ON A CONVERSE THEOREM FOR G_2 OVER FINITE FIELDS

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ABSTRACT. In this paper, we prove certain multiplicity one theorems and define twisted gamma factors for irreducible generic cuspidal representations of split G_2 over finite fields k of odd characteristic. Then we prove the first converse theorem for exceptional groups, namely, GL_1 and GL_2-twisted gamma factors will uniquely determine an irreducible generic cuspidal representation of $\mathrm{G}_2(k)$.

1. INTRODUCTION

In the theory of automorphic representations, the global converse problems aim to recover automorphic forms from their Fourier coefficients. Global converse theorems have played crucial roles in establishing global Langlands functoriality ([CKPSS01, CKPSS04, CPSS11]) and are very important for the Langlands Program. In the theory of representations of reductive groups over local and finite fields, the converse problems aim to find a minimal complete set of invariants of twisted gamma factors uniquely determining irreducible generic representations. Local converse theorems have been used to prove the uniqueness of the generic local Langlands functoriality and the local Langlands correspondence ([JS03, H93]). While converse problems have been extensively studied for general linear and classical groups, they have not been studied for exceptional groups. The goal of this paper is to prove the converse theorem for the split exceptional group of type G_2 over finite fields of odd characteristic, which seems to be the first converse theorem for exceptional groups. In the following, we first introduce the recent progress on the study of the converse problems for general linear and classical groups over local and finite fields.

Let F be a p-adic field. Let π be an irreducible generic representation of $\mathrm{GL}_n(F)$. The family of local twisted gamma factors $\gamma(s, \pi \times \tau, \psi)$, for τ any irreducible generic representation of $\mathrm{GL}_r(F)$, ψ an additive character of F and $s \in \mathbb{C}$, can be defined using Rankin–Selberg convolution [JPSS83] or the Langlands–Shahidi method [S84]. The local converse problem is that which family of local twisted gamma factors will uniquely determine π? The following is the famous Jacquet's conjecture on the local converse problem.

Conjecture 1.1 (Jacquet's conjecture on the local converse problem)**.** *Let π_1, π_2 be irreducible generic representations of $\mathrm{GL}_n(F)$. Suppose that they have the same central character. If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s, for all irreducible generic representations τ of $\mathrm{GL}_r(F)$ with $1 \le r \le [\frac{n}{2}]$, then $\pi_1 \cong \pi_2$.

Conjecture 1.1 has recently been proved by Chai ([Ch19]), and by Jacquet and the first-named author ([JL18]), independently, using different analytic methods.

One can propose a more general family of conjectures as follows (see [ALSX16]). Let π_1, π_2 be irreducible generic representations of $\mathrm{GL}_n(F)$. We say that π_1 and π_2 satisfy hypothesis $\mathcal{H}_0$ if they have the same central character. For $m \in \mathbb{Z}_{\ge 1}$, we say that they satisfy hypothesis $\mathcal{H}_m$ if they satisfy hypothesis $\mathcal{H}_0$ and satisfy

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

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as functions of the complex variable s, for all irreducible generic representations τ of $\mathrm{GL}_m(F)$. For $r \in \mathbb{Z}_{\geq 0}$, we say that π_1, π_2 satisfy hypothesis $\mathcal{H}_{\leq r}$ if they satisfy hypothesis $\mathcal{H}_m$, for $0 \leq m \leq r$.

Conjecture $\mathcal{J}(n, r)$. *If π_1, π_2 are irreducible generic representations of $\mathrm{GL}_n(F)$ which satisfy hypothesis $\mathcal{H}_{\leq r}$, then $\pi_1 \simeq \pi_2$.*

Conjecture 1.1 was exactly Conjecture $\mathcal{J}(n, [\frac{n}{2}])$. Conjecture $\mathcal{J}(2, 1)$ was first proved by Jacquet and Langlands [JL70]. Conjecture $\mathcal{J}(3, 1)$ was first proved by Jacquet, Piatetski-Shapiro, and Shalika [JPSS79]. For general n, Conjecture $\mathcal{J}(n, n-1)$ was proved by Henniart in [H93], and by Cogdell-Piatetski-Shapiro using a global method in [CPS94]. Conjecture $\mathcal{J}(n, n-2)$ (for $n \geq 3$) is a theorem due to Chen [Ch96, Ch06], to Cogdell and Piatetski-Shapiro [CPS99], and to Hakim and Offen [HO15].

In [JNS15], Jiang, Nien and Stevens showed that Conjecture 1.1 is equivalent to the same conjecture with the adjective "generic" replaced by "unitarizable supercuspidal" as follows:

Conjecture 1.2. *Let π_1, π_2 be irreducible unitarizable supercuspidal representations of $\mathrm{GL}_n(F)$. Suppose that they have the same central character. If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s, for all irreducible supercuspidal representations τ of $\mathrm{GL}_r(F)$ with $1 \leq r \leq [\frac{n}{2}]$, then $\pi_1 \cong \pi_2$.

Making use of the construction of supercuspidal representations of $\mathrm{GL}_n(F)$ in [BK93] and properties of Whittaker functions of supercuspidal representations constructed in [PS08], Jiang, Nien and Stevens introduced the notion of a special pair of Whittaker functions for a pair of irreducible unitarizable supercuspidal representations π_1, π_2 of $\mathrm{GL}_n(F)$. They proved that if there is such a pair, and π_1, π_2 satisfy hypothesis $\mathcal{H}_{\leq [\frac{n}{2}]}$, then $\pi_1 \cong \pi_2$. They also found special pairs of Whittaker functions in many cases, in particular the case of depth zero representations. In [ALSX16], Adrian, the first-named author, Stevens and Xu proved part of the case left open in [JNS15]. In particular, the results in [JNS15] and [ALSX16] together imply that Conjecture 1.2 is true for GL_n, n prime.

It is easy to find pairs of generic representations showing that in Conjecture 1.1, $[\frac{n}{2}]$ is sharp for the generic dual of $\mathrm{GL}_n(F)$. In [ALST18], Adrian, the first-named author, Stevens and Tam showed that, in Conjecture 1.2, $[\frac{n}{2}]$ is sharp for the supercuspidal dual of $\mathrm{GL}_n(F)$, for n prime, in the tame case. It is believed that in Conjecture 1.2, $[\frac{n}{2}]$ is sharp for the supercuspidal dual of $\mathrm{GL}_n(F)$, for any n, in all cases. However, it is expected that for certain families of supercuspidal representations, $[\frac{n}{2}]$ may not be sharp, for example, for simple supercuspidal representations (of depth $\frac{1}{n}$), the upper bound may be lowered to 1 (see [BH14, Proposition 2.2] and [AL16, Remark 3.18] in general, and [X13] in the tame case).

For general reductive groups, one can consider analogue converse problems whenever the twisted gamma factors have been defined, for example, using either the Rankin-Selberg convolution method or the Langlands-Shahidi method if available.

Nien in [N14] proved the finite fields analogue of Conjecture 1.1 for cuspidal representations of GL_n, using special properties of normalized Bessel functions and the twisted gamma factors defined by Roditty ([Ro10]). Similar local converse theorems were extended to certain classical groups in [LZ21]. Adrian and Takeda in [AT18] proved a local converse theorem for GL_n over archimedean local fields using L-functions. In [M16], Moss defined the twisted gamma factors for ℓ-adic families of smooth representations of $\mathrm{GL}_n(F)$, where F is a finite extension of $\mathbb{Q}_p$ and ℓ is different from p, and proved an analogue of Conjecture $\mathcal{J}(n, n-1)$. In [LM20], joint with Moss, the first-named author proved an analogue of Conjecture $\mathcal{J}(n, [\frac{n}{2}])$ for ℓ-adic families, using the idea in [JL18]. In [NZ21], Nien and Zhang verified a converse theorem for Gauss sum of characters of finite fields $\mathbb{F}_{q^n}$ and showed that such a character is determined by Gauss sum twisted by characters of $\mathrm{GL}_1(\mathbb{F}_q)$, for $n \leq 5$, or for $n < \frac{q-1}{2\sqrt{q}} + 1$ in the appendix by Zhiwei Yun.

For p-adic groups other than GL_n, in particular classical groups, twisted gamma factors have been defined in many cases and the local converse problems have been vastly studied: $\mathrm{U}(2, 1)$ and $\mathrm{GSp}(4)$ (Baruch, [B95] and [B97]); $\mathrm{SO}(2n+1)$ (Jiang and Soudry, [JS03]); U_{2n} (Morimoto [Mo18], and the second named author [Zh17a, Zh17b, Zh18]); $\mathrm{Sp}(2n)$ and U_{2n+1}(the second named author, [Zh18], and [Zh19]).

The ideas of converse theorems have been extended to distinction problems, namely, using special values of twisted gamma factors to characterize representations of $\mathrm{GL}_n(E)$ distinguished by $\mathrm{GL}_n(F)$, where E/F is a quadratic extension, see Hakim and Offen [HO15] (over p-adic fields), and Nien [N19] (over finite fields).

In this paper, we consider the first case (split G_2) of converse problems for generic representations of exceptional groups. Let k be a finite field of odd characteristic p and let ψ be a fixed non-trivial additive character of k. We define GL_1 and GL_2-twisted gamma factors for irreducible generic cuspidal representations of $\mathrm{G}_2(k)$, which are denoted using Γ rather than γ since we do not consider the normalization issue here, see Propositions 5.8 and 6.6, and prove the first converse theorem for exceptional groups as follows:

Theorem 1.3 (Theorem 7.3)**.** *Let Π_1, Π_2 be two irreducible generic cuspidal representation of $\mathrm{G}_2(k)$. If*

$$\Gamma(\Pi_1 \times \chi, \psi) = \Gamma(\Pi_2 \times \chi, \psi),$$
$$\Gamma(\Pi_1 \times \tau, \psi) = \Gamma(\Pi_2 \times \tau, \psi),$$

for all characters χ of $k^\times$ and all irreducible generic representations τ of $\mathrm{GL}_2(k)$, we have $\Pi_1 \cong \Pi_2$.

To define GL_1-twisted gamma factors, we use the finite fields analogue of Ginzburg's local zeta integral in [Gi93] and we prove the following multiplicity one theorem to deduce the functional equation:

Theorem 1.4 (Theorem 2.1)**.** *Let Π be an irreducible cuspidal representation of $\mathrm{G}_2(k)$, then*

$$\dim \mathrm{Hom}_J(\Pi, I(\chi) \otimes \omega_\psi) \leq 1,$$

where J is the Jacobi group contained in the maximal parabolic subgroup of $\mathrm{G}_2(k)$ with the long root in the Levi (see § 2.3 for definitions), χ is a character of $k^\times$, $I(\chi)$ is the induced representation of $\mathrm{SL}_2(k)$ from χ, and ω_ψ is the Weil representation of J with central character ψ.

To prove the above theorem, we need to use the classification of representations of $\mathrm{G}_2(k)$ given by Chang and Ree ([CR74], for $p > 3$) and by Enomoto ([En76], for $p = 3$), and compute the dimension of the Hom space for each irreducible cuspidal representation.

We remark that in [L84] Lusztig gave the classification of representations of connective reductive groups using the virtual character theory of Deligne-Lusztig [DL76]. For convenience, in this paper we follow the classification of Chang-Ree and Enomoto.

To define GL_2-twisted gamma factors, we embed G_2 into SO_7 and use the finite fields analogue of the local zeta integral developed by Piatetski-Shapiro, Rallis and Schiffmann ([PSRS92]). The functional equation in this case follows from the following multiplicity one result

Proposition 1.5 (Proposition 6.4)**.** *Let Π be an irreducible generic cuspidal representation of $\mathrm{G}_2(k)$ and let τ be an irreducible generic representation of $\mathrm{GL}_2(k)$. Then we have*

$$\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(I(\tau)|_{\mathrm{G}_2(k)}, \Pi) = 1,$$

where $I(\tau) = \mathrm{Ind}_{\widetilde{P}}^{\mathrm{SO}_7(k)}(\tau \otimes 1_{\mathrm{SO}_3})$, $\widetilde{P}$ is a parabolic subgroup of $\mathrm{SO}_7(k)$ with the Levi subgroup isomorphic to $\mathrm{GL}_2(k) \times \mathrm{SO}_3(k)$ (see §6 for definitions).

The existence of gamma factors for $\mathrm{G}_2(k) \times \mathrm{GL}_2(k)$ follows from the above proposition and is given in Proposition 6.6.

Over p-adic fields F, given an irreducible generic cuspidal representation π of $\mathrm{G}_2(F)$ and an irreducible generic representation τ of $\mathrm{GL}_i(F)$, $i = 1, 2$, assuming the gamma factor $\gamma(s, \pi \times \tau, \psi)$ has been defined, it is expected that if the gamma factor $\gamma(s, \pi \times \tau, \psi)$ has a pole at $s = 1$, then τ should occur in the conjectural local Langlands parameter of π. Similar results are also expected for general reductive groups, and have been proved for classical groups (see for example [JS12] and references given there). However, over finite fields k, the analogue meaning of gamma factors is not clear, noting that the gamma factors over finite fields are just complex numbers.

For the convenience of readers, we summarize the proof of Theorem 1.3 briefly as follows. Let $\mathcal{B}_i := \mathcal{B}_{\Pi_i} \in \mathcal{W}(\Pi_i, \psi)$ be the Bessel function of Π_i for $i = 1, 2$, namely, the Whittaker function associated with a Whittaker vector, normalized by $\mathcal{B}_i(1) = 1$ (see §5.1 and Lemma 5.2 for the basic properties of

$\mathcal{B}_i$). We will prove that $\mathcal{B}_1(g) = \mathcal{B}_2(g)$ for all $g \in \mathrm{G}_2(k)$ under the assumption of Theorem 1.3. Since $\mathrm{G}_2(k) = \coprod_{w \in \mathrm{W}(\mathrm{G}_2)} BwB$, where B is a fixed Borel subgroup of G_2, it suffices to show that $\mathcal{B}_1$ agrees with $\mathcal{B}_2$ on various cells BwB. Let $\mathrm{B}(\mathrm{G}_2) = \{w \in \mathrm{W}(\mathrm{G}_2) : \forall \gamma \in \Delta, w\gamma > 0 \implies w\gamma \in \Delta\}$, where $\Delta = \{\alpha, \beta\}$ is the set of simple roots of G_2 with α being the short root and β being the long root. Let s_α (resp. s_β) be the simple reflection defined by α (resp. β). Then $\mathrm{B}(\mathrm{G}_2) = \{1, w_1, w_2, w_\ell\}$, where w_ℓ is the longest Weyl element, $w_1 = w_\ell s_\alpha$ and $w_2 = w_\ell s_\beta$. By Lemma 7.1, if $w \notin \mathrm{B}(\mathrm{G}_2)$, we have $\mathcal{B}_1(g) = \mathcal{B}_2(g) = 0$ for $g \in BwB$. If $w = 1$, we also have $\mathcal{B}_1(g) = \mathcal{B}_2(g), \forall g \in B$ by Lemma 5.2. Thus it suffices to show that $\mathcal{B}_1(g) = \mathcal{B}_2(g), \forall g \in BwB$ with $w = w_1, w_2, w_\ell$. It turns out that the equality of GL_1-twisted gamma factors implies that $\mathcal{B}_1(g) = \mathcal{B}_2(g), \forall g \in Bw_1B$, and the equality of GL_2-twisted gamma factors implies that $\mathcal{B}_1(g) = \mathcal{B}_2(g), \forall g \in BwB$ with $w = w_2, w_\ell$. This completes the proof of Theorem 1.3.

Theorem 1.3 inspires us to consider the local converse problem for $\mathrm{G}_2(F)$ when F is a p-adic field. In this case, our proof in §6 is actually valid for an analogue of Proposition 1.5 without the restriction that Π is cuspidal, which gives us the local functional equation of the local zeta integral of Piatetski-Shapiro-Rallis-Schiffmann ([PSRS92]) and hence the existence of the GL_2-twisted local gamma factors. However, the existence of the GL_1-twisted local gamma factors relies on the following

Conjecture 1.6. *Let F be a p-adic field and Π be an irreducible generic representation of $\mathrm{G}_2(F)$. Let ψ be a nontrivial additive character of F. Let $\widetilde{I}(\chi, \psi)$ be the genuine induced representation on the double cover $\widetilde{\mathrm{SL}}_2(F)$ for a character χ of $F^\times$. Then if $\widetilde{I}(\chi, \psi)$ is irreducible, we have*

$$\dim_J(\Pi, \widetilde{I}(\chi, \psi) \otimes \omega_\psi) \le 1.$$

Note that both $\widetilde{I}(\chi, \psi)$ and ω_ψ are genuine representations on a double cover of J and the thus the tensor product $\widetilde{I}(\chi, \psi)$ is a representation on J.

In the above conjecture, we keep the requirement minimal so that it is enough to deduce the local functional equation of Ginzburg's local zeta integral ([Gi93]). We do expect that the following generalized conjecture is true

Conjecture 1.7. *Let F be a p-adic field and Π be an irreducible (selfdual) representation of $\mathrm{G}_2(F)$. Let ψ be a nontrivial additive character of F. Let $\widetilde{\pi}$ be an irreducible genuine representation on the double cover $\widetilde{\mathrm{SL}}_2(F)$. Then we have*

$$\dim_J(\Pi, \widetilde{\pi} \otimes \omega_\psi) \le 1.$$

As explained in [LZ19, §6], Conjecture 1.7 is an analogue of the uniqueness problem of Fourier-Jacobi models for Sp_{2n}, which was proved in [BR00] (for $n = 2$) and in [GGP12, Su12](for general n). Once Conjecture 1.6 is established, we then have the local gamma factors for irreducible generic representations of $\mathrm{G}_2(F) \times \mathrm{GL}_1(F)$ using Ginzburg's local zeta integral. Inspired by Theorem 1.3, we propose the following conjecture on the local converse problem for $\mathrm{G}_2(F)$.

Conjecture 1.8. *Let F be a p-adic field. Suppose that Conjecture 1.6 is true. Let Π_1, Π_2 be two irreducible generic representations of $\mathrm{G}_2(F)$. If*

$$\gamma(s, \Pi_1 \times \chi, \psi) = \gamma(s, \Pi_2 \times \chi, \psi),$$
$$\gamma(s, \Pi_1 \times \tau, \psi) = \gamma(s, \Pi_2 \times \tau, \psi),$$

for all characters χ of $\mathrm{GL}_1(F)$ and all irreducible generic representations τ of $\mathrm{GL}_2(F)$, then $\Pi_1 \cong \Pi_2$.

Conjectures 1.6–1.8 are current work in progress of the authors.

Again, by Langlands philosophy of functoriality, representations of $\mathrm{G}_2(F)$ are expected to be lifted to representations of $\mathrm{GL}_7(F)$ and this lifting is expected to preserve GL-twisted local gamma factors. Then the Jacquet's local converse conjecture for GL_n, which was recently proved in [Ch19, JL18], implies that two irreducible generic representations $\Pi_i, i = 1, 2$ of $\mathrm{G}_2(F)$ would be isomorphic if the twisted local gamma factors $\gamma(s, \Pi_1 \times \tau, \psi)$, $\gamma(s, \Pi_2 \times \tau, \psi)$, are the same for all irreducible generic representations τ of $\mathrm{GL}_n(F)$ for all $n = 1, 2, 3$ (once they are all defined). Theorem 1.3 says that we only need GL_1 and GL_2-twisted gamma factors over finite fields, and we expect the same is true over p-adic fields as in Conjecture 1.8.

The paper is organized as follows. In §2, we introduce the group G_2, the Fourier-Jacobi group J, the Weil representations ω_ψ, and the Multiplicity One Theorem 1.4. Theorem 1.4 is proved in §3 (for $p > 3$) and §4 (for $p = 3$). We define GL_1 and GL_2-twisted gamma factors for irreducible generic cuspidal representations in §5 and §6, respectively. Finally, Theorem 1.3 is proved in §7. In Appendix A, we compute certain Gauss sums which are used in the proof of Theorem 1.4. In Appendix B, we describe the embedding of G_2 into SO_7 used in this paper.

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2. The Fourier-Jacobi group and a multiplicity one theorem

2.1. **Some notations and conventions.** Throughout this paper, unless specified otherwise, we fix the following notations. Let p be an odd prime and q is a power of p. Let $k = \mathbb{F}_q$, the finite field with q elements. Let $\epsilon(x) = \left(\frac{x}{q}\right)$, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. Let $\epsilon_0 = \epsilon(-1)$. Then we have $\epsilon_0 = 1$ if $q \equiv 1 \bmod 4$, and $\epsilon_0 = -1$ if $q \equiv 3 \bmod 4$. Let $k^{\times,2} = \left\{x^2 : x \in k^\times\right\}$, and $k^{\times,3} = \left\{x^3 : x \in k^\times\right\}$. Let k_2 be the unique quadratic extension of k, i.e., $k_2 = \mathbb{F}_{q^2}$. We fix a generator κ of the multiplicative group $k^\times$. Then we have $\kappa \in k^\times - k^{\times,2}$. Let ψ be a fixed non-trivial additive character of k. Then there exists a 4-th root of unity ϵ_ψ such that for any $a \in k^\times$, we have

$$\sum_{x\in k} \psi(ax^2) = \epsilon_\psi \epsilon(a)\sqrt{q}. \tag{2.1}$$

Moreover, we have $\epsilon_\psi^2 = \epsilon_0$. See [Bu97, Ex.4.1.14] for example. By abuse of notation, we write ϵ_ψ as $\sqrt{\epsilon_0}$.

We usually don't distinguish a representation and its space. Thus for a representation π of a group G, a vector $v \in \pi$ means that a vector v in the space of π.

2.2. **Weil representations of** $SL_2(k)$**.** Let $W = k^2$, endowed with the symplectic structure $\langle\ ,\ \rangle$ defined by

$$\langle (x_1, y_1), (x_2, y_2)\rangle = -2x_1y_2 + 2x_2y_1. \tag{2.2}$$

Let $\mathscr{H}$ be the Heisenberg group associated with the symplectic space W. Explicitly, $\mathscr{H} = W \oplus k$ with addition

$$[x_1, y_1, z_1] + [x_2, y_2, z_2] = [x_1 + x_2, y_1 + y_2, z_1 + z_2 - x_1y_2 + x_2y_1].$$

Let $SL_2(k)$ act on $\mathscr{H}$ such that it acts on W from the right and act on the third component k in $\mathscr{H}$ trivially. Then we can form the semi-direct product $SL_2(k) \ltimes \mathscr{H}$. The product in $SL_2(k) \ltimes \mathscr{H}$ is given by

$$(g_1, v_1)(g_2, v_2) = (g_1g_2, v_1.g_2 + v_2).$$

Here $v_1.g_2$ is the action of g_2 on v_1 from the right and v_1 is viewed as a row vector. By [Ge77], there is a Weil representation ω_ψ of on $\mathcal{S}(k)$, the space of $\mathbb{C}$-valued functions on k. The Weil representation ω_ψ is determined by several formula, which can be found in [GH17]. Note that the symplectic form in [GH17] is a little bit different from ours. Thus the formulas in [GH17] should be adapted to our slightly different symplectic structure on W. One can consult [Ku96] for the dependence on the symplectic structure. The Weil representation in our case is determined by the following formulas:

$$
\begin{aligned}
\omega_\psi([x,0,z])\phi(\xi) &= \psi(z)\phi(\xi+x),\\
\omega_\psi([0,y,0])\phi(\xi) &= \psi(-2\xi y)\phi(\xi),\\
\omega_\psi(\mathrm{diag}(a,a^{-1}))\phi(\xi) &= \epsilon(a)\phi(a\xi),\\
\omega_\psi\left(\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\right)\phi(\xi) &= \psi(-b\xi^2)\phi(\xi),\\
\omega_\psi\left(\begin{pmatrix} & b\\ -b^{-1} & \end{pmatrix}\right)\phi(\xi) &= \frac{1}{\gamma(b,\psi)}\sum_{x\in k}\phi(x)\psi(-2xb\xi),
\end{aligned}
\tag{2.3}
$$

for $\phi \in \mathcal{S}(k), x, y, z, \xi, b \in k, a \in k^\times$. Here $\gamma(b,\psi) = \sum_{x\in k}\psi(-bx^2)$, which can be computed using (2.1).

2.3. **The group G_2 and its Fourier-Jacobi subgroup.** In this subsection, we give a very brief review of some definitions and notations related to the group $\mathrm{G}_2(k)$. More details can be found in [LZ19, §5].

Let G_2 be the split exceptional algebraic group of type G_2 over the field k. The group G_2 has two simple roots α, β, where α is the short root and β is the long root, and has 6 positive roots $\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta$. Let s_α, s_β be the reflections determined by α, β respectively. One has $s_\alpha(\beta) = 3\alpha+\beta, s_\beta(\alpha) = \alpha+\beta$. We use the standard notations of Chevalley groups [St67]. For a root γ, let $U_\gamma \subset \mathrm{G}_2(k)$ be the corresponding root space and let $\mathbf{x}_\gamma : k \to \mathrm{G}_2(k)$ be a fixed isomorphism which satisfies various Chevalley relations, see [St67, Chapter 3]. The explicit commutator relations can be found in [Ch68, p.192]. A matrix realization of $\mathbf{x}_\gamma(r), r \in k$, is given in Appendix B. The calculations in this paper could be performed using this explicit matrix realization.

For a root γ, let $w_\gamma(t) = \mathbf{x}_\gamma(t)\mathbf{x}_{-\gamma}(-t^{-1})\mathbf{x}_\gamma(t)$, $w_\gamma = w_\gamma(1)$, and $h_\gamma(t) = w_\gamma(t)w_\gamma^{-1}$. Note that w_γ is a representative of the Weyl group element s_γ. Let $h(t_1,t_2) = h_\alpha(t_1t_2)h_\beta(t_1^2t_2)$. One can check that $h(t_1,t_2)$ agrees with the notation $h(t_1,t_2,t_1^{-1}t_2^{-1})$ in [Ch68, CR74] and the notation $h(t_1,t_2)$ in [Gi93]. Let $T = \{h(t_1,t_2) : t_1, t_2 \in k^\times\}$ be the maximal torus of $\mathrm{G}_2(k)$ and let U be the subgroup generated by U_γ, γ positive. Then $B = TU$ is a Borel subgroup of $\mathrm{G}_2(k)$. It is known that G_2 has trivial center.

