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Generalized parafermions of orthogonal type

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ABSTRACT

There is an embedding of affine vertex algebras $V^k(\mathfrak{gl}_n) \hookrightarrow V^k(\mathfrak{sl}_{n+1})$, and the coset $\mathcal{C}^k(n) = \mathrm{Com}(V^k(\mathfrak{gl}_n), V^k(\mathfrak{sl}_{n+1}))$ is a natural generalization of the parafermion algebra of $\mathfrak{sl}_2$. It was called the algebra of generalized parafermions by the third author and was shown to arise as a one-parameter quotient of the universal two-parameter $\mathcal{W}_\infty$-algebra of type $\mathcal{W}(2, 3, \dots)$. In this paper, we consider an analogous structure of orthogonal type, namely $\mathcal{D}^k(n) = \mathrm{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}$. We realize this algebra as a one-parameter quotient of the two-parameter even spin $\mathcal{W}_\infty$-algebra of type $\mathcal{W}(2, 4, \dots)$, and we classify all coincidences between its simple quotient $\mathcal{D}_k(n)$ and the algebras $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$ and $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$. As a corollary, we show that for the admissible levels $k = -(2n-2) + \frac{1}{2}(2n + 2m - 1)$ for $\widehat{\mathfrak{so}}_{2n}$ the simple affine algebra $L_k(\mathfrak{so}_{2n})$ embeds in $L_k(\mathfrak{so}_{2n+1})$, and the coset is strongly rational. As a consequence, the category of ordinary modules of $L_k(\mathfrak{so}_{2n+1})$ at such a level is a braided fusion category.

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1. Introduction

For $n \geq 1$, the natural embedding of Lie algebras $\mathfrak{gl}_n \hookrightarrow \mathfrak{sl}_{n+1}$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & -\mathrm{tr}(a) \end{pmatrix},$$

induces a vertex algebra homomorphism

$$V^k(\mathfrak{gl}_n) \hookrightarrow V^k(\mathfrak{sl}_{n+1}). \tag{1.1}$$

The coset vertex algebra

$$\mathcal{C}^k(n) = \mathrm{Com}(V^k(\mathfrak{gl}_n), V^k(\mathfrak{sl}_{n+1})) \tag{1.2}$$

was called the algebra of *generalized parafermions* in [30]. The reason for this terminology is that for $n = 1$, $\mathcal{C}^k(1)$ is isomorphic to the parafermion algebra $N^k(\mathfrak{sl}_2) = \mathrm{Com}(\mathcal{H}, \mathfrak{sl}_2)$, where $\mathcal{H}$ denotes the Heisenberg algebra corresponding to the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}_2$.

By Theorem 8.1 of [30], $\mathcal{C}^k(n)$ is of type $\mathcal{W}(2, 3, \dots, n^2+3n+1)$, i.e., it has a minimal strong generating set consisting of one field in each weight $2, 3, \dots, n^2 + 3n + 1$. This generalizes the case $n = 1$, which appears in [20]. When k is a positive integer, (1.1) descends to a map of simple affine vertex algebras $L_k(\mathfrak{gl}_n) \hookrightarrow L_k(\mathfrak{sl}_{n+1})$, and the coset $\mathrm{Com}(L_k(\mathfrak{gl}_n), L_k(\mathfrak{sl}_{n+1}))$ coincides with the simple quotient $\mathcal{C}_k(n)$ of $\mathcal{C}^k(n)$. By Theorem 13.1 of [6], we have an isomorphism

$$\mathcal{C}_k(n) \cong \mathcal{W}_\ell(\mathfrak{sl}_k), \qquad \ell = -k + \frac{k+n}{k+n+1}. \tag{1.3}$$

In particular, $\mathcal{C}_k(n)$ is *strongly rational*, that is, C_2-cofinite and rational. This generalizes the case $n = 1$, which was proved earlier in [8].

A useful perspective on $\mathcal{C}^k(n)$ is that these algebras all arise in a uniform way as quotients of the universal two-parameter $\mathcal{W}_\infty$-algebra $\mathcal{W}(c, \lambda)$ of type $\mathcal{W}(2, 3, \dots)$; see Theorem 8.2 of [30]. This realization gives a nice conceptual explanation for the isomorphisms appearing in (1.3). Each one-parameter quotient of $\mathcal{W}(c, \lambda)$ corresponds to an ideal in $\mathbb{C}[c, \lambda]$, or equivalently, a curve in the parameter space $\mathbb{C}^2$ called the *truncation curve*. The truncation curves for $\mathcal{W}^\ell(\mathfrak{sl}_m)$ and $\mathcal{C}^k(n)$ are given by Equations 7.8 and 8.4 of [30], and the above isomorphisms correspond to intersection points on these curves.

The algebras $\mathcal{C}^k(n)$ appear naturally as building blocks for affine vertex algebras of type A. It is convenient to replace $\mathcal{C}^k(n)$ with $\tilde{\mathcal{C}}^k(n) = \mathcal{H} \otimes \mathcal{C}^k(n)$, where $\mathcal{H}$ is a rank one Heisenberg vertex algebra. Then we have

$$\mathrm{Com}(V^k(\mathfrak{gl}_{n-1}), V^k(\mathfrak{gl}_n)) \cong \tilde{\mathcal{C}}^k(n-1),$$

so $V^k(\mathfrak{gl}_n)$ can be regarded as an extension of $V^k(\mathfrak{gl}_{n-1}) \otimes \tilde{\mathcal{C}}^k(n-1)$. Iterating this procedure, we see that $V^k(\mathfrak{gl}_n)$ is an extension of

$$\mathcal{H} \otimes \tilde{\mathcal{C}}^k(1) \otimes \tilde{\mathcal{C}}^k(2) \otimes \cdots \otimes \tilde{\mathcal{C}}^k(n-1). \tag{1.4}$$

Note that if k is a positive integer, the simple quotient $L_k(\mathfrak{gl}_n)$ is then an extension of

$$\mathcal{H} \otimes \tilde{\mathcal{C}}_k(1) \otimes \tilde{\mathcal{C}}_k(2) \otimes \cdots \otimes \tilde{\mathcal{C}}_k(n-1) \cong \mathcal{W}_{\ell_1}(\mathfrak{gl}_k) \otimes \mathcal{W}_{\ell_2}(\mathfrak{gl}_k) \otimes \cdots \otimes \mathcal{W}_{\ell_n}(\mathfrak{gl}_k),$$

where $\ell_i = -k + \frac{k+n-i}{k+n+1-i}$. In [6], this was regarded as a noncommutative analogue of the Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_n)$. Similarly, we may regard the subalgebra (1.4) as the universal version of this structure.

