CSS Codes that are Oblivious to Coherent Noise

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***Abstract*—Physical platforms such as trapped ions suffer from coherent noise that does not follow a simple stochastic model. We view coherent errors as rotations about a particular axis, and observe that since they can accumulate coherently over time, they can be more damaging. It is natural to consider coherent noise acting transversally giving rise to an effective error, which is a Z-rotation on each qubit by some angle θ.**

Rather than addressing coherent noise through active error correction, we instead investigate passive mitigation through decoherence free subspaces. In the language of stabilizer codes, we require the noise to preserve the code space, and to act trivially (as the logical identity operator) on the protected information. Thus, we develop necessary and sufficient conditions for all transversal Z-rotations to preserve the code space of a stabilizer code.

These conditions require the existence of a large number of weight 2 Z-stabilizers, and together, these weight 2 Z-stabilizers generate a direct product of single-parity-check codes. By adjusting the size of these components, we are able to construct a large family of CSS codes, oblivious to coherent noise, that includes the $[[4L^2, 1, 2L]]$ Shor codes. Given m even and given any $[[n, k, d]]$ CSS code, we can construct an $[[mn, k, d' \geq d]]$ CSS code that is oblivious to coherent noise. This result is generalized to stabilizer codes in [Hu, Liang, Rengaswamy, and Calderbank 2020].

The MacWilliams Identities play a central role in the technical analysis, and classical coding theorists may be interested in connections to classical codes with all weights divisible by some integer d.

***Index Terms* — coherent noise, DFS, transversal Z-rotations, Clifford hierarchy, MacWilliams identities**

I. Introduction

In quantum systems, noise can broadly be classified into two types – stochastic and coherent errors. Stochastic errors occur randomly and do not accumulate over time along a particular direction. Coherent errors may be viewed as rotations about a particular axis, and can be more damaging, since they can accumulate coherently over time [1]. As quantum computers move out of the lab and become generally programmable, the research community is paying more attention to coherent errors, and especially to the decay in coherence of the effective induced logical channel [2], [3]. It is natural to consider coherent noise acting *transversally*, where the effect of the noise is to implement a separate unitary on each qubit. Consider, for example, an n-qubit physical system with a uniform background magnetic field acting on the system according to the Hamiltonian $H = \sigma_Z^{(1)} + \sigma_Z^{(2)} + \ldots + \sigma_Z^{(n)}$, where $\sigma_Z^{(i)}$ denotes the Pauli Z operator on the i^{th} qubit. Then the effective error is a (unitary) Z-rotation on each qubit by some (small) angle θ.

While it is possible to address coherent noise through active error correction, it can be more economical to passively mitigate such noise through decoherence free subspaces (DFSs) [4]. In such schemes, one designs a computational subspace of the full n-qubit Hilbert space which is unperturbed by the noise. In the language of stabilizer codes, we require the noise to preserve the code space, and to act trivially (as the logical identity operator) on the protected information. Inspired by the aforementioned Hamiltonian, which is physically motivated by technologies such as trapped-ion systems, we develop necessary and sufficient conditions for *all* transversal Z-rotations to preserve the code space of a stabilizer code, i.e., $\exp(i\theta H)\rho\exp(i\theta H)^\dagger = \rho$ for all code states ρ in the stabilizer code. When all angles preserve the code space, the logical action must be trivial for any error-detecting stabilizer code [5]. The conditions we derive build upon previous work specifying conditions for a given transversal Z-rotation in the Clifford hierarchy [6], [7], [8] to preserve the code space of a stabilizer code [9]. The key challenge is handling the trigonometric constraints, and we exploit the celebrated MacWilliams identities [10] in classical coding theory for this purpose. The conditions we derive lead to the construction of a family of CSS codes with constant rate or growing distance. A product structure with DFS components provides resilience to coherent noise. Note that while our Z-DFS family is CSS, all our conditions apply to general stabilizer codes.

Ouyang [11] provided a method of addressing coherent phase errors by pairing two qubits to convert their collective interactions to a global phase, in which a $[[2n, k, d]]$ non-stabilizer constant-excitation code is formed by concatenation of an $[[n, k, d]]$ stabilizer outer code with dual-rail inner code. This approach has the disadvantage of producing a non-stabilizer code which makes syndrome extraction and decoding difficult. We avoid these decoding issues by deriving necessary and sufficient conditions for a stabilizer code to be oblivious to coherent phase errors. Based on the conditions and given any $[[n, k, d]]$ CSS code, we are able to construct a new $[[2n, k, d']]$ CSS code, oblivious to coherent noise, with $d' \geq d$.

II. Preliminaries and Notations

A. The Pauli Group

There are four single qubit Pauli operators

$$I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and $\sigma_Y := \imath\sigma_X\sigma_Z$, where $\imath = \sqrt{-1}$. $\sigma_X^2 = \sigma_Y^2 = \sigma_Z^2 = I_2, \sigma_X\sigma_Y = -\sigma_Y\sigma_X, \sigma_X\sigma_Z = -\sigma_Z\sigma_X$, and $\sigma_Y\sigma_Z = -\sigma_Z\sigma_Y$.

Let $A \otimes B$ denote the Kronecker product (tensor product) of two matrices A and B. For any binary vectors $\mathbf{a} =$

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$[\alpha_1, \alpha_2, \cdots, \alpha_n]$ and $\mathbf{b} = [\beta_1, \beta_2, \cdots, \beta_n]$ in $\mathbb{F}_2^n$, where $\mathbb{F}_2 = \{0, 1\}$ is the finite field of size 2, we define the operators

$$D(\mathbf{a}, \mathbf{b}) := \sigma_X^{\alpha_1}\sigma_Z^{\beta_1} \otimes \sigma_X^{\alpha_2}\sigma_Z^{\beta_2} \otimes \cdots \otimes \sigma_X^{\alpha_n}\sigma_Z^{\beta_n},$$
$$E(\mathbf{a}, \mathbf{b}) := \imath^{\mathbf{a}\mathbf{b}^T \bmod 4} D(\mathbf{a}, \mathbf{b}).$$