Let $P' = M'V'$ be the parabolic subgroup with Levi M' and unipotent V' such that $U_\alpha \subset M'$. Then $M' \cong \mathrm{GL}_2(k)$ and V' is the group generated by $U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta}$.

Let $P = MV$ be the parabolic subgroup with Levi M and unipotent V such that $U_\beta \subset M$. Note that V is generated by $U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta}$. Let $Z \subset V$ be the subgroup generated by $U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta}$. We still have $M \cong \mathrm{GL}_2(k)$ and the isomorphism can be realized by

$$
\begin{aligned}
\mathbf{x}_\beta(x) &\mapsto \begin{pmatrix}1 & x\\ & 1\end{pmatrix},\\
h(a,b) &\mapsto \begin{pmatrix}a & \\ & b\end{pmatrix}.
\end{aligned}
$$

Let $J \subset P$ be the subgroup $\mathrm{SL}_2(k) \ltimes V$, where $\mathrm{SL}_2(k)$ is viewed as a subgroup of M via $\mathrm{SL}_2(k) \subset \mathrm{GL}_2(k) \cong M$. A typical element in V is of the form

$$(r_1,r_2,r_3,r_4,r_5) := \mathbf{x}_\alpha(r_1)\mathbf{x}_{\alpha+\beta}(r_2)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5).$$

There is a group homomorphism

$$\overline{\mathrm{pr}} : J = \mathrm{SL}_2(k) \ltimes V \to \mathrm{SL}_2(k) \ltimes \mathscr{H}$$

given by

$$(g,(r_1,r_2,r_3,r_4,r_5)) \mapsto (d_1gd_1,(r_1,r_2,r_3-r_1r_2)),$$

where $d_1 = \mathrm{diag}(-1,1)$, see [LZ19, §5] for more details. Thus the Weil representation ω_ψ can be viewed as a representation of J via the above group homomorphism. By (2.3) and the description of the above group homomorphism, we have the following formulas

$$\begin{aligned}\omega_\psi((r_1,0,r_3,r_4,r_5))\phi(\xi)&=\psi(r_3)\phi(\xi+r_1),\\ \omega_\psi((0,r_2,0,0,0))\phi(\xi)&=\psi(-2\xi r_2)\phi(\xi),\\ \omega_\psi(h(a,a^{-1}))\phi(\xi)&=\epsilon(a)\phi(a\xi),\\ \omega_\psi(\mathbf{x}_\beta(b))\phi(\xi)&=\psi(b\xi^2)\phi(\xi),\\ \omega_\psi\left(\begin{pmatrix} & b\\ -b^{-1} & \end{pmatrix}\right)\phi(\xi)&=\frac{1}{\gamma(b,\psi)}\sum_{x\in k}\phi(x)\psi(-2xb\xi).\end{aligned}\tag{2.4}$$

Let χ be a character of $k^\times$ and we view χ as a character of the upper triangular subgroup $B_{\mathrm{SL}_2}=A_{\mathrm{SL}_2}N_{\mathrm{SL}_2}$ of $\mathrm{SL}_2(k)$ by

$$\chi\left(\begin{pmatrix} a & b\\ & a^{-1}\end{pmatrix}\right)=\chi(a).$$

Here A_{SL_2} is the diagonal torus of SL_2 and N_{SL_2} is the upper triangular unipotent subgroup of SL_2. Consider the induced representation $I(\chi):=\mathrm{Ind}_{B_{\mathrm{SL}_2}}^{\mathrm{SL}_2(k)}(\chi)$. We view $I(\chi)$ as a representation of J via the natural quotient map $J\to\mathrm{SL}_2(k)$. The first main result of this paper is the following

Theorem 2.1. *Let Π be an irreducible cuspidal representation of $\mathrm{G}_2(k)$, then*

$$\dim\mathrm{Hom}_J(\Pi, I(\chi)\otimes\omega_\psi)\le 1.$$

Remark 2.2. By Frobenius reciprocity, the above theorem can be restated as

$$\dim\mathrm{Hom}_{\mathrm{G}_2(k)}(\Pi,\mathrm{Ind}_J^{\mathrm{G}_2(k)}(I(\chi)\otimes\omega_\psi))\le 1$$

for all irreducible cuspidal representations Π of $\mathrm{G}_2(k)$. The representation $\mathrm{Ind}_J^{\mathrm{G}_2(k)}(I(\chi)\otimes\omega_\psi)$ is a special case of Generalized Gelfand-Graev representation considered by Kawanaka, see [K85, K86, K87]. There are many results on the computation of multiplicities of irreducible representations ρ in the Alvis-Curtis dual of certain Generalised Gelfand-Graev representations which are associated to the unipotent support of ρ, for example see [L92, Gec99, GeH08, T13]. In particular, in [L92], Lusztig gave a bound on such multiplicities under the assumption that p,q large. However, in this paper, we are mainly interested in the multiplicities of irreducible generic cuspidal representations of $\mathrm{G}_2(k)$ in the Generalized Gelfand-Graev representations which is associated to the next-to-minimal unipotent orbit $\widetilde{A}_1$ of $\mathrm{G}_2(\overline{k})$ (see [Ca85, p.401]). Note that the unipotent support of generic representations is the regular unipotent orbit, which does not contain $\widetilde{A}_1$.

Remark 2.3. (1). Our proof of Theorem 2.1 is by brute force and our main tool is the character table of $\mathrm{G}_2(k)$. Note that the characters of a finite reductive group are given in [L84] after the seminal work [DL76]. But prior to that, the detailed character table of $\mathrm{G}_2(k)$ was given by Chang-Ree [CR74] (when $p>3$) and Enomoto [En76] (when $p=3$). We will prove Theorem 2.1 for $p>3$ and $p=3$ separately in the next two sections.

(2). Note that, in the above theorem, we don't require that $I(\chi)$ is irreducible. One should compare Theorem 2.1 with [LZ19, Remark 7.2], where we have shown that the dimension of the Hom space may be bigger than 1 for general irreducible representations of $\mathrm{G}_2(k)$ even when $I(\chi)$ is irreducible, however, here we show that if we consider irreducible cuspidal representations of $\mathrm{G}_2(k)$, then the dimension of the Hom space is indeed less than or equal to 1.

Corollary 2.4. *Let Π be an irreducible cuspidal representation of $\mathrm{G}_2(k)$ and π be an irreducible representation of $\mathrm{SL}_2(k)$, then we have*

$$\dim\mathrm{Hom}_J(\Pi,\pi\otimes\omega_\psi)\le 1.$$

Proof. If π is an irreducible representation of $\mathrm{SL}_2(k)$ which is not of the form $I(\chi)$, the assertion follows from the main theorem of [LZ19]. If π is of the form $I(\chi)$, the assertion follows from Theorem 2.1. $\square$

3. Proof of Theorem 2.1 when $p > 3$

In this section, we prove Theorem 2.1 when $p > 3$. The character table of $\mathrm{G}_2(k)$ when $p > 3$ is given in [CR74].

3.1. **Character table of $I(\chi) \otimes \omega_\psi$.** As a preparation for the proof of Theorem 2.1, in this subsection, we give the character table of the representation $I(\chi) \otimes \omega_\psi$ of J. Given a representation π of a finite group, denote by Ch_π the character of π. It is well-known that

$$\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}(g) = \mathrm{Ch}_{I(\chi)}(g)\mathrm{Ch}_{\omega_\psi}(g). \tag{3.1}$$

We first record the conjugacy classes of $\mathrm{SL}_2(k)$:

Representative	Number of elements in class	Number of classes
$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	1	1
$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	1	1
$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$(q^2-1)/2$	1
$\begin{pmatrix} 1 & \kappa \\ & 1 \end{pmatrix}$	$(q^2-1)/2$	1
$\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$	$(q^2-1)/2$	1
$\begin{pmatrix} -1 & \kappa \\ & -1 \end{pmatrix}$	$(q^2-1)/2$	1
$\begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}, x \neq \pm 1$	$q(q+1)$	$(q-3)/2$
$\begin{pmatrix} x & y \\ \kappa y & x \end{pmatrix}, x \neq \pm 1, y \neq 0$	$q(q-1)$	$(q-1)/2$

The above table could be found in [FH91], for example. As a representation of $\mathrm{SL}_2(k)$, the character tables of $I(\chi)$ and ω_ψ are given in [LZ19]. In particular, we know that

$$\mathrm{Ch}_{I(\chi)}\left(\begin{pmatrix} x & y \\ \kappa y & x \end{pmatrix}\right) = 0.$$

Thus by (3.1), it suffices to consider elements of the form gv with $v \in V$, and $g \in \mathrm{SL}_2(k)$ not of the form $\begin{pmatrix} x & y \\ \kappa y & x \end{pmatrix}$. Recall that Z is the group generated by $U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta}$.

Proposition 3.1 ([Ge77, Theorem 4.4, (b)]). *If the function $\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}$ is nonzero on $j \in J$, then j is conjugate to an element in $\mathrm{SL}_2(k) \ltimes Z$.*

By the above proposition, we need to consider elements in J which are J-conjugate to elements of the form $g(0,0,r_3,r_4,r_5), g \in \mathrm{SL}_2(k)$. For a group H and $h_1, h_2 \in H$, we write $h_1 \sim_H h_2$ if $h_1 = h_0 h_2 h_0^{-1}$ for some $h_0 \in H$.

Lemma 3.2. *The following is a set of representatives of $j \in J$ such that j is conjugate to an element of the form $g(0,0,r_3,r_4,r_5)$ with $g \in \mathrm{SL}_2(k)$ and not of the form $\begin{pmatrix} x & y \\ \kappa y & x \end{pmatrix}$, $y \neq 0$:*

1. $1; (0,0,0,0,1); (0,0,r_3,0,0), r_3 \in k^\times$;
2. $\mathbf{x}_\beta(b)(0,0,r_3,0,0); \mathbf{x}_\beta(b)(0,0,r_3,r_4,0), b \in \{1,\kappa\}, r_3 \in k, r_4 \in k^\times/\{\pm 1\}$;
3. $h(-1,-1)\mathbf{x}_\beta(b)(0,0,r_3,0,0), r_3 \in k, b \in \{1,\kappa\}$;
4. $h(x,x^{-1})(0,0,r_3,0,0), x \neq \pm 1, r_3 \in k$.

Proof. First we notice that if $g \sim_{\mathrm{SL}_2(k)} g'$, then $g(0,0,r_3,r_4,r_5) \sim_J g'(0,0,r_3',r_4',r_5')$ for some $r_3', r_4', r_5' \in k$. Thus we only need to consider the case when g runs over a set of representatives of $\mathrm{SL}_2(k)$-conjugacy classes.

TABLE 3.1. Conjugacy classes of J which are J-conjugate to elements of the form $g(0,0,r_3,r_4,r_5)$. Here $*$ in the last row means the details of the corresponding entries are omitted since they are not used.

Representative t	$C_J(t)$	$\lvert J(t)\rvert$	$No.$	$\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}$
$\begin{pmatrix}1 & \\ & 1\end{pmatrix}$	J	1	1	$q(q+1)$
$\begin{pmatrix}1 & \\ & 1\end{pmatrix}(0,0,0,0,1)$	$N_{\mathrm{SL}_2}\ltimes V$	q^2-1	1	$q(q+1)$
$(0,0,r_3,0,0), r_3\neq 0$	$\mathrm{SL}_2\ltimes Z$	q^2	$q-1$	$q(q+1)\psi(r_3)$
$\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}(0,0,r_3,0,0)$, $r_3\in k$	$\mu_2\langle U_\beta,U_{\alpha+\beta},U_{2\alpha+\beta},U_{3\alpha+2\beta}\rangle$	$\frac{q^2-1}{2}q^2$	q	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\begin{pmatrix}1 & \kappa\\ & 1\end{pmatrix}(0,0,r_3,0,0)$, $r_3\in k$	$\mu_2\langle U_\beta,U_{\alpha+\beta},U_{2\alpha+\beta},U_{3\alpha+2\beta}\rangle$	$\frac{q^2-1}{2}q^2$	q	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}(0,0,r_3,r_4,0)$, $r_3\in k, r_4\in k^\times/\langle\pm1\rangle$	$\langle U_\beta,U_{\alpha+\beta},U_{2\alpha+\beta},U_{3\alpha+2\beta}\rangle$	$(q^2-1)q^2$	$\frac{q(q-1)}{2}$	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\begin{pmatrix}1 & \kappa\\ & 1\end{pmatrix}(0,0,r_3,r_4,0)$, $r_3\in k, r_4\in k^\times/\langle\pm1\rangle$	$\langle U_\beta,U_{\alpha+\beta},U_{2\alpha+\beta},U_{3\alpha+2\beta}\rangle$	$(q^2-1)q^2$	$\frac{q(q-1)}{2}$	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$h(-1,-1)(0,0,r_3,0,0)$, $r_3\in k$	$\mathrm{SL}_2\ltimes U_{2\alpha+\beta}$	q^4	q	$(q+1)\chi(-1)\epsilon_0\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$	$\mu_2\ltimes U_\beta\times U_{2\alpha+\beta}$	$\frac{q^2-1}{2}q^4$	q	$\epsilon_0\chi(-1)\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$	$\mu_2\ltimes U_\beta\times U_{2\alpha+\beta}$	$\frac{q^2-1}{2}q^4$	q	$\epsilon_0\chi(-1)\psi(r_3)$
$h(x,x^{-1})(0,0,r_3,0,0)$, $x\neq\pm1$	$A_{\mathrm{SL}_2}\ltimes U_{2\alpha+\beta}$	$q^5(q+1)$	$\frac{q(q-3)}{2}$	$\epsilon(\chi(x)+\chi(x^{-1}))\psi(r_3)$
$\begin{pmatrix}x & y\\ \kappa y & x\end{pmatrix}(0,0,r_3,r_4,r_5)$, $x\neq\pm1$	$*$	$*$	$*$	0

We first consider the case when $g=1$. If $r_3\neq 0$, we have

$$(-r_4/(3r_3),-r_5/(3r_3),0,0,0)(0,0,r_3,r_4,r_5)(-r_4/(3r_3),-r_5/(3r_3),0,0,0)^{-1}=(0,0,r_3,0,0). \tag{3.2}$$

Thus for any $r_4,r_5\in k$, we have $(0,0,r_3,r_4,r_5)\sim_J(0,0,r_3,0,0)$. If $r_3=0, r_4\neq 0$, then $(0,0,0,r_4,r_5)\sim_J(0,0,0,0,r_5)$. In fact, we have

$$w_\beta\mathbf{x}_\beta(-r_5/r_4)(0,0,0,r_4,r_5)(w_\beta\mathbf{x}_\beta(-r_5/r_4))^{-1}=(0,0,0,0,r_5).$$

Moreover, if $r_5\neq 0$, then $(0,0,0,0,r_5)\sim_J(0,0,0,0,1)$ by considering the action of $h(x,x^{-1})$.

Next, we consider the case when $g=h(x,x^{-1}), x\neq 1$. We have

$$\begin{aligned}&h(x,1/x)(0,0,r_3,0,0)\\ =&(0,0,0,-r_4/(x-1),r_5x/(x-1))h(x,x^{-1})(0,0,r_3,r_4,r_5)(0,0,0,-r_4/(x-1),r_5x/(x-1))^{-1}.\end{aligned}$$

Thus for any r_3,r_4,r_5, we have $h(x,x^{-1})(0,0,r_3,r_4,r_5)\sim_J h(x,x^{-1})(0,0,r_3,0,0)$.

Next, we consider the case when $g=\mathbf{x}_\beta(b), b=1$ or κ. If $r_5\neq 0$, one can check that $g(0,0,r_3,r_4,r_5)\sim_J g(0,0,r_3,r_4,0)$. In fact, we have

$$g(0,0,r_3,r_4,0)=\mathbf{x}_{3\alpha+\beta}(t)g(0,0,r_3,r_4,r_5)\mathbf{x}_{3\alpha+\beta}(-t),$$

with $t=r_5/b$. Finally, we consider the action of $h(a,a^{-1})$ on $g(0,0,r_3,r_4,0)$. To preserve r_3, a should be ±1. On the other hand, we have $h(-1,-1)g(0,0,r_3,r_4,0)h(-1,-1)=g(0,0,r_3,-r_4,0)$.

Finally, if $g=h(x,x^{-1})\mathbf{x}_\beta(b), x=-1$, then one can check that

$$(0,0,0,s,t).g(0,0,r_3,r_4,r_5).(0,0,0,s,t)^{-1}=g.(0,0,r_3,0,0),$$

with $s=r_4/(1-x), t=-r_5x/(1-x)+br_4x^2/(1-x)^2$. This completes the proof of the lemma. $\square$

TABLE 3.2. Character table of $X_i(\pi_i)$. The missing part in rows 5-8 (i.e., those $*$s) depends on q and the details are given in Table 3.3 and Table 3.4.

Representative t	$X_2(\pi_2)$	$X_3(\pi_3)$	$X_6(\pi_6)$	$\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}$
$\begin{pmatrix}1 & \\ & 1\end{pmatrix}$	$\frac{(q^2-1)(q^6-1)}{(q+1)^2}$	$\frac{(q^2-1)(q^6-1)}{q^2+q+1}$	$\frac{(q^2-1)(q^6-1)}{q^2-q+1}$	$q(q+1)$
$\begin{pmatrix}1 & \\ & 1\end{pmatrix}(0,0,0,0,1)$	$-(q-1)(q^2-q+1)$	$(q-1)(q^2-1)$	$-(q+1)(q^2-1)$	$q(q+1)$
$(0,0,r_3,0,0), r_3\neq 0$	$(q-1)(2q-1)$	$-(q^2-1)$	$-(q^2-1)$	$q(q+1)\psi(r_3)$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k$	$*$	$*$	$*$	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k$	$*$	$*$	$*$	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}(0,0,r_3,r_4,0)$, $r_3\in k, r_4\in k^\times/\langle\pm1\rangle$	$*$	$*$	$*$	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\begin{pmatrix}1 & \kappa\\ & 1\end{pmatrix}(0,0,r_3,r_4,0)$, $r_3\in k, r_4\in k^\times/\langle\pm1\rangle$	$*$	$*$	$*$	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$h(-1,-1)$	$(q-1)^2\epsilon(\pi_2)$	0	0	$(q+1)\chi(-1)\epsilon_0$
$h(-1,-1)(0,0,r_3,0,0)$, $r_3\in k^\times$	$-(q-1)\epsilon(\pi_2)$	0	0	$(q+1)\chi(-1)\epsilon_0\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(1)$	$-(q-1)\epsilon(\pi_2)$	0	0	$\epsilon_0\chi(-1)$
$h(-1,-1)\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^\times$	$\epsilon(\pi_2)$	0	0	$\epsilon_0\chi(-1)\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(\kappa)$	$-(q-1)\epsilon(\pi_2)$	0	0	$\epsilon_0\chi(-1)$
$h(-1,-1)\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^\times$	$\epsilon(\pi_2)$	0	0	$\epsilon_0\chi(-1)\psi(r_3)$
$h(x,x^{-1})(0,0,r_3,0,0)$, $x\neq\pm1$	0	0	0	$\epsilon(\chi(x)+\chi(x^{-1}))\psi(r_3)$
$\begin{pmatrix}x & y\\ \kappa y & x\end{pmatrix}(0,0,r_3,r_4,r_5)$, $x\neq\pm1$	$*$	$*$	$*$	0

Table 3.1 gives the conjugacy classes of J. In Table 3.1, for an element $t\in J$, the set $C_J(t)$ is the centralizer of t in J and $J(t)$ is the set of J-congugacy classes of t. The centralizer $C_J(t)$ is essentially computed in [Ch68]. We have $|J(t)| = |J|/|C_J(t)|$. Note that $|J| = q^6(q^2-1)$. The column "No." means the number of classes of a given form. Note that the last column is given by (3.1) using the character tables of $I(\chi)$ and ω_ψ, which can be found in [LZ19, §2].

3.2. **Proof of Theorem 2.1 when** $p>3$. Following [CR74], let $\mathfrak{H}_i, i=2,3,6$ be the 3 anisotropic torus of $\mathrm{G}_2(k)$ such that $\mathfrak{H}_2\cong\mathbb{Z}_{q+1}\times\mathbb{Z}_{q+1}$, $\mathfrak{H}_3\cong\mathbb{Z}_{q^2+q+1}$ and $\mathfrak{H}_6\cong\mathbb{Z}_{q^2-q+1}$. For $i=2,3,6$, in [CR74], Chang-Ree associated a class function $X_i(\pi_i)$ of $\mathrm{G}_2(k)$ for each character π_i of $\mathfrak{H}_i$.

Proposition 3.3. *Let* Π *be a representation of* $\mathrm{G}_2(k)$ *of the form* $X_i(\pi_i)$ *with* $i=2,3,$ *or* 6*, and* χ *be a character of* $k^\times$*. Then we have*

$$\langle \Pi|_J, I(\chi)\otimes\omega_\psi\rangle = 1.$$

Remark 3.4. Here we do not require that Π or $I(\chi)$ is irreducible. Note that

$$\dim\mathrm{Hom}_J(\Pi, I(\chi)\otimes\omega_\psi) = \langle\Pi|_J, I(\chi)\otimes\omega_\psi\rangle.$$

Thus the above proposition shows that $\dim\mathrm{Hom}_J(\Pi, I(\chi)\otimes\omega_\psi)\leq 1$.

Proof of Proposition 3.3 relies on a brute force computation. We first give the character table of $X_i(\pi_i)$ when restricted to J, Table 3.2, which follows from results in [CR74]. Here $\epsilon(\pi_2)$ is a number depending on the character π_2.

The missing part in rows 5-8 of Table 3.2 depends on $q\equiv 1 \bmod 3$ or $q\equiv -1 \bmod 3$, which will be described separately below. Note that if $q\equiv 1 \bmod 3$, then κ is a non-cube in $\mathbb{F}_q$ since it is assumed to be a generator of $\mathbb{F}_q^\times$. If $q \equiv -1$, we fix an element $\zeta \in \mathbb{F}_q$ such that $x^3 - 3x - \zeta$ is irreducible over $\mathbb{F}_q$.

Let u be a unipotent element in rows 5-8 in Table 3.2. The value of the characters $X_i(\pi_i)$ depends on $|C_{\mathrm{G}_2(k)}(u)|$, the size of the centralizer of u in $\mathrm{G}_2(k)$. The detailed information of $|C_{\mathrm{G}_2(k)}(u)|$ is given in [Ch68], which we will give a brief review below.

We first consider the case when $q \equiv 1 \bmod 3$. If $rr_3 \in k^{\times,2}$, then $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(1)$, whose centralizer in $\mathrm{G}_2(k)$ has size $6q^4$, see [Ch68, p.202]. If $rr_3 \in \kappa k^{\times,2}$, then $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(1)$, whose centralizer has size $2q^4$, see [Ch68, p.202]. Then, if $r/r_4 \in k^{\times,3}$, one has that $\mathbf{x}_\beta(r)\mathbf{x}_{3\alpha+\beta}(r_4) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(1) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(1)$ (see [Ch68, p.197]), whose centralizer has size $6q^4$. If $r/r_4 \notin k^{\times,3}$, then $\mathbf{x}_\beta(r)\mathbf{x}_{3\alpha+\beta}(r_4) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(\kappa)$, whose centralizer in $\mathrm{G}_2(k)$ has size $3q^4$, see [Ch68, p.202]. Finally, we consider the conjugacy class $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4), rr_3r_4 \neq 0$. We first have

$$\begin{aligned}&\mathbf{x}_\beta(r)(0,0,r_3,r_4,r_5)\\ =&h(-r_3^2/r_4, r_4/r_3)w_\alpha^{-1}\mathbf{x}_\alpha(-1)\mathbf{x}_\beta(1)(0,0,-1,z,0)(h(-r_3^2/r_4,r_4/r_3)w_\alpha^{-1}\mathbf{x}_\alpha(-1))^{-1}\end{aligned}$$

for some appropriate r_5, where $z = -2 - \frac{rr_4^2}{r_3^3}$. Thus we have

$$\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(r)(0,0,r_3,r_4,r_5) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)(0,0,-1,-2-rr_4^2/r_3^3,0).$$

For $r, r_3, r_4 \in k^\times$, we define $t = t(r, r_3, r_4) \in k_2 = \mathbb{F}_{q^2}$ as a solution of

$$t + t^{-1} = -2 - rr_4^2/r_3^3. \tag{3.3}$$

Note that $t \neq -1$ since $rr_3r_4 \neq 0$. If $t(r, r_3, r_4) = 1$, then according the calculation in [Ch68, p.196-197], one can check that [1]

$$\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4) \sim_{\mathrm{G}_2(k)} \mathbf{x}_{\alpha+\beta}(1) \sim_{\mathrm{G}_2(k)} \mathbf{x}_{2\alpha+\beta}(1),$$

whose centralizer has order $q^4(q^2-1)$. If $t \in k = \mathbb{F}_q$, $t \neq \pm 1$, then by [Ch68, p.197-198], we have

$$\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(t^{-1}),$$

whose centralizer has size $6q^4$ if $t \in k^{\times,3}$, and $3q^4$ if $t \in k^\times - k^{\times,3}$. If $t \in \mathbb{F}_{q^2} - \mathbb{F}_q$, then $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(\kappa)$ (see [Ch68, p.198]), whose centralizer has size $2q^4$. From [CR74], rows 5-8 of Table 3.2 when $q \equiv 1 \bmod 3$ are given in Table 3.3.

Using Table 3.1, Table 3.2 and Table 3.3, we can compute the pair $\langle \Pi|_J, I(\chi) \otimes \omega_\psi \rangle$. Recall that

$$\begin{aligned}|J|\langle \Pi|_J, I(\chi)\otimes\omega_\psi\rangle &= \sum_{g\in J} \overline{\mathrm{Ch}_\Pi}(g)\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}(g)\\ &= \sum_t |J(t)|\overline{\mathrm{Ch}_\Pi}(t)\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}(t),\end{aligned} \tag{3.4}$$

where in (3.4), t runs over a complete set of representatives of conjugacy classes of J and $|J(t)|$ is the number of elements in the conjugacy class $J(t)$.