The algebras $\mathcal{C}^k(n)$ also appear as building blocks for various $\mathcal{W}$-(super)algebras. For example, an important conjecture of Ito [24] asserts that the principal $\mathcal{W}$-algebra $\mathcal{W}^\ell(\mathfrak{sl}_{n+1|n})$ has a coset realization as

$$\mathrm{Com}(V^{k+1}(\mathfrak{gl}_n), V^k(\mathfrak{sl}_{n+1}) \otimes \mathcal{F}(2n)), \tag{1.5}$$

where $\mathcal{F}(2n)$ denotes the rank $2n$ free fermion algebra, and $(\ell+1)(k+n+1) = 1$. Ito's conjecture was stated in this form in [16], and these algebras have the same strong generating type by Lemma 7.12 of [16]. In the case $n = 1$, the conjecture clearly holds because both sides are isomorphic to the $N = 2$ superconformal algebra. The first non-trivial case $n = 2$ was proven in [22]. It was also shown in [22] that the coset (1.5) is naturally an extension of $\mathcal{W}^r(\mathfrak{gl}_n) \otimes \mathcal{C}^k(n)$ for $r = -n + \frac{n+k}{n+k+1}$. An important ingredient in the proof of Ito's conjecture will be to show that $\mathcal{W}^\ell(\mathfrak{sl}_{n+1|n})$ is indeed an extension of $\mathcal{W}^r(\mathfrak{gl}_n) \otimes \mathcal{C}^k(n)$. Note that $\mathcal{C}^k(n)$ is itself a subalgebra of a $\mathcal{W}$-superalgebra of $\mathfrak{sl}_{n+1|n}$ corresponding to a *small hook-type* nilpotent element [17].

Generalized parafermion algebras of orthogonal type There are two different analogues of $\mathcal{C}^k(n)$ in the orthogonal setting. We have natural embeddings $\mathfrak{so}_{2n} \hookrightarrow \mathfrak{so}_{2n+1} \hookrightarrow \mathfrak{so}_{2n+2}$, which induce homomorphisms of affine vertex algebras

$$V^k(\mathfrak{so}_{2n}) \hookrightarrow V^k(\mathfrak{so}_{2n+1}) \hookrightarrow V^k(\mathfrak{so}_{2n+2}). \tag{1.6}$$

The cosets $\mathrm{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}))$ and $\mathrm{Com}(V^k(\mathfrak{so}_{2n+1}), V^k(\mathfrak{so}_{2n+2}))$ both have actions of $\mathbb{Z}_2$, and we define

$$\begin{aligned} \mathcal{D}^k(n) &= \mathrm{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}, \\ \mathcal{E}^k(n) &= \mathrm{Com}(V^k(\mathfrak{so}_{2n+1}), V^k(\mathfrak{so}_{2n+2}))^{\mathbb{Z}_2}. \end{aligned} \tag{1.7}$$

Both these algebras arise as one-parameter quotients of the universal even spin $\mathcal{W}_\infty$-algebra $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ constructed recently by Kanade and the third author in [26]. Such quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ are in bijection with a family of ideals I in the polynomial ring $\mathbb{C}[c,\lambda]$, or equivalently, the truncation curves $V(I) \subseteq \mathbb{C}^2$. The main result in this paper is the explicit description of the truncation curve for $\mathcal{D}^k(n)$ for all n; see Theorem 3.3.

The proof is based on the coset realization of principal $\mathcal{W}$-algebras of type D and a certain level-rank duality appearing in [6], which implies that

$$\mathcal{D}_{2m}(n) \cong \mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}, \qquad \ell = -(2m-2) + \frac{2m+2n-2}{2m+2n-1}. \tag{1.8}$$

Here $\mathcal{D}_{2m}(n)$ denotes the simple quotient of $\mathcal{D}^{2m}(n)$. This is analogous to the isomorphisms (1.3) in type A. Since a similar coset realization of type B principal $\mathcal{W}$-algebras is not available, we are currently unable to obtain an explicit description of $\mathcal{E}^k(n)$, and in this paper we only study $\mathcal{D}^k(n)$.

As in type A, there is a similar description of affine vertex algebras of orthogonal type as extensions of Gelfand-Tsetlin type subalgebras. Clearly $V^k(\mathfrak{so}_{2n+2})$ is an extension of

$$\mathcal{H} \otimes \mathcal{D}^k(1) \otimes \mathcal{E}^k(1) \otimes \mathcal{D}^k(2) \otimes \mathcal{E}^k(2) \otimes \cdots \otimes \mathcal{D}^k(n-1) \otimes \mathcal{E}^k(n-1) \otimes \mathcal{D}^k(n) \otimes \mathcal{E}^k(n),$$

and similarly, $V^k(\mathfrak{so}_{2n+1})$ is an extension of

$$\mathcal{H} \otimes \mathcal{D}^k(1) \otimes \mathcal{E}^k(1) \otimes \mathcal{D}^k(2) \otimes \mathcal{E}^k(2) \otimes \cdots \otimes \mathcal{D}^k(n-1) \otimes \mathcal{E}^k(n-1) \otimes \mathcal{D}^k(n).$$

Additionally, $\mathcal{D}^k(n)$ is a building block for various $\mathcal{W}$-(super)algebras. For example, consider the principal $\mathcal{W}$-superalgebra $\mathcal{W}^\ell(\mathfrak{osp}_{2n|2n})$ where $(\ell+1)(k+2n-1) = 1$. Note that 1 and $2n-1$ are the dual Coxeter numbers of $\mathfrak{osp}_{2n|2n}$ and $\mathfrak{so}_{2n+1}$, respectively. The free fermion algebra $\mathcal{F}(2n)$ carries an action of $L_1(\mathfrak{so}_{2n})$, and it is expected that

$$\mathcal{W}^\ell(\mathfrak{osp}_{2n|2n}) \cong \mathrm{Com}(V^{k+1}(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n)). \tag{1.9}$$

