcal{T}_n', l_n')$ are at least $\epsilon_0 > 0$. Therefore, the discrete conformal maps f_n are K–quasiconformal for a constant K independent of n. By Theorem 6.3, it follows that every limit function g of a convergent subsequence $\{f_{n_i}\}$ is a K–quasiconformal map from $\mathrm{int}(\Delta ABC)$ to Ω which extends continuously to ΔABC, sending A, B and C to p, q and r, respectively. We claim that the limit map g is conformal. Indeed, by Lemma 4.7, the discrete conformal map f_n^{-1}, when restricted to a fixed compact set R of Ω, maps equilateral triangles in $\mathcal{T}_n$ which are inside R to triangles of $\mathcal{T}_n'$ that become arbitrarily close to equilateral triangles as $n \to \infty$. Therefore, the limit map g of the subsequence f_{n_i} is 1–conformal and therefore conformal in $\mathrm{int}(\Delta ABC)$. The continuous extension of g sends A, B and C to p, q and r, respectively, by Theorem 6.3. On the other hand, there is only one Riemann mapping $f\colon \mathrm{int}(\Delta) \to \Omega$ whose continuous extension sends A, B and C to p, q and r, respectively. Therefore, $g = f$. This shows all limits of convergent subsequences of $\{f_n\}$ are equal f. Therefore $\{f_n\}$ converges to f uniformly on compact sets in $\mathrm{int}(\Delta ABC)$. By Theorem 6.2, $\bar{f}_n$ converges uniformly to $\bar{f}$. □

7 A convergence conjecture on discrete uniformization maps

We discuss a general approximation conjecture and the related topics of discrete conformal equivalence of polyhedral metrics.

7.1 A strong version of convergence of discrete conformal maps

As discussed before, the main drawback of the vertex scaling operation on polyhedral metrics is the lacking of an existence theorem. For instance, given a PL metric on a closed triangulated surface $(S, \mathcal{T}, l)$, there is in general no discrete conformal factor $w\colon V \to \mathbb{R}$ such that the new PL metric $(S, \mathcal{T}, w * l)$ has constant discrete curvature.

The recent work of [10] established an existence and a uniqueness theorem for polyhedral metrics by allowing the triangulations to be changed.

Definition 7.1 (discrete conformality of PL metrics [10]) Two PL metrics d and d' on (S, V) are discrete conformal if there exist sequences of PL metrics $d_1 = d, \dots, d_m = d'$ on (S, V) and triangulations $\mathcal{T}_1, \dots, \mathcal{T}_m$ of (S, V) satisfying:

(a) **Delaunay** Each $\mathcal{T}_i$ is Delaunay in d_i.

(b) **Vertex scaling** If $\mathcal{T}_i = \mathcal{T}_{i+1}$, there exists a function $w\colon V \to \mathbb{R}$ such that, if e is an edge in $\mathcal{T}_i$ with endpoints v and v', then the lengths $l_{d_{i+1}}(e)$ and $l_{d_i}(e)$ of e in d_i and d_{i+1} are related by

$$l_{d_{i+1}}(e) = e^{w(v)+w(v')} l_{d_i}(e). \tag{47}$$

(c) If $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then (S, d_i) is isometric to (S, d_{i+1}) by an isometry homotopic to the identity in (S, V).

The main theorem proved in [10] is the following:

Theorem 7.2 *Suppose (S, V) is a closed connected marked surface and d is a PL metric on (S, V). Then, for any $K^*\colon V \to (-\infty, 2\pi)$ with $\sum_{v \in V} K^*(v) = 2\pi\chi(S)$, there exists a PL metric d^*, unique up to scaling and isometry homotopic to the identity, on (S, V) such that d^* is discrete conformal to d and the discrete curvature of d^* is K^*. Furthermore, the metric d^* can be found using a finite-dimensional (convex) variational principle.*