Lemma 3.5. *Let Π be one of $X_i(\pi_i)$ for $i = 2, 3, 6$. Then we have*

1. *The contribution of conjugacy classes of the form $h(-1,-1)u$, where u is an unipotent element, to (3.4) is zero.*
2. *The contribution of conjugacy classes of the form $\mathbf{x}_\beta(r), r \in k^\times$, to (3.4) is zero.*

---

[1] This relation is not explicitly given in [CR74]. Due to its importance for our calculation, we give some details in this footnote. Let $\varphi_\alpha : \mathrm{SL}_2(k) \to \mathrm{G}_2(k)$ be the embedding such that $\varphi_\alpha\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right) = \mathbf{x}_\alpha(x)$ and $\varphi_\alpha(\mathrm{diag}(a, a^{-1})) = h_\alpha(a, a^{-1})$. For $g, h \in \mathrm{G}_2(k)$, denote the conjugation $g^{-1}hg$ by $h.g$. The conjugation of $\varphi_\alpha(g)$ for $g \in \mathrm{SL}_2(k)$ on $\mathbf{x}_\beta(r_0)(0, r_2, r_3, r_4, 0)$ is given in [Ch68, p.196, (3.5)]. From that description, one can check the following relations $\mathbf{x}_\beta(1)(0,0,-1,2,0).\varphi_\alpha\begin{pmatrix}1 & 1/2\\ -1 & 1/2\end{pmatrix} = (0,0,1,1/2,0)$, $(0,0,1,1/2,0).\varphi_\alpha\begin{pmatrix} & 1\\ -1 & \end{pmatrix} = \mathbf{x}_\beta(1/2)(0,1,0,0,0)$, and $\mathbf{x}_\beta(1/2)(0,1,0,0,0).\varphi_\alpha\begin{pmatrix}1 & 0\\ -1/6 & 1\end{pmatrix} = \mathbf{x}_{\alpha+\beta}(1)$. This shows that $\mathbf{x}_\beta(1)(0,0,-1,2,0) \sim_{\mathrm{G}_2(k)} \mathbf{x}_{\alpha+\beta}(1)$.

TABLE 3.3. Missing part in rows 5-8 of Table 3.2 when $q \equiv 1 \bmod 3$

u	$\lvert C_{\mathrm{G}_2(k)}(u)\rvert$	$X_2(\pi_2)$	$X_3(\pi_3)$	$X_6(\pi_6)$
$\mathbf{x}_\beta(r), r \neq 0$	$*$	$-(q-1)(q^2-q+1)$	$(q-1)(q^2-1)$	$-(q+1)(q^2-1)$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k^{\times,2}$	$6q^4$	$-4q+1$	$q+1$	$-q+1$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in \kappa k^{\times,2}$	$2q^4$	$-2q+1$	$-q+1$	$q+1$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k^{\times,2}$	$2q^4$	$-2q+1$	$-q+1$	$q+1$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in \kappa k^{\times,2}$	$6q^4$	$-4q+1$	$q+1$	$-q+1$
$\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4 \in k^{\times,3}$	$6q^4$	$-4q+1$	$q+1$	$-q+1$
$\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4 \notin k^{\times,3}$	$3q^4$	$-q+1$	$-2q+1$	$2q+1$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4 \in \kappa k^{\times,3}$	$6q^4$	$-4q+1$	$q+1$	$-q+1$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4 \notin \kappa k^{\times,3}$	$3q^4$	$-q+1$	$-2q+1$	$2q+1$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4)=\pm 1$	$q^4(q^2-1)$	$(q-1)(2q-1)$	$-q^2+1$	$-q^2+1$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in k^{\times,3}-\{\pm 1\}$	$6q^4$	$-4q+1$	$q+1$	$-q+1$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in k^\times - k^{\times,3}$	$3q^4$	$-q+1$	$-2q+1$	$2q+1$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \notin k$	$2q^4$	$-2q+1$	$-q+1$	$q+1$

(3) *The contribution of conjugacy classes of the form* $\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(r_4), r_4 \in k^{\times,3}$, *and the contribution of* $\mathbf{x}_\beta(\kappa)\mathbf{x}_{3\alpha+\beta}(r_4), r_4 \in \kappa k^{\times,3}$, *to* (3.4) *are cancelled out. Similarly, the contribution of* $\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(r_4)$, $r_4 \in k^\times - k^{\times,3}$, *and the contribution of* $\mathbf{x}_\beta(\kappa)\mathbf{x}_{3\alpha+\beta}(r_4)$, $r_4 \in k^\times - \kappa k^{\times,3}$, *to* (3.4) *are cancelled out.*

Proof. We only give some details for the proof of (1) when $\Pi = X_2(\pi_2)$, and the proofs of the other cases are similar or just follow from a simple observation. By Table 3.1 and Table 3.2, the contribution of conjugacy classes of the form $h(-1,-1)u$ to (3.4) is

$$\begin{aligned}
&q^4(q+1)\chi(-1)\epsilon_0(q-1)^2\epsilon(\pi_2)\\
&- q^4(q+1)\chi(-1)\epsilon_0\left(\sum_{r_3\in k^\times}\psi(r_3)\right)(q-1)\epsilon(\pi_2)\\
&+ \frac{q^2-1}{2}q^4\epsilon_0\chi(-1)(-(q-1))\epsilon(\pi_2)\cdot 2\\
&+ \frac{q^2-1}{2}q^4\epsilon_0\chi(-1)\epsilon(\pi_2)\left(\sum_{r_3\in k^\times}\psi(r_3)\right).
\end{aligned}$$

A simple calculation shows that the above summation is zero. $\square$

The following lemma is Proposition 3.3 when $q \equiv 1 \bmod 3$.

Lemma 3.6. *Let* Π *be one of* $X_i(\pi_i)$ *for* $i = 2, 3, 6$. *If* $q \equiv 1 \bmod 3$, *then*

$$\langle \Pi|_J, I(\chi) \otimes \omega_\psi \rangle = 1.$$

Proof. We compute (3.4) for $\Pi = X_2(\pi_2), X_3(\pi_3), X_6(\pi_6)$, separately. If $\Pi = X_2(\pi_2)$, by Tables 3.1, 3.2, 3.3 and Lemma 3.5, we have

$$\begin{aligned}
&|J|\langle X_2(\pi_2), I(\chi)\otimes\omega_\psi\rangle\\
&= q(q+1)(q^2-1)(q^6-1)/(q+1)^2 - (q^2-1)q(q+1)(q-1)(q^2-q+1)\\
&+ q^2(q-1)(2q-1)q(q+1)\left(\sum_{r_3\in k^\times}\psi(r_3)\right)\\
&+ \frac{q^2-1}{2}q^2\sqrt{\epsilon_0 q}(-4q+1)(A_1-A_\kappa) - \frac{q^2-1}{2}q^2\sqrt{\epsilon_0 q}(-2q+1)(A_1-A_\kappa)\\
&+ q^2(q^2-1)\sqrt{\epsilon_0 q}((q-1)(2q-1)(B_1^0-B_\kappa^0))\\
&+ q^2(q^2-1)\sqrt{\epsilon_0 q}\left[(-4q+1)(B_1^1-B_\kappa^1)+(-q+1)(B_1^2-B_\kappa^2)+(-2q+1)(B_1^3-B_\kappa^3)\right],
\end{aligned}$$

where

$$A_1 = \sum_{r_3\in k^{\times,2}}\psi(r_3),\quad A_\kappa = \sum_{r_3\in\kappa k^{\times,2}}\psi(r_3), \tag{3.5}$$

and

$$\begin{aligned}
B_r^0 &= \sum_{r_3\in k^\times, r_4\in k^\times/\{\pm1\}, t(r,r_3,r_4)=1}\psi(r_3),\\
B_r^1 &= \sum_{r_3\in k^\times, r_4\in k^\times/\{\pm1\}, t(r,r_3,r_4)\in k^{\times,3}-\{\pm1\}}\psi(r_3),\\
B_r^2 &= \sum_{r_3\in k^\times, r_4\in k^\times/\{\pm1\}, t(r,r_3,r_4)\in k^\times-k^{\times,3}}\psi(r_3),\\
B_r^3 &= \sum_{r_3\in k^\times, r_4\in k^\times/\{\pm1\}, t(r,r_3,r_4)\notin k^\times}\psi(r_3),
\end{aligned}\tag{3.6}$$

for $r = 1,\kappa$. Here recall that $t = t(r, r_3, r_4)$ is a solution of the equation (3.3). The computations of $A_1 - A_\kappa, B_1^i - B_\kappa^i$ for $i = 0, 1, 2, 3$ are given in the appendix, see Lemmas A.1 and A.2, and the results read as

$$\begin{aligned}
A_1 - A_\kappa &= \sqrt{\epsilon_0 q},\\
B_1^0 - B_\kappa^0 &= \epsilon_0\sqrt{\epsilon_0 q},\\
B_1^1 - B_\kappa^1 &= -\frac{1}{2}(1+\epsilon_0)\sqrt{\epsilon_0 q},\\
B_1^2 - B_\kappa^2 &= 0,\\
B_1^3 - B_\kappa^3 &= \frac{1}{2}(1-\epsilon_0)\sqrt{\epsilon_0 q}.
\end{aligned}\tag{3.7}$$

Plugging these formulas into the computation of $\langle X_2(\pi_2), I(\chi)\otimes\omega_\psi\rangle$, it follows that

$$|J|\langle X_2(\pi_2), I(\chi)\otimes\omega_\psi\rangle = q^6(q^2-1).$$

Thus we have

$$\langle X_2(\pi_2), I(\chi)\otimes\omega_\psi\rangle = 1.$$

Similarly, we have

$$\begin{aligned}
&|J|\langle X_3(\pi_3), I(\chi)\otimes\omega_\psi\rangle\\
&= q(q+1)(q^2-1)(q^6-1)/(q^2+q+1) + (q^2-1)q(q+1)(q-1)(q^2-1)\\
&- q^2q(q+1)(q^2-1)\left(\sum_{r_3\in k^\times}\psi(r_3)\right)\\
&+ \frac{q^2-1}{2}q^2\sqrt{\epsilon_0 q}((q+1)(A_1-A_\kappa)-(-q+1)(A_1-A_\kappa))\\
&+ q^2(q^2-1)\sqrt{\epsilon_0 q}((-q^2+1)(B_1^0-B_\kappa^0)\\
&+ q^2(q^2-1)\sqrt{\epsilon_0 q}((q+1)(B_1^1-B_\kappa^1)+(-2q+1)(B_1^2-B_\kappa^2)+(-q+1)(B_1^3-B_\kappa^3)),
\end{aligned}$$

where A_r, B_r^i for $r=1,\kappa, i=0,1,2,3$ are defined in (3.5) and (3.6). Plugging the formulas (3.7) into the computation of $\langle X_3(\pi_3), I(\chi)\otimes\omega_\psi\rangle$, we obtain that

$$\langle X_3(\pi_3), I(\chi)\otimes\omega_\psi\rangle = 1.$$

A similar calculation shows that

$$\begin{aligned}
&|J|\langle X_6(\pi_6), I(\chi)\otimes\omega_\psi\rangle\\
&= q(q+1)(q^2-1)(q^6-1)/(q^2-q+1) - (q^2-1)(q+1)(q^2-1)q(q+1)\\
&- q^2(q^2-1)q(q+1)\left(\sum_{r_3\in k^\times}\psi(r_3)\right)\\
&+ \frac{q^2-1}{2}q^2\sqrt{\epsilon_0 q}((-q+1)(A_1-A_\kappa)-(q+1)(A_1-A_\kappa))\\
&+ q^2(q^2-1)\sqrt{\epsilon_0 q}(-q^2+1)(B_1^0-B_\kappa^0)\\
&+ q^2(q^2-1)\sqrt{\epsilon_0 q}((-q+1)(B_1^1-B_\kappa^1)+(-2q+1)(B_1^2-B_\kappa^2)+(q+1)(B_1^3-B_\kappa^3)).
\end{aligned}$$

By formulas (3.7), we also obtain that $\langle X_6(\pi_6), I(\chi)\otimes\omega_\psi\rangle = 1$. □

We next consider the case when $q\equiv -1 \bmod 3$. In this case, we have $k^\times = k^{\times,3}$. If $rr_3\in k^{\times,2}$, then $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(1)$, whose centralizer in $\mathrm{G}_2(k)$ has size $2q^4$, see [Ch68, p.202]. If $rr_3\in\kappa k^{\times,2}$, then $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(1)$, whose centralizer has size $6q^4$, see [Ch68, p.202]. For any r_4, we have $\mathbf{x}_\beta(r)\mathbf{x}_{3\alpha+\beta}(r_4)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(1)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(1)$ (see [Ch68, p.197]), whose centralizer has size $2q^4$. Finally, we consider the conjugacy class $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4), rr_3r_4\neq 0$. As in the previous case, we still have

$$\mathbf{x}_\beta(r)(0,0,r_3,r_4,r_5)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)(0,0,-1,z,0),$$

with some appropriate r_5, where $z=-2-\frac{rr_4^2}{r_3^3}$. Thus we have

$$\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(r)(0,0,r_3,r_4,r_5)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)(0,0,-1,-2-rr_4^2/r_3^3,0).$$

We also write $t=t(r,r_3,r_4)\in k_2=\mathbb{F}_{q^2}$ as a solution of (3.3). For $t\in k_2^\times - k^\times$ with $t+t^{-1}\in k$, one can check that $t^{1+q}=1$, i.e., $t\in k_2^1$, the norm 1 subgroup of $k_2^\times$. Thus $t=t(r,r_3,r_4)$ is either in $k^\times$ or in k_2^1. If $t=1$, then,

$$\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)\sim_{\mathrm{G}_2(k)}\mathbf{x}_{2\alpha+\beta}(1),$$

as in the previous case. If $t\in k^\times-\{\pm1\}$, then by [Ch68, p.197-198], we have

$$\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(t^{-1}),$$

whose centralizer has size $2q^4$. If $t\in(k_2^1-\{\pm1\})\cap k_2^{\times,3}$, then $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(\kappa)$ (see [Ch68, p.198]), whose centralizer has size $6q^4$. If $t\in k_2^1-k_2^{\times,3}$, the centralizer of $\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)$ has size $3q^4$.

From [CR74], for $q\equiv -1 \bmod 3$, the missing part in rows 5-8 of Table 3.2 is given in Table 3.4.

The following lemma is Proposition 3.3 when $q\equiv -1 \bmod 3$.

TABLE 3.4. Missing part in rows 5-8 of Table 3.2 when $q \equiv -1 \bmod 3$

u	$\lvert C_{G_2(k)}(u)\rvert$	$X_2(\pi)$	$X_3(\pi_3)$	$X_6(\pi_6)$
$\mathbf{x}_\beta(r), r \neq 0$	$*$	$-(q-1)(q^2-q+1)$	$(q-1)(q^2-1)$	$-(q+1)(q^2-1)$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k^{\times,2}$	$2q^4$	$-2q+1$	$-q+1$	$q+1$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in \kappa k^{\times,2}$	$6q^4$	$-4q+1$	$q+1$	$-q+1$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k^{\times,2}$	$6q^4$	$-4q+1$	$q+1$	$-q+1$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in \kappa k^{\times,2}$	$2q^4$	$-2q+1$	$-q+1$	$q+1$
$\mathbf{x}_\beta(u)\mathbf{x}_{3\alpha+\beta}(r_4)$ $u=1,\kappa, r_4 \in k^\times$	$2q^4$	$-2q+1$	$-q+1$	$q+1$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in \{\pm 1\}$	$q^4(q^2-1)$	$(q-1)(2q-1)$	$-q^2+1$	$-q^2+1$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in k^\times - \{\pm 1\}$	$2q^4$	$-2q+1$	$-q+1$	$q+1$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in (k_2^1 - \{\pm 1\}) \cap k_2^{\times,3}$	$6q^4$	$-4q+1$	$q+1$	$-q+1$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in k_2^1 - k_2^{\times,3}$	$3q^4$	$-q+1$	$-2q+1$	$2q+1$

Lemma 3.7. *Let Π be one of $X_i(\pi_i)$ for $i = 2, 3, 6$. If $q \equiv -1 \bmod 3$, then we have*

$$\langle \Pi, I(\chi) \otimes \omega_\psi \rangle = 1.$$

Proof. From Tables 3.1, 3.2, 3.4 and Lemma 3.5, we have

$$\begin{aligned}
&|J|\langle X_2(\pi_2), I(\chi) \otimes \omega_\psi \rangle \\
&= q(q+1)(q^2-1)(q^6-1)/(q+1)^2 - (q^2-1)q(q+1)(q-1)(q^2-q+1) \\
&+ q^2 q(q+1)(q-1)(2q-1)\left(\sum_{r_3 \in k^\times} \psi(r_3)\right) \\
&+ \frac{q^2-1}{2} q^2 \sqrt{\epsilon_0 q}((-2q+1)(A_1 - A_\kappa) - (-4q+1)(A_1 - A_\kappa)) \\
&+ q^2(q^2-1)\sqrt{\epsilon_0 q}(q-1)(2q-1)(C_1^0 - C_\kappa^1) \\
&+ q^2(q^2-1)\sqrt{\epsilon_0 q}((-2q+1)(C_1^1 - C_\kappa^1) + (-4q+1)(C_1^2 - C_\kappa^2) + (-q+1)(C_1^3 - C_\kappa^3)),
\end{aligned}$$

where for $r = 1, \kappa$, A_r are defined as before, and

$$\begin{aligned}
C_r^0 &= \sum_{r_3 \in k^\times, r_4 \in k^\times/\{\pm 1\}, t(r,r_3,r_4)=1} \psi(r_3), \\
C_r^1 &= \sum_{r_3 \in k^\times, r_4 \in k^\times/\{\pm 1\}, t(r,r_3,r_4) \in k^\times - \{\pm 1\}} \psi(r_3), \\
C_r^2 &= \sum_{r_3 \in k^\times, r_4 \in k^\times/\{\pm 1\}, t(r,r_3,r_4) \in (k_2^1 - \{\pm 1\}) \cap k_2^{\times,3}} \psi(r_3), \\
C_r^3 &= \sum_{r_3 \in k^\times, r_4 \in k^\times/\{\pm 1\}, t(r,r_3,r_4) \in k_2^1 - k_2^{\times,3}} \psi(r_3).
\end{aligned} \tag{3.8}$$

Recall that $t = t(r, r_3, r_4)$ is a solution of (3.3), which actually cannot be -1. The quantities $C_1^i - C_\kappa^i$ are computed in Lemma A.4. Applying those formulas in Lemma A.4, a straightforward calculation shows that

$$\langle X_2(\pi_2), I(\chi) \otimes \omega_\psi \rangle = 1.$$

It is similar to show that

$$\langle X_i(\pi_i), I(\chi) \otimes \omega_\psi \rangle = 1,$$

for $i = 3, 6$ as well. We omit the details here. $\square$

Proof of Theorem* 2.1 *when $p > 3$. The irreducible representations of $\mathrm{G}_2(k)$ have been classified in [CR74]. Let $\mathfrak{H}_1$ be the maximal split torus,

$$\mathfrak{H}_a = \left\{ h(z^q, z^{1-q}) : z^{q^2-1} = 1 \right\},$$

and

$$\mathfrak{H}_b = \left\{ h(z, z^q) : z^{q^2-1} = 1 \right\}.$$

For $i = 1, 2, a, b, 3, 6$, and a character π_i of $\mathfrak{H}_i$, there is an associated character $X_i(\pi_i)$ of $\mathrm{G}_2(k)$, and when π_i is in general position, see [CR74, p.398] for the precise definition, $X_i(\pi_i)$ is irreducible. There are several other isolated classes of irreducible representations of $\mathrm{G}_2(k)$ constructed using linear combinations of $X_i(\pi_i)$ (when π_i is not in general position) and 4 other class functions $Y_i, i = 1, 2, 3, 4$. If a representation Π is a component of $X_i(\pi_i)$ for $i = 1, a, b$, then Π is not cuspidal. From the list given in [CR74], it is not hard to see that if Π is an irreducible cuspidal representation of $\mathrm{G}_2(k)$, then it has to be one of following form:

$$X_i(\pi_i), (i = 2, 3, 6), X_{33} = -\frac{1}{3}X_2(\pi_2) + \frac{1}{3}X_6(\pi_6), X_{17}, X_{18}, X_{19}, \overline{X}_{19},$$

where π_i for $i = 2, 3, 6$ are in general positions, X_{33} appears when $q \equiv -1 \bmod 3$, and $X_{17}, X_{18}, X_{19}, \overline{X}_{19}$ are defined in [CR74, p.402].

We have shown that

$$\langle X_i(\pi_i), I(\chi) \otimes \omega_\psi \rangle = 1,$$

for $i = 2, 3, 6$, no matter π_i is in general position or not. Thus, we get

$$\langle X_{33}, I(\chi) \otimes \omega_\psi \rangle = 0.$$

Finally, to deal with the last 4 isolated cases, we need to compute $\langle Y_i, I(\chi) \otimes \omega_\psi \rangle$. According to the table given in [CR74, p.411], we have

Representative t	Y_1	Y_2	Y_3	Y_4	$\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}$
$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	0	0	0	0	$q(q+1)$
$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}(0,0,0,0,1)$	0	0	0	0	$q(q+1)$
$(0,0,r_3,0,0), r_3 \neq 0$	0	0	0	0	$q(q+1)\psi(r_3)$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k$	$*$	0	0	0	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k$	$*$	0	0	0	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}(0,0,r_3,r_4,0)$, $r_3 \in k, r_4 \in k^\times/\langle \pm 1 \rangle$	$*$	0	0	0	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\begin{pmatrix} 1 & \kappa \\ & 1 \end{pmatrix}(0,0,r_3,r_4,0)$, $r_3 \in k, r_4 \in k^\times/\langle \pm 1 \rangle$	$*$	0	0	0	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$h(-1,-1)$	0	0	0	0	$(q+1)\chi(-1)\epsilon_0$
$h(-1,-1)(0,0,r_3,0,0)$, $r_3 \in k^\times$	0	0	0	0	$(q+1)\chi(-1)\epsilon_0\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(r), r \in k^\times$	0	0	0	0	$\epsilon_0\chi(-1)$
$h(-1,-1)\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)$, $rr_3 \in k^{\times,2}$	0	q	0	0	$\epsilon_0\chi(-1)\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(r)\mathbf{x}_{2\alpha+\beta}(r_3)$, $rr_3 \in \kappa k^{\times,2}$	0	$-q$	0	0	$\epsilon_0\chi(-1)\psi(r_3)$
$h(x,x^{-1})(0,0,r_3,0,0)$, $x \neq \pm 1$	0	0	0	0	$\epsilon(\chi(x)+\chi(x^{-1}))\psi(r_3)$
$\begin{pmatrix} x & y \\ \kappa y & x \end{pmatrix}(0,0,r_3,r_4,r_5)$, $x \neq \pm 1$	$*$	$*$	$*$		0

The missing part for Y_1 (from 5th row to 8th row) depends on the residue of q mod 3. If $q \equiv 1 \bmod 3$, then one has

u	$\lvert C_G(u)\rvert$	Y_1
$\mathbf{x}_\beta(r), r \neq 0$	$*$	0
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k^{\times,2}$	$6q^4$	q^2
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in \kappa k^{\times,2}$	$2q^4$	$-q^2$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k^{\times,2}$	$2q^4$	$-q^2$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in \kappa k^{\times,2}$	$6q^4$	q^2
$\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4 \in k^{\times,3}$	$6q^4$	q^2
$\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4 \notin k^{\times,3}$	$3q^4$	q^2
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4 \in \kappa k^{\times,3}$	$6q^4$	q^2
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4 \notin \kappa k^{\times,3}$	$3q^4$	q^2
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in \{\pm 1\}$	$q^4(q^2-1)$	0
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in k^{\times,3} - \{\pm 1\}$	$6q^4$	q^2
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in k^\times - k^{\times,3}$	$3q^4$	q^2
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \notin k$	$2q^4$	$-q^2$

and when $q \equiv -1 \bmod 3$, one has

u	$\lvert C_G(u)\rvert$	Y_1
$\mathbf{x}_\beta(r), r \neq 0$	$*$	0
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k^{\times,2}$	$2q^4$	q^2
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in \kappa k^{\times,2}$	$6q^4$	$-q^2$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in k^{\times,2}$	$6q^4$	$-q^2$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3 \in \kappa k^{\times,2}$	$2q^4$	q^2
$\mathbf{x}_\beta(u)\mathbf{x}_{3\alpha+\beta}(r_4)$ $u=1,\kappa, r_4 \in k^\times$	$2q^4$	q^2
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in \{\pm 1\}$	$q^4(q^2-1)$	0
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in k^\times - \{\pm 1\}$	$2q^4$	q^2
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in (k_2^1 - \{\pm 1\}) \cap k_2^{\times,3}$	$6q^4$	$-q^2$
$\mathbf{x}_\beta(r)(0,0,r_3,r_4,0)$, $t(r,r_3,r_4) \in k_2^1 - k_2^{\times,3}$	$3q^4$	$-q^2$

It is easy to see that

$$\langle Y_i, I(\chi) \otimes \omega_\psi \rangle = 0,$$

for $i = 2, 3, 4$. We next compute $\langle Y_1, I(\chi) \otimes \omega_\psi \rangle$ when $q \equiv 1 \bmod 3$. We have

$$\begin{aligned}
&|J| \langle Y_1, I(\chi) \otimes \omega_\psi \rangle \\
= &\frac{q^2-1}{2} q^2 \sqrt{\epsilon_0 q}(q^2(A_1 - A_\kappa) - (-q^2)(A_1 - A_\kappa)) \\
+ &q^2(q^2-1)\sqrt{\epsilon_0 q}(q^2(B_1^1 - B_\kappa^1) + q^2(B_1^2 - B_\kappa^1) + (-q^2)(B_1^3 - B_\kappa^3)) \\
= &q^4(q^2-1)\sqrt{\epsilon_0 q}((A_1 - A_\kappa) + (B_1^1 - B_\kappa^1) - (B_1^3 - B_\kappa^3)).
\end{aligned}$$

From the computation of $A_1 - A_\kappa$ and $B_1^i - B_\kappa^i$ for $i = 0, 1, 2, 3$, in Lemma A.1 and Lemma A.2, one can see that

$$\langle Y_1, I(\chi) \otimes \omega_\psi \rangle = 0.$$

Similarly, when $q \equiv -1 \bmod 3$, we also have

$$\langle Y_1, I(\chi) \otimes \omega_\psi \rangle = 0.$$

From the definitions of $X_{17}, X_{18}, X_{19}, \overline{X}_{19}$, given in [CR74, p.402], one can check that

$$\langle \Pi, I(\chi) \otimes \omega_\psi \rangle = 0,$$

TABLE 4.1. Conjugacy class of J when $p = 3$. The $*$s in the last row mean that the corresponding entries are omitted.