This algebra appears in physics in the duality of $N=1$ superconformal field theories and higher spin supergravities [11,18], and this conjecture appeared in this context. Note that central charges coincide. It is apparent that the coset appearing in (1.9) is an extension of $\mathcal{W}^r(\mathfrak{so}_{2n}) \otimes \mathcal{D}^k(n)$ where $r = -(2n-2) + \frac{k+2n-2}{k+2n-1}$. As in the case of Ito's conjecture, an important step in the proof of (1.9) will be to show that $\mathcal{W}^\ell(\mathfrak{osp}_{2n|2n})$ is also an extension of this structure.

Applications The first application of our main result is to classify all isomorphisms between the simple quotient $\mathcal{D}_k(n)$ and the simple algebras $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$ and $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$. Using results of [26], this can be achieved by finding the intersection points between the truncation curve for $\mathcal{D}^k(n)$, and the truncation curves for $\mathcal{W}^\ell(\mathfrak{so}_{2m+1})$ and $\mathcal{W}^\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$, respectively. In the type A case, we find only one family of points where $\mathcal{C}_k(n)$ is isomorphic to a strongly rational $\mathcal{W}$-algebra of type A; these appear in (1.3). In the orthogonal setting, the situation is more interesting. In addition to the isomorphisms (1.8) when k is a positive integer, we also find that for $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$, we have an embedding of simple affine vertex algebras $L_k(\mathfrak{so}_{2n}) \to L_k(\mathfrak{so}_{2n+1})$, and an isomorphism

$$\mathcal{D}_k(n) = \mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2} \cong \mathcal{W}_\ell(\mathfrak{so}_{2m+1}),$$
$$\ell = -(2m-1) + \frac{2m+2n-1}{2m+2n+1}.$$

Since ℓ is a nondegenerate admissible level for $\mathfrak{so}_{2m+1}$, $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$ is strongly rational [1,2]. These are new examples of cosets of non-rational vertex algebras by admissible level affine vertex algebras, which are strongly rational.

This coset is also closely related to level-rank duality. Recall that $2n(2m+1)$ free fermions carry an action of $L_{2n}(\mathfrak{so}_{2m+1}) \otimes L_{2m+1}(\mathfrak{so}_{2n})$. The levels shifted by the respective dual Coxeter numbers are $2n+2m-1$ in both cases. Therefore $L_k(\mathfrak{so}_{2n+1})$ is an extension of $L_k(\mathfrak{so}_{2n}) \otimes \mathcal{W}_\ell(\mathfrak{so}_{2m+1})$, where $\ell = -(2m-1) + \frac{2m+2n-1}{2m+2n+1}$, i.e., both levels k and ℓ shifted by the respective dual Coxeter numbers are of the form $(2m+2n-1)/v$ for $v = 2$ and $v = 2+2m+2n-1$. In particular, the shifted levels have the same numerator as the original level-rank duality and the two denominators only differ by a multiple of the numerator. Note that under certain vertex tensor category assumptions the tensor product of two vertex algebras can be extended to a larger vertex algebra with a certain multiplicity freeness condition if and only if the two vertex algebras have subcategories that are braid-reversed equivalent, see [14, Main Thm. 3] for the precise statement. Applied to our setting, this means that there are vertex algebra extensions of $L_k(\mathfrak{so}_{2n})$ and $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$ that have subcategories of modules that are braid-reversed equivalent.

The theory of vertex algebra extensions, especially [14, Thm. 5.12], then implies that the category of ordinary modules of $L_k(\mathfrak{so}_{2n+1})$ at level $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$ is fusion, i.e. a rigid braided semisimple tensor category. This proves special cases of Conjecture 1.1 of [12].

Finally, our rationality results for $\mathcal{D}_k(n)$ suggest the existence of a new series of principal $\mathcal{W}$-superalgebras of $\mathfrak{osp}_{2n|2n}$ which are strongly rational. By Corollary 14.2 of [6], the coset $\mathrm{Com}(L_{k+1}(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n))$ is strongly rational when k is a positive integer. In view of the conjectured isomorphism (1.9), this implies that for k a positive integer and ℓ satisfying $(\ell+1)(k+2n-1) = 1$, $\mathcal{W}_\ell(\mathfrak{osp}_{2n|2n})$ is strongly rational. Similarly, it follows from Corollary 1.1 of [14] that for $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$ and ℓ satisfying $(\ell+1)(k+2n-1) = 1$, the coset $\mathrm{Com}(L_{k+1}(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n))$ is again strongly rational. This motivates the following

Conjecture 1.1. *For $k = -(2n-2)+\frac{1}{2}(2n+2m-1)$ and ℓ satisfying $(\ell+1)(k+2n-1) = 1$, $\mathcal{W}_\ell(\mathfrak{osp}_{2n|2n})$ is strongly rational.*

The conjecture is true for the $N = 2$ super Virasoro algebra, i.e. the case $n = 1$ [4]. Otherwise strong rationality for principal $\mathcal{W}$-superalgebras of orthosymplectic type is completely open. There is, however, a C_2-cofiniteness results in the case of $\mathfrak{osp}_{2|2n}$ [10, Cor. 5.19].

2. Vertex algebras

We shall assume that the reader is familiar with vertex algebras, and we use the same notation and terminology as the papers [26,30]. We first recall the universal two-parameter vertex algebra $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ of type $\mathcal{W}(2,4,\dots)$, which was recently constructed in [26]. It is defined over the polynomial ring $\mathbb{C}[c,\lambda]$ and is generated by a Virasoro field L of central charge c, and a weight 4 primary field W^4, and is strongly generated by fields $\{L, W^{2i}|\ i \geq 2\}$ where $W^{2i} = W^4_{(1)}W^{2i-2}$ for $i \geq 3$. The idea of the construction is as follows.