Note that $D(\mathbf{a}, \mathbf{b})$ can have order $1, 2$ or 4, but $E(\mathbf{a}, \mathbf{b})^2 = \imath^{2\mathbf{a}\mathbf{b}^T} D(\mathbf{a}, \mathbf{b})^2 = \imath^{2ab^T}(\imath^{2\mathbf{a}\mathbf{b}^T} I_N) = I_N$ ($N = 2^n$). The n-qubit *Pauli group* is defined as

$$\mathcal{HW}_N := \{\imath^\kappa D(\mathbf{a}, \mathbf{b}) | \mathbf{a}, \mathbf{b} \in \mathbb{F}_2^n, \kappa = 0, 1, 2, 3\}.$$

The basis states of a single qubit in $\mathbb{C}^2$ are represented by *Dirac notation*, $|\cdot\rangle$. For any $\mathbf{v} = [v_1, v_2, \cdots, v_n] \in \mathbb{F}_2^n$, we define $|\mathbf{v}\rangle = |v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle$, which is the standard basis vector in $\mathbb{C}^N$ with 1 in the position indexed by $\mathbf{v}$ and 0 elsewhere. An arbitrary n-qubit quantum state can be written as $|\psi\rangle = \sum_{\mathbf{v} \in \mathbb{F}_2^n} \alpha_\mathbf{v} |\mathbf{v}\rangle \in \mathbb{C}^N$, where $\alpha_\mathbf{v} \in \mathbb{C}$ and $\sum_{\mathbf{v} \in \mathbb{F}_2^n} |\alpha_\mathbf{v}|^2 = 1$. The Pauli matrices act on a single qubit as

$$\sigma_X|0\rangle = |1\rangle, \sigma_X|1\rangle = |0\rangle, \sigma_Z|0\rangle = |0\rangle, \text{ and } \sigma_Z|1\rangle = -|1\rangle.$$

Define $\langle [\mathbf{a}, \mathbf{b}], [\mathbf{c}, \mathbf{d}] \rangle_S = \mathbf{a}\mathbf{d}^T + \mathbf{b}\mathbf{c}^T \pmod 2$ and using the relation $\sigma_X\sigma_Z = -\sigma_Z\sigma_X$ we have (see [12])

$$E(\mathbf{a}, \mathbf{b})E(\mathbf{c}, \mathbf{d}) = (-1)^{\langle [\mathbf{a}, \mathbf{b}], [\mathbf{c}, \mathbf{d}] \rangle_S} E(\mathbf{c}, \mathbf{d})E(\mathbf{a}, \mathbf{b}).$$

B. The Clifford Hierarchy

The *Clifford hierarchy* of unitary operators was defined in [6]. The first level of the hierarchy is defined to be the Pauli group $\mathcal{C}^{(1)} = \mathcal{HW}_N$. For $l \geq 2$, the levels l are defined recursively as

$$\mathcal{C}^{(l)} := \{U \in \mathbb{U}_N : UE(\mathbf{a}, \mathbf{b})U^\dagger \in \mathcal{C}^{(l-1)}, \forall\, E(a, b) \in \mathcal{HW}_N\},$$

where $\mathbb{U}_N$ is the group of $N \times N$ unitary matrices. The second level is the Clifford Group, $\mathcal{C}^{(2)}$, which can be generated using the unitaries *Hadamard*, *Phase*, and Controlled-NOT (CX) defined respectively as

$$H := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, P := \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix}, CNOT := \begin{bmatrix} I_2 & 0 \\ 0 & \sigma_X \end{bmatrix}.$$

It is well-known that Clifford unitaries along with any unitary from a higher level can be used to approximate any unitary operator arbitrarily well [13]. Hence, they form a universal set for quantum computation. A widely used choice for the non-Clifford unitary is the T gate defined as

$$T := \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{\imath\pi}{4}} \end{bmatrix} = \sqrt{P} = \sigma_Z^{\frac{1}{4}} \equiv \begin{bmatrix} e^{-\frac{\imath\pi}{8}} & 0 \\ 0 & e^{\frac{\imath\pi}{8}} \end{bmatrix} = e^{-\frac{\imath\pi}{8}\sigma_Z}.$$

C. Stabilizer Codes

We define a stabilizer group $\mathcal{S}$ to be a commutative subgroup of the Pauli group $\mathcal{HW}_N$ with Hermitian elements that does not include $-I_N$. We say $\mathcal{S}$ has dimension r if it can be generated by r independent elements as $\mathcal{S} = \langle \mu_i E(\mathbf{c_i}, \mathbf{d_i}) : i = 1, 2, \ldots, r \rangle$, where $\mu_i \in \{\pm 1\}$ and $\mathbf{c_i}, \mathbf{d_i} \in \mathbb{F}_2^n$. Since $\mathcal{S}$ is commutative, we must have $\langle [\mathbf{c_i}, \mathbf{d_i}], [\mathbf{c_j}, \mathbf{d_j}] \rangle_S = \mathbf{c_i}\mathbf{d_j}^T + \mathbf{d_i}\mathbf{c_j}^T = 0 \pmod 2$.

Given a stabilizer group $\mathcal{S}$, the corresponding *stabilizer code* is defined as $\mathcal{V}(\mathcal{S}) := \{|\psi\rangle \in \mathbb{C}^N : g|\psi\rangle = |\psi\rangle \text{ for all } g \in \mathcal{S}\}$, which is the subspace spanned by all eigenvectors in the common eigenbasis of $\mathcal{S}$ that have eigenvalue $+1$. The subspace $\mathcal{V}(\mathcal{S})$ is called an $[[n, k, d]]$ stabilizer code because it encodes $k := n - r$ logical qubits into n *physical* qubits. The minimum distance d is defined to be the minimum weight of any operator in $\mathcal{N}_{\mathcal{HW}_N}(\mathcal{S}) \setminus \mathcal{S}$. Here, the weight of a Pauli operator is the number of qubits on which it acts non-trivially (i.e., as σ_X, σ_Y or σ_Z) and $\mathcal{N}_{\mathcal{HW}_N}(\mathcal{S})$ denotes the normalizer of $\mathcal{S}$ in $\mathcal{HW}_N$ as $\mathcal{N}_{\mathcal{HW}_N}(\mathcal{S}) := \{\imath^\kappa E(\mathbf{a}, \mathbf{b}) \in \mathcal{HW}_N : E(\mathbf{a}, \mathbf{b}) E(\mathbf{c}, \mathbf{d}) E(\mathbf{a}, \mathbf{b}) = E(\mathbf{c}, \mathbf{d}) \text{ for all } E(\mathbf{c}, \mathbf{d}) \in \mathcal{S}, \kappa \in \{0, 1, 2, 3\}\}$.