Representative t	$C_J(t)$	$\lvert J(t)\rvert$	$No.$	$\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}$
1	J	1	1	$q(q+1)$
$\mathbf{x}_{3\alpha+2\beta}(1)$	$N_{\mathrm{SL}_2} \ltimes V$	q^2-1	1	$q(q+1)$
$\mathbf{x}_{2\alpha+\beta}(r_3), r_3 \neq 0$	$\mathrm{SL}_2(k) \ltimes V$	1	$q-1$	$q(q+1)\psi(r_3)$
$\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+2\beta}(1), r_3 \neq 0$	$N_{\mathrm{SL}_2} \ltimes V$	q^2-1	$q-1$	$q(q+1)\psi(r_3)$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3), r_3 \in k$	$\mu_2\langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+2\beta}\rangle$	$\frac{q^2-1}{2}q^2$	q	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3), r_3 \in k$	$\mu_2\langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+2\beta}\rangle$	$\frac{q^2-1}{2}q^2$	q	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_3\in k, r_4\in k^\times/\langle\pm1\rangle$	$\langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+2\beta}\rangle$	$(q^2-1)q^2$	$\frac{q(q-1)}{2}$	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)$, $r_3\in k, r_4\in k^\times/\langle\pm1\rangle$	$\langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+2\beta}\rangle$	$(q^2-1)q^2$	$\frac{q(q-1)}{2}$	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$h(-1,-1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k$	$\mathrm{SL}_2 \ltimes U_{2\alpha+\beta}$	q^4	q	$(q+1)\chi(-1)\epsilon_0\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$	$\mu_2 \ltimes U_\beta \times U_{2\alpha+\beta}$	$\frac{q^2-1}{2}q^4$	q	$\epsilon_0\chi(-1)\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$	$\mu_2 \ltimes U_\beta \times U_{2\alpha+\beta}$	$\frac{q^2-1}{2}q^4$	q	$\epsilon_0\chi(-1)\psi(r_3)$
$h(x,x^{-1})\mathbf{x}_{2\alpha+\beta}(r_3)$, $x\neq\pm1$	$A_{\mathrm{SL}_2} \ltimes U_{2\alpha+\beta}$	$q^5(q+1)$	$\frac{q(q-3)}{2}$	$\epsilon(\chi(x)+\chi(x^{-1}))\psi(r_3)$
$\begin{pmatrix} x & y \\ \kappa y & x \end{pmatrix}(0,0,r_3,r_4,r_5)$, $x\neq\pm1$	$*$	$*$	$*$	0

if $\Pi = X_{17}, X_{18}, X_{19}$, or $\overline{X}_{19}$. For example, we have

$$X_{17} = -\frac{1}{6}X_2(1) + \frac{1}{6}X_6(1) - \frac{1}{2}Y_1 + \frac{1}{2}Y_2.$$

Since $\langle X_i(1), I(\chi)\otimes\omega_\psi\rangle = 1$ for $i = 2, 3, 6$, and $\langle Y_i, I(\chi)\otimes\omega_\psi\rangle = 0$, we get

$$\langle X_{17}, I(\chi)\otimes\omega_\psi\rangle = -\frac{1}{6} + \frac{1}{6} = 0.$$

The other 3 cases can be checked similarly. This completes the proof Theorem 2.1. $\square$

4. PROOF OF THEOREM 2.1 WHEN $p = 3$

In this section let $k = \mathbb{F}_{3^f}$ for some integer f. The character table of $\mathrm{G}_2(k)$ is given in [En76], which will be used to prove Theorem 2.1.

Lemma 4.1. *The following is a complete set of representatives of $j \in J$ (up to J-conjugacy) of the form $j = gz$ with $z \in Z$ and $g \in \mathrm{SL}_2(k)$ such that g is not conjugate to an element of the form* $\begin{pmatrix} x & \kappa y \\ y & x \end{pmatrix}$, $y \neq 0$:

(1) 1; $\mathbf{x}_{3\alpha+2\beta}(1)$; $\mathbf{x}_{2\alpha+\beta}(r_3), r_3 \in k^\times$; $\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+2\beta}(1), r_3 \neq 0$;
(2) $\mathbf{x}_\beta(b)\mathbf{x}_{2\alpha+\beta}(r_3), \mathbf{x}_\beta(b)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4), b \in \{1,\kappa\}, r_3 \in k, r_4 \in k^\times/\{\pm1\}$;
(3) $h(-1,-1)\mathbf{x}_\beta(b)\mathbf{x}_{2\alpha+\beta}(r_3), r_3 \in k, b \in \{1,\kappa\}$;
(4) $h(x,x^{-1})\mathbf{x}_{2\alpha+\beta}(r_3), x \in k^\times - \{\pm1\}, r_3 \in k$.

Proof. The proof of this lemma is similar to the proof of Lemma 3.2. One difference is that here we have $3 = 0$ in k and thus (3.2) is not valid. Hence, $\mathbf{x}_{2\alpha+\beta}(r_3)$ and $\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+2\beta}(1)$ are no longer in the same J-conjugacy class. On the other hand, if $r_3 \neq 0, r_4 \neq 0$, we have

$$w_\beta\mathbf{x}_\beta(-r_5/t_4)(0,0,r_3,r_4,r_5)(w_\beta\mathbf{x}_\beta(-r_5/r_4))^{-1} = (0,0,r_3,0,r_4).$$

Thus any element of the form $(0,0,r_3,r_4,r_5)$ is J-conjugate to an element of the form $(0,0,r_3,0,r_4)$. The other parts of the proof is exactly the same as that of Lemma 3.2. We omit the details here. $\square$

TABLE 4.2. Character table of $X_i(\pi_i)$ when $p = 3$. The missing part (i.e., those $*$s) in 10th and 11th row are given in Table 4.3.

Representative t	$\chi_{12}(k,l)$	$\chi_{13}(k)$	$\chi_{14}(k)$	$\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}$
1	$\frac{(q^2-1)(q^6-1)}{(q+1)^2}$	$\frac{(q^2-1)(q^6-1)}{q^2+q+1}$	$\frac{(q^2-1)(q^6-1)}{q^2-q+1}$	$q(q+1)$
$\mathbf{x}_{3\alpha+2\beta}(1)$	$-(q-1)(q^2-q+1)$	$(q-1)(q^2-1)$	$-(q+1)(q^2-1)$	$q(q+1)$
$\mathbf{x}_{2\alpha+\beta}(r_3), r_3 \neq 0$	$-(q-1)(q^2-q+1)$	$(q-1)(q^2-1)$	$-(q+1)(q^2-1)$	$q(q+1)\psi(r_3)$
$\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+2\beta}(1)$, $r_3\neq0$	$2q^2-2q+1$	$-(q^2+q-1)$	$-(q^2-q-1)$	$q(q+1)\psi(r_3)$
$\mathbf{x}_\beta(1)$	$-(q-1)(q^2-q+1)$	$(q-1)(q^2-1)$	$-(q+1)(q^2-1)$	$\sqrt{\epsilon_0 q}$
$\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^\times$	$-(2q-1)$	$-(q-1)$	$q+1$	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\mathbf{x}_\beta(\kappa)$	$-(q-1)(q^2-q+1)$	$(q-1)(q^2-1)$	$-(q+1)(q^2-1)$	$-\sqrt{\epsilon_0 q}$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^\times$	$-(2q-1)$	$-(q-1)$	$q+1$	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$\mathbf{x}_\beta(1)(0,0,r_3,r_4,0)$, $r_3\in k, r_4\in k^\times/\langle\pm1\rangle$	$*$	$*$	$*$	$\sqrt{\epsilon_0 q}\psi(r_3)$
$\mathbf{x}_\beta(\kappa)(0,0,r_3,r_4,0)$, $r_3\in k, r_4\in k^\times/\langle\pm1\rangle$	$*$	$*$	$*$	$-\sqrt{\epsilon_0 q}\psi(r_3)$
$h(-1,-1)$	$(q-1)^2\epsilon(k,l)$	0	0	$(q+1)\chi(-1)\epsilon_0$
$h(-1,-1)(0,0,r_3,0,0)$, $r_3\in k^\times$	$-(q-1)\epsilon(k,l)$	0	0	$(q+1)\chi(-1)\epsilon_0\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(1)$	$-(q-1)\epsilon(k,l)$	0	0	$\epsilon_0\chi(-1)$
$h(-1,-1)\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^\times$	$\epsilon(k,l)$	0	0	$\epsilon_0\chi(-1)\psi(r_3)$
$h(-1,-1)\mathbf{x}_\beta(\kappa)$	$-(q-1)\epsilon(k,l)$	0	0	$\epsilon_0\chi(-1)$
$h(-1,-1)\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^\times$	$\epsilon(k,l)$	0	0	$\epsilon_0\chi(-1)\psi(r_3)$
$h(x,x^{-1})(0,0,r_3,0,0)$, $x\neq\pm1$	0	0	0	$\epsilon(\chi(x)+\chi(x^{-1}))\psi(r_3)$
$\begin{pmatrix} x & y \\ \kappa y & x\end{pmatrix}(0,0,r_3,r_4,r_5)$, $x\neq\pm1$	$*$	$*$	$*$	0

The conjugacy classes in J and the character of $I(\chi)\otimes\omega_\psi$ are given in Table 4.1. Note that in Table 4.1, the element $\mathbf{x}_{2\alpha+\beta}(r_3)$ is in fact in the center of J, see the commutator relation in [En76, p.192].

As in §3, we still let $\mathfrak{H}_i, i = 2, 3, 6$, be the 3 anisotropic torus of $\mathrm{G}_2(k)$ such that $\mathfrak{H}_2 \cong \mathbb{Z}_{q+1}$, $\mathfrak{H}_3 \cong \mathbb{Z}_{q^2+q+1}$ and $\mathfrak{H}_6 \cong \mathbb{Z}_{q^2-q+1}$. Then given a character π_i of $\mathfrak{H}_i$, there is a class function $X_i(\pi_i)$ on $\mathrm{G}_2(k)$ as in the case of $p > 3$. The notation in [En76] is different from that of [CR74]. If $i = 2$, the character π_2 of $\mathfrak{H}_2$ is determined by two integers $(k, l) \in {}^{12}S_6$ and the associated class function is denoted by $\chi_{12}(k, l)$ in [En76, p.246]. Here ${}^{12}S_6$ is a set of pairs of integers modulo certain relations introduced in [En76, p.194] with size $|{}^{12}S_6| = \frac{1}{12}(q-1)(q-3)$; here we don't recall its precise meaning since we don't use it. If $i = 3, 6$, the character π_i of $\mathfrak{H}_i$ is determined by a single integer k, and the corresponding class functions are denoted by $\chi_{13}(k)$ and $\chi_{14}(k)$ respectively in [En76, p.247]. As in the χ_{12} case, the integer k appeared in χ_{13} and χ_{14} depends only on k modulo certain relations and ranges over finite sets of sizes $\frac{1}{6}q(q+1)$ and $\frac{1}{6}q(q-1)$ respectively, see [En76, p.194 and p.205] for the details. The restrictions of the characters $\chi_{12}(k, l)$, $\chi_{13}(k)$ and $\chi_{14}(k)$ to J can be read out directly from the table in [En76, p.246-247] and are given in Table 4.2. In Table 4.2, $\epsilon(k, l) = (-1)^k + (-1)^l + (-1)^{k+l}$.

The missing part of the Table 4.2 (10th row and 11th row) is determined as follows. If $r_3 = 0, r_4 \neq 0$, then

$$\mathbf{x}_\beta(r)(0, 0, r_3, r_4, 0) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(1),$$

since every element in k has a cubic root, see the calculation in [Ch68, p.197] or the discussion in the previous section. As in the previous section, if $r_3 \neq 0$, we have

$$\mathbf{x}_\beta(r)(0, 0, r_3, r_4, 0) \sim_{\mathrm{G}_2(k)} \mathbf{x}_\beta(1)(0, 0, -1, z, 0),$$

TABLE 4.3. Missing part of rows 10-11 in Table 4.2.

representative	$\chi_{12}(k,l)$	$\chi_{13}(k)$	$\chi_{14}(k)$
$\mathbf{x}_\beta(1)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4\neq 0$	$-(2q-1)$	$-(q-1)$	$q+1$
$\mathbf{x}_\beta(1)(0,0,r_3,r_4,0)$ $t(1,r_3,r_4)\in\{\pm 1\}$	$2q^2-2q+1$	$-(q^2+q-1)$	$-(q^2-q-1)$
$\mathbf{x}_\beta(1)(0,0,r_3,r_4,0)$ $t(1,r_3,r_4)\notin\{\pm 1\}$	$-(2q-1)$	$-(q-1)$	$q+1$
$\mathbf{x}_\beta(\kappa)\mathbf{x}_{3\alpha+\beta}(r_4)$, $r_4\neq 0$	$-(2q-1)$	$-(q-1)$	$q+1$
$\mathbf{x}_\beta(\kappa)(0,0,r_3,r_4,0)$ $t(\kappa,r_3,r_4)\in\{\pm 1\}$	$2q^2-2q+1$	$-(q^2+q-1)$	$-(q^2-q-1)$
$\mathbf{x}_\beta(\kappa)(0,0,r_3,r_4,0)$ $t(\kappa,r_3,r_4)\notin\{\pm 1\}$	$-(2q-1)$	$-(q-1)$	$q+1$

with $z=-2-rr_4^2/r_3^3$. As in the previous section, we set $t=t(r,r_3,r_4)$ as a solution of (3.3). If $t=1$, then we have

$$\mathbf{x}_\beta(1)(0,0,r_3,r_4,0)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1/2)\mathbf{x}_{\alpha+\beta}(1)\sim_{\mathrm{G}_2(k)}\mathbf{x}_{2\alpha+\beta}(1)\mathbf{x}_{3\alpha+2\beta}(1),$$

see Footnote 1 for the first relation. Note that $1/6$ is undefined in the last equation of Footnote 1 and thus we cannot obtain that $\mathbf{x}_\beta(1/2)\mathbf{x}_{\alpha+\beta}(1)\sim_{\mathrm{G}_2(k)}\mathbf{x}_{\alpha+\beta}(1)$ here. On the other hand, we have $w_\beta w_\alpha\mathbf{x}_\beta(-1)\mathbf{x}_{\alpha+\beta}(1)(w_\beta w_\alpha)^{-1}=\mathbf{x}_{2\alpha+\beta}(1)\mathbf{x}_{3\alpha+2\beta}(-1)$. By considering a conjugation of the torus, we then get $\mathbf{x}_\beta(1/2)\mathbf{x}_{\alpha+\beta}(1)\sim_{\mathrm{G}_2(k)}\mathbf{x}_{2\alpha+\beta}(1)\mathbf{x}_{3\alpha+2\beta}(1)$.

If $t\neq\pm 1$, using the description in [Ch68] and the fact that any element in $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ has a cubic root, one can check that $\mathbf{x}_\beta(1)(0,0,r_3,r_4,0)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(1)$ if $t\in k^\times-\{\pm 1\}$, and

$$\mathbf{x}_\beta(1)(0,0,r_3,r_4,0)\sim_{\mathrm{G}_2(k)}\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(\kappa)$$

if $t\in\mathbb{F}_{q^2}-\mathbb{F}_q$. Thus we obtain Table 4.3 following the table in [En76, p.246-247].

Lemma 4.2. *Let Π be $\chi_{12}(k,l),\chi_{13}(k)$ or $\chi_{14}(k)$. Then we have*

$$\langle\Pi,I(\chi)\otimes\omega_\psi\rangle=1.$$

We have

$$|J|\langle\Pi,I(\chi)\otimes\omega_\psi\rangle=\sum_t|J(t)|\overline{\mathrm{Ch}_\Pi}(t)\mathrm{Ch}_{I(\chi)\otimes\omega_\psi}(t),\tag{4.1}$$

where t runs over a complete set of representatives of conjugacy classes of J and $|J(t)|$ is the number of elements in the conjugacy class $J(t)$. Before proving Lemma 4.2, we first record the following result.

Lemma 4.3. *Let Π be $\chi_{12}(k,l),\chi_{13}(k)$ or $\chi_{14}(k)$.*

(1) *The contribution of conjugacy classes of the form $h(-1,-1)u$, with u in the unipotent, to (4.1) is zero.*
(2) *The contribution of conjugacy classes of the form $\mathbf{x}_\beta(1)\mathbf{x}_{2\alpha+\beta}(r_3),r_3\in k$, and the contribution of conjugacy classes of the form $\mathbf{x}_\beta(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3),r_3\in k$, to (4.1) are cancelled out.*

The proof of Lemma 4.3 is the same as that of Lemma 3.5 and we omit the details.

Proof of Lemma 4.2. This lemma can be checked case by case and we only give the details when $\Pi=\chi_{12}(k,l)$ and omit the details of the other two cases. We suppose that $\Pi=\chi_{12}(k,l)$. By Tables

4.1, 4.2 and 4.3, we have

$$\begin{aligned}
&|J|\langle \Pi, I(\chi)\otimes\omega_\psi\rangle \\
=\ &\frac{(q^2-1)(q^6-1)}{(q+1)^2}q(q+1)-(q^2-1)(q-1)(q^2-q+1)q(q+1) \\
&-(q-1)(q^2-q+1)q(q+1)\left(\sum_{r_3\in k^\times}\psi(r_3)\right) \\
&+(q^2-1)(2q^2-2q+1)q(q+1)\left(\sum_{r_3\in k^\times}\psi(r_3)\right) \\
&+(q^2-1)q^2\sqrt{\epsilon_0 q}((2q^2-2q+1)(D_1^0-D_\kappa^0)) \\
&+(q^2-1)q^2\sqrt{\epsilon_0 q}(-(2q-1)(D_1^1+D_1^2-D_\kappa^1-D_\kappa^2)),
\end{aligned}$$

where

$$\begin{aligned}
D_r^0&=\sum_{r_3\in k^\times, r_4\in k^\times/\{\pm1\}, t(r,r_3,r_4)\in\{\pm1\}}\psi(r_3),\\
D_r^1&=\sum_{r_3\in k^\times, r_4\in k^\times/\{\pm1\}, t(r,r_3,r_4)\in k^\times-\{\pm1\}}\psi(r_3),\\
D_r^2&=\sum_{r_3\in k^\times, r_4\in k^\times/\{\pm1\}, t(r,r_3,r_4)\in \mathbb{F}_{q^2}-\mathbb{F}_q}\psi(r_3),
\end{aligned}\tag{4.2}$$

for $r=1,\kappa$. The computation of $D_1^i-D_\kappa^i$ is given in Appendix A.4. By Lemma A.5, we have $D_1^0-D_\kappa^0=\epsilon_0\sqrt{\epsilon_0 q}$ and $D_1^1+D_1^2-D_\kappa^1-D_\kappa^2=-\epsilon_0\sqrt{\epsilon_0 q}$. Plugging these formulas into the computation of $|J|\langle \Pi, I(\chi)\otimes\omega_\psi\rangle$, we can get

$$|J|\langle \Pi, I(\chi)\otimes\omega_\psi\rangle = q^6(q^2-1).$$

Thus we have $\langle \Pi, I(\chi)\otimes\omega_\psi\rangle=1$. □

We can now start the proof of Theorem 2.1 in the case $p=3$.

Proof of Theorem* 2.1 *when $p=3$. Irreducible representations of $\mathrm{G}_2(k)$ when $k=\mathbb{F}_{3^f}$ are classified in [En76]. Using the notation of [En76], there are 12 isolated irreducible representations θ_i $0\le i\le 11$ and 15 families of irreducible representations $\theta_{12}(k)$, $\chi_i(k), 1\le i\le 11$, $\chi_{12}(k,l)$, $\chi_{13}(k)$ and $\chi_{14}(k)$, where k,l are integers. From the definitions given in [En76, Section 5], the representations $\chi_i(k), 1\le i\le 11$, are not cuspidal. By Lemma 4.2, we only need to consider cuspidal representations among $\theta_i, 0\le i\le 11$, and $\theta_{12}(k)$. From the definitions of θ_i in [En76, Section 5], one can check that $\theta_0,\theta_1,\theta_2,\theta_3,\theta_4,\theta_6,\theta_7,\theta_8,\theta_9$, are components of parabolic induced representations and thus cannot be cuspidal. Precisely, first, from the definitions in [En76, p.204], one has that $\theta_1+\theta_2=\mu_5=\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_1(0))-\theta_0-\theta_3$, and hence $\theta_i, 0\le i\le 3$, are components of $\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_1(0))$. Here $\chi_1(0)$ is a character of P' and μ_5 is an auxiliary representation. Note that the notation in [En76] is a little bit different from ours. In particular, the group P in [En76] is our P'. Moreover, one has $\theta_4=\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_3(0))-\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_1(0))+\theta_3$, see [En76, p.204]. Since $\theta_0+\theta_1+\theta_2+\theta_3=\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_1(0))$, we can get, $\theta_0+\theta_1+\theta_2+\theta_4=\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_3(0))$. Hence θ_4 is a component of $\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_3(0))$ and thus not cuspidal. Here $\chi_3(0)$ is a character on P'. Furthermore, from the description in [En76, p.201], we have $\theta_6+\theta_9=\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_1(\frac{1}{2}(q-1)))$ and $\theta_7+\theta_8=\mathrm{Ind}_{P'}^{\mathrm{G}_2(k)}(\chi_3(\frac{1}{2}(q-1)))$. Thus $\theta_6,\theta_7,\theta_8,\theta_9$ are not cuspidal either. Consequently, it suffices to consider the cases when $\Pi=\theta_5,\theta_{10},\theta_{11},\theta_{12}(k)$.

Following [En76], the character table of $\theta_5,\theta_{10},\theta_{11},\theta_{12}(k)$, is given in Table 4.4. Recall that U is the maximal unipotent subgroup of $\mathrm{G}_2(k)$. From the character table, we see that for $u\in U$, we

TABLE 4.4. Character table of $\theta_5, \theta_{10}, \theta_{11}, \theta_{12}(k)$.