There is a close relation between the discrete conformal equivalence in Definition 7.1 and convex geometry in hyperbolic 3–space. The first work relating the vertex scaling operation and hyperbolic geometry is by Bobenko, Pinkall and Springborn [3]. They associated to each polyhedral metric on $(S, \mathcal{T}, l)$ a hyperbolic metric with cusp end on the punctured surface $S - V(\mathcal{T})$. However, the Delaunay condition on the triangulation $\mathcal{T}$ was missing in their definition. The discrete conformal equivalence in Definition 7.1 is equivalent to the following hyperbolic geometry construction: Let (S, V, d) be a PL surface. Take a Delaunay triangulation $\mathcal{T}$ of (S, V, d) and consider the PL metric d as isometric gluing of Euclidean triangles $\tau \in \mathcal{T}$. Consider each triangle τ in $\mathcal{T}$ as the Euclidean convex hull of three points v_1, v_2 and v_3 in the complex plane $\mathbb{C}$. Let τ^* be the convex hull of $\{v_1, v_2, v_3\}$ in the upper-half-space model of the hyperbolic 3–space $\mathbb{H}^3$. Thus, τ^* is an ideal hyperbolic triangle having the same vertices as those of τ. If σ and τ are two Euclidean triangles in $\mathcal{T}$ glued isometrically along two edges by an isometry f considered as an isometry of the Euclidean plane, we glue τ^* and σ^* along the corresponding edges using the *same* map f considered as an isometry of $\mathbb{H}^3$. Here we have used the fact that each isometry of the complex plane extends naturally to an isometry of the hyperbolic 3–space $\mathbb{H}^3$. The result of the gluing of these τ^* produces a hyperbolic metric d^* on the punctured surface $S - V$. It is easy to see that d^* is independent of the choices of Delaunay triangulations. It is shown in [10] (see also [11]) that two PL metrics d_1 and d_2 on (S, V) are discrete conformal in the sense of Definition 7.1 if and only if the associated hyperbolic metrics d_1^* and d_2^* are isometric by an isometry homotopic to the identity on $S - V$.

Using this hyperbolic geometry interpretation, one defines the discrete conformal map between two discrete conformally equivalent PL metrics d_1 and d_2 as follows (see [3; 10]). The vertical projection of the ideal triangle τ^* to τ induces a homeomorphism $\phi_d \colon (S - V, d^*) \to (S - V, d)$. Suppose d_1 and d_2 are two discrete conformally equivalent PL metrics on (S, V). Then the *discrete conformal map* from (S, V, d_1) to (S, V, d_2) is given by $\phi_{d_2} \circ \psi \circ \phi_{d_1}^{-1}$ where $\psi \colon (S, V, d_1^*) \to (S, V, d_2^*)$ is the hyperbolic isometry. Note that in this new setting, discrete conformal maps are piecewise projective instead of piecewise linear.

Theorem 7.2 can be used for approximating Riemann mappings for Jordan domains. Given a simply connected polygonal disk with a PL metric (D, V, d) and three boundary vertices $p, q, r \in V$, let the metric double of (D, V, d) along the boundary be the polyhedral 2–sphere $(\mathbb{S}^2, V', d')$. Using Theorem 7.2, one produces a new polyhedral surface $(\mathbb{S}^2, V', d^*)$ such that

(1) $(\mathbb{S}^2, V', d^*)$ is discrete conformal to $(\mathbb{S}^2, V', d')$,

(2) the discrete curvatures of d^* at p, q and r are $\frac{4\pi}{3}$,

(3) the discrete curvatures of d^* at all other vertices are zero, and

(4) the area of $(\mathbb{S}^2, V', d^*)$ is $\frac{\sqrt{3}}{2}$.

Therefore, $(\mathbb{S}^2, V', d^*)$ is isometric to the metric double $(\mathcal{D}(\Delta ABC), V'', d'')$ of an equilateral triangle ΔABC of edge length 1. Let F be the discrete conformal map from $(\mathcal{D}(\Delta ABC), V'', d'')$ to $(\mathbb{S}^2, V', d')$ such that F sends A, B and C to p, q and r, respectively. Due to the uniqueness part of Theorem 7.2, we may assume that $f = F|\colon \Delta ABC \to D$ and f sends A, B and C to p, q and r, respectively. We call f the *discrete uniformization map* associated to $(D, V, d, (p, q, r))$.

A strong form of the convergence is the following:

Conjecture 7.3 *Let $(\Omega, (p, q, r))$ be a Jordan domain in the complex plane with three marked boundary points and $(\Omega_n, \mathcal{T}_n, d_{\mathrm{st}}, (p_n, q_n, r_n))$ be any sequence of triangulated flat polygonal disks with three marked boundary vertices such that*

(a) *$\mathcal{T}_n$ is an equilateral triangulation,*

(b) *$\partial\Omega_n$ converges uniformly to $\partial\Omega$,*

(c) *the edge length of $\mathcal{T}_n$ goes to zero,*

(d) $\lim_n p_n = p$, $\lim_n q_n = q$ *and* $\lim_n r_n = r$.