For any Hermitian Pauli matrix $E(\mathbf{c}, \mathbf{d})$ and $\nu \in \{\pm 1\}$, $\frac{I_N + \nu E(\mathbf{c}, \mathbf{d})}{2}$ is the projector on to the ν-eigenspace of $E(\mathbf{c}, \mathbf{d})$. Thus, the projector on to the codespace $\mathcal{V}(\mathcal{S})$ of the stabilizer code defined by $\mathcal{S} = \langle \mu_i E(\mathbf{c_i}, \mathbf{d_i}) : i = 1, 2, \ldots, r \rangle$ is

$$\Pi_s = \prod_{i=1}^{r} \frac{(I_N + \nu_i E(\mathbf{c_i}, \mathbf{d_i}))}{2} = \frac{1}{2^r} \sum_{j=1}^{2^r} \epsilon_j E(\mathbf{a_j}, \mathbf{b_j}),$$

where $\epsilon_j \in \{\pm 1\}$ is a character of the group $\mathcal{S}$, and is determined by the signs of the generators that produce $E(\mathbf{a_j}, \mathbf{b_j})$: $\epsilon_j E(\mathbf{a_j}, \mathbf{b_j}) = \prod_{t \in J \subset \{1, 2, \ldots, r\}} \nu_t E(\mathbf{c_t}, \mathbf{d_t})$ for a unique J.

D. CSS Codes

A *CSS (Calderbank-Shor-Steane) code* is a special type of stabilizer code defined by a stabilizer $\mathcal{S}$ whose generators can be split into strictly X-type and Z-type operators. Consider two classical binary codes $\mathcal{C}_1, \mathcal{C}_2$ such that $\mathcal{C}_2 \subset \mathcal{C}_1$, and let $\mathcal{C}_1^\perp, \mathcal{C}_2^\perp$ denote the dual spaces of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Note that $\mathcal{C}_1^\perp \subset \mathcal{C}_2^\perp$. The corresponding CSS code has the stabilizer group

$$\mathcal{S} = \langle \nu_c E(\mathbf{c}, \mathbf{0}), \nu_d E(\mathbf{0}, \mathbf{d}), c \in \mathcal{C}_2, d \in \mathcal{C}_1^\perp \rangle$$

for some suitable $\nu_c, \nu_d \in \{\pm 1\}$. If $\mathcal{C}_1$ is an $[n, k_1]$ code and $\mathcal{C}_2$ is an $[n, k_2]$ code such that $\mathcal{C}_1$ and $\mathcal{C}_2^\perp$ can correct up to t errors, then $\mathcal{S}$ defines an $[[n, k_1 - k_2, d]]$ CSS code with $d \geq 2t + 1$, which we will represent as CSS$(X, \mathcal{C}_2; Z, \mathcal{C}_1^\perp)$. If G_2 and $G_1^\perp$ are the generator matrices for $\mathcal{C}_2$ and $\mathcal{C}_1^\perp$ respectively, then a binary generator matrix for $\mathcal{S}$ can be written as the $(n - k_1 + k_2) \times (2n)$ matrix

$$G_\mathcal{S} = \left[\begin{array}{c|c} G_2 & \\ \hline & G_1^\perp \end{array}\right].$$

E. The MacWilliams Identities

We denote the Hamming weight of a binary vector $\mathbf{v}$ by $\text{wgt}(\mathbf{v})$. The weight enumerator of a binary linear code $\mathcal{C} \subset \mathbb{F}_2^m$ is the polynomial

$$P_\mathcal{C}(x, y) = \sum_{\mathbf{v} \in \mathcal{C}} x^{m - \text{wgt}(\mathbf{v})} y^{\text{wgt}(\mathbf{v})}.$$

The MacWilliams Identities [10] relate the weight enumerator of a code $\mathcal{C}$ to that of the dual code $\mathcal{C}^\perp$

$$P_\mathcal{C}(x, y) = \frac{1}{|\mathcal{C}^\perp|} P_{\mathcal{C}^\perp}(x + y, x - y).$$

We frequently make the substitution $x = \cos\frac{2\pi}{2^l}$ and $y = \imath \sin\frac{2\pi}{2^l}$ for some integer l, and we define

$$P[\mathcal{C}] := P_\mathcal{C}\left(\cos\frac{2\pi}{2^l}, \imath \sin\frac{2\pi}{2^l}\right)$$

$$= \sum_{\mathbf{v}\in\mathcal{C}} \left(\cos\frac{2\pi}{2^l}\right)^{m-\mathrm{wgt}(\mathbf{v})} \left(\imath\sin\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})}.$$

III. Main Contributions

Recall that we want to find conditions that render stabilizer codes oblivious to coherent errors. Note that if an error-detecting code can implement the transversal $\exp(\imath\theta\sigma_Z)$ for a sequence of θ approaching 0, then it must implement the logical identity [5], since we treat infinitesimally small transversal rotations as a sum of single-qubit errors. For each $l \geq 3$, Rengaswamy et al. [9] provided necessary and sufficient conditions for a stabilizer code to be invariant under a transversal $\frac{\pi}{2^l}$ Z-rotation, and the conditions are expressed as two trigonometric constraints on the binary code formed by Z-stabilizers supported on the non-zero X-component ($\mathbf{a_j}$) of any stabilizer (denoted as Z_j) and its cosets. In order to be oblivious to coherent noise, we need to design stabilizer codes satisfying these two trigonometric conditions for all $l \geq 3$.