Representative t	θ_5	θ_{10}	θ_{11}	$\theta_{12}(k)$
1	q^6	$\frac{1}{6}q(q-1)^2(q^2-q+1)$	$\frac{1}{2}q(q-1)(q^3-1)$	$\frac{1}{3}q(q^2-1)^2$
$\mathbf{x}_{3\alpha+2\beta}(1)$	0	$\frac{1}{6}q(q-1)(2q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{3}q(q^2-1)$
$\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\neq 0$	0	$\frac{1}{6}q(q-1)(2q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{3}q(q^2-1)$
$\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+2\beta}(1)$, $r_3\neq 0$	0	$-\frac{1}{6}q(3q-1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{3}q$
$\mathbf{x}_{\beta}(1)$	0	$\frac{1}{6}q(q-1)(2q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{3}q(q^2-1)$
$\mathbf{x}_{\beta}(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^{\times,2}$	0	$\frac{1}{6}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{3}q(q+1)$
$\mathbf{x}_{\beta}(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in \kappa k^{\times,2}$	0	$-\frac{1}{6}q(q-1)$	$\frac{1}{2}q(q+1)$	$-\frac{1}{3}q(q-1)$
$\mathbf{x}_{\beta}(\kappa)$	0	$\frac{1}{6}q(q-1)(2q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{3}q(q^2-1)$
$\mathbf{x}_{\beta}(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^{\times,2}$	0	$-\frac{1}{6}q(q-1)$	$\frac{1}{2}q(q+1)$	$-\frac{1}{3}q(q-1)$
$\mathbf{x}_{\beta}(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in \kappa k^{\times,2}$	0	$\frac{1}{6}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{3}q(q+1)$
$\mathbf{x}_{\beta}(1)\mathbf{x}_{3\alpha+\beta}(r_4)$, $r_4\neq 0$	0	$\frac{1}{6}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{3}q(q+1)$
$\mathbf{x}_{\beta}(1)(0,0,r_3,r_4,0), r_3\neq 0$, $r_4\in k^\times/\{\pm 1\}, t(1,r_3,r_4)\in\{\pm 1\}$	0	$-\frac{1}{6}q(3q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{3}q$
$\mathbf{x}_{\beta}(1)(0,0,r_3,r_4,0), r_3\neq 0$, $r_4\in k^\times/\{\pm 1\}, t(1,r_3,r_4)\in k^\times-\{\pm 1\}$	0	$\frac{1}{6}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{3}q(q+1)$
$\mathbf{x}_{\beta}(1)(0,0,r_3,r_4,0), r_3\neq 0$, $r_4\in k^\times/\{\pm 1\}, t(1,r_3,r_4)\in\mathbb{F}_{q^2}-\mathbb{F}_q$	0	$-\frac{1}{6}q(q-1)$	$\frac{1}{2}q(q+1)$	$-\frac{1}{3}q(q-1)$
$\mathbf{x}_{\beta}(\kappa)\mathbf{x}_{3\alpha+\beta}(r_4)$ $r_4\neq 0$	0	$\frac{1}{6}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{3}q(q+1)$
$\mathbf{x}_{\beta}(\kappa)(0,0,r_3,r_4,0), r_3\neq 0$, $r_4\in k^\times/\{\pm 1\}, t(\kappa,r_3,r_4)\in k^\times-\{\pm 1\}$	0	$\frac{1}{6}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{3}q(q+1)$
$\mathbf{x}_{\beta}(\kappa)(0,0,r_3,r_4,0), r_3\neq 0$, $r_4\in k^\times/\{\pm 1\}, t(\kappa,r_3,r_4)\in\mathbb{F}_{q^2}-\mathbb{F}_q$	0	$-\frac{1}{6}q(q-1)$	$\frac{1}{2}q(q+1)$	$-\frac{1}{3}q(q-1)$
$\mathbf{x}_{\beta}(\kappa)(0,0,r_3,r_4,0), r_3\neq 0$, $r_4\in k^\times/\{\pm 1\}, t(\kappa,r_3,r_4)\in\{\pm 1\}$	0	$-\frac{1}{6}q(3q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{3}q$
$h(-1,-1)$	q^2	$-\frac{1}{2}(q-1)^2$	$-\frac{1}{2}(q-1)^2$	0
$h(-1,-1)(0,0,r_3,0,0)$, $r_3\in k^\times$	0	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	0
$h(-1,-1)\mathbf{x}_{\beta}(1)$	0	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	0
$h(-1,-1)\mathbf{x}_{\beta}(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^{\times,2}$	0	$-\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	0
$h(-1,-1)\mathbf{x}_{\beta}(1)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in \kappa k^{\times,2}$	0	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q+1)$	0
$h(-1,-1)\mathbf{x}_{\beta}(\kappa)$	0	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	0
$h(-1,-1)\mathbf{x}_{\beta}(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in k^\times$	0	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q+1)$	0
$h(-1,-1)\mathbf{x}_{\beta}(\kappa)\mathbf{x}_{2\alpha+\beta}(r_3)$, $r_3\in \kappa k^\times$	0	$-\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	0
$h(x,x^{-1})$, $x\neq\pm 1$	q	0	0	0
$h(x,x^{-1})\mathbf{x}_{2\alpha+\beta}(r_3)$, $x\neq\pm 1, r_3\neq 0$	0	0	0	0
$\begin{pmatrix} x & y \\ \kappa y & x \end{pmatrix}(0,0,r_3,r_4,r_5)$, $x\neq\pm 1$	$*$	$*$	$*$	$*$

have $\theta_5(u)\neq 0$ if and only if $u=1$. In particular, we have

$$\sum_{u\in U}\theta_5(u)=\theta_5(1)=q^6.$$

This implies that θ_5 is not a cupidal character.[2] Thus it suffices to show that $\langle \Pi, I(\chi)\otimes\omega_\psi\rangle \le 1$, when $\Pi = \theta_{10}, \theta_{11}, \theta_{12}(k)$. We now compute $\langle \theta_{10}, I(\chi)\otimes\omega_\psi\rangle$. Similar to Lemma 4.3, the contribution of terms of the form $h(-1,-1)u$ to $\langle \theta_{10}, I(\chi)\otimes\omega_\psi\rangle$ is zero. Thus we get

$$\begin{aligned}
&|J|\langle \theta_{10}, I(\chi)\otimes\omega_\psi\rangle\\
=&\frac{1}{6}q(q-1)^2(q^2-q+1)q(q+1)+(q^2-1)q(q+1)\frac{1}{6}q(q-1)(2q-1)\\
&+q(q+1)\frac{1}{6}q(q-1)(2q-1)\left(\sum_{r_3\in k^\times}\psi(r_3)\right)+(q^2-1)q(q+1)(-\frac{1}{6})q(3q-1)\left(\sum_{r_3\in k^\times}\psi(r_3)\right)\\
&+\frac{q^2-1}{2}q^2\sqrt{\epsilon_0 q}\left(\frac{1}{6}q(q+1)(A_1(1)-A_\kappa(1))+\frac{1}{6}q(q-1)(A_1(1)-A_\kappa(1))\right)\\
&+q^2(q^2-1)\sqrt{\epsilon_0 q}\left(-\frac{1}{6}q(3q-1)(D_1^0-D_\kappa^0)+\frac{1}{6}q(q+1)(D_1^1-D_\kappa^1)-\frac{1}{6}q(q-1)(D_1^2-D_\kappa^2)\right).
\end{aligned}$$

Plugging the formula of $A_1(1)-A_\kappa(1)$ from Lemma A.1 and and the formulas of $D_1^i - D_\kappa^i$ for $i = 0,1,2$, from Lemma A.5, into the above equation, a simple calculation shows that $\langle \theta_{10}, I(\chi)\otimes\omega_\psi\rangle = 0$. Similarly, one can check that $\langle \theta_{11}, I(\chi)\otimes\omega_\psi\rangle = 0$ and $\langle \theta_{12}(k), I(\chi)\otimes\omega_\psi\rangle = 0$. We omit the details. $\square$

5. Gamma factors for $\mathrm{G}_2(k)\times \mathrm{GL}_1(k)$

5.1. **Generic representations and Bessel functions.** Recall that U is the maximal unipotent subgroup of $\mathrm{G}_2(k)$. Let ψ_U be the character of U defined by

$$\psi_U(\mathbf{x}_\alpha(x)\mathbf{x}_\beta(y)u') = \psi(x+y), x, y\in k, u'\in [U,U].$$

We will write ψ_U as ψ by abuse of notation. An irreducible representation Π of $\mathrm{G}_2(k)$ is called ψ-generic if

$$\mathrm{Hom}_U(\Pi,\psi)\neq 0.$$

It is well-known that $\dim \mathrm{Hom}_U(\Pi,\psi)\le 1$.

Remark 5.1. A character ψ' of U is called generic if $\psi'|_{U_a}$ is nontrivial for $a=\alpha,\beta$. There is only one T-conjugacy class of generic characters of U. Thus if Π is ψ-generic, then it is generic with respect to any generic character of U.

Let Π be an irreducible generic representation of $\mathrm{G}_2(k)$. We fix a nonzero element $l\in \mathrm{Hom}_U(\Pi,\psi)$. For a vector v in the space of Π, we consider the function

$$W_v(g) := l(\Pi(g)v).$$

Then the space $\mathcal{W}(\Pi,\psi) := \{W_v : v\in\Pi\}$ is called the ψ-Whittaker model of Π.

Let $\Pi(U,\psi)$ be the subspace of Π generated by elements of form $\Pi(u)v-\psi(u)v$ for $u\in U, v\in\Pi$. Let $\Pi_{U,\psi}=\Pi/\Pi(U,\psi)$ be the twisted Jacquet module. By Jacquet-Langlands Lemma [BZ76, Lemma 2.33], an element $v\in\Pi(U,\psi)$ if and only if $\sum_{u\in U}\psi^{-1}(u)\Pi(u)v=0$. Note that for an irreducible generic representation Π, we have $\dim \Pi_{U,\psi}=1$. For a vector $v\in\Pi, v\notin \Pi(U,\psi)$, we consider the vector

$$v_0 = \frac{1}{|U|}\sum_{u\in U}\psi^{-1}(u)\Pi(u)v,$$

[2]Recall that an irreducible character θ of a reductive group H over a finite field is cuspidal if and only if for any proper parabolic subgroup $Q=M_QU_Q$ with Levi M_Q and unipotent U_Q, one has $\sum_{u\in U_Q}\theta(uh)=0$ for all $h\in H$, see [Ca85, Corollary 9.1.2] for example. In fact, from the character table 4.4, one can check that

$$\langle \theta_5, I(\chi)\otimes\omega_\psi\rangle = \begin{cases} 1, & \text{if } \epsilon\chi\neq 1,\\ 2, & \text{if } \epsilon\chi = 1.\end{cases}$$

Thus θ_5 indeed does not satisfy the conclusion of Theorem 2.1 if $\chi=\epsilon^{-1}=\epsilon$.

where $|U|$ is the number of elements in U. From the choice of v and Jacquent-Langlands Lemma [BZ76, Lemma 2.33], we have $v_0 \neq 0$. On the other hand, we have

$$\Pi(u)v_0 = \psi(u)v_0, \forall u \in U. \tag{5.1}$$

A vector which satisfies the above condition is called a *Whittaker vector*. Let $\langle\ ,\ \rangle$ be a nontrivial $\mathrm{G}_2(k)$-invariant bilinear form $\Pi \times \widetilde{\Pi} \to \mathbb{C}$, where $\widetilde{\Pi}$ is the dual representation of Π. Let v_0 be a Whittaker vector of Π. Then $\tilde{v} \mapsto \langle v_0, \tilde{v}\rangle$ defines a nonzero element in $\mathrm{Hom}_U(\widetilde{\Pi}, \psi^{-1})$. Conversely, a nonzero element in $\mathrm{Hom}_U(\widetilde{\Pi}, \psi^{-1})$ can be viewed as a Whittaker vector of Π via the natural isomorphism $\widetilde{\widetilde{\Pi}} \cong \Pi$. By the uniqueness of Whittaker model, the Whittaker vectors are unique up to scalar.

Let $\mathcal{B}_\Pi \in \mathcal{W}(\Pi, \psi)$ be the Whittaker function associated with a Whittaker vector, normalized by $\mathcal{B}_\Pi(1) = 1$. By the above discussion, the function $\mathcal{B}_\Pi$ is unique. The function $\mathcal{B}_\Pi$ is called the Bessel function of Π.

Lemma 5.2. *We have*

$$\mathcal{B}_\Pi(u_1 g u_2) = \psi(u_1 u_2)\mathcal{B}_\Pi(g), \forall u_1, u_2 \in U, g \in \mathrm{G}_2(k).$$

Proof. This a direct consequence of the definition of $\mathcal{B}_\Pi$. □

Lemma 5.3. (1) *Let $t = h(a,b) \in T$. If $\mathcal{B}_\Pi(t) \neq 0$, then $a = b = 1$.*
(2) *If $r \neq 0$, then $\mathcal{B}_\Pi(h(a,1)\mathbf{x}_{-\beta}(r)) = 0$ for all $a \in k^\times$.*

Proof. (1) Let a be the simple root α or β. First, we have

$$t\mathbf{x}_a(r) = \mathbf{x}_a(a(t)r)t, \forall r \in k.$$

Then, by Lemma 5.2, we have

$$\mathcal{B}_\Pi(t)\psi(r) = \psi(a(t)r)\mathcal{B}_\Pi(t).$$

Thus if $B_\Pi(t) \neq 0$, we have $\psi(r) = \psi(a(t)r)$ for all $r \in k$. Since ψ is a nontrivial character, we must have $a(t) = 1$. Since $\alpha(t) = b, \beta(t) = a/b$, we get $a = b = 1$ if $\mathcal{B}_\Pi(t) \neq 0$.

(2) Take $s \in k$. We have $\mathbf{x}_{-\beta}(r) = w_\beta \mathbf{x}_\beta(-r) w_\beta^{-1}$ and $\mathbf{x}_{\alpha+\beta}(s) = w_\beta \mathbf{x}_\alpha(s) w_\beta^{-1}$. Thus from the commutator relations, we have

$$\begin{aligned}\mathbf{x}_{-\beta}(r)\mathbf{x}_{\alpha+\beta}(s) &= w_\beta \mathbf{x}_\beta(-r)\mathbf{x}_\alpha(s) w_\beta^{-1}\\ &= w_\beta u_1 \mathbf{x}_{\alpha+\beta}(-rs)\mathbf{x}_\alpha(s)\mathbf{x}_\beta(-r) w_\beta^{-1}\\ &= u_2 \mathbf{x}_\alpha(-rs)\mathbf{x}_{\alpha+\beta}(s)\mathbf{x}_{-\beta}(r),\end{aligned}$$

where $u_2 = w_\beta u_1 w_\beta^{-1} \in [U, U]$. Thus, we get

$$h(a,1)\mathbf{x}_{-\beta}(r)\mathbf{x}_{\alpha+\beta}(s) = u_3 \mathbf{x}_\alpha(-rs)\mathbf{x}_{\alpha+\beta}(as)h(a,1)\mathbf{x}_{-\beta}(r),$$

where $u_3 = h(a,1)u_2 h(a^{-1},1)$. Note that $\psi(\mathbf{x}_{\alpha+\beta}(s)) = 1$ and $\psi(u_2\mathbf{x}_\alpha(-rs)\mathbf{x}_{\alpha+\beta}(as)) = \psi(-rs)$. By Lemma 5.2, we get

$$\mathcal{B}_\Pi(h(a,1)\mathbf{x}_{-\beta}(r)) = \psi(-rs)\mathcal{B}_\Pi(h(a,1)\mathbf{x}_{-\beta}(r)), \forall s \in k.$$

Thus if $\mathcal{B}_\Pi(h(a,1)\mathbf{x}_{-\beta}(r)) \neq 0$, we have $\psi(-rs) = 1$ for all $s \in k$. Since ψ is nontrivial, we then get $r = 0$. □

5.2. **Ginzburg's local zeta integral.** Let Π be an irreducible generic representation of $\mathrm{G}_2(k)$ and let χ be a character of $k^\times$. For $W \in \mathcal{W}(\Pi, \psi)$, $f \in I(\chi)$, $\phi \in \mathcal{S}(k)$, we consider the following sum

$$\begin{aligned}\Psi(W,\phi,f) &= \sum_{g \in N_{\mathrm{SL}_2}\backslash \mathrm{SL}_2(k)} \sum_{x,y \in k} W(\mathbf{x}_{-\beta}(y)\mathbf{x}_{-(\alpha+\beta)}(x)j(g))(\omega_{\psi^{-1}}(g)\phi)(x)f(g)\\ &= \sum_{g \in N_{\mathrm{SL}_2}\backslash \mathrm{SL}_2(k)} \sum_{x,y \in k} W(j(\mathbf{x}_{3\alpha+\beta}(y)\mathbf{x}_\alpha(x)g))(\omega_{\psi^{-1}}(g)\phi)(x)f(g),\end{aligned} \tag{5.2}$$

where we embed $\mathrm{SL}_2(k)$ in $\mathrm{G}_2(k)$ by embedding it in the Levi subgroup of P, and $j(g) = w_\beta w_\alpha g w_\alpha^{-1} w_\beta^{-1}$ for $g \in \mathrm{G}_2(k)$. One can easily check that the above sum on the quotient $N_{\mathrm{SL}_2}\backslash\mathrm{SL}_2(k)$ is well-defined using the commutator relations of G_2. The above sum is the finite fields analogue of Ginzburg's local zeta integral [Gi93].

Lemma 5.4. *Let $\delta_0 \in \mathcal{S}(k)$ be the function $\delta_0(x) = 0$ if $x \neq 0$ and $\delta_0(0) = 1$. Let $f_0 \in I(\chi)$ be the function such that* $\mathrm{supp}(f_0) \subset N_{\mathrm{SL}_2} A_{\mathrm{SL}_2}$ *and $f_0(1) = 1$. Then*

$$\Psi(\mathcal{B}_\Pi, \delta_0, f_0) = 1.$$

Proof. We have $\mathrm{SL}_2(k) = N_{\mathrm{SL}_2} A_{\mathrm{SL}_2} \coprod N_{\mathrm{SL}_2} A_{\mathrm{SL}_2} w^1 N_{\mathrm{SL}_2}$, where $w^1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Since f_0 is zero on $N_{\mathrm{SL}_2} A_{\mathrm{SL}_2} w^1 N_{\mathrm{SL}_2}$, we have

$$\Psi(\mathcal{B}_\Pi, \delta_0, f_0) = \sum_{a \in k^\times} \sum_{x, y \in k} \mathcal{B}_\Pi(\mathbf{x}_{-\beta}(y)\mathbf{x}_{-(\alpha+\beta)}(x) j(t(a)))\omega_{\psi^{-1}}(t(a))\delta_0(x)\chi(a),$$

where $t(a) = \mathrm{diag}(a, a^{-1}) \in \mathrm{SL}_2(k)$. Note that $j(t(a)) = h(a, 1)$. On the other hand, we have $\omega_{\psi^{-1}}(t(a))\delta_0(x) = \epsilon(a)\delta_0(ax)$, which is zero if $x \neq 0$ since $a \neq 0$. Thus we have

$$\Psi(\mathcal{B}_\Pi, \delta_0, f_0) = \sum_{a \in k^\times} \sum_{y \in k} \mathcal{B}_\Pi(\mathbf{x}_{-\beta}(y) h(a, 1))\epsilon(a)\chi(a).$$

By Lemma 5.3 (2) and (1), we have

$$\begin{aligned} \Psi(\mathcal{B}_\Pi, \delta_0, f_0) &= \sum_{a \in k^\times} \mathcal{B}_\Pi(h(a, 1))\epsilon(a)\chi(a) \\ &= \mathcal{B}_\Pi(1) \\ &= 1. \end{aligned}$$

This completes the proof of the Lemma. □

Lemma 5.5. *The trilinear form $(W, \phi, f) \mapsto \Psi(W, \phi, f)$ on $\mathcal{W}(\Pi, \psi) \times \omega_{\psi^{-1}} \times I(\chi)$ satisfies the property*

$$\Psi((\Pi(j(h)))W, \omega_{\psi^{-1}}(\overline{\mathrm{pr}}(h))\phi, r(\overline{\mathrm{pr}}(h))f) = \Psi(W, \phi, f), \forall h \in J, \tag{5.3}$$

where r denotes the right translation action, and for $(g, h) \in \mathrm{SL}_2(k) \ltimes \mathscr{H}$, $r(g, h)f := r(g)f$. Recall that $\overline{\mathrm{pr}}$ is the projection map $J \to \mathrm{SL}_2(k) \ltimes \mathscr{H}$ in §2.3.

Proof. Note that for $h \in \mathrm{SL}_2(k)$, Eq.(5.3) follows from a simple changing of variables. Hence, we only need to check formula (5.3) when $h \in V$. Suppose that $h = (s_1, s_2, s_3, s_4, s_5)$. Since $\overline{\mathrm{pr}}(h) = (1, (s_1, s_2, s_3 - s_1 s_2)) \in \mathrm{SL}_2(k) \ltimes \mathscr{H}$, $r(\overline{\mathrm{pr}}(h))f = f$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$h' := ghg^{-1} = (s_1', s_2', s_3', s_4', s_5'),$$

with $s_1' = ds_1 + cs_2, s_2' = bs_1 + as_2, s_3' - s_1' s_2' = s_3 - s_1 s_2$. Then, $\overline{\mathrm{pr}}(h') = (1, (s_1', s_2', s_3' - s_1' s_2')) \in \mathrm{SL}_2(k) \ltimes \mathscr{H}$. Thus

$$\omega_{\psi^{-1}}(g\overline{\mathrm{pr}}(h))\phi(x) = \omega_{\psi^{-1}}(\overline{\mathrm{pr}}(h')g)\phi(x) = \psi^{-1}(s_3' - 2s_1' s_2' - 2xs_2')(\omega_{\psi^{-1}}(g)\phi)(x + s_1'), \tag{5.4}$$

by (2.4).

Next, we compute $W(j(\mathbf{x}_{3\alpha+\beta}(r_1)\mathbf{x}_\alpha(r_2)gh))$. Using the commutator relations, see [Ch68, p.192], one can check that:

$$\begin{aligned} &\mathbf{x}_{3\alpha+\beta}(y)\mathbf{x}_\alpha(x)gh \\ &= \mathbf{x}_{3\alpha+\beta}(y)\mathbf{x}_\alpha(x)h'g \\ &= \mathbf{x}_{3\alpha+2\beta}(s_5')\mathbf{x}_{3\alpha+\beta}(y + s_4')\mathbf{x}_\alpha(x + s_1')\mathbf{x}_{\alpha+\beta}(s_2')\mathbf{x}_{2\alpha+\beta}(s_3')g \\ &= \mathbf{x}_{3\alpha+2\beta}(s_5'')\mathbf{x}_{3\alpha+\beta}(y + s_4'')\mathbf{x}_{\alpha+\beta}(s_2')\mathbf{x}_\alpha(x + s_1')\mathbf{x}_{2\alpha+\beta}(s_3' - 2(r_2 + s_1')s_2')g \\ &= \mathbf{x}_{3\alpha+2\beta}(s_5'')\mathbf{x}_{3\alpha+\beta}(y + s_4''')\mathbf{x}_{\alpha+\beta}(s_2')\mathbf{x}_{2\alpha+\beta}(s_3' - 2(r_2 + s_1')s_2')\mathbf{x}_\alpha(x + s_1')g \\ &= \mathbf{x}_{3\alpha+2\beta}(s_5'')\mathbf{x}_{\alpha+\beta}(s_2')\mathbf{x}_{2\alpha+\beta}(s_3' - 2(r_2 + s_1')s_2')\mathbf{x}_{3\alpha+\beta}(y + s_4''')\mathbf{x}_\alpha(x + s_1')g, \end{aligned}$$

where $s_5''=s_5'+3(x+s_1')(s_2')^2$, $s_4''=s_4'+3(x+s_1')^2s_2'$, and $s_4'''=s_4''+3(x+s_1')(s_3'-2(x+s_1')s_2')$. Note that $j(\mathbf{x}_{3\alpha+2\beta}(s_5''))\in U_{3\alpha+\beta}$, $j(\mathbf{x}_{\alpha+\beta}(s_2'))\in U_{2\alpha+\beta}$, $j(\mathbf{x}_{2\alpha+\beta}(s_3'-2(x+s_1')s_2'))=\mathbf{x}_\alpha(s_3'-2(x+s_1')s_2')$ and $W\in\mathcal{W}(\Pi,\psi)$, we get

$$W(j(\mathbf{x}_{3\alpha+\beta}(y)\mathbf{x}_\alpha(x)gh))=\psi(s_3'-2(x+s_1')s_2')W(j(\mathbf{x}_{3\alpha+\beta}(y+s_4''')\mathbf{x}_\alpha(x+s_1')g)). \tag{5.5}$$

Plugging (5.4) and (5.5) into the left hand side of (5.3), we get

$$\begin{aligned}&\Psi(\Pi(j(h))W,\omega_{\psi^{-1}}(\overline{\mathrm{pr}}(h))\phi,r(\overline{\mathrm{pr}}(h))f)\\ =&\sum_{g\in N_{\mathrm{SL}_2}\backslash\mathrm{SL}_2(k)}\sum_{x,y\in k}W(j(\mathbf{x}_{3\alpha+\beta}(y+s_4''')\mathbf{x}_\alpha(x+s_1')g))(\omega_\psi(g)\phi)(x+s_1')f(g).\end{aligned}$$

By changing variables, we get

$$\Psi(\Pi(j(h))W,\omega_{\psi^{-1}}(\overline{\mathrm{pr}}(h))\phi,r(\overline{\mathrm{pr}}(h))f)=\Psi(W,\phi,f).$$

The completes the proof of the lemma. □

Corollary 5.6. *If Π is an irreducible generic representation of $\mathrm{G}_2(k)$, then we have*

$$\dim\mathrm{Hom}_J(\Pi,\omega_\psi\otimes I(\chi))\geq 1.$$

Proof. Let Π^j be the representation defined by $\Pi^j(g)=\Pi(j(g))$. Note that $\Pi^j\cong\Pi$ since j is an inner automorphism. The assertion then follows from Lemma 5.4 and Lemma 5.5 directly. □

Remark 5.7. In the proof of Theorem 2.1 when $p>3$ in §3, we showed that if $\Pi=X_{33},X_{17},X_{18},X_{19},\overline{X}_{19}$, then $\langle\Pi,I(\chi)\otimes\omega_\psi\rangle=0$. Thus by Corollary 5.6, the representations $X_{33},X_{17},X_{18},X_{19},\overline{X}_{19}$ can not be generic. As pointed out by the referees, this has already been known, for example, the last 4 representations are the 4 unipotent cuspidal representation of $\mathrm{G}_2(k)$ as in [Ca85, p.460] hence are not generic. In particular, the irreducible generic cuspidal representations of $\mathrm{G}_2(k)$ when $p>3$ must be in the families of the representations $X_i(\pi_i)$ for $i=2,3,6$ when π_i are in general positions. Similarly, the irreducible generic cuspidal representations of $\mathrm{G}_2(k)$ when $p=3$ must be in the families of the representations $\chi_{12}(k,l),\chi_{13}(k),\chi_{14}(k)$.

5.3. **GL_1-twisted gamma factors for generic cuspidal representations.** Consider the standard intertwining operator $M:I(\chi)\to I(\chi^{-1})$ defined by

$$M(f)(g)=\sum_{x\in k}f((w^1)^{-1}n(x)g),$$

where $w^1=\begin{pmatrix}&1\\-1&\end{pmatrix}$ and $n(x)=\begin{pmatrix}1&x\\&1\end{pmatrix}$. Note that under the embedding $\mathrm{SL}_2(k)\hookrightarrow\mathrm{GL}_2(k)\cong M\hookrightarrow\mathrm{G}_2(k)$, w^1 is mapped to w_β.

Proposition 5.8. *Let Π be an irreducible generic cuspidal representation of $\mathrm{G}_2(k)$ and χ be a character of $k^\times$. Then there is a number $\Gamma(\Pi\times\chi,\psi)\in\mathbb{C}$ such that*

$$\Psi(W,\phi,M(f))=\Gamma(\Pi\times\chi,\psi)\Psi(W,\phi,f),$$

for all $W\in\mathcal{W}(\Pi,\psi),\phi\in\mathcal{S}(k),f\in I(\chi)$.

Proof. Note that $(W,\phi,f)\mapsto\Psi(W,\phi,f)$ and $(W,\phi,f)\mapsto\Psi(W,\phi,M(f))$ define two trilinear forms in $\mathrm{Hom}_J(\Pi^j\otimes\omega_{\psi^{-1}}\otimes I(\chi),\mathbb{C})$. Then the assertion follows from Theorem 2.1 directly. □

Lemma 5.9. *We have*

$$\Gamma(\Pi\times\chi,\psi)=\frac{q^{5/2}}{\sqrt{\epsilon_0}}\sum_{a\in k^\times}\mathcal{B}_\Pi(h(a,1)w_1)\epsilon\chi^{-1}(a),$$

where $w_1=w_\beta w_\alpha w_\beta w_\alpha^{-1}w_\beta^{-1}=j(w_\beta)$.