(1) All structure constants in the OPEs of $L(z)W^{2i}(w)$ and $W^{2j}(z)W^{2k}(w)$ for $2i \leq 12$ and $2j+2k \leq 14$, are uniquely determined as elements of $\mathbb{C}[c,\lambda]$ by imposing the Jacobi identities among these fields.
(2) This data uniquely and recursively determines all OPEs $L(z)W^{2i}(w)$ and $W^{2j}(z)W^{2k}(w)$ over the ring $\mathbb{C}[c,\lambda]$ if a certain subset of Jacobi identities are imposed.
(3) By showing that the algebras $\mathcal{W}^k(\mathfrak{sp}_{2m})$ all arise as one-parameter quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ after a suitable localization, we show that all Jacobi identities hold. Equivalently, $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ is freely generated by the fields $\{L, W^{2i}|\ i \geq 2\}$, and is the universal enveloping algebra of the corresponding nonlinear Lie conformal algebra [19].

$\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ is simple as a vertex algebra over $\mathbb{C}[c,\lambda]$, but there is a certain discrete family of prime ideals $I = (p(c,\lambda)) \subseteq \mathbb{C}[c,\lambda]$ for which the quotient

$$\mathcal{W}^{\mathrm{ev},I}(c,\lambda) = \mathcal{W}^{\mathrm{ev}}(c,\lambda)/I \cdot \mathcal{W}^{\mathrm{ev}}(c,\lambda),$$

is not simple as a vertex algebra over the ring $\mathbb{C}[c,\lambda]/I$. We denote by $\mathcal{W}^{\mathrm{ev}}_I(c,\lambda)$ the simple quotient of $\mathcal{W}^{\mathrm{ev},I}(c,\lambda)$ by its maximal proper graded ideal $\mathcal{I}$. After a suitable localization, all one-parameter vertex algebras of type $\mathcal{W}(2,4,\dots,2N)$ for some N satisfying some mild hypotheses, can be obtained as quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ in this way. This includes the principal $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{so}_{2m+1})$ and the orbifolds $\mathcal{W}^k(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$. The generators $p(c,\lambda)$ for such ideals arise as irreducible factors of Shapovalov determinants, and are in bijection with such one-parameter vertex algebras.

We also consider $\mathcal{W}^{\mathrm{ev},I}(c,\lambda)$ for maximal ideals

$$I = (c - c_0, \lambda - \lambda_0), \qquad c_0, \lambda_0 \in \mathbb{C}.$$

Then $\mathcal{W}^{\mathrm{ev},I}(c,\lambda)$ and its quotients are vertex algebras over $\mathbb{C}$. Given maximal ideals $I_0 = (c - c_0, \lambda - \lambda_0)$ and $I_1 = (c - c_1, \lambda - \lambda_1)$, let $\mathcal{W}_0$ and $\mathcal{W}_1$ be the simple quotients of $\mathcal{W}^{\mathrm{ev},I_0}(c,\lambda)$ and $\mathcal{W}^{\mathrm{ev},I_1}(c,\lambda)$. Theorem 8.1 of [26] gives a simple criterion for $\mathcal{W}_0$ and $\mathcal{W}_1$ to be isomorphic. Aside from a few degenerate cases, we must have $c_0 = c_1$ and $\lambda_0 = \lambda_1$. This implies that aside from the degenerate cases, all other coincidences among the simple quotients of one-parameter vertex algebras $\mathcal{W}^{\mathrm{ev},I}(c,\lambda)$ and $\mathcal{W}^{\mathrm{ev},J}(c,\lambda)$, correspond to intersection points of their truncation curves $V(I)$ and $V(J)$.

We shall need the following result which is analogous to Theorem 6.2 of [30].

Theorem 2.1. *Let $\mathcal{W}$ be a vertex algebra of type $\mathcal{W}(2,4,\dots,2N)$ which is defined over some localization R of $\mathbb{C}[c,\lambda]/I$, for some prime ideal I. Suppose that $\mathcal{W}$ is generated by the Virasoro field L and a weight 4 primary field W^4. If in addition, the graded character of $\mathcal{W}$ agrees with that of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ up to weight 13, then $\mathcal{W}$ is a quotient of $\mathcal{W}^I(c,\lambda)$ after localization.*

Proof. First, note that Theorem 3.10 of [26] holds without the simplicity assumption; see Remark 5.1 of [30] for a similar statement in the case of the algebra $\mathcal{W}(c,\lambda)$ of type $\mathcal{W}(2,3,\dots)$. By Theorem 3.10 of [26], it suffices to prove that the OPEs $L(z)W^{2i}(w)$ and $W^{2j}(z)W^{2k}(w)$ for $2i \leq 12$ and $2j+2k \leq 14$ in $\mathcal{W}$ are the same as the corresponding OPEs in $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ if the structure constants are replaced with their images in R. In this notation, $W^{2i} = W^4_{(1)}W^{2i-2}$ for $i \geq 3$. But this is automatic because the graded character assumption implies that there are no null vectors of weight $w \leq 13$ in the (possibly degenerate) nonlinear conformal algebra corresponding to $\{L, W^{2i} |\ 2 \leq i \leq N\}$. $\square$

3. Generalized parafermions of orthogonal type

For $n \geq 1$, the natural embedding $\mathfrak{so}_{2n} \hookrightarrow \mathfrak{so}_{2n+1}$ induces a vertex algebra homomorphism

$$V^k(\mathfrak{so}_{2n}) \to V^k(\mathfrak{so}_{2n+1}).$$

The action of $\mathfrak{so}_{2n}$ on $V^k(\mathfrak{so}_{2n+1})$ given by the zero modes of the generating fields integrates to an action of the orthogonal group O_{2n}. Therefore the coset

$$\mathrm{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1})) = V^k(\mathfrak{so}_{2n+1})^{\mathfrak{so}_{2n}[t]}$$

has a nontrivial action of $\mathbb{Z}_2$. We define

$$\mathcal{D}^k(n) = \mathrm{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}. \tag{3.1}$$

It has Virasoro element $L^{\mathfrak{so}_{2n+1}} - L^{\mathfrak{so}_{2n}}$ with central charge

$$c = \frac{kn(2k+2n-3)}{(k+2n-2)(k+2n-1)}. \tag{3.2}$$

Note that in the case $n = 1$, $\mathcal{D}^k(n) \cong N^k(\mathfrak{sl}_2)^{\mathbb{Z}_2}$ which is of type $\mathcal{W}(2,4,6,8,10)$ by Theorem 10.1 of [26].