Theorem 1. *[9, Theorem 17] Let $\mathcal{S} = \langle \nu_i E(\mathbf{c_i}, \mathbf{d_i}); i = 1, \ldots, r\rangle$ define an $[[n, n-r]]$ stabilizer code, where $\nu_i \in \{\pm 1\}$. For any $\epsilon_j E(\mathbf{a_j}, \mathbf{b_j}) \in S$ with non-zero a_j, we define*

$$\mathcal{Z}_j := \{\mathbf{z} \in \mathbb{F}_2^{\mathrm{wgt}(\mathbf{a_j})} \colon \epsilon_{\tilde{\mathbf{z}}} E(\mathbf{0}, \tilde{\mathbf{z}}) \in S \text{ and } \tilde{\mathbf{z}} \preceq \mathbf{a_j}\}, \quad (1)$$

where $\tilde{\mathbf{z}} \in \mathbb{F}_2^n$ with $\tilde{\mathbf{z}}|_{\mathrm{supp}(\mathbf{a_j})} = \mathbf{z}$ and constantly zeros outside the support of $\mathbf{a_j}$. Then the transversal application of the $\exp\left(\frac{\imath\pi}{2^l}\sigma_Z\right)$ gate ($l \geq 3$) realizes a logical operation on $\mathcal{V}(\mathcal{S})$ if and only if the following are true for all such $\mathbf{a_j} \neq 0$:

$$\sum_{\mathbf{v}\in Z_j} \epsilon_{\mathbf{v}} \left(\imath\tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} = \left(\sec\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{a_j})}, \quad (2)$$

$$\sum_{\mathbf{v}\in Z_j} \epsilon_{\mathbf{v}} \left(\imath\tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v}\oplus\boldsymbol{w})} = 0 \quad \text{for all } \boldsymbol{w} \in \mathcal{O}_j \quad (3)$$

where $\epsilon_{\mathbf{v}} = \epsilon_{\tilde{\mathbf{v}}} \in \{\pm 1\}$ is the sign of $E(0, \tilde{\mathbf{v}})$ in the stabilizer group $\mathcal{S}$ and $\mathcal{O}_j := \mathbb{F}_2^{\mathrm{wgt}(\mathbf{a_j})} \setminus \mathcal{Z}_j$.

If the signs of the Z-stabilizers in Z_j are all one, the first trigonometric condition states that the weight enumerator polynomial evaluated at $x = \cos\frac{2\pi}{2^l}$ and $y = \imath\sin\frac{2\pi}{2^l}$ is equal to 1. We now use the MacWilliams Identities [10] to translate the trigonometric constraints into divisibility conditions on Hamming weights of vectors in $Z_j^{\perp}$. We denote the length m vector whose entries are all 0 (resp. 1) by $\underline{\mathbf{0}}_m$ (resp. $\underline{\mathbf{1}}_m$).

Lemma 1. *Let $\mathcal{C}$ be a binary linear code with block length M, where all weights are even. Let $l \geq 3$. Then,*

$$\sum_{\mathbf{v}\in\mathcal{C}} \left(\imath\tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} = \left(\sec\frac{2\pi}{2^l}\right)^{m} \quad (4)$$

if and only if $(m - 2\,\mathrm{wgt}(\mathbf{w}))$ is divisible by 2^l for all $\mathbf{w} \in \mathcal{C}^{\perp}$.

Proof. We rewrite (4) as

$$P[\mathcal{C}] = \sum_{\mathbf{v}\in\mathcal{C}} \left(\cos\frac{2\pi}{2^l}\right)^{m-\mathrm{wgt}(\mathbf{v})} \left(\imath\sin\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} = 1. \quad (5)$$

After applying the MacWilliams Identities, (5) becomes

$$\frac{1}{|\mathcal{C}^{\perp}|} P_{\mathcal{C}^{\perp}}(x_{\mathrm{new}}, y_{\mathrm{new}}) = 1, \quad (6)$$

where $x_{\mathrm{new}} = \cos\frac{2\pi}{2^l} + \imath\sin\frac{2\pi}{2^l}$ and $y_{\mathrm{new}} = \cos\frac{2\pi}{2^l} - \imath\sin\frac{2\pi}{2^l}$.

We may rewrite (6) as

$$\frac{1}{|\mathcal{C}^{\perp}|} \sum_{\mathbf{w}\in\mathcal{C}^{\perp}} x_{\mathrm{new}}^{m-\mathrm{wgt}(\mathbf{w})} y_{\mathrm{new}}^{\mathrm{wgt}(\mathbf{w})} = 1, \quad (7)$$

which can be further simplifed as

$$\frac{1}{|\mathcal{C}^{\perp}|} \sum_{\mathbf{w}\in\mathcal{C}^{\perp}} x_{\mathrm{new}}^{m-2\,\mathrm{wgt}(\mathbf{w})} = 1, \quad (8)$$

since $(\cos\theta + \imath\sin\theta)(\cos\theta - \imath\sin\theta) = 1$ for all θ. Note that $\underline{\mathbf{1}}_m \in \mathcal{C}^{\perp}$, so the complement of a codeword is again a codeword in $\mathcal{C}$, and we may rewrite (8) as

$$\frac{1}{|\mathcal{C}^{\perp}|} \left[\sum_{\mathbf{w}\in\mathcal{C}^{\perp}} x_{\mathrm{new}}^{m-2\,\mathrm{wgt}(\mathbf{w})} + \sum_{\mathbf{w}\in\mathcal{C}^{\perp}} x_{\mathrm{new}}^{-(m-2\,\mathrm{wgt}(\mathbf{w}))}\right] = 2. \quad (9)$$

Since $(\cos\theta + \imath\sin\theta)^n = e^{\imath n\theta}$, for all θ, (9) reduces to

$$\frac{1}{|\mathcal{C}^{\perp}|} \sum_{\mathbf{w}\in\mathcal{C}^{\perp}} \cos\left(\frac{2\,(m - 2\,\mathrm{wgt}\,(\mathbf{w}))\,\pi}{2^l}\right) = 1. \quad (10)$$

We observe that (10) is satisfied if and only if each term contributes 1 to the sum, and this is equivalent to 2^l dividing $m - 2\,\mathrm{wgt}(\mathbf{w})$ for all codewords $\mathbf{w}$ in $\mathcal{C}^{\perp}$. ∎

If half the signs are positive and half negative, then the trigonometric condition is a linear combination of weight enumerators, and the same method of analysis applies.