Proof. As in Lemma 5.4, let $\delta_0 \in \mathcal{S}(k)$ be the function $\delta_0(x) = 0$ if $x \neq 0$ and $\delta_0(0) = 1$. And let $f_0 \in I(\chi)$ be the function such that $\mathrm{supp}(f_0) \subset N_{\mathrm{SL}_2} A_{\mathrm{SL}_2}$ and $f_0(1) = 1$. Then, by Lemma 5.4, we have $\Psi(B_\Pi, \delta_0, f_0) = 1$. Thus we get

$$\Gamma(\Pi \times \chi, \psi) = \Psi(\mathcal{B}_\Pi, \delta_0, M(f_0)),$$

where,

$$M(f_0)(g) = \sum_{x \in k} f_0\left((w^1)^{-1} n(x) g\right).$$

Since $M(f_0) \in I(\chi^{-1})$ and $\mathrm{SL}_2(k) = N_{\mathrm{SL}_2} A_{\mathrm{SL}_2} \coprod N_{\mathrm{SL}_2} A_{\mathrm{SL}_2} w^1 N_{\mathrm{SL}_2}$, we need to determine the value of $M(f_0)$ at 1 and at $w^1 n(r), r \in k$. Since for any $x \in k$, we have $(w^1)^{-1} n(x) \notin B_{\mathrm{SL}_2}$, we have $M(f_0)(1) = 0$. Since $(w^1)^{-1} n(x) w^1 n(r) \in B_{\mathrm{SL}_2}$ iff $x = 0$, $M(f_0)(w^1 n(r)) = f_0(n(r)) = 1, \forall r \in k$. Thus we get

$$\begin{aligned}
\Gamma(\Pi \times \chi, \psi) &= \sum_{g \in N_{\mathrm{SL}_2} \backslash \mathrm{SL}_2(k)} \sum_{x,y \in k} \mathcal{B}_\Pi(\mathbf{x}_{-\beta}(y)\mathbf{x}_{-(\alpha+\beta)}(x) j(g)) \omega_{\psi^{-1}}(g) \delta_0(x) M(f_0)(g) \\
&= \sum_{a \in k^\times, r \in k} \sum_{x,y \in k} \mathcal{B}_\Pi(\mathbf{x}_{-\beta}(y)\mathbf{x}_{-(\alpha+\beta)}(x) j(t(a) w^1 n(r))) \\
&\qquad \cdot \omega_{\psi^{-1}}(t(a) w^1 n(r)) \delta_0(x) M(f_0)(t(a) w^1 n(r)) \\
&= \sum_{a \in k^\times, r \in k} \sum_{x,y \in k} \mathcal{B}_\Pi(\mathbf{x}_{-\beta}(y)\mathbf{x}_{-(\alpha+\beta)}(x) j(t(a) w^1 n(r))) \\
&\qquad \cdot \omega_{\psi^{-1}}(w^1 n(r)) \delta_0(ax) \epsilon\chi^{-1}(a).
\end{aligned}$$

Note that $w_1 = j(w^1)$ and $j(n(r)) = \mathbf{x}_{3\alpha+2\beta}(r)$, and

$$\mathbf{x}_{-\beta}(y)\mathbf{x}_{-(\alpha+\beta)}(x) h(a,1) w_1 \mathbf{x}_{3\alpha+2\beta}(r) = h(a,1) w_1 \mathbf{x}_{3\alpha+\beta}(ay) \mathbf{x}_{2\alpha+\beta}(ax) \mathbf{x}_{3\alpha+2\beta}(r).$$

By Lemma 5.2, we have $\mathcal{B}_\Pi(\mathbf{x}_{-\beta}(y)\mathbf{x}_{-(\alpha+\beta)}(x) j(t(a) w^1 n(r))) = \mathcal{B}_\Pi(h(a,1) w_1)$. Thus we get

$$\begin{aligned}
\Gamma(\Pi \times \chi, \psi) &= \sum_{a \in k^\times, r \in k} \sum_{x,y \in k} \mathcal{B}_\Pi(h(a,1) w_1) \omega_{\psi^{-1}}(w^1 n(r)) \delta_0(ax) \epsilon\chi^{-1}(a) \\
&= q \sum_{a \in k^\times} \sum_{x,r \in k} \mathcal{B}_\Pi(h(a,1) w_1)(\omega_{\psi^{-1}}(w^1 n(r)) \delta_0)(ax) \epsilon\chi^{-1}(a).
\end{aligned}$$

We have

$$\begin{aligned}
(\omega_{\psi^{-1}}(w^1 n(r)) \delta_0)(ax) &= \frac{1}{\gamma(1, \psi^{-1})} \sum_{y \in k} \psi^{-1}(-2axy)(\omega_{\psi^{-1}}(n(r))) \delta_0(y) \\
&= \frac{1}{\gamma(1, \psi^{-1})} \sum_{y \in k} \psi^{-1}(-2axy) \psi(ry^2) \delta_0(y) \\
&= \frac{1}{\gamma(1, \psi^{-1})}.
\end{aligned}$$

Recall that

$$\gamma(1, \psi^{-1}) = \sum_{x \in k} \psi^{-1}(-x^2) = \sum_{x \in k} \psi(x^2) = \sqrt{\epsilon_0 q},$$

see (2.1). Thus we get

$$\Gamma(\Pi \times \chi, \psi) = \frac{1}{\sqrt{\epsilon_0}} q^{5/2} \sum_{a \in k^\times} \mathcal{B}_\Pi(h(a,1) w_1) \epsilon\chi^{-1}(a).$$

This completes the proof of the lemma. $\square$

Remark 5.10. We use $\Gamma(\pi \times \chi, \psi)$ instead of $\gamma(\pi \times \chi, \psi)$ to denote the gamma factor defined from the functional equation in Proposition 5.8 because it is not normalized in any way. In fact, if one compare the formula in Lemma 5.9 and the corresponding gamma factor formula in the GL_n-case in [N14, Proposition 2.16], it seems that $q^{-5/2}\Gamma(\Pi \times (\chi^{-1}\epsilon))$ is certain normalized gamma factor. Over p-adic fields, Lapid and Rallis [LR05] formulated a series of properties of local gamma factors which can characterize them uniquely. It is an interesting question that how to normalize gamma factors in a canonical way in the finite field case.

6. Gamma factors for $\mathrm{G}_2 \times \mathrm{GL}_2$

In this section, we review the integral for $\mathrm{G}_2 \times \mathrm{GL}_2$ (similar to §5, in our case, it is a sum rather than an integral) developed by Piatetski-Shapiro, Rallis and Schiffmann in [PSRS92] and define the GL_2-twisted gamma factors.

In this section, k is a finite field of odd characteristic unless in subsection 6.4, where k can be either a finite field or a p-adic field.

6.1. **Embedding of G_2 into SO_7.** To introduce the integral of Piatetski-Shapiro, Rallis and Schiffmann, we need to embed G_2 into SO_7. We use the embedding of G_2 into SO_7 given in [RS89]. Let $\mathcal{H}$ be a quaternion algebra over k. We can write $\mathcal{H} = ke_0 \oplus \mathcal{H}^0$, where e_0 is the neutral element and $\mathcal{H}^0$ is the three dimensional subspace of pure quaternions. We denote by e_1, e_2, e_3 a basis of $\mathcal{H}^0$ such that $e_ie_j = -e_je_i$ and $e_3 = (e_1e_2 - e_2e_1)/2 = e_1e_2$. Let $\lambda = e_1e_1$ and $\mu = e_2e_2$. Then $e_3e_3 = -\lambda\mu$. For $x = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathcal{H}$ with $a_i \in k$, we define $\bar{x} = a_0e_0 - a_1e_1 - a_2e_2 - a_3e_3$.

Let $\mathcal{C} = \mathcal{H} \times \mathcal{H}$. We consider the non-associative product on $\mathcal{C}$ given by

$$(a, b)(c, d) = (ac + \bar{d}b, da + b\bar{c}).$$

With this product, $\mathcal{C}$ is called a Cayley or octonion algebra. The conjugate of $(a, b) \in \mathcal{C}$ is defined by $\overline{(a, b)} = (\bar{a}, -b)$ and its norm is $Q((a, b)) = (a, b)\overline{(a, b)} = a\bar{a} - b\bar{b}$.

Note that k can be embedded in to $\mathcal{C}$ by the map $a \mapsto (ae_0, 0)$. We have a decomposition $\mathcal{C} = k \oplus \mathcal{C}^0$, where $\mathcal{C}^0$ is the space of pure Cayley numbers. Note that $\dim_k \mathcal{C}^0 = 7$. We put

$$X^+ = (1/2, 1/2), X^- = (1/2, -1/2), X^0 = X^+ - X^- = (0, -1).$$

One can check that $\mathcal{H}^0X^+, \mathcal{H}^0X^-$ are totally isotropic subspaces of $\mathcal{C}^0$, and we have

$$\mathcal{C}^0 = \mathcal{H}^0X^+ \oplus kX^0 \oplus \mathcal{H}^0X^-,$$

cf. [RS89, p.805]. The group $\mathrm{G}_2(k)$ can be defined to be the automorphism group of the algebra $\mathcal{C}$. Note that if $g \in \mathrm{G}_2(k)$, then $g(1) = 1$, where $1 = (e_0, 0) \in \mathcal{C}$ is the unit element, and $(gu)(gv) = g(uv)$ for $u, v \in \mathcal{C}$. In particular, $g \in \mathrm{G}_2(k)$ preserves the norm form Q. Consider the bilinear form $(v_1, v_2)_Q := Q(v_1) + Q(v_2) - Q(v_1 + v_2)$. Then $g \in \mathrm{G}_2(k)$ preserves the bilinear form $(\ ,\ )_Q$. One can check that the decomposition $\mathcal{C} = k \oplus \mathcal{C}^0$ is an orthogonal decomposition with respect to $(\ ,\ )_Q$. Thus $g \in \mathrm{G}_2(k)$ preserves $\mathcal{C}^0$ and $Q|_{\mathcal{C}^0}$. In particular, we have $\mathrm{G}_2(k) \subset \mathrm{O}(\mathcal{C}^0, Q|_{\mathcal{C}^0}) = \left\{g \in \mathrm{GL}(\mathcal{C}^0) : (gv_1, gv_2)_Q = (v_1, v_2)_Q, \forall v_1, v_2 \in \mathcal{C}^0\right\}$. By [RS89, Corollary 4, p.810], one has

$$\mathrm{G}_2(k) \subset \mathrm{SO}(\mathcal{C}^0, Q|_{\mathcal{C}^0}) = \left\{g \in \mathrm{O}(\mathcal{C}^0, Q|_{\mathcal{C}^0}), \det(g) = 1\right\}.$$

Note that the quaternion algebra over finite fields alway splits. Thus we can assume that $\lambda = \mu = 1$.

A basis of $\mathcal{C}^0$ is given by $e_1^+ := e_1X^+, e_2^+ := e_2X^+, e_3^+ := e_3X^+, e_0 := X^0, e_3^- = e_3X^-, e_2^- := e_2X^-, e_1^- := e_1X^-$. From the formulas given in [RS89, p.805], we can check that the bilinear form $(\ ,\ )_Q$ with respect to the basis $(e_1^+, e_2^+, e_3^+, e_0, e_3^-, e_2^-, e_1^-)$ is given by the following matrix (which is still denoted by Q by abuse of notation)

$$Q = \begin{pmatrix} & & s_3 \\ & 2 & \\ {}^ts_3 & & \end{pmatrix},$$

where

$$s_3 = \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}.$$

Thus $\mathrm{SO}(\mathcal{C}^0, Q|_{\mathcal{C}^0}) = \{g \in \mathrm{GL}_7(k) : {}^tgQg = Q, \det(g) = 1\}$, where we view elements in $\mathcal{C}^0$ as column vectors and $\mathrm{SO}(\mathcal{C}^0, Q|_{\mathcal{C}^0})$ acts on them from the left hand side. In the following, we will fix $\mathrm{SO}(\mathcal{C}^0, Q|_{\mathcal{C}^0})$ as the above form and write it as $\mathrm{SO}_7(k)$. We then get our desired embedding $\mathrm{G}_2(k) \to \mathrm{SO}_7(k)$.

Let $T_{\mathrm{SO}(Q)}$ be the diagonal torus of $\mathrm{SO}_7(k)$. A typical element in $T_{\mathrm{SO}(Q)}$ has the form $t = \mathrm{diag}(a_1, a_2, a_3, 1, a_3^{-1}, a_2^{-1}, a_1^{-1})$. Let ϵ_i be the character of $T_{\mathrm{SO}(Q)}$ of the form $\epsilon_i(t) = a_i$ for $1 \le i \le 3$, where $t = \mathrm{diag}(a_1, a_2, a_3, 1, a_3^{-1}, a_2^{-1}, a_1^{-1})$. Then the positive roots of $\mathrm{SO}_7(k)$ relative to the upper triangular Borel subgroup is the set $\{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, \epsilon_i \text{ with } 1 \le i \le 3, 1 \le j \le 3, i < j\}$. Under the above embedding $\mathrm{G}_2(k) \to \mathrm{SO}_7(k)$, one has $\alpha = \epsilon_2, \beta = \epsilon_1 - \epsilon_2, \alpha + \beta = \epsilon_1, 2\alpha + \beta = -\epsilon_3, 3\alpha + \beta = \epsilon_2 - \epsilon_3, 3\alpha + 2\beta = \epsilon_1 - \epsilon_3$. See [RS89, p.812].

The embedding $\mathrm{G}_2(k) \to \mathrm{SO}_7(k)$ can be explicitly realized by giving matrix realizations of $\mathbf{x}_\gamma(r)$ for all roots γ of G_2, which is given in Appendix B. From this explicit realization, one can see how subgroups of G_2 are embedded in SO_7. For example, the Levi subgroup $M \cong \mathrm{GL}_2(k)$ is embedded into $\mathrm{SO}_7(k)$ by the map

$$m \mapsto \mathrm{diag}(m, \det(m)^{-1}, 1, \det(m), m^*), m \in \mathrm{GL}_2(k),$$

where $m^* = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^t m^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.

6.2. **The Piatetski-Shapiro-Rallis-Schiffmann local zeta integral for $\mathrm{G}_2 \times \mathrm{GL}_2$.** Let $\widetilde{P}$ be the parabolic subgroup of $\mathrm{SO}_7(k)$ which is isomorphic to $(\mathrm{GL}_2(k) \times \mathrm{SO}_3(k)) \ltimes \widetilde{U}$ where $\mathrm{GL}_2(k) \times \mathrm{SO}_3(k)$ is the Levi factor of the form

$$\left\{ \begin{pmatrix} a & & \\ & b & \\ & & a^* \end{pmatrix}, a \in \mathrm{GL}_2(k), b \in \mathrm{SO}_3(k) \right\}.$$

Here $\mathrm{SO}_3(k)$ is the special orthogonal group realized by the matrix $\begin{pmatrix} & & -1 \\ & 2 & \\ -1 & & \end{pmatrix}$. Note that the $\mathrm{GL}_2(k)$ part in the Levi of $\widetilde{P}$ is exactly the Levi subgroup M of $P \subset \mathrm{G}_2(k)$. A typical element of $\widetilde{P}$ will be written as (x, y, u), where $x \in \mathrm{GL}_2(k), y \in \mathrm{SO}_3(k), u \in \widetilde{U}$. Denote $H = M \ltimes Z \subset \mathrm{G}_2(k)$. Under our fixed embedding $\mathrm{G}_2(k) \to \mathrm{SO}_7(k)$, we have $H = \mathrm{G}_2(k) \cap \widetilde{P}$, see [PSRS92, Lemma 1.2, p.1273] and its proof there. One can also see this from the matrix realizations of the embedding in Appendix B.

Let (τ, V_τ) be an irreducible generic representation of $\mathrm{GL}_2(F) \cong M$, we consider the induced representation

$$I(\tau) = \mathrm{Ind}_{\widetilde{P}}^{\mathrm{SO}_7(k)}(\tau \otimes 1_{\mathrm{SO}_3}).$$

A section $\xi \in I(\tau)$ is a map $\xi : \mathrm{SO}_7(k) \to V_\tau$ such that

$$\xi((x, y, u)g) = \tau(x)\xi(g), x \in \mathrm{GL}_2(k), y \in \mathrm{SO}_3(k), u \in \widetilde{U}.$$

We fix a nontrivial ψ-Whittaker functional $\Lambda \in \mathrm{Hom}_{N_{\mathrm{GL}_2}}(\tau, \psi^{-1})$ of τ, where N_{GL_2} is the upper triangular unipotent subgroup of $\mathrm{GL}_2(F)$. We then consider the $\mathbb{C}$-valued function f_ξ on $\mathrm{SO}_7(k) \times \mathrm{GL}_2(k)$ by

$$f_\xi(g, a) = \Lambda(\tau(a)\xi(g)), g \in \mathrm{SO}_7(k), a \in \mathrm{GL}_2(k).$$

We denote by $I'(\mathcal{W}(\tau, \psi^{-1}))$ the space consisting of all functions of the form f_ξ, $\xi \in I(\tau)$.

Let U_H be the subgroup of H generated by root spaces of $\beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$. We have $U_H \subset H = G \cap \widetilde{P}$. Let ψ_{U_H} be the character of U_H such that $\psi_{U_H}|_{U_\beta} = \psi$ and $\psi_{U_H}|_{U_\gamma} = 1$ for $\gamma = 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$. For $u \in U_H$, we have

$$f_\xi(ug, \mathrm{I}_2) = \psi_{U_H}^{-1}(u) f_\xi(g, \mathrm{I}_2), \tag{6.1}$$

where I_2 is the 2×2 identity matrix.

Let Π be an irreducible $\psi = \psi_U$-generic representation of $\mathrm{G}_2(k)$ and τ be an irreducible generic representation of $\mathrm{GL}_2(k)$. For $W \in \mathcal{W}(\Pi, \psi)$, and $f \in I'(\mathcal{W}(\tau, \psi^{-1}))$, we consider the following Piatetski-Shapiro-Rallis-Schiffmann local zeta integral

$$\Psi(W, f) = \sum_{g \in U_H \backslash \mathrm{G}_2(k)} W(g) f(g, \mathrm{I}_2). \tag{6.2}$$

Note that by (6.1), the above sum $\Psi(W, \xi)$ is well-defined.

6.3. **Decomposition of $I(\tau)|_{\mathrm{G}_2(k)}$.** Recall that $P' = M'V'$ denotes the standard parabolic subgroup of $\mathrm{G}_2(k)$ such that U_α is included in the Levi subgroup M'. Let $\widetilde{H} = w_\alpha w_\beta P'(w_\alpha w_\beta)^{-1}$, which is a conjugate of P' and thus still a parabolic subgroup of $\mathrm{G}_2(k)$. It is clear that $\widetilde{H} = \mathrm{GL}_2(k) \ltimes U'$, where $\mathrm{GL}_2(F) = w_\alpha w_\beta M'(w_\alpha w_\beta)^{-1}$ and $U' = w_\alpha w_\beta V'(w_\alpha w_\beta)^{-1}$. Note that U' is generated by the root subgroups of $\beta, 3\alpha+2\beta, -(3\alpha+\beta), \alpha+\beta, -\alpha$, see [PSRS92, Corollary to Lemma 1.2, p.1276].

The double coset $\widetilde{P}\backslash \mathrm{SO}_7(k)/\mathrm{G}_2(k)$ has two elements. From [PSRS92, Lemma 1.2 and its Corollary] and Mackey's theory, we have the following decomposition

$$0 \to \mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau \otimes 1_Z) \to I(\tau)|_{\mathrm{G}_2(k)} \to \mathrm{ind}_{\widetilde{H}}^{\mathrm{G}_2(k)}(\tau \otimes 1_{U'}) \to 0. \tag{6.3}$$

See [PSRS92, p.1287] for local fields analogue of the above decomposition. Note that over finite fields, the above exact sequence splits.

For $\xi \in \mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau \otimes 1_Z)$, we denote $f_\xi(g,a) = \Lambda(\tau(a)\xi(g))$ for $g \in \mathrm{G}_2(k), a \in \mathrm{GL}_2(k)$. We denote $I(\mathcal{W}(\tau,\psi^{-1}))$ the space spanned by f_ξ for $\xi \in \mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau \otimes 1_Z)$, which is a subspace of $I'(\mathcal{W}(\tau,\psi^{-1}))$. In particular, we can consider $\Psi(W,f)$ for $W \in \mathcal{W}(\Pi,\psi)$ and $f \in I(\mathcal{W}(\tau,\psi^{-1}))$.

Remark 6.1. In the exact sequence 6.3, ind also denotes the induced representation. Note that over finite fields, the notations ind and Ind have no difference, but over local fields, they are different. Here, we try to keep the notations the same as in the literature [PSRS92] and thus we used two different notations (ind and Ind) to denote the same object (induced representation). Hopefully, this won't cause any confusion.

6.4. **On the Jacquet functor Π_Z.** In general, let (Π, V_Π) be a representation of a group L and let χ be a character of a subgroup $K \subset L$, then the twisted Jacquet functor $\Pi_{K,\chi}$ is defined to be $V_\Pi/\langle \Pi(k)v - \chi(k)v, k \in K, v \in V_\Pi\rangle$. If $\chi = 1$ is the trivial character, then we write $\Pi_{K,1}$ as Π_K.

In this subsection, let k be either a finite field or a p-adic field. We go back to our G_2 notation. Let Π be an irreducible generic smooth representation of $\mathrm{G}_2(k)$. We consider the Jacquet functor Π_Z. Let $P^1 = \left\{\begin{pmatrix} * & * \\ & 1\end{pmatrix}\right\}$ be the mirabolic subgroup of $\mathrm{GL}_2(k)$. Let ψ_V be the character of V such that $\psi_V|_{U_\alpha} = \psi$ and $\psi_V|_{U_\gamma} = 1$ for $\gamma = \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta$.

Lemma 6.2. *We have the following exact sequences*

$$0 \to \mathrm{ind}_{P^1}^{\mathrm{GL}_2(k)}(\Pi_{V,\psi_V}) \to \Pi_Z \to \Pi_V \to 0, \tag{6.4}$$

and

$$0 \to \mathrm{ind}_{N_{\mathrm{GL}_2}}^{P^1}(\psi) \to \Pi_{V,\psi_V} \to \Pi_{U,\psi'_U} \to 0, \tag{6.5}$$

where ind *means compact induction when k is a local field, ψ'_U is the degenerate character of U defined by $\psi'_U|_{U_\alpha} = \psi$ and $\psi'_U|_{U_\beta} = 1$, and N_{GL_2} is the upper triangular unipotent subgroup of $\mathrm{GL}_2(k)$.*

Remark 6.3. Lemma 6.2 is the finite and p-adic fields analogue of [RS89, Theorem 5, p.824] and its proof given in the following is also parallel to the one given in [RS89]. Note that over finite fields, the above exact sequences split and the topology is discrete.

***Proof of Lemma* 6.2.** Note that $\mathrm{G}_2(k)$ is an ℓ-group in the sense of [BZ76]. Note that when k is a finite field, the topology on $\mathrm{G}_2(k)$ is discrete. We use the language of sheaf theory on ℓ-spaces, see [BZ76].

Note that the parabolic subgroup P normalizes Z, and thus Π_Z can be viewed as a representation of P. Since Z acts on Π_Z trivially, we can view Π_Z as a representation of $P/Z = M \ltimes (V/Z)$. Note that $V/Z \cong k^2$. Moreover, as a representation of V/Z, Π_Z is smooth. Denote the space of Π_Z by V_{Π_Z}. The smoothness of Π_Z implies that $\mathcal{S}(V/Z).V_{\Pi_Z} = V_{\Pi_Z}$, see [BZ76, §2.5] for example. Let $\widehat{V/Z}$ be the dual group of V/Z, i.e., the set of characters on V/Z. The Fourier transform defines an isomorphism $\mathcal{S}(V/Z) \cong \mathcal{S}(\widehat{V/Z})$. Under this isomorphism, we view V_{Π_Z} as a module over $\mathcal{S}(\widehat{V/Z})$. By [BZ76, Proposition 1.14], up to isomorphism, there is a unique sheaf $\mathcal{V}_{\Pi_Z}$ on $\widehat{V/Z}$ such that as a $\mathcal{S}(\widehat{V/Z})$-module, V_{Π_Z} is isomorphic to the finite cross sections $(\mathcal{V}_{\Pi_Z})_c$. For the definition of finite cross sections, see [BZ76, §1.13].

Note that V/Z is generated by the root spaces of α and $\alpha+\beta$. The set $\widehat{V/Z}$ is consisting of characters of the form $\psi_{\kappa_1,\kappa_2}, \kappa_1, \kappa_2 \in k$, where

$$\psi_{\kappa_1,\kappa_2}(\mathbf{x}_{\alpha+\beta}(r_1)\mathbf{x}_\alpha(r_2)) = \psi(\kappa_1 r_1 + \kappa_2 r_2).$$

The map $(\kappa_1,\kappa_2) \mapsto \psi_{\kappa_1,\kappa_2}$ defines a bijection $k^2 \cong \widehat{V/Z}$. Under this bijection, we consider the action of $M \cong \mathrm{GL}_2(k)$ on $\widehat{V/Z}$ given by

$$(g,(\kappa_1,\kappa_2)) = {}^t g^{-1} \cdot {}^t(\kappa_1,\kappa_2).$$

This action has two orbits: the open orbit $O = \left\{\psi_{\kappa_1,\kappa_2} : (\kappa_1,\kappa_2) \in k^2 - \{0\}\right\}$ and the closed orbit $C = \{\psi_{0,0}\}$. We then have the exact sequence

$$0 \to (\mathcal{V}_{\Pi_Z})_c(O) \to (\mathcal{V}_{\Pi_Z})_c \to (\mathcal{V}_{\Pi_Z})_c(C) \to 0, \tag{6.6}$$

see [BZ76, §1.16].

We consider $\psi_{0,1} \in O$. The stabilizer of $\psi_{0,1}$ in $M \cong \mathrm{GL}_2(k)$ is P^1, and the map $g \mapsto g.\psi_{0,1}$ defines a bijection $P^1\backslash \mathrm{GL}_2(k) \to O$. A simple calculation shows that the stalk of the sheaf $\mathcal{V}_{\Pi_Z}$ at the point $\psi_{0,1} \in O$ is

$$(\Pi_Z)_{V/Z,\psi_{0,1}} = \Pi_{V,\psi_V},$$

see [BZ76, Lemma 5.10] for a similar calculation in the GL_n-case. Thus by [BZ76, Proposition 2.23], we have

$$(\mathcal{V}_{\Pi_Z})_c(O) \cong \mathrm{ind}_{P^1}^{\mathrm{GL}_2(k)}(\Pi_{V,\psi_V}).$$

Similarly, consider the stalk of the sheaf $\mathcal{V}_{\Pi_Z}$ at $\psi_{0,0}$, we have

$$(\mathcal{V}_{\Pi_Z})_c(C) = (\Pi_Z)_{V/Z,\psi_{0,0}} = (\Pi_Z)_{V/Z} = \Pi_V.$$

Now the exact sequence (6.4) follows from the exact sequence (6.6).