Lemma 3.1. *For all $n \geq 1$, $\mathcal{D}^k(n)$ is of type $\mathcal{W}(2,4,\dots,2N)$ for some N satisfying $N \geq 2n^2+3n$. We conjecture, but do not prove, that $N = 2n^2+3n$. Moreover, for generic values of k, $\mathcal{D}^k(n)$ is generated by the weight 4 primary field W^4.*

Proof. By Theorem 6.10 of [16], we have

$$\lim_{k\to\infty} \mathcal{D}^k(n) \cong \mathcal{H}(2n)^{\mathrm{O}_{2n}},$$

and a strong generating set for $\mathcal{H}(2n)^{\mathrm{O}_{2n}}$ corresponds to a strong generating set for $\mathcal{D}^k(n)$ for generic values of k. Here $\mathcal{H}(2n)$ denotes the rank $2n$ Heisenberg vertex algebra. It was shown in [29], Theorem 6.5, that $\mathcal{H}(2n)^{\mathrm{O}_{2n}}$ has the above strong generating type. By Lemma 4.2 of [28], the weights 2 and 4 fields generate $\mathcal{H}(2n)^{\mathrm{O}_{2n}}$. In fact, it is easy to check that only the weight 4 field is needed, and that it can be replaced with a primary field which also generates the algebra. Finally, the statement that $\mathcal{D}^k(n)$ inherits these properties of $\mathcal{H}(2n)^{\mathrm{O}_{2n}}$ for generic values of k is also clear; the argument is similar to the proof of Corollary 8.6 of [15]. $\square$

Corollary 3.2. *For all $n \geq 1$, there exists an ideal $K_n \subseteq \mathbb{C}[c,\lambda]$ and a localization R_n of $\mathbb{C}[c,\lambda]/K_n$ such that $\mathcal{D}^k(n)$ is the simple quotient of $\mathcal{W}^{\mathrm{ev},K_n}_{R_n}(c,\lambda)$.*

Proof. This holds for $n=1$ by Theorem 10.1 of [26]. For $n>1$, the simplicity of $\mathcal{D}^k(n)$ as a vertex algebra over a localization of $\mathbb{C}[k]$ follows from the simplicity of $\mathcal{H}(2n)^{\mathrm{O}_{2n}}$, which follows from [21]. In view of Theorems 2.1 and 3.1, it then suffices to show that the graded characters of $\mathcal{D}^k(n)$ and $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ agree up to weight 13. This follows from Weyl's second fundamental theorem of invariant theory for O_{2n} [31], since there are no relations among the generators of weight less than $4n^2+6n+2$. $\square$

Theorem 3.3. *For all $n \geq 2$, $\mathcal{D}^k(n)$ is isomorphic to a localization of the quotient $\mathcal{W}^{ev}_{K_n}(c,\lambda)$, where the ideal $K_n \subseteq \mathbb{C}[c,\lambda]$ is described explicitly via the parametrization $k \mapsto (c_n(k),\lambda_n(k))$ given by*

$$\begin{aligned}
c_n(k) &= \frac{kn(2k+2n-3)}{(k+2n-2)(k+2n-1)}, \quad \lambda_n(k) = \frac{(k+2n-2)(k+2n-1)p_n(k)}{7(k-2)(k+n-1)(2n-1)q_n(k)r_n(k)},\\
p_n(k) &= -112+188k-62k^2-26k^3+12k^4+744n-1336kn+857k^2n-252k^3n\\
&\quad +36k^4n-1720n^2+2534kn^2-1198k^2n^2+188k^3n^2+1632n^3-1544kn^3\\
&\quad +304k^2n^3-544n^4+152kn^4,\\
q_n(k) &= 20-19k+6k^2-42n+28kn+28n^2,\\
r_n(k) &= 44-66k+22k^2-132n+73kn+10k^2n+88n^2+10kn^2.
\end{aligned} \tag{3.3}$$

Proof. Let n be fixed. In view of Corollary 3.2 and the fact that all structure constants in $\mathcal{D}^k(n)$ are rational functions of k, there is some rational function $\lambda_n(k)$ of k such that

$\mathcal{D}^k(n)$ is obtained from $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ by setting $c = c_n(k)$ and $\lambda = \lambda_n(k)$, and then taking the simple quotient. It is not obvious yet that $\lambda_n(k)$ is a rational function of n as well.

For k a positive integer, it is well known [25] that the map $V^k(\mathfrak{so}_{2n}) \to V^k(\mathfrak{so}_{2n+1})$ descends to a homomorphism of simple algebras $L_k(\mathfrak{so}_{2n}) \to L_k(\mathfrak{so}_{2n+1})$. Letting $\mathcal{D}_k(n)$ denote the simple quotient of $\mathcal{D}^k(n)$, it is apparent from Lemma 2.1 of [7] and Theorem 8.1 of [16] that $\mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))$ is simple and coincides with the simple quotient of $\mathrm{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}))$. Moreover, taking $\mathbb{Z}_2$-invariants preserves simplicity, hence

$$\mathcal{D}_k(n) \cong \mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}.$$

Next, by Corollary 1.3 of [6], for all $n \geq 1$ and $m \geq 2$, we have an isomorphism

$$\begin{aligned} &\left(\left(L_{2m}(\mathfrak{so}_{2n+1}) \oplus \mathbb{L}_{2m}(2m\omega_1)\right)^{\mathfrak{so}_{2n}[t]}\right)^{\mathbb{Z}_2\times\mathbb{Z}_2} \cong \mathcal{W}_\ell(\mathfrak{so}_{2m}), \\ &\ell = -(2m-2) + \frac{2n+2m-2}{2n+2m-1}. \end{aligned} \tag{3.4}$$

In this notation, ω_1 denotes the first fundamental weight of $\mathfrak{so}_{2n+1}$ and $\mathbb{L}_{2m}(2m\omega_1)$ denotes the simple quotient of the corresponding Weyl module.