Lemma 2. *If $\mathcal{W}$ is the $[m, m-1]$ code consisting of all vectors with even weight, and if $\epsilon_{\mathbf{v}} = (-1)^{\mathbf{v}\mathbf{y}^T}$ is a character on $\mathcal{W}$, then*

$$\sum_{\mathbf{v}\in\mathcal{W}} \epsilon_{\mathbf{v}} \left(\imath\tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} = \cos\gamma \cdot \left(\sec\frac{2\pi}{2^l}\right)^{m}, \quad (11)$$

where $\gamma = \frac{2\pi(M - 2\,\mathrm{wgt}(\mathbf{y}))}{2^l}$.

Proof. If ϵ is the trivial character, then $\mathbf{y} = \underline{\mathbf{0}}_m$, we have

$$\frac{\sum_{\mathbf{v}\in\mathcal{W}} \left(\imath\tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})}}{\left(\sec\frac{2\pi}{2^l}\right)^m} = P[\mathcal{W}]. \quad (12)$$

We apply the MacWilliams Identities to obtain

$$\begin{aligned} P[\mathcal{W}] &= \frac{1}{|\mathcal{W}^{\perp}|} P_{\mathcal{W}^{\perp}}\left(\cos\frac{2\pi}{2^l} + \imath\sin\frac{2\pi}{2^l}, \cos\frac{2\pi}{2^l} - \imath\sin\frac{2\pi}{2^l}\right) \\ &= \frac{1}{|\mathcal{W}^{\perp}|} P_{\mathcal{W}^{\perp}}\left(e^{\imath\frac{2\pi}{2^l}}, e^{-\imath\frac{2\pi}{2^l}}\right) \\ &= \frac{1}{2}\left[\left(e^{\imath\frac{2\pi}{2^l}}\right)^m \left(e^{-\imath\frac{2\pi}{2^l}}\right)^0 + \left(e^{\imath\frac{2\pi}{2^l}}\right)^0 \left(e^{-\imath\frac{2\pi}{2^l}}\right)^m\right] \\ &= \cos\frac{2\pi m}{2^l}, \end{aligned} \quad (13)$$

which means

$$\sum_{\mathbf{v}\in\mathcal{W}} \left(\imath\tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} = \cos\frac{2\pi M}{2^l} \left(\sec\frac{2\pi}{2^l}\right)^m. \quad (14)$$

If ϵ is a non-trivial character, then there exists $\mathbf{y} \in \mathbb{F}_2^m$ with $\mathbf{y} \neq \underline{\mathbf{0}}_m$ or $\underline{\mathbf{1}}_m$ such that

$$\mathcal{B} = \{\mathbf{v} \in W | \epsilon_{\mathbf{v}} = 1\} = \langle \underline{\mathbf{1}}_m, \mathbf{y}\rangle^{\perp}, \quad (15)$$

and

$$\mathcal{B}^{\perp} = \langle \underline{\mathbf{1}}_m, \mathbf{y} \rangle = \{\underline{\mathbf{0}}_m, \underline{\mathbf{1}}_m, \mathbf{y}, \underline{\mathbf{1}}_m \oplus \mathbf{y}\}. \quad (16)$$

Note that $|\mathcal{B}| = \frac{|\mathcal{W}|}{2}$ and $|\mathcal{B}^{\perp}| = 2|\mathcal{W}^{\perp}|$. We rewrite

$$\sum_{\mathbf{v}\in\mathcal{W}} \epsilon_{\mathbf{v}} \left(\imath \tan \frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} \quad (17)$$

$$= \sum_{\mathbf{v}\in\mathcal{B}} \left(\imath \tan \frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} - \sum_{\mathbf{v}\in\mathcal{W}\setminus\mathcal{B}} \left(\imath \tan \frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} \quad (18)$$

$$= 2\sum_{\mathbf{v}\in\mathcal{B}} \left(\imath \tan \frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} - \sum_{\mathbf{v}\in\mathcal{W}} \left(\imath \tan \frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})}, \quad (19)$$

so that

$$\frac{\sum_{\mathbf{v}\in\mathcal{W}} \epsilon_{\mathbf{v}} \left(\imath \tan \frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})}}{\left(\sec \frac{2\pi}{2^l}\right)^m} = 2P[\mathcal{B}] - P[\mathcal{W}]. \quad (20)$$

We apply the MacWilliams Identities to obtain

$$P[\mathcal{B}] = \frac{1}{|\mathcal{B}^{\perp}|} P_{\mathcal{B}^{\perp}}\left(e^{\imath\frac{2\pi}{2^l}}, e^{-\imath\frac{2\pi}{2^l}}\right) \quad (21)$$

$$= \frac{1}{2}\left[\cos\frac{2\pi m}{2^l} + \cos\frac{2\pi(m - 2\,\mathrm{wgt}(\mathbf{y}))}{2^l}\right]. \quad (22)$$

We combine with (14) to obtain

$$2P[\mathcal{B}] - P[\mathcal{W}] = \cos\frac{2\pi\,(m - 2\,\mathrm{wgt}\,(\mathbf{y}))}{2^l} \quad (23)$$

as required. ■

Lemma 3. *[5, Lemma 6] Let ϵ be a non-trivial character of $\mathbb{F}_2^m$, $\mathcal{B} = \{\mathbf{v} \in \mathcal{W} | \epsilon_{\mathbf{v}} = 1\} = \langle \underline{\mathbf{1}}_m, \mathbf{y}\rangle^{\perp}$, and $\mathcal{B}' = \{\mathbf{x} \in \mathbb{F}_2^m | \epsilon_{\mathbf{x}} = 1\}$. If $\mathcal{W}$ is the $[m, m-1]$ code consisting of all vectors with even weight, then*

$$\sum_{\mathbf{v}\in\mathbb{F}_2^m\setminus\mathcal{W}} \epsilon_{\mathbf{v}} \left(\imath \tan \frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} = \imath \sin\gamma \cdot \left(\sec\frac{2\pi}{2^l}\right)^m, \quad (24)$$

where $\gamma = \frac{2\pi(m-2\,\mathrm{wgt}(\mathbf{y}))}{2^l}$.