In general, given any smooth representation ρ of P^1, we have an exact sequence

$$0 \to \mathrm{ind}_{N_{\mathrm{GL}_2}}^{P^1}(\rho_{N_{\mathrm{GL}_2},\psi}) \to \rho \to \rho_{N_{\mathrm{GL}_2}} \to 0, \tag{6.7}$$

see [BZ76, Proposition 5.12] for example. We now apply the exact sequence (6.7) to the representation $\rho = \Pi_{V,\psi_V}$. Note that $(\Pi_{V,\psi_V})_{N_{\mathrm{GL}_2}} = \Pi_{U,\psi'_U}$ and $(\Pi_{V,\psi_V})_{N_{\mathrm{GL}_2},\psi} = \Pi_{U,\psi_U}$ (transitivity of Jacquet functors, see [BZ76, Lemma 2.32, p.24]). By the uniqueness of Whittaker model, we get $\dim \Pi_{U,\psi_U} = 1$. Thus, as a representation of N_{GL_2}, we have $(\Pi_{V,\psi_V})_{N_{\mathrm{GL}_2},\psi} = \psi$. Then, the exact sequence (6.5) follows. □

6.5. **Intertwining operator.** Denote $w_2 = h(1,-1)w_\alpha w_\beta w_\alpha w_\beta^{-1} w_\alpha^{-1}$, one can check that $w_2^2 = 1$. For an irreducible representation τ of $\mathrm{GL}_2(k)$, we consider the representation τ^* of $\mathrm{GL}_2(k)$ defined by $\tau^*(a) := \tau(a^*)$, where $a^* = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^t a^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. The realization of $\mathrm{GL}_2(k) \cong M$ is given by $a \mapsto m(a) := \mathrm{diag}(a, \det(a)^{-1}, 1, \det(a), a^*)$ with $a \in \mathrm{GL}_2(k)$. Note that M normalizes Z and $w_2 m(a) w_2^{-1} = m(a^*)$. Thus we can define an intertwining operator $M_{w_2} : \mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau \otimes 1_Z) \to \mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau^* \otimes 1_Z)$ by the formula

$$M_{w_2}(\xi)(g) = \sum_{z\in Z} \xi(w_2 z g), g \in \mathrm{G}_2(k).$$

For $f \in I(\mathcal{W}(\tau,\psi^{-1}))$, we define

$$M_{w_2}(f)(g,a) = \sum_{z\in Z} f(w_2 z g, d_1 a^*), g \in \mathrm{G}_2(k), a \in \mathrm{GL}_2(k),$$

where $d_1 = \mathrm{diag}(-1,1)$. Here the factor d_1 is added to make sure that the function $a \mapsto M_{w_2}(f)(g,a)$ is a ψ^{-1}-Whittaker function on $\mathrm{GL}_2(k)$. Hence, $M_{w_2}(f) \in I(\mathcal{W}(\tau^*,\psi^{-1}))$, and for $W \in \mathcal{W}(\Pi,\psi)$, one can consider the sum

$$\Psi(W, M_{w_2}(f)) = \sum_{g\in U_H\backslash \mathrm{G}_2(k)} W(g) M_{w_2}(f)(g).$$

6.6. GL_2**-twisted gamma factors for generic cuspidal representations.**

Proposition 6.4. *Let* Π *be an irreducible generic cuspidal representation of* $\mathrm{G}_2(k)$ *and let* τ *be an irreducible generic representation of* $\mathrm{GL}_2(k)$*. Then we have*

$$\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(\mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau\otimes 1_Z),\Pi)=1$$

and

$$\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(I(\tau)|_{\mathrm{G}_2(k)},\Pi)=1.$$

Proof. We use the decomposition $I(\tau)|_{\mathrm{G}_2(k)}$ given in (6.3). By Frobenius reciprocity, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{G}_2(k)}(\mathrm{ind}_{\widetilde{H}}^{\mathrm{G}_2(k)}(\tau\otimes 1_{U'}),\Pi)=&\mathrm{Hom}_{\widetilde{H}}(\tau\otimes 1_{U'},\Pi|_{\widetilde{H}})\\ =&\mathrm{Hom}_{\mathrm{GL}_2(k)}(\tau,\Pi_{U'}).\end{aligned}$$

Since U' is the unipotent of a nontrivial parabolic subgroup and Π is cuspidal, we get $\Pi_{U'}=0$. Thus we have

$$\mathrm{Hom}_{\mathrm{G}_2(k)}(\mathrm{ind}_{\widetilde{H}}^{\mathrm{G}_2(k)}(\tau\otimes 1_{U'}),\Pi)=0.$$

From the decomposition of $I(\tau)|_{\mathrm{G}_2(k)}$ in (6.3) and Frobenius reciprocity, we have

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(I(\tau)|_{\mathrm{G}_2(k)},\Pi)&=\mathrm{Hom}_{\mathrm{G}_2(k)}(\mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau\otimes 1_Z),\Pi)\\ &=\mathrm{Hom}_H(\tau\otimes 1_Z,\Pi)\\ &=\mathrm{Hom}_{\mathrm{GL}_2(k)}(\tau,\Pi_Z).\end{aligned}$$

We now apply exact sequences in Lemma 6.2. Note that V is a unipotent subgroup of a nontrivial parabolic, we have $\Pi_V=0$. Thus we have $\Pi_Z\cong \mathrm{ind}_{P^1}^{\mathrm{GL}_2(k)}(\Pi_{V,\psi_V})$ by (6.4). By Frobenius reciprocity again, we get

$$\begin{aligned}\mathrm{Hom}_{\mathrm{G}_2(k)}(I(\tau)|_{\mathrm{G}_2(k)},\Pi)&=\mathrm{Hom}_{\mathrm{GL}_2(k)}(\tau,\Pi_Z)\\ &=\mathrm{Hom}_{\mathrm{GL}_2(k)}(\tau,\mathrm{ind}_{P^1}^{\mathrm{GL}_2(k)}(\Pi_{V,\psi_V}))\\ &=\mathrm{Hom}_{P^1}(\tau|_{P^1},\Pi_{V,\psi_V}).\end{aligned}$$

Since $\psi'_U|_{U_\beta}=1$, we have $\Pi_{U,\psi'_U}=(\Pi_{V'})_{U_\alpha,\psi}=0$ since V' is the unipotent of the nontrivial parabolic subgroup P' and Π is cuspidal. Thus (6.5) shows that $\Pi_{V,\psi_V}\cong \mathrm{ind}_{N_{\mathrm{GL}_2}}^{P^1}(\psi)$. We then have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{G}_2(k)}(I(\tau)|_{\mathrm{G}_2(k)},\Pi)&=\mathrm{Hom}_{P^1}(\tau|_{P^1},\Pi_{V,\psi_V})\\ &=\mathrm{Hom}_{P^1}(\tau,\mathrm{ind}_{N_{\mathrm{GL}_2}}^{P^1}(\psi))\\ &=\mathrm{Hom}_{N_{\mathrm{GL}_2}}(\tau,\psi).\end{aligned}$$

Since τ is irreducible generic, we have $\mathrm{Hom}_{N_{\mathrm{GL}_2}}(\tau,\psi)=1$ by the uniqueness of Whittaker model for $\mathrm{GL}_2(k)$. The above proof also shows that $\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(\mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau\otimes 1_Z),\Pi)=1$. This completes the proof. □

Remark 6.5. Note that if Π is not cuspidal, from the above proof, we cannot expect that

$$\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(I(\tau)|_{\mathrm{G}_2(k)},\Pi)=1$$

in general. This is because the *tale* terms, say, $\Pi_{U'},\Pi_V,\Pi_{U,\psi'_U}$ can cause some trouble. For example, if $\Pi_{U'}\neq 0$ and $\mathrm{Hom}_{\mathrm{GL}_2(k)}(\tau,\Pi_{U'})\neq 0$, then the above proof shows that

$$\begin{aligned}&\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(I(\tau)|_{\mathrm{G}_2(k)},\Pi)\\ =&\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(\mathrm{ind}_{\widetilde{H}}^{G}(\tau\otimes 1_{U'}),\Pi)+\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(\mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau\otimes 1_Z),\Pi)\\ \geq&\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(\mathrm{ind}_{\widetilde{H}}^{G}(\tau\otimes 1_{U'}),\Pi)+1\\ \geq&\, 2.\end{aligned}$$

Note that over a p-adic field k, we can introduce a complex number parameter in the induced representation $I(\tau)$ and consider the induced representation

$$I(s,\tau):=\mathrm{Ind}_{\widetilde{P}}^{\mathrm{SO}_7(k)}(\tau|\det|^s\otimes 1_{\mathrm{SO}_3})$$

on $\mathrm{SO}_7(k)$. Then the same strategy can show that, *except for a finite number of* q^s, where q is the number of residue field of k, we have

$$\dim \mathrm{Hom}_{\mathrm{G}_2(k)}(I(s,\tau)|_{\mathrm{G}_2(k)}, \Pi) = 1,$$

for any irreducible generic representation Π of $\mathrm{G}_2(k)$. Here we don't need the cuspidal condition on Π, because the *tale* terms $\Pi_{U'}, \Pi_V$, all have finite length as a representation of $\mathrm{GL}_2(k)$, and Π_{U,ψ'_U} has finite dimension, and thus the corresponding Hom spaces are still zero if we exclude a finite number of q^s.

Proposition 6.6. *Let Π be an irreducible generic cuspidal representation of $\mathrm{G}_2(k)$ and τ be an irreducible generic representation of $\mathrm{GL}_2(k)$. Then there exists a number $\Gamma(\Pi\times\tau,\psi)$ such that*

$$\Psi(W, M_{w_2}(f)) = \Gamma(\Pi\times\tau,\psi)\Psi(W,f)$$

for all $W\in\mathcal{W}(\Pi,\psi)$ and $f\in I(\mathcal{W}(\tau,\psi^{-1}))$.

Proof. Note that $(W,f)\mapsto \Psi(W,f)$ and $(W,f)\mapsto\Psi(W,M(f))$ define two elements in $\mathrm{Hom}_{\mathrm{G}_2(k)}(\Pi\otimes \mathrm{ind}_H^{G_2(k)}(\tau\otimes 1_Z),\mathbb{C})$. Proposition 6.4 implies the existence of $\Gamma(\Pi\times\tau,\psi)$. ☐

As in the $\mathrm{G}_2\times\mathrm{GL}_1$ case, see Remark 5.10, the gamma factor $\Gamma(\Pi\times\tau,\psi)$ defined in Proposition 6.6 depends on M_{w_2}, and it is not canonically normalized in any sense.

7. A converse theorem

In this section k is a finite field of odd characteristic.

7.1. **Weyl elements supporting Bessel functions.** Let $\Delta=\{\alpha,\beta\}$ be the set of simple roots of G_2 and let $\mathrm{W}(\mathrm{G}_2)$ be the Weyl group of G_2. The group $\mathrm{W}(\mathrm{G}_2)$ is generated by s_α, s_β and has 12 elements. Let $\mathrm{B}(\mathrm{G}_2)=\{w\in\mathrm{W}(\mathrm{G}_2):\forall\gamma\in\Delta, w\gamma>0 \implies w\gamma\in\Delta\}$. The set $\mathrm{B}(\mathrm{G}_2)$ is called the set of Weyl elements which support Bessel functions and the name is justified by the following

Lemma 7.1. *Let Π be an irreducible generic representation of $\mathrm{G}_2(k)$ and $\mathcal{B}_\Pi\in\mathcal{W}(\Pi,\psi)$ be the Bessel function. If $w\in\mathrm{W}(\mathrm{G}_2)-\mathrm{B}(\mathrm{G}_2)$ and $\dot w\in\mathrm{G}_2(k)$ is a representative of w, then*

$$\mathcal{B}_\Pi(t\dot w)=0, \forall t\in T.$$

Proof. Since $w\notin\mathrm{B}(\mathrm{G}_2)$, there exists an element $\gamma\in\Delta$ such that $w\gamma>0$ but $w\gamma$ is not simple. For any $r\in k$, we consider the element $\mathbf{x}_\gamma(r)\in U_\gamma\subset U$. We have

$$t\dot w\mathbf{x}_\gamma(r) = t\mathbf{x}_{w\gamma}(cr)\dot w = \mathbf{x}_{w\gamma}(w\gamma(t)cr)t\dot w,$$

where $c\in\{\pm1\}$. Note that $\psi_U(\mathbf{x}_{w\gamma}(w\gamma(t)cr))=1$ since $w\gamma$ is not a simple root. By Lemma 5.2, we have

$$\psi(r)\mathcal{B}_\Pi(t\dot w)=\mathcal{B}_\Pi(t\dot w), \forall r\in k.$$

Since ψ is not trivial, we must have $\mathcal{B}_\Pi(t\dot w)=0$. ☐

Let $w_\ell=(s_\alpha s_\beta)^3$, which is the longest Weyl element in $\mathrm{W}(\mathrm{G}_2)$. One can check that $\mathrm{B}(\mathrm{G}_2)=\{1, w_\ell s_\alpha, w_\ell s_\beta, w_\ell\}$. Note that $w_1=w_\beta w_\alpha w_\beta w_\alpha^{-1}w_\beta^{-1}$ is a representative of $w_\ell s_\alpha$ and $w_2=h(1,-1)w_\alpha w_\beta w_\alpha w_\beta^{-1}w_\alpha^{-1}$ is a representative of $w_\ell s_\beta$.

7.2. **An auxiliary lemma.** Let t be a positive integer and N_t be the upper triangular unipotent subgroup of $\mathrm{GL}_t(k)$. Let ψ_t be a generic character of N_t.

Lemma 7.2 ([N14, Lemma 3.1])**.** *Let ϕ be a function on $\mathrm{GL}_t(k)$ such that $\phi(ng)=\psi_t(n)\phi(g)$ for all $n\in N_t$ and $g\in\mathrm{GL}_t(k)$. If*

$$\sum_{g\in N_t\backslash\mathrm{GL}_t(k)}\phi(g)W(g)=0,$$

for all $W\in\mathcal{W}(\pi,\psi_t^{-1})$ and all irreducible generic representations π of $\mathrm{GL}_t(k)$, then $\phi\equiv0$.

Note that in the above lemma, when $t=1$, N_t is trivial. We will only use the above lemma for $t=1,2$.

7.3. **The converse theorem and twisting by** GL_1**.** The following theorem is the main result of this paper.

Theorem 7.3. *Let k be a finite field with odd characteristic. Let Π_1, Π_2 be two irreducible generic cuspidal representation of $\mathrm{G}_2(k)$. If*

$$\Gamma(\Pi_1 \times \chi, \psi) = \Gamma(\Pi_2 \times \chi, \psi),$$
$$\Gamma(\Pi_1 \times \tau, \psi) = \Gamma(\Pi_2 \times \tau, \psi),$$

for all characters χ of $k^\times$ and all irreducible generic representations τ of $\mathrm{GL}_2(k)$, then $\Pi_1 \cong \Pi_2$.

The proof of Theorem 7.3 will be given in the following subsections. The strategy of the proof is as follows. Let $\mathcal{B}_i := \mathcal{B}_{\Pi_i} \in \mathcal{W}(\Pi_i, \psi)$ be the Bessel function of Π_i for $i = 1, 2$. We will prove that $\mathcal{B}_1(g) = \mathcal{B}_2(g)$ for all $g \in \mathrm{G}_2(k)$ under the assumption of Theorem 7.3. Since $\mathrm{G}_2(k) = \coprod_{w \in \mathrm{W}(\mathrm{G}_2)} BwB$, it suffices to show that $\mathcal{B}_1$ agrees with $\mathcal{B}_2$ on various cells BwB. By Lemma 7.1 and Lemma 5.2, if $w \notin \mathrm{B}(\mathrm{G}_2)$, we have $\mathcal{B}_1(g) = \mathcal{B}_2(g) = 0$ for $g \in BwB$. If $w = 1$, we also have $\mathcal{B}_1(g) = \mathcal{B}_2(g), \forall g \in B$ by Lemma 5.2 and Lemma 5.3. Thus it suffices to show that $\mathcal{B}_1(g) = \mathcal{B}_2(g), \forall g \in BwB$ with $w = w_1, w_2, w_\ell$. Here we do not distinguish a Weyl element and its representative. We start from w_1.

Lemma 7.4. *If $\Gamma(\Pi_1 \times \chi, \psi) = \Gamma(\Pi_2 \times \chi, \psi)$ for all characters χ of $\mathrm{GL}_1(k)$, then $\mathcal{B}_1(g) = \mathcal{B}_2(g)$, for all $g \in Bw_1B$.*

Proof. By Lemma 5.9, we have

$$\Gamma(\Pi_i \times \chi, \psi) = \frac{q^{5/2}}{\sqrt{\epsilon_0}} \sum_{a \in k^\times} \mathcal{B}_i(h(a,1)w_1)\epsilon\chi^{-1}(a).$$

Thus the assumplition implies that

$$\sum_{a \in k^\times} (\mathcal{B}_1(h(a,1)w_1) - \mathcal{B}_2(h(a,1)w_1))\epsilon\chi^{-1}(a) = 0,$$

for all character χ of $k^\times$. Then we get

$$\mathcal{B}_1(h(a,1)w_1) - \mathcal{B}_2(h(a,1)w_1) = 0$$

for all $a \in k^\times$ by Lemma 7.2.

On the other hand, for any $a, b \in k^\times$, one can check the following identity

$$\mathbf{x}_\alpha(br)h(a,b)w_1 = h(a,b)w_1\mathbf{x}_\alpha(r), \forall r \in k.$$

Thus by Lemma 5.2, we have

$$\psi(br)\mathcal{B}_i(h(a,b)w_1) = \psi(r)\mathcal{B}_i(h(a,b)w_1), \forall r \in k.$$

Since ψ is nontrivial, we get

$$\mathcal{B}_i(h(a,b)w_1) = 0, \text{ if } b \neq 1.$$

Therefore, we get $\mathcal{B}_1(tw_1) = \mathcal{B}_2(tw_1)$ for all $t \in T$. Since $Bw_1B = UTw_1U$, we get

$$\mathcal{B}_1(g) = \mathcal{B}_2(g), \forall g \in Bw_1B$$

by Lemma 5.2. □

7.4. **Sections in the induced representation** $\mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau \otimes 1_Z)$**.** Let (τ, V_τ) be an irreducible generic representation of $\mathrm{GL}_2(k)$. Fix a nonzero $v \in V_\tau$, we consider the function ξ_v on $\mathrm{G}_2(k)$ defined by $\mathrm{supp}(\xi_v) = H$ and

$$\xi_v(az) = \tau(a)v, a \in M \cong \mathrm{GL}_2(k), z \in Z.$$

Note that $\xi_v \in \mathrm{ind}_H^{\mathrm{G}_2(k)}(\tau \otimes 1_Z)$. Following §6.2, we fix a nonzero Whittaker functional $\Lambda \in \mathrm{Hom}_{N_{\mathrm{GL}_2}}(\tau, \psi^{-1})$ and consider the following function in $I(\mathcal{W}(\tau, \psi^{-1}))$,

$$f_{\xi_v}(g, a) = \Lambda(\tau(a)\xi_v(g)), g \in \mathrm{G}_2(k), a \in \mathrm{GL}_2(k).$$

Let

$$\tilde{f}_v(g, a) = M_{w_2}(f_{\xi_v})(g, a) = \sum_{z \in Z} f_{\xi_v}(w_2 z g, d_1 a).$$

Lemma 7.5. *Let $g = m\mathbf{x}_\alpha(r_1)\mathbf{x}_{\alpha+\beta}(r_2)w_2\mathbf{x}_\alpha(s_1)\mathbf{x}_{\alpha+\beta}(s_2)s'$ with $m \in M, s' \in Z$. Then if $\widetilde{f}_v(g, I_2) \neq 0$, we have $r_1 = r_2 = s_1 = s_2 = 0$. Moreover, if $r_1 = r_2 = s_1 = s_2 = 0$, we have*

$$\widetilde{f}_v(g, I_2) = W_v^*(m),$$

where $W_v^*(m) := \Lambda(\tau(d_1 m^*)v)$, recall that $d_1 = \mathrm{diag}(1, -1)$.

Proof. By the definition of intertwining operator in §6.5, we have

$$\begin{aligned}\widetilde{f}_v(g, \mathrm{I}_2) &= \sum_{z\in Z} f_{\xi_v}(w_2 z g, d_1) \\ &= \sum_{\bar{z}\in\overline{Z}} f_{\xi_v}(\bar{z}m^*\mathbf{x}_{-(\alpha+\beta)}(-r_1)\mathbf{x}_{-\alpha}(-r_2)\mathbf{x}_\alpha(s_1)\mathbf{x}_{\alpha+\beta}(s_2)s', d_1),\end{aligned} \tag{7.1}$$

where $\overline{Z}$ is the opposite of Z. If $\widetilde{f}_v(g, I_2) \neq 0$, then there exists $\bar{z} \in \overline{Z}$, such that

$$\bar{z}m^*\mathbf{x}_{-(\alpha+\beta)}(-r_1)\mathbf{x}_{-\alpha}(-r_2)\mathbf{x}_\alpha(s_1)\mathbf{x}_{\alpha+\beta}(s_2)s' = h \in H.$$

Then we have

$$\mathbf{x}_{-(\alpha+\beta)}(-r_1)\mathbf{x}_{-\alpha}(-r_2)\mathbf{x}_\alpha(s_1)\mathbf{x}_{\alpha+\beta}(s_2) = m_1\bar{z}^{-1}h(s')^{-1}, \tag{7.2}$$

where $m_1 = (m^*)^{-1}$. Suppose that $h = m_2 z_2$ with $m_2 \in M, z_2 \in Z$, and write $z_1 = z_2(s')^{-1} \in Z$.

Note that a typical element in Z has the form

$$\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5) = \begin{pmatrix} 1 & 0 & r_5 & 0 & 0 & -r_3 & 0 \\ 0 & 1 & r_4 & 0 & 0 & 0 & r_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & r_3^2 & -2r_3 & 1 & r_4 & r_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the matrix form can be computed using the matrix realization of $\mathrm{G}_2(k)$ given in Appendix B. For simplicity, we write the element $z_1 \in Z$ as

$$z = \begin{pmatrix} \mathrm{I}_2 & x_1 & x_2 \\ 0 & b & x_1^* \\ & & \mathrm{I}_2 \end{pmatrix},$$

with $b \in \mathrm{SO}_3(k), x_2 \in \mathrm{Mat}_{2\times 2}(k)$ and $x_1 \in \mathrm{Mat}_{2\times 3}(k)$ of the form

$$x_1 = \begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

We write $m_i = \mathrm{diag}(a_i, I_3, a_i^*)$ with $a_i \in \mathrm{GL}_2(k)$ for $i = 1, 2$, and

$$\bar{z}^{-1} = \begin{pmatrix} \mathrm{I}_2 & & \\ \bar{u}_1 & \mathrm{I}_3 & \\ \bar{u}_2 & \bar{u}_1^* & \mathrm{I}_2 \end{pmatrix},$$

where $\bar{u}_1 \in \mathrm{Mat}_{3\times 2}(k), \bar{u}_2 \in \mathrm{Mat}_{2\times 2}(k)$, and $\bar{u}_1^*$ is determined by $\bar{u}_1$.

Then we have

$$\begin{aligned} m_1\bar{z}^{-1}h(s')^{-1} &= m_1\bar{z}^{-1}m_2 z_1 \\ &= \begin{pmatrix} a_1a_2 & a_1a_2x_1 & a_1a_2x_2 \\ \bar{u}_1a_2 & * & * \\ a_1^*\bar{u}_2a_2 & * & * \end{pmatrix}. \end{aligned}$$

On the other hand, from the matrix realization given in Appendix B, we have

$$\mathbf{x}_{-(\alpha+\beta)}(-r_1)\mathbf{x}_{-\alpha}(-r_2)\mathbf{x}_\alpha(s_1)\mathbf{x}_{\alpha+\beta}(s_2) = \begin{pmatrix} b_1 & y_1 & y_2 \\ u_1' & * & * \\ u_2' & * & * \end{pmatrix},$$

where

$$b_1 = \begin{pmatrix} 1-r_2s_1 & r_2s_2 \\ r_1s_1 & 1-r_1s_2 \end{pmatrix},$$
$$y_1 = \begin{pmatrix} 0 & -2s_2(1-r_2s_1) & r_2 \\ -s_1^2s_2 & -2s_1(1+r_1s_2) & -r_1 \end{pmatrix},$$
$$u_1' = \begin{pmatrix} 0 & r_1r_2^2 \\ r_1-r_1r_2s_1 & r_2+r_1r_2s_2 \\ -s_1 & s_2 \end{pmatrix},$$
$$u_2' = \begin{pmatrix} 0 & -r_2^2 \\ -r_1^2(1-r_2s_1) & -r_1r_2(2+r_1s_2) \end{pmatrix}.$$

From the identity (7.2), we have

$$b_1 = a_1a_2, \quad y_1 = a_1a_2x_1, \quad y_2 = a_1a_2x_2, \quad u_1' = \bar{u}_1a_2, \quad u_2' = a_1^*\bar{u}_2a_2.$$

Since $a_1a_2x_1$ is still of the form

$$\begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \end{pmatrix},$$

the equation $y_1 = a_1a_2x_1$ implies that

$$\begin{pmatrix} -2s_2(1-r_2s_1) & r_2 \\ -2s_1(1+r_1s_2) & -r_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which then implies that $r_1 = r_2 = s_1 = s_2 = 0$ since $2 \neq 0$ in k.

If $r_1 = r_2 = s_1 = s_2 = 0$, we then have $u_1' = 0, u_2' = 0$ and thus $\bar{u}_1 = 0, \bar{u}_2 = 0$. Hence $\bar{z} = 1$. Thus, if $r_1 = r_2 = s_1 = s_2 = 0$, by (7.1), we have

$$\widetilde{f}_v(g, \mathrm{I}_2) = f_{\xi_v}(m^*s', d_1) = \Lambda(\tau(d_1)\xi_v(m^*s')) = \Lambda(\tau(d_1m^*)v) = W_v^*(m).$$

This completes the proof of the lemma. $\square$

7.5. **Proof of Theorem 7.3.** Denote by $\mathcal{B}(g) = \mathcal{B}_1(g) - \mathcal{B}_2(g)$. By the discussion in §7.3 and Lemma 7.4, we see that $\mathcal{B}$ is supported on $Bw_2B \coprod Bw_\ell B$.