Note that $\left(L_{2m}(\mathfrak{so}_{2n+1})^{\mathfrak{so}_{2n}[t]}\right)^{\mathbb{Z}_2} = \mathcal{D}_{2m}(n)$ is manifestly a subalgebra of the left hand side of (3.4). Also, the lowest-weight component of $\mathbb{L}_{2m}(2m\omega_1)$ has conformal weight m. If $m > 4$, the left-hand side then has a unique primary weight 4 field which lies in $\mathcal{D}_{2m}(n)$. Similarly, since $\mathcal{W}_\ell(\mathfrak{so}_{2m})$ has strong generators in weights $2, 4, \dots, 2m$ and m, for $m > 4$ the right hand side has a unique primary weight 4 field, which lies in the $\mathbb{Z}_2$-orbifold $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$.

Since $\mathcal{D}^k(n)$ is generated by the weight 4 field as a one-parameter vertex algebra, the weight 4 field must generate $\mathcal{D}_{2m}(n)$ for all m sufficiently large. By Corollary 6.1 of [26], $\mathcal{W}^\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$ is generated by the weight 4 field as a one-parameter vertex algebra; equivalently, this holds for generic values of ℓ. By the same argument as Proposition A.4 of [8], the vertex Poisson structure on the associated graded algebra $\mathrm{gr}\, \mathcal{W}^\ell(\mathfrak{so}_{2m})$ with respect to Li's canonical filtration, is independent of ℓ for all noncritical values of ℓ. In particular this holds for the subalgebra $(\mathrm{gr}\, \mathcal{W}^\ell(\mathfrak{so}_{2m}))^{\mathbb{Z}_2} = \mathrm{gr}(\mathcal{W}^\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2})$. It follows from the same argument as Proposition A.3 of [8] that $\mathcal{W}^\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$ is generated by the weights 2 and 4 fields for all noncritical values of ℓ, and the same therefore holds for the simple quotient $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$. Finally, for $\ell = -(2m-2) + \frac{2n+2m-2}{2n+2m-1}$, it is straightforward to verify that the Virasoro field can be generated from the weight 4 field, so the weight 4 field generates the whole algebra.

Therefore if m is sufficiently large, we obtain

$$\mathcal{D}_{2m}(n) \cong \mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}, \qquad \ell = -(2m-2) + \frac{2m+2n-2}{2m+2n-1}. \tag{3.5}$$

In fact, we will see later (Theorem 4.1) that this holds for all $m \geq 2$.

Finally, the truncation curve that realizes $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$ as a quotient of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ is given by Theorem 6.3 of [26], and in parametric form by Equation (B.1) of [26]. In view of (3.5), we must have $\lambda_n(2m) = \lambda_m(\ell)$ for $\ell = -(2m-2) + \frac{2n+2m-2}{2n+2m-1}$ for m sufficiently large, where $\lambda_m(\ell)$ is given by Equation (B.1) of [26]. It follows that for infinitely many values of k, $\lambda_n(k)$ is given by the above formula (3.3). Since $\lambda_n(k)$ is a rational function of k, this equality holds for all k where it is defined. This completes the proof. $\square$

4. Coincidences

In this section, we shall use Theorem 3.3 to classify all coincidences between the simple quotient $\mathcal{D}_k(n)$ and the $\mathbb{Z}_2$-orbifold $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$, as well as $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$. We also classify all coincidences between $\mathcal{D}_k(n)$ and $\mathcal{D}_\ell(m)$ for $m \neq n$.

Theorem 4.1. *For $n \geq 1$ and $m \geq 2$, aside from the critical levels $k = -2n+2$ and $k = -2n+1$, and the degenerate cases $c = \frac{1}{2}, -24$, all isomorphisms $\mathcal{D}_k(n) \cong \mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$ appear on the following list:*

(1) $k = 2m, \qquad \ell = -(2m-2) + \dfrac{2n+2m-2}{2n+2m-1},$

(2) $k = -(2n-2) - \dfrac{2n-1}{2(m-1)}, \qquad \ell = -(2m-2) + \dfrac{2m-2n-1}{2(m-1)},$

(3) $k = -(2n-2) + \dfrac{n-m}{m}, \qquad \ell = -(2m-2) + \dfrac{m-n}{m}.$

Proof. Recall first that $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$ is realized as the simple quotient of $\mathcal{W}^{\mathrm{ev},J_m}(c,\lambda)$, where the ideal $J_m \subseteq \mathbb{C}[c,\lambda]$ is given in parametrized form by Equation (B.1) of [26]. First, we exclude the values of k and ℓ which are poles of the functions $\lambda_n(k)$ given by (3.3), and $\lambda_m(\ell)$ given by Equation (B.1) of [26], since at these values, $\mathcal{D}^k(n)$ and $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$ are not quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$. For all other noncritical values of k and ℓ, $\mathcal{D}^k(n)$ and $\mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$ are obtained as quotients of $\mathcal{W}^{\mathrm{ev},I_n}(c,\lambda)$ and $\mathcal{W}^{\mathrm{ev},J_m}(c,\lambda)$, respectively. By Corollary 8.2 of [26], aside from the degenerate cases given by Theorem 8.1 of [26], all other coincidences $\mathcal{D}_k(n) \cong \mathcal{W}_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$ correspond to intersection points on the truncation curves $V(K_n)$ and $V(J_m)$. A calculation shows that $V(K_n) \cap V(J_m)$ consists of exactly five points (c,λ), namely,

$$\begin{aligned}&\Big(-24, -\frac{1}{245}\Big), \qquad \Big(\frac{1}{2}, -\frac{2}{49}\Big), \qquad \Big(\frac{mn(4m+2n-3)}{(m+n-1)(2m+2n-1)},\ \lambda_1\Big),\\ &\Big(-\frac{2mn(3-4m-2n+4mn)}{2m-2n-1},\ \lambda_2\Big), \qquad \Big(-\frac{(2mn+m-2n)(2mn-m-n)}{m-n},\ \lambda_3\Big).\end{aligned} \tag{4.1}$$