Suppose that every qubit is in the support of some stabilizer $\epsilon_j E(\mathbf{a_j}, \mathbf{b_j})$. When the trigonometric conditions are satisfied for all $l \geq 3$, we show that the weight 2 Z-stabilizers cover all the qubits. We define a graph Γ with n vertices representing n qubits, where two vertices are joined by an edge if there exists a weight 2 Z-stabilizer involving those two qubits.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ be the connected components of Γ, and let $N_k = |\Gamma_k|$ be even for $k = 1, 2, \ldots, t$. We observe that each Γ_k is a complete graph. Hence, the weight 2 Z-stabilizers in each Γ_k span the $[N_k, N_k - 1, 2]$ binary single-parity-check code $\mathcal{W}_k$, which contains all vectors of even weight. Note that the signs are multiplicative, and if half of them are positive and half negative, then ϵ_v, the sign of Z-stabilizer $E(\underline{\mathbf{0}}_{N_k}, \mathbf{v})$, takes the form $\epsilon_{\mathbf{v}} = (-1)^{\mathbf{v}\mathbf{u}^T}$ for some $\mathbf{u} \in \mathbb{F}_2^n$. We write $\mathbf{u} = \sum_{k=1}^t \tilde{\mathbf{u}}_{\mathbf{k}}$ where $\tilde{\mathbf{u}}_{\mathbf{k}} \in \mathbb{F}_2^n$ is supported on the qubits in Γ_k, and we use $\mathbf{u_k} \in \mathbb{Z}_2^{N_k}$ to denote the projection of $\tilde{\mathbf{u}}_{\mathbf{k}}$ to Γ_k. We calculate the trigonometric conditions on each Γ_k for $k = 1, \cdots, t$ separately, and then glue them together.

Let $\mu_i = \pm 1$ for $i = 1, \ldots, r$ and let $\mathcal{S} = \langle \nu_i E(\mathbf{c_i}, \mathbf{d_i}); i = 1, \ldots, r\rangle$ define an $[[n, n-r]]$ stabilizer code. Let $\epsilon_j E(\mathbf{a_j}, \mathbf{b_j}) \in \mathcal{S}$ be a stabilizer with $\mathbf{a_j} \neq \underline{\mathbf{0}}_n$. We define $(\Delta^j)_k := 1$ if $\Gamma_k \subseteq \mathrm{supp}(\mathbf{a_j})$ and $(\Delta^j)_k := 0$ if $\Gamma_k \cap \mathrm{supp}(\mathbf{a_j}) = \emptyset$.

Theorem 2. *Transversal $\frac{\pi}{2^l}$ Z-rotation preserves the stabilizer code for all $l \geq 3$ if and only if*

1)
$$\bigcup_{k:(\Delta^j)_k=1} \Gamma_k = \mathrm{supp}(\mathbf{a_j}) \quad (25)$$

2) *N_k is even and $\mathrm{wgt}(\mathbf{u_k}) = \frac{N_k}{2}$ for all k such that $(\Delta^j)_k = 1$.*

Proof of Necessity. We divide the weight 2 stabilizers in Γ_k into two classes of sizes P_k and Q_k where $P_k = |\{\mathbf{v} \in \Gamma_k | \mathrm{wgt}(\mathbf{v}) = 2 \text{ and } \epsilon_{\mathbf{v}} = 1\}|$ and $Q_k = |\{\mathbf{v} \in \Gamma_k | \mathrm{wgt}(\mathbf{v}) = 2 \text{ and } \epsilon_{\mathbf{v}} = -1\}|$. Setting $\mathrm{wgt}(\mathbf{u_k}) = s$, we have

$$Q_k - P_k = \binom{s}{1}\binom{N_k - s}{1} - \left(\binom{s}{2} + \binom{N_k - s}{2}\right) \quad (26)$$

$$= -2\left(s - \frac{N_k}{2}\right)^2 + \frac{N_k}{2}. \quad (27)$$

Thus, $Q_k - P_k \leq \frac{N_k}{2}$, and equality holds if and only if $\mathrm{wgt}(\mathbf{u_k}) = \frac{N_k}{2}$. Theorem 1 implies all $\mathrm{wgt}(\mathbf{a_j})$ are even and

$$\sum_{\mathbf{v}\in\mathcal{Z}_j} \epsilon_{\mathbf{v}} \left(\imath \tan\theta\right)^{\mathrm{wgt}(\mathbf{v})} = (\sec\theta)^{\mathrm{wgt}(\mathbf{a_j})} = (1+(\tan\theta)^2)^{\frac{\mathrm{wgt}(\mathbf{a_j})}{2}} \quad (28)$$

for all $\theta = \frac{\pi}{2^l}$ with $l \geq 3$. Let $\mathcal{Z}_j(2t) = \{\mathbf{z} \in \mathcal{Z}_j | \mathrm{wgt}(\mathbf{z}) = 2t\}$. We have

$$\sum_{t=0}^{\frac{\mathrm{wgt}(\mathbf{a_j})}{2}} \sum_{\mathbf{v}\in\mathcal{Z}_j(2t)} \epsilon_{\mathbf{v}}(-1)^t (\tan\theta)^{2t} = \left(1 + (\tan\theta)^2\right)^{\frac{\mathrm{wgt}(\mathbf{a_j})}{2}}. \quad (29)$$

for all $\theta = \frac{\pi}{2^l}$ with $l \geq 3$. Since this polynomial has infinitely many roots, it is identically zero and we may equate the coefficients of $(\tan\theta)^2$ to obtain

$$\frac{\mathrm{wgt}(\mathbf{a_j})}{2} = \sum_{\mathbf{v}\in Z_j(2)} \epsilon_{\mathbf{v}} \cdot (-1) = \sum_{k:(\Delta^j)_k=1} (Q_k - P_k). \quad (30)$$