Then the discrete uniformization maps f_n associated to $(\Omega_n, \mathcal{T}_n, d_{\mathrm{st}}, (p_n, q_n, r_n))$ converge uniformly to the Riemann mapping associated to $(\Omega, (p, q, r))$.

7.2 Discrete conformal equivalence and convex sets in the hyperbolic 3–space

We now discuss the relationship between the discrete conformal equivalence given in Definition 7.1, ideal convex sets in the hyperbolic 3–space $\mathbb{H}^3$ and the motivation for Conjectures 1.5 and 1.6.

The classical uniformization theorem for Riemann surfaces follows from the special case that every simply connected Riemann surface is biholomorphic to $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{S}^2$. The discrete analogous should be the statement that each noncompact simply connected polyhedral surface is discrete conformal to either $(\mathbb{C}, V, d_{\mathrm{st}})$ or $(\mathbb{D}, V, d_{\mathrm{st}})$, where V is a discrete set and d_{st} is the standard Euclidean metric. Furthermore, the set V is unique up to Möbius transformations. For a noncompact polyhedral surface (S, V, d) with an infinite set V, the hyperbolic geometric viewpoint of discrete conformality is a better approach. Namely, discrete conformal equivalence between two PL metrics is the same as the Teichmüller equivalence between their associated hyperbolic metrics. For instance, if we take a Delaunay triangulation $\mathcal{T}$ of the complex plane $(\mathbb{C}, d_{\mathrm{st}})$

with vertex set V, then the associated hyperbolic metric d^*_{st} on $\mathbb{C} - V$ is isometric to the boundary of the convex hull $\partial C_{\mathbb{H}}(V)$ in $\mathbb{H}^3$. Therefore, a PL surface (S, V', d) is discrete conformal to $(\mathbb{C}, V, d_{\mathrm{st}})$ for some discrete subset $V \subset \mathbb{C}$ if and only if the associated hyperbolic metric d^* is isometric to the boundary of the convex hull $\partial C_{\mathbb{H}}(V)$. It shows discrete uniformization is the same as realizing hyperbolic metrics as the boundaries of convex hulls (in $\mathbb{H}^3$) of closed sets in $\partial\mathbb{H}^3$. One can formulate the conjectural discrete uniformization theorem as follows. Given a discrete set V' in $\mathbb{C}$ or $\mathbb{D}$, let $\hat{d}$ be the unique conformal complete hyperbolic metric on $\mathbb{C} - V'$ or $\mathbb{D} - V'$. Then $\hat{d}$ is isometric to the boundary of the convex hull of a discrete set $V \subset \mathbb{C}$ or $(\mathbb{C} \cup \{\infty\} - \mathbb{D}) \cup V$, where V is discrete and unique up to Möbius transformations. This is the original motivation for proposing Conjectures 1.5 and 1.6.

These two conjectures bring discrete uniformization close to the classical Weyl problem on realizing surfaces of nonnegative Gaussian curvature as the boundaries of convex bodies in 3–space. In the hyperbolic 3–space $\mathbb{H}^3$, convex surfaces have curvature at least -1. The work of Alexandrov [2] and Pogorelov [22] shows that, for each path metric d on the 2–sphere $\mathbb{S}^2$ of curvature ≥ -1, there exists a compact convex body, unique up to isometry, in $\mathbb{H}^3$ whose boundary is isometric to $(\mathbb{S}^2, d)$. The interesting remaining cases are noncompact surfaces of genus zero in the hyperbolic 3–space $\mathbb{H}^3$. A theorem of Alexandrov [2] states that any complete surface of genus zero whose curvature is at least -1 is isometric to the boundary of a closed convex set in $\mathbb{H}^3$. On the other hand, given a closed set $X \subset \mathbb{C}$, Thurston proved that the intrinsic metric on $\partial C_{\mathbb{H}}(X)$ is complete hyperbolic (see [8] for a proof). Putting these two theorems together, one sees that each complete hyperbolic metric on a surface of genus zero is isometric to the boundary of the convex hull of a closed set in the Riemann sphere. However, in this generality, the uniqueness of the convex surface is false. Conjectures 1.5 and 1.6 say that one has both the existence and uniqueness if one restricts to the boundaries of the convex hulls of closed sets.