Here

$$\begin{aligned}
\lambda_1 &= \frac{(m+n-1)(2m+2n-1)g}{7(m-1)(2m+n-1)(2n-1)gh}, \\
f &= -28+94m-62m^2-52m^3+48m^4+186n-668mn+857m^2n-504m^3n \\
&\quad +144m^4n-430n^2+1267mn^2-1198m^2n^2+376m^3n^2+408n^3-772mn^3 \\
&\quad +304m^2n^3-136n^4+76mn^4, \\
g &= 10-19m+12m^2-21n+28mn+14n^2, \\
h &= 22-66m+44m^2-66n+73mn+20m^2n+44n^2+10mn^2.
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\lambda_2 &= \frac{(1-2m+2n)f}{7(1-2m+2mn)(-1-2n+4mn)gh}, \\
f &= 14-33m-2m^2+24m^3+74n-404mn+873m^2n-696m^3n+144m^4n \\
&\quad +80n^2-178mn^2-260m^2n^2+452m^3n^2-112m^4n^2-24n^3+264mn^3 \\
&\quad -348m^2n^3+256m^3n^3-64m^4n^3+72mn^4-128m^2n^4 \\
&\quad -48m^3n^4+32m^4n^4, \\
g &= -10+19m-12m^2-2n+22mn-8m^2n-12n^2-8mn^2+8m^2n^2, \\
h &= 11-22m+22n+15mn-20m^2n-10mn^2+20m^2n^2.
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\lambda_3 &= \frac{(n-m)f}{7(m-1)(2n-1)(m-n+2mn)gh}, \\
f &= -34m^3+19m^4+68m^2n-38m^3n-22mn^2-185m^2n^2+302m^3n^2 \\
&\quad -80m^4n^2 \\
&\quad -12n^3+204mn^3-302m^2n^3+80m^3n^3-36n^4+100mn^4-40m^2n^4 \\
&\quad -40m^3n^4+16m^4n^4, \\
g &= -7m^2+7mn-6n^2-4mn^2+4m^2n^2, \\
h &= -22m-5m^2+22n+5mn+10n^2-30mn^2+20m^2n^2.
\end{aligned} \tag{4.4}$$

By Theorem 8.1 of [26], the first two intersection points occur at degenerate values of c. By replacing the parameter c with the levels k and ℓ, we see that the remaining intersection points yield the nontrivial isomorphisms in Theorem 4.1. Moreover, by Corollary 8.2 of [26], these are the only such isomorphisms except possibly at the values of k, ℓ excluded above.

Finally, suppose that k is a pole of the function $\lambda_n(k)$ given by (3.3). It is not difficult to check that the corresponding values of ℓ for which $c_n(k) = c_m(\ell)$, are not poles of $\lambda_m(\ell)$. As above, $c_n(k)$ and $\lambda_n(k)$ are given by (3.3), and $c_m(\ell)$ and $\lambda_m(\ell)$ are given by Equation (B.1) of [26]. It follows that there are no additional coincidences at the excluded points. $\square$

Next, we classify the coincidences between $\mathcal{D}_k(n)$ and $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$.

Theorem 4.2. *For $n \geq 1$ and $m \geq 2$, aside from the critical levels $k = -2n+2$ and $k = -2n+1$, and the degenerate cases $c = \frac{1}{2}, -24$, all isomorphisms $\mathcal{D}_k(n) \cong \mathcal{W}_\ell(\mathfrak{so}_{2m+1})$ appear on the following list:*

(1) $k = -(2n-2) + \frac{1}{2}(2n+2m-1), \qquad \ell = -(2m-1) + \frac{2m+2n-1}{2m+2n+1},$

(2) $k = -(2n-2) + \frac{2n-2m-1}{2m+2}, \qquad \ell = -(2m-1) + \frac{2m-2n+1}{2m+2},$

(3) $k = -(2n-2) - \frac{n}{m}, \qquad \ell = -(2m-1) + \frac{m-n}{m},$

(4) $k = -(2n-2) - \frac{2(n-1)}{2m-1}, \qquad \ell = -(2m-1) + \frac{2m-1}{2m-2n+1}.$

(5) $k = -(2n-2) + \frac{2(n-m-1)}{2m+1}, \qquad \ell = -(2m-1) + \frac{2m+1}{2(m-n+1)}.$

Proof. The argument is the same as the proof of Theorem 4.1. First, $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$ is realized as the simple quotient of $\mathcal{W}^{\mathrm{ev},I_m}(c,\lambda)$ where the ideal $I_m \subseteq \mathbb{C}[c,\lambda]$ is parametrized explicitly by Equation (A.3) of [26]. The above isomorphisms all arise from the intersection points between the truncation curves $V(K_n)$ for $\mathcal{D}^k(n)$ and $V(I_m)$ for $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$. A calculation shows that there are exactly 7 intersection points: the degenerate points $(\frac{1}{2}, -\frac{2}{49})$ and $(-24, -\frac{1}{245})$, and the five nontrivial ones appearing above. One then has to rule out additional coincidences at the points where $\mathcal{D}_k(n)$ does not arise as a quotient of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$, namely, the poles of $\lambda_n(k)$. The details are straightforward and are left to the reader. $\square$

Finally, we classify all isomorphisms $\mathcal{D}_k(m) \cong \mathcal{D}_\ell(n)$ for $n \neq m$.

Theorem 4.3. *For $m, n \geq 1$ and $n \neq m$, aside from the degenerate cases $c = \frac{1}{2}, -24$ and poles of $c_n(k)$, $\lambda_n(k)$ and $c_m(k)$, $\lambda_m(k)$ the complete list of isomorphisms $\mathcal{D}_k(m) \cong \mathcal{D}_\ell(n)$ is the following:*

(1) $k = -(2m-2) + \frac{2(m-1)}{1+2n}, \qquad \ell = -(2n-2) - \frac{2m+2n-1}{2(m-1)},$

(2) $k = -(2m-2) - \frac{2m+2n-1}{2(n-1)}, \qquad \ell = -(2n-2) + \frac{2(n-1)}{1+2m}.$

The proof is similar to the proof of Theorem 4.1 and is omitted.