It follows from (27) that

$$\frac{\mathrm{wgt}(\mathbf{a_j})}{2} \leq \sum_{k:(\Delta^j)_k=1} \frac{N_k}{2} \leq \frac{\mathrm{wgt}(\mathbf{a_j})}{2}. \quad (31)$$

Therefore equality holds in (31) and $Q_k - P_k = \frac{N_k}{2}$ for all k such that $(\Delta^j)_k = 1$, which complete the proof.

Proof of Sufficiency. Let $\mathcal{W}_k^0$ be the $[N_k, N_k - 1]$ single-parity-check code and let $\mathcal{W}_k^1 = \mathbb{F}_2^{N_k} \setminus \mathcal{W}_k^0$. Let $\mathcal{W}_j = \bigoplus_{k:(\Delta^j)_k=1} \mathcal{W}_k^0$. Then, we observe for $\mathbf{r} \in \mathbb{F}_2^{\mathrm{wgt}(\mathbf{a_j})}/\mathcal{Z}_j$,

$$\sum_{\mathbf{v}\in\mathcal{Z}_j\oplus\mathbf{r}} \epsilon_{\mathbf{v}} \left(\imath \tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} = \sum_{\boldsymbol{\delta}\in(\mathcal{Z}_j/\mathcal{W}_j)\oplus\mathbf{r}} \prod_{\substack{k \\ (\Delta^j)_k=1}} f_{j,k}(\boldsymbol{\delta}), \quad (32)$$

where

$$f_{j,k}(\boldsymbol{\delta}) = \sum_{\boldsymbol{\eta}\in\mathcal{W}_k^{\alpha_{\boldsymbol{\delta}}^k}} (-1)^{\mathbf{u_k}\boldsymbol{\eta}^T} \left(\imath \tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\boldsymbol{\eta})}, \quad (33)$$

and $\alpha_{\boldsymbol{\delta}}^k = 0$ or 1 according as $\mathrm{wgt}(\boldsymbol{\delta}|_{\Gamma_k})$ is even or odd.

Let $\gamma = \frac{2\pi(N_k - 2\,\mathrm{wgt}(\mathbf{u_k}))}{2^l}$. We apply (14) and (24) to simplify (33) as

$$f_{j,k}(\boldsymbol{\delta}) = \begin{cases} \cos\gamma \cdot \left(\sec\frac{2\pi}{2^l}\right)^{N_k} & \text{if } \alpha_{\boldsymbol{\delta}}^k = 0, \\ \imath \sin\gamma \cdot \left(\sec\frac{2\pi}{2^l}\right)^{N_k} & \text{if } \alpha_{\boldsymbol{\delta}}^k = 1, \end{cases}$$
$$= \begin{cases} \left(\sec\frac{2\pi}{2^l}\right)^{N_k} & \text{if } \alpha_{\boldsymbol{\delta}}^k = 0, \\ 0 & \text{if } \alpha_{\boldsymbol{\delta}}^k = 1. \end{cases} \quad (34)$$

To verify (2) in Theorem 1, we see that if $r = \underline{\mathbf{0}}_{\mathrm{wgt}(a_j)}$, the only term that contributes to the outer sum of (32) is the trivial $\boldsymbol{\delta}$, so for all $l \geq 3$

$$\sum_{\mathbf{v}\in\mathcal{Z}_j} \epsilon_v \left(\imath \tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v})} = \prod_{\substack{k \\ (\Delta^j)_k = 1}} \left(\sec\frac{2\pi}{2^l}\right)^{N_k}$$
$$= \left(\sec\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{a_j})}. \quad (35)$$

To verify the second condition, let $\boldsymbol{\omega} \in \mathcal{O}_j = \mathbb{F}_2^{\mathrm{wgt}(\mathbf{a_j})} \setminus \mathcal{Z}_j$ and we change variables to $\boldsymbol{\beta} = \mathbf{v} \oplus \boldsymbol{\omega}$ and $\boldsymbol{\omega}$ on the right hand side (note that we have extended the $\epsilon_{\mathbf{v}}$ to all binary vectors)

$$\sum_{\mathbf{v}\in\mathcal{Z}_j} \epsilon_{\mathbf{v}} \left(\imath \tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\mathbf{v}\oplus\boldsymbol{\omega})} = \epsilon_{\boldsymbol{\omega}} \sum_{\boldsymbol{\beta}\in\boldsymbol{\omega}\oplus\mathcal{Z}_j} \epsilon_{\boldsymbol{\beta}} \left(\imath \tan\frac{2\pi}{2^l}\right)^{\mathrm{wgt}(\boldsymbol{\beta})}$$
$$= \epsilon_{\boldsymbol{\omega}} \sum_{\boldsymbol{\delta}\in(\mathcal{Z}_j\oplus\boldsymbol{\omega})/W_j} \prod_{\substack{k \\ (\Delta^j)_k = 1}} f_{j,k}(\boldsymbol{\delta})$$
$$= 0, \quad (36)$$

for all $l \geq 3$ since the last step follows from $\boldsymbol{\omega} \neq \underline{\mathbf{0}}_{\mathrm{wgt}(a_j)}$ and there is at least one zero factor in the product. ■

Once the code space is preserved by transversal Z rotations from all levels l of the Clifford hierarchy, it is easy to see that the transversal Z rotation of *any* angle preserves the code space as well [5]. Furthermore, for error-detecting stabilizer codes, it can also be seen that this implies that every such transversal Z rotation acts trivially on the code space. Thus, any code that satisfies the above theorem acts as a DFS for a coherent error that acts via the Hamiltonian $H = \sigma_Z^{(1)} + \sigma_Z^{(2)} + \ldots + \sigma_Z^{(n)}$. The code can be seen as the product of all connected components Γ_k, which act as DFS components for this noise.