Let (τ, V_τ) be an irreducible generic representation of $\mathrm{GL}_2(k)$, $v \in V_\tau$, and $f_{\xi_v} \in I(\mathcal{W}(\tau, \psi^{-1}))$ be the section constructed in §7.4. We now compute $\Psi(\mathcal{B}_i, f_{\xi_v})$ for $i = 1, 2$. Since the function $\mathcal{B}_i(g)f_{\xi_v}(g)$ is supported on H, we have

$$\begin{aligned}
\Psi(\mathcal{B}_i, f_{\xi_v}) &= \sum_{g \in U_H \backslash \mathrm{G}_2(k)} \mathcal{B}_i(g) f_{\xi_v}(g, \mathrm{I}_2) \\
&= \sum_{g \in U_H \backslash H} \mathcal{B}_i(g) f_{\xi_v}(g, \mathrm{I}_2) \\
&= \sum_{g \in U_\beta \backslash M} \mathcal{B}_i(g) f_{\xi_v}(g, \mathrm{I}_2) \\
&= \sum_{g \in N_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(k)} \mathcal{B}_i(g) W_v(g),
\end{aligned} \tag{7.3}$$

where an element $g \in \mathrm{GL}_2(k)$ is identified with an element of $\mathrm{G}_2(k)$ via the embedding $\mathrm{GL}_2(k) \cong M \to \mathrm{G}_2(k)$, and $W_v(g) = \Lambda(\tau(g)v)$, which is the Whittaker function of τ associated with $v \in V_\tau$. Note that $M \subset B \cup Bs_\beta B$, which has empty intersection with $Bw_2B \coprod Bw_\ell B$. Since $\mathcal{B}$ is supported on $Bw_2B \coprod Bw_\ell B$, it vanishes on M. We then have

$$\Psi(\mathcal{B}_1, f_{\xi_v}) - \Psi(\mathcal{B}_2, f_{\xi_v}) = \sum_{g \in N_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(k)} \mathcal{B}(g) W_v(g) = 0.$$

Thus the assumption $\Gamma(\Pi_1 \times \tau, \psi) = \Gamma(\Pi_2 \times \tau, \psi)$ and the functional equation, see Theorem 6.6, imply that

$$\Psi(\mathcal{B}_1, \widetilde{f}_v) = \Psi(\mathcal{B}_2, \widetilde{f}_v),$$

or

$$\Psi(\mathcal{B}, \widetilde{f}_v) = \Psi(\mathcal{B}_1, \widetilde{f}_v) - \Psi(\mathcal{B}_2, \widetilde{f}_v) = 0. \tag{7.4}$$

On the other hand, we have

$$\Psi(\mathcal{B}, \widetilde{f}_v) = \sum_{g \in U_H \backslash \mathrm{G}_2(k)} \mathcal{B}(g)\widetilde{f}_v(g, \mathrm{I}_2).$$

Note that $\mathrm{G}_2(k)$ has the following decomposition

$$\mathrm{G}_2(k) = P \coprod P w_\alpha P \coprod P w_\alpha w_\beta w_\alpha P \coprod P w_2 P. \tag{7.5}$$

Since $\mathcal{B}$ is supported on $Bw_2B \cup Bw_\ell B \subset Pw_2P$, it vanishes on

$$P \coprod P w_\alpha P \coprod P w_\alpha w_\beta w_\alpha P.$$

Furthermore, we have

$$U_H \setminus P w_2 P = U_H \setminus (MVw_2V) = U_\beta \backslash M \times U_\alpha U_{\alpha+\beta} \times w_2 V.$$

By the above discussion and Lemma 7.5, we have

$$\begin{aligned}
\Psi(\mathcal{B}, \widetilde{f}_v) &= \sum_{m \in U_\beta \backslash M} \sum_{r_1, r_2, s_1, s_2 \in k, s' \in Z} \mathcal{B}(m\mathbf{x}_\alpha(r_1)\mathbf{x}_{\alpha+\beta}(r_2)w_2\mathbf{x}_\alpha(s_1)\mathbf{x}_{\alpha+\beta}(s_2)s') \\
&\qquad \cdot \widetilde{f}_v(m\mathbf{x}_\alpha(r_1)\mathbf{x}_{\alpha+\beta}(r_2)w_2\mathbf{x}_\alpha(s_1)\mathbf{x}_{\alpha+\beta}(s_2)s', \mathrm{I}_2) \\
&= q^3 \sum_{m \in U_\beta \backslash M} \mathcal{B}(mw_2)W_v^*(m). \qquad (7.6)
\end{aligned}$$

Then the equation (7.4) implies that

$$\sum_{m \in U_\beta \backslash M} \mathcal{B}(mw_2)W_v^*(m) = 0, \tag{7.7}$$

which holds for all $v \in V_\tau$ and all irreducible generic representations τ of $\mathrm{GL}_2(k)$. Thus by Lemma 7.2, we have

$$\mathcal{B}(mw_2) = 0, \forall m \in M. \tag{7.8}$$

If we take $m = h(x, y) \in M$ in (7.8), we get

$$\mathcal{B}(h(x, y)w_2) = 0, \forall x, y \in k^\times. \tag{7.9}$$

If we take $m = h(x, y)w_\beta$ in (7.8), we then get

$$\mathcal{B}(h(x, y)w_\beta w_2) = 0, \forall x, y \in k^\times. \tag{7.10}$$

Denote $\dot{w}_\ell = w_\beta w_2$. Note that $\dot{w}_\ell$ is a representative of w_ℓ. Together with Lemma 5.2, equations (7.9) (7.10) imply that $\mathcal{B}$ vanishes on the cells Bw_2B and $Bw_\ell B$. This shows that $\mathcal{B}$ is identically zero. Thus

$$\mathcal{B}_1(g) = \mathcal{B}_2(g), \forall g \in \mathrm{G}_2(k).$$

By the uniqueness of Whittaker model and irreducibility of Π_1, Π_2, we get $\Pi_1 \cong \Pi_2$.

This completes the proof of Theorem 7.3.

Appendix A. Computation of certain Gauss sums

A.1. **Basic Gauss sum.** Let ψ be a nontrivial additive character of $k = \mathbb{F}_q$. Recall that we have fixed a square root $\sqrt{\epsilon_0}$ of ϵ_0 such that

$$\sum_{x \in k} \psi(ax^2) = \epsilon(a)\sqrt{\epsilon_0 q}.$$

For $a \in k^\times$, let

$$A_r(a) = \sum_{x \in k^{\times,2}} \psi(arx), r = 1, \kappa.$$

We then have

$$1 + 2A_1(a) = \sum_{x \in k} \psi(ax^2) = \epsilon(a)\sqrt{\epsilon_0 q},$$

and

$$1 + 2A_\kappa(a) = \sum_{x \in k} \psi(a\kappa x^2) = -\epsilon(a)\sqrt{\epsilon_0 q}.$$

Thus we get the following

Lemma A.1. *We have* $A_1(a) - A_\kappa(a) = \epsilon(a)\sqrt{\epsilon_0 q}$.

We write $A_r(1)$ as A_r for simplicity, for $r = 1, \kappa$.

A.2. **Computation of B_r^i.** We now compute the sums B_r^i for $r = 1, \kappa$ and $i = 0, 1, 2, 3$ in (3.6) used in §3. We assume $q \equiv 1 \bmod 3$. Given $r \in \{1, \kappa\}, r_3 \in k^\times, r_4 \in k^\times/\pm 1$, let $z(r, r_3, r_4) = -2 - \frac{rr_4^2}{r_3^3} \in k$. Note that for any $a \in k$, the equation $t + t^{-1} = a$ for t is solvable over k_2. Given r, r_3, r_4 as above, and recall that $t = t(r, r_3, r_4)$ denotes a solution of the equation $t + t^{-1} = z(r, r_3, r_4)$. Note that $rr_3r_4 \neq 0$ implies that $t \neq -1$. Although there are two choices of $t(r, r_3, r_4)$ in general, one can check that the condition $t(r, r_3, r_4) \in \{\pm 1\}$ (resp. $t(r, r_3, r_4) \in k^{\times,3} - \{\pm 1\}$, $t(r, r_3, r_4) \in k^\times - k^{\times,3}$, $t(r, r_3, r_4) \in k_2 - k^\times$) is independent on the choice of $t(r, r_3, r_4)$.

Lemma A.2. *We have*

$$\begin{aligned}
B_1^0 - B_\kappa^0 &= \epsilon_0\sqrt{\epsilon_0 q}, \\
B_1^1 - B_\kappa^1 &= -\frac{1}{2}(1 + \epsilon_0)\sqrt{\epsilon_0 q}, \\
B_1^2 - B_\kappa^2 &= 0, \\
B_1^3 - B_\kappa^3 &= \frac{1}{2}(1 - \epsilon_0)\sqrt{\epsilon_0 q}.
\end{aligned}$$

Proof. Notice that the condition $-2 - rr_4^2/r_3^3 = t + t^{-1}$ implies that $t \neq -1$ and

$$(-r_3)^3 = rt\left(\frac{r_4}{t+1}\right)^2. \tag{A.1}$$

We first compute B_r^0. We first assume that $r = 1$. When $t = 1$, (A.1) becomes $(-r_3)^3 = (r_4/2)^2$. Since $k^\times$ is a cyclic group generated by κ, the condition $(-r_3)^3 = (r_4/2)^2$ implies that $-r_3 \in k^{\times,2}$. Moreover, for each $-r_3 \in k^{\times,2}$, there exists a unique $r_4 \in k^\times/\{\pm 1\}$ such that the equation $(-r_3)^3 = (r_4/2)^2$ holds. Thus we get

$$B_1^0 = \sum_{-r_3 \in k^{\times,2}} \psi(r_3) = A_1(-1).$$

Similarly, we have $B_\kappa^0 = A_\kappa(-1)$. Thus we have $B_1^0 - B_\kappa^1 = A_1(-1) - A_\kappa(-1) = \epsilon_0\sqrt{\epsilon_0 q}$ by Lemma A.1.

We next compute B_r^1, $r = 1, \kappa$. Let $t = t(r, r_3, r_4) \in k^{\times,3} - \{\pm 1\}$. Let $a \in k^\times$ with $t = a^3$. We first assume that $r = 1$. From (A.1), we have $-a^{-1}r_3 \in k^{\times,2}$. Thus the contribution of each fixed $t = t(1, r_3, r_4)$ to the sum B_1^1 is

$$\sum_{x \in k^{\times,2}} \psi(-t^{1/3}x),$$

where $t^{1/3}$ is any cubic root of t in $k^\times$. Because t and t^{-1} contributes the same to the sum B_1^1, we have

$$B_1^1 = \frac{1}{2}\sum_{t \in k^{\times,3} - \{\pm 1\}} \sum_{x \in k^{\times,2}} \psi(-t^{1/3}x).$$

Similarly, we have

$$B_\kappa^1 = \frac{1}{2}\sum_{t \in k^{\times,3} - \{\pm 1\}} \sum_{x \in k^{\times,2}} \psi(-t^{1/3}\kappa x).$$

Thus by Lemma A.1, we have

$$\begin{aligned}B_1^1 - B_\kappa^1 &= \frac{1}{2}\sum_{t\in k^{\times,3}-\{\pm 1\}} (A_1(-t^{1/3}) - A_\kappa(-t^{1/3}))\\ &= \frac{1}{2}\epsilon_0\sqrt{\epsilon_0 q}\sum_{t\in k^{\times,3}-\{\pm 1\}} \epsilon(t^{1/3}).\end{aligned}$$

We have $k^{\times,3} = \left\{\kappa^{3i} : 1 \le i \le \frac{q-1}{3}\right\}$. Thus we get

$$\sum_{t\in k^{\times,3}} \epsilon(t^{1/3}) = \sum_{i=1}^{\frac{q-1}{3}} \epsilon(\kappa)^i = 0,$$

where the last equality follows from the fact that $\epsilon(\kappa) = -1$ and $\frac{q-1}{3}$ must be even. Thus we get

$$B_1^1 - B_\kappa^1 = -\frac{1}{2}\epsilon_0\sqrt{\epsilon_0 q}(1+\epsilon_0) = -\frac{1}{2}(1+\epsilon_0)\sqrt{\epsilon_0 q}.$$

We next consider B_r^2. Note that $k^\times - k^{\times,3} = \kappa k^{\times,3}\coprod \kappa^2 k^{\times,3}$. For $j = 1, 2$, we define

$$B_r^{2,j} = \sum_{r_3\in k^\times, r_4\in k^\times/\{\pm 1\}, t(r,r_3,r_4)\in \kappa^j k^{\times,3}} \psi(r_3).$$

We have $B_r^2 = B_r^{2,1} + B_r^{2,2}$. Take an element $t \in k^\times - k^{\times,3}$ with $t(r, r_3, r_4) = t$. Then the condition $-2 - \frac{rr_4^2}{r_3^3} = t + t^{-1}$ implies (A.1). Note that if $r = 1$ and $t \in \kappa k^{\times,3}$, equation (A.1) implies that $-r_3 \in \kappa k^{\times,2}$, and for such an r_3, there is a unique r_4 satisfying that equation. Thus we get

$$B_1^{2,1} = \sum_{t\in \kappa k^{\times,3}}\sum_{x\in k^{\times,2}} \psi(-\kappa x) = \frac{q-1}{3}A_\kappa(-1).$$

For $t \in \kappa^2 k^{\times,2}$ and $r = \kappa$, we also have that $-r_3 \in \kappa k^{\times,2}$ and a unique r_4 determined by these datum. This shows that

$$B_1^{2,1} = \sum_{t\in \kappa k^{\times,3}}\sum_{x\in k^{\times,2}} \psi(-\kappa x) = \frac{q-1}{3}A_\kappa(-1) = B_\kappa^{2,2}.$$

Similarly, we have $B_\kappa^{2,1} = B_1^{2,2}$. Thus we get $B_1^2 = B_\kappa^2$.

Finally, we consider B_r^3. We have

$$B_r^0 + B_r^1 + B_r^2 + B_r^3 = \sum_{r_3\in k^\times, r_4\in k^\times/\{\pm 1\}} \psi(r_3) = -\frac{q-1}{2}.$$

Thus, from the previous results, we get

$$B_1^3 - B_\kappa^3 = -(B_1^0 - B_\kappa^0) - (B_1^1 - B_\kappa^1).$$

This concludes the proof of the lemma. □

A.3. **Computation of C_r^i.** In this subsection, we compute the sums C_r^i for $r = 1, \kappa$, and $i = 0, 1, 2, 3$ defined in (3.8). Note that in this case, $q \equiv -1 \bmod 3$. Recall that k_2 is the unique quadratic extension of $k = \mathbb{F}_q$. We can realize k_2 as $k[\sqrt{\kappa}]$. Let $\mathrm{Nm} : k_2 \to k$ be the norm map. We have $\mathrm{Nm}(x + y\sqrt{\kappa}) = x^2 - y^2\kappa$. Recall that k_2^1 is the norm 1 subgroup of $k_2^\times$.

Lemma A.3. (1) *If an element $u \in k_2^1$ has a cubic root $v \in k_2^\times$, then we must have $v \in k_2^1$.*

(2) *Let $t \in k_2^1$ and $t \ne -1$. Then $t + t^{-1} + 2$ is a square in $k^\times$ if and only if t is a square in k_2^1.*

Proof. (1) Since $u = v^3 \in k_2^1$, we have $v^{3q+3} = 1$. On the other hand, we have $v^{q^2-1} = 1$ since $v \in k_2^\times$. Since $q \equiv -1 \bmod 3$, the greatest common divisor of $q^2 - 1$ and $3q + 3$ is $q + 1$. Thus $v^{q+1} = 1$, which means that $v \in k_2^1$.

(2) Suppose that $t = \beta^2$ with $\beta \in k_2^1$. We write $\beta = a + b\sqrt{\kappa}$ with $a, b \in k$. Then $\beta \in k_2^1$ means that $a^2 - b^2\kappa = 1$, which implies that $b^2\kappa = a^2 - 1$. We have $t = \beta^2 = a^2 + b^2\kappa + 2ab\sqrt{\kappa}$. Thus

$$t + t^{-1} + 2 = 2(a^2 + b^2\kappa) + 2 = 4a^2 \in k^{\times,2}.$$

Conversely, suppose that $t+t^{-1}+2 \in k^{\times,2}$. Suppose that $t = x+y\sqrt{\kappa}$ with $x, y \in k$ and $t+t^{-1}+2 = a^2$ with $a \in k^\times$. Note that $t + t^{-1} + 2 = 2x + 2$. Thus $a^2 = 2x + 2$. On the other hand, we have

$$a^2 = t + t^{-1} + 2 = t^{-1}(t + 1)^2.$$

Thus, we have $t = (a^{-1}(t + 1))^2$. It suffices to show that $a^{-1}(t + 1) \in k^1_2$. We have

$$\mathrm{Nm}(t + 1) = (x + 1)^2 - y^2\kappa = 2 + 2x = a^2,$$

where we used $x^2 - y^2\kappa = 1$. Thus $\mathrm{Nm}(a^{-1}(t + 1)) = 1$. □

Lemma A.4. *We have*

$$\begin{aligned}
C^0_1 - C^0_\kappa &= \epsilon_0\sqrt{\epsilon_0 q},\\
C^1_1 - C^1_\kappa &= -\frac{1}{2}(1 + \epsilon_0)\sqrt{\epsilon_0 q},\\
C^2_1 - C^2_\kappa &= \frac{1}{2}(1 - \epsilon_0)\sqrt{\epsilon_0 q},\\
C^3_1 - C^3_\kappa &= 0.
\end{aligned}$$

Proof. Note that $C^0_r = B^0_r$ and thus $C^0_1 - C^0_2 = \epsilon_0\sqrt{\epsilon_0 q}$ follows from Lemma A.2. To compute C^2_r, we take an element $t \in k^\times - \{\pm1\}$ and let $t(r, r_3, r_4) = t$, which implies

$$(-r_3)^3 = rt\left(\frac{r_4}{t + 1}\right)^2,$$

see (A.1). Note that any $t \in k^\times$ is has a cubic root in $k^\times$. Let $t^{1/3} \in k^\times$ be one cubic root of t. Then the above equation implies that

$$(-r_3/t^{1/3})^3 = r\left(\frac{r_4}{t + 1}\right)^2.$$

If $r = 1$, this implies that $r_3 \in -t^{1/3}k^{\times,2}$, and for such an r_3 (and a fixed t), there is a unique $r_4 \in k^\times/\{\pm1\}$ such that $(-r_3/t^{1/3})^3 = r\left(\frac{r_4}{t+1}\right)^2$. Thus the contribution of a single t with $t(1, r_3, r_4)$ to the sum C^1_1 is

$$\sum_{k^{\times,2}} \psi(-t^{1/3}x).$$

Since t and t^{-1} have the same contribution, we have

$$C^1_1 = \frac{1}{2}\sum_{t\in k^\times-\{\pm1\}}\sum_{x\in k^{\times,2}} \psi(-t^{1/3}x).$$

Since $t \mapsto t^3$ is a bijection from $k^\times - \{\pm1\}$ to itself, we get

$$C^1_1 = \frac{1}{2}\sum_{t\in k^\times-\{\pm1\}}\sum_{x\in k^{\times,2}} \psi(-tx) = \frac{1}{2}\sum_{t\in k^\times-\{\pm1\}} A_1(-t).$$

Similarly, we have

$$C^1_\kappa = \frac{1}{2}\sum_{t\in k^\times-\{\pm1\}} A_\kappa(-t).$$

Thus by Lemma A.1, we have

$$\begin{aligned}
C^1_1 - C^1_\kappa &= \frac{1}{2}\sum_{t\in k^\times-\{\pm1\}} (A_1(-t) - A_\kappa(-t))\\
&= \frac{1}{2}\sum_{t\in k^\times-\{\pm1\}} \epsilon_0\epsilon(t)\sqrt{\epsilon_0 q}.
\end{aligned}$$

Since ϵ is a nontrivial character on $k^\times$, we have $\sum_{t\in k^\times} \epsilon(t) = 0$. Thus we have

$$C^1_1 - C^1_\kappa = -\frac{1}{2}(1 + \epsilon_0)\sqrt{\epsilon_0 q}.$$

We next consider C_r^3. Let α be a generator of k_2^1. Note that α has no cubic root in k_2^1. By Lemma A.3 (1), we have

$$k_2^1 - k_2^{\times,3} = \left\{\alpha^i : 0 \le i \le q, 3 \nmid i\right\}.$$

Consider the subsets S_1, S_2 of $k_2^1 - k_2^{\times,3}$:

$$S_1 = \left\{\alpha^i : 0 \le i \le q, 3 \nmid i, 2 \nmid i\right\}, S_2 = \left\{\alpha^i : 0 \le i \le q, 3 \nmid i, 2|i\right\}.$$

Note that $|S_1| = |S_2| = \frac{q+1}{3}$. For $i = 1, 2$, let

$$C_r^{3,i} = \sum_{r_3 \in k^\times, r_4 \in k^\times/\{\pm 1\}, t(r, r_3, r_4) \in S_i} \psi(r_3).$$

We have $C_r^3 = C_r^{3,1} + C_r^{3,2}$. Take $t \in S_i$, the condition $t(r, r_3, r_4) = t$ implies that

$$(-r_3)^3 = \frac{r r_4^2}{t + t^{-1} + 2}.$$

If $t \in S_1$, by Lemma A.3, we have $t + t^{-1} + 2 \in \kappa k^{\times,2}$. Thus for $r = 1, t \in S_1$, we have $-r_3 \in \kappa k^{\times,2}$, and for each $-r_3 \in \kappa k^{\times,2}$, there is a unique $r_4 \in k^\times/\{\pm 1\}$ such that $t(1, r_3, r_4) = t$ (for fixed t). Thus, we get

$$C_1^{3,1} = \frac{1}{2}\sum_{t \in S_1}\sum_{x \in k^{\times,2}} \psi(-\kappa x) = \frac{q+1}{6} A_\kappa(-1),$$

where the 1/2 was appeared since t and t^{-1} have the same contribution to the above sum. Similarly, we have

$$C_\kappa^{3,2} = \frac{1}{2}\sum_{t \in S_2}\sum_{x \in k^{\times,2}} \psi(-\kappa x) = \frac{q+1}{6} A_\kappa(-1).$$

In particular, we have $C_1^{3,1} = C_\kappa^{3,2}$. Similarly, we have $C_1^{3,2} = C_\kappa^{3,1}$. Thus we have $C_1^3 - C_\kappa^3 = 0$.

Finally, to compute $C_1^2 - C_\kappa^2$, it suffices to notice that

$$\sum_{i=0}^{3} C_1^i = \sum_{i=0}^{3} C_\kappa^i,$$

and thus

$$C_1^2 - C_\kappa^2 = -(C_1^0 - C_\kappa^0) - (C_1^1 - C_\kappa^1) - (C_1^3 - C_\kappa^3).$$

One can also compute $C_1^2 - C_\kappa^2$ directly from Lemma A.3. □

A.4. **Computation of D_r^i.** In this subsection, let $q = 3^f$ and $k = \mathbb{F}_q$. We compute the Gauss sums in (4.2).

Lemma A.5. *We have*

$$\begin{aligned} D_1^0 - D_\kappa^0 &= \epsilon_0\sqrt{\epsilon_0 q}, \\ D_1^1 - D_\kappa^1 &= -\frac{1}{2}(1+\epsilon_0)\sqrt{\epsilon_0 q}, \\ D_1^2 - D_\kappa^2 &= \frac{1}{2}(1-\epsilon_0)\sqrt{\epsilon_0 q}. \end{aligned}$$

Proof. Note that we have $D_r^0 = B_r^0$. Thus the first identity follows from Lemma A.2. The second identity can be computed similarly as the computation of $C_1^1 - C_\kappa^1$. Since $D_1^0 + D_1^1 + D_1^2 = D_\kappa^0 + D_\kappa^1 + D_\kappa^2$, the last identity follows from the first one. □

Appendix B. Embedding of G_2 into SO_7

In this appendix, based on [RS89], we give an explicit matrix realization of $\mathbf{x}_\gamma(r)$ for each root γ of G_2, which gives an explicit embedding of $G_2(k)$ into $SO_7(k)$. Here $SO_7(k) = \{g \in GL_7(k) : {}^t gQg = Q\}$, with $Q = \begin{pmatrix} & & s_3 \\ & 2 & \\ {}^t s_3 & & \end{pmatrix}$, where $s_3 = \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}$. The explicit realization of $\mathbf{x}_\gamma(r)$ is given as follows.

$$\mathbf{x}_\alpha(r) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2r & 0 & -r^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & r & 0 \\ -r & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -r & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x}_{-\alpha}(r) = \begin{pmatrix} 1 & 0 & 0 & 0 & -r & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -r \\ 0 & -r & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -r^2 & 0 & 2r & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{x}_{\alpha+\beta}(r) = \begin{pmatrix} 1 & 0 & 0 & -2r & 0 & 0 & -r^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & r \\ 0 & r & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x}_{-(\alpha+\beta)}(r) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & r & 0 \\ -r & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -r^2 & 0 & 0 & 2r & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{x}_{2\alpha+\beta}(r) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -r & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & -2r & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x}_{-(2\alpha+\beta)}(r) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2r & r^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -r & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & r & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{x}_{\beta}(r) = \begin{pmatrix} 1 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x}_{3\alpha+\beta}(r) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{x}_{3\alpha+2\beta}(r) = \begin{pmatrix} 1 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\mathbf{x}_{-\beta}(r) = {}^t\mathbf{x}_\beta(r), \mathbf{x}_{-(3\alpha+\beta)}(r) = {}^t\mathbf{x}_{3\alpha+\beta}(r), \mathbf{x}_{-(3\alpha+2\beta)}(r) = {}^t\mathbf{x}_{3\alpha+\beta}(r).$$

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