There is some evidence supporting Conjectures 1.5 and 1.6. The work of Rivin [24] and Schlenker [26] shows that Conjectures 1.5 and 1.6 hold if Ω has finite area (ie X is a finite set) or if Ω is conformal to the 2–sphere with a finite number of disjoint disks removed (ie X is a finite disjoint union of round disks). Our recent work [19] shows that Conjecture 1.5 holds for Ω having countably many topological ends using the work of He and Schramm on the Köbe conjecture. In particular, we prove that every noncompact simply connected polyhedral surface is discrete conformal to a marked plane or a marked disk.

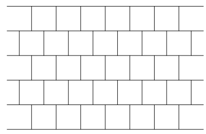

Figure 11: Regular hexagonal square tiling.

One should compare Conjectures 1.5 and 1.6 with the Köbe circle domain conjecture, which states that each genus zero Riemann surface is biholomorphic to the complement of a circle type closed set in the Riemann sphere. The work of He and Schramm [12] shows that Köbe conjecture holds for surfaces with countably many ends and the circle type set is unique up to Möbius transformations. Uniqueness is known to be false for the Köbe conjecture in general. Our recent work [19] shows that the Köbe conjecture is equivalent to Conjecture 1.5.

We end by proposing the following conjecture. The work of Rodin and Sullivan [25] and Theorem 1.3 show the rigidity phenomena for the two most regular patterns (regular hexagonal circle packing and regular hexagonal triangulation) in the plane. These rigidity results can be used to approximate the Riemann mappings and the uniformization metrics. The third regular pattern in the plane is the hexagonal square tiling in which each square of side length 1 interests exactly six others. See Figure 11.

Conjecture 7.4 *Suppose $\{S_i \mid i \in I\}$ is a locally finite square tiling of the complex plane $\mathbb{C}$ such that each square intersects exactly six others. Then all squares S_i have the same size.*

References

[1] **L V Ahlfors**, *Lectures on quasiconformal mappings*, 2nd edition, Univ. Lect. Ser. 38, Amer. Math. Soc., Providence, RI (2006) MR Zbl

[2] **A D Alexandrov**, *Selected works*, *II*: *Intrinsic geometry of convex surfaces*, Chapman & Hall, Boca Raton, FL (2006) MR Zbl

[3] **A I Bobenko**, **U Pinkall**, **B A Springborn**, *Discrete conformal maps and ideal hyperbolic polyhedra*, Geom. Topol. 19 (2015) 2155–2215 MR Zbl

[4] **U Bücking**, *Approximation of conformal mappings by circle patterns*, Geom. Dedicata 137 (2008) 163–197 MR Zbl

[5] **U Bücking**, *Approximation of conformal mappings using conformally equivalent triangular lattices*, from "Advances in discrete differential geometry" (A I Bobenko, editor), Springer (2016) 133–149 MR Zbl

[6] **F R K Chung**, *Spectral graph theory*, CBMS Region. Conf. Ser. Math. 92, Amer. Math. Soc., Providence, RI (1997) MR Zbl

[7] **S Dai**, **H Ge**, **S Ma**, *Rigidity of the hexagonal Delaunay triangulated plane*, Peking Math. J. 5 (2022) 1–20

[8] **D B A Epstein**, **A Marden**, *Convex hulls in hyperbolic space*, *a theorem of Sullivan*, *and measured pleated surfaces*, from "Analytical and geometric aspects of hyperbolic space" (D B A Epstein, editor), Lond. Math. Soc. Lect. Note Ser. 111, Cambridge Univ. Press (1987) 113–253 MR Zbl

[9] **D Glickenstein**, *Discrete conformal variations and scalar curvature on piecewise flat two- and three-dimensional manifolds*, J. Differential Geom. 87 (2011) 201–237 MR Zbl

[10] **X D Gu**, **F Luo**, **J Sun**, **T Wu**, *A discrete uniformization theorem for polyhedral surfaces*, J. Differential Geom. 109 (2018) 223–256 MR Zbl

[11] **X Gu**, **R Guo**, **F Luo**, **J Sun**, **T Wu**, *A discrete uniformization theorem for polyhedral surfaces, II*, J. Differential Geom. 109 (2018) 431–466 MR Zbl

[12] **Z-X He**, **O Schramm**, *Fixed points, Koebe uniformization and circle packings*, Ann. of Math. 137 (1993) 369–406 MR Zbl

[13] **Z-X He**, **O Schramm**, *On the convergence of circle packings to the Riemann map*, Invent. Math. 125 (1996) 285–305 MR Zbl

[14] **S Hersonsky**, *Discrete harmonic maps and convergence to conformal maps, I: Combinatorial harmonic coordinates*, Comment. Math. Helv. 90 (2015) 325–364 MR Zbl