Remark 1. Given any CSS code, Theorem 2 forces a product structure on the code and provides constraints on the signs. This enables the construction of a family of new CSS codes that is oblivious to coherent noise.

Let $\mathcal{A} \subset \mathcal{B}$ be two classical codes with length t. Let r_1, r_2 be the rates of $\mathcal{A}$, $\mathcal{B}$ respectively. Then, by choosing X-stabilizers to be $\mathcal{A}$ and Z-stabilizers to be $\mathcal{B}^\perp$, we have a $[[t, (r_2 - r_1)t, \geq \min(d_{\min}(\mathcal{B}), d_{\min}(\mathcal{A}^\perp))]]$ CSS code. Let $m \geq 2$ be even, and let W to be the sigle-parity-check $[m, m-1]$ code. Define $\mathcal{C}_2 = \mathcal{A} \otimes \underline{1}_m$ and

$$\mathcal{C}_1^\perp = \{\mathbf{b} \otimes \mathbf{e_1} + \mathbf{w} \mid \mathbf{w} \in \bigoplus_{i=1}^t W \text{ and } \mathbf{b} \in \mathcal{B}^\perp\} \quad (37)$$

to be the X-stabilizers and Z-stabilizers respectively in the new family of CSS codes. By this construction, we ensure that $\mathcal{C}_1^\perp$ includes the direct sum of t single-parity-check codes $\mathcal{W}$ (Condition 1 in Theorem 2). Thus, we can choose $\mathbf{y}$ such that

$$\epsilon_{\mathbf{z_i}} = (-1)^{\mathbf{z_i}\mathbf{y}^T}, \text{ where } \mathrm{wgt}(\mathbf{y}) = \frac{m}{2}. \quad (38)$$

on each component (Condition 2 in Theorem 2). Note that the choice of $\mathbf{y}$ is not unique. Observe that $\dim(\mathcal{C}_1^\perp) = (m-1)t + \dim(\mathcal{B}^\perp) = (m-1)t + (1 - r_2)t$ and $\dim(\mathcal{C}_2) = \dim(\mathcal{A}) = r_1 t$. The number of logical qubits in this new family is $k = mt - \dim(\mathcal{C}_1^\perp) - \dim(\mathcal{C}_2) = (r_2 - r_1)t$. If $\mathbf{x}$ is orthogonal to all Z-stabilizers, then $\mathbf{x}$ has weight at least $m d_{\min}(\mathcal{B})$. If $\mathbf{z}$ is a vector of minimum weight that is orthogonal to all X-stabilizers, then either $\mathbf{z}$ is a Z-stabilizer or $\mathbf{z}$ is a vector from $\mathcal{A}^\perp$ interspersed with appropriate zeros. Thus, the minimum distance of the CSS code is at least $\min(m d_{\min}(\mathcal{B}), d_{\min}(\mathcal{A}^\perp))$. Thus, we have a $[[mt, (r_2 - r_1)t, \geq \min(m d_{\min}(\mathcal{B}), d_{\min}(\mathcal{A}^\perp))]]$ (CSS) QECC family that is oblivious to coherent noise.

Increasing the number of qubits by a factor m makes it possible to design a CSS code that is oblivious to coherent noise. In particular, if we choose $m = 2$, then we generalize Ouyang's construction [11] and are able to provide stabilizer codes with increasing distance. Please see [5] for the generalized construction for stabilizer codes.

An extremal example is to take $\mathcal{B}^\perp = \{\underline{\mathbf{0}}_t\}$ and $\mathcal{A}$ a $[t, t-1]$ single-parity-check code. For fixed t, this pair of $\mathcal{A}$ and $\mathcal{B}$ leads to the maximum $(r_2 - r_1) = t - 1$ logical qubits of the new CSS code, which achieves the maximal rate $(t-1)/2$ by choosing $m = 2$. On the other hand, for $t = 2L$, we may choose the maximal $m = t = 2L$ to obtain the well-known family of $[[4L^2, 1, 2L]]$ Shor codes.

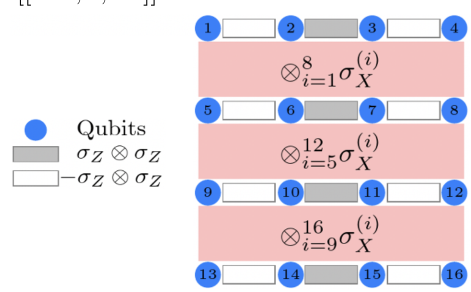

Fig. 1. The $[[16, 1, 4]]$ Shor code constructed by concatenating the $[[4, 1]]$ bit-flip code and the $[[4, 1]]$ phase-flip code. The filled circles represent physical qubits, the white (resp. gray filled) squares represent weight 2 Z-stabilizers with negative (resp. positive) sign, and the three large filled rectangles represent weight 8 X-stabilizers.

Example 1. The connected components $\Gamma_1, \ldots, \Gamma_4$ of this $[[16, 1, 4]]$ Shor code correspond to the 4 qubits in some row. For each component Γ_k, we see that $\mathbf{u_k} = [0, 1, 1, 0]$ satisfies the aforesaid necessary and sufficient condition. Hence, all transversal Z rotations on this code fix the code space and induce the trivial logical identity operation on the single encoded qubit. Moreover, we can see that the $[[16, 1, 4]]$ Shor code is included in the $[[mt, (r_2 - r_1)t, \geq \min(m d_{\min}(\mathcal{B}), d_{\min}(\mathcal{A}^\perp))]]$ CSS family, oblivious to coherent noise, with $t = m = 4$ and $\mathcal{B}^\perp = \{\underline{\mathbf{0}}_4\}$ and $\mathcal{A} = [4, 3]$ single-parity-check code.

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