5. Some rational cosets

By composing the map $V^k(\mathfrak{so}_{2n}) \to V^k(\mathfrak{so}_{2n+1})$ with the quotient map $V^k(\mathfrak{so}_{2n+1}) \to L_k(\mathfrak{so}_{2n+1})$, we obtain an embedding

$$\tilde{V}^k(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n+1}),$$

where $\tilde{V}^k(\mathfrak{so}_{2n})$ denotes the quotient of $V^k(\mathfrak{so}_{2n})$ by the kernel $\mathcal{J}_k$ of the above composition. In general, it is a difficult and important problem to determine when $\mathcal{J}_k$ is the maximal proper graded ideal, or equivalently, when $\tilde{V}^k(\mathfrak{so}_{2n}) = L_k(\mathfrak{so}_{2n})$. In the case where k is an admissible level for $\widehat{\mathfrak{so}_{2n}}$, Lemma 2.1 of [7] would then imply that $\mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))$ is simple, and hence its orbifold $\mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}$ would be simple as well [21]. Additionally, Theorem 8.1 of [16] would imply that $\mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}$ coincides with the simple quotient $\mathcal{D}_k(n)$ of $\mathcal{D}^k(n)$. This is particularly interesting in the cases where $\mathcal{D}_k(n)$ is strongly rational.

We conclude by proving this for first family in Theorem 4.2. These are new examples of cosets of non-rational vertex algebras by admissible level affine vertex algebras, which are strongly rational.

Lemma 5.1. *For $n \geq 2$ and $m \geq 0$, we have an embedding of simple affine vertex algebras*

$$L_k(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n+1}), \qquad k = -(2n-2) + \frac{1}{2}(2n+2m-1).$$

Proof. We proceed by induction on m. In the case $m = 0$, we have $k = -n + \frac{3}{2}$, and it is well known that there exists a conformal embedding $L_k(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n+1})$, see e.g. Section 3 of [5]. Next, we assume the result for $m-1$, so that $k = -n + \frac{3}{2} + m - 1$. Recall that the rank $2n+1$ free fermion algebra $\mathcal{F}(2n+1)$ admits an action of $L_1(\mathfrak{so}_{2n+1})$, as well as an action of $L_1(\mathfrak{so}_{2n})$ via the embedding $L_1(\mathfrak{so}_{2n}) \hookrightarrow L_1(\mathfrak{so}_{2n+1})$. The image of $L_1(\mathfrak{so}_{2n})$ lies in the subalgebra $\mathcal{F}(2n) \subseteq \mathcal{F}(2n+1)$.

Since k is admissible for $\mathfrak{so}_{2n+1}$, it is known [25] that we have a diagonal embedding of simple affine vertex algebras

$$L_{k+1}(\mathfrak{so}_{2n+1}) \hookrightarrow L_k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n+1). \tag{5.1}$$

By induction, we have the map $L_k(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n+1})$. Then we have an embedding

$$L_{k+1}(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n}) \otimes \mathcal{F}(n) \hookrightarrow L_k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n+1), \tag{5.2}$$

where $\mathcal{F}(2n) \hookrightarrow \mathcal{F}(2n+1)$ is the isomorphism onto the first $2n$ copies. Since the image of (5.2) lies in the image of (5.1), it follows that $L_{k+1}(\mathfrak{so}_{2n})$ embeds in $L_{k+1}(\mathfrak{so}_{2n+1})$. $\square$

This has the following immediate consequence.

Corollary 5.2. *For $n \geq 2$, $m \geq 0$, and $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$, we have an isomorphism*

$$\mathit{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2} \cong \mathcal{W}_\ell(\mathfrak{so}_{2m+1}), \qquad \ell = -(2m-1) + \frac{2m+2n-1}{2m+2n+1}.$$

In particular, $\mathit{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}$ is strongly rational.

Proof. This follows from Theorem 4.2 together with the fact that $\mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}$ is simple, and the map $\mathcal{D}^k(n) \to \mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}$ is surjective. □

Recall that the category of ordinary modules of an affine vertex algebra at admissible level is semisimple [3] and a vertex tensor category [12]. Conjecturally, this category is fusion [12] and this has been proven for simply-laced Lie algebras [9]. For type $\mathfrak{so}_{2n+1}$ and level $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$ this conjecture is also true. First, $\mathrm{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))$ is a simple current extension, call it $\mathcal{V}_\ell(\mathfrak{so}_{2m+1})$, of $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$ and thus rational as well [27]. It follows that $L_k(\mathfrak{so}_{2n+1})$ is a simple $\mathbb{Z}$-graded extension of $L_k(\mathfrak{so}_{2n}) \otimes \mathcal{V}_\ell(\mathfrak{so}_{2m+1})$ in a rigid vertex tensor category $\mathcal{C}$ of $L_k(\mathfrak{so}_{2n}) \otimes \mathcal{V}_\ell(\mathfrak{so}_{2m+1})$-modules, namely the Deligne product of the categories of ordinary $L_k(\mathfrak{so}_{2n})$-modules and $\mathcal{V}_\ell(\mathfrak{so}_{2m+1})$-modules. Every ordinary module for $L_k(\mathfrak{so}_{2n+1})$ must be an object in this category $\mathcal{C}$. This means that as a braided tensor category the category of ordinary modules of $L_k(\mathfrak{so}_{2n+1})$ is equivalent to the category of local modules for $L_k(\mathfrak{so}_{2n+1})$ viewed as an algebra object in $\mathcal{C}$ [13,23]. All assumptions of Theorem 5.12 of [14] are satisfied (with $U = \mathcal{V}_\ell(\mathfrak{so}_{2m+1})$ and $V = L_k(\mathfrak{so}_{2n})$) and so

Corollary 5.3. *The category of ordinary modules of $L_k(\mathfrak{so}_{2n+1})$ at level $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$ is fusion.*

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