[15] **S Hersonsky**, *Electrical networks and Stephenson's conjecture*, preprint (2015) arXiv 1506.07549

[16] **S-Y Lan**, **D-Q Dai**, *The C^∞–convergence of SG circle patterns to the Riemann mapping*, J. Math. Anal. Appl. 332 (2007) 1351–1364 MR Zbl

[17] **O Lehto**, **K I Virtanen**, *Quasiconformal mappings in the plane*, 2nd edition, Grundl. Math. Wissen. 126, Springer (1973) MR Zbl

[18] **F Luo**, *Combinatorial Yamabe flow on surfaces*, Commun. Contemp. Math. 6 (2004) 765–780 MR Zbl

[19] **F Luo**, **T Wu**, *Koebe conjecture and the Weyl problem for convex surfaces in hyperbolic 3–space*, preprint (2019) arXiv 1910.08001

[20] **D Matthes**, *Convergence in discrete Cauchy problems and applications to circle patterns*, Conform. Geom. Dyn. 9 (2005) 1–23 MR Zbl

[21] **B Palka**, *Fréchet distance and the uniform convergence of quasiconformal mappings*, Duke Math. J. 39 (1972) 289–296 MR Zbl

[22] **A V Pogorelov**, *Extrinsic geometry of convex surfaces*, Transl. Math. Monogr. 35, Amer. Math. Soc., Providence, RI (1973) MR Zbl

[23] **C Pommerenke**, *Univalent functions*, Studia Math. 25, Vandenhoeck & Ruprecht, Göttingen (1975) MR Zbl

[24] **I Rivin**, *Intrinsic geometry of convex ideal polyhedra in hyperbolic* 3*–space*, from "Analysis, algebra, and computers in mathematical research" (M Gyllenberg, L-E Persson, editors), Lect. Notes Pure Appl. Math. 156, Dekker, New York (1994) 275–291 MR Zbl

[25] **B Rodin**, **D Sullivan**, *The convergence of circle packings to the Riemann mapping*, J. Differential Geom. 26 (1987) 349–360 MR Zbl

[26] **J-M Schlenker**, *Réalisations de surfaces hyperboliques complètes dans* H^3, Ann. Inst. Fourier (Grenoble) 48 (1998) 837–860 MR Zbl

[27] **W Thurston**, *Geometry and topology of three-manifolds*, lecture notes, Princeton Univ. (1980) Available at `http://library.msri.org/books/gt3m/`

[28] **T Wu**, **X Gu**, **J Sun**, *Rigidity of infinite hexagonal triangulation of the plane*, Trans. Amer. Math. Soc. 367 (2015) 6539–6555 MR Zbl

Department of Mathematics, Rutgers University
Piscataway, NJ, United States

Mathematical Science Center, Tsinghua University
Beijing, China

Center of Mathematical Sciences and Applications, Harvard University
Cambridge, MA, United States

`fluo@math.rutgers.edu`, `sunjian0813@gmail.com`,
`tianqi@cmsa.fas.harvard.edu`

Proposed: David Gabai — Received: 8 April 2018
Seconded: Tobias H Colding, Dmitri Burago — Revised: 28 December 2020

GEOMETRY & TOPOLOGY

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2022 is US $685/year for the electronic version, and $960/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 26 Issue 3 (pages 937–1434) 2022

Discrete conformal geometry of polyhedral surfaces and its convergence 937

FENG LUO, JIAN SUN and TIANQI WU

Pseudoholomorphic curves relative to a normal crossings symplectic divisor: compactification 989

MOHAMMAD FARAJZADEH-TEHRANI

Effective bilipschitz bounds on drilling and filling 1077

DAVID FUTER, JESSICA S PURCELL and SAUL SCHLEIMER

Twisted Brin–Thompson groups 1189

JAMES BELK and MATTHEW C B ZAREMSKY

A homological model for $U_q\mathfrak{sl}2$ Verma modules and their braid representations 1225

JULES MARTEL

Counting hyperbolic multigeodesics with respect to the lengths of individual components and asymptotics of Weil–Petersson volumes 1291

FRANCISCO ARANA-HERRERA

Combinatorial Ricci flows and the hyperbolization of a class of compact 3–manifolds 1349

KE FENG, HUABIN GE and BOBO HUA

Asymptotically rigid mapping class groups, I: Finiteness properties of braided Thompson's and Houghton's groups 1385

ANTHONY GENEVOIS, ANNE LONJOU and CHRISTIAN URECH